Annals of Operations Research
https://doi.org/10.1007/s10479-025-06719-0

ORIGINAL RESEARCH

A position value for multigraphs

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Received: 6 September 2023 / Accepted: 16 June 2025

Abstract
Multicommunication is one of the methods at our disposal to address the significant challenges of reliability and latency in today's communications. This paper presents a game-theoretical analysis of multicommunication, where we assume that players in a TU-game are restricted in their communication by a multigraph, whose links may or may not have limitations on capacity, frequency, or transmission speed. We generalize the classical position value defined for graph games to assess the importance of different communication providers. Extending to this context the classical properties of component efficiency and balanced link contributions we can characterize the new value. For a wide variety of multicommunications including but not limited to all those where the connections are in series or parallel, the value can be characterized by substituting balanced link contributions for the more restrictive balanced link pack contributions. In the case of limitations in capacity, parallel characterizations are obtained in which component efficiency must be substituted by a certain inefficiency that takes these limitations into account.

Keywords Game theory · Multinetwork · Multicommunication situations · Position value

1 Introduction

Applications that come with a promising future, such as intelligent transport, industrial automation, communication between vehicles or tele-surgery, pose great challenges in terms of reliability and latency in communication systems. Current communication capabilities do

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Published online: 15 July 2025

not meet these predictable demands for latency and reliability. Different approaches have been proposed to achieve this goal. One of the most promising is the so-called multicommunication. Roughly speaking, multicommunication can be defined as the possibility of establishing simultaneous connection between two points with several paths. In principle, paths do not have to be used all at once and information can be sent by each path independently of other paths.

It seems interesting to analyze the importance of link owners in multicommunication and how this relevance is affected by different forms or restrictions in communication (in series, in parallel, etc).

In this paper, we introduce a game-theoretic analysis of multicommunication walking in the footsteps of the rich existing literature on communication situations (i.e., TU-games with cooperation restricted by a graph or graph games) which already has a long tradition.

From the seminal work of Myerson (1977), we can assume that nodes in a network are simultaneously players in a TU-game that models economic, social interests or information transmission needs that motivate their interactions. Myerson defined a new TU-game, the graph-restricted game, in which the value of a coalition is obtained by adding the worth of its maximally connected subcoalitions. As the Shapley value is one of the most prominent allocation rules for TU-games, Myerson assigned to players their Shapley value payoffs in the graph restricted game. This rule is now known as the Myerson value.

Meessen (1988) and Borm et al. (1992) introduced a new perspective in the analysis of TU-games with cooperation restricted by networks. They define the link game, a new TU-game in which the players are the links of the graph, the coalitions are the subgraphs, and the worth of a subgraph is obtained from the value of the grand coalition in the Myerson restricted game to this subgraph. Again the Shapley value is used to assign value to the links in this game. Then, a new rule, the position value, is obtained for players by equally apportioning the value of each link between the two incident nodes.

As mentioned, both the Myerson value and the position value are based on the Shapley value, possibly the most relevant allocation rule for TU games. The Myerson value is even a generalization of the Shapley value. Algaba et al. (2019) provide an updated review of the multiple scenarios in which the Shapley value has inspired both theoretical and applied works.

From its introduction the position value has received a lot of attention: Algaba et al. (2000) introduced the position value for union stable systems; Algaba et al. (2004) defined the position value for communications structures; Gómez et al. (2004), Casajus (2007), Kongo (2010) and Wang and Shan (2021) related the position value to the Myerson value in different settings; Slikker (2005) obtained the first characterization in the set of all communication situations; Ghintran et al. (2012) extended it to probabilistic networks; Ghintran (2013) introduced a weighted version; Cesari and Ferrari (2016) and Li and Shan (2019) particularized it for some special geometries of graphs; and Gavilán et al. (2022) introduced a family of position values for TU-games with cooperation restricted by digraphs. Algaba and Saavedra-Nieves (2024) introduce the position value as a centrality measure for both players and links in a network. They also estimate (a crucial problem) the position value based on statistical sampling theory, which allows them to obtain centralities in various relevant networks.

In this paper we extend the position value for the case of games with multicommunication restrictions, which can have (or not) weights representing capacities, frequencies or transmission speed (latency) in the links.

When multiple paths are present, it is necessary to specify how the information is transmitted through the paths. In this paper we will frequently refer to two special and frequent connectivity types: links connected in parallel (load balancing if the objective of multicommunication is the transmission of information packets) and links in series (packet splitting).

Finally we will characterize the position value for multigraphs in terms of the extension to this setting of the axioms of component efficiency (Myerson, 1977) and balanced link contributions (Slikker, 2005). For specific relevant cases such as series connection or parallel connection, but not limited to these, the value can be characterized replacing the property of balanced link contributions with balanced link pack contributions. These characterizations also apply to the broader class of weighted multicommunication situations, replacing efficiency in components with $\boldsymbol{\lambda}$-efficiency in components (a certain inefficiency) as the limitations in the capacity of the links generate inefficiency.

This paper is organized as follows: After some preliminaries on cooperative TU-games, graphs and communication situations, in Sect. 3 we introduce the position value for multicommunication situations. In Sect. 4 we characterize this value. Section 5 is devoted to the introduction of the weighted position value and its characterization. The paper concludes with a final remarks and conclusions section, followed by the references.

2 Preliminaries

2.1 Cooperative TU-games

An n-person TU-game (cooperative game with transferable utility) is an ordered pair (N, v) where $N = \{1, 2, ..., n\}$ is the set of players and v, the characteristic function, is a map $v : 2^N \to \mathbb{R}$, satisfying $v(\emptyset) = 0$. Each $S \in 2^N$ is a possible coalition and $v(S)$ represents its worth. We will note $s = |S|$, $t = |T|$, etc. A TU-game is zero-normalized if $v(\{i\}) = 0$ for all $i \in N$.

We will denote by G^N the vector space of all n-person TU-games. The null vector in G^N is $(N, \mathbf{0})$ with $\mathbf{0}(S) = 0$ for all $S \subseteq N$. The game $(N, \mathbf{1})$ is given by $\mathbf{1}(S) = 1$ for all $\emptyset \neq S \subseteq N$, and, of course, $\mathbf{1}(\emptyset) = 0$.

The family $\{(N, u_S) | \emptyset \neq S \subseteq N\}$, with

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}.$$

is the class of unanimity games that forms a basis of G^N. The coordinates of $(N, v) \in G^N$ in such a basis $\{\Delta_v(S)\}_{\emptyset \neq S \subseteq N}$ are known as the Harsanyi dividends (Harsanyi, 1959), and can be calculated as:

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T). \tag{1}$$

G^N also has an algebraic ring structure in which:

i) The sum of two TU-games (*N*, *v*) and (*N*, *w*), denoted by $(N, v+w)$, has characteristic function given by, $(v+w)(S) = v(S) + w(S)$, for all $S \subseteq N$.
ii) The product of two TU-games (*N*, *v*) and (*N*, *w*), denoted by (*N*, *vw*), has characteristic function given by, $(vw)(S) = v(S)w(S)$, for all $S \subseteq N$. For games (N, v_1), ..., (N, v_k) $\in G^N$ we will denote the product of (N, v_1), ..., (N, v_k) by $(N, \prod_{i=1}^{k} v_i)$. And thus, for $S \subseteq N$, $(\prod_{i=1}^{k} v_i)(S) = \prod_{i=1}^{k} v_i(S)$.

An allocation rule or point solution on G^N assigns a specific payoff to each player in the game. The Shapley value (Shapley, 1953), *Sh*, is a well-known allocation rule that can be obtained, for each $i \in N$, as:

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!}[v(S \cup \{i\}) - v(S)] = \sum_{S \subseteq N : i \in S} \frac{\Delta_v(S)}{s}.$$

2.2 Graphs

A *network* (*graph*) is a pair (N, γ) in which $N = \{1, 2, \ldots, n\}$ is the set of nodes and $\gamma \subseteq \gamma_N = \big\{\{i, j\} | i, j \in N, i \neq j\big\}$. Each $l = \{i, j\} \in \gamma$ is a link representing a direct communication between *i* and *j*.

We will say that $i, j \in N$ are *connected* in γ if there exists a sequence of nodes (intermediaries) $i_1, i_2, \ldots i_k$ with $i_1 = i$ and $i_k = j$ such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \ldots, k-1$. A set $S \subseteq N$ is connected in γ if every pair of nodes in *S* is connected, with all the intermediaries in *S*. We will assume that *S* is connected whenever $s = 1$.

A maximal connected subset *C* in (N, γ) is a connected component, i.e., *C* is connected in the graph and, for all $C' \subseteq N$, if $C \subsetneq C'$ then, C' is not connected. We will denote by N/γ the partition of *N* in connected components induced by (N, γ). For $\emptyset \neq S \subseteq N$, S/γ will denote the partition of *S* in connected components induced by $(S, \gamma_{|_S})$, i.e., induced by the restriction of (N, γ) to *S*.

For $i \in N$, γ_i is the set of all links incident on *i*, i.e., $\gamma_i = \{l \in \gamma | i \in l\}$. Given (N, γ) and $l \in \gamma$, $(N, \gamma \setminus \{l\})$ is the subgraph of (N, γ) obtained when the link *l* is severed.

2.3 Communication situations

A communication situation is modeled by means of a triple (N, v, γ) where (*N*, *v*) is a TU-game and (N, γ) is a network. We will denote by $\mathcal{CS}^N$ the set of all communication situations with *N* as the set of nodes/players. $\mathcal{CS}_0^N$ is the subset of those communication situations in $\mathcal{CS}^N$ for which the game is zero-normalized.

For communication situations (N, v, γ) with a zero-normalized game, Meessen (1988) and Borm et al. (1992) introduced the link game[1], $(\gamma, r_\gamma^v) \in G^\gamma$(the set of TU games on γ where players are links and coalitions are subgraphs) in which the characteristic function is given by[2]:

$$r_\gamma^v(\eta) = v^\eta(N) = \sum_{C \in N/\eta} v(C) \text{ for } \text{ all } \eta \subseteq \gamma$$

and they also introduced the position value, π, an allocation rule for players in such a communication situations, defined by:

$$\pi_i(N, v, \gamma) = \sum_{l \in \gamma_i} \frac{1}{2} Sh_l(\gamma, r_\gamma^v),\ i \in N.$$

Slikker (2005) characterized the position value on $\mathcal{CS}_0^N$ in terms of component efficiency and balanced link contributions. An allocation rule ψ on $\mathcal{CS}^N$ satisfies *component efficiency* (Myerson, 1977) if, for all $(N, v, \gamma) \in \mathcal{CS}^N$ and all $C \in N/\gamma$,

$$\sum_{i \in C} \psi_i(N, v, \gamma) = v(C).$$

An allocation rule satisfies balanced link contributions (Slikker, 2005) if for all players $i, j \in N$, the sum of the marginal contributions of the links of player *j* to the payoff of *i* is the same as the sum of the marginal contributions of the links of *i* to the payoff of *j*. Formally, an allocation rule ψ, on $\mathcal{CS}^N$ satisfies balanced link contributions, if given $(N, v, \gamma) \in \mathcal{CS}^N$, and $i, j \in N$, it holds that

$$\sum_{l \in \gamma_j} [\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma \setminus \{l\})] = \sum_{l \in \gamma_i} [\psi_j(N, v, \gamma) - \psi_j(N, v, \gamma \setminus \{l\})].$$

3 The position value for multicommunication situations

3.1 Multigraphs and connectedness

A multigraph, multinetwork or a pseudo-graph is a graph in which the existence of multiple links is admitted, i.e., links that have the same extreme nodes.

Formally, a multigraph is a pair (N, γ) in which N is the set of nodes (vertices) and γ is a multiset of links (a modification of the concept of set in which it is allowed for multiple instances for each element), $\gamma = \{l_1, l_2, ..., l_{|\gamma|}\}$($|\gamma|$ be the cardinality of γ) with

[1] The reader may note that, in the case of the link game, there is a deviation from the classical notation for TU-games since the set of players is the set of links that are usually not labeled with natural numbers.

[2] As each characteristic function gives zero to the empty coalition, $r_\gamma^v(\emptyset) = \sum_{i \in N} v(\{i\})$ must be zero. For this it is sufficient that the game is zero-normalized.

$l_i = \{k_1, k_2\}$, $k_1, k_2 \in N$, $k_1 \neq k_2$, $i = 1, ..., |\gamma|$. We will assume that the multigraph has no loops, i.e., links in which the two vertices coincide. For $i, j \in N$, $i \neq j$, we will denote $\gamma_{ij} = \{l \in \gamma | l = \{i, j\}\}$ the set of links with vertices *i* and *j*.

When multiple communication links are present, we need to know the type of the restrictions in the connection considered. In alternative terms, connectedness methods are required to determine the allocation of information packets to specific transmission paths. In a graph it is usually assumed that direct communication between two nodes is equivalent to the existence of an arc of communication between them. In a multigraph, different forms of direct communication between two nodes can be considered based on the different multilinks between them. Two direct connectedness methods that immediately come to mind are the connection in series and in parallel:

(1) We will say that *i* and *j* are direct connected in series in the multigraph (N, γ) if $|\gamma_{ij}| \neq 0$ and all the links in γ_{ij} are needed to communicate them. We can think of a segmented encryption in which all channels are necessary for the information to reach its destination.
(2) We will say that $i, j \in N$ are direct connected in parallel in the multigraph (N, γ) if $\gamma_{ij} \neq \emptyset$ and it is necessary and sufficient at least one of these links to communicate them. You can think of an international telephonic communication in which it is needed at least one of the operators in the different countries to communicate two persons.

But we can have more general situations in which series and parallel connections are combined.

In the following definition we introduce the concept of minimal connection multigraph for each pair of nodes, having a set of links between them, as the minimum subset of these links allowing them to communicate directly with each other.

Definition 3.1 Given a multigraph (N, γ) and $i, j \in N$, $i \neq j$ with $\gamma_{ij} \neq \emptyset$, we will say that $\eta \subseteq \gamma_{ij}$ is a minimal connection multigraph of *i* and *j* if $(\{i, j\}, \eta)$ is connected, i.e., *i* and *j* can communicate using the links in η, but *i* and *j* can not communicate using only the links in δ, for all $\delta \subsetneq \eta$.

For a multigraph (N, γ) and $i, j \in N$, $i \neq j$ with $\gamma_{ij} \neq \emptyset$[3], the direct connectedness set of *i* and *j*, denoted by $\mathcal{DC}_{ij}(N, \gamma)$, is defined as

$$\mathcal{DC}_{ij}(N, \gamma) = \{\eta \subseteq \gamma_{ij} | \eta \text{ is a minimal connection multigraph of i and j }\}.$$

Remark 3.1 *If* (N, γ) *is a graph, and* $\{i, j\} \in \gamma$ *then* $\mathcal{DC}_{ij} = \{\{i, j\}\}$.

To clarify this definition let us consider the following examples.

Example 3.1 *Consider the multigraph* (N, γ) *with* $N = \{1, 2, 3\}$ *and* $\gamma = \{a, b, c\}$ *as represented in Fig. 1*

[3] Clearly, this definition can be extended to pairs of nodes *i*, *j* such that $\gamma_{ij} = \emptyset$, simply by setting $\mathcal{DC}_{ij}(N, \gamma) = \emptyset$.

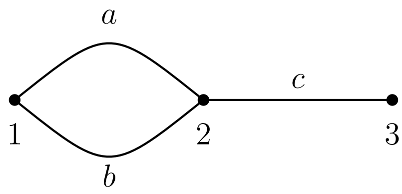


Fig. 1 Multigraph of the Example 3.1

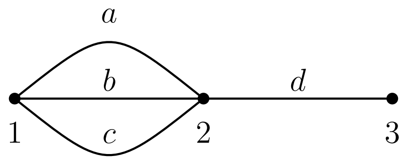


Fig. 2 Multigraph of the Example 3.3

Suppose that 1 and 2 are connected in series. As mentioned before, this situation can model the case in which the messages between 1 and 2 are sent encrypted in two different packages. Players 1 and 2 have only one minimal connection multigraph which is $\{a, b\}$, whereas 2 and 3 have $\{c\}$ as the unique minimal connection multigraph. Then, $\mathcal{DC}_{12} = \{\{a, b\}\}$ and $\mathcal{DC}_{23} = \{\{c\}\}$.

Example 3.2 *Consider again the multigraph in the Example 3.1 and suppose that 1 and 2 are now connected in parallel. Players 1 and 2 have two minimal connection multigraphs which are $\{a\}$ and $\{b\}$. In this case, $\mathcal{DC}_{12} = \{\{a\}, \{b\}\}$ and $\mathcal{DC}_{23} = \{\{c\}\}$.*

Example 3.3 *Consider now the multigraph (N, γ) with $N = \{1, 2, 3\}$ and $\gamma = \{a, b, c, d\}$ as represented in Fig. 2.*

Let us suppose that 1 and 2 need links a and b simultaneously, or link c, to communicate. In other words, a and b are in series, while c is in parallel with them. Players 1 and 2 have two minimal connection multigraphs which are $\{a, b\}$ and $\{c\}$. And thus, $\mathcal{DC}_{12} = \{\{a, b\}, \{c\}\}$ and $\mathcal{DC}_{23} = \{\{d\}\}$.

In the following definition we introduce the concept of direct connectivity set for a multigraph which provides us with all different forms of direct communication possible between the pairs of nodes.

Definition 3.2 Given a multigraph (N, γ), the direct connectedness set of (N, γ), denoted by $\mathcal{DC}(N, \gamma)$, is the family of minimal connection multigraphs for the pairs of nodes $i, j \in N$ such that $\gamma_{ij} \neq \emptyset$, i.e.,

$$\mathcal{DC}(N, \gamma) = \bigcup_{i,j \in N : \gamma_{ij} \neq \emptyset} \mathcal{DC}_{ij}(N, \gamma).$$ [4]

4

Remark 3.2 *If (N, γ) is a graph then, $\mathcal{DC}(N, \gamma) = \gamma$.*

[4] For consistency, if $\eta \subseteq \gamma$, $\mathcal{DC}(N, \eta) = \{\delta \in \mathcal{DC}(N, \gamma) | \delta \subseteq \eta\}$.

Example3.4 *Forthegraph*(N,γ)*intheExample3.1*$\mathcal{DC}(N,\gamma)=\{\{a,b\},\{c\}\}$.*IntheExample 3.2,* $\mathcal{DC}(N,\gamma)=\{\{a\},\{b\},\{c\}\}$, *and in the Example 3.3* $\mathcal{DC}(N,\gamma)=\{\{a,b\},\{c\},\{d\}\}$.

At this point, we will assume that every multigraph has associated (exogenously) a direct connectivity set. This allows us to introduce the definition of connectivity (not necessarily direct) in a multigraph that clearly rests on the idea of direct communication between nodes. Therefore, from now on, we will include in the notation of the multigraph the direct connectedness set exogenously associated to it. And thus, a multigraph will be denoted by $(N,\gamma,\mathcal{DC}(N,\gamma))$, and when there is no possibility of confusion, to lighten the notation, we will simply write $(N,\gamma,\mathcal{DC})$.

Definition 3.3 Given a multigraph $(N,\gamma,\mathcal{DC})$ and $i,j\in N$ we will say, that *i* and *j* are connected in (N,γ) if there exists a sequence $\eta_{i_1,i_2},\ldots,\eta_{i_{r-1},i_r}$ with $i_1=i$ and $i_r=j$ such that $\eta_{i_k,i_{k+1}}\in\mathcal{DC}(N,\gamma)$ for $k=1,\ldots,r-1$. We will admit that singletons in *N* are connected.

Definition 3.4 Given a multigraph $(N,\gamma,\mathcal{DC})$ we will say that C is a connected component in $(N,\gamma,\mathcal{DC})$ if C is a maximally connected set of $(N,\gamma,\mathcal{DC})$. We will denote by $N/(\gamma,\mathcal{DC}(N,\gamma))$ the set of the connected components of N in the multigraph $(N,\gamma,\mathcal{DC}(N,\gamma))$. And where there is no possibility of confusion we will simply denote this set by $N/(\gamma,\mathcal{DC})$. Similarly, for subsets $\eta\subseteq\gamma$ the notation $N/(\eta,\mathcal{DC})$ always will refer to $N/(\eta,\mathcal{DC}(N,\eta))$.

We will say that $(N,\gamma,\mathcal{DC})$ is a connected multigraph if it has only one connected component, i.e., $|N/(\gamma,\mathcal{DC})|=1$.

The Definition 3.1 can be extended to subsets of *N* with more than two players.

Definition 3.5 Given a multigraph $(N,\gamma,\mathcal{DC})$ and $S\subseteq N$, we will say that the set of links $\eta\subseteq\gamma$ is a minimal connection multigraph of *S* in $(N,\gamma,\mathcal{DC})$ if for all $i,j\in S$, *i* and *j* are connected in $(N,\eta,\mathcal{DC}(N,\eta))$ and for all $\delta\subsetneq\eta$ there exists $i,j\in S$ such that *i* and *j* are not connected in $(N,\delta,\mathcal{DC}(N,\delta))$. We will denote with $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$ the (occasionally empty) family of the minimal connection multigraph sets of *S* in $(N,\gamma,\mathcal{DC})$.

In this paper we will use the concept of set of link packs for each player in a multigraph. Each pack represents a communication possibility from one player to another, where all links in the pack must be used at the same time. This concept will permit us to introduce a property that it is useful to characterize the value that we will propose for TU-games with communication restricted by multigraphs having some particular direct connectedness sets.

Definition 3.6 Given $(N,\gamma,\mathcal{DC})$ a multigraph and $i\in N$, the set of link packs of *i* is given by

$$\mathcal{LP}_i=\{\eta\in\mathcal{DC}(N,\gamma)\text{ and }\eta\subseteq\gamma_{ij},j\in N,j\neq i\}.$$

Each element of this set is a link pack.

Example 3.5 *For the multigraph in the Example 3.1 with the direct connectivity set* $\mathcal{DC}(N,\gamma)=\{\{a,b\},\{c\}\}$*, the family of the minimal connection multigraphs for different* $S\subseteq N$ *is*

$$\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})=\begin{cases}\emptyset & for S=\emptyset or S=\{i\} i\in N,\\ \{\{a,b\}\} & for S=\{1,2\},\\ \{\{c\}\} & for S=\{2,3\},\\ \{\{a,b,c\}\} & for S=\{1,3\} or S=N.\end{cases} \tag{2}$$

The set of link packs of player 1 is $\mathcal{LP}_1=\{\{a,b\}\}$. For player 2 and 3 we have $\mathcal{LP}_2=\{\{a,b\},\{c\}\}$ and $\mathcal{LP}_3=\{\{c\}\}$.

For $(N,\gamma,\mathcal{DC})$ *with* (N,γ) *as in the Example 3.2 and* $\mathcal{DC}=\{\{a\},\{b\},\{c\}\}$ *the family of the minimal connection multigraphs for different* $S\subseteq N$ *is*

$$\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})=\begin{cases}\emptyset & for S=\emptyset or S=\{i\} i\in N,\\ \{\{a\},\{b\}\} & for S=\{1,2\},\\ \{\{c\}\} & for S=\{2,3\},\\ \{\{a,c\},\{b,c\}\} & for S=\{1,3\} or S=N.\end{cases} \tag{3}$$

Here, $\mathcal{LP}_1=\{\{a\},\{b\}\}$, $\mathcal{LP}_2=\{\{a\},\{b\},\{c\}\}$ and $\mathcal{LP}_3=\{\{c\}\}$.

For $(N,\gamma,\mathcal{DC})$ *with* (N,γ) *as in the Example 3.3 and* $\mathcal{DC}=\{\{a,b\},\{c\},\{d\}\}$*, the family of the minimal connection multigraphs for different* $S\subseteq N$ *is*

$$\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})=\begin{cases}\emptyset & for S=\emptyset or S=\{i\} i\in N,\\ \{\{a,b\},\{c\}\} & for S=\{1,2\},\\ \{\{d\}\} & for S=\{2,3\},\\ \{\{a,b,d\},\{c,d\}\} & for S=\{1,3\} or S=N.\end{cases} \tag{4}$$

In this case $\mathcal{LP}_1=\{\{a,b\},\{c\}\}$, $\mathcal{LP}_2=\{\{a,b\},\{c\},\{d\}\}$ and $\mathcal{LP}_3=\{\{d\}\}$.

3.2 Definition of the value

A multicommunication situation or a pseudo-communication situation is a 4-tuple $(N,v,\gamma,\mathcal{DC})$ in which (N, v) is a TU-game and $(N,\gamma,\mathcal{DC})$ is a multinetwork with a direct connectedness set. Then, in a multicommunication situation, the connection among players is restricted by a multigraph. The set of all multicommunication situations with N as the set of nodes/players is denoted by $\mathcal{MCS}^N$. $\mathcal{MCS}^N_0$ will denote the subset of those elements in $\mathcal{MCS}^N$ in which the game is zero-normalized.

In this paper we are interested in allocation rules on $\mathcal{MCS}^N_0$, i.e., functions, $\varphi:\mathcal{MCS}^N_0\to\mathbb{R}^n$ assigning to every $(N,v,\gamma,\mathcal{DC})\in\mathcal{MCS}^N_0$ a vector $\varphi(N,v,\gamma,\mathcal{DC})$ containing the payoffs for the players in the multicommunication situation.

Let $(N,v,\gamma,\mathcal{DC})$ be a multicommunication situation in $\mathcal{MCS}^N_0$. Following Meessen (1988) and Borm et al. (1992), we can define the multilink game $(\gamma,r^v_{\gamma,\mathcal{DC}})$ with

$$r^v_{\gamma,\mathcal{DC}}(\eta)=v^\eta(N)=\sum_{C\in N/(\eta,\mathcal{DC})} v(C) \text{ for all } \eta\subseteq\gamma, \tag{5}$$

where $N/(\eta, \mathcal{DC})$ is the set of all the connected components of N in the multigraph $(N, \eta, \mathcal{DC}(N, \eta))$. As mentioned, connectedness and connected components in a multigraph are crucial concepts in this paper. This is so as they depend on the type of connection considered, which is also another relevant element in this setting. The notation $N/(\eta, \mathcal{DC}(N, \eta))$, lightened to $N/(\eta, \mathcal{DC})$, reflects the fact that the connected components in $(N, \eta, \mathcal{DC}(N, \eta))$ depend not only on the links in η but also on the type of direct communications established between the nodes.

As a consequence, fixed γ, -the multiset of links-, the expression (5) generates different multilink games which depend on the connectedness used in the transmission of the information.

The position value for multicommunication situations in $\mathcal{MCS}_0^N$ is given by

$$\pi_i(N, v, \gamma, \mathcal{DC}) = \sum_{l \in \gamma_i} \frac{1}{2} Sh_l(\gamma, r^v_{\gamma,\mathcal{DC}}),\ i \in N.$$

To clarify the previous definitions consider the following examples.

Example 3.6 *Consider the multicommunciation situation $(N, v, \gamma, \mathcal{DC})$ with $N = \{1, 2, 3\}$, (N, v) the messages game with characteristic function given by*

$$v(S) = \begin{cases} \frac{s(s-1)}{2} & if s \geq 2 \\ 0 & otherwise, \end{cases}$$

and $(N, \gamma, \mathcal{DC})$ as given in Example 3.1 (connection in series).

Then,

$$N/(\eta, \mathcal{DC}) = \begin{cases} \big\{\{1\}, \{2\}, \{3\}\big\}, & if \eta = \{\emptyset\}, \eta = \{a\} or \eta = \{b\}, \\ \big\{\{1\}, \{2, 3\}\big\}, & if \eta = \{c\}, \eta = \{a, c\} or \eta = \{b, c\}, \\ \big\{\{1, 2\}, \{3\}\big\}, & if \eta = \{a, b\}, \\ \big\{\{1, 2, 3\}\big\}, & if \eta = \{a, b, c\}, \end{cases}$$

and

$$r^v_{\gamma,\mathcal{DC}}(\eta) = \begin{cases} 0, & if \eta = \{\emptyset\}, \eta = \{a\} or \eta = \{b\}, \\ 1, & if \eta = \{c\}, \eta = \{a, c\}, \eta = \{b, c\} or \eta = \{a, b\} \\ 3, & if \eta = \{a, b, c\}, \end{cases}$$

which in terms of the unanimity games can be written, using (1), as

$$r^v_{\gamma,\mathcal{DC}} = u_{\{c\}} + u_{\{a,b\}} + u_{\{a,b,c\}}.$$

The position value for this connection in series is

$$\pi(N, v, \gamma, \mathcal{DC}) = \left(\frac{5}{6}, \frac{9}{6}, \frac{4}{6}\right)$$

Example 3.7 *Consider the multicommunication situation* $(N, v, \gamma, \mathcal{DC})$ *with* $N = \{1, 2, 3\}$, *(N, v) the messages game and* $(N, \gamma, \mathcal{DC})$ *as in the Example 3.2 (connection in parallel). Then,*

$$N/(\eta, \mathcal{DC}) = \begin{cases} \{\{1\},\{2\},\{3\}\}, & if \eta = \emptyset, \\ \{\{1,2\},\{3\}\}, & if \eta = \{a\}, \eta = \{b\} or \eta = \{a,b\}, \\ \{\{1\},\{2,3\}\}, & if \eta = \{c\}, \\ \{N\}, & if \eta = \{a,c\}, \eta = \{b,c\} or \eta = \{a,b,c\}, \end{cases}$$

and

$$r^v_{\gamma,\mathcal{DC}}(\eta) = \begin{cases} 0, & if \eta = \emptyset, \\ 1, & if \eta = \{a\}, \eta = \{b\}, \eta = \{c\} or \eta = \{a,b\}, \\ 3, & if \eta = \{a,c\}, \eta = \{b,c\} or \eta = \{a,b,c\}. \end{cases}$$

Then,

$$r^v_{\gamma,\mathcal{DC}} = u_{\{a\}} + u_{\{b\}} + u_{\{c\}} - u_{\{a,b\}} + u_{\{a,c\}} + u_{\{b,c\}} - u_{\{a,b,c\}}.$$

The position value for this connection in parallel is[5]

$$\pi(N, v, \gamma, \mathcal{DC}) = \left(\frac{4}{6}, \frac{9}{6}, \frac{5}{6}\right).$$

Example 3.8 *Consider now the multicommunication situation* $(N, v, \gamma, \mathcal{DC})$ *with* $N = \{1, 2, 3\}$, *(N, v) the messages game* $(N, \gamma, \mathcal{DC})$ *as in the Example 3.3 (mixed connection). In this case,*

$$N/(\eta, \mathcal{DC}) = \begin{cases} \{\{1\},\{2\},\{3\}\}, & if \eta = \emptyset, \eta = \{a\} or \eta = \{b\}, \\ \{\{1,2\},\{3\}\}, & if \eta = \{c\}, \eta = \{a,b\}, \eta = \{a,c\}, \\ & \eta = \{b,c\} or \eta = \{a,b,c\}, \\ \{\{1\},\{2,3\}\}, & if \eta = \{d\}, \eta = \{a,d\} or \eta = \{b,d\}, \\ \{1,2,3\}, & if \eta = \{c,d\}, \eta = \{a,b,d\}, \eta = \{a,c,d\}, \\ & \eta = \{b,c,d\} or \eta = \{a,b,c,d\}, \end{cases}$$

and

[5] As can be seen, the competition between the parallel links weakens them. Think again of two telephone companies. While in the series connection the value of each of the links *a* and *b* is $\frac{5}{6}$, in the parallel connection it becomes $\frac{4}{6}$. This means that player 1, in particular, sees his payoff in parallel reduced compared to his payoff in series. Efficiency makes link *c* more valuable in the parallel connection than in the series connection, which benefits player 3. Player 2 compensates what he loses from links *a* and *b* with what he gains from *c*.

$$r^{v}_{\gamma,\mathcal{DC}}(\eta)=\begin{cases}0, & if \eta=\emptyset,\eta=\{a\} or \eta=\{b\},\\ 1, & if \eta=\{c\},\eta=\{d\},\eta=\{a,b\},\eta=\{a,c\},\eta=\{a,d\},\\ & \eta=\{b,c\},\eta=\{b,d\} or \eta=\{a,b,c\},\\ 3, & if \eta=\{c,d\},\eta=\{a,b,d\},\eta=\{a,c,d\},\\ & \eta=\{b,c,d\} or \eta=\{a,b,c,d\}.\end{cases}$$

Then,

$$r^{v}_{\gamma,\mathcal{DC}}=u_{\{c\}}+u_{\{d\}}+u_{\{a,b\}}+u_{\{c,d\}}-u_{\{a,b,c\}}+u_{\{a,b,d\}}-u_{\{a,b,c,d\}}.$$

The position value for this connection is

$$\pi(N,v,\gamma,\mathcal{DC})=\left(\frac{17}{24},\frac{36}{24},\frac{19}{24}\right).$$

3.3 The dividends of the multilink game

This section is devoted to obtain the Harsanyi dividends of the multilink games with regard to their decomposition through unanimity multilink games.

In the following proposition we obtain the expression, in terms of the unanimity basis, for the multilink game corresponding to u_S, $\emptyset\neq S\subseteq N$. This result is useful to determine the Harsanyi dividends of the multilink game.

Proposition 3.1 *Given the multicommunication situation* $(N,u_S,\gamma,\mathcal{DC})$ *with* $\emptyset\neq S\subseteq N$, *we have*

$$r^{u_S}_{\gamma,\mathcal{DC}}\equiv\mathbf{0} if \mathcal{MCMS}(S,N,\gamma,\mathcal{DC})=\emptyset$$

and

$$r^{u_S}_{\gamma,\mathcal{DC}}=\mathbf{1}-\prod_{i=1}^{r(S)}(\mathbf{1}-u_{\eta_i}) \tag{6}$$

where $u_{\eta_1},u_{\eta_2},...,u_{\eta_{r(S)}}$ are the characteristic functions of the unanimity games of the multigraphs in the set $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$ when the cardinal of this set is $r(S)\geq 1$.

Proof For $\emptyset\neq S\subseteq N$ such that $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})=\emptyset$ we have that $r^{u_S}_{\gamma,\mathcal{DC}}(\eta)=\sum_{C\in N/(\eta,\mathcal{DC})}u_S(C)=0$ as for all $\eta\subseteq\gamma$ and all $C\in N/(\eta,\mathcal{DC})$, it holds that $S\subsetneq C$, given that the restriction of $(N,\eta,\mathcal{DC}(N,\eta))$ to coalition S, $(S,\eta_{|S},\mathcal{DC}(S,\eta_{|S}))$, is not a connected multigraph.

To prove the equality (6) let us demonstrate that for $\emptyset \neq S \subseteq N$ with $\mathcal{MCMS}(S, N, \gamma, \mathcal{DC}) = \{\eta_1, ..., \eta_{r(S)}\}$, and $\eta \subseteq \gamma$ it holds that

$$r^{u_S}_{\gamma,\mathcal{DC}}(\eta) = \left[\mathbf{1} - \prod_{i=1}^{r(S)} (\mathbf{1} - u_{\eta_i})\right](\eta).$$

We have that $r^{u_S}_{\gamma,\mathcal{DC}}(\eta) = \sum_{C \in N/\eta} u_S(C) = 1$ if and only if S is contained in a connected component of $(N, \eta, \mathcal{DC})$. Otherwise, $r^{u_S}_{\gamma,\mathcal{DC}}(\eta) = 0$.

On the other hand, $\left[\mathbf{1} - \prod_{i=1}^{r(s)} (\mathbf{1} - u_{\eta_i})\right](\eta) = 1$ if and only if there exists $i \in \{1, 2, ..., r(S)\}$ such that $\eta_i \subseteq \eta$, which is equivalent to S is contained in a connected component of $(N, \eta, \mathcal{DC})$[6]. Otherwise $\left[\mathbf{1} - \prod_{i=1}^{r(s)} (\mathbf{1} - u_{\eta_i})\right](\eta) = 0$.

This completes the proof. □

In the following example we apply the result obtained in the previous proposition to the multicommunication situations considered in the Examples 3.6, 3.7 and 3.8, from the minimal connection multigraph sets obtained in Example 3.5.

Example 3.9 *For the multicommunication situation of Example 3.6 we have*

$$r^{u_S}_{\gamma,\mathcal{DC}} = \begin{cases} \mathbf{0} & if S = \{i\}, i \in N, \\ \mathbf{1} - (\mathbf{1} - u_{\{a,b\}}) = u_{\{a,b\}} & if S = \{1,2\}, \\ \mathbf{1} - (\mathbf{1} - u_{\{c\}}) = u_{\{c\}} & if S = \{2,3\}, \\ \mathbf{1} - (\mathbf{1} - u_{\{a,b,c\}}) = u_{\{a,b,c\}} & if S = \{1,3\} or S = N. \end{cases}$$

In the case of Example 3.7 we have[7]

$$r^{u_S}_{\gamma,\mathcal{DC}} = \begin{cases} \mathbf{0} & for S = \{i\}, i \in N, \\ \mathbf{1} - (\mathbf{1} - u_{\{a\}})(\mathbf{1} - u_{\{b\}}) & \\ \quad = u_{\{a\}} + u_{\{b\}} - u_{\{a,b\}} & for S = \{1,2\}, \\ \mathbf{1} - (\mathbf{1} - u_{\{c\}}) = u_{\{c\}} & for S = \{2,3\}, \\ \mathbf{1} - (\mathbf{1} - u_{\{a,c\}})(\mathbf{1} - u_{\{b,c\}}) & \\ \quad = u_{\{a,c\}} + u_{\{b,c\}} - u_{\{a,b,c\}} & for S = \{1,3\} or S = N, \end{cases}$$

and finally, for the multicommunication situation of the Example 3.8

[6] In other words, S has a minimal connection multigraph contained in η if and only if S is contained in a connected component of $(N, \eta, \mathcal{DC})$. If it is not contained in a component it is not connectable.

[7] Let us recall that for unanimity games (γ, u_η) and $(\gamma, u_{\eta'}) \in G^\gamma$ and $\delta \subseteq \gamma$ we have

$$u_\eta u_{\eta'}(\delta) = u_\eta(\delta) u_{\eta'}(\delta) = \begin{cases} 1 & \text{if } \eta \subseteq \delta \text{ and } \eta' \subseteq \delta \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} 1 & \text{if } \eta \cup \eta' \subseteq \delta \\ 0 & \text{otherwise.} \end{cases} = u_{\eta \cup \eta'}(\delta)$$

and thus $u_\eta u_{\eta'} = u_{\eta \cup \eta'}$.

$$r^{u_S}_{\gamma,\mathcal{DC}} = \begin{cases} \mathbf{0} & for S=\{i\}, i\in N,\\ \mathbf{1}-(\mathbf{1}-u_{\{a,b\}})(\mathbf{1}-u_{\{c\}}) & \\ = u_{\{a,b\}}+u_{\{c\}}-u_{\{a,b,c\}} & for S=\{1,2\},\\ \mathbf{1}-(\mathbf{1}-u_{\{d\}})=u_{\{d\}} & for S=\{2,3\},\\ \mathbf{1}-(\mathbf{1}-u_{\{a,b,d\}})(\mathbf{1}-u_{\{c,d\}}) & \\ = u_{\{a,b,d\}}+u_{\{c,d\}}-u_{\{a,b,c,d\}} & for S=\{1,3\} or S=N. \end{cases}$$

In the next proposition we obtain a general expression for the Harsanyi dividends of the multilink game with regard to its decomposition through unanimity multilink games. In the special case of a graph this expression also permits to obtain the dividends of the classical link game (in terms of unanimity link games). To do this we need some additional notation.

Given a multigraph $(N,\gamma,\mathcal{DC})$ and $S\subseteq N$, if $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})\neq\emptyset$, we will denote by $[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]$ the *union hull* in $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$, i.e.:

$$[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]$$

$$=\big\{\eta\subseteq\gamma|\eta=\bigcup_{j=1}^{r}\eta_j \text{ with } \eta_1,...,\eta_r\in\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})\big\}.$$

If $\eta\in[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]$ and $\eta=\bigcup_{j=1}^{r}\eta_j$ with $\eta_1,...,\eta_r\in\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$ we will say that *r* is a *signature* of η in $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$. Of course, η can have several signatures in $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$. Let us denote by $[\eta,S,N,\gamma,\mathcal{DC}]$ the set of signatures of η in $\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})$.

Proposition 3.2 *Given the multicommunication situation $(N,v,\gamma,\mathcal{DC})$ and $\emptyset\neq\eta\subseteq\gamma$ the dividend of η in the multilink game $r^v_{\gamma,\mathcal{DC}}$ is given by:*

$$\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta)=\begin{cases} 0 & if for all S\subseteq N\ \ \eta\notin[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]\\ \sum\limits_{S\subseteq N:\,\eta\in[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]}\Delta_v(S)\sum\limits_{k\in[\eta,S,N,\gamma,\mathcal{DC}]}(-1)^{k-1} & otherwise. \end{cases}$$

Proof Clearly the multilink game is linear with respect to the original game. Then,

$$r^v_{\gamma,\mathcal{DC}}=\sum_{\emptyset\neq S\subseteq N}\Delta_v(S)r^{u_S}_{\gamma,\mathcal{DC}}.$$

Using the Proposition 3.1,

$$r^v_{\gamma,\mathcal{DC}}=\sum_{\emptyset\neq S\subseteq N}\Delta_v(S)\big[\mathbf{1}-\prod_{i=1}^{r(S)}(\mathbf{1}-u_{\eta_i})\big]. \tag{7}$$

The expression in (6) can be rewritten as

$$\mathbf{1}-\prod_{i=1}^{r(S)}(\mathbf{1}-u_{\eta_i})=\mathbf{1}-(\mathbf{1}-u_{\eta_1})(\mathbf{1}-u_{\eta_2})\ldots(\mathbf{1}-u_{\eta_{r(S)}})$$

$$=\mathbf{1}-[\mathbf{1}-u_{\eta_1}-u_{\eta_2}+u_{\eta_1\cup\eta_2}](\mathbf{1}-u_{\eta_3})\ldots(\mathbf{1}-u_{\eta_{r(S)}})$$

$$=\mathbf{1}-[\mathbf{1}-u_{\eta_1}-u_{\eta_2}-u_{\eta_3}+u_{\eta_1\cup\eta_2}+u_{\eta_1\cup\eta_3}+u_{\eta_2\cup\eta_3}-u_{\eta_1\cup\eta_2\cup\eta_3}]$$

$$(\mathbf{1}-u_{\eta_4})\ldots(\mathbf{1}-u_{\eta_{r(S)}})$$

$$=\mathbf{1}-\Big[\mathbf{1}-\sum_{i=1}^{r(S)}u_{\eta_i}-\sum_{i=1}^{r(S)-1}\sum_{j=i+1}^{r(S)}u_{\eta_i\cup\eta_j}+\sum_{i=1}^{r(S)-2}\sum_{j=i+1}^{r(S)-1}\sum_{k=j+1}^{r(S)}u_{\eta_i\cup\eta_j\cup\eta_k}$$

$$+\ldots+(-1)^{r(S)-1}u_{\cup_{i=1}^{r(S)}\eta_i}\Big]$$

$$=\sum_{i=1}^{r(S)}u_{\eta_i}-\sum_{i=1}^{r(S)-1}\sum_{j=i+1}^{r(S)}u_{\eta_i\cup\eta_j}+\sum_{i=1}^{r(S)-2}\sum_{j=i+1}^{r(S)-1}\sum_{k=j+1}^{r(S)}u_{\eta_i\cup\eta_j\cup\eta_k}$$

$$+\ldots+(-1)^{r(S)-1}u_{\cup_{i=1}^{r(S)}\eta_i}.$$

And thus, a fixed u_η, with $\eta\subseteq\gamma$, is multiplied in (7) by each of the $\Delta_v(S)$ such that u_η is one of the summand in the expression (6) for this particular *S*. Then, η must be in the union hull $[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]$. Moreover, this fixed u_η is multiplied by $(-1)^{k-1}$ for each signature *k* in $[\eta,S,N,\gamma,\mathcal{DC}]$,. Then, adding in the different signatures and in the different $S\subseteq N$ we have the result. □

Example 3.10 *In the Example 3.6 when 1 and 2 are connected in series, using the expression (2), the union hulls are*

$$[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]=\begin{cases}\emptyset & for S=\emptyset or S=\{i\}, where i\in N,\\ \{\{a,b\}\} & for S=\{1,2\},\\ \{\{c\}\} & for S=\{2,3\},\\ \{\{a,b,c\}\} & for S=\{1,3\} or S=N\end{cases}$$

and the sets of signatures and dividends are:

$$[\{a,b\},\{1,2\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b\})=\Delta_v(\{1,2\})(-1)^{1-1}=1$$

$$[\{c\},\{2,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{c\})=\Delta_v(\{2,3\})(-1)^{1-1}=1$$

$$[\{a,b,c\},\{1,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b,c\})=\Delta_v(\{1,3\})(-1)^{1-1}=1,$$

These dividends coincide with those obtained in the Example 3.6.

In the connection in parallel of Example 3.7, using the expression (3), the union hulls are

$$[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]=\begin{cases}\emptyset & for S=\emptyset or S=\{i\} i\in N,\\ \{\{a\},\{b\},\{a,b\}\} & for S=\{1,2\},\\ \{\{c\}\} & for S=\{2,3\},\\ \{\{a,c\},\{b,c\},\{a,b,c\}\} & for S=\{1,3\} or S=N.\end{cases}$$

and the sets of signatures and dividends are:

$$[\{a\},\{1,2\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a\})=\Delta_v(\{1,2\})(-1)^{1-1}=1,$$

$$[\{b\},\{1,2\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{b\})=\Delta_v(\{1,2\})(-1)^{1-1}=1,$$

$$[\{a,b\},\{1,2\},N,\gamma,\mathcal{DC}]=\{2\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b\})=\Delta_v(\{1,2\})(-1)^{2-1}=-1,$$

$$[\{c\},\{2,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{c\})=\Delta_v(\{2,3\})(-1)^{1-1}=1,$$

$$[\{a,c\},\{1,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,c\})=\Delta_v(\{1,3\})(-1)^{1-1}=1,$$

$$[\{b,c\},\{1,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{b,c\})=\Delta_v(\{1,3\})(-1)^{1-1}=1,$$

$$[\{a,b,c\},\{1,3\},N,\gamma,\mathcal{DC}]=\{2\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b,c\})=\Delta_v(\{1,3\})(-1)^{2-1}=-1.$$

These dividends coincide with those obtained in the Example 3.7.

In the Example 3.8 using the expression (4), the union hulls are

$$[\mathcal{MCMS}(S,N,\gamma,\mathcal{DC})]=\begin{cases}\emptyset & for S=\emptyset or S=\{i\} i\in N,\\ \{\{a,b\},\{c\},\{a,b,c\}\} & for S=\{1,2\},\\ \{\{d\}\} & for S=\{2,3\},\\ \{\{a,b,d\},\{c,d\},\{a,b,c,d\}\} & for S=\{1,3\} or S=N.\end{cases}$$

the signatures are:

$$[\{a,b\},\{1,2\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b\})=\Delta_v(\{1,2\})(-1)^{1-1}=1,$$

$$[\{c\},\{1,2\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{c\})=\Delta_v(\{1,2\})(-1)^{1-1}=1,$$

$$[\{a,b,c\},\{1,2\},N,\gamma,\mathcal{DC}]=\{2\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b,c\})=\Delta_v(\{1,2\})(-1)^{2-1}=-1,$$

$$[\{a,b,d\},\{1,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b,d\})=\Delta_v(\{1,3\})(-1)^{1-1}=1,$$

$$[\{c,d\},\{1,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{c,d\})=\Delta_v(\{1,3\})(-1)^{1-1}=1,$$

$$[\{a,b,c,d\},\{1,3\},N,\gamma,\mathcal{DC}]=\{2\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a,b,c,d\})=\Delta_v(\{1,3\})(-1)^{2-1}=-1,$$

$$[\{d\},\{2,3\},N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^v_{\gamma,\mathcal{DC}}}(\{a\})=\Delta_v(\{2,3\})(-1)^{1-1}=1.$$

These dividends coincide with those obtained in the Example 3.8.

The following example illustrates the fact that the set of signatures is not always a singleton.

Example 3.11 *Consider the multicommunication situation* $(N,v,\gamma,\mathcal{DC})$ *with* $N=\{1,2,3\}$*,* $v=u_N$ *and* $\gamma=\{a,b,c,d\}$ *as represented in Fig. 3.*

Both links connecting 1 to 2 and 2 to 3, are in parallel i.e., $\mathcal{DC}=\{\{a\},\{b\},\{c\},\{d\}\}$*, and thus*

$$\mathcal{MCMS}(N,N,\gamma,\mathcal{DC}))=\big\{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\big\},$$

and its union hull is

$$[\mathcal{MCMS}(N,N,\gamma,\mathcal{DC}))]=\big\{\{a,c\}\{a,d\},\{b,c\},\{b,d\},\{a,c,d\},\{a,b,c\},$$

$$\{a,b,d\},\{b,c,d\},\{a,b,c,d\}\big\}.$$

Then, using expression (6)

$$r^{u_N}_{\gamma,\mathcal{DC}}=u_{\{a,c\}}+u_{\{a,d\}}+u_{\{b,c\}}+u_{\{b,d\}}-u_{\{a,c,d\}}-u_{\{a,b,c\}}-u_{\{a,b,d\}}-u_{\{b,c,d\}}+u_{\{a,b,c,d\}}.$$

In this case

$$[\{a,c\},N,N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,c\})=\Delta_{u_N}(N)(-1)^{1-1}=1,$$

$$[\{a,d\},N,N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,d\})=\Delta_{u_N}(N)(-1)^{1-1}=1,$$

$$[\{b,c\},N,N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{b,c\})=\Delta_{u_N}(N)(-1)^{1-1}=1,$$

$$[\{b,d\},N,N,\gamma,\mathcal{DC}]=\{1\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{b,d\})=\Delta_{u_N}(N)(-1)^{1-1}=1,$$

$$[\{a,c,d\},N,N,\gamma,\mathcal{DC}]=\{2\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,c,d\})=\Delta_{u_N}(N)(-1)^{1-1}=-1,$$

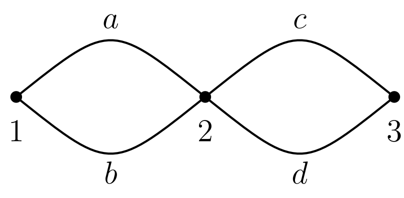


Fig. 3 Multigraph of the Example 3.11

$$[\{a,b,c\},N,N,\gamma,\mathcal{DC}] = \{2\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,b,c\}) = \Delta_{u_N}(N)(-1)^{1-1} = -1,$$

$$[\{a,b,d\},N,N,\gamma,\mathcal{DC}] = \{2\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,b,d\}) = \Delta_{u_N}(N)(-1)^{1-1} = -1,$$

$$[\{b,c,d\},N,N,\gamma,\mathcal{DC}] = \{2\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{b,c,d\}) = \Delta_{u_N}(N)(-1)^{1-1} = -1,$$

$$[\{a,b,c,d\},N,N,\gamma,\mathcal{DC}] = \{2,2,3,3,3,3,4\} and \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,b,c,d\}) = \Delta_{u_N}(N)$$

$$[(-1)^{2-1} + (-1)^{2-1} + (-1)^{3-1} + (-1)^{3-1} + (-1)^{3-1} + (-1)^{3-1} + (-1)^{4-1}] = 1.$$

Finally,

$$\Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a\}) = \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{b\}) = \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{c\}) = \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{d\})$$

$$= \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{a,b\}) = \Delta_{r^{u_N}_{\gamma,\mathcal{DC}}}(\{c,d\}) = 0$$

as $\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{c,d\}$ are not in $[\mathcal{MCMS}(N,N,\gamma,\mathcal{DC})]$.

4 Characterization of the position value for multigraph games

In this section, we will characterize the position value for multicommunication situations. The characterization will use the extensions to this setting of the component efficiency (Myerson, 1977) and the balanced link contributions (Slikker, 2005). For a certain subset of multicommunication situations, those in which all the link packs of each player are disjoint, the property of balanced link contributions can be substituted by that of balanced link pack contributions. The latter property is more restrictive than the classic axiom of balanced link contributions. In this property it is assumed that removing a link between two players is the same as removing the link pack to which that link belongs. This must be so because when several links are necessary to connect two players, the disappearance of one of them prevents such communication. We introduce these properties in the following.

Definition 4.1 An allocation rule φ on $\mathcal{MCS}^N$ satisfies *component efficiency* if, for all $(N,v,\gamma,\mathcal{DC}) \in \mathcal{MCS}^N$ and all $C \in N/(\gamma,\mathcal{DC})$, $\sum_{i\in C} \varphi_i(N,v,\gamma,\mathcal{DC}) = v(C)$.

Definition 4.2 An allocation rule, φ on $\mathcal{MCS}^N$ satisfies balanced link contributions if given $(N,v,\gamma,\mathcal{DC}) \in \mathcal{MCS}^N$, and $i,j \in N$, it holds that[8]

$$\sum_{l\in\gamma_j} \left[\varphi_i(N,v,\gamma,\mathcal{DC}) - \varphi_i(N,v,\gamma\setminus\{l\},\mathcal{DC})\right]$$

[8]As mentioned, to lighten the notation, the reader should keep in mind that $(N,v,\gamma,\mathcal{DC})$ and $(N,v,\gamma\setminus\{l\},\mathcal{DC})$ refers to $(N,v,\gamma,\mathcal{DC}(N,\gamma))$ and $(N,v,\gamma\setminus\{l\},\mathcal{DC}(N,\gamma\setminus\{l\}))$, respectively.

$$= \sum_{l \in \gamma_i} \left[\varphi_j(N, v, \gamma, \mathcal{DC}) - \varphi_j(N, v, \gamma \setminus \{l\}, \mathcal{DC})\right].$$

Definition 4.3 An allocation rule, φ on $\mathcal{MCS}^N$ satisfies balanced link pack contributions if given $(N, v, \gamma, \mathcal{DC}) \in \mathcal{MCS}^N$, and $i, j \in N$, it holds that[9]

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\varphi_i(N, v, \gamma, \mathcal{DC}) - \varphi_i(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC})\right]$$

$$= \sum_{k=1}^{r_i} |\gamma_i^k| \left[\varphi_j(N, v, \gamma, \mathcal{DC}) - \varphi_j(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC})\right],$$

where $\mathcal{LP}_i = \{\gamma_i^1, \gamma_i^2, ..., \gamma_i^{r_i}\}$ and $\mathcal{LP}_j = \{\gamma_j^1, \gamma_j^2, ..., \gamma_j^{r_j}\}$ are respectively the sets of link packs of players *i* and *j*.

Remark 4.1 *For multicommunication situations $(N, v, \gamma, \mathcal{DC})$ in which $\mathcal{DC}$ is given by the communication in parallel between nodes i and j for all $i, j \in N$, $i \neq j$, the balanced link pack contributions coincides with the balanced link contributions. This is so as, in such a case, each link pack of $i \in N$ contains only one link.*

Let us first analyze the extent to which the defined properties are satisfied by the position value in this setting.

Proposition 4.1 *The position value for multicommunication situations in $\mathcal{MCS}_0^N$ satisfies component efficiency.*

Proof Let $C \in N/(\gamma, \mathcal{DC})$, then,

$$\sum_{i \in C} \pi_i(N, v, \gamma, \mathcal{DC}) = \sum_{i \in C} \sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \pi_i(N, u_\eta, \gamma, \mathcal{DC}) = \sum_{i \in C} \sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{2|\eta|}$$

$$= \sum_{\eta \subseteq \gamma_{|C}} \Delta_{r^{v_{|C}}_{\gamma_{|C},\mathcal{DC}}}(\eta) \frac{\sum_{i \in C} |\eta_i|}{2|\eta|} = \sum_{\eta \subseteq \gamma_{|C}} \Delta_{r^{v_{|C}}_{\gamma_{|C},\mathcal{DC}}}(\eta) = r^{v_{|C}}_{\gamma_{|C},\mathcal{DC}}(\gamma_{|C}) = v_{|C}^{\gamma_{|C}}(C) = v(C),$$

where the first equality holds using the linearity in the game of the position value, the second one using that for unanimity games of links the position value of each player is proportional to his degree, the fifth as the sum of the dividends is the value of the grand coalition and the sixth using the definition of the link game. □

Proposition 4.2 *The position value for multicommunication situations in $\mathcal{MCS}_0^N$ satisfies balanced link contributions.*

[9] A similar comment to the one in the previous footnote applies here.

Proof For $(N, v, \gamma, \mathcal{DC}) \in \mathcal{MCS}_0^N$ and $i, j \in N$, $i \neq j$, we have that

$$\sum_{l\in\gamma_j} [\pi_i(N, v, \gamma, \mathcal{DC}) - \pi_i(N, v, \gamma \setminus \{l\}, \mathcal{DC})]$$

$$= \frac{1}{2}\sum_{l\in\gamma_j}\left[\sum_{\eta\subseteq\gamma}\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta)\frac{|\eta_i|}{|\eta|} - \sum_{\eta\subseteq\gamma\setminus l}\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta)\frac{|\eta_i|}{|\eta|}\right]$$

$$= \frac{1}{2}\sum_{l\in\gamma_j}\sum_{\eta\subseteq\gamma, l\in\eta}\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta)\frac{|\eta_i|}{|\eta|} = \frac{1}{2}\sum_{\eta\subseteq\gamma}\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta)\frac{|\eta_j||\eta_i|}{|\eta|}.$$

This last expression is symmetric in *i* and *j* and thus the result is proved. ■

However, the position value does not satisfy balanced link pack contributions.

Counterexample 4.1 *Consider now the multicommunication situation* $(N, v, \gamma, \mathcal{DC})$ *with* $N = \{1, 2, 3\}$, $v = u_{\{1,3\}}$, $\gamma = \{a, b, c, d\}$ *as represented in Fig. 4 and* $\mathcal{DC} = \{\{a, c\}, \{b, c\}, \{d\}\}$*, i.e., a and b are in parallel and c is in series with a and b.*

In this model, $\mathcal{LP}_1 = \{\{a, c\}, \{b, c\}\}$*,* $\mathcal{LP}_2 = \{\{a, c\}, \{b, c\}, \{d\}\}$ *and* $\mathcal{LP}_3 = \{\{d\}\}$*. Then, the associated multilink game is given by:*

$$r^v_{\gamma,\mathcal{DC}} = u_{\{a,c,d\}} + u_{\{b,c,d\}} - u_{\{a,b,c,d\}}.$$

The position value for this multicommunication situation is

$$\pi(N, v, \gamma, \mathcal{DC}) = \left(\frac{7}{24}, \frac{12}{24}, \frac{5}{24}\right).$$

If player 3 removes his link pack, then

$$r^v_{\gamma\setminus\{d\},\mathcal{DC}} = \mathbf{0},$$

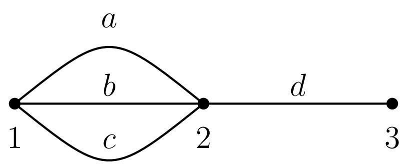
and $\pi_1(N, v, \gamma, \mathcal{DC}) - \pi_1(N, v, \gamma \setminus \{d\}, \mathcal{DC}) = \frac{7}{24} - 0 = \frac{7}{24}$.

If, on the other hand, player 1 does not use link pack $\{a, c\}$*(but he can use the link pack* $\{b, c\}$*), then*

$$r^v_{\gamma\setminus\{a,c\},\mathcal{DC}} = u_{\{b,c,d\}}$$

and thus $\pi_3(N, v, \gamma, \mathcal{DC}) - \pi_3(N, v, \gamma \setminus \{a, c\}, \mathcal{DC}) = \frac{5}{24} - \frac{1}{6} = \frac{1}{24}$.

Fig. 4 Multigraph of the Example 4.1



Similarly, by symmetry, when the player 1 does not use link pack $\{b, c\}$*(but he can use the link pack* $\{a, c\}$*),*

$$\pi_3(N, v, \gamma, \mathcal{DC}) - \pi_3(N, v, \gamma \setminus \{b, c\}, \mathcal{DC}) = \frac{5}{24} - \frac{1}{6} = \frac{1}{24}.$$

Then,

$$2[\pi_3(N, v, \gamma, \mathcal{DC}) - \pi_3(N, v, \gamma \setminus \{a, c\}, \mathcal{DC})]$$
$$+2[\pi_3(N, v, \gamma, \mathcal{DC}) - \pi_3(N, v, \gamma \setminus \{b, c\}, \mathcal{DC})] = \frac{4}{24}$$

which is different from

$$\pi_1(N, v, \gamma, \mathcal{DC}) - \pi_1(N, v, \gamma \setminus \{d\}, \mathcal{DC}) = \frac{7}{24}.$$

And thus, the defined value does not satisfy balanced link pack contributions.

In the following, we will denote by $\mathcal{DMCS}_0^N$ the subset of those multicommunication situations in $\mathcal{MCS}_0^N$ in which, for each player, their link packs are disjoint. This family contains all multicommunication situations in which the connection between each pair of players is either in series or in parallel, and may be in series between some players and in parallel between others. Note that these are not the only situations where the packs are disjoint as can be seen in Example 3.8.

In the following proposition we show that the position value restricted to this family satisfies balanced link pack contributions.

Proposition 4.3 *The position value for multicommunication situations in* $\mathcal{DMCS}_0^N$ *satisfies balanced link pack contributions.*

Proof We have for $i, j \in N$, $i \neq j$, and $\gamma_i^1, ..., \gamma_i^{r_i}$ the family of link packs for player *i* and similarly $\gamma_j^1, ..., \gamma_j^{r_j}$ the link packs for player *j*, that

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\pi_i(N, v, \gamma, \mathcal{DC}) - \pi_i(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC})\right]$$
$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \left[\sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{|\eta|} - \sum_{\eta \subseteq \gamma \setminus \gamma_j^k} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{|\eta|}\right]$$
$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \sum_{\substack{\eta \subseteq \gamma \\ \eta \cap \gamma_j^k \neq \emptyset}} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{|\eta|}.$$

If $\eta \cap \gamma_j^k \neq \emptyset$ but $\eta \not\supseteq \gamma_j^k$ then $\Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) = 0$, and thus

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\pi_i(N, v, \gamma, \mathcal{DC}) - \pi_i(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC})\right]$$

$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \sum_{\substack{\eta \subseteq \gamma \\ \gamma_j^k \subseteq \eta}} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{|\eta|}.$$

Therefore,

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\pi_i(N, v, \gamma, \mathcal{DC}) - \pi_i(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC})\right]$$

$$= \frac{1}{2} \sum_{\eta \subseteq \gamma} \Big(\sum_{\gamma_j^k \subseteq \eta} |\gamma_j^k| \Big) \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_i|}{|\eta|} = \frac{1}{2} \sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\eta) \frac{|\eta_j||\eta_i|}{|\eta|},$$

the last equality holding since $\sum_{\gamma_j^k \subseteq \eta} |\gamma_j^k| = |\eta_j|$, given that the link packs of player *j* are all disjoint.

This last expression is symmetric in *i* and *j* and thus the result is proved. □

Then we can now introduce the following characterizations. The proof of the first one mimics the corresponding in Slikker (2005) and then it is omitted.

Theorem 4.1 *The position value is the unique allocation rule on $\mathcal{MCS}_0^N$ satisfying component efficiency and balanced link contributions.*

Theorem 4.2 *The position value is the unique allocation rule on $\mathcal{DMCS}_0^N$ satisfying component efficiency and balanced link pack contributions.*

Proof As it is already proved, the position value in $\mathcal{DMCS}_0^N$ for multicommunication situations satisfies component efficiency and balanced link pack contributions. Conversely, let us consider an allocation rule φ on $\mathcal{DMCS}_0^N$ satisfying component efficiency and balanced link pack contributions. We will prove that $\varphi_i(N, v, \gamma, \mathcal{DC}) = \pi_i(N, v, \gamma, \mathcal{DC})$ for all $(N, v, \gamma, \mathcal{DC}) \in \mathcal{DMCS}_0^N$ and all $i \in N$, using induction on the cardinality of γ.

Let $(N, v, \gamma, \mathcal{DC}) \in \mathcal{DMCS}_0^N$, $i \in N$ and C_i the connected component in $(N, \gamma, \mathcal{DC})$ to which *i* belongs. If $|\gamma| = 0$ then $|C_i| = 1$ and thus $\varphi_i(N, v, \gamma, \mathcal{DC}) = \pi_i(N, v, \gamma, \mathcal{DC})$ as both rules satisfy component efficiency.

Suppose uniqueness holds for all $(N, v, \gamma, \mathcal{DC})$ with $|\gamma| \leq k$ and consider $(N, v, \gamma, \mathcal{DC})$ with $|\gamma| = k + 1$. If $|C_i| = 1$, using again component efficiency, $\varphi_i(N, v, \gamma, \mathcal{DC}) = \pi_i(N, v, \gamma, \mathcal{DC})$. Otherwise, suppose that $C_i = \{i_1 = i, i_2, ..., i_{c_i}\}$. As φ satisfies balanced link pack contributions we have, for $j = 2, ..., c_i$,

$$\sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \left[\varphi_i(N, v, \gamma, \mathcal{DC}) - \varphi_i(N, v, \gamma \setminus \gamma_{i_j}^k, \mathcal{DC})\right]$$

$$= \sum_{k=1}^{r_i} |\gamma_i^k| \left[\varphi_{i_j}(N, v, \gamma, \mathcal{DC}) - \varphi_{i_j}(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC})\right],$$

or equivalently

$$\sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \varphi_i(N, v, \gamma, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \varphi_{i_j}(N, v, \gamma, \mathcal{DC})$$

$$= \sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \varphi_i(N, v, \gamma \setminus \gamma_{i_j}^k, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \varphi_{i_j}(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC}).$$

We have $|\gamma_{i_j}^k| \geq 1$, for all $j = 2, ..., c_i$, and for all $k = 1, ..., r_{i_j}$. Also $|\gamma_i^k| \geq 1$ for all $k = 1, ..., r_i$. This is so because all these nodes are in a connected component. Using the induction hypothesis it holds that

$$\sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \varphi_i(N, v, \gamma \setminus \gamma_{i_j}^k, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \varphi_{i_j}(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC})$$

$$= \sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \pi_i(N, v, \gamma \setminus \gamma_{i_j}^k, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \pi_{i_j}(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC}).$$

Then, the $c_i - 1$ linear equations

$$\sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \varphi_i(N, v, \gamma, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \varphi_{i_j}(N, v, \gamma, \mathcal{DC})$$

$$= \sum_{k=1}^{r_{i_j}} |\gamma_{i_j}^k| \pi_i(N, v, \gamma \setminus \gamma_{i_j}^k, \mathcal{DC}) - \sum_{k=1}^{r_i} |\gamma_i^k| \pi_{i_j}(N, v, \gamma \setminus \gamma_i^k, \mathcal{DC})$$

with the component efficiency equation

$$\sum_{j=1}^{c_i} \varphi_{i_j}(N, v, \gamma, \mathcal{DC}) = v(C_i)$$

form a linear system of c_i independent equations which has a unique solution $(\pi_{i_1}(N, v, \gamma, \mathcal{DC}), ..., \pi_{i_{c_i}}(N, v, \gamma, \mathcal{DC}))$. $\square$

5 A weighted position value

In this section we take another step enlarging the set of multicommunication situations. We will assume that the links in the multigraph can have asymmetries due to different capacities, frequencies or latencies (slowness in the transmission of the information). For example, in the transmission of information over the long distance the link with the smallest capacity determines the size of the flow. Then, given a multigraph $(N, \gamma, \mathcal{DC})$, we will assign a weight $\lambda_l \in [0,1]$ to each $l \in \gamma$, this weight representing such a capacity or celerity in the link. The value of $\lambda_l = 1$ is associated with links with full capacity or null latency. A value of $\lambda_l = 0$ indicates null capacity or celerity in the link. In fact, it is equivalent to the non-existence of the link.

Now we can define the weighted multicommunication situations.

Definition 5.1 A weighted multicommunication situation is $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda})$ with (N, v) a TU-game, $(N, \gamma, \mathcal{DC})$ a multigraph and $\boldsymbol{\lambda} \in [0,1]^{|\gamma|}$. The family of all weighted multicommunication situations with players/nodes set N will be denoted by $\mathcal{WMCS}^N$. $\mathcal{WMCS}_0^N$ will be the subset of all those weighted multicommunciation situations in $\mathcal{WMCS}^N$ for which the game is zero-normalized.

Remark 5.1 *The set of all multicommunication situations, $\mathcal{MCS}^N$, can be identified with the subset of those $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}^N$ in which $\boldsymbol{\lambda} = (1, ..., 1) \in \mathbb{R}^{|\gamma|}$.*

For weighted multicommunication situations, the value of each coalition of links (given by the multilink game) must take into account the new restrictions in the communication. In the following definition we propose the value of a coalition of links to be the sum of the dividends of the multilink game of all its subcoalitions, each of them multiplied by the minimum of the weights of the links in the subcoalition. This proposal also appears in Manuel & Martín (2020, 2021). Under this idea, in the definition of the new multilink game, *the weighted multilink game*, the value of each subgraph is obtained by discounting each original dividend by a factor which is the minimum of the capacities of the links involved in the dividends[10] (under the assumption that you can not transmit more packages of information or to have more transmission speed than the corresponding to the link with the lesser capacity or speed).

Definition 5.2 Given a weighted multicommunication situation $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}_0^N$, we define the weighted multilink game $(\gamma, r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}})$ as the TU-game with characteristic function

$$r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}(\eta) = \sum_{\delta \subseteq \eta} \Delta_{r^v_{\gamma,\mathcal{DC}}}(\delta) \min_{h \in \delta}\{\lambda_h\}, \text{ for all } \eta \subseteq \gamma, \delta \neq \emptyset.$$

[10] It might be thought that this discount factor could depend on the type of connectivity, and for example, in parallel connections, the maximum could be used. Unfortunately, this would lead to weighted games that are not necessarily superadditive. To see this, consider nodes 1 and 2 connected in parallel with two arcs a and b, of respective weights 0.1 and 0.8. If we multiply the dividend of $u_{\{a,b\}}$ by the maximum of the both previous weights, we would have that the weighted multilink game would be $0.1u_{\{a\}} + 0.8u_{\{b\}} - 0.8u_{\{a,b\}}$ which is clearly not superadditive.

Remark 5.2 *For* $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}_0^N$ *with* $\boldsymbol{\lambda} = (1, ..., 1)$ *the game* $(\gamma, r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}})$ *coincides with* $(\gamma, r^v_{\gamma,\mathcal{DC}})$.

Then, we propose as weighted position value for weighted multicommunication situations to apply the Shapley value to the previously defined weighted multilink game.

Definition 5.3 Given $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}_0^N$ we define a weighted position value as

$$\pi_i^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) = \frac{1}{2} \sum_{l \in \gamma_i} Sh_l(\gamma, r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}).$$

Remark 5.3 *For* $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}_0^N$ *with* $\boldsymbol{\lambda} = (1, ..., 1)$, $\pi_i^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda})$ *coincides with* $\pi_i(N, v, \gamma, \mathcal{DC})$.

Example 5.1 *Consider the weighted multicommunication situation* $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda})$ *with* $(N, v, \gamma, \mathcal{DC})$ *as in the Example 3.6 and* $\boldsymbol{\lambda} = (0.6, 0.4, 0.7)$. *Then, the characteristic function of* $(\gamma, r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}})$ *is*

$$r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}} = 0.4u_{\{a,b\}} + 0.4u_{\{a,b\,c\}} + 0.7u_{\{c\}}.$$

Finally, the values obtained in the allocation rule π^W are,

$$\pi^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) = \left(\frac{0.4}{2} + \frac{0.4}{3}, \frac{0.4}{2} + \frac{0.4}{3}, \frac{0.4}{3} + 0.7\right) = \left(\frac{2}{6}, \frac{2}{6}, \frac{5}{6}\right).$$

Example 5.2 *Consider the weighted multicommunication situation* $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda})$ *with* $(N, v, \gamma, \mathcal{DC})$ *as in the Example 3.7 and* $\boldsymbol{\lambda} = (0.6, 0.4, 0.7)$. *Then, the characteristic function is*

$$r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}} = 0.6u_{\{a\}} + 0.4u_{\{b\}} + 0.7u_{\{c\}} + 0.6u_{\{a,c\}} + 0.4u_{\{b,c\}} - 0.4u_{\{a,b\}} - 0.4u_{\{a,b,c\}}.$$

Finally, the values obtained in the allocation rule π^W are,

$$\pi^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda})$$

$$= \left(0.6 + \frac{0.6}{2} - \frac{0.4}{2} - \frac{0.4}{3}, 0.4 + \frac{0.4}{2} - \frac{0.4}{2} - \frac{0.4}{3}, 0.7 + \frac{0.6}{2} + \frac{0.4}{2} - \frac{0.4}{3}\right)$$

$$= \left(\frac{17}{30}, \frac{8}{30}, \frac{32}{30}\right).$$

In the following we characterize the defined value by tracing a path parallel to the one we have used for the unweighted position value. First of all we introduce a modification of the component efficiency property, given the possible inefficiency of the weighted multinetwork.

Definition 5.4 Given $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}^N$, we will say that i and $j \in N$ are $\boldsymbol{\lambda}$ -connected if there exist a sequence of nodes $i_1 = i, i_2, ..., i_k = j$ with $\gamma_{i_k, i_{k+1}} \neq \emptyset$ for $k = 1, ..., r-1$ and all the links l in at least one minimal connection multigraph of i_k, i_{k+1} with $\lambda_l > 0$ for $k = 1, ..., r-1$.

The $\boldsymbol{\lambda}$-connection is an equivalence relation in the set of nodes N. We will note $N/(\gamma, \mathcal{DC}, \boldsymbol{\lambda})$ the partition of N generated by this relation. The elements of $N/(\gamma, \mathcal{DC}, \boldsymbol{\lambda})$ will be the $\boldsymbol{\lambda}$-connected components of the weighted multicommunication situation $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}^N$.

Definition 5.5 An allocation rule φ defined on $\mathcal{WMCS}^N$ satisfies $\boldsymbol{\lambda}-$component efficiency if, for all $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}^N$, and for all $C \in N/(\gamma, \mathcal{DC}, \boldsymbol{\lambda})$,

$$\sum_{i \in C} \varphi_i(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) = r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}(\gamma_{|C}) = \sum_{\emptyset \neq \delta \subseteq \gamma_{|C}} \Delta_{r^v_{\gamma, \mathcal{DC}}}(\delta) \min_{h \in \delta}\{\lambda_h\}.$$

Proposition 5.1 *The weighted position value on $\mathcal{WMCS}_0^N$ satisfies $\boldsymbol{\lambda}-$component efficiency and balanced link contributions.*

Proof To prove that π^W satisfies $\boldsymbol{\lambda}$-component efficiency, consider $(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) \in \mathcal{WMCS}_0^N$ and $C \in N/(\gamma, \mathcal{DC}, \lambda)$. Then,

$$\sum_{i \in C} \pi_i^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) = \sum_{i \in C} \left[\frac{1}{2} \sum_{l \in \gamma_i} Sh_l(\gamma, r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}) \right] = \sum_{l \in \gamma_{|C}} Sh_l(\gamma, r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}})$$

$$= \sum_{l \in \gamma_{|C}} Sh_l(\gamma_{|C}, r^v_{\gamma_{|C}, \mathcal{DC}, \boldsymbol{\lambda}}) = r^v_{\gamma_{|C}, \mathcal{DC}, \boldsymbol{\lambda}}(\gamma_{|C}) = r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}(\gamma_{|C}).$$

In order to prove that π^W satisfies balanced link pack contributions, consider $i, j \in N$ and $i \neq j$

$$\sum_{l \in \gamma_j} [\pi_i(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) - \pi_i(N, v, \gamma \setminus \{l\}, \mathcal{DC}, \boldsymbol{\lambda})]$$

$$= \frac{1}{2} \sum_{l \in \gamma_j} \left[\sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|} - \sum_{\eta \subseteq \gamma \setminus l} \Delta_{r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|} \right]$$

$$= \frac{1}{2} \sum_{l \in \gamma_j} \sum_{\eta \subseteq \gamma, l \in \eta} \Delta_{r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|}$$

$$= \frac{1}{2} \sum_{l \in \gamma_j} \sum_{\eta \subseteq \gamma, l \in \eta} \Delta_{r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|} = \frac{1}{2} \sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma, \mathcal{DC}, \boldsymbol{\lambda}}}(\eta) \frac{|\eta_j||\eta_i|}{|\eta|}.$$

□

In the following, we will denote by $\mathcal{WDMCS}_0^N$ the subset of those multicommunication situations in $\mathcal{WMCS}_0^N$ in which the link packs of each player are disjoint.

For this family, $\mathcal{WDMCS}_0^N$, we can assure that the balanced link pack contributions is satisfied.

Proposition 5.2 *The weighted position value on $\mathcal{WMCS}_0^N$ satisfies balanced link pack contributions.*

Proof Consider $i, j \in N$ and $i \neq j$. Then,

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\pi_i^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) - \pi_i^W(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC}, \boldsymbol{\lambda})\right]$$

$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \left[\sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|} - \sum_{\eta \subseteq \gamma \setminus \gamma_j^k} \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|}\right]$$

$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \sum_{\substack{\eta \subseteq \gamma \\ \eta \cap \gamma_j^k \neq \emptyset}} \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|}$$

$$= \frac{1}{2} \sum_{k=1}^{r_j} |\gamma_j^k| \sum_{\substack{\eta \subseteq \gamma \\ \gamma_j^k \subseteq \eta}} \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|}.$$

If $\eta \cap \gamma_j^k \neq \emptyset$ but $\eta \not\supseteq \gamma_j^k$ then $\Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) = 0$.

Therefore,

$$\sum_{k=1}^{r_j} |\gamma_j^k| \left[\pi_i^W(N, v, \gamma, \mathcal{DC}, \boldsymbol{\lambda}) - \pi_i^W(N, v, \gamma \setminus \gamma_j^k, \mathcal{DC}, \boldsymbol{\lambda})\right]$$

$$= \frac{1}{2} \sum_{\eta \subseteq \gamma} \Big(\sum_{\gamma_j^k \subseteq \eta} |\gamma_j^k| \Big) \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_i|}{|\eta|} = \frac{1}{2} \sum_{\eta \subseteq \gamma} \Delta_{r^v_{\gamma,\mathcal{DC},\boldsymbol{\lambda}}}(\eta) \frac{|\eta_j||\eta_i|}{|\eta|},$$

the last equality holding since $\sum_{\gamma_j^k \subseteq \eta} |\gamma_j^k| = |\eta_j|$, given that the link packs of player *j* are all disjoint. This last expression is symmetric in *i* and *j* and thus the result is proved. □

The proofs of the following characterizations are parallel to the proofs of Theorem 4.1 and Theorem 4.2, respectively, and so they are omitted.

Theorem 5.1 *The weighted position value is the unique allocation rule defined on the family of the weighted multicommunication situations $\mathcal{WMCS}_0^N$, satisfying $\boldsymbol{\lambda}-$ component efficiency and balanced link contributions.*

Theorem 5.2 *The weighted position value is the unique allocation rule defined on the family of the weighted multicommunication situations $\mathcal{DWMCS}_0^N$, satisfying $\boldsymbol{\lambda}-$component efficiency and balanced link pack contributions.*

6 Final remarks and conclusions

In this paper, we have analyzed the issue of multicommunication from a game-theoretical perspective. Multicommunication is one of the proposed methods to address the challenges of reliability and speed posed by the transmission of information in our days. In our proposal, it is assumed that multicommunication among players itself aims to facilitate cooperation in a game of transferable utility.

We have expanded the classical context of communication situations, games with communication restricted by a graph or graph games, to incorporate multicommunication. In the initial phase, it was assumed that communication channels have no restrictions on their capacity, frequency, or latency, and in a subsequent phase, weights were introduced on the links to incorporate reduced capacities. In both scenarios, we addressed the generalization of the classical position value for communication situations to establish a ranking of the importance of different communication providers. This generalization of the position value has been characterized using the classical properties of component efficiency and balanced link contributions, extended to this setting. For the special case of multicommunication situations in which the link packs of the players are disjoint, we can substitute the balanced link contributions with the more restrictive property of balanced link pack contributions. Both characterizations can be extended for the corresponding weighted classes substituting in this case the component efficiency by a certain inefficiency which we have call λ-component efficiency.

We have also obtained an expression for the dividends of the multilink game, expression that also applies to the particular case of the link game.

In Fig. 5, we depicts the various possible scenarios under multicommunication. It shows that classical communication situations lie at the intersection of multicommunication in series, $\mathcal{MCS}_{0,s}^N$, and in parallel, $\mathcal{MCS}_{0,p}^N$, which, on the other hand, are subfamilies of $\mathcal{DMCS}_0^N$. However, there are other more general alternatives, and for all of them, the possibility of considering channel capacity limitations exists. As mentioned, in all of these frameworks, balanced link contributions shows its importance to characterize the position value, with the component efficiency (or the adapted sub-efficiency when limitations of capacity, frequency or speed are present). But in the case of $\mathcal{DMCS}_0^N$ or $\mathcal{WDMCS}_0^N$, balanced link contributions can be replaced by balanced link pack contributions which is a more restrictive property.

Given the extensive existing literature on game-theoretical analysis of cooperation constrained by communication graphs, it seems reasonable to explore similar problems when restrictions are defined through multicommunication.

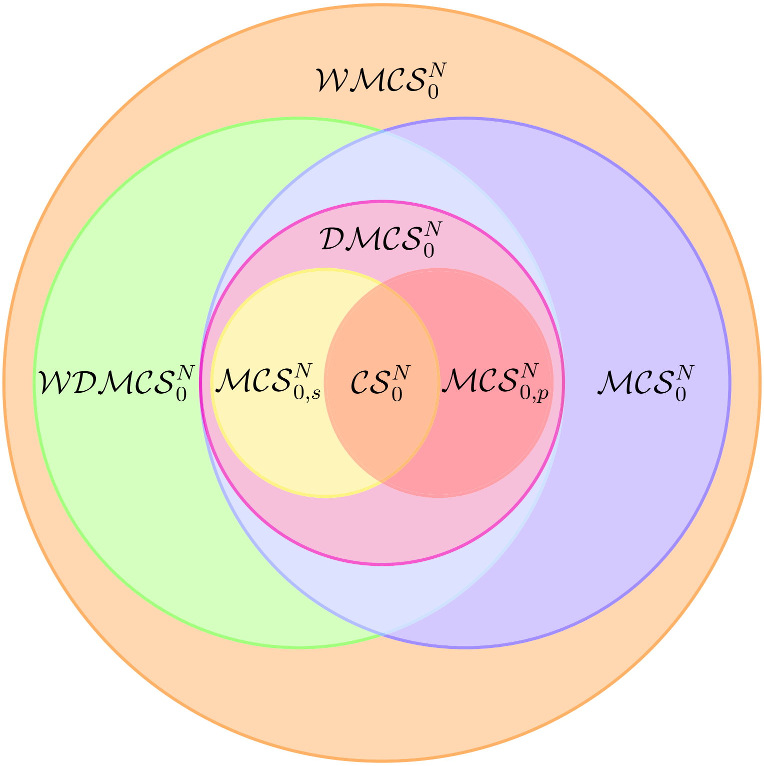


Fig. 5 Venn diagram of the multicommunication situations

Acknowledgements The authors would like to thank the editor and four anonymous reviewers for all their comments which have greatly helped to improve the article. We would also like to thank René van den Brink for his interesting comments, corrections and help.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. This study has been partially supported by the *Plan Nacional de I+D+i* of the Spanish Government under the project PID2020-116884GB-I00.

Declarations

Conflict of Interest Authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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