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Bilinear Yano's extrapolation theory
Teoría de extrapolación de Yano bilineal

MEMORIA PARA OPTAR AL GRADO DE DOCTORA

PRESENTADA POR

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La misma noche que hace blanquear los mismos árboles.
Nosotros, los de entonces, ya no somos los mismos.

Ya no la quiero, es cierto, pero cuánto la quise.
Mi voz buscaba el viento para tocar su oído.

De otro. Será de otro. Como antes de mis besos.
Su voz, su cuerpo claro. Sus ojos infinitos.

Ya no la quiero, es cierto, pero tal vez la quiero.
Es tan corto el amor, y es tan largo el olvido.

Porque en noches como ésta la tuve entre mis brazos,
mi alma no se contenta con haberla perdido.

Aunque éste sea el último dolor que ella me causa,
y estos sean los últimos versos que yo le escribo.

Pablo Neruda, *Poema 20,*
Veinte poemas de amor y una canción desesperada

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Contents

Resumen

Existen diversos problemas interesantes en el análisis matemático cuya solución implica la acotación de un operador. De hecho, en el contexto de los espacios de Lebesgue, es común que un operador T sea acotado cuando $p \in (p_0, p_1)$, para ciertos p_0 y p_1, y no sea acotado fuera de este rango. A estos dos números se les denomina *puntos extremos* y estas son las hipótesis de la teoría de extrapolación de Yano, que será el tema principal de este trabajo, titulado *Teoría de extrapolación de Yano bilineal*. Más específicamente, el teorema de Yano aborda el problema de encontrar estimaciones en el punto extremo $p_0 = 1$, suponiendo que la norma de T en L^p para $p > 1$ se comporta como $(p-1)^{-\alpha}$ para algún $\alpha > 0$.

Desde su formulación original, diversas versiones del teorema de Yano han aparecido en la literatura, lo que ha llevado a una teoría de extrapolación muy completa que incluye diferentes tipos de espacios en las hipótesis (hipótesis fuertes, débiles, etc.), distintos tipos de puntos extremos ($p_0 > 1$, $p_1 < \infty$ y $p_1 = \infty$), y varios comportamientos de la norma del operador.

La primera contribución de este trabajo, presentada en el Capítulo 3, es proporcionar una colección completa de resultados de la teoría lineal de Yano en el contexto de los espacios de Lorentz, abordando algunos casos no considerados previamente. Estos resultados incluyen demostraciones lo más directas posibles, evitando las dificultades técnicas de la teoría abstracta. En particular, dedicamos el Capítulo 2 a estandarizar y generalizar la técnica ODR, a partir de la cual se obtienen fácilmente la mayoría de las demostraciones de este trabajo. También extendemos todos estos resultados al caso en el que la norma del operador explota como una función *admisible*, una clase más amplia que las funciones de tipo potencia, que incluye los distintos comportamientos de la norma que hemos observado en la literatura. Apli-

camos estos teoremas una gran variedad de operadores para obtener estimaciones en los puntos extremos, incluyendo multiplicadores de Fourier como los multiplicadores de Marcinkiewicz y Mihlin-Hörmander, y operadores clásicos del análisis armónico como la transformada de Hilbert, la función maximal esférica, el operador maximal con núcleo no suave, el operador de Carleson y la función cuadrática intrínseca.

El Capítulo 4 está dedicado a nuestro segundo (y principal) objetivo, que es extender la teoría de extrapolación de Yano al contexto bilineal en el marco de los espacios de Lorentz. Siguiendo la estructura del caso lineal, consideramos diferentes tipos de hipótesis, normas que explotan como una función admisible, y varios tipos de puntos extremos. Sin embargo, a diferencia del caso lineal, este contexto requiere que también tengamos en cuenta la manera en que nos aproximamos al punto extremo. Para ello, empleamos la técnica ODR correspondiente, combinada con dos métodos: la linealización mediante un resultado fuera de la diagonal y la extrapolación a lo largo de rayos.

Por último, el Capítulo 5 explora la relación entre el teorema bilineal de Rubio de Francia y la extrapolación de Yano bilineal. En particular, aplicamos la extrapolación bilineal de Yano para obtener estimaciones con pesos A_p en los puntos extremos en los que no se obtienen acotaciones con el teorema bilineal de Rubio de Francia. También proporcionamos varios ejemplos de operadores para los cuales esta teoría produce nuevas estimaciones, como el operador bilineal de Bochner-Riesz en el índice crítico, operadores singulares con núcleo no suave, y multiplicadores bilineales de Fourier con variación acotada. Algunos de los resultados obtenidos para estos ejemplos aparecen en [6] y [31].

Abstract

There are many interesting problems in mathematical analysis whose solution involves the boundedness of an operator. It is also common that, in the setting of Lebesgue spaces, an operator T is bounded whenever $p \in (p_0, p_1)$ for some p_0 and p_1, and unbounded outside of this range. These two numbers are called *endpoints*. These are the assumptions of Yano's extrapolation theory, which will be the main subject of this manuscript, titled *Bilinear Yano's Extrapolation Theory*. To be more precise: Yano's theorem deals with the problem of finding estimates at the endpoint $p_0 = 1$, assuming that the norm of T in L^p for $p > 1$ behaves as $(p-1)^{-\alpha}$ for some $\alpha > 0$.

Since its original formulation, diverse versions of Yano's theorem have appeared in the literature, leading to a very complete extrapolation theory that includes different types of spaces appearing in the hypotheses (strong hypotheses, weak hypotheses, etc.), distinct types of endpoints ($p_0 > 1$, $p_1 < \infty$ and $p_1 = \infty$), and various operator norm behaviors.

The first contribution of this work, presented in Chapter 3, is to provide a comprehensive collection of results from Yano's linear theory within the context of Lorentz spaces, addressing some cases not previously considered. These results include proofs that are as direct as possible, avoiding technicalities from abstract theory. In particular, we devote Chapter 2 to standardize and generalize the ODR technique, allowing most proofs to be easily derived from it. We also extend all these results to the case where the operator norm blows up as an *admissible* function, a broader class than power functions, which includes the operator norm behaviors that we have observed in the literature. We apply the theorems to obtain endpoint estimates for a wide range of operators, including Fourier multipliers such as the

Marcinkiewicz and Mihlin-Hörmander multipliers, and classical harmonic analysis operators like the Hilbert transform, the spherical maximal function, the rough maximal operator, the Carleson operator, and the intrinsic square function.

Chapter 4 is dedicated to our second (and main) goal, which is to extend Yano's extrapolation theory to the bilinear setting in the context of Lorentz spaces. Following the structure of the linear case, we consider different types of hypotheses, norms that blow up as an admissible function, and various types of endpoints. However, in contrast to the linear case, this setting requires us to also consider the approach to the endpoint. To address this, we employ the corresponding ODR technique combined with two methods: linearization via an off-diagonal result and extrapolation along rays.

Finally, Chapter 5 explores the relationship between bilinear Rubio de Francia and bilinear Yano's extrapolation. Specifically, we apply bilinear Yano's extrapolation to obtain A_p-weighted estimates at endpoints not covered by the bilinear Rubio de Francia theorem. We also provide several examples of operators for which this theory yields new estimates, such as the bilinear Bochner-Riesz operator at the critical index, rough singular operators and bilinear Fourier multipliers of bounded variation. Some of the results we have obtained regarding these examples appear in [6] and [31].

Introduction

There are many interesting problems in mathematical analysis whose solution involves the boundedness of an operator. Let us think, for example, in the Lebesgue differentiation theorem and its connection with the boundedness of the Hardy-Littlewood maximal operator in L^p.

It is also common that in the setting of Lebesgue spaces, an operator T is bounded whenever $p \in (p_0, p_1)$ for some p_0 and p_1, and unbounded outside of this range. In this case, the norm of the operator T in L^p, given by

$$\|T\|_{L^p} = \sup_{\|f\|_{L^p}=1} \|Tf\|_{L^p},$$

blows up whenever p tends to p_0 or to p_1. These two numbers will be called *endpoints*. These are the assumptions of Yano's extrapolation theory, which will be the main subject of this work. To be more precise: Yano's theorem deals with the problem of finding estimates at the endpoint $p_0 = 1$ when the norm of T in L^p for $p > 1$ behaves like $(p-1)^{-\alpha}$ for some $\alpha > 0$. For example, the norm of the Hardy-Littlewood maximal operator M in $L^p(\mathbb{R}^n)$ behaves like $(p-1)^{-1}$, so it is unbounded at the endpoint $p_0 = 1$. Nevertheless, it is still interesting to determine for which integrable functions f, the function Mf is also integrable or at least locally integrable.

The origins of Yano's theory go back to 1928, when Titchmarsh [99] proved that if $\mathscr{C}$ is the conjugate function on the interval $(0, 1)$

$$\mathscr{C}f(x) := \text{p.v.} \int_0^1 f(x-y) \cot y \; dy,$$

then $\mathscr{C}f \in L^1(0,1)$, whenever $f\log(2+|f|)^{1+\varepsilon}$ is integrable for some $\varepsilon > 0$. Almost at the same time, Zygmund [104] claimed that the above result should remain true for $\varepsilon = 0$. One year later, Titchmarsh [100] published a second paper where, using that

$$\mathscr{C} : L^p(0,1) \longrightarrow L^p(0,1), \qquad \|\mathscr{C}\| \lesssim \frac{1}{p-1},$$

it was obtained Zygmund's conclusion. The same year, Zygmund [105] also proved the result using a different argument.

In 1951, inspired by the ideas of Titchmarsh, Yano [103] published the following theorem.

Theorem 1 (Yano, 1951)**.** *Let (X,μ) and (Y,ν) be finite measure spaces. If T is a sublinear operator such that, for every $1 < p \le 2$ and for some $\alpha > 0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

then,

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1(\nu),$$

where $L(\log L)^\alpha(\mu) \subset L^1(\mu)$ is the space of functions such that

$$\int_X \big|f(x)\big| \Big(1 + \log^+\big|f(x)\big|\Big)^\alpha \, d\mu(x) < \infty.$$

The relevance of Yano's theorem also relies on the fact that is optimal in the sense that we cannot expect to obtain a better endpoint estimate with these hypotheses. Indeed, in the particular case of the maximal operator M, for $1 < p \le 2$,

$$M : L^p(0,1) \longrightarrow L^p(0,1), \qquad \|M\| \lesssim \frac{1}{p-1},$$

and $Mf \in L^1(0,1)$ if and only if $f \in L\log L(0,1)$.

Since its original formulation and the extension to σ-finite measure spaces (see [14]), different versions of Yano's theorem have appeared in the literature, leading to a very complete extrapolation theory, even in the context of compatible couples of Banach spaces using techniques from interpolation theory (see [59–63], [81]). However, we shall restrict our work to the setting of weighted Lorentz spaces which includes, in a more precise way, the Lebesgue theory.

Many of the variants we are going to present have been motivated by questions related to the convergence of Fourier series. In fact, in 1966, Carleson [15] established the almost everywhere convergence of Fourier series for functions in $L^2(0,1)$

by proving the boundedness of the Carleson operator $\mathcal{C}$ from $L^2(0,1)$ to $L^{2,\infty}(0,1)$. At that time, it was already known (see [69]) that there exist functions in $L^1(0,1)$ whose Fourier series diverges almost everywhere. However, in 1967, Hunt [56] proved that

$$\mathcal{C} : L^{p,1}(0,1) \longrightarrow L^1(0,1), \qquad \|\mathcal{C}\| \lesssim \frac{1}{(p-1)^2},$$

for $1 < p < \infty$ and, therefore, the almost everywhere convergence of the Fourier series of functions in $L(\log L)^2(0,1)$ followed from a slight modification of Yano's theorem. To improve this space, a possible approach is to revisit Hunt's proof and observe that $L^p(0,1)$ estimates on $\mathcal{C}$ stem from the fact that

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(0,1)} \lesssim \frac{1}{p-1}|E|^{1/p}, \tag{1}$$

for every measurable set E. This type of estimate suggests that one might attempt to establish an analogue of Yano's theorem starting from restricted weak-type estimates of the form

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \qquad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

so that the norm of $\mathcal{C}$ grows like $(p-1)^{-1}$, rather than $(p-1)^{-2}$. In 1996, Antonov [3] used (1) to establish the almost everywhere convergence of the Fourier series for functions in $L \log L \log\log\log L(0,1)$. Following Antonov's ideas, Carro and Martín [23] proved an analogue of Yano's theorem with restricted weak-type estimates, concluding convergence in the same space. Other authors (see [91], [92], [17]) have also developed several versions of Yano's result using different types of spaces in the hypotheses.

Regarding the endpoint, there also exist variants of Yano's theorem in which $p_0 \neq 1$. For instance, the norm of the Hilbert transform on $L^p(\mathbb{R})$ blows up as p tends to ∞, and is unbounded on $L^\infty(\mathbb{R})$. To address this type of problem, in 1959, Zygmund [106, p. 119] proved a dual version of Yano's result for cases where the endpoint is $p_1 = \infty$.

Another interesting example is the Bochner-Riesz operator of order α, denoted by $\mathcal{B}^\alpha$. In particular, in $\mathbb{R}^2$ and for $0 < \alpha \leq 1/2$, Carleson and Sjölin [16] (see also [93]) showed in 1972 that

$$\mathcal{B}^\alpha : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2) \qquad \text{if and only if} \qquad \frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha}.$$

Hence, in this case, $p_0 = \frac{4}{3+2\alpha} > 1$ and $p_1 = \frac{4}{1-2\alpha} < \infty$.

Finally, there are also extensions of Yano's theorem concerning on how the operator norm behaves. Indeed, there exist examples of operators for which the best-known norm does not grow like a power function. One interesting case is the Carleson lacunary operator $\mathcal{C}_{\text{lac}}$ (see (3.13) for a definition), which satisfies that, for $1 < p \le 2$,

$$\mathcal{C}_{\text{lac}} : L^p(\mathbb{R}) \longrightarrow L^{p,\infty}(\mathbb{R}), \qquad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1 \left(\frac{1}{p-1} \right).$$

Generalizations of Yano's results with more general blow up behaviour of the operator norm can be found in [92], [1], [2].

Most of the proofs of Yano's extrapolation results in the context of Lorentz spaces follow a common structure, which we introduce here as the *ODR technique*. It consists of three steps.

Step 1. **Optimization of Hypotheses.** *Obtain an endpoint estimate for a subset $\mathscr{O}$ (to be determined) of the space $L^{p_0}(\mu)$.* For example, in the classical Yano's theorem,

$$\mathscr{O} = \left\{ f \in L^1(\mu) \cap L^\infty(\mu) : \|f\|_{L^\infty(\mu)} \le 1 \right\},$$

and the endpoint estimate obtained is

$$\|Tf\|_{L^1(\nu)} \lesssim \|f\|_{L^1(\mu)} \left(1 + \log^+ \frac{1}{\|f\|_{L^1(\mu)}} \right)^\alpha, \qquad \forall f \in \mathscr{O}.$$

Step 2. **Decomposition.** *Decompose a function $f \in L^{p_0}(\mu)$ as a linear combination (possibly infinite) of functions in $\mathscr{O}$.* In the case of Yano's theorem, one considers

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i, \quad \text{where} \quad f_i = \frac{f \chi_{\{2^{i-1} \le |f| < 2^i\}}}{2^i}.$$

Step 3. **Reconstruction.** *Combine properly the information from Steps 1 and 2.* In our example, since T is sublinear, for any $f \in L^1(\mu)$, applying the estimate from Step 1 to each f_i yields

$$\|Tf\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|Tf_i\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|f_i\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f_i\|_{L^1(\mu)}} \right)^\alpha \lesssim \|f\|_{L(\log L)^\alpha(\mu)}.$$

There exist several operators for which this theory yields endpoint estimates. In particular, the main source of most of these examples is Rubio de Francia's extrapolation theorem (see Theorem 1.5.2), which provides weighted estimates for $1 < p < \infty$.

For our purposes, we are mainly interested in the A_p classes of weights, which play an important role in harmonic analysis, as Muckenhoupt [82] proved in 1972 that they characterize the boundedness of the Hardy-Littlewood maximal operator M; i.e., for some $p \geq 1$, M is weak bounded on $L^p(w)$ if and only if $w \in A_p$. This characterization is significant because the Hardy-Littlewood maximal operator has often served as a model for the study of other operators. Indeed, it was shown that many other classical operators satisfy the same type of weighted inequalities, including the Hilbert transform or the Bochner-Riesz operator at the critical index.

If one is interested in studying this type of estimates, Rubio de Francia theorem (see [88], [89]) provides a useful method for obtaining A_p weighted estimates, based on information from a single point. More precisely, if for some fixed p and every $w \in A_p$, the operator T is bounded on $L^p(w)$, then for every $1 < p < \infty$ and every $w \in A_p$, T is bounded on $L^p(w)$. We have to mention here that, in general, the endpoints $p_0 = 1$ and $p_1 = \infty$ cannot be reached. However, a quantitative version of Rubio de Francia's theorem (see [20]) states that if, for some fixed $1 < q < \infty$ and every $w \in A_q$, the operator T is bounded on $L^q(w)$ with a norm essentially controlled by $[w]_{A_q}^{\alpha}$ for some $\alpha > 0$ (where $[w]_{A_q}$ is the A_q constant of the weight w as defined in (1.1)), then for every $1 < p < q$ and every $w \in A_1$, the norm of T in $L^p(w)$ can be bounded by $(p-1)^{-\alpha(q-1)}$. Therefore, by Yano's theorem, it follows that for every $w \in A_1$, the operator T is bounded from $L(\log L)^{\alpha(q-1)}(w)$ into $L^1 + L^\infty(w)$.

That is, Yano's theory can be used to obtain A_1 weighted estimates at the endpoint $p_0 = 1$. Using a similar argument involving Zygmund's extrapolation result, one can also obtain A_∞ weighted estimates at the endpoint $p_1 = \infty$, as shown in [29].

The main goal of this work is to extend Yano's extrapolation theory to the bilinear setting. The study of bilinear operators began in the 1970s with the work of Coifman and Meyer (see [36], [37]), who considered commutators of bilinear singular integrals, such as the bilinear Hilbert transform, defined as follows

$$B_H(f,g)(x) := \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(x-t)g(x+t)\frac{dt}{t}, \qquad x \in \mathbb{R}, \quad f,g \in \mathscr{S}(\mathbb{R}).$$

Years earlier, Calderón had posed the question of whether B_H satisfies an estimate of the form

$$B_H : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \longrightarrow L^p(\mathbb{R}), \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{2}$$

In particular, he conjectured that the case $p_1 = p_2 = 2$ holds. The first results concerning this problem were provided by Lacey and Thiele (see [71], [72]), who proved that (2) is true when $1 < p_1, p_2 \leq \infty$ and $2/3 < p < \infty$. Their work attracted significant attention to this research area, leading to the development of many results in the setting of multilinear operators, such as those by Kenig and Stein (see [66], [67]), Muscalu, Tao, and Thiele [83], Grafakos and Li [50], and Grafakos and Torres (see [52], [53]).

To obtain estimates for bilinear operators, multiple results regarding linear (or sublinear) operators have been extended to this new context. In particular, a bilinear version of Rubio de Francia extrapolation theory has been widely studied (see [51], [84], [76], [75]), and some attempts to extend Yano's theory to the bilinear context in the abstract setting appear in the work of Jawerth and Milman [63] (see also [5]). To develop a complete Yano's extrapolation theory for bilinear operators in the context of Lorentz spaces is the main goal of this work.

One of the main particularities of the bilinear theory is the fact that, if $p < 1$, the space L^p is quasi-Banach, and it is not true that, for any decomposition $f = \sum_i f_i$,

$$\Big\| \sum_i f_i \Big\|_{L^p} \leq \sum_i \|f_i\|_{L^p}.$$

As a consequence, Step 3 of the ODR technique entails some difficulties. Observe, however, that the condition $p < 1$ is common in the bilinear setting, as shown in Figure 1, where the square represents all possible points with $1 \leq p_1, p_2 \leq \infty$, and the shaded area indicates those points for which $p < 1$.

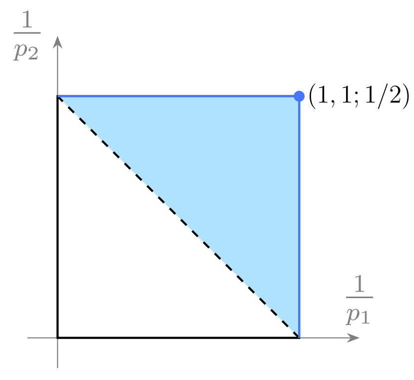


Figure 1: $(p_1, p_2; p)$ such that $1 \leq p_1, p_2 \leq \infty$ and $p < 1$.

As we will see later, another particularity of the bilinear context is that different results arise depending on how we reach the endpoint. Specifically, we may have hypotheses valid for all points near the endpoint, or only for those lying on a ray or even along a curve (as seen in [87]).

Concerning the operator norm, it may behave as a function of p alone or as a function of both p_1 and p_2. For example, the analogue of Yano's result in the bilinear context could involve hypotheses such as

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p, \qquad \|B\| \lesssim \frac{1}{(p_1-1)^{\alpha_1}(p_2-1)^{\alpha_2}},$$

for some $\alpha_1, \alpha_2 > 0$ and for every $1 < p_1, p_2 \leq 2$, or alternatively,

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p, \qquad \|B\| \lesssim \frac{1}{\left(p-\frac{1}{2}\right)^{\alpha}},$$

for some $\alpha > 0$ and for every $1 < p_1, p_2 \leq 2$, generating different endpoints in each case. Let us also observe that, if the norm of B follows the first form, there is not just a single endpoint $(1, 1; 1/2)$, but rather all points with $p_1 = 1$ and $1 < p_2 \leq 2$, or $p_2 = 1$ and $1 < p_1 \leq 2$, also act as endpoints.

Contributions of this manuscript. The first contribution of this work is to present a comprehensive collection of results from Yano's linear theory within the context of Lorentz spaces, with proofs as direct as possible, avoiding technicalities from the abstract theory. In particular, we have standardized and generalized the ODR technique, allowing most proofs to be easily derived from it. We extend all these results to the case where the operator norm blows up as an *admissible* function, a broader class than power functions which includes all operator norm behaviours observed in the literature.

Some of these cases we consider have not been previously addressed, such as the case when

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu), \quad p_0 < p < p_1,$$

and the endpoint is $p_0 > 1$ (the original proof was for $p_0 = 1$) or $p_1 < \infty$ (see Section 3.5).

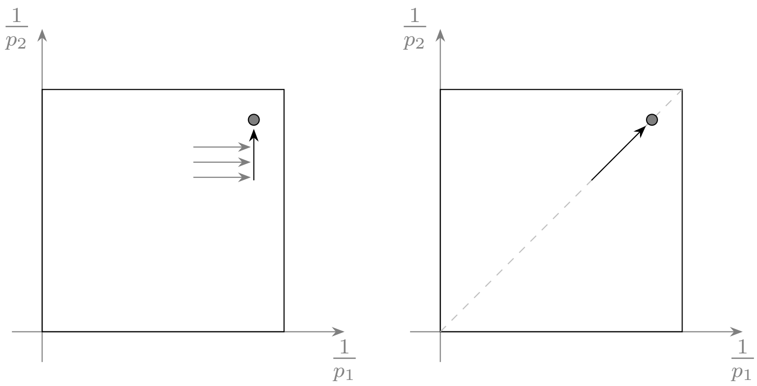
We also devote Section 3.6 to the case of having a fixed target space for all the hypothesis estimates; i.e., we consider estimates of the form

$$T : L^p(\mu) \longrightarrow \mathcal{X},$$

where $\mathcal{X}$ is fixed for all p.

We include applications of our theorems to obtain endpoint estimates for a broad range of operators, including Fourier multipliers such as the Marcinkiewicz and Mihlin–Hörmander multipliers, and classical harmonic operators like the Hilbert transform, the spherical maximal function, the rough maximal operator, the Carleson operator, and the intrinsic square function.

Our second (and main) goal in this work is to extend Yano's extrapolation theory to the bilinear setting, in the context of Lorentz spaces. Following the structure of the linear case, we have considered different types of hypotheses, norms that blow up like an admissible function, and various types of endpoints.



(a) (b)

Figure 2: Techniques to approach an endpoint in the bilinear setting.

In contrast to the linear case, in this setting, we must also take into account the approach to the endpoint. We have used two techniques: linearization via an off-diagonal result (Figure 2(a)) and extrapolation by rays (Figure 2(b)). We present, as examples, two of our theorems that illustrate how we can reach the point $(1, 1; 1/2)$ in the context of finite measure spaces.

Theorem 2. *Let (X_1, μ_1), (X_2, μ_2), and (Y, ν) be finite measure spaces. If B is a bisublinear operator such that, for every $(p_1, p_2; p)$ with $1 < p_1, p_2 \leq 2$ and for some $\alpha_1, \alpha_2 > 0$,*

$$B : L^{p_1}(\mu_1) \times L^{p_2}(\mu_2) \longrightarrow L^p(\nu), \qquad \|B\| \lesssim \frac{1}{(p_1 - 1)^{\alpha_1}(p_2 - 1)^{\alpha_2}},$$

then, for every $\varepsilon > 0$,

$$B : L(\log L)^{\alpha_1+1+\varepsilon}(\mu_1) \times L(\log L)^{\alpha_2+1+\varepsilon}(\mu_2) \longrightarrow L^{\frac{1}{2},\infty}(\nu).$$

Theorem 3. *Let (X_1, μ_1), (X_2, μ_2), and (Y, ν) be finite measure spaces. If B is a bisublinear operator such that, for every $1/2 < p \leq 1$, for some $\alpha > 0$,*

$$B : L^{2p}(\mu_1) \times L^{2p}(\mu_2) \longrightarrow L^p(\nu), \qquad \|B\| \lesssim \frac{1}{(p - \frac{1}{2})^{\alpha}},$$

then, for every $\varepsilon > 0$,

$$B : L(\log L)^{\alpha+1+\varepsilon}(\mu_1) \times L(\log L)^{\alpha+1+\varepsilon}(\mu_2) \longrightarrow L^{\frac{1}{2},\infty}(\nu).$$

In addition, we apply bilinear Yano's extrapolation to obtain A_p weighted estimates at the endpoints not covered by the bilinear Rubio de Francia theorem (see [51]), which provides estimates of the form

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w), \quad w_1 \in A_{p_1},\, w_2 \in A_{p_2},$$

where $w = w_1^{p/p_1} w_2^{p/p_2}$, at each point $(p_1, p_2; p)$ with $1 < p_1, p_2 < \infty$, based on information from a single point $(r_1, r_2; r)$ for each $w_1 \in A_{r_1}$ and $w_2 \in A_{r_2}$. By employing Yano's bilinear theory, we obtain weighted estimates at the remaining points: $(1, p_2; p)$, $(p_1, 1; p)$, $(1, \infty; 1)$, $(\infty, 1; 1)$, $(\infty, p_2; p)$, $(p_1, \infty; p)$, and $(\infty, \infty; \infty)$.

We also provide several examples of operators for which this theory yields new estimates, such as the bilinear Bochner-Riesz operator at the critical index, rough singular operators and bilinear Fourier multipliers of bounded variation. Some results we have obtained regarding these examples appear in [6] and [31].

Outline. We present a brief summary of the contents of each chapter.

▸ **Chapter 1.** In this chapter, we establish the theoretical framework used throughout the manuscript. We begin in Section 1.1 by setting the conventions and notations that will be assumed. Section 1.2 introduces the concepts of distribution function, decreasing rearrangement, and maximal function, along with their main properties. Our notion of admissible function is introduced in Section 1.3. Section 1.4 is devoted to defining the function spaces used in the manuscript. Finally, in Section 1.5, we define the classes of Muckenhoupt weights, and state the extrapolation theorems concerning them.

▸ **Chapter 2.** This chapter summarizes all the technical results related to the ODR technique that are needed for linear and bilinear Yano's extrapolation theory. In Section 2.1, we present the results concerning Step 1. In Section 2.2, we introduce the different types of decompositions we shall use, and Section 2.3 is devoted to reconstruction results.

▸ **Chapter 3.** The third chapter contains a review of Yano's linear extrapolation in the context of Lorentz spaces. The chapter is divided into six sections, according to the different type of spaces involved in the hypotheses: strong L^p hypotheses (Section 3.1), weak hypotheses (Section 3.2), restricted weak-type hypotheses (Section 3.3), hypotheses for characteristic functions (Section 3.4), strong $L^{p,\infty}$ hypotheses (Section 3.5), and hypotheses with fixed target space in Section 3.6. All the extrapolation proofs presented follow the ODR technique, except for those in Section 3.4 and one in Section 3.5.

▸ **Chapter 4.** In this chapter, we develop a bilinear Yano's extrapolation theory. As in the linear case, we distinguish different types of hypotheses. The chapter is structured as follows: in Section 4.1, we shall treat the case when the target space at the endpoint estimate is Banach and Section 4.2 will be dedicated to the quasi-Banach case. Our proofs will follow the ODR technique and, in both settings, we mainly use linearization in each variable (see Theorems 4.1.3, 4.1.6 and 4.2.4). We also present results where we reach the endpoint through rays (see Theorems 4.1.8, 4.1.9, 4.2.6 and 4.2.7), and we treat the endpoint $(1, \infty; 1)$ separately (see Theorem 4.1.10 and Theorem 4.2.8).

▸ **Chapter 5.** The final chapter explores the relationship between bilinear Rubio de Francia and Yano's extrapolation. We then use this technique to obtain new estimates for bilinear rough singular operators (see Theorem 5.1.1), the bilinear Bochner-Riesz operator at the critical index (see Theorem 5.2.1), and bilinear Fourier multipliers of bounded variation (see Theorem 5.3.1). Finally, in Theorems 5.4.1 and 5.4.2, we derive new estimates for bilinear Hardy operators with power weights using only bilinear Yano's theory.

Notations

$C,\ C_1,\ C_2$	positive universal constants
$a \lesssim b$	$a \leq Cb$ for some constant $C > 0$
$a \gtrsim b$	$a \geq Cb$ for some constant $C > 0$
$a \approx b$	$a \lesssim b$ and $b \lesssim a$
λ_f^μ	the distribution function of f in the measure space (X, μ)
f_μ^*	the decreasing rearrangement of f in the measure space (X, μ)
f_μ^{**}	the maximal function of f in the measure space (X, μ)
$f(s^+)$	$\lim_{t\to s^+} f(t)$
$\inf$	the infimum of a set
$\sup$	the supremum of a set
$\operatorname{ess\,sup}$	the essential supremum of a set
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^-$	the set of negative integers (without 0)
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of strictly positive real numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\mathbb{S}^{n-1}$	the $(n-1)$-dimensional unit sphere $\big\{x \in \mathbb{R}^n : \lvert x\rvert = 1\big\}$
$B(x, R)$	the ball with radius $R > 0$ centered at x
χ_E	the characteristic function of the set E
$\lvert E\rvert$	the Lebesgue measure of $E \subset \mathbb{R}^n$

$\log s$	the logarithm with base e of $s > 0$
$\log^+ s$	$\max\{0, \log s\}$ for $s > 0$
$\log_n s$	$1 + \underbrace{\log^+ \dots \log^+}_{n} s$
$\varphi,\ \varphi_1,\ \varphi_2$	admissible functions
$I_{\varphi,p}(s)$	$s^{1/p}\varphi\left(\log_1 \frac{1}{s}\right)$ for $s > 0$ and $0 < p < \infty$
$J_{\varphi,p}(s)$	$s^{1/p}\varphi\left(\log_1 s\right)$ for $s > 0$ and $0 < p < \infty$
$C^\infty(\mathbb{R}^n)$	the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$
$\partial_j^m f$	the m-th partial derivative of $f(x_1, \dots, x_n)$ with respect to x_j
$\partial^\beta f$	$\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$
$\mathscr{S}(\mathbb{R}^n)$	$\{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \left\lvert x^\alpha \partial^\beta f(x)\right\rvert < \infty\}$
$L^p(\mu)$	the Lebesgue space over the measure space (X, μ)
$L^p(X)$	the Lebesgue space over the measure space $(X, \lvert\cdot\rvert)$
$L^1_{\text{loc}}(\mu)$	the space of functions in $L^1(\mu)$ over (μ, K) for any compact set $K \subset X$
$L^1_{\text{loc}}(X)$	the space of functions in $L^1(K)$ for any compact set $K \subset X$
$L^{p,\infty}(\mu)$	the weak Lebesgue space over the measure space (X, μ)
$L^{p,\infty}(X)$	the weak Lebesgue space over the measure space $(X, \lvert\cdot\rvert)$
BMO	the space of bounded mean oscillation functions
v, v_0, v_1, v_2	weights defined in $\mathbb{R}^+$
$V(t)$	$\int_0^t v(s)ds$
w	a weight defined in $\mathbb{R}^n$
$w(E)$	$\int_E w(x)\,dx$
$[w]_{A_p}$	the A_p constant of the weight w
$[w]_{A_p^{\mathcal{R}}}$	the $A_p^{\mathcal{R}}$ constant of the weight w
T	an operator defined in one variable
T_m	the operator associated to the linear multiplier m
M	the Hardy-Littlewood maximal operator
M_Ω	the rough maximal operator
$\mathscr{M}$	the spherical maximal operator
H	the Hilbert transform
$\mathcal{C}$	the Carleson operator
$\mathcal{C}_{\text{lac}}$	the Carleson lacunary operator
G_α	the intrinsic square function
B	an operator defined in two variables
B_m	the operator associated the bilinear multiplier m
B_H	the bilinear Hilbert transform
$\mathcal{H}_v$	the weighted bilinear Hardy operator
$\mathcal{B}^\alpha$	the bilinear Bochner-Riesz operator of index α

CHAPTER 1

Preliminaries

In this chapter, we establish the theoretical framework used throughout the text. We begin in Section 1.1 by setting the conventions and notations that will be assumed. Section 1.2 introduces the concepts of distribution function, decreasing rearrangement, and maximal function, along with their main properties. Our notion of *admissible function* is introduced in Section 1.3. Section 1.4 is devoted to defining the function spaces used in the text. Finally, in Section 1.5, we define the classes of Muckenhoupt weights, and state the extrapolation theorems concerning them.

1.1 Conventions and notations

We use capital C to denote positive universal constants that do not depend on any important parameter. As usual, we write $a \lesssim b$ (or $a \gtrsim b$) if there exists a constant $C > 0$ such that $a \leq Cb$ (or $a \geq Cb$). If both $a \lesssim b$ and $a \gtrsim b$ hold, we write $a \approx b$. In particular, if f and g are functions, we write $f \lesssim g$ if $f(s) \lesssim g(s)$ for every s in the domain.

We denote by $\log s$ the logarithm of $s > 0$ with base e and $\log^+ s = \max\{0, \log s\}$. For $n \in \mathbb{N}$, we write

$$\log_n s = 1 + \underbrace{\log^+ \dots \log^+}_{n} s.$$

Recall that a function f defined on a subset of $\mathbb{R}$ is *essentially increasing* (respectively, *essentially decreasing*) if there exists an increasing function (respectively, a decreasing function) g, defined on the same subset, such that $f \approx g$.

Given a pair of quasi-normed function spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $T : \mathcal{X} \to \mathcal{Y}$ if the operator T is well-defined from $\mathcal{X}$ to $\mathcal{Y}$ and

$$\|T\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{f\in\mathcal{X}} \frac{\|Tf\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} < \infty.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are clear from the context, we simply write $\|T\|$. Here, $\mathcal{X}$ is referred to as the domain space, and $\mathcal{Y}$ as the target space. If $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{Y}$ are quasi-normed function spaces, we write $B : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}$ if the operator B is well-defined from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{Y}$ and

$$\|B\|_{\mathcal{X}_1\times\mathcal{X}_2\to\mathcal{Y}} := \sup_{\substack{f\in\mathcal{X}_1\\ g\in\mathcal{X}_2}} \frac{\big\|B(f,g)\big\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}_1}\|g\|_{\mathcal{X}_2}} < \infty.$$

Definition 1.1.1. An operator T is sublinear if, for every pair of functions f, g and every $a \in \mathbb{R}$, we have

$$\big|T(af)\big| = |a||Tf|,$$

and

$$\big|T(f+g)\big| \le |Tf| + |Tg|.$$

Similarly, we say that B is bisublinear if it is sublinear in each variable.

For the purpose of this work, we shall always assume that T also satisfies that, for every pair of functions f, g,

$$|Tf - Tg| \le \big|T(f-g)\big|,$$

and, if B is bisublinear, this condition must hold in each variable. We observe that this property is always fulfilled by linear (respectively bilinear) operators or by positive sublinear (respectively positive bisublinear) operators, which cover all our examples.

1.2 Distribution function and decreasing rearrangement

Given a non-atomic and σ-finite measure space (X, μ), we define the distribution function, the decreasing rearrangement, and the maximal function of a μ-measurable function f. We refer to [8, Chapter 2].

Definition 1.2.1. The *distribution function* of a μ-measurable function f is defined by

$$\lambda_f^\mu(s) := \mu\big(\{x \in X : \big|f(x)\big| > s\}\big), \quad s \ge 0.$$

Definition 1.2.2. The *decreasing rearrangement* of a μ-measurable function f is defined by

$$f_\mu^*(t) := \inf\{s \geq 0 : \lambda_f^\mu(s) \leq t\}, \quad t \geq 0.$$

Definition 1.2.3. The *maximal function* of a μ-measurable function f is defined by

$$f_\mu^{**}(t) := \frac{1}{t}\int_0^t f_\mu^*(s)\,ds, \quad t > 0.$$

When the measure μ is not relevant in the context, we omit it and simply write λ_f, f^*, and f^{**}, respectively. We now state some elementary properties of these functions.

Proposition 1.2.4. *Let f, g be two μ-measurable functions. Then:*

(i) λ_f^μ, f_μ^*, *and* f_μ^{**} *are nonnegative, decreasing functions.*

(ii) λ_f^μ *and* f_μ^* *are right-continuous functions, and* f_μ^{**} *is continuous whenever* $f_\mu^* \in L^1_{\text{loc}}(\mathbb{R}^+)$.

(iii) If $|g| \leq |f|$ *almost everywhere, then* $\lambda_f^\mu \leq \lambda_g^\mu$, $f_\mu^* \leq g_\mu^*$, *and* $f_\mu^{**} \leq g_\mu^{**}$.

(iv) For every $a \in \mathbb{R}$, $\lambda_{af}^\mu(s) = \lambda_f^\mu(s/|a|)$, $(af)_\mu^* = |a|\, f_\mu^*$, *and* $(af)_\mu^{**} = |a|\, f_\mu^{**}$.

(v) We have

$$\lambda_{f+g}^\mu(s_1+s_2) \leq \lambda_f^\mu(s_1) + \lambda_g^\mu(s_2), \quad s_1, s_2 \geq 0,$$

and

$$(f+g)_\mu^*(t_1+t_2) \leq f_\mu^*(t_1) + g_\mu^*(t_2), \quad t_1, t_2 \geq 0.$$

(vi) $(f+g)_\mu^{**} \leq f_\mu^{**} + g_\mu^{**}$.

(vii) $f_\mu^* \leq f_\mu^{**}$.

(viii) For every $0 < p < \infty$, $(|f|^p)^* = (f^*)^p$,

$$\int_X |f|^p \, d\mu = p\int_0^\infty s^{p-1}\lambda_f^\mu(s)\,ds = \int_0^\infty (f_\mu^*(t))^p\,dt,$$

and, for $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in X} \big|f(x)\big| = \inf\{s : \lambda_f^\mu(s) = 0\} = f_\mu^*(0).$$

The following formula from [28] will be useful for our purposes.

Proposition 1.2.5. *Let $\phi \in L^1_{\text{loc}}(\mathbb{R}^+)$. Then, for $0 < p < \infty$,*

$$\int_0^\infty (f^*_\mu(t))^p \phi(t)\, dt \approx \int_0^\infty s^{p-1} \left(\int_0^{\lambda^\mu_f(s)} \phi(t)\, dt \right) ds.$$

1.3 Admissible functions

We define the following set of functions as extensions of power functions.

Definition 1.3.1. An increasing, absolutely continuous, and derivable function

$$\varphi : [1, \infty) \longrightarrow [1, \infty)$$

is said to be *admissible* if:

i) $\varphi(1) = 1$,

ii) $\lim_{t\to\infty} \varphi(t) = \infty$,

iii) for every $s, t \geq 1$, $\varphi(st) \leq \varphi(s)\varphi(t)$.

Throughout the text, φ will always denote an admissible function.

Remark 1.3.2. For every admissible function φ, there exists $\beta > 0$ such that, for every $s \geq 1$,

$$\varphi(s) \lesssim s^\beta.$$

To see this, let β such that $\varphi(2) = 2^\beta$. Since, for every $s \geq 1$, there exists $n \geq 0$ such that $2^n \leq s < 2^{n+1}$, we get

$$\varphi(s) \leq \varphi(2^{n+1}) \leq (\varphi(2))^{n+1} = 2^{(n+1)\beta} \leq 2^\beta s^\beta.$$

For our purposes, we can replace iii) with the weaker conditions:

iv) for every $s, t \geq 1$, $\varphi(st) \lesssim \varphi(s)\varphi(t)$,

v) for some $\beta > 0$ and every $s \geq 1$, $\varphi(s) \lesssim s^\beta$,

and all the theory still holds.

Example 1.3.3. If $\gamma \geq 0$ and $\beta_1, \dots, \beta_k \geq 0$,

$$\varphi(s) = s^\gamma \prod_{n=1}^k (\log_n s)^{\beta_n}$$

is an admissible function.

1.4 Function spaces

In this section, we define the main function spaces that will appear in the text, explain their main properties and the relation between them. Let $0 < p < \infty$, unless indicated otherwise.

The *Lebesgue space* $L^p(\mu)$ is defined as the set of measurable functions such that

$$\|f\|_{L^p(\mu)} := \left(\int_X |f(s)|^p \, d\mu \right)^{1/p} < \infty.$$

For any weight (a non-negative and locally integrable function) w, if $d\mu = w(s)\, ds$, we write $L^p(w)$. When $w \equiv 1$, we omit it.

For every weight v defined in $\mathbb{R}^+$, the *classical Lorentz space* $\Lambda^p_\mu(v)$ is defined using Lebesgue spaces as $\|f\|_{\Lambda^p_\mu(v)} = \|f^*_\mu\|_{L^p(v)}$. These spaces were first studied by Lorentz when the underlying measure space is $((0,\ell),|\cdot|)$ with $\ell < \infty$ (see [77], [78]). In particular, if for some $0 < p, q < \infty$, $v(t) = t^{\frac{p}{q}-1}$, we have that $\Lambda^p_\mu(v) = L^{q,p}(\mu)$.

The space $\Lambda^{p,\infty}_\mu(v)$ is defined by

$$\|f\|_{\Lambda^{p,\infty}_\mu(v)} := \sup_{t>0} f^*_\mu(t) V(t)^{\frac{1}{p}} < \infty,$$

where, for every $t > 0$, $V(t) := \int_0^t v(s)\, ds$.

The *logarithmic spaces* that we define below are special cases of $\Lambda^1_\mu(v)$. They are an extension of the *Zygmund space* $L \log L(\mu)$.

Definition 1.4.1. Let $\alpha_1, \dots, \alpha_j \geq 0$ and let φ be an admissible function. The space $L \prod_{k=1}^j (\log_k L)^{\alpha_k} \varphi(\log_j L)(\mu)$ is the set of μ-measurable functions such that

$$\|f\|_{L \prod_{k=1}^j (\log_k L)^{\alpha_k} \varphi(\log_j L)(\mu)} := \int_0^\infty f^*_\mu(t) \prod_{k=1}^j \left(\log_k \frac{1}{t} \right)^{\alpha_k} \varphi\left(\log_j \frac{1}{t} \right) dt < \infty.$$

We also introduce the *weighted gamma space* $\Gamma^{p,\infty}_\mu(v)$ that was first studied in [28].

Definition 1.4.2. Let v be a weight in $\mathbb{R}^+$. The space $\Gamma^{p,\infty}_\mu(v)$ is defined by the set of μ-measurable functions such that

$$\|f\|_{\Gamma^{p,\infty}_\mu(v)} := \sup_{t>0} f^{**}_\mu(t) V(t)^{\frac{1}{p}} < \infty.$$

We observe that they are always Banach spaces.

Remark 1.4.3. From the fact that $f_\mu^* \le f_\mu^{**}$ and Hardy's inequality, it follows that, for every $1 < p < \infty$,

$$\|f\|_{L^{p,\infty}(\mu)} \le \|f\|_{\Gamma_\mu^{p,\infty}} \le \frac{p}{p-1}\|f\|_{L^{p,\infty}(\mu)}.$$

On some occasions, we shall also use the following notation:

$$\|f\|_{\Lambda_\mu^{p,\infty}[V]} := \sup_{t>0} f_\mu^*(t)V(t)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{\Gamma_\mu^{p,\infty}[V]} := \sup_{t>0} f_\mu^{**}(t)V(t)^{\frac{1}{p}}.$$

Remark 1.4.4. We observe that, if $V(t) = t^{1/p}/\varphi(\log_1 t)$, then we have

$$\Lambda^{1,\infty}[V] \subseteq L^{p,\infty}_{\text{loc}} \qquad \text{and, if } p=1,\ \Gamma^{1,\infty}[V] \subseteq L^1_{\text{loc}}.$$

Let us consider now the following extension of Zygmund exponential spaces.

Definition 1.4.5. Let $\alpha > 0$ and let w be a weight in $\mathbb{R}^n$. The space $E^\alpha(w)$ is the set of all μ-measurable functions such that, for some $\lambda = \lambda(f)$,

$$\sup_K \frac{\int_K \exp\Big(\lambda\big|f(x)\big|\Big)^{1/\alpha} w(x)dx}{\max\big(w(K),1\big)} < \infty,$$

where the supremum extends over all measurable sets K with $w(K) := \int_K w < \infty$.

Moreover, if $V(t) = \big(\log_1 \frac{1}{t}\big)^{-\alpha}$, $f \in E^\alpha(w)$ if and only if $f \in \Lambda_w^{1,\infty}[V]$ (see [29]).

Definition 1.4.6. Let w be a weight in $\mathbb{R}^n$, $L^\infty_\varphi(w)$ is the space of measurable functions such that

$$\|f\|_{L^\infty_\varphi(w)} = \|f\|_{L^\infty(w)} + \int_1^\infty f_w^*(s)\varphi\left(\log_1 s\right)\frac{ds}{s} < \infty.$$

If $\varphi(s) = s^\alpha$ for some $\alpha > 0$, we simply write $L^\infty_\alpha(w)$ (see [29]).

1.5 Muckenhoupt weights and extrapolation

We conclude this chapter by introducing the A_p classes of *Muckenhoupt weights*, which characterize the boundedness of the Hardy-Littlewood maximal function, defined, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing the point x. We refer to [45, Chapter 7] and [42, Chapter 7] for a complete statement of the results presented in this section. A weight w defined on $\mathbb{R}^n$ belongs to A_p for some $1 < p < \infty$, if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty, \tag{1.1}$$

where p' is the conjugate of p. The quantity $[w]_{A_p}$ is known as the A_p constant of w. For $p = 1$, a weight $w \in A_1$ if

$$[w]_{A_1} := \operatorname{ess\,sup} \frac{Mw(x)}{w(x)} < \infty.$$

The class of A_∞ weights is defined by

$$A_\infty = \bigcup_{1 \le p < \infty} A_p.$$

In 1972, Muckenhoupt [82] proved the following result.

Theorem 1.5.1. *For every $1 < p < \infty$ and every weight w,*

$$M : L^p(w) \longrightarrow L^p(w)$$

if and only if $w \in A_p$. Moreover,

$$\|M\|_{L^p(w) \longrightarrow L^p(w)} \lesssim pp'[w]_{A_p}^{\frac{1}{p-1}}. \tag{1.2}$$

Also, for $1 \le p < \infty$,

$$M : L^p(w) \to L^{p,\infty}(w)$$

if and only if $w \in A_p$ and

$$\|M\|_{L^p(w) \to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{1/p}.$$

The behaviour of the operator norm in (1.2) was proved in [13]. Additionally, it holds (see [20]) that, for every $1 < p < \infty$ and every $w \in A_1$,

$$\|M\|_{L^p(w) \to L^p(w)} \lesssim p'[w]_{A_1}. \tag{1.3}$$

For our purposes, Rubio de Francia extrapolation theorem will be fundamental. We state a modern version of the original theorem (see [88], [89], [41], [43]).

Theorem 1.5.2. *If for some* $1 \leq p_0 < \infty$ *and every* $w \in A_{p_0}$,

$$T : L^{p_0}(w) \to L^{p_0}(w), \quad \|T\| \lesssim \psi([w]_{A_{p_0}}), \tag{1.4}$$

where $\psi : (0, \infty) \to (0, \infty)$ *is an increasing function, then, for* $1 < p < \infty$ *and every* $w \in A_p$,

$$T : L^p(w) \to L^p(w),$$

with

$$\|T\| \lesssim \begin{cases} \psi\left(C_1 (p')^{(p_0-1)} [w]_{A_p}^{\frac{p_0-1}{p-1}}\right) & \text{if } \ 1 < p < p_0, \\ \psi\left(C_2 p^{\frac{p-p_0}{p-1}} [w]_{A_p}\right) & \text{if } \ p_0 < p < \infty. \end{cases}$$

Using (1.3), the following result was obtained in [20].

Corollary 1.5.3. *If* T *satisfies* (1.4), *then, for* $1 < p < p_0$ *and every* $w \in A_1$,

$$T : L^p(w) \longrightarrow L^p(w) \quad \|T\| \lesssim \psi\left(C (p')^{p_0-1} [w]_{A_1}^{p_0}\right).$$

CHAPTER 2

Optimization, Decomposition, Reconstruction

In 1951, inspired by the ideas of Titchmarsh, Yano [103] established the following theorem.

Theorem 2.0.1. *Let (X, μ) and (Y, ν) be finite measure spaces. If T is a sublinear operator such that, for every $1 < p \leq 2$ and for some fixed $\alpha > 0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1(\nu).$$

Yano's theorem is optimal, in the sense that we can not expect to get a better endpoint estimate with these hypotheses. Indeed, for $1 < p \leq 2$,

$$M : L^p(0,1) \longrightarrow L^p(0,1), \qquad \|M\| \lesssim \frac{1}{(p-1)},$$

and Stein [95] proved that $Mf \in L^1(0,1)$ if and only if $f \in L \log L(0,1)$.

Let us present a modern proof of Yano's theorem (see, for example, [26]) that illustrates the basic structure of most proofs of the extrapolation theorems in this text.

Step 1. **Optimization of Hypotheses.** *To obtain an endpoint estimate for a* subset $\mathscr{O}$ *(to be determined) of* $L^1(\mu)$. For example, in this case,

$$\mathscr{O} = \left\{ f \in L^1(\mu) \cap L^\infty(\mu) : \|f\|_{L^\infty(\mu)} \leq 1 \right\}.$$

Since $\mathscr{O} \subset L^p(\mu)$ for every $1 < p \leq 2$, we have that, for every $f \in \mathscr{O}$,

$$\|Tf\|_{L^1(\nu)} \leq \|Tf\|_{L^p(\nu)} \left(\nu(Y)\right)^{1/p'} \approx \|Tf\|_{L^p(\nu)} \lesssim \frac{\|f\|_{L^p(\mu)}}{(p-1)^\alpha} \lesssim \frac{\|f\|_{L^1(\mu)}^{1/p}}{(p-1)^\alpha}.$$

Hence, taking the infimum over $1 < p \leq 2$ on the right hand side, we obtain

$$\|Tf\|_{L^1(\nu)} \lesssim \|f\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f\|_{L^1(\mu)}}\right)^\alpha, \qquad \forall f \in \mathscr{O}. \tag{2.1}$$

Step 2. **Decomposition.** *To find a decomposition of a function $f \in L^1(\mu)$ as a linear combination of functions in $\mathscr{O}$.* In this case, we can consider

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i \quad \text{where} \quad f_i = \frac{f \chi_{\{2^{i-1} \leq |f| < 2^i\}}}{2^i}.$$

Observe that $f_i \in \mathscr{O}$ and $\|f_i\|_{L^1(\mu)} \leq \lambda_f^\mu(2^{i-1})$.

Step 3. **Reconstruction.** *To gather the information obtained in Steps 1 and 2.* As T is sublinear, for any $f \in L^1(\mu)$, applying (2.1) to each f_i yields

$$\begin{aligned}\|Tf\|_{L^1(\nu)} &\lesssim \sum_{i \in \mathbb{Z}} 2^i \|Tf_i\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|f_i\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f_i\|_{L^1(\mu)}}\right)^\alpha \\ &\lesssim \int_0^\infty \lambda_f^\mu(s) \left(\log_1 \frac{1}{\lambda_f^\mu(s)}\right)^\alpha ds \lesssim \|f\|_{L(\log L)^\alpha(\mu)},\end{aligned}$$

where Proposition 1.2.5 is used in the last inequality.

The above steps will be called the *ODR technique* (Optimization, Decomposition, and Reconstruction). This chapter summarizes all the technical results that we will need in Chapters 3 and 4. In Section 2.1, we present the results concerning Step 1. In Section 2.2, we introduce the different types of decompositions we shall use and Section 2.3 is devoted to reconstruction results. From now on, we omit the underlying measure spaces (X, μ) and (Y, ν), which are σ-finite and non-atomic.

2.1 Optimization of the hypotheses

In this section, we compute the various infima that will arise. Some of them already appear in [1], [2] and [26], but we included them all for the sake of completeness.

Lemma 2.1.1. *For each fixed* $1 \le p_0 < \infty$*, the following holds:*

$$\inf_{p_0 \le p < \infty} \varphi(p) s^{-\frac{1}{p}} \lesssim \begin{cases} \varphi(p_0)\, \varphi(\log_1 \frac{1}{s}), & 0 < s < 1, \\ \varphi(p_0) s^{-\frac{1}{p_0}}, & s \ge 1, \end{cases}$$

where the omitted constant does not depend on p_0*.*

Proof. When $0 < s < 1$, choose $p = p_0(\log_1 \frac{1}{s})$. Clearly $p_0 < p < \infty$, and

$$\varphi(p) s^{-\frac{1}{p}} = \varphi\left(p_0 \left(\log_1 \frac{1}{s}\right)\right) e^{\frac{\log \frac{1}{s}}{p_0(1+\log \frac{1}{s})}} \le \varphi\left(p_0 \left(\log_1 \frac{1}{s}\right)\right) e \lesssim \varphi(p_0)\varphi\left(\log_1 \frac{1}{s}\right).$$

In the case $s \ge 1$, it suffices to select $p = p_0$. □

The following functions $I_{\varphi,p}$ and $J_{\varphi,p}$ will be relevant for our purposes.

Lemma 2.1.2. *The function*

$$I_{\varphi,p}(s) := s^{\frac{1}{p}} \varphi\left(\log_1 \frac{1}{s}\right), \qquad s > 0, \tag{2.2}$$

satisfies the following properties:

a) $I_{\varphi,p}(0^+) = 0$*.*

b) $I_{\varphi,p}$ *is essentially increasing. In fact,*

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt \approx I_{\varphi,p}(s).$$

c) $I_{\varphi,1}$ *is subadditive.*

Proof. *a)* By Remark 1.3.2,

$$s^{\frac{1}{p}} \le I_{\varphi,p}(s) \le s^{\frac{1}{p}} \left(\log_1 \frac{1}{s}\right)^{\beta}.$$

Taking $s \to 0$, we obtain the result.

b) We observe that

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt \ge \varphi\left(\log_1 \frac{1}{s}\right) \int_0^s t^{\frac{1}{p}-1} dt \approx I_{\varphi,p}(s),$$

and, since $\varphi(\log_1 \cdot)$ is submultiplicative,

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt = \int_0^1 \frac{I_{\varphi,p}(sr)}{r} dr \lesssim I_{\varphi,p}(s) \int_0^1 r^{\frac{1}{p}-1} \varphi\left(\log_1 \frac{1}{r}\right) dr \approx I_{\varphi,p}(s).$$

c) For all $s, t > 0$, since φ is increasing, we have that

$$I_{\varphi,1}(s+t) = s\varphi\left(\log_1 \frac{1}{s+t}\right) + t\varphi\left(\log_1 \frac{1}{s+t}\right) \leq I_{\varphi,1}(s) + I_{\varphi,1}(t).$$

□

Lemma 2.1.3. *The function*

$$J_{\varphi,p}(s) := s^{\frac{1}{p}} \varphi(\log_1 s), \qquad s > 0, \tag{2.3}$$

satisfies the following properties:

a) $J_{\varphi,p}(0^+) = 0$.

b) $J_{\varphi,p}$ *is essentially increasing. In fact,*

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt \approx J_{\varphi,p}(s).$$

Proof. *a*) By Remark 1.3.2,

$$s^{\frac{1}{p}} \leq J_{\varphi,p}(s) \leq s^{\frac{1}{p}} (\log_1 s)^{\beta}.$$

Letting $s \to 0$, we are done.

b) We have that

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt \leq \varphi(\log_1 s) \int_0^s t^{\frac{1}{p}-1} dt \approx J_{\varphi,p}(s),$$

and, since

$$\varphi(\log_1 s) \leq \varphi(\log_1 sr) \varphi\left(\log_1 \frac{1}{r}\right),$$

we get

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt = \int_0^1 \frac{J_{\varphi,p}(sr)}{r} dr \gtrsim J_{\varphi,p}(s) \int_0^1 r^{\frac{1}{p}-1} \varphi\left(\log_1 \frac{1}{r}\right)^{-1} dr \approx J_{\varphi,p}(s).$$

□

Lemma 2.1.4. *Let* $1 \le p_0 < p_1 < \infty$. *For every* $s > 0$,

$$\inf_{p_0 < p \le p_1} \varphi\left(\frac{1}{p-p_0}\right) s^{\frac{1}{p}} \lesssim I_{\varphi,p_0}(s),$$

$$\inf_{p_0 \le p < p_1} \varphi\left(\frac{1}{p_1-p}\right) s^{\frac{1}{p}} \lesssim J_{\varphi,p_1}(s),$$

and

$$\inf_{p_0 \le p < \infty} \varphi(p) s^{\frac{1}{p}} \lesssim \varphi(\log_1 s).$$

Proof. By Lemma 2.1.1, setting $\frac{1}{q} = \frac{1}{p_0} - \frac{1}{p}$, we get

$$\begin{aligned}\inf_{p_0 < p \le p_1} \varphi\left(\frac{1}{p-p_0}\right) s^{\frac{1}{p}} &\approx s^{\frac{1}{p_0}} \inf_{\left(\frac{1}{p_0}-\frac{1}{p_1}\right)^{-1} \le q < \infty} \varphi(q) s^{-\frac{1}{q}} \\ &\lesssim s^{\frac{1}{p_0}} \varphi\left(\log_1 \frac{1}{s}\right) \chi_{(0,1)}(s) + s^{\frac{1}{p_1}} \chi_{[1,\infty)}(s) \qquad (2.4)\end{aligned}$$

and, since $\varphi(1) = 1$, we conclude

$$\inf_{p_0 < p \le p_1} \varphi\left(\frac{1}{p-p_0}\right) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_0}} \varphi\left(\log_1 \frac{1}{s}\right) = I_{\varphi,p_0}(s).$$

Similarly, if $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_1}$, we have

$$\begin{aligned}\inf_{p_0 \le p < p_1} \varphi\left(\frac{1}{p_1-p}\right) s^{\frac{1}{p}} &\approx s^{\frac{1}{p_1}} \inf_{\left(\frac{1}{p_0}-\frac{1}{p_1}\right)^{-1} \le q < \infty} \varphi(q) \left(\frac{1}{s}\right)^{-\frac{1}{q}} \\ &\lesssim s^{\frac{1}{p_0}} \chi_{(0,1]}(s) + s^{\frac{1}{p_1}} \varphi(\log_1 s) \chi_{(1,\infty)}(s),\end{aligned}$$

and, using that $\varphi(1) = 1$, we obtain

$$\inf_{p_0 \le p < p_1} \varphi\left(\frac{1}{p_1-p}\right) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_1}} \varphi(\log_1 s) = J_{\varphi,p}(s).$$

Finally,

$$\inf_{p_0 \le p < \infty} \varphi(p) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_0}} \chi_{(0,1]}(s) + \varphi(\log_1 s) \chi_{(1,\infty)}(s) \le \varphi(\log_1 s).$$

□

2.2 Decomposition results

We shall use the following decompositions.

a) The *dyadic* decomposition over $\mathbb{Z}$:

$$f = \sum_{i\in\mathbb{Z}} 2^i f_i \quad \text{where} \quad f_i = \frac{f\chi_{\{2^{i-1}\le|f|<2^i\}}}{2^i}.$$

b) The *k-dyadic* decomposition over $\mathbb{N}$:

$$f = \sum_{n\geq 0} d_n f_n \text{ where } d_n = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{k\text{-times } 2} \text{ and } f_n = \frac{f\chi_{\{d_{n-1}\le|f|<d_n\}}}{d_n}$$

with $d_{-1} = 0$. If $k = 1$, we refer to it as the *dyadic* decomposition over $\mathbb{N}$.

c) The f^**-decomposition*:

$$f = \sum_{n\geq 0} f_n \quad \text{where} \quad f_n = \begin{cases} f\chi_{\{|f|\geq f^*(1)\}} & n = 0, \\ f\chi_{\{f^*(2^n)\leq|f|<f^*(2^{n-1})\}} & n \geq 1. \end{cases} \tag{2.5}$$

d) The $K(A_0, A_1)$*-decomposition*, given below.

Recall that Peetre's K-functional (see [8, Chapter 5] or [9, Chapter 3]) for a pair of compatible quasi-normed function spaces $\bar{A} = (A_0, A_1)$, is defined, for $t > 0$, as

$$K(f, t; \bar{A}) = \inf\left\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\right\}.$$

The K-functional is an increasing concave function and, hence, is differentiable almost everywhere for $t > 0$.

Lemma 2.2.1 ([26])**.** *Let*

$$F(t) = K\left(f, t; \bar{A}\right),$$

and let, for every $i \in \mathbb{Z}$,

$$E_i := \{t \in (0, \infty) : F'(t) > 2^i\}.$$

Then, there exist functions $\{f_i\}_i$ *such that* $f = \sum_{i\in\mathbb{Z}} 2^i f_i$ *(convergence in* $A_0 + A_1$*) and*

$$K\left(f_i, t; \bar{A}\right) \lesssim \min\left\{t, |E_i|\right\}. \tag{2.6}$$

The proof is based on the *K-divisibility Theorem* (see [12, pp. 315-337]). The above decomposition of f is what we call $K(A_0, A_1)$-decomposition.

In this work, we shall only use the following K-functionals. For $1 \leq p < \infty$,

- $K(f,t;L^p,L^\infty) \approx \left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p}$,
- $K\left(f,t;L^{p,\infty},L^\infty\right) \approx \sup_{s\leq t^p} s^{\frac{1}{p}} f^*(s)$;

and, for $1 < p < \infty$,

- $K(f,t;L^1,L^p) \approx t\left(\int_{t^{p'}}^\infty f^{**}(s)^p ds\right)^{1/p}$,
- $K\left(f,t;L^1,L^{p,\infty}\right) \approx t\sup_{s\geq t^{p'}} s^{\frac{1}{p}} f^{**}(s)$.

Lemma 2.2.2. *Fix $1 \leq p_0 < \infty$, $1 < p_1 < \infty$, a set $E \subset \mathbb{R}$ and a function f. Suppose that, for every $t > 0$,*

$$K\left(f,t;\bar{A}\right) \lesssim \min\left\{t,|E|\right\}.$$

Then,

a) if $\bar{A} = (L^{p_0}, L^\infty)$,

$$\|f\|_{L^p} \lesssim |E|^{p_0/p}, \qquad \forall\, p_0 < p < \infty.$$

b) if $\bar{A} = (L^{p_0,\infty}, L^\infty)$,

$$\|f\|_{L^{p,\infty}} \lesssim |E|^{p_0/p}, \qquad \forall\, p_0 < p < \infty.$$

c) if $\bar{A} = \left(L^1, L^{p_1}\right)$,

$$\|f\|_{L^p} \lesssim |E|^{p_1'\left(\frac{1}{p}-\frac{1}{p_1}\right)}, \qquad \forall\, 1 \leq p < p_1.$$

d) if $\bar{A} = \left(L^1, L^{p_1,\infty}\right)$,

$$\|f\|_{L^{p,\infty}} \lesssim |E|^{p_1'\left(\frac{1}{p}-\frac{1}{p_1}\right)}, \qquad \forall\, 1 \leq p < p_1.$$

Proof. *a)* We have that

$$\left(\int_0^{t^{p_0}} f^*(s)^{p_0} ds\right)^{1/p_0} \lesssim \min\left\{t,|E|\right\},$$

which is equivalent to

$$\int_0^t f^*(s)^{p_0}ds \lesssim \min\left\{t,|E|^{p_0}\right\} = \int_0^t \chi_{[0,|E|^{p_0}]}(s)ds.$$

Hence, $(f^{p_0})^{**} \lesssim \left(\chi_{[0,|E|^{p_0}]}\right)^{**}$ and, [8, Corollary 4.7, p. 61] implies that, for every $p_0 < p < \infty$,

$$\|f\|_{L^p} = \|f^{p_0}\|_{L^{\frac{p}{p_0}}}^{1/p_0} \lesssim \left\|\chi_{[0,|E|^{p_0}]}\right\|_{L^{\frac{p}{p_0}}}^{1/p_0} = |E|^{p_0/p}.$$

b) In this case,

$$K\left(f,t;L^{p_0,\infty},L^\infty\right) \approx \sup_{s\le t^{p_0}} s^{\frac{1}{p_0}} f^*(s) \lesssim \min\left\{t,|E|\right\}.$$

Hence,

$$f^*(t) \lesssim \min\left\{1,\frac{|E|}{t^{\frac{1}{p_0}}}\right\},$$

and, therefore, for every $p_0 < p < \infty$,

$$\|f\|_{L^{p,\infty}} \lesssim \sup_{t>0} t^{\frac{1}{p}} \min\left\{1,\frac{|E|}{t^{\frac{1}{p_0}}}\right\} = \max\left\{\sup_{t\le|E|^{p_0}} t^{\frac{1}{p}}, \sup_{t>|E|^{p_0}} t^{\frac{1}{p}-\frac{1}{p_0}}|E|\right\} = |E|^{p_0/p}.$$

c) Since

$$\begin{aligned}&\int_0^t f^*(s)ds \approx t^{1/p_1'}\left(\int_t^\infty s^{-p_1}ds\right)^{1/p_1}\int_0^t f^*(s)ds\\ &\le t^{1/p_1'}\left(\int_t^\infty \frac{\left(\int_0^s f^*(r)dr\right)^{p_1}}{s^{p_1}}ds\right)^{1/p_1} \approx K\left(f,t^{1/p_1'};L^1,L^{p_1}\right) \lesssim \min\left\{t^{\frac{1}{p_1'}},|E|\right\},\end{aligned}$$

we get that $\|f\|_{L^1} \lesssim |E|$. Also,

$$\left(\int_{t^{p_1'}}^\infty f^*(s)^{p_1}ds\right)^{1/p_1} \le \left(\int_{t^{p_1'}}^\infty f^{**}(s)^{p_1}ds\right)^{1/p_1} \lesssim \min\left\{1,\frac{|E|}{t}\right\} \le 1,$$

and, therefore, $\|f\|_{L^{p_1}} \lesssim 1$. Now, for every $1 \le p < p_1$, let $0 \le \theta < 1$ such that $\frac{1}{p} = 1-\theta+\frac{\theta}{p_1}$. Then,

$$\|f\|_{L^p} = \left(\int |f|^{(1-\theta)p}|f|^{\theta p}\right)^{\frac{1}{p}} \le \left\||f|^{(1-\theta)p}\right\|_{L^{\frac{1}{(1-\theta)p}}}^{\frac{1}{p}} \left\||f|^{\theta p}\right\|_{L^{\frac{p_1}{\theta p}}}^{\frac{1}{p}} = \|f\|_{L^1}^{(1-\theta)}\|f\|_{L^{p_1}}^{\theta}.$$

Since $(1-\theta) = p_1'\left(\frac{1}{p} - \frac{1}{p_1}\right)$, we obtain

$$\|f\|_{L^p} \lesssim |E|^{p_1'\left(\frac{1}{p}-\frac{1}{p_1}\right)}.$$

d) Using that

$$f^{**}(t) \lesssim t^{-\frac{1}{p_1}} \min\left\{1, \frac{|E|}{t^{\frac{1}{p_1'}}}\right\},$$

we conclude that, for every $1 \leq p < p_1$,

$$\begin{aligned}\|f\|_{L^{p,\infty}} &\lesssim \sup_{t>0} t^{\frac{1}{p}-\frac{1}{p_1}} \min\left\{1, \frac{|E|}{t^{\frac{1}{p_1'}}}\right\} \\ &= \max\left\{\sup_{t\leq |E|^{p_1'}} t^{\frac{1}{p}-\frac{1}{p_1}}, \sup_{t>|E|^{p_1'}} t^{\frac{1}{p}-1}|E|\right\} = |E|^{p_1'\left(\frac{1}{p}-\frac{1}{p_1}\right)}.\end{aligned}$$

□

Parts *a)* and *c)* of the above result can be found in [26].

2.3 Reconstruction results

Here, we present the results related to Step 3. Let $I_{\varphi,p}$ and $J_{\varphi,p}$ be the functions described in (2.2) and (2.3).

Lemma 2.3.1. *Let $1 \leq p < \infty$. If $\{f_i\}_i$ are the functions obtained from f with the dyadic decomposition over $\mathbb{Z}$,*

$$\sum_{i\in\mathbb{Z}} 2^i I_{\varphi,p}\left(\|f_i\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)}, \quad v(t) = t^{\frac{1}{p}-1}\varphi\left(\log_1 \frac{1}{t}\right),$$

and

$$\sum_{i\in\mathbb{Z}} 2^i J_{\varphi,p}\left(\|f_i\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)}, \quad v(t) = t^{\frac{1}{p}-1}\varphi\left(\log_1 t\right).$$

Proof. For the first statement, we use that $\|f_i\|_{L^1} \leq \lambda_f(2^{i-1})$ and that $I_{\varphi,p} \circ \lambda_f$ is essentially decreasing, to get

$$\sum_{i\in\mathbb{Z}} 2^i I_{\varphi,p}\left(\|f_i\|_{L^1}\right) \lesssim \sum_{i\in\mathbb{Z}} 2^i I_{\varphi,p}\left(\lambda_f(2^i)\right) \approx \int_0^\infty I_{\varphi,p}\left(\lambda_f(s)\right) ds.$$

From Lemma 2.1.2 *b)* and Proposition 1.2.5 it follows that

$$\int_0^\infty I_{\varphi,p}\left(\lambda_f(s)\right) ds \approx \int_0^\infty f^*(t) t^{\frac{1}{p}-1} \varphi\left(\log_1 \frac{1}{t}\right) dt.$$

The proof of the second statement follows the same steps using Lemma 2.1.3 *b)*. □

Lemma 2.3.2. *For any $f \in L^\infty$, let $\{f_n\}_n$ be the functions obtained from f with the f^*-decomposition (2.5), then*

$$\sum_{n\geq 0} \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \lesssim \|f\|_{L^\infty_\varphi}.$$

Proof. By definition, it comes clear that $\|f_n\|_{L^1} \lesssim \|f_n\|_{L^\infty} 2^{n-1}$. Therefore,

$$\sum_{n\geq 0} \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \lesssim \|f\|_{L^\infty} + \sum_{n\geq 1} f^*(2^{n-1}) \varphi\left(\log_1 2^{n-1}\right)$$
$$\lesssim \|f\|_{L^\infty} + \sum_{n\geq 1} \int_{2^{n-2}}^{2^{n-1}} f^*(s) \varphi\left(\log_1 s\right) \frac{ds}{s} = \|f\|_{L^\infty_\varphi} + \int_{1/2}^1 f^*(s) \varphi\left(\log_1 s\right) \frac{ds}{s}.$$

The result follows from the fact that

$$\int_{1/2}^1 f^*(s) \varphi\left(\log_1 s\right) \frac{ds}{s} \lesssim \|f\|_{L^\infty}.$$

□

For the reconstruction results involving the $K(A_0, A_1)$-decomposition, we need to define first the following spaces (see [22]).

Definition 2.3.3. The spaces $K^+_\varphi(\bar{A})$ and $K^-_\varphi(\bar{A})$ are determined by the norms

$$\|f\|_{K^+_\varphi(\bar{A})} := \sup_{t>0} K(f,t;\bar{A}) + \int_0^1 \frac{K(f,t;\bar{A})}{t} \varphi'\left(\log_1 t\right) dt,$$

and

$$\|f\|_{K^-_\varphi(\bar{A})} := \sup_{t>0} \frac{K(f,t;\bar{A})}{t} + \int_1^\infty \frac{K(f,t;\bar{A})}{t^2} \varphi'\left(\log_1 t\right) dt.$$

Proposition 2.3.4. *The spaces $K^+_\varphi(\bar{A})$ and $K^-_\varphi(\bar{A})$ are complete.*

Proof. Let $(f_n)_n \subset K_\varphi^+(\bar{A})$ such that $\sum_{n\geq 1} \|f_n\|_{K_\varphi^+(\bar{A})} < \infty$. Since we have that $K_\varphi^+(\bar{A}) \subset A_0 + A_1$, which is a complete space, and

$$\sum_{n\geq 1} \|f_n\|_{A_0+A_1} \leq \sum_{n\geq 1} \|f_n\|_{K_\varphi^+(\bar{A})} < \infty,$$

there exists $f \in A_0 + A_1$ such that,

$$\lim_{N\to\infty} \sum_{n=1}^{N} f_n = f \quad \text{in } A_0 + A_1.$$

Hence, for every $t > 0$,

$$\lim_{N\to\infty} K\Big(\sum_{n>N} f_n, t; \bar{A}\Big) = 0. \tag{2.7}$$

Now, we need to prove that $\sum_{n\geq 1} f_n = f$ in $K_\varphi^+(\bar{A})$. Indeed, by the monotone convergence theorem and (2.7), we have

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{t>0} K\Big(\sum_{n>N} f_n, t; \bar{A}\Big) + \lim_{N\to\infty} \int_0^1 \frac{K\big(\sum_{n>N} f_n, t; \bar{A}\big)}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt \\
&\leq \lim_{N\to\infty} \sum_{n>N} \sup_{t>0} K(f_n, t; \bar{A}) + \lim_{N\to\infty} \sum_{n>N} \int_0^1 \frac{K(f_n, t; \bar{A})}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt = 0,
\end{aligned}$$

which concludes that

$$\begin{aligned}
&\lim_{N\to\infty} \Big\| f - \sum_{n\geq 1}^{N} f_n \Big\|_{K_\varphi^+(\bar{A})} = \lim_{N\to\infty} \Big\| \sum_{n>N} f_n \Big\|_{K_\varphi^+(\bar{A})} \\
&= \lim_{N\to\infty} \sup_{t>0} K\Big(\sum_{n>N} f_n, t; \bar{A}\Big) + \lim_{N\to\infty} \int_0^1 \frac{K\big(\sum_{n>N} f_n, t; \bar{A}\big)}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt = 0.
\end{aligned}$$

Using the same argument with the norm of the space $K_\varphi^-(\bar{A})$, we conclude the result. □

The above spaces allows us to state the following lemmas, inspired by the ideas in [26].

Lemma 2.3.5. *For any function f and $\{E_i\}_i$ as in Lemma 2.2.1, we have*

$$\sum_{i\in\mathbb{Z}} 2^i I_{\varphi,1}\left(|E_i|\right) \approx \|f\|_{K_\varphi^+(\bar{A})}.$$

Proof. Let $F(t) := K(f,t;\bar{A})$ and, therefore, since $I_{\varphi,p}\circ\lambda_{F'}$ is essentially decreasing, we have

$$\sum_{i\in\mathbb{Z}} 2^i I_{\varphi,1}\left(|E_i|\right) = \sum_{i\in\mathbb{Z}} 2^i I_{\varphi,1}\left(\lambda_{F'}(2^i)\right) \approx \int_0^\infty I_{\varphi,1}\left(\lambda_{F'}(s)\right) ds.$$

Now, in view of Lemma 2.1.2 *b)*, we get

$$\begin{aligned}\int_0^\infty I_{\varphi,1}\left(\lambda_{F'}(s)\right) ds &\approx \int_0^\infty \int_0^{\lambda_{F'}(s)} \frac{I_{\varphi,1}(t)}{t} dtds \\ &\approx \int_0^1 F'(s)\varphi\left(\log_1 \frac{1}{s}\right) ds + \int_1^\infty F'(s)ds = I_1 + I_2,\end{aligned}$$

where, in the second equivalence, we have used Proposition 1.2.5 and the fact that, since F is concave, $(F')^* = F'$. For I_1, we use Fubini to obtain

$$\begin{aligned}I_1 &= F(1) + \int_0^1 F'(s)\left(\varphi\Big(\log_1 \frac{1}{s}\Big) - 1\right) ds = F(1) + \int_0^1 F'(s)\int_s^1 \frac{\varphi'\Big(\log_1 \frac{1}{t}\Big)}{t} dt\, ds \\ &= F(1) + \int_0^1 \left(\int_0^t F'(s)ds\right) \frac{\varphi'\Big(\log_1 \frac{1}{t}\Big)}{t} dt = F(1) + \int_0^1 F(t)\frac{\varphi'\Big(\log_1 \frac{1}{t}\Big)}{t} dt.\end{aligned}$$

Finally, since $I_2 = F(\infty) - F(1)$, we conclude the desired result. $\square$

Lemma 2.3.6. *For any function f and $\{E_i\}_i$ as in Lemma 2.2.1, if we fix $N \in \mathbb{Z}$ such that*

$$2^{N-1} < \sup_{t>0} \frac{K(f,t;\bar{A})}{t} \leq 2^N,$$

and φ is an admissible function such that

$$\int_s^\infty \frac{\varphi'(\log_1 t)}{t^2} dt \lesssim \frac{\varphi(\log_1 s)}{s} \qquad \forall s \geq 1, \tag{2.8}$$

we have that

$$\sum_{i\leq N} 2^i \varphi\left(\log_1 |E_i|\right) \approx \|f\|_{K_\varphi^-(\bar{A})}.$$

Proof. Let $F(t) := K(f,t;\bar{A})$. Hence,

$$\sum_{i\leq N} 2^i \varphi\left(\log_1 |E_i|\right) = \sum_{i\leq N} 2^i \varphi\left(\log_1 \lambda_{F'}(2^i)\right) \lesssim \int_0^{2^N} \varphi\left(\log_1 \lambda_{F'}(s)\right) ds.$$

Since F is an increasing concave function, it follows that, for almost every $t > 0$,

$$F'(t) \leq \frac{1}{t}\int_0^t F'(s)ds \leq \frac{F(t)}{t} \leq 2^N,$$

and, for every $t > 2^N$,

$$\varphi\left(\log_1 \lambda_{F'}(t)\right) = \varphi(1) = 1.$$

Thus, if $H(s) := \varphi(\log_1 s) - 1$, from Proposition 1.2.5 we obtain

$$\begin{aligned}
&\int_0^{2^N} \varphi\left(\log_1 \lambda_{F'}(s)\right) ds = 2^N + \int_0^\infty \varphi\left(\log_1 \lambda_{F'}(s)\right) - 1 \; ds \\
&\approx \sup_{t>0} \frac{F(t)}{t} + \int_0^\infty H(\lambda_{F'}(s))ds \approx \sup_{t>0}\frac{F(t)}{t} + \int_1^\infty F'(t)H'(t)dt \\
&= \sup_{t>0}\frac{F(t)}{t} + \int_1^\infty F'(t)\varphi'\left(\log_1 t\right)\frac{dt}{t} \leq \sup_{t>0}\frac{F(t)}{t} + \int_1^\infty \frac{F(t)}{t}\varphi'\left(\log_1 t\right)\frac{dt}{t}.
\end{aligned}$$

Now, to get the inverse inequality, we observe that, by (2.6) and (2.8), for every i,

$$\begin{aligned}
\|f_i\|_{K_\varphi^-(\bar{A})} &\lesssim \sup_{t>0}\left(\min\left\{1, \frac{|E_i|}{t}\right\}\right) + \int_1^\infty \frac{\min\{t,|E_i|\}}{t^2}\varphi'\left(\log_1 t\right)dt \\
&= 1 + \int_1^{|E_i|}\frac{\varphi'\left(\log_1 t\right)}{t}dt + |E_i|\int_{|E_i|}^\infty \frac{\varphi'\left(\log_1 t\right)}{t^2}dt \\
&= \varphi\left(\log_1 |E_i|\right) + |E_i|\int_{|E_i|}^\infty \frac{\varphi'\left(\log_1 t\right)}{t^2}dt \lesssim \varphi\left(\log_1 |E_i|\right).
\end{aligned}$$

Consequently,

$$\Big\|\sum_{i\leq N} 2^i f_i\Big\|_{K_\varphi^-(\bar{A})} \leq \sum_{i\leq N} 2^i \|f_i\|_{K_\varphi^-(\bar{A})} \lesssim \sum_{i\leq N} 2^i \varphi\left(\log_1 |E_i|\right).$$

Thus, since $K_\varphi^-(\bar{A}) \subset A_0 + A_1$, we have that $\|f\|_{K_\varphi^-(\bar{A})} \lesssim \sum_{i\leq N} 2^i\varphi\left(\log_1 |E_i|\right)$. □

Remark 2.3.7. If for some $\gamma > 0$ and $\beta_1, \ldots, \beta_k \geq 0$,

$$\varphi(s) = s^\gamma \prod_{n=1}^k (\log_n s)^{\beta_n},$$

condition (2.8) is satisfied.

2.3.1 The quasi-Banach setting

Now, we introduce the results required for Step 3 when the target space is quasi-Banach. To overcome the lack of triangle inequality, we shall use the following proposition from [19].

Proposition 2.3.8. *Given $0 < a < 1$ and a weight $v \in \mathbb{R}^+$, let*

$$G_V(a) := \sup_{t>0} \left(\frac{1}{t} \int_{at}^{t} V(s)^{-1} ds \right) V(t).$$

If $f = \sum_{n\geq 1} c_n f_n$, where c_n are non-negative numbers and $\{f_n\}_n$ are functions such that $\|f_n\|_{\Lambda^{1,\infty}[V]} \leq 1$, then, for every sequence of positive numbers $(a_n)_n$ such that $\sum_{n\geq 1} a_n \leq 1$,

$$\|f\|_{\Lambda^{1,\infty}[V]} \lesssim \sum_{n\geq 1} \big(1 + G_V(a_n)\big)\, c_n.$$

Lemma 2.3.9. *Let $0 < p \leq 1$. If*

$$V(t) \approx \frac{t^{1/p}}{\varphi(\log_1 t)},$$

then,

i) for $0 < p < 1$, $G_V(a) \approx a^{-\frac{1}{p}+1}$,

ii) for $p = 1$, $G_V(a) \approx \log \frac{1}{a}$.

Proof. We need to compute

$$G_V(a) \approx \sup_{t>0} \int_{at}^{t} \frac{\varphi(\log_1 s)}{s^{1/p}} ds \; \frac{t^{\frac{1}{p}-1}}{\varphi(\log_1 t)}.$$

$i)$ Since $\frac{1}{p} > 1$ and φ is increasing, it follows

$$G_V(a) \lesssim \sup_{t>0} t^{\frac{1}{p}-1} \int_{at}^{t} s^{-1/p}\, ds = \sup_{t>0} t^{\frac{1}{p}-1} \left[\frac{s^{-\frac{1}{p}+1}}{-\frac{1}{p}+1} \right]_{at}^{t} = \frac{a^{-\frac{1}{p}+1} - 1}{\frac{1}{p} - 1} \approx a^{-\frac{1}{p}+1}.$$

Indeed, when $t \leq 1$, we can use that $\varphi(1) = 1$, to conclude

$$\int_{at}^{t} \frac{\varphi(\log_1 s)}{s^{1/p}} ds \; \frac{t^{\frac{1}{p}-1}}{\varphi(\log_1 t)} = t^{\frac{1}{p}-1} \int_{at}^{t} s^{-1/p}\, ds = \frac{a^{-\frac{1}{p}+1} - 1}{\frac{1}{p} - 1},$$

and the result follows.

ii) We can use again that φ is increasing to obtain

$$G_V(a) \lesssim \sup_{t>0} \int_{at}^{t} \frac{ds}{s} = \sup_{t>0} [\log s]_{at}^{t} = \log \frac{1}{a},$$

and $G_V(a) \approx \log \frac{1}{a}$ if $t \leq 1$. □

Lemma 2.3.10. *Let $0 < p \leq 1$. If*

$$V(t) \approx \frac{t^{1/p}}{\varphi\Big(\log_1 \frac{1}{t}\Big)},$$

then,

i) for $0 < p < 1$, $G_V(a) \lesssim a^{-\frac{1}{p}+1} \varphi\left(\log_1 \frac{1}{a}\right)$,

ii) for $p = 1$, $G_V(a) \lesssim \left(\log \frac{1}{a}\right) \varphi\left(\log_1 \frac{1}{a}\right)$.

Proof. Let us compute

$$G_V(a) \approx \sup_{t>0} \int_{at}^{t} \frac{\varphi\left(\log_1 \frac{1}{s}\right)}{s^{1/p}} ds \; \frac{t^{\frac{1}{p}-1}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

i) We have that, since $\frac{1}{p} > 1$,

$$G_V(a) \lesssim \sup_{t>0} t^{\frac{1}{p}-1} \int_{at}^{t} s^{-1/p} \; ds \; \varphi\left(\log_1 \frac{1}{a}\right) \approx a^{-\frac{1}{p}+1} \varphi\left(\log_1 \frac{1}{a}\right).$$

ii) In this case, we get

$$G_V(a) \lesssim \sup_{t>0} \left(\int_{at}^{t} \frac{ds}{s}\right) \varphi\left(\log_1 \frac{1}{a}\right) = \left(\log \frac{1}{a}\right) \varphi\left(\log_1 \frac{1}{a}\right).$$

□

Lemma 2.3.11. *If*

$$V(t) \approx \frac{t}{\varphi_1(\log_1 t)\, \varphi_2\left(\log_1 \frac{1}{t}\right)},$$

then

$$G_V(a) \lesssim \left(\log \frac{1}{a}\right) \varphi_2\left(\log_1 \frac{1}{a}\right).$$

Proof. It follows from the fact that

$$\int_{at}^{t} \frac{\varphi_1\left(\log_1 s\right)\varphi_2\left(\log_1 \frac{1}{s}\right)}{s} ds \frac{1}{\varphi_1\left(\log_1 t\right)\varphi_2\left(\log_1 \frac{1}{t}\right)} \lesssim \left(\log \frac{1}{a}\right) \varphi_2\left(\log_1 \frac{1}{a}\right).$$

□

The following proofs are inspired by the ideas of Carro and Martín [23].

Lemma 2.3.12. *Fix $f \in L^1$ such that $\|f\|_{L^1} = 1$ and let $\{f_n\}_n$ be the functions obtained with the 2-dyadic decomposition. Then,*

i) if $0 < p < 1$ and $\varepsilon > 0$,

$$I_{\varphi,1}\left(\|f_0\|_{L^1}\right) + \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} I_{\varphi,1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)},$$

where $v(t) = \varphi\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}$,

ii)

$$\sum_{n\geq 0} \left(\log_1 n\right) \varphi\left(\log_1 n\right) 2^{2^n} I_{\varphi_1,1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)},$$

where $v(t) = \varphi_1\left(\log_1 \frac{1}{t}\right)\left(\log_3 \frac{1}{t}\right)\varphi\left(\log_3 \frac{1}{t}\right)$.

Proof. $i)$ For every $n \geq 1$,

$$\|f_n\|_{L^1} = \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{2^{2^n}} \int_0^\infty f(s)\chi_{\{2^k \leq |f| < 2^{k+1}\}}(s) ds \lesssim \sum_{k=2^{n-1}}^{2^n-1} \frac{2^k}{2^{2^n}} \lambda_f(2^k).$$

Hence, since $I_{\varphi,1}$ is subadditive (Lemma 2.1.2 c)),

$$I_{\varphi,1}\left(\|f_n\|_{L^1}\right) \lesssim I_{\varphi,1}\left(\sum_{k=2^{n-1}}^{2^n-1} \frac{2^k}{2^{2^n}} \lambda_f(2^k)\right) \lesssim \sum_{k=2^{n-1}}^{2^n-1} I_{\varphi,1}\left(\frac{2^k}{2^{2^n}} \lambda_f(2^k)\right).$$

Thus, if

$$B_1 := \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} I_{\varphi,1}\left(\|f_n\|_{L^1}\right),$$

we have that

$$\begin{aligned} B_1 &\lesssim \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} \sum_{k=2^{n-1}}^{2^n-1} I_{\varphi,1}\left(\frac{2^k}{2^{2^n}}\lambda_f(2^k)\right) \\ &= \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \sum_{k=2^{n-1}}^{2^n-1} 2^k\lambda_f(2^k)\varphi\left(\log_1 \frac{2^{2^n}}{2^k\lambda_f(2^k)}\right). \end{aligned}$$

Using that, for every $2^{n-1} \leq k \leq 2^n$,

$$\frac{2^{2^n}}{2^k} = \frac{(2^{2^{n-1}})^2}{2^k} \leq 2^k \quad \text{and} \quad n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \lesssim \left(\log_2 2^k\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}, \tag{2.9}$$

we get

$$\begin{aligned} B_1 &\lesssim \sum_{n\geq 1}\sum_{k=2^{n-1}}^{2^n-1} (2^k)^2 I_{\varphi,1}\left(\frac{\lambda_f(2^k)}{2^k}\right)(\log_2 2^k)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \\ &\lesssim \int_1^\infty sI_{\varphi,1}\left(\frac{\lambda_f(s)}{s}\right)(\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}ds \\ &= \int_1^\infty \lambda_f(s)\varphi\left(\log_1 \frac{s}{\lambda_f(s)}\right)(\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}ds, \end{aligned}$$

where we have used that $I_{\varphi,1}\left(\frac{\lambda_f(s)}{s}\right)$ is essentially decreasing. Since, $s\lambda_f(s) \leq 1$ for every $s > 0$,

$$\lambda_f(s)\varphi\left(\log_1 \frac{s}{\lambda_f(s)}\right) \lesssim \lambda_f(s)\varphi\left(\log_1 \frac{1}{(\lambda_f(s))^2}\right) \approx I_{\varphi,1}\left(\lambda_f(s)\right),$$

and we obtain

$$\begin{aligned} B_1 &\lesssim \int_0^\infty I_{\varphi,1}\left(\lambda_f(s)\right)(\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}ds \\ &= \int_0^\infty \left(\int_0^{\lambda_f(s)} dI_{\varphi,1}(t)\right)(\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}ds. \end{aligned}$$

If we use Fubini, we conclude

$$\begin{aligned} B_1 &\lesssim \int_0^\infty \left(\int_0^{f^*(t)} (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds \right) dI_{\varphi,1}(t) \\ &\lesssim \int_0^\infty f^*(t)(\log_2 f^*(t))^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,1}(t) \\ &\lesssim \int_0^\infty f^*(t) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,1}(t) \\ &\approx \int_0^\infty f^*(t) \varphi \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dt, \end{aligned}$$

and the result follows because, since $\|f\|_{L^1} \leq 1$, $I_{\varphi,1}\left(\|f_0\|_{L^1}\right) \lesssim 1$.

ii) We can follow the same argument, taking into account that, instead of (2.9), we have

$$(\log_1 n) \, \varphi \, (\log_1 n) \lesssim \left(\log_3 2^k \right) \varphi \left(\log_3 2^k \right).$$

□

Lemma 2.3.13. *Let $1 \leq p_1 < \infty$. Fix $f \in L^{p_1}$ such that $\|f\|_{L^{p_1}} = 1$ and let $\{f_n\}_n$ be the functions obtained with the dyadic decomposition over $\mathbb{N}$. Then,*

i) if $0 < p < 1$ and $\varepsilon > 0$,

$$I_{\varphi,p_1}\left(\|f_0\|_{L^1}\right) + \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi,p_1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)},$$

where $v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi\left(\log_1 \frac{1}{t}\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}$,

ii)

$$\sum_{n \geq 0} (\log_1 n) \, 2^n I_{\varphi,p_1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1(v)},$$

where $v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)$.

Proof. We only prove *i*) and *ii*) follows similarly. Since $\|f_n\|_{L^\infty} \leq 1$,

$$\|f_n\|_{L^1} \leq \left|\{f_n \neq 0\}\right| \leq \left|\{2^{n-1} \leq |f|\}\right| = \lambda_f\left(2^{n-1}\right).$$

Therefore, if

$$B_2 := \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi,p_1}\left(\|f_n\|_{L^1}\right),$$

and we use that, for any $2^{n-1} < s < 2^n$,

$$(n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \lesssim (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)},$$

we get

$$\begin{aligned} B_2 &\lesssim \sum_{n\geq 0} (n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi,p_1}\left(\lambda_f\left(2^n\right)\right) \\ &\lesssim \sum_{n\geq 0} \int_{2^{n-1}}^{2^n} (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} I_{\varphi,p_1}\left(\lambda_f\left(2^n\right)\right) ds \\ &\lesssim \int_0^\infty (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} I_{\varphi,p_1}\left(\lambda_f(s)\right) ds. \end{aligned}$$

Finally, following the same argument as in Lemma 2.3.12, we conclude

$$\begin{aligned} B_2 &\lesssim \int_0^\infty f^*(t)(\log_1 f^*(t))^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,p_1}(t) \\ &\lesssim \int_0^\infty f^*(t)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,p_1}(t), \end{aligned}$$

and, since $I_{\varphi,p_1}\left(\|f_0\|_{L^1}\right) \lesssim 1$, the result follows. □

We have analogous results in the case of $J_{\varphi,p}$.

Lemma 2.3.14. *Let $1 < p_1 < \infty$. Fix $f \in L^{p_1}$ such that $\|f\|_{L^{p_1}} = 1$ and let $\{f_n\}_n$ be the functions obtained with the dyadic decomposition over $\mathbb{N}$. Then,*

i) if $0 < p < 1$ and $\varepsilon > 0$,

$$J_{\varphi,p_1}\left(\|f_0\|_{L^1}\right) + \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_1 n\right) 2^n J_{\varphi_1,p_1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1}(v),$$

where $v_1(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi_1\left(\log_1 t\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right)$,

ii)

$$\sum_{n\geq 0} \left(\log_1 n\right) \varphi\left(\log_1 n\right) 2^n J_{\varphi_1,p_1}\left(\|f_n\|_{L^1}\right) \lesssim \|f\|_{\Lambda^1}(v),$$

where $v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi_1\left(\log_1 t\right)\left(\log_2 \frac{1}{t}\right) \varphi\left(\log_2 \frac{1}{t}\right)$.

Proof. We only prove i), as ii) follows in a similar manner. Let

$$B_3 := \sum_{n\geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_1 n\right) 2^n J_{\varphi_1,p_1}\left(\|f_n\|_{L^1}\right).$$

Since, for every $2^{n-1} < s < 2^n$,

$$(n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) \lesssim (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_2 s),$$

we have

$$\begin{aligned} B_3 &\lesssim \sum_{n\geq 0} \int_{2^{n-1}}^{2^n} (n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) J_{\varphi_1,p_1}\left(\lambda_f\left(2^n\right)\right) ds. \\ &\lesssim \int_0^\infty (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_2 s) J_{\varphi_1,p_1}\left(\lambda_f\left(s\right)\right) ds. \end{aligned}$$

Finally, following the same argument as in Lemma 2.3.12, we conclude

$$J_{\varphi,p_1}\left(\|f_0\|_{L^1}\right) + B_3 \lesssim \int_0^\infty f^*(t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right) dJ_{\varphi_1,p_1}(t),$$

and the result follows from Proposition 1.2.5 and Lemma 2.1.3. □

Lemma 2.3.15. *Fix $f \in L^\infty$. If $\{f_n\}_n$ are the functions obtained from f with the f^*-decomposition,*

$$\sum_{n\geq 0} (\log_1 n)\, \varphi\left(\log_1 n\right) \|f_n\|_{L^\infty}\, \varphi_1\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \lesssim \|f\|_{L^\infty_{\widetilde{\varphi}}}$$

where $\widetilde{\varphi}(t) = \varphi_1(t)(\log_1 t)\varphi\left(\log_1 t\right)$.

Proof. It holds that $\|f_n\|_{L^1} \lesssim \|f_n\|_{L^\infty} 2^{n-1}$. Therefore,

$$\begin{aligned} &\sum_{n\geq 0} (\log_1 n)\, \varphi\left(\log_1 n\right) \|f_n\|_{L^\infty}\, \varphi_1\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \\ &\lesssim \|f\|_{L^\infty} + \sum_{n\geq 1} (\log_1 n)\, \varphi\left(\log_1 n\right) f^*(2^{n-1}) \varphi_1\left(\log_1 2^{n-1}\right) \\ &\lesssim \|f\|_{L^\infty} + \sum_{n\geq 1} \int_{2^{n-2}}^{2^{n-1}} f^*(s) \varphi_1\left(\log_1 s\right)\left(\log_2 s\right) \varphi\left(\log_2 s\right) \frac{ds}{s} \\ &= \|f\|_{L^\infty_{\widetilde{\varphi}}} + \int_{1/2}^1 f^*(s) \varphi_1\left(\log_1 s\right)\left(\log_2 s\right) \varphi\left(\log_2 s\right) \frac{ds}{s} \lesssim \|f\|_{L^\infty_{\widetilde{\varphi}}} + \|f\|_{L^\infty}, \end{aligned}$$

and we are done. □

2.3.2 Density

To establish our extrapolation theorems, we shall need the following density results.

Proposition 2.3.16. *Given any weight v defined on $\mathbb{R}^+$ with $v \notin L^1$, the set of bounded functions such that $|f| > \alpha$ on their support, for some $\alpha > 0$, is dense in $\Lambda^1(v)$. In particular, the set of bounded functions is dense in $\Lambda^1(v)$.*

Proof. Let $f \in \Lambda^1(v)$ and let $(g_n)_n$ be the sequence of functions with $g_n = f_{\{\frac{1}{n}<|f|<n\}}$. We have that

$$f - g_n = f\chi_{\{|f|\leq \frac{1}{n}\}} + f\chi_{\{|f|\geq n\}}$$

and

$$\begin{aligned}(f-g_n)^*(2t) &\leq \left(f\chi_{\{|f|\leq\frac{1}{n}\}}\right)^*(t) + \left(f\chi_{\{|f|\geq n\}}\right)^*(t)\\ &= f^*\left(t+\lambda_f\left(\frac{1}{n}\right)\right) + f^*(t)\,\chi_{(0,\lambda_f(n))}(t).\end{aligned}$$

Therefore, since $v \notin L^1$, $\lim_{t\to\infty} f^*(t) = 0$ and $\lim_{t\to\infty}\lambda_f(t) = 0$, which implies that

$$\begin{aligned}\lim_{n\to\infty}(f-g_n)^*(2t) &\leq \lim_{n\to\infty} f^*\left(t+\lambda_f\left(\frac{1}{n}\right)\right) + \lim_{n\to\infty} f^*(t)\,\chi_{(0,\lambda_f(n))}(t)\\ &= f^*\Big(t+|\mathrm{supp} f|\Big) + 0 = 0.\end{aligned}$$

Since $(f-g_n)^* \leq f^*$, the Dominated Convergence Theorem gives

$$\lim_{n\to\infty}\|f-g_n\|_{\Lambda^1(v)} = 0.$$

□

Proposition 2.3.17. *The set of functions with support of finite measure is dense in L^∞_φ.*

Proof. Let $f \in L^\infty_\varphi$ and let $(g_n)_n$ be the sequence with $g_n = f_{\{|f|>f^*(n)\}}$. Since $f \in L^\infty_\varphi$, necessarily $\lim_{t\to\infty} f^*(t) = 0$. Hence, by the dominated convergence theorem we get

$$\begin{aligned}\lim_{n\to\infty}\|f-g_n\|_{L^\infty_\varphi} &= \lim_{n\to\infty}\|f-g_n\|_{L^\infty} + \int_1^\infty (f-g_n)^*(s)\,\varphi(\log_1 s)\frac{ds}{s}\\ &= \lim_{n\to\infty} f^*(n) + \int_1^\infty f^*(s+n)\,\varphi(\log_1 s)\frac{ds}{s} = 0.\end{aligned}$$

□

Proposition 2.3.18. *The set of functions with a finite $K(A_0, A_1)$-decomposition is dense in $K^+_\varphi(\bar{A})$.*

Proof. For any $f \in K^+_\varphi(\bar{A})$, we can consider the functions f_i obtained with the $K(A_0, A_1)$-decomposition of f and set the sequence $(g_n)_n$ such that $g_n = \sum_{|i|\le n} 2^i f_i$. By (2.6), we have that, for every i,

$$\|f_i\|_{K^+_\varphi(\bar{A})} \lesssim \sup_{t>0}(\min\{t,|E_i|\}) + \int_0^1 \frac{\min\{t,|E_i|\}}{t}\varphi'\left(\log_1 \frac{1}{t}\right) dt.$$

If $|E_i| \le 1$,

$$\|f_i\|_{K^+_\varphi(\bar{A})} \lesssim |E_i| + \int_0^{|E_i|} \varphi'\left(\log_1 \frac{1}{t}\right) dt + |E_i| \int_{|E_i|}^1 \frac{\varphi'\left(\log_1 \frac{1}{t}\right)}{t} dt = |E_i| + I_1 + I_2.$$

First, we have

$$\begin{aligned} I_1 &= \int_0^{|E_i|} \frac{\varphi'\left(\log_1 \frac{1}{t}\right)}{t}\Big(\int_0^t ds\Big) dt = \int_0^{|E_i|}\int_s^{|E_i|} \frac{\varphi'\left(\log_1 \frac{1}{t}\right)}{t} dt\, ds \\ &= \int_0^{|E_i|} \varphi\left(\log_1 \frac{1}{s}\right) ds - I_{\varphi,1}(|E_i|) \lesssim I_{\varphi,1}(|E_i|), \end{aligned} \tag{2.10}$$

where the last inequality follows by Lemma 2.1.2 *b*). We also have

$$I_2 = |E_i| \int_{|E_i|}^1 \frac{\varphi'\left(\log_1 \frac{1}{t}\right)}{t} dt = I_{\varphi,1}(|E_i|) - |E_i| \le I_{\varphi,1}(|E_i|). \tag{2.11}$$

From (2.10) and (2.11) it follows that $\|f_i\|_{K^+_\varphi(\bar{A})} \lesssim I_{\varphi,1}(|E_i|)$. If $|E_i| > 1$, by (2.10), it follows that

$$\|f_i\|_{K^+_\varphi(\bar{A})} \lesssim |E_i| + \int_0^1 \varphi'\left(\log_1 \frac{1}{t}\right) dt \lesssim I_{\varphi,1}(|E_i|) + 1 \lesssim I_{\varphi,1}(|E_i|).$$

Now, if we use this fact and Lemma 2.3.5, we get that

$$\sum_{i\in\mathbb{Z}} 2^i \|f_i\|_{K^+_\varphi(\bar{A})} \lesssim \sum_{i\in\mathbb{Z}} 2^i I_{\varphi,1}\left(|E_i|\right) \approx \|f\|_{K^+_\varphi(\bar{A})} < \infty.$$

Since $K^+_\varphi(\bar{A})$ is complete and $K^+_\varphi(\bar{A}) \subset A_0 + A_1$, necessarily $\sum_{i\in\mathbb{Z}} 2^i f_i = f$ in $K^+_\varphi(\bar{A})$, that is,

$$\lim_{n\to\infty} \|f - g_n\|_{K^+_\varphi(\bar{A})} = \lim_{n\to\infty} \Big\| \sum_{|i|\ge n} f_i \Big\|_{K^+_\varphi(\bar{A})} = 0$$

and we are done. □

Proposition 2.3.19. *If the function φ satisfies* (2.8)*, the set of functions with a finite $K(A_0, A_1)$-decomposition is dense in $K^-_\varphi(\bar{A})$.*

Proof. We can follow the same argument as in the proof of Proposition 2.3.18, using Lemma 2.3.6 and the fact that in the proof of such lemma, we have concluded that $\|f_i\|_{K^-_\varphi(\bar{A})} \lesssim \varphi\left(\log_1 |E_i|\right)$ for every $i \in \mathbb{Z}$. □

CHAPTER 3

Review of linear Yano's extrapolation

The origins of *Yano's Extrapolation Theory* go back to 1928, when Titchmarsh [99] proved that, if $\mathscr{C}$ is the conjugate function on $(0,1)$

$$\mathscr{C}f(x) = \text{p.v.} \int_0^1 f(x-y) \cot y \, dy,$$

then $\mathscr{C}f \in L^1(0,1)$ whenever $f \log(2+|f|)^{1+\varepsilon}$ is integrable for some $\varepsilon > 0$. Almost at the same time, Zygmund announced in [104] that the above result is also true for $\varepsilon = 0$. One year later, Titchmarsh [100] published a second paper where, using that,

$$\mathscr{C} : L^p(0,1) \longrightarrow L^p(0,1), \qquad \|\mathscr{C}\| \lesssim \frac{1}{p-1},$$

he obtained Zygmund's conclusion. That same year, using a different argument, Zygmund [105] proved his announced result.

In 1951, inspired by the ideas of Titchmarsh, Yano [103] published the result that gave the theory its name (see Theorem 2.0.1). Capri and Fava [14], in 1987, extended Yano's theorem to the case where (X,μ) and (Y,ν) are σ-finite measure spaces, concluding

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1_{\text{loc}}(\nu).$$

The problem of the convergence of the Fourier series pushed forward the development of these extrapolation techniques. In 1966, Carleson [15] proved, using the boundedness of the *Carleson operator* $\mathcal{C}$ (see (3.11)), that, for every function in $L^2(0,1)$, its Fourier series converges almost everywhere. At that time, it was known that there exist functions in $L^1(0,1)$ whose Fourier series diverges almost everywhere (see [69]). In 1967, Hunt [56] extended Carleson's result to $L^p(0,1)$ functions with $1 < p \leq \infty$ by proving that, for every measurable set $E \subset (0,1)$,

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(0,1)} \lesssim \frac{p^2}{(p-1)}|E|^{1/p}. \tag{3.1}$$

From (3.1), we get that

$$\mathcal{C} : L^{p,1}(0,1) \longrightarrow L^1(0,1), \qquad \|\mathcal{C}\| \lesssim \frac{1}{(p-1)^2},$$

and, by a slight modification of Theorem 2.0.1, the almost everywhere convergence of the Fourier series of functions in the space $L(\log L)^2(0,1)$ follows. In 1969, Sjölin [91] also used (3.1) to improve the space up to $L\,(\log L)\,(\log_2 L)\,(0,1)$. These works inspired Soria [92] in 1989 to state the following extrapolation result for finite measure spaces. If (X,μ) is a finite measure space and T is a sublinear operator such that for some fixed $\alpha > 0$, for every $1 < p \leq 2$ and every measurable $E \subset X$,

$$\|T\chi_E\|_{L^{1,\infty}(\mu)} \lesssim \frac{\mu(E)^{1/p}}{(p-1)^\alpha},$$

then,

$$T : B^*_\psi \to L^{1,\infty}(\mu),$$

where the function ψ is defined by $\psi(t) = t(\log_1 \frac{1}{t})^\alpha$ and B^*_ψ is a block space such that $L\,(\log L)^\alpha\,(\log_2 L)\,(\mu) \subset B^*_\psi$.

Some years later, Carro [17], in 2000, improved Yano's original theorem for σ-finite measure spaces (X,μ) and (Y,ν) showing that, if, for every measurable $E \subset X$,

$$\|T\chi_E\|_{L^p(\nu)} \lesssim \frac{\mu(E)^{1/p}}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha(\mu) \longrightarrow R(\nu),$$

where $R(\nu)$ is the space of functions such that

$$\|f\|_{R(\nu)} := \sup_{t>0} \frac{\int_{1/t}^\infty \lambda_f^\nu(s)ds}{(\log_1 t)^\alpha} < \infty;$$

that is, assuming a weaker condition on T, she obtained the boundedness on $L(\log L)^\alpha(\mu)$ and improved the range space.

Coming back to the problem of the almost everywhere convergence of Fourier series, we have to mention that in 1996, Antonov [3] proved such convergence in the space $L(\log L)(\log_3 L)(0,1)$ and Arias de Reyna [4] improved Antonov's theorem in 2002, achieving the best result of this type known so far.

In fact, Antonov's proof inspired Carro and Martín [23] to develop an extrapolation technique (see Theorem 3.3.1) that uses restricted weak type estimates of the form

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \qquad \|T\| \lesssim \frac{1}{(p-1)^{\alpha}}.$$

Now, there are also operators in the literature for which the best-known operator norm does not grow up like a power function. For example, if $\mathcal{C}_{\text{lac}}$ is the *Carleson lacunary operator* (see (3.13) for a definition), Di Plinio and Lerner [39] showed that, for $1 < p \le 2$,

$$\mathcal{C}_{\text{lac}} : L^{p}(\mathbb{R}) \longrightarrow L^{p,\infty}(\mathbb{R}), \qquad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1\left(\frac{1}{p-1}\right).$$

In this case, an extension of the theorem by Carro and Martín allows to conclude an endpoint estimate. This example (among many others) motivates the extension of the extrapolation theory to cases where the blow up is not necessarily a power function (see [92], [1], [2]).

Going back to 1959, Zygmund [106, p. 119] proved a dual version of Yano's result for cases where the operator norm blows up when $p \to \infty$. Namely, if T is a sublinear operator acting between finite measure spaces such that, for every $1 < p < \infty$ and for some fixed $\alpha > 0$,

$$T : L^{p}(\mu) \longrightarrow L^{p}(\nu), \qquad \|T\| \lesssim p^{\alpha},$$

then

$$T : L^{\infty}(\mu) \longrightarrow E^{\alpha}(\nu),$$

where $E^{\alpha}(\nu)$ is the space of Definition 1.4.5.

Other endpoints can also be considered. Let, for example, $\mathcal{B}^{\alpha}$ be the *Bochner-Riesz operator of order* α in $\mathbb{R}^2$ (see (3.17) for a definition) with $0 < \alpha \le 1/2$. It is known (see [16], [46, Chapter 5]) that

$$\mathcal{B}^{\alpha} : L^{p}(\mathbb{R}^2) \longrightarrow L^{p}(\mathbb{R}^2) \qquad \text{if and only if} \qquad \frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha}.$$

In this case, if $p_0 := \frac{4}{3+2\alpha} > 1$ and $p_1 := \frac{4}{1-2\alpha} < \infty$, we have $\|\mathcal{B}^\alpha\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \to \infty$ when $p \to p_0$ or $p \to p_1$. It is then natural to study other extrapolation results to cover these cases (see, for example, [26], [1], [2]).

Summarizing, given an admissible function φ and any fixed endpoint $1 \leq q \leq \infty$, we are going to consider extrapolation results under hypothesis of the form

$$T : L^{p,r}(\mu) \longrightarrow L^{p,s}(\nu), \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-q}\right), \qquad \text{for} \quad p > q,$$

where $r \in \{1, p, \infty\}$ and $s \in \{p, \infty\}$, and similarly for the case where $p < q$. The chapter is divided into six sections, according to the different hypotheses: $r = p = s$ (Section 3.1), $r = p$ and $s = \infty$ (Section 3.2), $r = 1$ and $s = \infty$ (Section 3.3), hypotheses for characteristic functions (Section 3.4), $r = \infty = s$ (Section 3.5), and other types of estimates in Section 3.6. All the extrapolation proofs presented follow the ODR technique, except for those in Section 3.4 and one in Section 3.5.

This chapter will also contain applications of the extrapolation results to concrete operators, such as the maximal rough operator, the spherical maximal operator, the Hilbert transform or the Carleson operator. As we shall see, the main source of most of these examples is the Rubio de Francia's extrapolation theorem (see Theorem 1.5.2).

Finally, we should mention and emphasize that in the 1990s, Jawerth and Milman extended extrapolation theory to the setting of compatible couples of Banach spaces using interpolation theory (see [59–62], [81], [5]). Most of the results we will present are particular cases of this more general theory; however, since we are restricted to the case of Lorentz spaces, we aim to avoid, whenever possible, technicalities from the abstract setting.

In what follows, we omit the underlying measure spaces which are σ-finite and non-atomic.

3.1 Strong type theorems

In this section, we shall assume that T is strong type (p, p) for some range of p. We treat three different cases, depending on the value of the endpoint.

3.1.1 Case endpoint $p_0 = 1$

First, we extend Yano's result to σ-finite spaces (see [26]) but in the context of admissible functions.

Theorem 3.1.1. *If for some fixed* $1 < p_1 < \infty$, *and for every* $1 < p \le p_1$,

$$T : L^p \longrightarrow L^p, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right), \tag{3.2}$$

then

$$T : L\ \varphi(\log L) \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

Remark 3.1.2. We observe that, if the target measure space is finite and $\varphi(x) = x^\alpha$, $\Gamma^{1,\infty}[V] = L^1$, and we recover original Yano's theorem.

Proof. We have that if $f \in L^1 \cap L^\infty$ with $\|f\|_{L^\infty} \le 1$, for every $t > 0$ and $1 < p \le p_1$,

$$(Tf)^{**}(t) \le t^{-\frac{1}{p}} \|Tf\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right) \frac{\|f\|_{L^p}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p-1}\right) \left(\frac{\|f\|_{L^1}}{t}\right)^{1/p},$$

and, by Lemma 2.1.4 and (2.2) we obtain

$$(Tf)^{**}(t) \lesssim I_{\varphi,1}\left(\frac{\|f\|_{L^1}}{t}\right) \lesssim \frac{\varphi\left(\log_1 t\right)}{t} I_{\varphi,1}\left(\|f\|_{L^1}\right),$$

from which follows that

$$\|Tf\|_{\Gamma^{1,\infty}[V]} \lesssim I_{\varphi,1}\left(\|f\|_{L^1}\right).$$

Therefore, given any function $f \in L^1$ with a finite dyadic decomposition over $\mathbb{Z}$, we can use Lemma 2.3.1 to obtain the result. Finally, we extend the result to any $f \in L\,\varphi(\log L)$ using a density argument (see Proposition 2.3.16). □

Example 3.1.3. Given $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$, the *rough maximal operator* (see [42, Chapter 4]) is

$$M_\Omega f(x) := \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')||f(x-y)|\, dy, \qquad x \in \mathbb{R}^n, \quad y' = \frac{y}{|y|}. \tag{3.3}$$

It holds that, for $1 < p \le \infty$,

$$M_\Omega : L^p \longrightarrow L^p, \qquad \|M_\Omega\| \lesssim p'.$$

However, it is unknown whether M_Ω is weak type (1,1). From Theorem 3.1.1, we conclude that,

$$M_\Omega : L \log L \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t}{\log_1 t}.$$

On the other hand, if $\Omega \in L\log L(\mathbb{S}^{n-1})$, it is known (see [34]) that M_Ω is weak type (1,1). Although the proof of this is very technical, a weaker result can be proved easily using extrapolation. Let

$$M_1(\Omega, f)(x) := \sup_{R\geq 1} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')||f(x-y)|\, dy, \qquad x \in \mathbb{R}^n, \quad y' = \frac{y}{|y|}.$$

For every $p > 1$, $\Omega \in L^p(\mathbb{S}^{n-1})$ and $f \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} \|M_1(\Omega, f)\|_{L^p(\mathbb{R}^n)} &\lesssim \left\| \int_{B(0,1)} |\Omega(y')||f(\cdot - y)|\, dy \right\|_{L^p(\mathbb{R}^n)} \\ &+ \left\| \sup_{R\geq 1} \int_{B(0,R)/B(0,1)} \frac{|\Omega(y')|}{|y|^n} |f(\cdot - y)|\, dy \right\|_{L^p(\mathbb{R}^n)} = I + II. \end{aligned}$$

By Minkowski's integral inequality we get that $I \leq \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})}$ and that

$$II \leq \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})} \left(\int_1^\infty \frac{r^{n-1}}{r^{np}} dr \right)^{1/p} \lesssim \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})} \frac{1}{p-1}.$$

Thus, if we fix $f \in L^1(\mathbb{R}^n)$, we get that

$$M_1(\cdot, f) : L^p(\mathbb{S}^{n-1}) \longrightarrow L^p(\mathbb{R}^n), \qquad \|M_1(\cdot, f)\| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{p-1},$$

and, from Theorem 3.1.1, we conclude that

$$M_1(\cdot, f) : L\log L(\mathbb{S}^{n-1}) \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\log_1 t}, \quad \|M_1(\cdot, f)\| \lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

which implies that, for $\Omega \in L\log L(\mathbb{S}^{n-1})$,

$$M_1(\Omega, \cdot) : L^1(\mathbb{R}^n) \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t}{\log_1 t}.$$

Example 3.1.4. The function $m \in L^\infty(\mathbb{R})$ is a *Marcinkiewicz Fourier multiplier* if is derivable and

$$\sup_{k\in\mathbb{Z}} \int_{\Delta_k} |m'(t)|\, dt < \infty, \qquad \text{where} \quad \Delta_k = (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}).$$

It is known (see [79], [98]) that, for $1 < p < \infty$,

$$T_m : L^p \longrightarrow L^p, \qquad \|T_m\| \lesssim \left(\max\{p, p'\}\right)^{3/2}. \tag{3.4}$$

Then, by Theorem 3.1.1, we obtain the following estimate at the endpoint

$$T_m : L\,(\log L)^{3/2} \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t}{(\log_1 t)^{3/2}}.$$

We should mention that T_m locally map $L(\log L)^{1/2}$ into $L^{1,\infty}$ and, for every $r < 1/2$, there exist multipliers of this type for which T_m does not locally map $L(\log L)^r$ into $L^{1,\infty}$ (see [98]).

Concerning weighted estimates, it is known (see [70], [74]) that, for $1 < p \leq 2$ and for every $w \in A_p$,

$$T_m : L^p(w) \longrightarrow L^p(w), \qquad \|T_m\| \lesssim [w]_{A_p}^{\frac{1}{p-1}\left(\frac{p+2}{2}\right)}.$$

Thus, if we fix $1 < p_1 \leq 2$ and use Corollary 1.5.3, we get that, for $1 < p \leq p_1$ and every $w \in A_1$,

$$T_m : L^p(w) \longrightarrow L^p(w), \qquad \|T_m\| \lesssim \frac{[w]_{A_1}^{p_1'\left(\frac{p_1+2}{2}\right)}}{(p-1)^{\left(\frac{p_1+2}{2}\right)}}.$$

By Theorem 3.1.1, we obtain that, for every $1 < p_1 \leq 2$ and $w \in A_1$,

$$T_m : L\,(\log L)^{\left(\frac{p_1+2}{2}\right)}(w) \longrightarrow \Gamma_w^{1,\infty}[V], \qquad V(t) = \frac{t}{(\log_1 t)^{\left(\frac{p_1+2}{2}\right)}}.$$

Taking $p_1 = 1 + 2\varepsilon$ with $0 < \varepsilon \leq 1/2$, we have that, for every $w \in A_1$,

$$T_m : L\,(\log L)^{\left(\frac{3}{2}+\varepsilon\right)}(w) \longrightarrow \Gamma_w^{1,\infty}[V], \qquad V(t) = \frac{t}{(\log_1 t)^{\left(\frac{3}{2}+\varepsilon\right)}}.$$

Remark 3.1.5. If $V(t) = t/\varphi\,(\log_1 t)$, the space $\Gamma^{1,\infty}[V]$ is not comparable to $L^{1,\infty}$. Indeed, $L^{1,\infty} \not\subseteq \Gamma^{1,\infty}[V]$ because the function $f(t) = f^*(t) = t^{-1}\chi_{(0,\infty)} \in L^{1,\infty}$ but

$$\|f\|_{\Gamma^{1,\infty}[V]} \geq \frac{\int_0^1 f^*(s)ds}{\varphi(\log_1 1)} = \infty.$$

On the other hand, by [25], $\Gamma^{1,\infty}[V] \not\subseteq L^{1,\infty}$, since

$$\sup_{t>0} \frac{t}{t \sup_{s \geq t} s^{-1} V(s)} = \sup_{t>0} \varphi\,(\log_1 t) = \infty.$$

Hence, if T satisfies (3.2) and is weak type (1,1),

$$T : L\;\varphi\,(\log L) \longrightarrow \Gamma^{1,\infty}[V] \cap L^{1,\infty},$$

where $\Gamma^{1,\infty}[V] \cap L^{1,\infty} \subsetneq L^{1,\infty}$.

Example 3.1.6. *Mihlin-Hörmander multipliers* are functions such that, for all multi-indices α with $|\alpha| \le [n/2]+1$,

$$\left|\partial_\xi^\alpha m(\xi)\right| \le C_\alpha |\xi|^{-|\alpha|}, \qquad \xi \in \mathbb{R}^n \backslash \{0\}.$$

It is known ([80], [55], [45, Theorem 6.2.7]) that the operator T_m associated with these multipliers is of weak type (1,1) and, for $1 < p < \infty$,

$$T_m : L^p \longrightarrow L^p, \qquad \|T_m\| \lesssim \max\left\{p, \frac{1}{p-1}\right\}.$$

Hence, in view of Remark 3.1.5,

$$T_m : L \log L \longrightarrow \Gamma^{1,\infty}[V] \cap L^{1,\infty}, \qquad V(t) = \frac{t}{\log_1 t}.$$

3.1.2 Case left endpoint $p_0 > 1$

The ideas of this section are based in the results of [26]. We first introduce the function spaces we need.

Definition 3.1.7. The spaces $D^+_{p,\varphi}(\mu)$ and $R^+_{p,\varphi}(\nu)$ are defined by the norms

$$\|f\|_{D^+_{p,\varphi}(\mu)} = \|f\|_{L^p(\mu)} + \int_0^1 \frac{\left(\int_0^t [f^*_\mu(s)]^p ds\right)^{1/p}}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt$$

and

$$\|f\|_{R^+_{p,\varphi}(\nu)} = \sup_{t>0} \frac{\left(\int_0^t [f^*_\nu(s)]^p ds\right)^{1/p}}{\varphi(\log_1 t)}.$$

If $\varphi(s) = s^\alpha$ for some $\alpha > 0$, we simply write $D^+_{p,\alpha}(\mu)$ and $R^+_{p,\alpha}(\nu)$.

Remark 3.1.8. We observe that $D^+_{p,\varphi}$ corresponds precisely to the space $K^+_\varphi(\bar{A})$ (see Definition 2.3.3) in the specific case where $\bar{A} = (L^p, L^\infty)$.

Theorem 3.1.9. *If for any fixed $1 < p_0 < p_1 < \infty$, and for every $p_0 < p \le p_1$,*

$$T : L^p \longrightarrow L^p, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right),$$

then

$$T : D^+_{p_0,\varphi} \longrightarrow R^+_{p_0,\varphi}.$$

Proof. It is enough to get the estimate for functions with a finite $K(L^{p_0}, L^\infty)$-decomposition, since these functions are dense in $D^+_{p_0,\varphi}$ (see Proposition 2.3.18). Thus, let $f \in D^+_{p_0,\varphi}$ be a function with a finite $K(L^{p_0}, L^\infty)$-decomposition. We can use Lemma 2.2.2 *a*) to write $f = \sum_{|i|<N} 2^i f_i$ where, for every $p_0 < p \le p_1$, we have that $\|f_i\|_{L^p} \lesssim |E_i|^{p_0/p}$. Then, for every i,

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^{p_0} ds\right)^{1/p_0} \le t^{\frac{1}{p_0}-\frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{\frac{1}{p_0}-\frac{1}{p}} \varphi\left(\frac{1}{p-p_0}\right) \|f_i\|_{L^p}$$
$$\lesssim t^{1/p_0} \varphi\left(\frac{1}{p-p_0}\right) \left(\frac{|E_i|^{p_0}}{t}\right)^{1/p}. \tag{3.5}$$

By Lemma 2.1.4 we get that

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^{p_0} ds\right)^{1/p_0} \lesssim t^{1/p_0} I_{\varphi,p_0}\left(\frac{|E_i|^{p_0}}{t}\right) \lesssim \varphi(\log_1 t) I_{\varphi,1}\left(|E_i|\right). \tag{3.6}$$

Summing over i and taking the supremum over $t > 0$, we obtain that

$$\sup_{t>0} \frac{\left(\int_0^t \left[(Tf)^*(s)\right]^{p_0} ds\right)^{1/p_0}}{\varphi\left(\log_1 t\right)} \lesssim \sum_{i\in\mathbb{Z}} 2^i I_{\varphi,1}\left(|E_i|\right).$$

Finally, Lemma 2.3.5 implies that

$$\sup_{t>0} \frac{\left(\int_0^t \left[(Tf)^*(s)\right]^{p_0} ds\right)^{1/p_0}}{\varphi\left(\log_1 t\right)} \lesssim \|f\|_{L^{p_0}} + \int_0^1 \frac{\left(\int_0^s [f^*(r)]^{p_0} dr\right)^{1/p_0}}{s} \varphi'\left(\log_1 \frac{1}{s}\right) ds.$$

□

Remark 3.1.10. Theorem 3.1.9 and its proof are also valid for $p_0 = 1$. However, in that case, we just recover Theorem 3.1.1.

Example 3.1.11. For $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^n)$, the *spherical maximal operator* is defined as

$$\mathscr{M} f(x) = \sup_{t>0} \left| \int_{\mathbb{S}^{n-1}} f(x-ty) d\sigma(y) \right|, \qquad x \in \mathbb{R}^n,$$

where σ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$. It is known (see [96], [90], [45, Theorem 6.5.1]) that, for $n \ge 3$,

$$\mathscr{M} : L^p \longrightarrow L^p, \qquad \|\mathscr{M}\| \lesssim \frac{1}{p-\frac{n}{n-1}}, \qquad \frac{n}{n-1} < p \le 2. \tag{3.7}$$

As far as we are concerned, the best result at the endpoint $p_0 := \frac{n}{n-1}$ is the one obtained in [11], where it is proved that, for $n \geq 3$,

$$\mathscr{M} : L^{\frac{n}{n-1},1} \longrightarrow L^{\frac{n}{n-1},\infty}.$$

Using Theorem 3.1.9 and (3.7), we conclude that, for $n \geq 3$,

$$\mathscr{M} : D^+_{\frac{n}{n-1},1} \longrightarrow R^+_{\frac{n}{n-1},1}.$$

These estimates are independent because, as shown in [26], $L^{\frac{n}{n-1},1}$ is not comparable with $D^+_{\frac{n}{n-1},1}$.

To present our next example, we recall that, for every $1 < p < \infty$, the function $|x|^a$ is an A_p weight if and only if $-n < a < n(p-1)$, and, for every $p_0 < p < \infty$,

$$\left[|x|^{n(p_0-1)}\right]_{A_p} \approx \frac{1}{(p-p_0)^{p-1}} = \frac{1}{(p-p_0)^{p_0-1}(p-p_0)^{p-p_0}} \approx \frac{1}{(p-p_0)^{p_0-1}}. \tag{3.8}$$

Corollary 3.1.12. *Let T be such that, for some fixed $1 < p_0 < p_1 < \infty$, for every $p_0 < p \leq p_1$ and every $w \in A_p$,*

$$T : L^p(w) \longrightarrow L^p(w), \qquad \|T\| \lesssim \psi\left([w]_{A_p}\right),$$

where ψ is increasing. If $\psi(s^{p_0-1}) \approx \varphi(s)$ for some admissible function φ, then,

$$T : D^+_{p_0,\varphi}\left(|x|^{n(p_0-1)}\right) \longrightarrow R^+_{p_0,\varphi}\left(|x|^{n(p_0-1)}\right).$$

Proof. By (3.8), we get that, for $p_0 < p \leq p_1$,

$$T : L^p\left(|x|^{n(p_0-1)}\right) \longrightarrow L^p\left(|x|^{n(p_0-1)}\right), \quad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right).$$

The result follows by Theorem 3.1.9. □

Example 3.1.13. Let H denote the Hilbert transform defined, for every $f \in \mathscr{S}(\mathbb{R})$ as follows

$$H(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy, \qquad x \in \mathbb{R}.$$

It is known (see [57], [86]) that, for $1 < p < \infty$,

$$H : L^p(w) \longrightarrow L^p(w) \qquad \text{if and only if} \qquad w \in A_p,$$

and $\|H\| \lesssim [w]_{A_p}$ for $2 < p < \infty$. As a consequence, for $p_0 > 1$, since $|x|^{n(p_0-1)} \notin A_{p_0}$, H is not bounded on $L^{p_0}\big(|x|^{n(p_0-1)}\big)$, but Corollary 3.1.12 implies that

$$H : D^+_{p_0,p_0-1}\left(|x|^{n(p_0-1)}\right) \longrightarrow R^+_{p_0,p_0-1}\left(|x|^{n(p_0-1)}\right). \tag{3.9}$$

On the other hand (see [35]), since $|x|^{n(p_0-1)} \in A^{\mathcal{R}}_{p_0}$,

$$H : L^{p_0,1}\left(|x|^{n(p_0-1)}\right) \longrightarrow L^{p_0,\infty}\left(|x|^{n(p_0-1)}\right). \tag{3.10}$$

We have that (3.9) and (3.10) are independent because $D^+_{p_0,p_0-1}\big(|x|^{n(p_0-1)}\big)$ and $L^{p_0,1}\big(|x|^{n(p_0-1)}\big)$ are not comparable.

Example 3.1.14. For $1 < p < \infty$ and $f \in L^p(\mathbb{R})$, the *Carleson* operator is defined as

$$\mathcal{C}f(s) := \sup_{\xi\in\mathbb{R}}\left| H\left(e^{2\pi i\xi(\cdot)}f(\cdot)\right)(s)\right|, \qquad s \in \mathbb{R}. \tag{3.11}$$

It is known (see [58], [39]) that

$$\mathcal{C} : L^p(w) \longrightarrow L^p(w) \qquad \|\mathcal{C}\| \lesssim [w]_{A_p}^{\max\left\{p', \frac{2}{p-1}\right\}} \log_2 [w]_{A_p}. \tag{3.12}$$

Thus, for any fixed $1 < p_0 < \infty$, we have:

(i) If $1 < p_0 < 2$,

$$\|\mathcal{C}\| \lesssim [w]_{A_p}^{\frac{2}{p_0-1}} \log_2 [w]_{A_p}, \qquad p_0 < p \leq 2.$$

Hence, for $\varphi_1(s) := s^2 \log_2 s$, Corollary 3.1.12 implies

$$\mathcal{C} : D^+_{p_0,\varphi_1}\left(|x|^{p_0-1}\right) \longrightarrow R^+_{p_0,\varphi_1}\left(|x|^{p_0-1}\right).$$

(ii) If $2 \leq p_0 < \infty$,

$$\|\mathcal{C}\| \lesssim [w]_{A_p}^{p_0'} \log_2 [w]_{A_p}, \qquad p_0 < p < \infty.$$

In this case, if $\varphi_2(s) := s^{p_0} \log_2 s$, Corollary 3.1.12 implies

$$\mathcal{C} : D^+_{p_0,\varphi_2}\left(|x|^{p_0-1}\right) \longrightarrow R^+_{p_0,\varphi_2}\left(|x|^{p_0-1}\right).$$

Example 3.1.15. For $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, the *Carleson lacunary* operator is defined as

$$\mathcal{C}_{\text{lac}} f(s) := \sup_{a \in A} \left| H\left(e^{2\pi i a(\cdot)} f(\cdot)\right)(s)\right| \qquad s \in \mathbb{R}, \tag{3.13}$$

where $A \subset \mathbb{R}$ is lacunary in the sense that there exists a constant $C > 0$ such that

$$\inf_{a \neq a' \in A} \frac{|a - a'|}{|a|} = C.$$

In [39], the authors proved that, for $1 < p < \infty$,

$$\mathcal{C}_{\text{lac}} : L^p(w) \longrightarrow L^p(w) \qquad \|\mathcal{C}_{\text{lac}}\| \lesssim [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \log_1 [w]_{A_p}. \tag{3.14}$$

Hence, by Corollary 3.1.12, we get that:

(i) If $1 < p_0 < 2$, for $\varphi_3(s) := s \log_1 s$,

$$\mathcal{C}_{\text{lac}} : D^+_{p_0,\varphi_3}\left(|x|^{p_0-1}\right) \longrightarrow R^+_{p_0,\varphi_3}\left(|x|^{p_0-1}\right).$$

(ii) If $2 \leq p_0 < \infty$, for $\varphi_4(s) := s^{p_0-1} \log_1 s$,

$$\mathcal{C}_{\text{lac}} : D^+_{p_0,\varphi_4}\left(|x|^{p_0-1}\right) \longrightarrow R^+_{p_0,\varphi_4}\left(|x|^{p_0-1}\right).$$

Example 3.1.16. Consider the weight

$$w(x) := \frac{|x|^{p_0-1}}{\log_1 \dfrac{1}{|x|}},$$

where $1 < p_0 < \infty$. Going back to Example 3.1.13, since $w \notin A^{\mathcal{R}}_{p_0}$ (see [24]), H does not map $L^{p_0,1}(w)$ to $L^{p_0,\infty}(w)$. However, for every fixed $p_0 < p_1 < \infty$, and for every $p_0 < p \leq p_1$,

$$[w]_{A_p} \lesssim \frac{1}{(p - p_0)^{p_0-1}},$$

which implies, following the same argument as in Corollary 3.1.12, that

$$H : D^+_{p_0,p_0-1}(w) \longrightarrow R^+_{p_0,p_0-1}(w).$$

3.1.3 Case right endpoint $p_1 < \infty$

This section follows the ideas appearing in [26]. First, we define the function spaces that we need for the theorem.

Definition 3.1.17. The spaces $D^-_{p,\varphi}(\mu)$ and $R^-_{p,\varphi}(\nu)$ are defined by the following norms

$$\|f\|_{D^-_{p,\varphi}(\mu)} = \|f\|_{L^p(\mu)} + \int_1^\infty \frac{\left(\int_s^\infty [f^{**}_\mu(r)]^p dr\right)^{1/p}}{s} \varphi'(\log_1 s)\, ds$$

and

$$\|f\|_{R^-_{p,\varphi}(\nu)} = \sup_{t>0} \frac{\left(\int_t^\infty [f^{**}_\nu(s)]^p ds\right)^{1/p}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

If $\varphi(s) = s^\alpha$ for some $\alpha > 0$, we can simply write $D^-_{p,\alpha}(\mu)$ and $R^-_{p,\alpha}(\nu)$.

Remark 3.1.18. We observe that $D^-_{p,\varphi}$ corresponds precisely to the space $K^-_\varphi(\bar{A})$ (see Definition 2.3.3) in the specific case where $\bar{A} = (L^1, L^p)$.

Theorem 3.1.19. *If for any fixed $1 < p_0 < p_1 < \infty$ and for every $p_0 \le p < p_1$,*

$$T : L^p \longrightarrow L^p, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

where φ satisfies (2.8), then,

$$T : D^-_{p_1,\varphi} \longrightarrow R^-_{p_1,\varphi}.$$

Proof. It is enough to obtain the estimate just for functions which have a finite $K(L^1, L^{p_1})$-decomposition, since these functions are dense in $D^-_{p_1,\varphi}$ (see Proposition 2.3.19). Let $f \in D^-_{p_1,\varphi}$ be a function with a finite $K(L^1, L^{p_1})$-decomposition and such that $\|f\|_{L^{p_1}} \le 1/p_1'$. We can use Lemma 2.2.2 *c*) to write $f = \sum_{|i|<N} 2^i f_i$ where, for every $p_0 \le p < p_1$, we have that $\|f_i\|_{L^p} \lesssim |E_i|^{p_1'\left(\frac{1}{p} - \frac{1}{p_1}\right)}$.

Since $F(t) := K(f, t; L^1, L^{p_1})$ is an increasing concave function, it follows that, for almost every $t > 0$,

$$F'(t) \le \frac{1}{t}\int_0^t F'(s)ds = \frac{F(t) - F(0)}{t} \le \|f^{**}\|_{L^{p_1}} \le p_1'\|f\|_{L^{p_1}} \le 1. \tag{3.15}$$

Therefore, $E_i = \emptyset$ whenever $i \geq 0$ and, for every $-N < i < 0$ and every $p_0 \leq p < p_1$, we have

$$\left(\int_t^\infty [(Tf_i)^{**}(s)]^{p_1} ds\right)^{1/p_1} \lesssim \left(\int_t^\infty [(Tf_i)^{**}(s)]^{p} s^{\frac{p}{p_1}-1} ds\right)^{1/p}$$
$$\lesssim t^{\frac{1}{p_1}-\frac{1}{p}} \|Tf_i\|_{L^p} \lesssim \varphi\left(\frac{1}{p_1-p}\right)\left(\frac{|E_i|^{p_1'}}{t}\right)^{\frac{1}{p}-\frac{1}{p_1}}. \quad (3.16)$$

By Lemma 2.1.4 we get that

$$\left(\int_t^\infty [(Tf_i)^{**}(s)]^{p_1} ds\right)^{1/p_1} \lesssim \varphi\left(\log_1 \frac{|E_i|^{p_1'}}{t}\right) \lesssim \varphi\left(\log_1 \frac{1}{t}\right)\varphi\left(\log_1 |E_i|\right).$$

Hence, summing over i, we obtain

$$\|Tf\|_{R^-_{p_1,\varphi}} \lesssim \sum_{i\in\mathbb{Z}^-} 2^i \varphi\left(\log_1 |E_i|\right) \lesssim 1 + \int_1^\infty \frac{\left(\int_s^\infty [f^{**}(r)]^{p_1} dr\right)^{1/p_1}}{s} \varphi'\left(\log_1 s\right) ds,$$

where we have used Lemma 2.3.6 in the last inequality. Finally, we can extend the estimate by homogeneity to any function $f \in L^{p_1}$ with finite decomposition. □

Example 3.1.20. The *Bochner-Riesz operator* of order $\alpha \geq 0$ in $\mathbb{R}^2$ is the multiplier defined by

$$\mathcal{B}^\alpha(f)(x) := \int_{\mathbb{R}^2} (1-|\xi|^2)_+^\alpha \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi, \quad x \in \mathbb{R}^2. \quad (3.17)$$

It can be deduced from [38] (see also [16]) that, for $0 < \alpha < \frac{1}{2}$ and $4 < p < \frac{4}{1-2\alpha}$,

$$\mathcal{B}^\alpha : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2), \quad \|\mathcal{B}^\alpha\| \lesssim \frac{1}{\left(\frac{4}{1-2\alpha}-p\right)^{5/4}}.$$

Therefore, Theorem 3.1.19 implies that

$$\mathcal{B}^\alpha : D^-_{\frac{4}{1-2\alpha},\frac{5}{4}} \longrightarrow R^-_{\frac{4}{1-2\alpha},\frac{5}{4}}.$$

We should mention that it is also known (see [68]) that

$$\mathcal{B}^\alpha : L^{\frac{4}{1-2\alpha},1} \longrightarrow L^{\frac{4}{1-2\alpha},\infty},$$

but $D^-_{\frac{4}{1-2\alpha},\frac{5}{4}} \not\subseteq L^{\frac{4}{1-2\alpha},1}$. For example, the function

$$f(t) = \frac{1}{t^{\frac{1-2\alpha}{4}}\left(\log\frac{1}{t}\right)} \chi_{\left[0,e^{-\frac{4}{1-2\alpha}}\right)}(t)$$

belongs to $D^-_{\frac{4}{1-2\alpha},\frac{5}{4}}$ but not to $L^{\frac{4}{1-2\alpha},1}$.

Also, since for every measurable set E

$$\|\chi_E\|_{D^-_{\frac{4}{1-2\alpha},\frac{5}{4}}} = |E|^{\frac{1-2\alpha}{4}} (\log_1 |E|)^{5/4} \gtrsim \|\chi_E\|_{L^{\frac{4}{1-2\alpha},1}} = |E|^{\frac{1-2\alpha}{4}},$$

we have that $L^{\frac{4}{1-2\alpha},1} \not\subseteq D^-_{\frac{4}{1-2\alpha},\frac{5}{4}}$.

3.2 Weak type theorems

In this section, we shall assume that T is weak type (p,p) for some range of p. We treat three different cases, depending on the value of the endpoint.

3.2.1 Case endpoint $p_0 = 1$

In this case, the target space at the endpoint is quasi-Banach. The lack of a triangle inequality implies some difficulties in the reconstruction step. Thus, we need to use the *Galb space* (see [101], [19]), defined as follows

$$\text{Galb}(\mathcal{X}) = \left\{ (e_n)_n : \sum_n e_n f_n \in \mathcal{X}, \text{ whenever } \|f_n\|_{\mathcal{X}} \leq 1 \right\},$$

endowed with the norm

$$\left\|(e_n)_n\right\|_{\text{Galb}(\mathcal{X})} := \sup_{\|f_n\|_{\mathcal{X}} \leq 1} \left\| \sum_n e_n f_n \right\|_{\mathcal{X}}.$$

It is known that

- if $\mathcal{X}$ is a Banach space, then $\text{Galb}(\mathcal{X}) = \ell^1$,
- $\text{Galb}\left(L^{1,\infty}\right) = \ell \log \ell$ (see [97], [65]) where

$$\ell \log \ell = \left\{ (e_n)_n : \sum_n e_n \log_1 n < \infty \right\},$$

- if $V(t) = t/\varphi(\log_1 t)$, $\ell \log \ell \subseteq \text{Galb}\left(\Lambda^{1,\infty}[V]\right)$ (see [28]).

The next theorem is an extension of the work in [23] to the setting of admissible functions (see also [21]).

Theorem 3.2.1. *If for any fixed* $1 < p_1 < \infty$ *and, for every* $1 < p \leq p_1$,

$$T : L^p \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right),$$

then,

$$T : L\ \varphi(\log L) \log_3 L \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

Proof. Since bounded functions are dense in $L\ \varphi(\log L)\log_3 L$ (see Proposition 2.3.16), we only need to prove the result for such functions, and we can extend it by density. Let $f \geq 0$ be a bounded function in $L\ \varphi(\log L)\log_3 L$, then we have that

$$A := \int f\varphi\left(\log_1 f\right)\left(\log_3 f\right) < \infty. \tag{3.18}$$

If we use the 2-dyadic decomposition, we obtain that, for every $1 < p \leq p_1$ and every $t > 0$,

$$\left(Tf_n\right)^*(t) \leq t^{-\frac{1}{p}}\|Tf_n\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-1}\right)\left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p}, \tag{3.19}$$

and, by Lemma 2.1.4 we get

$$\left(Tf_n\right)^*(t) \lesssim I_{\varphi,1}\left(\frac{\|f_n\|_{L^1}}{t}\right) \lesssim \frac{\varphi\left(\log_1 t\right)}{t} I_{\varphi,1}\left(\|f_n\|_{L^1}\right).$$

Thus, for $V(t) = t/\varphi\left(\log_1 t\right)$,

$$\|Tf_n\|_{\Lambda^{1,\infty}[V]} \lesssim I_{\varphi,1}\left(\|f_n\|_{L^1}\right),$$

and, consequently,

$$\begin{aligned}\|Tf\|_{\Lambda^{1,\infty}[V]} &\leq \left\| \sum_{0\leq n\leq N} 2^{2^n}|Tf_n| \right\|_{\Lambda^{1,\infty}[V]} \lesssim \left\| \left(2^{2^n} I_{\varphi,1}\left(\|f_n\|_{L^1}\right)\right)_n \right\|_{\mathrm{Galb}(\Lambda^{1,\infty}[V])} \\ &\lesssim \left\| \left(2^{2^n} I_{\varphi,1}\left(\|f_n\|_{L^1}\right)\right)_n \right\|_{\ell\log\ell}.\end{aligned}$$

If we denote $c_n = \varphi\left(2^n\right)\log_1 n$ and $a_n = \int_{\{d_{n-1}\leq f<d_n\}} f\varphi\left(\log_1 f\right)\log_3 f$, we have

$$I_{\varphi,1}\left(\|f_n\|_{L^1}\right) \approx I_{\varphi,1}\left(\frac{1}{2^{2^n}c_n}\int_{\{d_{n-1}\leq f<d_n\}} f\varphi\left(\log_1 f\right)\left(\log_3 f\right)\right) = I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right).$$

Therefore, in view of (3.18), if we prove that

$$\left\| \left(2^{2^n} I_{\varphi,1}\Big(\frac{a_n}{2^{2^n}c_n}\Big)\right)_n \right\|_{\ell\log\ell} \lesssim A,$$

we are done.

Now, we get

$$\left\| \left(2^{2^n} I_{\varphi,1}\Big(\frac{a_n}{2^{2^n}c_n}\Big)\right)_n \right\|_{\ell\log\ell} \lesssim 1 + \sum_{n\geq 1} 2^{2^n} I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right)\log_1 n$$
$$\lesssim 1 + \sum_{n\in \mathrm{I}} 2^{2^n} I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right)\log_1 n + \sum_{n\in \mathrm{II}} 2^{2^n} I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right)\log_1 n,$$

where

$$\mathrm{I} = \left\{ n\geq 1 : a_n \leq 2^{-n}\right\} \quad \text{and} \quad \mathrm{II} = \left\{ n\geq 1 : a_n > 2^{-n}\right\}.$$

First, since $I_{\varphi,1}$ is essentially increasing and $\varphi(t)\lesssim t^\beta$ (see Remark 1.3.2), we get

$$\sum_{n\in\mathrm{I}} 2^{2^n} I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right)\log_1 n \lesssim \sum_{n\geq 1} 2^{2^n} I_{\varphi,1}\left(\frac{1}{2^{2^n}\varphi(2^n)2^n\log_1 n}\right)\log_1 n$$
$$= \sum_{n\geq 1}\frac{1}{\varphi(2^n)2^n}\varphi\left(\log_1(2^{2^n}\varphi(2^n)2^n\log_1 n)\right) \lesssim \sum_{n\geq 1}\frac{1}{2^n}\approx 1.$$

Also, since $\|(a_n)_n\|_{\ell_1} = \int f\varphi\,(\log_1 f)\,(\log_3 f) = A$,

$$\sum_{n\in\mathrm{II}} 2^{2^n} I_{\varphi,1}\left(\frac{a_n}{2^{2^n}c_n}\right)\log_1 n \lesssim \sum_{n\geq 1}\frac{a_n}{\varphi(2^n)\log_1 n}\varphi\left(\log_1\left(\frac{2^{2^n}\varphi(2^n)\log_1 n}{a_n}\right)\right)\log_1 n$$
$$\lesssim \sum_{n\geq 1}\frac{a_n}{\varphi(2^n)}\varphi\left(\log_1(2^{2^n}\varphi(2^n)2^n\log_1 n)\right) \lesssim \sum_{n\geq 1} a_n = A.$$

□

Remark 3.2.2. Moreover, if for some $\beta_k > 0,\ \beta_{k+1}\cdots\beta_{k+n}\geq 0$,

$$\varphi(s)\approx \prod_{j=k}^{k+n}\left(\log_j s\right)^{\beta_j},$$

we can refine the above argument using the $(k+2)$-dyadic decomposition. Choosing $c_n = \varphi(\log_1 d_n)\log_1 n$ and $a_n = \int_{\{d_{n-1}\leq f< d_n\}} f\varphi\,(\log_1 f)\log_{k+3} f$, we obtain that

$$T : L\ \varphi(\log L)\log_{3+k} L \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\,(\log_1 t)}.$$

Example 3.2.3. It is known (see [39], [20]) that, if $\mathcal{C}_{\text{lac}}$ is the Carleson lacunary operator (3.13), for every weight $w \in A_1$ and every $1 < p \leq 2$,

$$\mathcal{C}_{\text{lac}} : L^p(w) \longrightarrow L^{p,\infty}(w), \qquad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1\left(\frac{1}{p-1}\right).$$

Hence, by the above remark,

$$\mathcal{C}_{\text{lac}} : L(\log_2 L)(\log_4 L)(w) \longrightarrow \Lambda_w^{1,\infty}[V], \qquad V(t) = \frac{t}{(\log_2 t)}.$$

3.2.2 Case left endpoint $p_0 > 1$

Theorem 3.2.4. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^p \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right),$$

then

$$T : D^+_{p_0,\varphi} \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{1/p_0}}{\varphi\left(\log_1 t\right)}.$$

Proof. It follows the same steps than the proof of Theorem 3.1.9, taking into account that, for every $1 < p_0 < p$, instead of (3.5), we have

$$\begin{aligned}(Tf_i)^{**}(t) &\leq t^{-1/p}\|Tf_i\|_{\Gamma^{p,\infty}} \lesssim t^{-1/p}\|Tf_i\|_{L^{p,\infty}} \lesssim t^{-1/p}\varphi\left(\frac{1}{p-p_0}\right)\|f_i\|_{L^p} \\ &\lesssim t^{1/p_0}\varphi\left(\frac{1}{p-p_0}\right)\left(\frac{|E_i|^{p_0}}{t}\right)^{1/p}.\end{aligned}$$

□

Example 3.2.5. Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}^+$ and $\Gamma_\alpha(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\right\}$. The *intrinsic square function* (see [102]) is defined by

$$G_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} \left|A_\alpha f(y,t)\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \qquad x \in \mathbb{R}^n,$$

where

$$A_\alpha f(y,t) = \sup_{\phi \in \mathcal{C}^\alpha} \left|\left(\phi_t * f\right)(y)\right|,$$

with $\phi_t(y) = t^{-n}\phi(yt^{-n})$ and $\mathcal{C}^\alpha$ the family of functions supported in $B(0,1)$ such that $\int \phi = 0$ and

$$\left|\phi(x) - \phi(x')\right| < \left|x - x'\right|^\alpha, \qquad \forall x, x' \in \mathbb{R}^n.$$

It is known [40] that, for $2 < p \le 3$ and every weight $w \in A_p$,

$$G_\alpha : L^p(w) \longrightarrow L^{p,\infty}(w), \qquad \|G_\alpha\| \lesssim [w]_{A_p}^{\frac{1}{2}} \left(\log_1 [w]_{A_\infty}\right)^{\frac{1}{2}},$$

where $[w]_{A_\infty} := \sup_Q \frac{\int_Q M(w\chi_Q)}{w(Q)}$.

Hence, by (3.8) we get that

$$G_\alpha : L^p\left(|x|^n\right) \longrightarrow L^{p,\infty}\left(|x|^n\right), \qquad \|G_\alpha\| \lesssim \frac{1}{(p-2)^{\frac{1}{2}}},$$

and, by Theorem 3.2.4, we have

$$G_\alpha : D^+_{2,\frac{1}{2}}\left(|x|^n\right) \longrightarrow \Gamma^{1,\infty}_{(|x|^n)}[V], \qquad V(t) = \frac{t^{1/2}}{(\log_1 t)^{\frac{1}{2}}}.$$

3.2.3 Case right endpoint $p_1 < \infty$

Theorem 3.2.6. *If for any fixed $1 < p_0 < p_1 < \infty$, and, for every $p_0 \le p < p_1$,*

$$T : L^p \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

where φ satisfies (2.8), then,

$$T : D^-_{p_1,\varphi} \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{1/p_1}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

Proof. It follows the same steps than the proof of Theorem 3.1.19. Instead of (3.16), we now have that, for every $1 < p_0 \le p < p_1$,

$$t^{1/p_1}(Tf_i)^{**}(t) \lesssim t^{\left(\frac{1}{p_1} - \frac{1}{p}\right)} \|Tf_i\|_{L^{p,\infty}} \lesssim t^{\left(\frac{1}{p_1} - \frac{1}{p}\right)} \varphi\left(\frac{1}{p_1 - p}\right) \|f_i\|_{L^p}$$
$$\lesssim \varphi\left(\frac{1}{p_1 - p}\right) \left(\frac{|E_i|}{t^{1/p_1'}}\right)^{p_1'\left(\frac{1}{p} - \frac{1}{p_1}\right)}.$$

The ODR technique finishes the proof. □

3.3 Restricted weak type theorems

By restricted weak type, we mean that

$$T : L^{p,1} \longrightarrow L^{p,\infty}.$$

3.3.1 Case endpoint $p_0 = 1$

Theorem 3.3.1. *If, for any fixed $1 < p_1 < \infty$ and, for every $1 < p \leq p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right),$$

then,

$$T : L\,\varphi(\log L)\log_3 L \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

Remark 3.3.2. Observe that the above theorem improves Theorem 3.2.1. In this case, we can follow the same proof because, for $f \in L^1 \cap L^\infty$ with $\|f\|_{L^\infty} \leq 1$, it also holds that $\|f\|_{L^{p,1}} \leq \|f\|_{L^1}^{1/p}$.

3.3.2 Case left endpoint $p_0 > 1$

We adapt the ideas in [26] to the setting of admissible functions.

Theorem 3.3.3. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right),$$

then

$$T : \Lambda^1(v_0) \longrightarrow \Gamma^{1,\infty}[V_1]$$

where $v_0(t) = t^{\frac{1}{p_0}-1}\varphi\left(\log_1 \frac{1}{t}\right)$ and $V_1(t) = \dfrac{t^{\frac{1}{p_0}}}{\varphi\left(\log_1 t\right)}$.

Proof. It is enough to get the estimate for functions with a finite dyadic decomposition over $\mathbb{Z}$, since these functions are dense in $\Lambda^1(v_0)$ (see Proposition 2.3.16). Given $f \in L^{p_0,1}$, we use the dyadic decomposition over $\mathbb{Z}$ to get $f = \sum_{|i|<N} 2^i f_i$ where $\|f_i\|_{L^\infty} \leq 1$. For every $t > 0$ and every $p_0 < p \leq p_1$,

$$(Tf_i)^{**}(t) \lesssim t^{-1/p}\|Tf_i\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-p_0}\right)\frac{\|f_i\|_{L^{p,1}}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p-p_0}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p}.$$

By Lemma 2.1.4, we have

$$(Tf_i)^{**}(t) \lesssim I_{\varphi,p_0}\left(\frac{\|f_i\|_{L^1}}{t}\right) \lesssim \frac{\varphi\left(\log_1 t\right)}{t^{1/p_0}} I_{\varphi,p_0}\left(\|f_i\|_{L^1}\right).$$

Summing over i and taking the supremum in $t>0$, the result follows by Lemma 2.3.1. □

3.3.3 Case right endpoint $p_1 < \infty$

The ideas of this case appear in [26] and we extend them to the setting of admissible functions.

Theorem 3.3.4. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 \le p < p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

then

$$T : \Lambda^1(v_0) \longrightarrow \Gamma^{1,\infty}[V_1],$$

where $v_0(t) = t^{\frac{1}{p_1}-1}\varphi\left(\log_1 t\right)$ and $V_1(t) = \dfrac{t^{\frac{1}{p_1}}}{\varphi\left(\log_1 \frac{1}{t}\right)}$.

Proof. By Proposition 2.3.16, it suffices to prove the result functions with a finite dyadic decomposition over $\mathbb{Z}$ and extend the estimate with a density argument. Given $f \in L^{p_1,1}$ with a finite dyadic decomposition over $\mathbb{Z}$, we have that, for every i, for every $t>0$ and every $p_0 \le p < p_1$,

$$(Tf_i)^{**}(t) \lesssim t^{-1/p}\|Tf_i\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p_1-p}\right)\frac{\|f_i\|_{L^{p,1}}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p_1-p}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p}. \tag{3.20}$$

By Lemma 2.1.4, we obtain

$$(Tf_i)^{**}(t) \lesssim J_{\varphi,p_1}\left(\frac{\|f_i\|_{L^1}}{t}\right) \lesssim \frac{\varphi\left(\log_1 \frac{1}{t}\right)}{t^{1/p_1}} J_{\varphi,p_1}\left(\|f_i\|_{L^1}\right).$$

Summing over i and taking the supremum in $t>0$, the result follows by Lemma 2.3.1. □

3.3.4 Case endpoint $p_1 = \infty$

The case where the endpoint is $p_1 = \infty$ was considered first in [105] in the context of finite measure spaces, and the extension to σ-finite measure spaces was studied in [29]. Here, we extend it to the context of admissible functions.

Theorem 3.3.5. *If for a fixed $1 < p_0 < \infty$, and, for every $p_0 \leq p < \infty$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi(p),$$

then

$$T : L^\infty_\varphi \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{1}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

Proof. We only prove the estimate for functions with finite support (and thus a finite f^*-decomposition) and the result follows by density (see Proposition 2.3.17). Given $f \in L^\infty_\varphi$ with finite support, we use the f^*-decomposition to get that $f = \sum_{0 \leq n < N} f_n$ with $f_n \in L^1 \cap L^\infty$ and, for every $t > 0$ and every $p_0 \leq p < \infty$,

$$(Tf_n)^{**}(t) \lesssim t^{-\frac{1}{p}} \|Tf_n\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{p}} \varphi(p) \|f_n\|_{L^{p,1}} \lesssim \|f_n\|_{L^\infty} \varphi(p) \left(\frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right)^{1/p}. \tag{3.21}$$

By Lemma 2.1.4 we obtain

$$(Tf_n)^{**}(t) \lesssim \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right) \lesssim \varphi\left(\log_1 \frac{1}{t}\right) \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right).$$

The result follows from Lemma 2.3.2. □

Remark 3.3.6. Recall that, if $\varphi(t) = t^\alpha$ for some $\alpha > 0$ and $V(t) = \frac{1}{\varphi\left(\log_1 \frac{1}{t}\right)}$, we have that $\Gamma^{1,\infty}[V] = E^\alpha$ (see Definition 1.4.5).

Example 3.3.7. Let G_α be the intrinsic square function (see Example 3.2.5). It is known that (see [73]) that, for every $w \in A_3$,

$$G_\alpha : L^3(w) \longrightarrow L^3(w), \qquad \|G_\alpha\| \lesssim [w]_{A_3}^{\frac{1}{2}}.$$

By Rubio de Francia extrapolation (see Theorem 1.5.2), we obtain that, for every $3 < p < \infty$ and every $w \in A_p$,

$$G_\alpha : L^p(w) \longrightarrow L^p(w) \qquad \|G_\alpha\| \lesssim p^{\frac{1}{2}} [w]_{A_p}^{\frac{1}{2}}.$$

If we fix $w \in A_\infty$, we can take $3 < p_0 < \infty$ such that, $w \in A_{p_0}$ and, for every $p_0 \leq p < \infty$,

$$G_\alpha : L^p(w) \longrightarrow L^p(w) \qquad \|G_\alpha\| \lesssim p^{\frac{1}{2}}[w]_{A_{p_0}}^{\frac{1}{2}} \approx p^{\frac{1}{2}}.$$

Consequently, Theorem 3.3.5 implies that,

$$G_\alpha : L^\infty_{\frac{1}{2}}(w) \longrightarrow E^{\frac{1}{2}}(w).$$

Example 3.3.8. Let m be a Marcinkiewicz multiplier (see Example 3.1.4). By (3.4) and Theorem 3.3.5, we can deduce that

$$T_m : L^\infty_{\frac{3}{2}} \longrightarrow E^{\frac{3}{2}}.$$

Moreover, for every $\varepsilon > 0$, if we take $p_0 := 1 + \frac{1}{2\varepsilon}$, we have (see [74]) that, for every $w \in A_{p_0}$,

$$T_m : L^{p_0}(w) \longrightarrow L^{p_0}(w), \qquad \|T_m\| \lesssim [w]_{A_{p_0}}^{\frac{3}{2}+\varepsilon}.$$

By Rubio de Francia extrapolation, we get that, for every $p_0 \leq p < \infty$, and every $w \in A_p$,

$$T_m : L^p(w) \longrightarrow L^p(w), \qquad \|T_m\| \lesssim (p\,[w]_{A_p})^{\frac{3}{2}+\varepsilon},$$

and hence, for every $w \in A_\infty$,

$$T_m : L^\infty_{\frac{3}{2}+\varepsilon}(w) \longrightarrow E^{\frac{3}{2}+\varepsilon}(w).$$

Example 3.3.9. Let $\mathcal{C}$ be the Carleson operator (3.11). If we fix $0 < \varepsilon \leq 1$ and $p_0 := \frac{1+\varepsilon}{\varepsilon}$, (3.12) implies that, for every $w \in A_{p_0}$,

$$\mathcal{C} : L^{p_0}(w) \longrightarrow L^{p_0}(w) \qquad \|\mathcal{C}\| \lesssim [w]_{A_{p_0}}^{1+\varepsilon} \log_2 [w]_{A_{p_0}}.$$

Using Rubio de Francia extrapolation and also Theorem 3.3.5, we obtain that, if $\varphi(s) = s^{1+\varepsilon}\log_2 s$, for every $w \in A_\infty$,

$$\mathcal{C} : L^\infty_\varphi(w) \longrightarrow \Gamma^{1,\infty}_w[V], \qquad V(t) = \frac{1}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

In the case of the Carleson lacunary $\mathcal{C}_{\mathrm{lac}}$, for every $w \in A_\infty$, it follows from (3.14) that, if $\varphi(s) = s\log_1 s$,

$$\mathcal{C}_{\mathrm{lac}} : L^\infty_\varphi(w) \longrightarrow \Gamma^{1,\infty}_w[V], \qquad V(t) = \frac{1}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

Example 3.3.10. Since the Hilbert transform H is bounded on $L^p(w)$ for every $w \in A_p$ with $\|H\| \lesssim p[w]_{A_p}$ when $p \geq 2$ (see [86]), from Theorem 3.3.5 it follows

$$H : L_1^\infty(w) \longrightarrow E^1(w).$$

It is also known (see [94]) that

$$H : L^\infty \longrightarrow BMO,$$

where BMO is the space of bounded mean oscillation functions. However, E^1 is not comparable with BMO. Indeed, $\log|x| \in BMO$ (see [46, Example 3.1.3]), but $\log|x| \notin E^1$ because is an increasing function. On the other hand, we have that $h(s) = \chi_{(0,1)}(s) \log \frac{1}{s} \notin BMO$ (see [46, Example 3.1.4]) but $h(s) \in E^1$ because, for $0 < t \leq 1$,

$$h^{**}(t) = \frac{1}{t}\int_0^t \log \frac{1}{s} ds = \frac{1}{t}\left[s + s\log\frac{1}{s}\right]_0^t = \log_1 \frac{1}{t}$$

and, therefore, for $V(t) = \left(\log_1 \frac{1}{t}\right)^{-1}$, $\|h\|_{\Gamma^{1,\infty}[V]} \leq 1$. Consequently,

$$H : L_1^\infty \longrightarrow BMO \cap E^1,$$

where $BMO \cap E^1 \subsetneq BMO$.

3.4 Hypotheses for characteristic functions

Here, we improve Theorem 3.1.1 by assuming weaker hypotheses. The proof follows the same argument presented in [2].

Theorem 3.4.1. *If for some fixed* $1 < p_1 < \infty$*, and, for every* $1 < p \leq p_1$ *and every measurable set* E*,*

$$\|T\chi_E\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right)\mu(E)^{1/p},$$

then

$$T : L\,\varphi(\log L) \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

Proof. Fix a measurable set $E \subseteq X$ with $\mu(E) < \infty$. For every $1 < p \leq p_1$ and $t > 0$,

$$t^{1/p}(T\chi_E)^{**}(t) \leq \|T\chi_E\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right)\mu(E)^{1/p}.$$

Taking the infimum over $1 < p \le p_1$, we obtain by (2.4) that

$$(T\chi_E)^{**}(t) \lesssim \left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}} \chi_{(0,\mu(E)]}(t) + \left(\frac{\mu(E)}{t}\right) \varphi\left(\log_1 \frac{t}{\mu(E)}\right) \chi_{(\mu(E),\infty)}(t).$$

Now, since, for every $t < \mu(E)$,

$$\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}} \approx 1 + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\mu(E)} \frac{ds}{s^{1-\frac{1}{p_1}}} \lesssim \frac{1}{t}\int_0^t \varphi\left(\log_1 \frac{t}{s}\right) ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\mu(E)} \frac{ds}{s^{1-\frac{1}{p_1}}},$$

and for every $t > \mu(E)$,

$$\left(\frac{\mu(E)}{t}\right) \varphi\left(\log_1 \frac{t}{\mu(E)}\right) \lesssim \frac{1}{t}\int_0^{\mu(E)} \varphi\left(\log_1 \frac{t}{s}\right) ds,$$

we get that, for $t > 0$,

$$\begin{aligned}(T\chi_E)^{**}(t) &\lesssim \frac{1}{t}\int_0^t \varphi\left(\log_1 \frac{t}{s}\right) \chi_{(0,\mu(E)]}(s)ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\infty} \chi_{(0,\mu(E)]}(s)\frac{ds}{s^{1-\frac{1}{p_1}}} \\ &= R_{1,p_1,\varphi}((\chi_E)^*)(t), \end{aligned} \tag{3.22}$$

where, for every measurable function f,

$$R_{1,p_1,\varphi}(f^*)(t) := \frac{1}{t}\int_0^t \varphi\left(\log_1 \frac{t}{s}\right) f^*(s)ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\infty} f^*(s)\frac{ds}{s^{1-\frac{1}{p_1}}}.$$

To extend this inequality to simple functions, we take $f = \sum_{n=1}^{j} a_n\chi_{F_n}$, where the sets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_j$ have finite measure. Thus, $f^* = \sum_{n=1}^{j} a_n\chi_{[0,\mu(F_n))}$, and, by (3.22), we get

$$(Tf)^{**}(t) \lesssim \sum_{n=1}^{j} a_j(T\chi_{F_n})^{**}(t) \lesssim \sum_{n=1}^{j} a_n R_{1,p_1,\varphi}((\chi_{F_n})^*)(t) = R_{1,p_1,\varphi}(f^*)(t).$$

To extend the above inequality to arbitrary measurable functions we use a density argument. In fact, we observe that

$$R_{1,p_1,\varphi}(f^*)(t) = \|f\|_{\Lambda^1(v_t)}$$

where

$$v_t(s) := \begin{cases} \dfrac{1}{t}\varphi\left(\log_1 \dfrac{t}{s}\right), & 0 < s < t, \\[2ex] \dfrac{1}{t}\left(\dfrac{t}{s}\right)^{1-\frac{1}{p_1}}, & t \le s. \end{cases}$$

Therefore, given a sequence of simple functions such that $|f_n| \uparrow |f|$ at almost every point, we have that, for every $n, m \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{t} \sup_{\nu(E)=t} \int_E |Tf_n - Tf_m| \, d\nu &\leq \frac{1}{t} \sup_{\nu(E)=t} \int_E \big|T(f_n - f_m)\big| \, d\nu = \big(T(f_n - f_m)\big)^{**}(t) \\ &\lesssim R_{1,p_1,\varphi}((f_n - f_m)^*)(t) = \|f_n - f_m\|_{\Lambda^1(v_t)},\end{aligned}$$

where we have used [8, Prop. 3.3, p. 53] in the first equality.

Since $v_t \notin L^1$, [27, Theorem 2.3.4] implies that $\Lambda^1(v_t)$ has absolutely continuous norm and, in view of Proposition 2.3.3. of the same work,

$$\lim_{n,m\to\infty} \|f_n - f_m\|_{\Lambda^1(v_t)} = 0.$$

Consequently, for every set E with finite measure, Tf_n is a Cauchy sequence in $L^1(E)$, and therefore converges to a function $T_E f$. Since our measure space is σ-finite, let us take a sequence $(E_n)_n$ of sets with finite measure such that $\cup_{n\geq 1} E_n = X$. Define $Tf(x) := T_{E_n} f(x)$ for every $x \in E_n$. Let us see that Tf is well defined. Indeed, for any pair of sets E, F such that $\nu(E \cap F) > 0$, if we fix $\varepsilon > 0$, there exists $n > 0$ such that

$$\|T_E f - T_F f\|_{L^1(E\cap F)} \leq \|T_E f - Tf_n\|_{L^1(E)} + \|Tf_n - T_F f\|_{L^1(F)} \leq \varepsilon,$$

and, therefore, $T_E f = T_F f$ almost everywhere in $E \cap F$. Moreover, this also implies that the definition of Tf does not depend on the sequence of sets $(E_n)_n$ chosen.

Hence,

$$\begin{aligned}(Tf)^{**}(t) &= \sup_{\nu(E)=t} \frac{1}{t} \int_E |Tf| \, d\nu = \sup_{\nu(E)=t} \lim_{n\to\infty} \frac{1}{t} \int_E |Tf_n| \, d\nu \leq \lim_{n\to\infty} \frac{1}{t} \int_0^t (Tf_n)^* \, dt \\ &\lesssim \lim_{n\to\infty} R_{1,p_1,\varphi}(f_n^*)(t) \lesssim R_{1,p_1,\varphi}(f^*)(t),\end{aligned}$$

which implies that for all measurable functions and every $t > 0$,

$$(Tf)^{**}(t) \lesssim R_{1,p_1,\varphi}(f^*)(t) \lesssim \frac{\varphi(\log_1 t)}{t} \int_0^\infty \varphi\left(\log_1 \frac{1}{s}\right) f^*(s) ds,$$

and taking the supremum over $t > 0$ we are done. □

For our next result, we need an extra condition on the operator T; namely that T is (ε, δ)-atomic approximable (see [18]).

Theorem 3.4.2. *If for some fixed $1 < p_1 < \infty$, and, for every $1 < p \le p_1$ and every measurable set E,*

$$\|T\chi_E\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-1}\right)\mu(E)^{1/p},$$

and T is (ε,δ)-atomic approximable, then

$$T : L\ \varphi(\log L)\log_3 L \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

Proof. We have that, for every measurable set E,

$$(T\chi_E)^*(t) \lesssim \varphi\left(\frac{1}{p-1}\right) t^{-1/p}\mu(E)^{1/p}.$$

Hence, since T is (ε,δ)-atomic approximable, it follows from [18] that, for every $f \in L^1 \cap L^\infty$ with $\|f\|_{L^\infty} \le 1$,

$$(Tf)^*(t) \lesssim \varphi\left(\frac{1}{p-1}\right)\left(\frac{\|f\|_{L^1}}{t}\right)^{1/p}.$$

Thus, we can use the proof of Theorem 3.2.1 to conclude the statement, because the above inequality can be use to get (3.19). □

Example 3.4.3. Since the Carleson operator $\mathcal{C}$ is (ε,δ)-atomic approximable (see [18]). From (3.1), we obtain

$$\mathcal{C} : L\log L\log_3 L \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{(\log_1 t)},$$

recovering Antonov's result.

3.5 Extrapolation on $L^{p,\infty}$

Let us introduce now some results where the hypotheses are of the form

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}.$$

3.5.1 Case endpoint $p_0 \ge 1$

The particular case $p_0 = 1$ can be found in [30]. We extend the result to $p_0 > 1$ and to the setting of admissible functions. We first define the function spaces that will be used.

Definition 3.5.1. The spaces $\big[L\,\varphi(\log L)\log_3 L\big]$ and $D^{+}_{p,\infty,\varphi}$ are defined by the norms

$$\|f\|_{\left[L\,\varphi(\log L)\log_3 L\right]} = \|f\|_{L^{1,\infty}} + \int_0^1 \sup_{t\leq s} t\ f^*(t)\phi'(s)ds < \infty,$$

where $\phi(s) = -\varphi\Big(\log_1 \frac{1}{s}\Big)\log_3 \frac{1}{s}$, and

$$\|f\|_{D^{+}_{p,\infty,\varphi}} = \|f\|_{L^{p,\infty}} + \int_0^1 \frac{\sup_{t\leq s} t^{\frac{1}{p}} f^*(t)}{s}\varphi'\left(\log_1 \frac{1}{s}\right)ds.$$

We observe that, if $\varphi'(x) \approx \frac{\varphi(x)}{x}$,

$$\|f\|_{\left[L\,\varphi(\log L)\log_3 L\right]} \approx \|f\|_{L^{1,\infty}} + \int_0^1 \frac{\sup_{t\leq s} t\ f^*(t)}{s}\varphi'\Big(\log_1 \frac{1}{s}\Big)\log_3 \frac{1}{s}ds < \infty.$$

Theorem 3.5.2. *If for some fixed $1 \leq p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right),$$

then:

i) If $p_0 = 1$,

$$T : \big[L\ \varphi(\log L)\log_3 L\big] \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi\left(\log_1 t\right)}.$$

ii) If $p_0 > 1$,

$$T : D^{+}_{p_0,\infty,\varphi} \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{\frac{1}{p_0}}}{\varphi\left(\log_1 t\right)}.$$

Proof. Fix $g_0 \in L^{p_0,\infty}$ and define the function $H(s) = \sup_{t\leq s} t(g_0^*(t))^{p_0}$. Clearly, $(g_0^*(s))^{p_0} \leq H(s)/s$, which implies that there exists $h_0 \in L^\infty$ with $\|h_0\|_{L^\infty} \leq 1$ such that $g_0^*(s) = h_0(s)\big(\frac{H(s)}{s}\big)^{1/p_0}$.

Since $g_0 \in L^{p_0,\infty}$, $\lim_{t\to\infty} g_0^*(t) = 0$ and, by [8, Corollary 7.6, p.83], there exists a measure preserving transformation σ_0 such that $g_0(x) = g_0^*(\sigma_0(x))$ and hence,

$$g_0(x) = h_0(\sigma_0(x))\left(\frac{H(\sigma_0(x))}{\sigma_0(x)}\right)^{1/p_0} := h_0(\sigma_0(x))H_{\sigma_0}(x). \tag{3.23}$$

We have that $(H_{\sigma_0})^*(s) = (H(s)/s)^{1/p_0}$ where H is quasi-concave (because $H(s)$ is increasing and $H(s)/s$ decreasing) and, thus, there exists a decreasing right-continuous function $\tilde{g}$ such that

$$(H_{\sigma_0})^*(s) \approx \left(\frac{\int_0^s (\tilde{g}(t))^{p_0} dt}{s} \right)^{1/p_0}.$$

Indeed, since $\|H_{\sigma_0}\|_{L^{p_0,\infty}} = \|g_0\|_{L^{p_0,\infty}}$, we get that $\tilde{g} \in L^{p_0}$ and hence $\lim_{t\to\infty} \tilde{g}(t) = 0$. Thus, there exists f_0 such that $f_0^* = \tilde{g}$. Moreover, there also exists h_1 such that $h_1, h_1^{-1} \in L^\infty$ and a measure preserving transformation σ_1 such that

$$H_{\sigma_0}(x) = h_1(\sigma_1(x)) \left(\frac{\int_0^{\sigma_1(x)} (f_0^*(t))^{p_0} dt}{\sigma_1(x)} \right)^{1/p_0}.$$

Now, for every $f \in L^{p_0}_{\text{loc}}$, we can define the operator

$$Sf(x) := h_1(\sigma_1(x)) \left(\frac{\int_0^{\sigma_1(x)} (f^*(t))^{p_0} dt}{\sigma_1(x)} \right)^{1/p_0}.$$

We observe that

$$Sf_0 = H_{\sigma_0}, \tag{3.24}$$

and $(Sf)^*(t) \approx \left(\frac{1}{t} \int_0^t (f^*(s))^{p_0} ds \right)^{\frac{1}{p_0}}$. Therefore, for $p > p_0$,

$$\|Sf\|_{L^{p,\infty}} \approx \sup_{t>0} t^{\frac{1}{p} - \frac{1}{p_0}} \left(\int_0^t (f^*(s))^{p_0} ds \right)^{\frac{1}{p_0}} \le \|f\|_{L^p}.$$

On the other hand, the operator $T'g := T\left((h_0 \circ \sigma_0) \cdot g\right)$, satisfies the hypothesis of the theorem and hence,

$$T' \circ S : L^p \longrightarrow L^{p,\infty}, \qquad \|T' \circ S\| \lesssim \varphi\left(\frac{1}{p - p_0}\right).$$

By (3.23) and (3.24), we have

$$Tg_0(x) = T\left((h_0 \circ \sigma_0) \cdot H_{\sigma_0}\right)(x) = T'\left(H_{\sigma_0}\right)(x) = T'(Sf_0)(x).$$

If $p_0 > 1$, we can apply Theorem 3.2.4 to $T' \circ S$ and get

$$\begin{aligned}
\|Tg_0\|_{\Gamma^{1,\infty}[V]} &= \big\|(T' \circ S)(f_0)\big\|_{\Gamma^{1,\infty}[V]} \\
&\lesssim \|f_0\|_{L^{p_0}} + \int_0^1 \frac{\left(\int_0^t (f_0^*(s))^{p_0} ds\right)^{\frac{1}{p_0}}}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt \\
&\approx \|Sf_0\|_{L^{p_0,\infty}} + \int_0^1 (Sf_0)^*(t)\, t^{\frac{1}{p_0}-1} \varphi'\left(\log_1 \frac{1}{t}\right) dt \\
&= \|H_{\sigma_0}\|_{L^{p_0,\infty}} + \int_0^1 (H_{\sigma_0})^*(t)\, t^{\frac{1}{p_0}-1} \varphi'\left(\log_1 \frac{1}{t}\right) dt \\
&\approx \|g_0\|_{L^{p_0,\infty}} + \int_0^1 \frac{\sup_{s \le t} s^{\frac{1}{p_0}}\ g_0^*(s)}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt.
\end{aligned}$$

If $p_0 = 1$ and $\phi(s) = -\varphi\Big(\log_1 \frac{1}{s}\Big) \log_3 \frac{1}{s}$, by Theorem 3.3.1 and Fubini, we get

$$\begin{aligned}
&\|Tg_0\|_{\Lambda^{1,\infty}[V]} = \big\|(T' \circ S)(f_0)\big\|_{\Lambda^{1,\infty}[V]} \lesssim \|f_0\|_{L\,\varphi(\log L)\log_3 L} = \int_0^\infty f_0^*(t)(-\phi(t))dt \\
&\approx \|f_0\|_{L^1} + \int_0^1 f_0^*(t)(-\phi(t))\ dt \lesssim \|f_0\|_{L^1} + \int_0^1 f_0^*(t) \int_t^1 \phi'(s) ds\, dt \\
&= \|f_0\|_{L^1} + \int_0^1 \Big(\frac{1}{s}\int_0^s f_0^*(t)dt\Big) s\, \phi'(s)ds \approx \|Sf_0\|_{L^{1,\infty}} + \int_0^1 (Sf_0)^*(s) s\, \phi'(s)ds, \\
&= \|H_{\sigma_0}\|_{L^{1,\infty}} + \int_0^1 (H_{\sigma_0})^*(s) s\, \phi'(s)ds \approx \|g_0\|_{L^{1,\infty}} + \int_0^1 \sup_{t \le s} t^{\frac{1}{p_0}}\ g_0^*(t)\phi'(s)ds.
\end{aligned}$$

□

3.5.2 Case left endpoint $p_1 < \infty$

We observe that, Theorem 3.5.2 can also be proved with the ODR technique using the $K\left(L^{p_0,\infty}; L^\infty\right)$-decomposition.

Let us present the case where the endpoint is $p_1 < \infty$, using this method.

Definition 3.5.3. The space $D^-_{p,\infty,\varphi}$ is defined by the norm

$$\|f\|_{D^-_{p,\infty,\varphi}} = \|f\|_{L^{p,\infty}} + \int_1^\infty \frac{\sup_{t \ge s} t^{\frac{1}{p}} f^{**}(t)}{s} \varphi'\left(\log_1 s\right) ds.$$

Remark 3.5.4. We observe that $D^-_{p,\infty,\varphi}$ corresponds precisely to the space $K^-_\varphi(\bar{A})$ (see Definition 2.3.3) in the specific case where $\bar{A} = (L^1, L^{p,\infty})$.

Theorem 3.5.5. *If for some fixed* $1 < p_0 < p_1 < \infty$ *and, for every* $p_0 \leq p < p_1$,

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

where φ *satisfies* (2.8), *then,*

$$T : D^-_{p_1,\infty,\varphi} \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{\frac{1}{p_1}}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

Proof. If $f \in L^{p_1,\infty}$ with $\|f\|_{L^{p_1,\infty}} \leq 1/p_1'$ and a finite $K(L^1, L^{p,\infty})$-decomposition, we can follow the same steps than the proof of Theorem 3.2.6, using part *d*) of Lemma 2.2.2, Lemma 2.3.6 and Proposition 2.3.19. □

3.6 Fixed target space

Theorem 3.6.1. *If for any fixed* $1 < p_1 < \infty$, *any fixed quasi-Banach space* $\mathcal{X}$, *and for every* $1 < p \leq p_1$,

$$T : L^p \longrightarrow \mathcal{X}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right),$$

then

$$\|Tf\|_{\mathcal{X}} \lesssim \inf_{\sum_n e_n f_n = f} \left\| \Big(e_n I_{\varphi,1}\left(\|f_n\|_{L^1}\right)\Big)_n \right\|_{\mathrm{Galb}(\mathcal{X})},$$

where the infimum is taken over all possible decompositions of f *with* $\|f_n\|_{L^\infty} \leq 1$.

Proof. If $f \in L^1$ such that $f \geq 0$ and $f = \sum_n e_n f_n$ with $\|f_n\|_{L^\infty} \leq 1$, since $f_n \in L^1 \cap L^\infty$, we have by Lemma 2.1.4 that

$$\|Tf_n\|_{\mathcal{X}} \lesssim \inf_{1<p\leq p_1} \varphi\left(\frac{1}{p-1}\right) \|f_n\|_{L^1}^{\frac{1}{p}} \lesssim I_{\varphi,1}\left(\|f_n\|_{L^1}\right),$$

and, therefore,

$$\|Tf\|_{\mathcal{X}} \leq \left\| \sum_n e_n T f_n \right\|_{\mathcal{X}} = \left\| \sum_n e_n \|Tf_n\|_{\mathcal{X}} \frac{Tf_n}{\|Tf_n\|_{\mathcal{X}}} \right\|_{\mathcal{X}} \leq \left\| \big(e_n \|Tf_n\|_{\mathcal{X}}\big)_n \right\|_{\mathrm{Galb}(\mathcal{X})}.$$

□

Example 3.6.2. Let M_Ω be the rough maximal operator on $\mathbb{R}^2$ (see (3.3)). It follows from [33] that, for every $q > 1$ and $\Omega \in L^q(\mathbb{S}^1)$,

$$M_\Omega : L^1(\mathbb{R}^2) \longrightarrow L^{1,\infty}(\mathbb{R}^2), \qquad \|M_\Omega\| \lesssim \frac{1}{q-1} \|\Omega\|_{L^q(\mathbb{S}^1)} .$$

Hence, if we fix $f \in L^1(\mathbb{R}^2)$ and we denote $M_f(\Omega) := M_\Omega(f)$, then

$$M_f : L^q(\mathbb{S}^1) \longrightarrow L^{1,\infty}(\mathbb{R}^2), \qquad \|M_f\| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^2)}}{(q-1)}.$$

Therefore, using Theorem 3.6.1, the 2-dyadic decomposition and the fact that $\mathrm{Galb}(L^{1,\infty}) = \ell \log \ell$, we obtain

$$M_f : L\,(\log L) \log_3 L(\mathbb{S}^1) \longrightarrow L^{1,\infty}(\mathbb{R}^2),$$

which implies that, for every $\Omega \in L\,(\log L)\log_3 L(\mathbb{S}^1)$, M_Ω is weak type $(1,1)$. We have to emphasize that it is known (see [34]) that, if $\Omega \in L \log L(\mathbb{S}^1)$, then M_Ω is weak type (1,1), but the proof is much more technical.

CHAPTER 4

Bilinear Yano's extrapolation

The study of multilinear operators began in the 1970s with the work of Coifman and Meyer (see [36], [37]), who studied commutators of bilinear singular integrals, such as the bilinear Hilbert transform, defined as follows

$$B_H\left(f,g\right)(x) := \lim_{\varepsilon\to 0}\int_{|t|>\varepsilon} f(x-t)g(x+t)\frac{dt}{t}, \qquad x\in\mathbb{R}, \quad f,g\in\mathscr{S}(\mathbb{R}).$$

Years earlier Calderón had posed the question of whether B_H satisfies an estimate of the form

$$B_H : L^{p_1}\left(\mathbb{R}\right)\times L^{p_2}\left(\mathbb{R}\right)\longrightarrow L^{p}\left(\mathbb{R}\right), \qquad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}. \tag{4.1}$$

In particular, he conjectured that the case $p_1 = p_2 = 2$ holds. The first results concerning this problem were provided by Lacey and Thiele (see [71], [72]), who proved that (4.1) is true when $1 < p_1, p_2 \leq \infty$ and $2/3 < p < \infty$. Their work attracted significant attention to this research area, leading to the development of many results in the setting of multilinear operators, such as those by Kenig and Stein (see [66], [67]), Muscalu, Tao, and Thiele [83], Grafakos and Li [50], and Grafakos and Torres (see [52], [53]).

In this chapter, we develop a multilinear version of Yano's extrapolation theory. We present the results only in the bilinear context to simplify the notation. The extension to the multilinear case is straight forward.

For $1 \leq p_1, p_2 \leq \infty$ and $0 < p \leq \infty$, we denote by $(p_1, p_2; p)$ the triple that satisfies Hölder's condition

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}.$$

Thus, if we fix an endpoint $(q_1, q_2; q)$, we consider operators

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,r}$$

such that $\|B\| \to \infty$ as $p_j \to q_j$ for $j = 1, 2$, and we want to study the behaviour of B at the endpoint spaces. As done in the previous chapter, we distinguish different cases (strong, weak, etc.) depending on the values of r_1, r_2 and r where

$$r_1 \in \{1, p_1\}, \qquad r_2 \in \{1, p_2\} \quad \text{and} \quad r \in \{p, \infty\}.$$

In the following, let $(p_1, p_2; p)$ denote the points where we already have an estimate, and let $1 < p_1^+, p_2^+, p_1^-, p_2^- < \infty$ be fixed numbers. If we aim to reach the point $(q_1, q_2; q)$, we could have, for example, the hypotheses for all $(p_1, p_2; p)$ such that $q_1 < p_1 \le p_1^+$ and $p_2^- \le p_2 < q_2$ (see Figure 4.1).

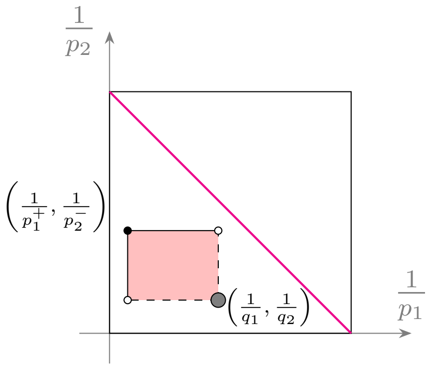


Figure 4.1: $(p_1, p_2; p)$ such that $q_1 < p_1 \le p_1^+$ and $p_2^- \le p_2 < q_2$.

The chapter is structured as follows: in Section 4.1, we shall treat the case when $L^{q,r}$ is Banach and Section 4.2 will be dedicated to the quasi-Banach context. Our proofs will follow the ODR technique and, in both settings, we mainly use linearization in each variable. For example, if

$$B : L^{p_1,r_1} \times L^{q_2,r_2} \longrightarrow L^{p,r}, \quad \|B\| \lesssim \varphi\left(\frac{1}{p_1 - q_1}\right),$$

and we fix $g \in L^{q_2,r_2}$, we shall consider the linear operator $T_1 := B(\cdot, g)$. Clearly, we have that

$$T_1 : L^{p_1,r_1} \longrightarrow L^{p,r}, \quad \|T_1\| \lesssim \varphi\left(\frac{1}{p_1 - q_1}\right) \|g\|_{L^{q_2,r_2}}.$$

Consequently, except in the case $q_2 = \infty$, we need to developed an off-diagonal extrapolation result.

Finally, we should mention that some partial results concerning the bilinear setting have been stated in the context of abstract extrapolation theory (see [63], [5]).

4.1 Banach setting

In this section we consider the case when $L^{q,r} = L^q$ with $q \geq 1$ or $L^{q,r} = L^{q,\infty}$ with $q > 1$.

4.1.1 Off-diagonal extrapolation

Proposition 4.1.1. *Suppose that* $1 \leq q_1, q < \infty$. *If, for every* $q_1 < p_1 \leq p_1^+$ *and* p *such that*

$$\frac{1}{p} - \frac{1}{p_1} = \frac{1}{q} - \frac{1}{q_1}, \tag{4.2}$$

$$T : L^{p_1,r_1} \longrightarrow L^{p,r}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - q_1}\right),$$

then

$$T : D_1^+ \longrightarrow R^+,$$

where D_1^+ *and* R^+ *are determined by the values described in Table 4.1 (see Definition 3.1.7).*

r_1	D_1^+
1	$\Lambda(v_1),\ v_1(t) = t^{\frac{1}{q_1}-1}\varphi(\log_1 \frac{1}{t})$
p_1	$D^+_{q_1,\varphi}$

r	R^+
p	$R^+_{q,\varphi}$
∞ $(q \neq 1)$	$\Gamma^{1,\infty}[V],\ V(t) = \dfrac{t^{1/q}}{\varphi(\log_1 t)}$

Table 4.1: Proposition 4.1.1.

Proof. $\boxed{r_1 = p_1}$ If $r = p$, given $f \in L^{q_1}$ with a finite $K(L^{q_1}, L^\infty)$-decomposition, by Lemma 2.2.2 *a*) we obtain that $f = \sum_{|i|<N} 2^i f_i$ with $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1/p_1}$ for every $q_1 < p_1 \leq p_1^+$.

Since $p > q$,

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^q ds\right)^{1/q} \leq t^{\frac{1}{q}-\frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{\frac{1}{q_1}-\frac{1}{p_1}} \varphi\left(\frac{1}{p_1-q_1}\right) \|f_i\|_{L^{p_1}}$$
$$\lesssim t^{1/q_1} \varphi\left(\frac{1}{p_1-q_1}\right)\left(\frac{|E_i|^{q_1}}{t}\right)^{1/p_1},$$

where we have used (4.2) in the second inequality. Therefore, Lemma 2.1.4 implies

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{1/q_1} I_{\varphi,q_1}\left(\frac{|E_i|^{q_1}}{t}\right) \lesssim \varphi(\log_1 t)\, I_{\varphi,1}\left(|E_i|\right).$$

The rest of the proof follows the same argument as Theorem 3.1.9 with $p_0 = q$ in (3.6).

If $r = \infty$, we can follow the same steps than the ones in Theorem 3.2.4 because

$$t^{\frac{1}{q}}(Tf_i)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}} \|Tf_i\|_{L^{p,\infty}} \lesssim t^{\frac{1}{q_1}-\frac{1}{p_1}} \varphi\left(\frac{1}{p_1-q_1}\right) \|f_i\|_{L^{p_1}}. \tag{4.3}$$

$\boxed{r_1 = 1}$ If $r = p$, by Proposition 2.3.16 it is enough to get the estimate for functions with a finite dyadic decomposition over $\mathbb{Z}$. Let $f \in L^{q_1,1}$ which has a finite dyadic decomposition over $\mathbb{Z}$. Thus, for every i, $p_1^- \leq p_1 < q_1$,

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^q ds\right)^{1/q} \leq t^{\frac{1}{q}-\frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{1/q_1} \varphi\left(\frac{1}{p_1-q_1}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p_1}.$$

By Lemma 2.1.4 we get

$$\left(\int_0^t \left[(Tf_i)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{1/q_1} I_{\varphi,q_1}\left(\frac{\|f_i\|_{L^1}}{t}\right) \lesssim \varphi(\log_1 t)\, I_{\varphi,q_1}\left(\|f_i\|_{L^1}\right).$$

Finally, by summing over i, using Lemma 2.3.1 and taking the supremum over $t > 0$, we obtain

$$\sup_{t>0} \frac{\left(\int_0^t \left[(Tf_i)^*(s)\right]^q ds\right)^{1/q}}{\varphi(\log_1 t)} \lesssim \sum_{|i|<N} 2^i I_{\varphi,q_1}\left(\|f_i\|_{L^1}\right) \lesssim \|f_i\|_{\Lambda^1(v_1)}.$$

If $r = \infty$, we can follow the same argument since, for every f_i obtained using the dyadic decomposition over $\mathbb{Z}$, $q_1 < p_1 \leq p_1^+$ and $t > 0$, we have (4.3). □

Using a similar argument, we can also obtain the following proposition.

Proposition 4.1.2. *Suppose that* $1 < q_1, q < \infty$*. If for every* $q_1 < p_1 \leq p_1^+$ *and* p *such that* (4.2) *is satisfied we have that*

$$T : L^{p_1,r_1} \longrightarrow \Gamma^{1,\infty}[V_p], \qquad V_p(t) = \frac{t^{1/p}}{\varphi_1\big(\log_1 t\big)}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - q_1}\right),$$

then

$$T : D_1^+ \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{(\varphi_1\varphi)\big(\log_1 t\big)},$$

where D_1^+ *is determined by the values described in Table 4.1 (see Definition 3.1.7).*

Proposition 4.1.1 and Proposition 4.1.2 yield the following theorem for the case where the operator blows up in both variables.

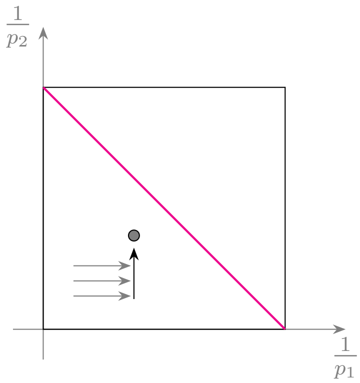


Figure 4.2: Banach setting. Theorem 4.1.3.

Theorem 4.1.3 (Figure 4.2)**.** *Suppose that* $1 \leq q_1, q_2, q < \infty$ *are fixed. If, for every* $q_1 < p_1 \leq p_1^+$ *and* $q_2 < p_2 \leq p_2^+$*,*

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,r}, \quad \|B\| \lesssim \varphi_1\left(\frac{1}{p_1 - q_1}\right)\varphi_2\left(\frac{1}{p_2 - q_2}\right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow R^+,$$

where D_1^+ *and* R^+ *are determined by the values described in Table 4.1, in which* $\varphi = \varphi_1$ *and* $\varphi = \varphi_1\varphi_2$*, respectively, and* D_2^+ *is determined by the values described in Table 4.2 (see Definition 3.1.7).*

r_2	D_2^+
1	$\Lambda(v_2),\ v_2(t) = t^{\frac{1}{q_2}-1}\varphi_2(\log_1 \frac{1}{t})$
p_2	$D^+_{q_2,\varphi_2}$

Table 4.2: Theorem 4.1.3.

Proof. We prove first the case when $r = p$. Let us fix p_2 such that $q_2 < p_2 \le p_2^+$, and $g \in L^{p_2,r_2}$. Thus, the operator $T_1 := B(\cdot, g)$ satisfies that, for every $(p_1, p_2; p)$ with $q_1 < p_1 \le p_1^+$,

$$T_1 : L^{p_1,r_1} \longrightarrow L^p, \qquad \|T_1\| \lesssim \varphi_1\left(\frac{1}{p_1 - q_1}\right)\varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,r_2}}.$$

Therefore, we can apply Proposition 4.1.1 to T_1 and obtain that, for every $(q_1, p_2; p)$,

$$T_1 : D_1^+ \longrightarrow R^+_{p,\varphi_1}, \qquad \|T_1\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,r_2}}.$$

Since we can do this reasoning for every $q_2 < p_2 \le p_2^+$, we have that,

$$B : D_1^+ \times L^{p_2,r_2} \longrightarrow R^+_{p,\varphi_1}, \quad \|B\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right).$$

In particular, if we fix $f \in D_1^+$, and we consider any measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have that, for every $q_2 < p_2 \le p_2^+$ where

$$\frac{1}{p} - \frac{1}{p_2} = \frac{1}{q} - \frac{1}{q_2} = \frac{1}{q_1},$$

the operator $T_2^E := B(f, \cdot)\chi_E$ satisfies

$$\begin{aligned}\frac{\|T_2^E g\|_{L^p}}{\varphi_1(\log_1|E|)} &= \frac{\left(\int_E [B(f,g)(s)]^p\,ds\right)^{1/p}}{\varphi_1(\log_1|E|)} \le \frac{\left(\int_0^{|E|}\left[(B(f,g))^*(s)\right]^p ds\right)^{1/p}}{\varphi_1(\log_1|E|)} \\ &\le \|B(f,g)\|_{R^+_{p,\varphi_1}} \lesssim \|f\|_{D_1^+}\,\varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,r_2}}.\end{aligned}$$

Thus, for every $q_2 < p_2 \le p_2^+$,

$$T_2^E : L^{p_2,r_2} \longrightarrow L^p, \qquad \left\|T_2^E\right\| \lesssim \varphi_1(\log_1|E|)\,\|f\|_{D_1^+}\,\varphi_2\left(\frac{1}{p_2 - q_2}\right).$$

Proposition 4.1.1 implies that

$$T_2^E : D_2^+ \longrightarrow R^+_{q,\varphi_2}, \quad \left\|T_2^E\right\| \lesssim \varphi_1\left(\log_1 |E|\right) \|f\|_{D_1^+}.$$

Finally, since this argument is valid for every measurable set E, if we fix $g \in D_2^+$ and $t > 0$, and we consider all the sets E such that $|E| = t$, it follows from [8, Proposition 3.3, p.53] that

$$\begin{aligned}
&\frac{\left(\int_0^t \left[\left(B(f,g)\right)^*(s)\right]^q ds\right)^{1/q}}{\varphi_1\left(\log_1 t\right)\varphi_2\left(\log_1 t\right)} = \frac{\sup_{|E|=t}\left(\int_E \left[\left(B(f,g)\right)(s)\right]^q ds\right)^{1/q}}{\varphi_1\left(\log_1 t\right)\varphi_2\left(\log_1 t\right)} \\
&= \frac{\sup_{|E|=t}\left(\int_E \left[\left(T_2^E g\right)(s)\right]^q ds\right)^{1/q}}{\varphi_1\left(\log_1 t\right)\varphi_2\left(\log_1 t\right)} \le \frac{\sup_{|E|=t}\left(\int_0^t \left[\left(T_2^E g\right)^*(s)\right]^q ds\right)^{1/q}}{\varphi_1\left(\log_1 t\right)\varphi_2\left(\log_1 t\right)} \\
&\le \frac{\sup_{|E|=t}\left\|T_2^E g\right\|_{R^+_{q,\varphi_2}}}{\varphi_1\left(\log_1 t\right)} \lesssim \|f\|_{D_1^+}\|g\|_{D_2^+}.
\end{aligned}$$

Taking the supremum over $t > 0$ we conclude the desired result.

For the case where $r = \infty$, we also fix the second variable and apply Proposition 4.1.1 to obtain that, for every $(q_1, p_2; p)$ with $q_2 < p_2 \le p_2^+$,

$$B : D_1^+ \times L^{p_2,r_2} \longrightarrow \Gamma^{1,\infty}[V_p], \quad V_p(t) = \frac{t^{1/p}}{\varphi_1\left(\log_1 t\right)}, \quad \|B\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right).$$

Hence, if we fix $f \in D_1^+$ and consider the operator $T_2 := B(f, \cdot)$, we have that, for every $q_2 < p_2 \le p_2^+$,

$$T_2 : L^{p_2,r_2} \longrightarrow \Gamma^{1,\infty}[V_p], \qquad \|T_2\| \lesssim \|f\|_{D_1^+}\varphi_2\left(\frac{1}{p_2 - q_2}\right),$$

and the result follows by Proposition 4.1.2. □

We can also consider the case where, the operator B satisfies that, for every $(p_1, q_2; p)$ with $p_1^- \le p_1 < q_1$,

$$B : L^{p_1,r_1} \times L^{q_2,r_2} \longrightarrow L^{p,r}, \quad \|B\| \lesssim \varphi\left(\frac{1}{q_1 - p_1}\right).$$

In this case we need to develop the following off-diagonal extrapolation result.

Proposition 4.1.4. *Suppose that* $1 < q_1 \leq \infty$ *and* $1 \leq q \leq \infty$*. If for every* $p_1^- \leq p_1 < q_1$ *and* p *such that* (4.2) *is satisfied we have that*

$$T : L^{p_1,r_1} \longrightarrow L^{p,r}, \qquad \|T\| \lesssim \begin{cases} \varphi\left(\dfrac{1}{q_1 - p_1}\right), & 1 < q_1 < \infty, \\ \varphi\left(p_1\right), & q_1 = \infty, \end{cases}$$

then

$$T : D_1^- \longrightarrow R^-,$$

where D_1^- *and* R^- *are determined by the values described in Table 4.3 (see Definition 3.1.17) and we assume that* φ *satisfies* (2.8) *when* $r_1 = p_1$.

r_1	$D_1^-,\ q_1 \neq \infty$	$D_1^-,\ q_1 = \infty$
1	$\Lambda^1(v_1),\ v_1(t) = t^{\frac{1}{q_1}-1}\varphi\left(\log_1 t\right)$	L^∞_φ
p_1	$D^-_{q_1,\varphi}$	

r	R^-
p $(q \neq \infty)$	$R^-_{q,\varphi}$
∞ $(q \neq 1)$	$\Gamma^{1,\infty}[V],\ V(t) = \dfrac{t^{1/q}}{\varphi(\log_1 \frac{1}{t})}$

Table 4.3: Proposition 4.1.4.

Proof. $\boxed{q_1 \neq \infty,\ r_1 = p_1}$ If $r = p$, fix $f \in L^{q_1}$ with $\|f\|_{L^{q_1}} \leq 1/q_1'$ and a finite $K(L^1, L^{q_1})$-decomposition. By Lemma 2.2.2 *c*) we get that $f = \sum_{|i|<N} 2^i f_i$ with f_i with $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1'\left(\frac{1}{p_1}-\frac{1}{q_1}\right)}$.

The rest of the proof follows the same steps than Theorem 3.1.19 with $p_1 = q_1$, using that, instead (3.16), since $p < q$, we have

$$\left(\int_t^\infty \left[(Tf_i)^{**}(s)\right]^q ds\right)^{1/q} \lesssim \left(\int_t^\infty \left[(Tf_i)^{**}(s)\right]^p s^{\frac{p}{q}-1} ds\right)^{1/p} \leq t^{\frac{1}{q}-\frac{1}{p}} \|Tf_i\|_{L^p}, \tag{4.4}$$

which yields

$$\left(\int_t^\infty \left[(Tf_i)^{**}(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q_1}-\frac{1}{p_1}} \varphi\left(\frac{1}{q_1-p_1}\right) \|f_i\|_{L^{p_1}}$$
$$\lesssim \varphi\left(\frac{1}{q_1-p_1}\right)\left(\frac{|E_i|^{q_1'}}{t}\right)^{\frac{1}{q_1}-\frac{1}{p_1}}.$$

If $r=\infty$, as seen in Theorem 3.2.6, we follow the same argument using that

$$t^{\frac{1}{q}}(Tf_i)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\|Tf_i\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{q_1-p_1}\right)\left(\frac{|E_i|^{q_1'}}{t}\right)^{\frac{1}{q_1}-\frac{1}{p_1}}$$

instead of (4.4).

$\boxed{q_1 \neq \infty,\ r_1 = 1}$ We proceed as in Theorem 3.3.4, taking into account that, instead of (3.20), we have by (4.4) that

$$\left(\int_t^\infty \left[(Tf_i)^{**}(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}-\frac{1}{p}}\|Tf_i\|_{L^p} \lesssim t^{-\frac{1}{q_1}}\varphi\left(\frac{1}{q_1-p_1}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p_1}$$

when $r=p$, and

$$t^{\frac{1}{q}}Tf_i^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\|Tf_i\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{q_1}}\varphi\left(\frac{1}{q_1-p_1}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p_1}$$

when $r=\infty$.

$\boxed{q_1 = \infty}$ We proceed as in Theorem 3.3.5, taking into account that, instead of (3.21), we get by (4.4) that

$$\left(\int_t^\infty \left[(Tf_n)^{**}(s)\right]^q ds\right)^{1/q} \lesssim t^{-\frac{1}{p_1}}\varphi(p_1)\|f_n\|_{L^{p_1,1}} \lesssim \|f_n\|_{L^\infty}\varphi(p_1)\left(\frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right)^{1/p_1}$$

when $r=p$, and

$$t^{\frac{1}{q}}Tf_n^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\|Tf_n\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{p_1}}\varphi(p_1)\|f_n\|_{L^{p_1,1}} \lesssim \|f_n\|_{L^\infty}\varphi(p_1)\left(\frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right)^{1/p_1}$$

when $r=\infty$. □

From a similar argument, we can also obtain the following proposition.

Proposition 4.1.5. *Suppose that* $1 < q_1 \leq \infty$ *and* $1 \leq q \leq \infty$. *If for every* $p_1^- \leq p_1 < q_1$ *and* p *such that* (4.2) *is satisfied we have that*

$$T : L^{p_1,r_1} \longrightarrow R^-(p), \qquad \|T\| \lesssim \begin{cases} \varphi\left(\dfrac{1}{q_1 - p_1}\right), & 1 < q_1 < \infty, \\ \varphi(p_1), & q_1 = \infty, \end{cases}$$

where $R^-(p) = R^-_{p,\varphi_1}$ *or* $R^-(p) = \Gamma^{1,\infty}[V_p]$ *with* $V_p(t) = \dfrac{t^{1/p}}{\varphi_1(\log_1 \frac{1}{t})}$, *then*

$$T : D_1^- \longrightarrow R^-,$$

where D_1^- *is determined by the values described in Table 4.3 and* R^- *by those in Table 4.4 (see Definition 3.1.17), and we assume that* φ *satisfies* (2.8) *if* $r_1 = p_1$.

$R^-(p)$	R^-
R^-_{p,φ_1} $(q \neq \infty)$	$R^-_{q,\varphi_1\varphi}$
$\Gamma^{1,\infty}[V_p],\ V_p(t) = \dfrac{t^{1/p}}{\varphi_1\left(\log_1 \frac{1}{t}\right)}$ $(q \neq 1)$	$\Gamma^{1,\infty}[V],\ V(t) = \dfrac{t^{1/q}}{(\varphi_1\varphi)(\log_1 \frac{1}{t})}$

Table 4.4: Proposition 4.1.5.

From Proposition 4.1.4 and Proposition 4.1.5 we obtain the following theorem.

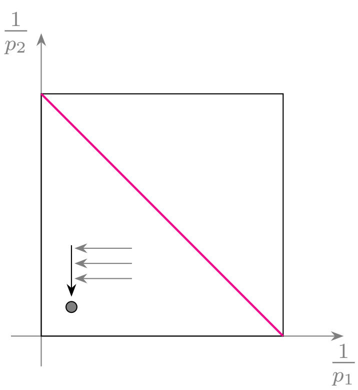


Figure 4.3: Banach setting. Theorem 4.1.6.

Theorem 4.1.6 (Figure 4.3)**.** *Suppose that* $1 < q_1, q_2, q \le \infty$*. If, for* $p_1^- \le p_1 < q_1$ *and* $p_2^- \le p_2 < q_2$*,*

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,\infty}$$

with

$$\|B\| \lesssim \begin{cases} \varphi_1\left(\dfrac{1}{q_1-p_1}\right)\varphi_2\left(\dfrac{1}{q_2-p_2}\right), & 1 < q_1, q_2 < \infty, \\ \varphi_1\left(\dfrac{1}{q_1-p_1}\right)\varphi_2\left(p_2\right), & 1 < q_1 < \infty,\ q_2 = \infty, \\ \varphi_1\left(p_1\right)\varphi_2\left(\dfrac{1}{q_2-p_2}\right), & q_1 = \infty,\ 1 < q_2 < \infty, \\ \varphi_1(p_1)\varphi_2(p_2), & q_1 = q_2 = \infty, \end{cases}$$

then

$$B : D_1^- \times D_2^- \longrightarrow R^-,$$

where D_1^+ *and* R^+ *are determined by the values described in Table 4.3, in which* $\varphi = \varphi_1$ *and* $\varphi = \varphi_1\varphi_2$*, respectively, and* D_2^+ *is determined by the values described in Table 4.5 (see Definition 3.1.17) and we assume that* φ_1 *satisfies* (2.8) *when* $r_1 = p_1$*, and* φ_2 *when* $r_2 = p_2$*.*

r_2	$D_2^-,\ q_2 \neq \infty$	$D_2^-,\ q_2 = \infty$
1	$\Lambda^1(v_2),\ v_2(t) = t^{\frac{1}{q_2}-1}\varphi_2(\log_1 t)$	$L^\infty_{\varphi_2}$
p_2	$D^-_{q_2,\varphi_2}$	

Table 4.5: Theorem 4.1.6.

Proof. We only prove the case $1 < q_1, q_2 < \infty$. The remaining cases follows using the same argument.

Fix p_2 such that $p_2^- \le p_2 < q_2$, and $g \in L^{p_2,r_2}$. Thus, the operator $T_1 := B(\cdot, g)$ satisfies that, for every $(p_1, p_2; p)$ such that $p_1^- \le p_1 < q_1$,

$$T_1 : L^{p_1,r_1} \longrightarrow L^{p,r}, \qquad \|T_1\| \lesssim \varphi_1\left(\frac{1}{q_1-p_1}\right)\varphi_2\left(\frac{1}{q_2-p_2}\right)\|g\|_{L^{p_2,r_2}}.$$

Therefore, we can apply Proposition 4.1.4 to T_1 and get that, for $(q_1, p_2; p)$,

$$T_1 : D_1^- \longrightarrow R^-(p), \qquad \|T_1\| \lesssim \varphi_2\left(\frac{1}{q_2 - p_2}\right)\|g\|_{L^{p_2,r_2}},$$

where $R^-(p) = R^-_{p,\varphi_1}$ when $r = p$ and $R^-(p) = \Gamma^{1,\infty}[V_p]$ with $V_p(t) = \dfrac{t^{1/p}}{\varphi_1(\log_1 \frac{1}{t})}$, when $r = \infty$.

Since this argument is valid for every $q_2 < p_2 \leq p_2^+$, we have that, for every $(q_1, p_2; p)$,

$$B : D_1^- \times L^{p_2,r_2} \longrightarrow R^-(p), \quad \|B\| \lesssim \varphi_2\left(\frac{1}{q_2 - p_2}\right).$$

If we now fix $f \in D_1^-$, the operator $T_2 := B(f, \cdot)$, satisfies that

$$T_2 : L^{p_2,r_2} \longrightarrow R^-(p), \quad \|T_2\| \lesssim \varphi_2\left(\frac{1}{q_2 - p_2}\right)\|f\|_{D_1^-}.$$

At this point, it is enough to use Proposition 4.1.5 to conclude the result. □

From Proposition 4.1.1 and Proposition 4.1.4, we can conclude the following theorem.

Theorem 4.1.7. *Suppose that $1 < q_1, q \leq \infty$. If, for every $q_1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < q_2$,*

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,\infty}$$

with

$$\|B\| \lesssim \begin{cases} \varphi_1\left(\dfrac{1}{p_1 - q_1}\right)\varphi_2\left(\dfrac{1}{q_2 - p_2}\right), & 1 < q_2 < \infty, \\ \varphi_1\left(\dfrac{1}{p_1 - q_1}\right)\varphi_2(p_2), & q_2 = \infty, \end{cases}$$

then

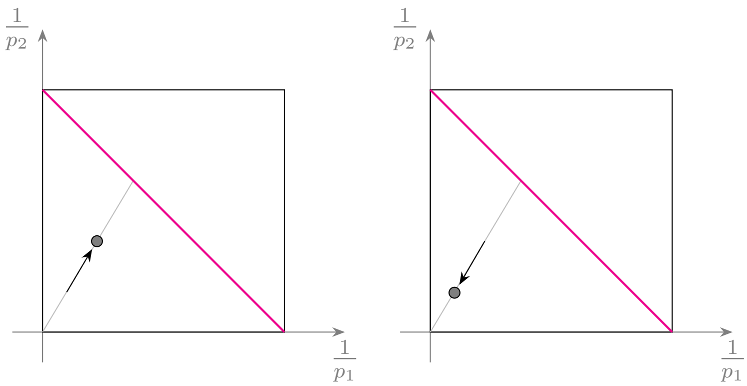
$$B : D_1^+ \times D_2^- \longrightarrow \Gamma^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{\varphi_1(\log_1 t)\,\varphi_2\left(\log_1 \frac{1}{t}\right)},$$

where D_1^+ is determined by the values described in Table 4.1 in which $\varphi = \varphi_1$ and D_2^- by those in Table 4.5 (see Definitions 3.1.7 and 3.1.17) and we assume that φ_2 satisfies (2.8) when $r_2 = p_2$.

4.1.2 Rays extrapolation

Let us consider now the case where the operator norm blows up in both variables simultaneously along a ray, that is, every $(p_1, p_2; p)$ satisfies

$$\frac{p}{q} = \frac{p_1}{q_1} = \frac{p_2}{q_2}. \tag{4.5}$$



(a) (b)

Figure 4.4: Banach setting: rays. Theorems 4.1.8 and 4.1.9.

Theorem 4.1.8 (Figure 4.4 (a)). *Suppose that* $1 \leq q_1, q_2, q < \infty$*. If for every* $q_1 < p_1 \leq p_1^+$ *and* $q_2 < p_2 \leq p_2^+$ *such that* (4.5) *is satisfied we have that*

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,r}, \qquad \|B\| \lesssim \varphi\left(\frac{1}{p-q}\right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow R^+,$$

where D_1^+ *and* R^+ *are determined by the values described in Table 4.1 and* D_2^+ *by those in Table 4.6 (see Definition 3.1.7).*

Proof. We prove all the cases for functions with finite decomposition, and the extension to the remaining functions follows from density.

$\boxed{r_1 = p_1,\ r_2 = p_2}$ When $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2}$, we decompose f with the $K(L^{q_1}, L^\infty)$-decomposition and g with the $K(L^{q_2}, L^\infty)$-decomposition to obtain $f = \sum_{|i| \leq N} 2^i f_i$ and $g = \sum_{|k| \leq M} 2^k g_k$ where $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1/p_1}$ and also $\|g_k\|_{L^{p_2}} \lesssim |F_k|^{q_2/p_2}$.

r_2	D_2^+
1	$\Lambda(v_1),\ v_1(t) = t^{\frac{1}{q_2}-1}\varphi(\log_1 \frac{1}{t})$
p_2	$D^+_{q_2,\varphi}$

Table 4.6: Theorem 4.1.8.

Thus,

$$\left(\int_0^t \left[\left(B(f_i,g_k)\right)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)\|f_i\|_{L^{p_1}}\|g_k\|_{L^{p_2}}$$
$$\lesssim t^{\frac{1}{q}}\ \varphi\left(\frac{1}{p-q}\right)\left(\frac{|E_i|^q|F_k|^q}{t}\right)^{1/p}.$$

By Lemma 2.1.4, we get

$$\left(\int_0^t \left[\left(B(f_i,g_k)\right)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}} I_{\varphi,q}\left(\frac{|E_i|^q|F_k|^q}{t}\right)$$
$$\lesssim \varphi\left(\log_1 t\right) I_{\varphi,1}\left(|E_i|\right) I_{\varphi,1}\left(|F_k|\right),$$

and, by Lemma 2.3.5, the result follows.

If $r=\infty$, we can follow the same argument using now that

$$\left(B(f_i,g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)\|f_i\|_{L^{p_1}}\|g_k\|_{L^{p_2}} \lesssim t^{\frac{1}{q}}\ \varphi\left(\frac{1}{p-q}\right)\left(\frac{|E_i|^q|F_k|^q}{t}\right)^{\frac{1}{p}}.$$

$\boxed{r_1=r_2=1}$ If $r=p$, given $f\in L^{q_1,1}$ and $g\in L^{q_2,1}$, we decompose them with the dyadic decomposition over $\mathbb{Z}$ and we get

$$\left(\int_0^t \left[\left(B(f_i,g_k)\right)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)\|f_i\|_{L^1}^{1/p_1}\|g_k\|_{L^1}^{1/p_2}$$
$$= t^{\frac{1}{q}}\ \varphi\left(\frac{1}{p-q}\right)\left(\frac{\|f_i\|_{L^1}^{q/q_1}\|g_k\|_{L^1}^{q/q_2}}{t}\right)^{1/p}.$$

By Lemma 2.1.4 and Lemma 2.3.5 the result follows.

If $r=\infty$, we have that

$$t^{\frac{1}{q}}\left(B(f_i,g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)\|f_i\|_{L^1}^{1/p_1}\|g_k\|_{L^1}^{1/p_2}.$$

$\boxed{r_1 = p_1,\ r_2 = 1}$ If $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2,1}$, we decompose f using the $K\left(L^{q_1}, L^{\infty}\right)$-decomposition and g with the dyadic decomposition over $\mathbb{Z}$ and we have that,

$$\left(\int_0^t \left[\left(B(f_i, g_k)\right)^*(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)|E_i|^{q_1/p_1}\|g_k\|_{L^1}^{1/p_2}$$
$$= t^{\frac{1}{q}}\ \varphi\left(\frac{1}{p-q}\right)\left(\frac{|E_i|^q\|g_k\|_{L^1}^{q/q_2}}{t}\right)^{1/p}.$$

By Lemma 2.1.4, Lemma 2.3.5 and Lemma 2.3.1, the result follows.

If $r = \infty$, we shall use that

$$t^{\frac{1}{q}}\left(B(f_i, g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\ \varphi\left(\frac{1}{p-q}\right)|E_i|^{q_1/p_1}\|g_k\|_{L^1}^{1/p_2}.$$

□

If we have the hypotheses for $p < q$, we have the following theorem.

Theorem 4.1.9 (Figure 4.4 (b)). *Suppose that $1 < q_1, q_2 \le \infty$ and $1 \le q \le \infty$. If for every $p_1^- \le p_1 < q_1$ and $p_2^- \le p_2 < q_2$ such that (4.5) is satisfied we have that*

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,r}, \qquad \|B\| \lesssim \begin{cases} \varphi\left(\dfrac{1}{q-p}\right), & 1 < q < \infty, \\ \varphi(p), & q = \infty, \end{cases}$$

then

$$B : D_1^- \times D_2^- \longrightarrow R^-,$$

where D_1^- and R^- are determined by the values described in Table 4.3 and D_2^- by those in 4.7 (see Definition 3.1.17) and we assume that φ satisfies (2.8) when $r_1 = p_1$ or $r_2 = p_2$.

Proof. The proof is done for functions with finite decomposition, and the extension to the rest follows by density.

$\boxed{q_1 \ne \infty,\ q_2 \ne \infty,\ r_1 = p_1,\ r_2 = p_2}$ If $r = p$, given any $f \in L^{q_1}$ such that $\|f\|_{L^{q_1}} \le 1/q_1'$ and $g \in L^{q_2}$ with $\|g\|_{L^{q_2}} \le 1/q_2'$, we decompose f with the $K\left(L^1; L^{q_1}\right)$-decomposition and g using the $K\left(L^1; L^{q_2}\right)$-decomposition. Hence, we have that $f = \sum_{|i|\le N} 2^i f_i$ and $g = \sum_{|k|\le M} 2^k g_k$ where,

$$\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1'\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \quad \text{and} \quad \|g_k\|_{L^{p_2}} \lesssim |F_k|^{q_2'\left(\frac{1}{p_2}-\frac{1}{q_2}\right)}.$$

r_2	$D_2^-,\ q_2 \neq \infty$	$D_2^-,\ q_2 = \infty$
1	$\Lambda^1(v_2),\ v_2(t) = t^{\frac{1}{q_2}-1}\varphi\left(\log_1 t\right)$	L_φ^∞
p_2	$D_{q_2,\varphi}^-$	

Table 4.7: Theorem 4.1.9.

From (3.15) follows that $E_i = F_k = \emptyset$ whenever $i, k \geq 0$. Since $\frac{p}{q} = \frac{p_1}{q_1}$, we have

$$q_1'\left(\frac{1}{p_1} - \frac{1}{q_1}\right) = q_1'\frac{q}{q_1}\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{q}{q_1 - 1}\left(\frac{1}{p} - \frac{1}{q}\right)$$

and the same happens with q_2', q_2, p_2. Therefore,

$$\left(\int_t^\infty \left[\left(B(f_i, g_k)\right)^{**}(s)\right]^q ds\right)^{1/q} \lesssim \varphi\left(\frac{1}{q-p}\right)\left(\frac{|E_i|^{\frac{q}{q_1-1}}|F_k|^{\frac{q}{q_2-1}}}{t}\right)^{\frac{1}{p}-\frac{1}{q}}$$

and by Lemma 2.1.4, we get

$$\left(\int_0^t \left[\left(B(f_i, g_k)\right)^{**}(s)\right]^q ds\right)^{1/q} \lesssim \varphi\left(\log_1 \frac{1}{t}\right)\varphi\left(\log_1 |E_i|\right)\varphi\left(\log_1 |F_k|\right).$$

Finally, by Lemma 2.3.6, the result follows.

If $r = \infty$, we use

$$t^{\frac{1}{q}}\left(B(f_i, g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\varphi\left(\frac{1}{q-p}\right)\|f_i\|_{L^{p_1}}\|g_k\|_{L^{p_2}}.$$

$\boxed{q_1 \neq \infty,\ q_2 \neq \infty,\ r_1 = r_2 = 1}$ If $r = p$, given $f \in L^{q_1,1}$ and $g \in L^{q_2,1}$, we can decompose f, g using the dyadic decomposition over $\mathbb{Z}$ and we have that

$$\begin{aligned}\left(\int_0^t \left[\left(B(f_i, g_k)\right)^{**}(s)\right]^q ds\right)^{1/q} &\lesssim t^{\frac{1}{q}-\frac{1}{p}}\varphi\left(\frac{1}{q-p}\right)\|f_i\|_{L^1}^{1/p_1}\|g_k\|_{L^1}^{1/p_2} \\ &= t^{\frac{1}{q}}\varphi\left(\frac{1}{q-p}\right)\left(\frac{\|f_i\|_{L^1}^{q/q_1}\|g_k\|_{L^1}^{q/q_2}}{t}\right)^{1/p}.\end{aligned}$$

By Lemma 2.1.4 and Lemma 2.3.1 the result follows.

If $r = \infty$, we can follow the same argument using now that,

$$t^{\frac{1}{q}}\left(B(f_i, g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}}\varphi\left(\frac{1}{q-p}\right)\|f_i\|_{L^1}^{1/p_1}\|g_k\|_{L^1}^{1/p_2}.$$

$\boxed{q_1 \neq \infty,\ q_2 \neq \infty,\ r_1 = p_1,\ r_2 = 1}$ If $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2,1}$, we decompose f using the $K\left(L^{q_1}, L^{\infty}\right)$-decomposition and g with the dyadic decomposition over $\mathbb{Z}$ and we have that

$$\left(\int_0^t \left[\left(B(f_i, g_k)\right)^{**}(s)\right]^q ds\right)^{1/q} \lesssim t^{\frac{1}{q}} |E_i|^{\frac{1}{1-q_1}} \varphi\left(\frac{1}{q-p}\right) \left(\frac{|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2}}{t}\right)^{1/p}.$$

By Lemma 2.1.4, we get

$$\begin{aligned}\left(\int_0^t \left[\left(B(f_i, g_k)\right)^{**}(s)\right]^q ds\right)^{1/q} &\lesssim t^{\frac{1}{q}} |E_i|^{\frac{1}{1-q_1}} J_{\varphi,q}\left(\frac{|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2}}{t}\right) \\ &\lesssim \varphi\left(\log_1 \frac{1}{t}\right) \varphi\left(\log_1 |E_i|\right) J_{\varphi,q_2}\left(\|g_k\|_{L^1}\right).\end{aligned}$$

The result follows from Lemma 2.3.6 and Lemma 2.3.1.

If $r = \infty$, we shall use that

$$t^{\frac{1}{q}} \left(B(f_i, g_k)\right)^{**}(t) \lesssim t^{\frac{1}{q}} |E_i|^{\frac{1}{1-q_1}} \varphi\left(\frac{1}{q-p}\right) \left(\frac{|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2}}{t}\right)^{1/p}.$$

$\boxed{q_1 = q_2 = \infty}$ Given $f, g \in L^{\infty}$, we decompose them using the f^*-decomposition and g^*-decomposition, and, we have that

$$\left(B(f_n, g_m)\right)^{**}(t) \lesssim \|f_n\|_{L^\infty} \|g_m\|_{L^\infty} \inf_{p^- \le p < \infty} \varphi(p) \left[\frac{1}{t} \left(\frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right)^{\frac{p^-}{p_1^-}} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right)^{\frac{p^-}{p_2^-}}\right]^{\frac{1}{p}}.$$

By Lemma 2.1.4, we get

$$\begin{aligned}\left(B(f_n, g_m)\right)^{**}(t) &\lesssim \|f_n\|_{L^\infty} \|g_m\|_{L^\infty} \varphi\left(\log_1\left[\frac{1}{t} \left(\frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right)^{\frac{p^-}{p_1^-}} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right)^{\frac{p^-}{p_2^-}}\right]\right) \\ &\lesssim \varphi\left(\log_1 \frac{1}{t}\right) \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \|g_m\|_{L^\infty} \varphi\left(\log_1 \frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right).\end{aligned}$$

The result follows summing over $n, m \in \mathbb{N}$ using Lemma 2.3.2. □

4.1.3 The endpoint $(1, \infty; 1)$

Now, we introduce the case where we reach the endpoint $(1, \infty; 1)$ by only using hypotheses where the target space is in the Banach setting.

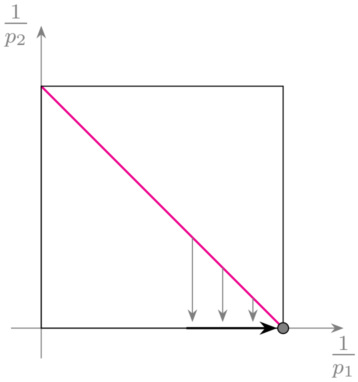


Figure 4.5: Banach setting. Theorem 4.1.10.

Theorem 4.1.10 (Figure 4.5)**.** *If, for every* $1 < p_1 \leq p_1^+$ *and* $(p_1^+)' \leq p_2 < \infty$,

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^p, \qquad \|B\| \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \varphi_2(p_2),$$

then

$$B : \Lambda^1(v_1) \times L^\infty_{\varphi_2} \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{(\varphi_1\varphi_2)(\log_1 t)\varphi_2\left(\log_1 \frac{1}{t}\right)},$$

where $v_1(t) = (\varphi_1\varphi_2)\left(\log_1 \frac{1}{t}\right)$.

Proof. We prove it for functions with finite decomposition and the result follows from density. First, given $f \in L^1$ and $g \in L^\infty$, we decompose f with the dyadic decomposition over $\mathbb{Z}$ and g using the g^*-decomposition. Hence, if we fix $i \in \mathbb{Z}$ and p_1, we get

$$\begin{aligned} B(f_i, g_m)^{**}(t) &\lesssim t^{-\frac{1}{p}} \left\| B(f_i, g_m) \right\|_{L^p} \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^{p_1,1}}}{t^{1/p_1}} \right) \varphi_2(p_2) \left(\frac{\|g_m\|_{L^{p_2,1}}}{t^{1/p_2}} \right) \\ &\lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1} \varphi_2(p_2) \|g_m\|_{L^\infty} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty} t} \right)^{1/p_2}. \end{aligned}$$

Thus, we can take the infimum over $p_1' \le p_2 < \infty$ using Lemma 2.1.1 to obtain

$$B(f_i,g_m)^{**}(t) \lesssim \varphi_1\left(\frac{1}{p_1-1}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p_1}\|g_m\|_{L^\infty}\inf_{p_1'\le p_2<\infty}\varphi_2(p_2)\left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}\,t}\right)^{1/p_2}$$

$$\approx(\varphi_1\varphi_2)\left(\frac{1}{p_1-1}\right)\left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p_1}\|g_m\|_{L^\infty}\,\varphi_2\left(\log_1\frac{1}{t}\right)\varphi_2\left(\log_1\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right).$$

Now, we can take the infimum over $1 < p_1 \le p_1^+$ using Lemma 2.1.4 to get

$$B(f_i,g_m)^{**}(t) \lesssim I_{\varphi_1\varphi_2,1}\left(\frac{\|f_i\|_{L^1}}{t}\right)\|g_m\|_{L^\infty}\,\varphi_2\left(\log_1\frac{1}{t}\right)\varphi_2\left(\log_1\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right),$$

which implies

$$\frac{tB(f_i,g_m)^{**}(t)}{(\varphi_1\varphi_2)\left(\log_1 t\right)\varphi_2\left(\log_1\frac{1}{t}\right)} \lesssim I_{\varphi_1\varphi_2,1}\left(\|f_i\|_{L^1}\right)\|g_m\|_{L^\infty}\,\varphi_2\left(\log_1\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}}\right).$$

Summing over i, m, using Lemma 2.3.1 and Lemma 2.3.2, the result follows. $\square$

4.2 Quasi-Banach setting

In this section, we treat the cases where $L^{q,r}$ is a quasi-Banach space. We only consider restricted-weak type hypotheses.

4.2.1 Off-diagonal extrapolation

As we did in the Banach setting, we shall use a linearization argument. Hence, we develop the corresponding off-diagonal extrapolation result.

Proposition 4.2.1. *Suppose that $1 \le q_1 < \infty$ and $0 < q \le 1$. If, for every $q_1 < p_1 \le p_1^+$ and p such that*

$$\frac{1}{p}-\frac{1}{p_1}=\frac{1}{q}-\frac{1}{q_1}, \tag{4.6}$$

$$T: L^{p_1,1}\longrightarrow L^{p,\infty}, \qquad \|T\|\lesssim\varphi\left(\frac{1}{p_1-q_1}\right),$$

then

$$T: D_1^+\longrightarrow \Lambda^{1,\infty}[V], \qquad V(t)=\frac{t^{1/q}}{\varphi\left(\log_1 t\right)}$$

where D_1^+ is determined by the values described in Table 4.8 in which $\varepsilon > 0$ is arbitrary.

		D_1^+
$0<q<1$	$q_1=1$	$\Lambda^1(v_1),\ v_1(t)=\varphi\left(\log_1\frac{1}{t}\right)\left(\log_2\frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1<q_1<\infty$	$\Lambda^1(v_1),\ v_1(t)=t^{\left(\frac{1}{q_1}-1\right)}\varphi\left(\log_1\frac{1}{t}\right)\left(\log_1\frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q=1$	$q_1=1$	$\Lambda^1(v),\ v_1(t)=\varphi\left(\log_1\frac{1}{t}\right)\left(\log_3\frac{1}{t}\right)$
	$1<q_1<\infty$	$\Lambda^1(v_1),\ v_1(t)=t^{\left(\frac{1}{q_1}-1\right)}\varphi\left(\log_1\frac{1}{t}\right)\left(\log_2\frac{1}{t}\right)$

Table 4.8: Proposition 4.2.1

Remark 4.2.2. Whenever it is needed we will denote $D_1^+(q)=D_1^+$ and we observe that, if $r\geq q$, we have that $D_1^+(q)\subset D_1^+(r)$.

Proof. We assume that the functions have a finite decomposition, and the result straightforwardly follows by density.

$\boxed{q_1=1,\ 0<q<1}$ Given $f\in L^1$ such that $\|f\|_{L^1}=1$, we use the 2-dyadic decomposition. Then,

$$t^{1/q}\left(Tf_n\right)^*(t)\lesssim t^{\frac{1}{q}-\frac{1}{p}}\|Tf_n\|_{L^{p,\infty}}\lesssim t\,\varphi\left(\frac{1}{p_1-q_1}\right)\left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p_1}.$$

By Lemma 2.1.4, we obtain

$$t^{1/q}\left(Tf_n\right)^*(t)\lesssim t\ I_{\varphi,1}\left(\frac{\|f_n\|_{L^1}}{t}\right)\lesssim\varphi\left(\log_1 t\right)I_{\varphi,1}\left(\|f_n\|_{L^1}\right).$$

Thus, if $V(t)=\dfrac{t^{1/q}}{\varphi\left(\log_1 t\right)}$, we get that

$$\|Tf_n\|_{\Lambda^{1,\infty}[V]}\lesssim I_{\varphi,1}\left(\|f_n\|_{L^1}\right).$$

Since

$$|Tf|\leq\sum_{0\leq n\leq N}2^{2^n}\|Tf_n\|_{\Lambda^{1,\infty}[V]}\frac{|Tf_n|}{\|Tf_n\|_{\Lambda^{1,\infty}[V]}},$$

Proposition 2.3.8 and Lemma 2.3.9 i) imply that, for every sequence $(a_n)_n$ of positive numbers such that $\sum_n a_n\leq 1$,

$$\|Tf\|_{\Lambda^{1,\infty}[V]}\lesssim\sum_{0\leq n\leq N}a_n^{-\frac{1}{q}+1}2^{2^n}\|Tf_n\|_{\Lambda^{1,\infty}[V]}\lesssim\sum_{0\leq n\leq N}a_n^{-\frac{1}{q}+1}2^{2^n}I_{\varphi,1}\left(\|f_n\|_{L^1}\right).$$

If we fix $\varepsilon > 0$ and choose $a_n \approx n^{-(1+\varepsilon)}$, the result follows by Lemma 2.3.12 i).

$\boxed{q_1 = q = 1}$ We follow the same steps but using Lemma 2.3.9 ii) to obtain

$$\|Tf\|_{\Lambda^{1,\infty}[V]} \lesssim \sum_{0\le n\le N} \left(\log \frac{1}{a_n}\right) 2^{2^n} \|Tf_n\|_{\Lambda^{1,\infty}[V]} \lesssim \sum_{0\le n\le N} \left(\log \frac{1}{a_n}\right) 2^{2^n} I_{\varphi,1}\left(\|f_n\|_{L^1}\right).$$

If we fix $\varepsilon > 0$ and we now choose $a_n \approx n^{-2}$, the result follows by Lemma 2.3.12 ii).

$\boxed{1 < q_1 < \infty,\ 0 < q < 1}$ We can use the dyadic decomposition over $\mathbb{N}$ to obtain

$$\|Tf_n\|_{\Lambda^{1,\infty}[V]} \lesssim I_{\varphi,q_1}\left(\|f_n\|_{L^1}\right).$$

The result follows using Lemma 2.3.9 i), choosing $a_n \approx n^{-(1+\varepsilon)}$ and applying Lemma 2.3.13 i).

$\boxed{1 < q_1 < \infty,\ q = 1}$ In this case, we apply Lemma 2.3.9 ii) with $a_n \approx n^{-2}$, and then Lemma 2.3.13 ii). $\square$

Similarly, we have the following proposition.

Proposition 4.2.3. *Suppose that* $1 \le q_1 < \infty$ *and* $0 < q \le 1$. *If for every* $q_1 < p_1 \le p_1^+$ *and* p *such that* (4.6) *is satisfied we have that*

$$T : L^{p_1,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 t)}, \qquad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - q_1}\right),$$

then

$$T : D_1^+ \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{(\varphi_1\varphi)(\log_1 t)},$$

where D_1^+ *is determined by the values described in Table 4.8 in which* $\varepsilon > 0$ *is arbitrary.*

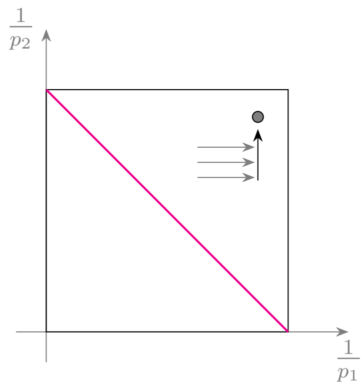


Figure 4.6: Quasi-Banach setting. Theorem 4.2.4.

Theorem 4.2.4 (Figure 4.6)**.** *Suppose that* $1 \le q_1, q_2 < \infty$ *and* $0 < q \le 1$*. If, for every* $q_1 < p_1 \le p_1^+$ *and* $q_2 < p_2 \le p_2^+$*,*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \quad \|B\| \lesssim \varphi_1\left(\frac{1}{p_1 - q_1}\right)\varphi_2\left(\frac{1}{p_2 - q_2}\right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{(\varphi_1\varphi_2)(\log_1 t)},$$

where D_1^+ *is determined by the values described in Table 4.8 in which* $\varphi = \varphi_1$ *and* $\varepsilon > 0$ *is arbitrary, and* D_2^+ *by those in Table 4.9 in which* $\varepsilon > 0$ *is also arbitrary.*

		D_2^+
$0 < q < 1$	$q_2 = 1$	$\Lambda^1(v_2),\ v_2(t) = \varphi_2\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1 < q_2 < \infty$	$\Lambda^1(v_2),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi_2\left(\log_1 \frac{1}{t}\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q = 1$	$1 < q_2 < \infty$	$\Lambda^1(v_2),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi_2\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)$

Table 4.9: Theorem 4.2.4

Proof. Let us fix p_2 and $g \in L^{p_2,1}$. Thus, the operator $T_1 := B(\cdot, g)$ satisfies that

$$T_1 : L^{p_1,1} \longrightarrow L^{p,\infty}, \qquad \|T_1\| \lesssim \varphi_1\left(\frac{1}{p_1 - q_1}\right)\varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,1}}.$$

Therefore, if we can apply Proposition 4.2.1 to T_1 to obtain (see Remark 4.2.2)

$$T_1 : D_1^+(r) \longrightarrow \Lambda^{1,\infty}[V_r], \quad V_r = \frac{t^{1/r}}{\varphi_1(\log_1 t)}, \quad \|T_1\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,1}},$$

where $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{p_2}$. Since $r \ge q$, $D_1^+ \subseteq D_1^+(r)$, and we get that

$$T_1 : D_1^+ \longrightarrow \Lambda^{1,\infty}[V_r], \qquad \|T_1\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right)\|g\|_{L^{p_2,1}}.$$

Thus, for every $(q_1, p_2; p)$ such that $q_2 < p_2 \le p_2^+$,

$$B : D_1^+ \times L^{p_2,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad \|B\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right).$$

Hence, if we now fix $f \in D_1^+$, the operator $T_2 := B(f, \cdot)$ satisfies that

$$T_2 : L^{p_2,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad \|T_2\| \lesssim \|f\|_{D_1^+} \varphi_2\left(\frac{1}{p_2 - q_2}\right),$$

and, by Proposition 4.2.3, the result follows. □

We also study the corresponding extrapolation result where we have the hypotheses for $p < q$.

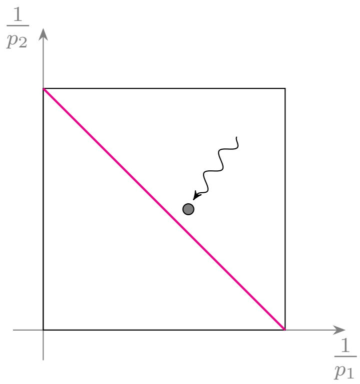


Figure 4.7: Quasi-Banach setting. Theorem 4.2.5.

Theorem 4.2.5 (Figure 4.7)**.** *Suppose that* $1 < q_1, q_2 < \infty$ *and* $0 < q \leq 1$*. If, for every* $p_1^- \leq p_1 < q_1$ *and* $p_2^- \leq p_2 < q_2$*,*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \qquad \|B\| \lesssim \varphi_1\left(\frac{1}{q_1 - p_1}\right) \varphi_2\left(\frac{1}{q_2 - p_2}\right),$$

then

$$B : D_1^- \times D_2^- \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{(\varphi_1\varphi_2)\left(\log_1 \frac{1}{t}\right)}$$

where D_1^-*,* D_2^- *are determined by the values described in Table 4.10 in which* $\varepsilon > 0$ *is arbitrary.*

Proof. The estimate is proved only for functions with a finite decomposition and the result follows by density. We prove the case where $0 < q < 1$. Given $f \in L^{q_1,1}$ and $g \in L^{q_2,1}$ such that $\|f\|_{L^{q_1,1}} = \|g\|_{L^{q_2,1}} = 1$, we decompose them with the dyadic decomposition over $\mathbb{N}$. Thus,

$$\left(B(f_n, g_m)\right)^*(t) \lesssim \varphi_1\left(\frac{1}{q_1 - p_1}\right)\left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p_1} \varphi_2\left(\frac{1}{q_2 - p_2}\right)\left(\frac{\|g_m\|_{L^1}}{t}\right)^{1/p_2}.$$

	D_1^-
$0<q<1$	$\Lambda^1(v_1),\ v_1(t)=t^{\left(\frac{1}{q_1}-1\right)}\varphi_1\left(\log_1 t\right)\left(\log_1\frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}\left(\varphi_1\varphi_2\right)\left(\log_2\frac{1}{t}\right)$
$q=1$	$\Lambda^1(v_1),\ v_1(t)=t^{\left(\frac{1}{q_1}-1\right)}\varphi_1\left(\log_1 t\right)\left(\log_2\frac{1}{t}\right)\left(\varphi_1\varphi_2\right)\left(\log_2\frac{1}{t}\right)$

	D_2^-
$0<q<1$	$\Lambda^1(v_2),\ v_2(t)=t^{\left(\frac{1}{q_2}-1\right)}\varphi_2\left(\log_1 t\right)\left(\log_1\frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}\left(\varphi_1\varphi_2\right)\left(\log_2\frac{1}{t}\right)$
$q=1$	$\Lambda^1(v_2),\ v_2(t)=t^{\left(\frac{1}{q_2}-1\right)}\varphi_2\left(\log_1 t\right)\left(\log_2\frac{1}{t}\right)\left(\varphi_1\varphi_2\right)\left(\log_2\frac{1}{t}\right)$

Table 4.10: Theorem 4.2.5

Taking the infimum over $p_1^-\le p_1<q_1$ and over $p_2^-\le p_2<q_2$ independently with Lemma 2.1.4, we get

$$\begin{aligned}\left(B(f_n,g_m)\right)^*(t)&\lesssim J_{\varphi_1,q_1}\left(\frac{\|f_n\|_{L^1}}{t}\right)J_{\varphi_2,q_2}\left(\frac{\|g_m\|_{L^1}}{t}\right)\\&\lesssim\frac{\left(\varphi_1\varphi_2\right)\left(\log_1\frac{1}{t}\right)}{t^{\frac{1}{q}}}J_{\varphi_1,q_1}\left(\|f_n\|_{L^1}\right)J_{\varphi_2,q_2}\left(\|g_m\|_{L^1}\right).\end{aligned}$$

Hence, if $V(t)=\dfrac{t^{\frac{1}{q}}}{\left(\varphi_1\varphi_2\right)\left(\log_1\frac{1}{t}\right)}$, we obtain that

$$\left\|B(f,g)\right\|_{\Lambda^{1,\infty}[V]}\lesssim J_{\varphi_1,q_1}\left(\|f\|_{L^1}\right)J_{\varphi_2,q_2}\left(\|g\|_{L^1}\right).\qquad(4.7)$$

On the other hand, since

$$\left|B(f,g)\right|\le\sum_{n\ge0}2^n\left\|B(f_n,g)\right\|_{\Lambda^{1,\infty}[V]}\frac{\left|B(f_n,g)\right|}{\left\|B(f_n,g)\right\|_{\Lambda^{1,\infty}[V]}}.$$

Proposition 2.3.8 and Lemma 2.3.10 i) imply that, for every sequence $(a_n)_n$ of positive numbers such that $\sum_n a_n\le1$,

$$\left\|B(f,g)\right\|_{\Lambda^{1,\infty}[V]}\lesssim\sum_{n\ge0}a_n^{-\frac{1}{q}+1}\left(\varphi_1\varphi_2\right)\left(\log_1\frac{1}{a_n}\right)2^n\left\|B(f_n,g)\right\|_{\Lambda^{1,\infty}[V]}.$$

We can repeat this argument with the other variable to get

$$\begin{aligned}&\left\|B(f,g)\right\|_{\Lambda^{1,\infty}[V]}\\&\lesssim \sum_{n\geq 0} a_n^{-\frac{1}{q}+1}(\varphi_1\varphi_2)\left(\log_1\frac{1}{a_n}\right)2^n\sum_{k\geq 0} a_k^{-\frac{1}{q}+1}(\varphi_1\varphi_2)\left(\log_1\frac{1}{a_k}\right)2^k\left\|B(f_n,g_k)\right\|_{\Lambda^{1,\infty}[V]}.\end{aligned}$$

By applying (4.7) to each term we get that

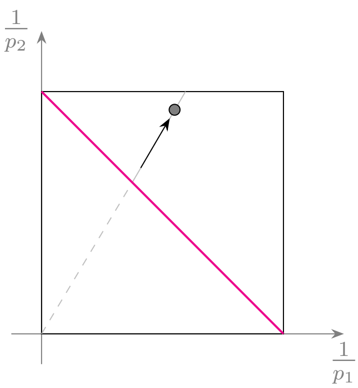
$$\begin{aligned}\left\|B(f,g)\right\|_{\Lambda^{1,\infty}[V]} &\lesssim \sum_{n\geq 0} a_n^{-\frac{1}{q}+1}(\varphi_1\varphi_2)\left(\log_1\frac{1}{a_n}\right)2^n J_{\varphi_1,q_1}\left(\|f_n\|_{L^1}\right)\\&\quad\times\sum_{k\geq 0} a_k^{-\frac{1}{q}+1}(\varphi_1\varphi_2)\left(\log_1\frac{1}{a_k}\right)2^k J_{\varphi_2,q_2}\left(\|g_k\|_{L^1}\right).\end{aligned}$$

Thus, we fix $\varepsilon > 0$ and we choose $a_n \approx n^{-(1+\varepsilon)}$ and $a_k \approx k^{-(1+\varepsilon)}$ and, by Lemma 2.3.14 i), the proof is complete. □

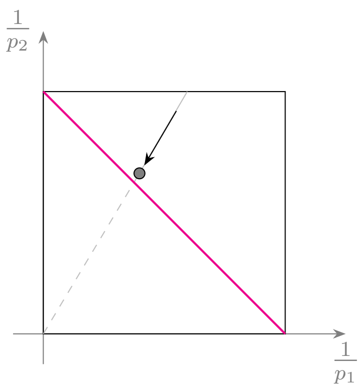
4.2.2 Rays extrapolation

We consider now the case where the operator blows up in both variables simultaneously along a ray. We recall that this means that every $(p_1, p_2; p)$ satisfies

$$\frac{p}{q} = \frac{p_1}{q_1} = \frac{p_2}{q_2}. \tag{4.8}$$



(a)



(b)

Figure 4.8: Quasi-Banach setting: rays. Theorems 4.2.6 and 4.2.7.

Theorem 4.2.6 (Figure 4.8 (a)). *Suppose that* $1 \le q_1, q_2 < \infty$ *and* $0 < q \le 1$. *If for every* $q_1 < p_1 \le p_1^+$ *and* $q_2 < p_2 \le p_2^+$ *such that* (4.8) *is satisfied we have that*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \qquad \|B\| \lesssim \varphi\left(\frac{1}{p-q}\right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)},$$

where D_1^+ *is determined by the values described in Table 4.8 in which* $\varepsilon > 0$ *is arbitrary and* D_2^+ *by those in Table 4.11 in which* $\varepsilon > 0$ *is also arbitrary.*

		D_2^+
$0 < q < 1$	$q_2 = 1$	$\Lambda^1(v_2),\ v_2(t) = \varphi\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1 < q_2 < \infty$	$\Lambda^1(v_2),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi\left(\log_1 \frac{1}{t}\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q = 1$	$1 < q_1 < \infty$	$\Lambda^1(v_2),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi\left(\log_1 \frac{1}{t}\right)\left(\log_2 \frac{1}{t}\right)$

Table 4.11: Theorem 4.2.6

Proof. We assume that our decompositions are finite and we can use later a density argument. Given $f \in L^{q_1,1}$ and $g \in L^{q_2,1}$ such that $\|f\|_{L^{q_1,1}} = \|g\|_{L^{q_2,1}} = 1$, we use the dyadic decomposition over $\mathbb{N}$. Thus,

$$\left(B(f_n, g_m)\right)^*(t) \lesssim t^{-\frac{1}{p}} \left\|B(f_n, g_m)\right\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-q}\right)\left(\frac{\|f_n\|_{L^1}^{\frac{q}{q_1}}\|g_m\|_{L^1}^{\frac{q}{q_2}}}{t}\right)^{1/p}.$$

By Lemma 2.1.4, we obtain

$$\left(B(f_n, g_m)\right)^*(t) \lesssim I_{\varphi,q}\left(\frac{\|f_n\|_{L^1}^{\frac{q}{q_1}}\|g_m\|_{L^1}^{\frac{q}{q_2}}}{t}\right) \lesssim \frac{\varphi(\log_1 t)}{t^{\frac{1}{q}}} I_{\varphi,q_1}\left(\|f_n\|_{L^1}\right) I_{\varphi,q_2}\left(\|g_m\|_{L^1}\right).$$

Now, if $V(t) = \dfrac{t^{1/q}}{\varphi(\log_1 t)}$, we conclude

$$\left\|B(f_n, g_m)\right\|_{\Lambda^{1,\infty}[V]} \lesssim I_{\varphi,q_1}\left(\|f_n\|_{L^1}\right) I_{\varphi,q_2}\left(\|g_m\|_{L^1}\right).$$

The rest of the proof follows the same ideas than Theorem 4.2.5 but using the 2-dyadic decomposition, Lemma 2.3.9 and Lemma 2.3.12 when $q_1 = 1$ and Lemma 2.3.13 when $1 < q_1 < \infty$. □

Theorem 4.2.7 (Figure 4.8 (b)). *Suppose that $1 < q_1, q_2 < \infty$ and $0 < q \leq 1$. If for every $p_1^- \leq p_1 < q_1$ and $p_2^- \leq p_2 < q_2$ such that (4.8) is satisfied we have that*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \qquad \|B\| \lesssim \varphi\left(\frac{1}{q-p}\right),$$

then

$$B : D_1^- \times D_2^- \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t^{1/q}}{\varphi\left(\log_1 \frac{1}{t}\right)},$$

where D_1^- and D_2^- are determined by the values described in Table 4.12 in which $\varepsilon > 0$ is arbitrary.

	D_1^-
$0 < q < 1$	$\Lambda^1(v_1),\ v_1(t) = t^{\left(\frac{1}{q_1}-1\right)}\varphi\left(\log_1 t\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}\varphi\left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_1),\ v_1(t) = t^{\left(\frac{1}{q_1}-1\right)}\varphi\left(\log_1 t\right)\left(\log_2 \frac{1}{t}\right)\varphi\left(\log_2 \frac{1}{t}\right)$

	D_2^-
$0 < q < 1$	$\Lambda^1(v_2),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi\left(\log_1 t\right)\left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}\varphi\left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_1),\ v_2(t) = t^{\left(\frac{1}{q_2}-1\right)}\varphi\left(\log_1 t\right)\left(\log_2 \frac{1}{t}\right)\varphi\left(\log_2 \frac{1}{t}\right)$

Table 4.12: Theorem 4.2.7

Proof. We follow the ideas of Theorem 4.2.6 but using Lemma 2.3.10 and Lemma 2.3.14. □

4.2.3 The endpoint $(1, \infty; 1)$

Theorem 4.2.8 (Figure 4.9). *If, for every $1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < \infty$*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \qquad \|B\| \lesssim \varphi_1\left(\frac{1}{p_1 - 1}\right)\varphi_2(p_2),$$

then

$$B : \Lambda^1(v_1) \times L^\infty_{\widetilde{\varphi_2}} \longrightarrow \Lambda^{1,\infty}[V], \qquad V(t) = \frac{t}{\varphi_1(\log_1 t)\varphi_2\left(\log_1 \frac{1}{t}\right)},$$

where $v_1(t) = \varphi_1(\log \frac{1}{t})(\log_3 \frac{1}{t})\varphi_2(\log_3 \frac{1}{t})$ and $\widetilde{\varphi_2}(t) := \varphi_2(t)\left(\log_1 t\right)\varphi_2\left(\log_1 t\right)$.

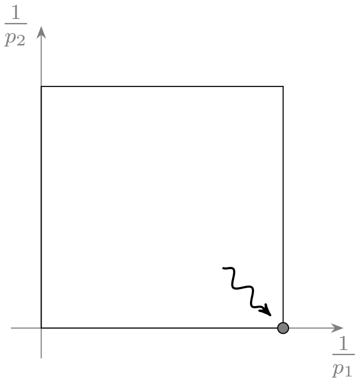


Figure 4.9: Quasi-Banach setting. Theorem 4.2.8.

Proof. We prove the result for functions with finite decomposition and the remaining cases follows by density. First, given $f \in L^1$ and $g \in L^\infty$, we decompose f using the 2-dyadic decomposition and g using the g^*-decomposition. Hence,

$$B(f_n, g_m)^*(t) \lesssim \varphi_1 \left(\frac{1}{p_1 - 1}\right) \left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p_1} \varphi_2(p_2) \left(\frac{\|g_m\|_{L^1}}{t}\right)^{1/p_2}.$$

Taking the infimum in each variable, by Lemma 2.1.4, we obtain

$$\sup_{t>0} \frac{t\ B(f_n, g_m)^*(t)}{\varphi_1(\log_1 t)\varphi_2\left(\log_1 \frac{1}{t}\right)} \lesssim I_{\varphi,1}\left(\|f_n\|_{L^1}\right) \left[\|g_m\|_{L^\infty}\, \varphi_2\left(\log_1 \frac{\|g_k\|_{L^1}}{\|g_m\|_{L^\infty}}\right)\right],$$

and now, we can proceed as in Theorem 4.2.5, using Lemmas 2.3.11, 2.3.12 and 2.3.15. □

CHAPTER 5

Endpoint estimates for particular operators

In this chapter, we apply bilinear Yano's theory to obtain endpoint estimates for certain classes of operators, such as rough singular operators and the bilinear Bochner-Riesz multiplier at the critical index.

As in the linear case, Rubio de Francia extrapolation theory will be a fundamental tool to find interesting examples. In particular, we shall use the following result (see [51], [43]).

Theorem 5.0.1. *If, for some* $(r_1, r_2; r)$ *with* $1 \leq r_1, r_2 < \infty$ *and, for every* $w_1 \in A_{r_1}$ *and* $w_2 \in A_{r_2}$*,*

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \psi\left([w_1]_{A_{r_1}}, [w_2]_{A_{r_2}}\right), \tag{5.1}$$

where ψ *is an increasing function in each variable and* $w = w_1^{r/r_1} w_2^{r/r_2}$*, then, for* $1 < p_1, p_2 < \infty$ *and, for every* $w_1 \in A_{p_1}$*,* $w_2 \in A_{p_2}$ *and* $w = w_1^{p/p_1} w_2^{p/p_2}$*,*

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(w),$$

with

$$\|B\| \lesssim \psi\left(C_1 p_1 [w_1]_{A_{p_1}}^{\max\left\{1, \frac{r_1-1}{p_1-1}\right\}}, C_2 p_2 [w_2]_{A_{p_2}}^{\max\left\{1, \frac{r_2-1}{p_2-1}\right\}}\right). \tag{5.2}$$

We shall denote by $(r_1, r_2; r)$ the first point where we have a weighted estimate, and by $(p_1, p_2; p)$ with $1 < p_1, p_2 < \infty$ all the points obtained by Rubio de Francia extrapolation theorem. Thus, we will study the boundedness of the operator at the

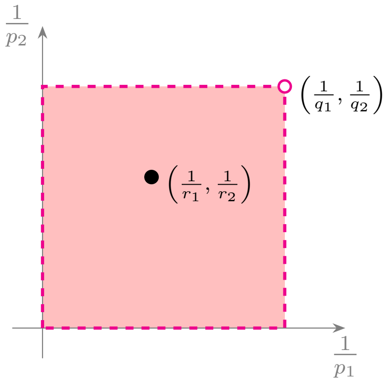


Figure 5.1: Bilinear Rubio de Francia Theorem 5.0.1.

endpoints $(q_1, q_2; q)$ which lie in the boundary of the shaded square (see Figure 5.1). Following carefully the behaviour of the constants in the proof of Theorem 5.0.1 (see [43]), one can easily obtain the following result.

Corollary 5.0.2. *If B satisfies (5.1), then, for $1 < p_1, p_2 < \infty$ and, for every $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$, and $w = w_1^{p/p_1} w_2^{p/p_2}$,*

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w),$$

with the following operator norm:

i) If $1 < p_1 < r_1$, $1 < p_2 < r_2$ and $w_1, w_2 \in A_1$,

$$\|B\| \lesssim \psi\left(C_1\left(\frac{1}{p_1 - 1}\right)^{r_1 - p_1} [w_1]_{A_1}^{1+r_1-p_1}, C_2\left(\frac{1}{p_2 - 1}\right)^{r_2 - p_2} [w_2]_{A_1}^{1+r_2-p_2}\right).$$

ii) If $1 < p_1 < r_1$, $1 < p_2 < \infty$ and $w_1 \in A_1$, $w_2 \in A_{p_2}$,

$$\|B\| \lesssim \psi\left(C_1\left(\frac{1}{p_1 - 1}\right)^{r_1 - p_1} [w_1]_{A_1}^{1+r_1-p_1}, C_2 p_2 [w_2]_{A_{p_2}}^{\max\left\{1, \frac{r_2-1}{p_2-1}\right\}}\right).$$

We observe that, in order to apply our Yano's extrapolation results after using Rubio de Francia theory, we need $w = w_1^{p/p_1} w_2^{p/p_2}$ to remain fixed for all $(p_1, p_2; p)$. This can be achieved by either setting $w_1 = w_2 = w$ (which will provide one-weight estimates at the endpoints) or by assuming that $\frac{p_1}{r_1} = \frac{p_2}{r_2} = \frac{p}{r}$ (which will provide two-weight estimates at the endpoints).

Let us start first with the one-weight case (see Figure 5.2).

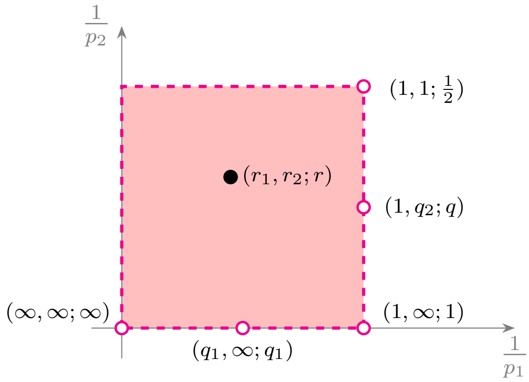


Figure 5.2: Theorem 5.0.3.

Theorem 5.0.3. *If, for some* $(r_1, r_2; r)$ *with* $1 < r_1, r_2 < \infty$ *and, for* $w_1 \in A_{r_1}$, $w_2 \in A_{r_2}$, *and* $w = w_1^{r/r_1} w_2^{r/r_2}$,

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \varphi_1\left([w_1]_{A_{r_1}}\right) \varphi_2\left([w_2]_{A_{r_2}}\right),$$

then, B satisfies the estimates described in Table 5.1 in which $\varepsilon > 0$ *is arbitrary.*

Proof. $\boxed{(1,1;\frac{1}{2})}$ If we set $w \in A_1$, by Corollary 5.0.2 follows that, for $1 < p_1 < r_1$ and $1 < p_2 < r_2$,

$$B : L^{p_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1\left(\left(\frac{1}{p_1 - 1}\right)^{r_1 - 1}\right) \varphi_2\left(\left(\frac{1}{p_2 - 1}\right)^{r_2 - 1}\right) \varphi_1\left([w]_{A_1}^{r_1}\right) \varphi_2\left([w]_{A_1}^{r_2}\right).$$

Hence, we can use Theorem 4.2.4 to obtain the estimate.

$\boxed{(1,q_2;q)}$ Fix $1 < q_2 < \infty$ and $w \in A_1$. Then, for every $1 < p_1 < r_1$,

$$B : L^{p_1}(w) \times L^{q_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1\left(\left(\frac{1}{p_1 - 1}\right)^{r_1 - 1}\right) \varphi_1\left([w]_{A_1}^{r_1}\right) \varphi_2\left([w]_{A_{q_2}}^{\max\left\{1, \frac{r_2 - 1}{q_2 - 1}\right\}}\right).$$

Thus, we can fix $g \in L^{q_2}(w)$ and use Proposition 4.2.1 to get the result.

Endpoint	Weights	One-weight estimate	$\|B\| \lesssim$
$(1,1;\frac{1}{2})$	$w \in A_1$	$B:\Lambda^1_w(v_1)\times\Lambda^1_w(v_2)\to\Lambda^{1,\infty}_w[V]$, $v_j(t)=\varphi_j\left((\log_1\frac{1}{t})^{r_j-1}\right)\left(\log_2\frac{1}{t}\right)^{1+\varepsilon}$, $V(t)=\dfrac{t^2}{\varphi_1\left((\log_1 t)^{r_1-1}\right)\varphi_2\left((\log_1 t)^{r_2-1}\right)}$	$\varphi_1([w]^{r_1}_{A_1})\varphi_2([w]^{r_2}_{A_1})$
$(1,q_2;q)$	$w \in A_1$	$B:\Lambda^1_w(v_1)\times L^{q_2}(w)\to\Lambda^{1,\infty}_w[V]$, $v_1(t)=\varphi_1\left((\log_1\frac{1}{t})^{r_1-1}\right)\left(\log_2\frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$, $V(t)=\dfrac{t^{1/q}}{\varphi_1\left((\log_1 t)^{r_1-1}\right)}$	$\varphi_1([w]^{r_1}_{A_1})\varphi_2\left([w]_{A_{q_2}}^{\max\{1,\frac{r_2-1}{q_2-1}\}}\right)$
$(1,\infty;1)$	$w \in A_1$	$B:\Lambda^1_w(v_1)\times L^\infty_{\varphi_2}(w)\to\Gamma^{1,\infty}_w[V]$, $v_1(t)=\varphi_1\left((\log_1\frac{1}{t})^{r_1-1}\right)\varphi_2\left(\log_1\frac{1}{t}\right)$, $V(t)=\dfrac{t}{\varphi_1\left((\log_1 t)^{r_1-1}\right)\varphi_2(\log_1 t)\,\varphi_2\left(\log_1\frac{1}{t}\right)}$	$\varphi_1([w]^{r_1}_{A_1})\varphi_2([w]_{A_{r_2}})$
$(q_1,\infty;q_1)$	$w \in A_{q_1}$	$B:L^{q_1}(w)\times L^\infty_{\varphi_2}(w)\to\Gamma^{1,\infty}_w[V]$, $V(t)=\dfrac{t^{1/q_1}}{\varphi_2\left(\log_1\frac{1}{t}\right)}$	$\varphi_1\left([w]_{A_{q_1}}^{\max\{1,\frac{r_1-1}{q_1-1}\}}\right)\varphi_2([w]_{A_{q_1}})$
$(\infty,\infty;\infty)$	$w \in A_\infty$	$B:L^\infty_{\varphi_1}(w)\times L^\infty_{\varphi_2}(w)\to\Gamma^{1,\infty}_w[V]$, $V(t)=\dfrac{1}{\varphi_1\left(\log_1\frac{1}{t}\right)\varphi_2\left(\log_1\frac{1}{t}\right)}$	

Table 5.1: Theorem 5.0.3

$\boxed{(1,\infty;1)}$ Let $w\in A_1$. By Corollary 5.0.2, we get that, for every $1<p_1<r_1$ and $r_2<p_2<\infty$,

$$B : L^{p_1}(w)\times L^{p_2}(w)\longrightarrow L^p(w),$$

with

$$\|B\|\lesssim \varphi_1\left(\left(\frac{1}{p_1-1}\right)^{r_1-1}\right)\varphi_2(p_2)\,\varphi_1\left([w]^{r_1}_{A_1}\right)\varphi_2\left([w]_{A_{p_2}}\right).$$

The endpoint estimate follows by Theorem 4.1.10.

$\boxed{(q_1,\infty;q_1)}$ Fix $w\in A_{q_1}$. For every $\max\{r_2,q_1\}<p_2<\infty$, by (5.2), we get

$$B : L^{q_1}(w)\times L^{p_2}(w)\longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1\left([w]_{A_{q_1}}^{\max\left\{1,\frac{r_1-1}{q_1-1}\right\}}\right)\varphi_2\left(p_2\right)\varphi_2\left([w]_{A_{q_1}}\right).$$

Thus, we can fix $f \in L^{q_1}(w)$ and use Proposition 4.1.4 to get the result.

$\boxed{(\infty,\infty;\infty)}$ If $w \in A_\infty$, there exists $\max\{r_1, r_2\} \leq p_0 < \infty$ such that $w \in A_{p_0}$. Thus, it follows from Corollary 5.0.2 that, for every $p_0 < p_1, p_2 < \infty$,

$$B : L^{p_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1\left(p_1\right)\varphi_2\left(p_2\right)\varphi_1\left([w]_{A_{p_0}}\right)\varphi_2\left([w]_{A_{p_0}}\right),$$

and by Theorem 4.1.6 the result follows. □

Remark 5.0.4. At the endpoint $(1,\infty;1)$, we can also use Theorem 4.2.8 to conclude that, for every $w \in A_1$,

$$B : \Lambda^1_w(v_1) \times L^\infty_{\widetilde{\varphi}_2}(w) \to \Lambda^{1,\infty}_w[V], \qquad V(t) = \frac{t}{\varphi_1\left((\log_1 t)^{r_1-1}\right)\varphi_2\left(\log_1 \frac{1}{t}\right)}$$

where

$$v_1(t) = \varphi_1\left((\log_1 \frac{1}{t})^{r_1-1}\right)\left(\log_3 \frac{1}{t}\right)\varphi_2\left(\log_3 \frac{1}{t}\right)$$

and $\widetilde{\varphi}_2(t) = \varphi_2(t)\left(\log_1 t\right)\varphi_2\left(\log_1 t\right).$

In the points $(q_1, q_2; q)$ on the boundary of the shaded square that can be reached through a ray, we obtain two-weights estimates.

Theorem 5.0.5. *If, for some $(r_1, r_2; r)$ with $1 < r_1, r_2 < \infty$, for every $w_1 \in A_{r_1}$, $w_2 \in A_{r_2}$ and $w = w_1^{r/r_1} w_2^{r/r_2}$,*

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \varphi_1\left([w_1]_{A_{r_1}}\right)\varphi_2\left([w_2]_{A_{r_2}}\right),$$

then, B satisfies the estimates described in Table 5.2, in which $\varepsilon > 0$ is arbitrary.

Proof. $\boxed{(1,1;\frac{1}{2})}$ If we set $w_1, w_2 \in A_1$, it follows from Corollary 5.0.2 that, for every $1 < p_1 < r_1$ and $1 < p_2 < r_2$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^{p/p_1} w_2^{p/p_2}),$$

Endpoint	Weights	Two-weight estimate	$\|B\| \lesssim$
$(1,1;\frac{1}{2})$	$w_1, w_2 \in A_1$	$B : \Lambda^1_{w_1}(v_0) \times \Lambda^1_{w_2}(v_0) \to \Lambda^{1,\infty}_{\left(w_1^{1/2} w_2^{1/2}\right)}[V]$, $v_0(t) = \varphi_1\left((\log_1 \frac{1}{t})^{r_1-1}\right) \varphi_2\left((\log_1 \frac{1}{t})^{r_2-1}\right) \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \frac{t^2}{\varphi_1\left((\log_1 t)^{r_1-1}\right) \varphi_2\left((\log_1 t)^{r_2-1}\right)}$	$\varphi_1([w_1]^{r_1}_{A_1})\varphi_2([w_2]^{r_2}_{A_1})$
$(1,q_2;q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$B : \Lambda^1_{w_1}(v_1) \times \Lambda^1_{w_2}(v_2) \to \Lambda^{1,\infty}_{\left(w_1^q w_2^{q/q_2}\right)}[V]$, $v_1(t) = \varphi_1\left(\left(\log_1 \frac{1}{t}\right)^{r_1-1}\right) \left(\log_2 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$ $v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_1\left(\left(\log_1 \frac{1}{t}\right)^{r_1-1}\right) \left(\log_1 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$ $V(t) = \frac{t^{1/q}}{\varphi_1\left((\log_1 t)^{r_1-1}\right)}$,	$\varphi_1([w_1]^{r_1}_{A_1})\varphi_2\left([w_2]_{A_{q_2}}^{\max\left\{1, \frac{r_2-1}{q_2-1}\right\}}\right)$
$(\infty,\infty;\infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha + \beta = 1$	$B : L^\infty_\varphi(w_1) \times L^\infty_\varphi(w_2) \to \Gamma^{1,\infty}_{\left(w_1^\alpha w_2^\beta\right)}[V]$, $V(t) = \frac{1}{\varphi\left(\log_1 \frac{1}{t}\right)}$, $\varphi(s) = \varphi_1(s)\varphi_2(s)$	

Table 5.2: Theorem 5.0.5

with

$$\|B\| \lesssim \varphi_1\left(\left(\frac{1}{p_1-1}\right)^{r_1-1}\right) \varphi_2\left(\left(\frac{1}{p_2-1}\right)^{r_2-1}\right) \varphi_1\left([w_1]^{r_1}_{A_1}\right) \varphi_2\left([w_2]^{r_2}_{A_1}\right).$$

In particular, for every $1 < p_1 < \min\{r_1, r_2\}$

$$B : L^{p_1}(w_1) \times L^{p_1}(w_2) \longrightarrow L^p(w_1^{1/2} w_2^{1/2}),$$

with

$$\|B\| \lesssim \varphi_1\left(\left(\frac{1}{p-\frac{1}{2}}\right)^{r_1-1}\right) \varphi_2\left(\left(\frac{1}{p-\frac{1}{2}}\right)^{r_2-1}\right) \varphi_1\left([w_1]^{r_1}_{A_1}\right) \varphi_2\left([w_2]^{r_2}_{A_1}\right).$$

Hence, from Theorem 4.2.6 the result follows.

$\boxed{(1, q_2; q)}$ Let us fix $1 < q_2 < \infty$, $w_1 \in A_1$ and $w_2 \in A_{q_2}$. It follows from Corollary 5.0.2 that, for any $1 < p_1 < r_1$ and $1 < p_2 < \infty$ such that $p_1 = \frac{p_2}{q_2} = \frac{p}{q} > 1$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^q w_2^{q/q_2}),$$

with

$$\|B\| \lesssim \varphi_1\left(\left(\frac{1}{p-q}\right)^{r_1-1}\right)\varphi_1\left([w]_{A_1}^{r_1}\right)\varphi_2\left([w]_{A_{q_2}}^{\max\left\{1,\frac{r_2-1}{q_2-1}\right\}}\right).$$

Thus, we can use Theorem 4.2.6 to obtain the endpoint estimate.

$\boxed{(\infty,\infty;\infty)}$ We observe that this point can be reached through different rays (see Figure 5.3). Let us fix $\alpha,\beta>0$ such that $\alpha+\beta=1$ and let us move along the ray

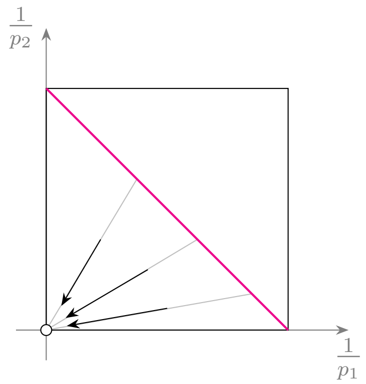


Figure 5.3: Endpoint $(\infty,\infty;\infty)$. Theorem 5.0.5.

where $\frac{p}{p_1}=\alpha$ and $\frac{p}{p_2}=\beta$. If we set $w_1,w_2\in A_\infty$, there exists $\max\{r_1,r_2\}\le p_0<\infty$ such that $w_1,w_2\in A_{p_0}$. Thus, by (5.2) we get that, for $p_0<p_1,p_2<\infty$ such that $\alpha p_1=\beta p_2=p$,

$$B: L^{p_1}(w_1)\times L^{p_2}(w_2)\longrightarrow L^p(w_1^\alpha w_2^\beta),$$

with

$$\|B\|\lesssim(\varphi_1\varphi_2)(p)\,\varphi_1\left([w_1]_{A_{p_0}}\right)\varphi_2\left([w_2]_{A_{p_0}}\right),$$

and the endpoint estimate follows by Theorem 4.1.9. □

5.1 Bilinear rough singular operators

Given a function Ω defined on $\mathbb{S}^{2n-1}$ such that $\int_{\mathbb{S}^{2n-1}}\Omega=0$, the *bilinear rough singular operator* associated to Ω (see [36]) is defined by

$$T_\Omega(f,g)(x)=\text{p.v.}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\Omega\left((x,y)'\right)}{\left|(y,z)\right|^{2n}}f(x-y)g(x-z)dydz,\quad x\in\mathbb{R}^n,$$

where $(y,z)' := \frac{(y,z)}{|y,z|}$. In [47] and [48] (see also [49]), the boundedness of the bilinear Hilbert transform has been used to obtain estimates for T_Ω. In fact, in [48] it is proved that, for every $(p_1, p_2; p)$ such that $1 < p_1, p_2 < \infty$,

$$T_\Omega : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

as long as $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. On the other hand, in [7] sparse domination is used to conclude that, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$,

$$\|T_\Omega\|_{L^3(w_1)\times L^3(w_2)\to L^{3/2}(w)} \lesssim C(\|\Omega\|_{L^\infty})[w_1]_{A_3}[w_2]_{A_3} \qquad \forall w_1, w_2 \in A_3. \tag{5.3}$$

Also, from [32] it can be deduced that,

$$\|T_\Omega\|_{L^2(w_1)\times L^2(w_2)\to L^1(w)} \lesssim \|\Omega\|_{L^\infty} [w_1]_{A_2}^{3/2}[w_2]_{A_2}^{3/2}, \qquad w_1, w_2 \in A_2. \tag{5.4}$$

Some results are known at the endpoints. Indeed, by [75] and (5.3), it follows the weighted boundedness of T_Ω for every $(q_1, \infty; q_1)$ with $1 < q_1 < \infty$. In the unweighted case, it is also known (see [54]) that

$$T_\Omega : L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \longrightarrow L^{1,\infty}(\mathbb{R}^n),$$

and

$$T_\Omega : L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \longrightarrow BMO.$$

As far as we know, the cases $(1, 1; 1/2)$ and $(1, q_2; q)$ with $1 < q_2 < \infty$ are open. We shall use estimates (5.3) and (5.4) to obtain new weighted estimates at these endpoints.

Theorem 5.1.1. *If $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ and $\int_{\mathbb{S}^{2n-1}} \Omega = 0$, the operator T_Ω satisfies the endpoint estimates described in Table 5.3 and Table 5.4, in which $\varepsilon > 0$ is arbitrary.*

Proof. We apply Theorem 5.0.3 and Theorem 5.0.5, using estimate (5.4) in the cases where the endpoint is $(1, 1; \frac{1}{2})$ or $(1, q_2; q)$, and (5.3) in when the endpoint is $(1, \infty; 1)$ or $(\infty, \infty; \infty)$. □

5.2 Bilinear Bochner-Riesz operators

The bilinear Bochner-Riesz operator of index $\alpha \geq 0$ is defined, for $f, g \in \mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{B}^\alpha(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1 - |\xi|^2 - |\eta|^2)_+^\alpha \hat{f}(\xi)\hat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta, \quad x \in \mathbb{R}^n.$$

Endpoint	Weights	One-weight estimate	$\|T_\Omega\| \lesssim$
$(1,1;\frac{1}{2})$	$w \in A_1$	$T_\Omega : \Lambda^1_w(v_1) \times \Lambda^1_w(v_2) \to \Lambda^{1,\infty}_w[V]$, $v_j(t) = \left(\log_1 \frac{1}{t}\right)^{3/2} \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \dfrac{t^2}{(\log_1 t)^3}$	$[w]^6_{A_1}$
$(1,q_2;q)$	$w \in A_1$	$T_\Omega : \Lambda^1_w(v_1) \times L^{q_2}(w) \to \Lambda^{1,\infty}_w[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^{3/2} \left(\log_2 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$, $V(t) = \dfrac{t^{1/q}}{(\log_1 t)^{3/2}}$	$[w]^3_{A_1}\left([w]^{\max\left\{1,\frac{1}{q_2-1}\right\}}_{A_{q_2}}\right)^{3/2}$
$(1,\infty;1)$	$w \in A_1$	$T_\Omega : \Lambda^1_w(v_1) \times L^\infty_1(w) \to \Gamma^{1,\infty}_w[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^3$, $V(t) = \dfrac{t}{(\log_1 t)^3 \left(\log_1 \frac{1}{t}\right)}$	$[w]^3_{A_1}[w]_{A_3}$
$(\infty,\infty;\infty)$	$w \in A_\infty$	$T_\Omega : L^\infty_1(w) \times L^\infty_1(w) \to E^2(w)$	

Table 5.3: Theorem 5.1.1

As in the linear case, it is useful to work with the kernel associated to this operator. Indeed,

$$\mathcal{B}^\alpha(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_\alpha(y_1,y_2) f(x-y_1) g(x-y_2)\,dy_1 dy_2, \quad x \in \mathbb{R}^n,$$

where, for every $y_1, y_2 \in \mathbb{R}^n$,

$$K_\alpha(y_1,y_2) = \frac{\Gamma(\alpha+1)}{\pi^\alpha} \frac{J_{\alpha+n}\left(2\pi\left|(y_1,y_2)\right|\right)}{\left|(y_1,y_2)\right|^{\alpha+n}}$$

and $J_{\alpha+n}$ denotes the standard Bessel function of order $\alpha+n$. It is known (see [10]) that for $\alpha > n - \frac{1}{2}$ and $1 \le p_1, p_2 \le \infty$,

$$\mathcal{B}^\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

Endpoint	Weights	Two-weight estimate	$\|T_\Omega\| \lesssim$
$(1,1;\frac{1}{2})$	$w_1, w_2 \in A_1$	$T_\Omega : \Lambda^1_{w_1}(v_0) \times \Lambda^1_{w_2}(v_0) \to \Lambda^{1,\infty}_{\left(w_1^{1/2} w_2^{1/2}\right)}[V]$, $v_0(t) = \left(\log_1 \frac{1}{t}\right)^3 \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \dfrac{t^2}{(\log_1 t)^3}$	$[w_1]^3_{A_1}[w_2]^3_{A_1}$
$(1, q_2; q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$T_\Omega : \Lambda^1_{w_1}(v_1) \times \Lambda^1_{w_2}(v_2) \to \Lambda^{1,\infty}_{\left(w_1^q w_2^{q/q_2}\right)}[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^{3/2} \left(\log_2 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$ $v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \left(\log_1 \frac{1}{t}\right)^{\frac{3}{2}+\frac{1+\varepsilon}{q_2}}$ $V(t) = \dfrac{t^{1/q}}{(\log_1 t)^{3/2}}$,	$[w_1]^3_{A_1}\left([w_2]_{A_{q_2}}^{\max\left\{1, \frac{1}{q_2-1}\right\}}\right)^{3/2}$
$(\infty,\infty;\infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha+\beta=1$	$T_\Omega : L^\infty_2(w_1) \times L^\infty_2(w_2) \to E^2(w_1^\alpha w_2^\beta)$	

Table 5.4: Theorem 5.1.1

and $\alpha = n - \frac{1}{2}$ is the so-called critical index.

In the weighted context, it has been proved in [64] that

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \longrightarrow L^1(w) \qquad \forall\, w_1, w_2 \in A_2. \tag{5.5}$$

Hence, we need to find φ_1, φ_2 such that, for every $w_1, w_2 \in A_2$, (5.5) is satisfied with

$$\left\| \mathcal{B}^{n-\frac{1}{2}} \right\| \leq \varphi_1\left([w_1]_{A_2}\right) \varphi_2\left([w_2]_{A_2}\right).$$

We observe (see [13], [85]) that there exists a constant $C_n := 1/2^{n+5}$ such that, for every $w_1, w_2 \in A_2$, if $\delta_1 = [w_1]^{-1}_{A_2}$ and $\delta_2 = [w_2]^{-1}_{A_2}$, then,

$$[w_1^{1+C\delta_1}]_{A_2} \lesssim [w_1]^{1+C\delta_1}_{A_2} \quad \text{and} \quad [w_2^{1+C\delta_2}]_{A_2} \lesssim [w_2]^{1+C\delta_2}_{A_2}.$$

Using this fact, a detailed study of the proof in [64] shows that, for every $n \geq 2$ and $w_1, w_2 \in A_2$, we have

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \longrightarrow L^1(w), \quad \|\mathcal{B}^{n-\frac{1}{2}}\| \lesssim \max\left\{[w_1]^2_{A_2}, [w_2]^2_{A_2}\right\}[w_1]_{A_2}[w_2]_{A_2}.$$

In particular,

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \longrightarrow L^1(w), \quad \|\mathcal{B}^{n-\frac{1}{2}}\| \lesssim [w_1]_{A_2}^3 [w_2]_{A_2}^3.$$

Thus, by Theorem 5.0.3 and Theorem 5.0.5, we obtain the following new estimates.

Theorem 5.2.1. *The operator $\mathcal{B}^{n-\frac{1}{2}}$ satisfies the endpoint estimates described in Table 5.5 and Table 5.6, in which $\varepsilon > 0$ is arbitrary.*

Endpoint	Weights	One-weight estimate	$\|\mathcal{B}^{n-\frac{1}{2}}\| \lesssim$
$(1, 1; \frac{1}{2})$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times \Lambda_w^1(v_2) \to \Lambda_w^{1,\infty}[V]$, $v_j(t) = \left(\log_1 \frac{1}{t}\right)^3 \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \dfrac{t^2}{(\log_1 t)^6}$	$[w]_{A_1}^{12}$
$(1, q_2; q)$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times L^{q_2}(w) \to \Lambda_w^{1,\infty}[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^3 \left(\log_2 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$, $V(t) = \dfrac{t^{1/q}}{(\log_1 t)^3}$	$[w]_{A_1}^6 [w]_{A_{q_2}}^{\max\left\{1, \frac{1}{q_2-1}\right\}}$
$(1, \infty; 1)$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times L_1^{\infty}(w) \to \Gamma_w^{1,\infty}[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^6$, $V(t) = \dfrac{t}{(\log_1 t)^6 \left(\log_1 \frac{1}{t}\right)^3}$	$[w]_{A_1}^9$
$(\infty, \infty; \infty)$	$w \in A_\infty$	$\mathcal{B}^{n-\frac{1}{2}} : L_3^{\infty}(w) \times L_3^{\infty}(w) \to E^6(w)$	

Table 5.5: Theorem 5.2.1

5.3 Bilinear Fourier multipliers of bounded variation

Let us consider now the following extension of bounded variation functions on $\mathbb{R}$. Set

$$m(\xi, \eta) := \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} d\mu(t, s), \qquad \xi, \eta \in \mathbb{R}, \tag{5.6}$$

Endpoint	Weights	Two-weight estimate	$\|\mathcal{B}^{n-\frac{1}{2}}\| \lesssim$
$(1,1;\frac{1}{2})$	$w_1, w_2 \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda^1_{w_1}(v_0) \times \Lambda^1_{w_2}(v_0) \to \Lambda^{1,\infty}_{\left(w_1^{1/2} w_2^{1/2}\right)}[V]$, $v_0(t) = \left(\log_1 \frac{1}{t}\right)^6 \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \frac{t^2}{(\log_1 t)^6}$	$[w_1]^6_{A_1}[w_2]^6_{A_1}$
$(1,q_2;q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda^1_{w_1}(v_1) \times \Lambda^1_{w_2}(v_2) \to \Lambda^{1,\infty}_{\left(w_1^q w_2^{q/q_2}\right)}[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^3 \left(\log_2 \frac{1}{t}\right)^{\frac{1+\varepsilon}{q_2}}$ $v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \left(\log_1 \frac{1}{t}\right)^{3+\frac{1+\varepsilon}{q_2}}$ $V(t) = \frac{t^{1/q}}{(\log_1 t)^3}$,	$[w_1]^6_{A_1}[w_2]^{\max\left\{1,\frac{1}{q_2-1}\right\}}_{A_{q_2}}$
$(\infty,\infty;\infty)$	$w_1, w_2 \in A_\infty$ $\alpha,\beta > 0$ $\alpha+\beta = 1$	$\mathcal{B}^{n-\frac{1}{2}} : L^\infty_6(w_1) \times L^\infty_6(w_2) \to E^6(w_1^\alpha w_2^\beta)$	

Table 5.6: Theorem 5.2.1

where $d\mu$ is a finite measure on $\mathbb{R}^2$ and without loss of generality we may assume that $\mu(\mathbb{R}^2) = 1$. We observe this is the case if $m(\xi,\eta) = m_1(\xi)m_2(\eta)$ with m_j, $j = 1,2$, two normalized ($dm_j(\mathbb{R}) = 1$, $m_j(-\infty) = 0$) bounded variation functions. For functions f and g in C_c^∞, the following operator associated to the multiplier m was introduced in [87],

$$B_m(f,g)(x) := \int_{\mathbb{R}^2} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x(\xi+\eta)}d\xi d\eta, \qquad x \in \mathbb{R}.$$

In particular, it is easy to see that

$$B_m(f,g)(x) = \int_{\mathbb{R}^2} H_t f(x) H_s g(x)\, d\mu(t,s), \qquad \forall x \in \mathbb{R},$$

where

$$H_t f(x) := \int_t^\infty \hat{f}(\xi)e^{2\pi i x\xi}d\xi, \qquad x \in \mathbb{R}.$$

Some endpoint estimates for this operator are known. Indeed, using techniques different from those presented in this work, we established in [6] the following one-weight estimate at the endpoint $(1,1;\frac{1}{2})$:

$$B_m : L^1(w) \times L^1(w) \longrightarrow L^{1/2,\infty}(w), \qquad \forall\, w \in A_1,$$

and, by [87], a two-weight estimate at the endpoint $(1, q_2; q)$ with $1 < q_2 < \infty$ can also be deduced, namely

$$B_m : L^1(w_1) \times L^{q_2}(w_2) \longrightarrow L^{q,\infty}(w), \qquad \forall\, w_1 \in A_1,\, w_2 \in A_{q_2}.$$

For the remaining endpoints, we can use Theorem 5.0.3 and Theorem 5.0.5 to obtain new estimates.

Theorem 5.3.1. *The operator B_m, associated to the multiplier m defined in (5.6), satisfies the endpoint estimates described in Table 5.7 and Table 5.8, in which $\varepsilon > 0$ is arbitrary.*

Endpoint	Weights	One-weight estimate	$\|B_m\| \lesssim$
$(1,\infty;1)$	$w \in A_1$	$B_m : \Lambda^1_w(v_1) \times L^\infty_{\varphi_2}(w) \to \Gamma^{1,\infty}_w[V]$, $v_1(t) = \left(\log_1 \frac{1}{t}\right)^2$, $V(t) = \dfrac{t}{(\log_1 t)^2 \left(\log_1 \frac{1}{t}\right)}$	$[w]^2_{A_1}[w]_{A_2}$
$(q_1,\infty;q_1)$	$w \in A_{q_1}$	$B_m : L^{q_1}(w) \times L^\infty_1(w) \to \Gamma^{1,\infty}_w[V]$, $V(t) = \dfrac{t^{1/q_1}}{\left(\log_1 \frac{1}{t}\right)}$	$[w]_{A_{q_1}}^{1+\max\left\{1,\frac{1}{q_1-1}\right\}}$
$(\infty,\infty;\infty)$	$w \in A_\infty$	$B_m : L^\infty_1(w) \times L^\infty_1(w) \to E^2(w)$	

Table 5.7: Theorem 5.3.1

Proof. It is immediate to see that, for every $w \in A_2$,

$$\|H_t\|_{L^2(w)\to L^2(w)} \le C_1 \left(1 + \|H\|_{L^2(w)\to L^2(w)}\right) \le C_2 [w]_{A_2},$$

with C_2 independent of t. Therefore, for every $w_1, w_2 \in A_2$ and $w = w_1^{1/2} w_2^{1/2}$,

$$\begin{aligned}\left\|B_m(f,g)\right\|_{L^1(w)} &\le \int_{\mathbb{R}^2} \|H_t f H_s g\|_{L^1(w)} |d\mu|\,(t,s) \\ &\le \int_{\mathbb{R}^2} \|H_t f\|_{L^2(w_1)} \|H_s g\|_{L^2(w_2)} |d\mu|\,(t,s) \le C_2^2 [w_1]_{A_2}[w_2]_{A_2} \|f\|_{L^2(w_1)} \|g\|_{L^2(w_2)}.\end{aligned}$$

Finally, we can apply Theorem 5.0.3 and Theorem 5.0.5 to this estimate. □

Endpoint	Weights	Two-weight estimate	$\|B_m\| \lesssim$
$\left(1,1;\frac{1}{2}\right)$	$w_1, w_2 \in A_1$	$B_m : \Lambda^1_{w_1}(v_0) \times \Lambda^1_{w_2}(v_0) \to \Lambda^{1,\infty}_{\left(w_1^{1/2} w_2^{1/2}\right)}[V]$, $v_0(t) = \left(\log_1 \frac{1}{t}\right)^2 \left(\log_2 \frac{1}{t}\right)^{1+\varepsilon}$, $V(t) = \dfrac{t^2}{(\log_1 t)^2}$	$[w_1]^2_{A_1}[w_2]^2_{A_1}$
$(\infty,\infty;\infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha+\beta=1$	$B_m : L^\infty_2(w_1) \times L^\infty_2(w_2) \to E^2\left(w_1^\alpha w_2^\beta\right)$,	

Table 5.8: Theorem 5.3.1

5.4 The weighted bilinear Hardy operator

In [44], it has been considered the following weighted bilinear Hardy operator. For any weight

$$v : [0,1] \times [0,1] \longrightarrow [0,\infty),$$

$$\mathcal{H}_v\,(f,g)\,(x) := \int_{0<t_1,t_2<1} f(t_1x)g(t_2x)v(t_1,t_2)dt_1dt_2, \qquad x \in \mathbb{R}^n.$$

When $p>1$, it has been proved (see [44]) that, if $1<p_1,p_2,p<\infty$, then

$$\mathcal{H}_v : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

with

$$\|\mathcal{H}_v\| = \int_{0<t_1,t_2<1} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2}} v(t_1,t_2)dt_1dt_2.$$

Hence, it follows that, if for some $\alpha_1,\alpha_2>-1$, we set $v(t_1,t_2)=t_1^{\alpha_1}t_2^{\alpha_2}$, then, for every $\frac{n}{1+\alpha_1}<p_1<\infty$ and $\frac{n}{1+\alpha_2}<p_2<\infty$ such that $p>1$, we have

$$\mathcal{H}_v : L^{p_1} \times L^{p_2} \longrightarrow L^p, \qquad \|\mathcal{H}_v\| \approx \frac{p_1p_2}{\left(p_1-\frac{n}{1+\alpha_1}\right)\left(p_2-\frac{n}{1+\alpha_2}\right)}.$$

Thus, from Proposition 4.1.1 and Theorem 4.1.3, the following endpoint estimates can be deduced.

Theorem 5.4.1. *If, for some* $\alpha_1, \alpha_2 > -1$*, we set* $v(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}$*, then, for every* $(\frac{n}{1+\alpha_1}, q_2; q)$ *such that* $\frac{n}{1+\alpha_2} < q_2 < \infty$ *and* $q > 1$*, we have that*

$$\mathcal{H}_v : D^+_{\frac{n}{1+\alpha_1},1} \times L^{q_2}(\mathbb{R}^n) \longrightarrow R^+_{q,1}.$$

In particular, if $n < q_2 < \infty$*, the unweighted bilinear Hardy operator satisfies that*

$$\mathcal{H} : D^+_{n,1} \times L^{q_2}(\mathbb{R}^n) \longrightarrow R^+_{q,1}.$$

Theorem 5.4.2. *If for some* $\alpha_1, \alpha_2 > -1$ *such that* $\dfrac{n}{2+\alpha_1+\alpha_2} > 1$*, we set* $v(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}$*, then*

$$\mathcal{H}_v : D^+_{\frac{n}{1+\alpha_1},1} \times D^+_{\frac{n}{1+\alpha_2},1} \longrightarrow R^+_{\frac{n}{2+\alpha_1+\alpha_2},2}.$$

In particular, if $n > 2$*, the unweighted bilinear Hardy operator satisfies*

$$\mathcal{H} : D^+_{n,1} \times D^+_{n,1} \longrightarrow R^+_{\frac{n}{2},2}.$$

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