FUZZY PREFERENCES IN DECISION-MAKING

F.J. Montero and J. Tejada
Dept. Estadística e I.O.
Fac. CC. Matemáticas
Universidad Complutense
28040 - Madrid (Spain)

Summary. In this paper, Orlovsky's concept of decision making with fuzzy preference relation is studied. On the one hand, the special significance of max-min transitivity inside the family of max-* transitivities is stablished. On the other hand, a necessary and sufficient condition for the existence of a non empty set of unfuzzy nondominated alternatives is proved. Moreover, other alternative methods are proposed in order to solve some practical difficulties.
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1. Introduction

Since L.A.Zadeh |8| introduced the concept of Fuzzy Subset in order to formalize vagueness, the problem of choice by an agent who has fuzzy preference has received an special attention. As it is well known, a fuzzy preference relation on a set of alternatives X is a fuzzy subset of the cartesian product $X \times X$, and it can be identified with a membership function $\mu : X \times X \to [0,1]$ in such a way that $\mu(x,y)$ is understood as the degree of preference of alternative x over alternative y. Through out this article such a set of alternatives X will be supposed finite with three elements at least.

The purpose of this paper is to formulate some properties on Orlovsky's concept of decision making |6|. In section 2 such a concept is related to the notion of max-* transitivity, as introduced by Zadeh |9| and Bezdek-Harris |2|. In section 3, the concept of fuzzy acyclity is proposed in order to characterize the existence of Orlovsky's choice set. Finally, the problem of emptiness of such a choice set is analysed.

2. Some results on Orlovsky's choice set

The initial idea behind transitivity is that the strenght of the link between two elements must be greater than or equal to the strength of any chain involving other elements, in such a way that the shorter the chain, the stronger the relation (see Dubois-Prade |3|). In particular, given any binary operation $* : [0,1] \times [0,1] \to [0,1]$, a fuzzy relation μ on X is said max-* transitive if and only if $\mu(x,z) \geq \mu(x,y) * \mu(y,z)$ for all $x,y,z \in X$. But other concepts of transitivy can be defined (for example, strict transitiveness is verified if and only if $\mu(x,z) \geq \mu(z,x)$ holds when $\mu(x,y) \geq \mu(y,x)$ and $\mu(y,z) \geq \mu(z,y)$ hold).

Given a fuzzy preference relation μ on X, Orlovsky's idea [6] is to define its fuzzy strict preference relation μ^s such that $\mu^s(x,y) = \max(\mu(x,y) - \mu(y,x), 0)$, in such a way that we can look for the choice set of unfuzzy nondominated alternatives $X^{UND} = \{x \in X / \mu^{ND}(x) = 1\}$, being $\mu^{ND}(x) = 1 - \mu^D(x)$, where $\mu^D(x) = \sup\{\mu^s(y,x) / y \in X\}$ represents the degree to which alternative x is dominated in X. His main result is that max-min transitivity $(\mu(x,z) \geq \min\{\mu(x,y), \mu(y,z)\}\ \forall x,y,z \in X)$ is enough to assure non-emptiness of X^{UND}. Some subsets of X^{UND} will be of special significance in decision making:

$$X^{UNDud} = \{x \in X^{UND} / \mu^s(x,y) > 0 \quad \forall y \notin X^{UND}\}$$

$$X^{UNDd} = \{x \in X^{UND} / \exists\, y \notin X^{UND},\ \mu^s(x,y) > 0\}$$

$$X^{UNDnd} = \{x \in X^{UND} / \mu^s(x,y) = 0 \quad \forall y \in X\}$$

and it is clear that $X^{UND} = X^{UNDd} \cup X^{UNDnd}$.

Theorem 1. *Let μ be a fuzzy relation on X verifying non-symmetry (i.e., $\mu(x,y) \neq \mu(y,x)$ for some $x,y \in X$). Then the following implications hold:*

i) μ max-min transitive $\Rightarrow X^{UNDd} \neq \emptyset$

ii) μ strictly transitive $\Rightarrow X^{UND} = X^{UNDud} \neq \emptyset$.

Proof: Let us consider $Y = X - X^{UNDnd}$, which is non-empty due to non-symmetry. It is clear that μ defines a max-min transitive fuzzy relation on Y, in such a way that $Y^{UND} \neq \emptyset$. Since $Y^{UND} = X^{UNDd}$ it follows i). Property ii) is obvious since a complete crisp preorder can be defined under strict transitiveness xRy if and only if $\mu(x,y) \geq \mu(y,x)$).

Moreover, max-min transitivity plays a threshold role inside the family of max-* transitivities:

Theorem 2. *Let us define the following families of binary operation in $[0,1]$:*

$$O^{min} = \{* / a*b \leq \min(a,b) \quad \forall a,b \in [0,1]\}$$

$$O_{min} = \{* / a*b \geq \min(a,b) \quad \forall a,b \in [0,1]\}$$

and let us suppose $* \in O^{min} \cup O_{min}$ being monotonous (i.e., $a*b \geq c*d$ $\forall a \geq c$, $b \geq d$). If $X^{UNDd} \neq \emptyset$ for any reflexive ($\mu(x,y) = 1\ \ \forall x$) non-symmetric and max- transitive fuzzy relation on X, then it must be* $a*b = \min(a,b)$ $\forall a,b \in [0,1]$.

Proof: On the one hand, if $* \in O^{min}$ and we suppose $a*b = \min(a,b)$ $\forall a \neq b$, then $a*b = \min(a,b)$ $\forall a,b$ holds ($0*0 = 0$ obviously, and $a \geq a*a \geq a*b = b$ $\forall b < a$ in such a way that $a*a = a$ $\forall a$). Therefore, if there exist $a \neq b$ verifying $a*b < \min(a,b)$, denoting $X = \{x_1 \ldots x_n\}$ we can define

$$\mu(x_i,x_j) = \begin{cases} \min(a,b) & \text{if } j < i \\ 1 & \text{if } j = i \\ a*b & \text{if } j = i+1 \\ \max(a,b) & \text{if } j > i+1 \end{cases}$$

in such a way that the fuzzy relation μ verifies reflexivity and max-* transitivity, but $X^{UND} = \emptyset$ (see Montero-Tejada [5] for a proof). On the other hand, if $* \in O_{min}$ and we suppose $\mu(x,y) = a < 1$ for some $x,y \in X$ ($X^{UNDd} = \emptyset$ otherwise) then $a \geq \mu(x,x) * \mu(x,y) = 1*a \geq b*a \geq a$ $\forall b \geq a$, and therefore $b*a = a$ $\forall b \geq a$. Analogously, $a*b = a$ $\forall b \geq a$, in such a way that $a*b = \min(a,b)$ $\forall a,b$.

3. Fuzzy Acyclity

The idea of assigning an unfuzzy set of nondominated alternatives is clearly related to the concept of Choice Function. A choice function on a set of alternatives X is a mapping that defines a non-empty choice set for every non-empty subset of X. A classical result (see for example Pattanaik [7]) states that a necessary and sufficient condition for a given weak crisp relation R defined on a finite set X to generate a non-empty maximal set for every non-empty subset of X is that the strict crisp preference P (xPy if and only if xRy but not yRx) is acyclic (i.e., there is no cycle $x_1Px_2Px_3 \ldots x_nPx_1$ with elements in X). And quasitransitivity (xRz holds if xRy and yPz hold) is a sufficient condition. Such crisp concepts suggest us the concepts of fuzzy acyclity and fuzzy quasitransitivity:

Definition 1. Let μ be a fuzzy relation on X. We will say that μ is acyclic if and only if for any given chain $(x_1,\ldots,x_n)$ of elements in X verifying $\mu(x_i,x_{i+1}) > \mu(x_{i+1},x_i)$ $\forall i$, then $\mu(x_1,x_n) \geq (x_n,x_1)$ holds.

In other words, if $\mu^s(x_i,x_{i+1}) > 0$ holds for each i in a given chain, then $\mu^s(x_n,x_1) = 0$ must be true. Such a concept plays an special role when the unfuzzy set of nondominated alternatives is to be defined:

Theorem 3. *Let μ be a fuzzy relation defined on X. Then $Y^{UND} \neq \emptyset$ for all non-empty subset Y if and only if μ is acyclic.*

Proof: On the one hand, let us suppose μ non-acyclic. Then we can find a chain $\{x_1,\ldots,x_n\}$ of elements in X such that $\min\{\mu^s(x_1,x_2),\ldots,\mu^s(x_{n-1},x_n)\} > 0$ and $\mu^s(x_n,x_1) > 0$. Therefore $\{x_1,\ldots,x_n\}^{UND} = \emptyset$ holds. On the other hand, let us suppose a non-empty subset Y such that $Y^{UND} = \emptyset$. Then for any given $x \in Y$ there is an element $y \in Y$ such that $\mu^s(y,x) > 0$. Since Y is finite, fixed an arbitrary $x_1 \in Y$ we can define a chain $\{x_1,\ldots,x_n\}$ of elements in Y such that $\mu^s(x_i,x_{i+1}) > 0$ for each i and $\mu^s(x_n,x_k) > 0$ for some $k \in \{1,\ldots,n-1\}$. Therefore, $\min\{\mu^s(x_k,x_{k+1}),\ldots,\mu^s(x_{n-1},x_n)\} > 0$ and $\mu^s(x_n,x_k) > 0$, against acyclity.

Definition 3. A fuzzy relation μ on X is said quasitransitive if and only if $\mu(x,z) \geq \mu(z,x)$ holds when $\mu(x,y) \geq \mu(y,x)$ and $\mu(y,z) > \mu(z,y)$ hold.

Theorem 4. *Given a fuzzy relation μ on X, a sufficient condition for assuring $X^{UND} \neq \emptyset$ is that quasitransitivity is verified.*

Proof: Trivial, since quasitransitivity implies acyclity.

It must be pointed out that the crisp weak relation R such that xRy if and only if $\mu^s(y,x) = 0$ (i.e., $\mu(x,y) \geq \mu(y,x)$) can be defined, in such a way that strict crisp relation xPy holds if and only if $\mu^s(x,y) > 0$ (i.e., $\mu(x,y) > \mu(y,x)$). Therefore, a fuzzy relation verifies (fuzzy) acyclity if and only if its associated strict crisp relation P verifies (crisp) acyclity.

4. Other concepts of choice set

Since acyclity may be violated by individuals, other treatments are needed in order to get a choice set. Some families of choice sets have been proposed in the literature:

i) $X^{ND} = \{x \in X / \mu^{ND}(x) = \max\{\mu^{ND}(y)/y \in X\}\}$ (Orlovsky [6]).

ii) $X^{D(\alpha)} = \{x \in X/\mu(x,y) \geq \alpha,\ \forall y \in X\}$ (Dutta et al. [4]).

iii) $X^G = \{x \in X/\min\{\mu(x,z)/z \in X\} \geq \min\{\mu(y,z)/z \in X\}\ \ \forall y \in X\}$ (Basu [1]).

The set described on i) is assured to be non-empty, but it can be improved by considering

$$X^{ND\delta} = \{x \in X/\mu^{ND}(x) \geq \max\{\mu^{ND}(y)/y \in X\} - \delta\}$$

for a fixed $\delta \in [0,1]$. In this way, slight errors in the definition of μ are taken into account. ii) is easily empty for significative values of α, and on the contrary, iii) is easily the hole set X. In this sense, the last two choice sets are often not useful to the decision-maker (it mus be pointed out that they are underestimating much of the information given by μ).

In any case, when there is no unfuzzy nondominated alternative, the degrees to which each alternative is dominated can be applied in order to discard successively alternatives. In other words, given a set of alternatives we propose to reject the non-acceptable alternatives, and only the remaining alternatives must be considered in the next step. Here is the proposed algorithm, for a given set X of alternatives:

STEP 1: Consider the initial remaining set of alternatives
$Y = X$

STEP 2: Define the set D to be discarded in Y:
$D = \{x \in Y/\mu^D(x) = \max\{\mu^D(y)/y \in Y\}\}$

STEP 3: Repeat step 2, replacing Y by the new remaining set $Y-D$, until such a set is empty.

By this method, we obtain a finite family of successive discarded sets $\{D_1,D_2,\ldots,D_m\}$, and the last one $X^{NSD} = D_m$ will be our choice set.

This successive discarding method can be improved by associating to each non-discarded alternative the decreasing family of degrees to which each alternative is dominated by other elements in the remaining set, in such a way that we can define succesive new discarded sets containing only maximum alternatives under the lexicographic order (in other words, given the remaining set $Y = \{y_1,\ldots,y_n\}$, for each alternative $y_i \in Y$ we define the vector $\mu^i = (\mu_1^i,\ldots,\mu_n^i)$ where $\mu_j^i = \mu^s(y_j,y_i)$, and its associated vector $\mu^{(i)} = (\mu_1^{(i)},\ldots,\mu_n^{(i)})$ whose components are arranged in decreasing order; then the vector $\mu^{(i)}$ is lexicographically smaller than a vector $\mu^{(j)}$ if there is some integer $1 \leq m \leq n$ such that $\mu_k^{(i)} = \mu_k^{(j)}$ for $1 \leq k < m$ and $\mu_m^{(i)} < \mu_m^{(j)}$).

Example

Let us considerer the decision-making problem (X,μ) where $X = \{x_1,x_2,x_3,x_4\}$ and μ is a fuzzy relation with its strict fuzzy preference given by the following matrix:

$$(\mu^s_{ij})_{i,j} = (\mu^s(x_i,x_j))_{i,j} = \begin{pmatrix} 0 & 0 & 1 & 0.3 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0 \end{pmatrix}$$

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...y orderings. Inf. Sci. 3,

It is clear that $X^{UND} = \emptyset$ and $X^{ND} = \{x_1, x_2\}$. But when the above algorithm is applied, we obtain x_3 as the first discarded alternative $(\mu^S(x_1, x_3) = 1)$ and therefore it must be removed $(D_1 = \{x_3\})$; if we consider only comparisons between elements in $\{x_1, x_2, x_4\}$, we get $D_2 = \{x_1\}$ as the following discarded set $(\mu^S(x_2, x_1) = 0.5)$. When the set $\{x_2, x_4\}$ is considered, $D_3 = \{x_2\}$ $(\mu^S(x_4, x_2) = 0.2)$, and obviously $D_4 = \{x_4\}$. Therefore, $X^{NSD} = \{x_4\}$ is the proposed choice set under the successive discarding method.

5. Final remarks

Another alternative methods have been studied by the authors in |5|. For example, we can consider randomization between alternatives in $X = \{x_1, \ldots, x_n\}$ and define a fuzzy preference relation over the set of probability distributions

$$\bar{X} = \{p = (p_1, \ldots, p_n)' / \sum_{i=1}^{n} p_i = 1 \ , \ p_i \geq 0 \quad \forall i\} \ .$$

As above, we are interested in assuring the existence of some unfuzzy nondominated distribution. In particular, we can consider fuzzy preference relations $\bar{\mu}$ on $\bar{X}$ such that $\bar{\mu}(p^i, q^j) = \mu(x_i, x_j)$ for each pair of degenerate distributions $(p_i^i = q_j^j = 1)$. In this way $\bar{\mu}$ becomes an extension of μ.

Although other extensions are possible, in Montero- Tejada |5| was proposed the linear extension

$$\mu_L(p,q) = \sum_{i,j} p_i \cdot q_j \cdot \mu(x_i, x_j)$$

for all $p, q \in \bar{X}$. It can be understood as a natural extension, $\mu_L(p,q)$ meaning the expected degree of preference of distribution p over distribution q, and it can be axiomatically justified. The main result obtained in |5| is that such a linear extension allows us to assure the non-emptiness of the set of unfuzzy nondominated distributions, in such a way that this approach can be applied to solve many of the decision-making problems.

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