

# On the varieties of nilpotent Lie algebras of dimension 7 and 8

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## Abstract

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Let  $N^n$  be the variety of  $n$ -dimensional complex nilpotent Lie algebras. We know that this algebraic variety is reducible for  $n \geq 11$  and irreducible for  $n \leq 6$ . In this work we prove that  $N^7$  is composed of two algebraic components and that  $N^8$  is also reducible.

## 1. The variety $N^n$ . Generalities

### 1.1.

Let  $L^n$  (resp.  $N^n$ ) be the set of Lie algebra laws (resp. nilpotent Lie algebra laws) in dimension  $n$ . We identify each law  $\mu$  with its structural constant  $C_{jk}^i$  in a fixed basis  $(e_i)$  of  $\mathbb{C}^n$ . The Jacobi identities

$$\sum_{l=1}^n C_{ij}^l C_{kl}^s + C_{jk}^l C_{il}^s + C_{ki}^l C_{jl}^s = 0 \quad (1 \leq i \leq j < k \leq n, 1 \leq s \leq n)$$

give us  $L^n$ , parametrized by the  $(n^3 - n^2)/2$  constants  $C_{jk}^i$ , an algebraic variety structure. Hence  $N^n$  is closed in  $L^n$ . The linear group  $\text{Gl}(n, \mathbb{C})$  acts on  $L^n$ , this action corresponds with a change of basis of the space  $\mathbb{C}^n$ .

Let  $\mathcal{O}(\mu)$  be the orbit of  $\mu$ . It is a regular subvariety of  $L^n$  (or of  $N^n$  if  $\mu \in N^n$ ), and its tangent space in the point  $\mu$  is just the linear space  $B^2(\mu, \mu)$  of the

2-coboundaries in the Chevalley cohomology associated to  $\mu$ . We endow  $L^n$  with the metric topology (or the induced topology as subset of  $\mathbb{C}^{(n^3-n^2)/2}$ ).

**Definition.** If the orbit  $\theta(\mu)$  of the law  $\mu$  is open in  $L^n$  (resp.  $N^n$ ), then the law  $\mu$  is rigid in  $L^n$  (resp.  $N^n$ ).

Since we work in the complex field, the closure of  $\theta(\mu)$  in the metric topology coincides with the closure in the Zariski topology of  $L^n$ . We see that *if  $\mu$  is rigid, the closure of  $\theta(\mu)$  is an irreducible component of  $L^n$  (or of  $N^n$  if  $\mu \in N^n$ , but in this case  $\mu$  may not be rigid in  $L^n$ )*.

### 1.2.

Take  $\mu_0 \in L^n$  and a sequence  $(f_p)_{p \in \mathbb{N}}$  in  $\text{Gl}(n, \mathbb{C})$ . We define a new sequence  $\mu_{0,p} = f_p^{-1} \circ \mu_0 \circ (f_p, f_p)$ .

If the limit  $\lim_{p \rightarrow \infty} \mu_{0,p}$  exists, we get a point in  $L^n$  which is in the closure of  $\theta(\mu_0)$ . We call it the *contraction* of  $\mu_0$ .

We use now the nonstandard setting in the theory I.S.T. of internal sets and we suppose  $n$  standard. Consider  $\mu_0$  standard in  $L^n$ .

**Definition.** A *perturbation*  $\mu$  of  $\mu_0$  is a law  $\mu \in L^n$  such that

$$\mu(e_i, e_j) \sim \mu_0(e_i, e_j) \quad i \leq j \leq n,$$

where  $(e_i)$  is the standard fixed basis of  $\mathbb{C}^n$ .

One checks easily that:

(a) If  $\mu_1$  is standard in  $L^n$  and  $\mu$  is a perturbation of  $\mu_1$ , then  $\mu_1$  is a contraction of  $\mu_0$  for all  $\mu_0$  standard isomorphic to  $\mu$ .

(b) Consider  $\mu_0$  standard in  $L^n$ . Then  $\mu_0$  is rigid if and only if all perturbations  $\mu$  of  $\mu_0$  is in  $\theta(\mu_0)$  (see [7]).

(c) The concept of perturbation corresponds to the classic concept of deformation when this is represented by a convergent series with small parameter.

### 1.3.

Consider a complex nilpotent Lie algebra  $g = (\mathbb{C}^n, \mu)$  whose law  $\mu$  is in  $N^n$ . For each  $X \in \mathbb{C}^n$  we denote by  $s(X)$  the ordered sequence of degrees of irreducible factors of the minimal polynomial of the nilpotent operator  $\text{ad}_\mu(X)$ .

**Definition.** The *characteristic sequence* of  $g$  is an isomorphism invariant  $s(g)$  (or  $s(\mu)$ ) defined by

$$s(g) = \sup \{s(X) \mid X \in g - \mathcal{C}^1(g)\},$$

where  $\mathcal{C}^1(g)$  is the first derived algebra of  $g$  and we order the  $s(X)$  lexicographically.

**Lemma.** *For each standard  $\mu_0 \in N^n$  and each  $\mu$  perturbation of  $\mu_0$ , we get  $s(\mu) \geq s(\mu_0)$ .  $\square$*

In fact, take  $X$  standard in  $g_0 - \mathcal{C}^1(g_0)$ , where  $g_0$  (resp.  $g$ ) is  $(\mathbb{C}^n, \mu_0)$  (resp.  $(\mathbb{C}^n, \mu)$ ). If  $X \in g - \mathcal{C}^1(g)$ , since  $\text{ad}_\mu(X)$  is a perturbation of  $\text{ad}_{\mu_0}(X)$ , we get  $s(\mu) \geq s(\mu_0)$ . If  $X \notin g - \mathcal{C}^1(g)$ , there is  $\tilde{X}$  in this cone such that  $\tilde{X}$  is close enough to  $X$  and the characteristic sequence of  $\text{ad}_\mu(\tilde{X})$  is at least equal to that of  $\text{ad}_{\mu_0}X$  and the lemma follows.

**Corollary.** *The set  $U_s^n = \{\mu \in N^n \mid s(\mu) \geq s\}$  is open in  $N^n$ .  $\square$*

For instance, if  $s$  is maximal, that is,  $s = (n - 1, 1)$ , then  $U_s^n$  is the open set of filiform laws in  $N^n$ .

We will use the notation  $N_s^n = \{\mu \in N^n \mid s(\mu) = s\} = \delta U_s^n$ .

## 2. The variety $N^7$

### 2.1. The reducibility of $N^7$

Let us consider two families  $\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}}$  and  $\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}}$  of laws in  $N^7$  given by

$$\begin{cases} \mu_\alpha^1(X_1, X_i) = X_{i-1}, & i = 3, 4, 5, 6, 7, \\ \mu_\alpha^1(X_4, X_7) = \alpha X_2, & \mu_\alpha^1(X_5, X_6) = X_2, \\ \mu_\alpha^1(X_5, X_7) = (1 + \alpha)X_3, & \mu_\alpha^1(X_6, X_7) = (1 + \alpha)X_4, \end{cases} \quad \alpha \in \mathbb{C},$$

and

$$\begin{cases} \mu_\alpha^2(X_1, X_i) = X_{i-1}, & i = 4, 5, 6, 7, \\ \mu_\alpha^2(X_2, X_6) = X_3, & \mu_\alpha^2(X_2, X_7) = X_3 + X_4, \\ \mu_\alpha^2(X_5, X_7) = \alpha X_3, & \mu_\alpha^2(X_6, X_7) = \alpha X_4 + X_2, \end{cases} \quad \alpha \in \mathbb{C},$$

where the nondefined products are supposed to be zero.

In [3] the following proposition was shown:

**Proposition.** *If  $\alpha \neq \alpha'$  (resp.  $\alpha \neq \pm \alpha'$ ), then the laws  $\mu_\alpha^1$  and  $\mu_{\alpha'}^1$  (resp.  $\mu_\alpha^2$  and  $\mu_{\alpha'}^2$ ) are not isomorphic.  $\square$*

**Main Theorem.** *The closures of the orbits  $\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})$  and  $\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})$  are the two only irreducible components in  $N^7$ . These components have dimension 40.*

The following subsection gives the proof. Now, let us state, as an immediate consequence of the preceding subsection the following result:

**Proposition.** *The families  $\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}}$  and  $\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}}$  are rigid in the following sense: Any perturbation  $\mu$  of a standard law  $\mu_\alpha^1$  (resp.  $\mu_\alpha^2$ ) is isomorphic to a law in  $\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}}$  (resp.  $\{\mu_\alpha^2\}$ ). Moreover, there is no contraction between different standard members of these families.  $\square$*

## 2.2. Proof of the main theorem

(i) We study the open set of filiform laws  $N_{(6,1)}^7 = U_{(6,1)}^7$ .

Take  $\mu \in N_{(6,1)}^7$ . There is  $X_1 \in g - \mathcal{C}^1(g)$ , where  $g = (\mathbb{C}^7, \mu)$ , such that  $\mu(X_1, X_i) = X_{i-1}$ ,  $i = 3, 4, 5, 6, 7$ , the vectors  $X_i$  being a basis of  $\mathbb{C}^7$ . We deduce that  $N_{(6,1)}^7$  is the orbit of the 4-space given by the laws

$$\begin{cases} \mu(X_1, X_i) = X_{i-1}, & 3 \leq i \leq 7, \\ \mu(X_4, X_7) = \beta_1 X_2, \\ \mu(X_5, X_6) = \beta_2 X_2, \\ \mu(X_5, X_7) = \beta_3 X_2 + (\beta_1 + \beta_2) X_3, \\ \mu(X_6, X_7) = \beta_4 X_2 + \beta_3 X_3 + (\beta_1 + \beta_2) X_4. \end{cases}$$

Since the laws corresponding to  $\beta_2 \neq 0$  are in  $\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})$  we get  $N_{(6,1)}^7 \subset \theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})$  and  $\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}}$  is rigid.

**Consequence.** The closure of  $\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})$  is an irreducible component containing the open set  $N_{(6,1)}^7$ . The dimension equals  $\dim(B^2(\mu_\alpha^1, \mu_\alpha^1)) + 1$ , that is, 40.

**Remark.** We have used Personal Computer for the cohomological calculus [8].

(ii) Study of  $N_{(5,1,1)}^7$ .

Every law in  $N_{(5,1,1)}^7$  is isomorphic to a law given by

$$\begin{cases} \mu(X_1, X_i) = X_{i-1}, & 4 \leq i \leq 7, \\ \mu(X_2, X_5) = \beta_1 X_3, \\ \mu(X_2, X_6) = \beta_1 X_4 + \beta_2 X_3, \\ \mu(X_2, X_7) = \beta_1 X_5 + \beta_2 X_4 + \beta_3 X_3, \\ \mu(X_4, X_7) = \beta_4 X_2, \\ \mu(X_5, X_6) = \beta_5 X_3 - \beta_4 X_2, \\ \mu(X_5, X_7) = \beta_5 X_4 + \beta_6 X_3, \\ \mu(X_6, X_7) = \beta_5 X_5 + \beta_6 X_4 + \beta_7 X_3 + \beta_8 X_2, \end{cases}$$

with conditions

$$\beta_1\beta_4 = \beta_2\beta_4 = \beta_3\beta_4 = \beta_1\beta_8 = \beta_1\beta_5 = 0.$$

**Proposition.** *The set  $N_{(5,1,1)}^7$  is the union of orbits of three subspaces with dimension 5 or 6.*

If we denote by  $\mu(\beta_1, \dots, \beta_8)$  a law in the preceding family, we can define three subspaces in  $N^7$  by

$$P_1 = \{ \mu(\beta_1, \dots, \beta_8) \mid \beta_1 = \beta_2 = \beta_3 = 0 \},$$

$$P_2 = \{ \mu(\beta_1, \dots, \beta_8) \mid \beta_1 = \beta_4 = 0 \},$$

$$P_3 = \{ \mu(\beta_1, \dots, \beta_8) \mid \beta_4 = \beta_5 = \beta_8 = 0 \}.$$

We have  $N_{(5,1,1)}^7 = \bigcup_{i=1}^3 P_i$ .

One checks that:

$$(a) \theta(P_1) \subset \overline{\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})}, \theta(P_3) \subset \overline{\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})},$$

$$(b) \{\mu_\alpha^2\}_{\alpha \in \mathbb{C}} \subset P_2,$$

$$(c) \theta(P_2) \subset \overline{\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})},$$

$$(d) \dim B^2(\mu_\alpha^2, \mu_\alpha^2) = 39, \text{ for all } \alpha \in \mathbb{C} \text{ and } \{\mu_\alpha^2\}_{\alpha \in \mathbb{C}} \text{ is rigid.}$$

This implies that the closure of  $\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})$  is an irreducible component of dimension 40 and that

$$U_{(5,1,1)}^7 \subset \overline{\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})} \cup \overline{\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})}.$$

**Remark.** For all  $\alpha \in \mathbb{C}$ , the laws  $\mu_\alpha^2$  are characteristically nilpotent (i.e. all derivations are nilpotents).

(iii) *Study of  $N_{(4,1,1,1)}^7$ .*

Every law in  $N_{(4,1,1,1)}^7$  is isomorphic to a law given by

$$\left\{ \begin{array}{l} \mu(X_1, X_i) = X_{i-1}, \quad i = 5, 6, 7, \\ \mu(X_2, X_3) = \alpha_1 X_4, \\ \mu(X_2, X_6) = \alpha_2 X_4, \\ \mu(X_2, X_7) = \alpha_3 X_4 + \alpha_2 X_5, \\ \mu(X_3, X_7) = \alpha_4 X_4, \\ \mu(X_6, X_7) = \alpha_5 X_2 + \alpha_6 X_3 + \alpha_7 X_4, \end{array} \right.$$

with Jacobi conditions

$$\alpha_1\alpha_5 = \alpha_1\alpha_6 = \alpha_2\alpha_5 = \alpha_2\alpha_6 = 0.$$

If we denote by  $\mu(\alpha_1, \dots, \alpha_7)$  a member of this family, we have:

$$N_{(4,1,1,1)}^7 = \theta(P_4) \cup \theta(P_5)$$

with

$$P_4 = \{ \mu(\alpha_1, \dots, \alpha_7) \mid \alpha_1 = \alpha_2 = 0 \} ,$$

$$P_5 = \{ \mu(\alpha_1, \dots, \alpha_7) \mid \alpha_5 = \alpha_6 = 0 \} .$$

One can also check that  $\theta(P_4) \subset \overline{\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})}$  and  $\theta(P_5) \subset \overline{\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})}$ .

(iv) *Study of  $N_{(4,2,1)}^7$ .*

Every law in this set is isomorphic to the law

$$\left\{ \begin{array}{l} \mu(X, X_i) = X_{i-1} , \quad i = 3, 4, 5, 7 , \\ \mu(X_3, X_4) = \alpha_1 X_2 , \\ \mu(X_3, X_5) = \alpha_1 X_3 + \alpha_2 X_6 , \\ \mu(X_3, X_7) = \alpha_3 X_2 , \\ \mu(X_4, X_5) = \alpha_1 X_4 + \alpha_2 X_7 + \alpha_4 X_2 + \alpha_5 X_6 , \\ \mu(X_4, X_6) = \alpha_6 X_2 , \\ \mu(X_4, X_7) = (\alpha_3 + \alpha_6) X_3 + \alpha_7 X_2 , \\ \mu(X_5, X_6) = \alpha_6 X_3 + \alpha_8 X_2 + \alpha_1 X_6 , \\ \mu(X_5, X_7) = (\alpha_3 + 2\alpha_6) X_4 + (\alpha_7 + \alpha_8) X_3 + \alpha_1 X_7 + \alpha_9 X_2 + \alpha_{10} X_6 , \\ \mu(X_6, X_7) = \alpha_{11} X_2 , \end{array} \right.$$

with

$$\begin{aligned} \alpha_1^2 - \alpha_2 \alpha_6 &= \alpha_2(\alpha_6 + \alpha_3) = \alpha_1(\alpha_3 + \alpha_6) = 2\alpha_1^2 + \alpha_2(\alpha_3 - \alpha_6) \\ &= \alpha_1 \alpha_6 - \alpha_2 \alpha_{11} = 2\alpha_1 \alpha_6 + \alpha_3 \alpha_1 - \alpha_{11} \alpha_2 \\ &= 2\alpha_1^2 - 2\alpha_{11} \alpha_1 = \alpha_1(2\alpha_7 + \alpha_8) - \alpha_6 \alpha_{10} - \alpha_7 \alpha_1 + \alpha_{11} \alpha_5 = 0 . \end{aligned}$$

Therefore,  $N_{(4,2,1)}^7$  is the union of the orbits of the following subspaces

$$P_6 = \{ \mu(\alpha_1, \dots, \alpha_{11}) \mid \alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = 0 \} ,$$

$$P_7 = \{ \mu(\alpha_1, \dots, \alpha_{11}) \mid \alpha_1 = \alpha_2 = \alpha_6 = \alpha_{11} = 0 \} ,$$

$$P_8 = \{ \mu(\alpha_1, \dots, \alpha_{11}) \mid \alpha_1 = \alpha_3 = \alpha_6 = \alpha_{11} = 0 \} ,$$

and the orbit of the variety

$$V_1 = \left\{ \mu(\alpha_1, \dots, \alpha_{11}) \mid \alpha_6 \neq 0, \alpha_2 = \frac{\alpha_1^2}{\alpha_6}, \alpha_5 = \frac{\alpha_1}{\alpha_6} \alpha_{10} - \frac{\alpha_1^2}{\alpha_6^2} (\alpha_7 + \alpha_8), \right. \\ \left. \alpha_{11} = \frac{\alpha_6^2}{\alpha_1}, \alpha_1 \neq 0 \right\}.$$

One can show by perturbation that  $\theta(P_7)$  and  $\theta(P_6)$  are subsets of  $\overline{N_{(6,1)}^7}$ , and also that  $\theta(P_8) \subset \overline{U_{(5,1,1)}^7}$  and  $V_1 \subset \overline{U_{(5,1,1)}^7}$ .

(v) *Study of  $N_{(3,3,1)}^7$ .*

This set is the orbit of a subspace of dimension 8 given by

$$\left\{ \begin{array}{l} \mu(X_1, X_i) = X_{i-1}, \quad i = 3, 4, 6, 7, \\ \mu(X_3, X_4) = \beta_1 X_2 + \beta_2 X_5, \\ \mu(X_6, X_7) = \beta_7 X_2 + \beta_8 X_5, \\ \mu(X_4, X_6) = \beta_4 X_2, \\ \mu(X_4, X_7) = \beta_5 X_2 + \beta_6 X_5 + \beta_4 X_3 + \beta_3 X_6, \\ \mu(X_3, X_7) = \beta_3 X_5. \end{array} \right.$$

This subspace is contained in the union of the closures of the laws corresponding to  $(\beta_1, \dots, \beta_8) = (0, 0, 0, 0, 1, 0, 0, 0)$ ,  $(1, 1, 1, 0, 0, 0, 0, 1)$  and  $(0, 1, 0, 0, 0, 0, 1, 1)$ . Since the first of these laws can be perturbed to an element of  $N_{(6,1)}^7$  and the other two in  $N_{(4,2,1)}^7$ , we get that

$$N_{(3,3,1)}^7 \subset \overline{\theta(\{\mu_\alpha^1\}_{\alpha \in \mathbb{C}})} \cup \overline{\theta(\{\mu_\alpha^2\}_{\alpha \in \mathbb{C}})}.$$

(vi) *Study of  $N_{(3,2,1,1)}^7$ ,  $N_{(3,1,1,1,1)}^7$ ,  $N_{(2,2,2,1)}^7$ ,  $N_{(2,2,1,1,1)}^7$ ,  $N_{(2,1,1,1,1,1)}^7$  and  $N_{(1,1,1,1,1,1,1)}^7$ .*

Opposite to the preceding cases, it is harder sometimes to describe these sets as union of orbits of an algebraic subvariety with a simple system of equations. We shall give for each representative of an isomorphism class in this set a perturbation and the associated characteristic sequence.

**Lemma.** Every standard law  $\mu_0$  in  $\bigcup_{i \leq (3,2,1,1)} N_i^7$  admits a perturbation  $\mu$  such that  $s(\mu) \geq (3, 3, 1)$ .

**Proof.** The notations are those of [3]. If  $\mu_0 \in \bigcup_{i \leq (3,2,1,1)} N_i^7$ , we define a perturbation  $\mu$  of length 1 (see [6]) by  $\mu = \mu_0 + \varepsilon \varphi$ , where  $\varepsilon$  is a small infinitesimal. We recall [3] the elements  $\mu_0 \in \bigcup_{i \leq (3,2,1,1)} N_i^7$  corresponding to laws  $n_j^7$  with  $j \geq 87$ ; see Table 1.

Table 1

law	perturbation $\mu = \mu_0 + \varepsilon\varphi$	$s(\mu)$
$n_{87}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{88}^7$	$\varphi(X_1, X_3) = X_2$	(3, 3, 1)
$n_{89}^7$	$\varphi(X_1, X_5) = X_4$	(6, 1)
$n_{90}^7 \rightarrow n_{93}^7$	$\varphi(X_1, X_3) = X_2$	(3, 3, 1)
$n_{94}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{95}^7$	$\varphi(X_1, X_5) = X_4$	(5, 1, 1)
$n_{96}^7$	$\varphi(X_1, X_2) = X_7$	(4, 2, 1)
$n_{97}^7 \rightarrow n_{99}^7$	$\varphi(X_1, X_2) = X_4$	(3, 3, 1)
$n_{100}^7$	$\varphi(X_1, X_2) = X_7$	(4, 2, 1)
$n_{101}^7$	$\varphi(X_1, X_4) = X_2$	(3, 3, 1)
$n_{102}^7$	$\varphi(X_1, X_2) = X_3$	(3, 3, 1)
$n_{103}^7, n_{104}^7$	$\varphi(X_1, X_4) = X_2$	(3, 3, 1)
$n_{105}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{106}^7, n_{107}^7$	$\varphi(X_2, X_3) = X_5$	(4, 1, 1, 1)
$n_{108}^7$	$\varphi(X_1, X_2) = X_7$	(4, 2, 1)
$n_{109}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{118}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{123}^7 \rightarrow n_{126}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
$n_{127}^7, n_{128}^7$	$\varphi(X_2, X_4) = X_6$	(4, 1, 1, 1)
$n_{133}^7$	$\varphi(X_1, X_4) = X_7$	(4, 1, 1, 1)
	$\varphi(X_1, X_5) = X_3$	(3, 2, 1, 1)
	$\varphi(X_1, X_2) = X_7$	(3, 2, 1, 1)
	$\varphi(X_1, X_2) = X_7$	(3, 2, 1, 1)

The nonwritten laws are products and so, by [5], they are among the components previously determined.

This ends the proof of the theorem.  $\square$

### 3. The variety $N^8$

The classification of nilpotent laws in dimension  $\leq 7$  is known but this is not the case in dimension 8. In fact, at this moment, only the classification of filiform laws has been studied. Nevertheless, we try to determine the irreducible component in  $N^8$  intersecting  $U_{(7,1)}^8$ .

**Lemma.** *The open  $U_{(7,1)}^8$  is the orbit of the family*



$$\left\{ \begin{array}{l} \mu(X_1, X_i) = X_{i-1}, \quad 3 \leq i \leq 8, \\ \mu(X_3, X_8) = a_1 X_2, \\ \mu(X_4, X_7) = a_2 X_2, \\ \mu(X_4, X_8) = (a_1 + a_2) X_3 + a_3 X_2, \\ \mu(X_5, X_6) = a_4 X_2, \\ \mu(X_5, X_7) = (a_2 + a_4) X_3 + a_5 X_2, \\ \mu(X_5, X_8) = (a_1 + 2a_2 + a_4) X_4 + (a_3 + a_5) X_3 + a_6 X_2, \\ \mu(X_6, X_7) = (a_2 + a_4) X_4 + a_5 X_3 + a_7 X_2, \\ \mu(X_6, X_8) = (a_1 + 3a_2 + 2a_4) X_5 + (a_3 + 2a_5) X_4 + (a_6 + a_7) X_3 \\ \quad + a_8 X_2, \\ \mu(X_7, X_8) = (a_1 + 3a_2 + 2a_4) X_6 + (a_3 + 2a_5) X_5 + (a_6 + a_7) X_4 \\ \quad + a_8 X_3 + a_9 X_2, \end{array} \right.$$

with the Jacobi conditions  $a_2 + a_4 = 0$  and  $a_2(5a_5 + 2a_3) = 0$ .  $\square$

We find this lemma by a simple computation.

This shows that  $U_{(7,1)}^8$  is a union of orbits of two subspaces defined by  $a_2 = a_4 = 0$  and by  $a_2 + a_4 = 5a_5 + 2a_3 = 0$ .

**Theorem.** *The open set of filiforms  $N_{(7,1)}^8$  is contained in the union of two irreducible components.*

**Proof.** One considers the two families  $\mu^1(\alpha)$  and  $\mu^2(\alpha)$  defined by

$$\left\{ \begin{array}{l} \mu^1(X_1, X_i) = X_{i-1}, \quad 3 \leq i \leq 8, \\ \mu^1(X_4, X_8) = \alpha X_2, \\ \mu^1(X_5, X_7) = X_2, \\ \mu^1(X_5, X_8) = (1 + \alpha) X_3 + X_2, \\ \mu^1(X_6, X_7) = X_3, \\ \mu^1(X_6, X_8) = (2 + \alpha) X_4 + X_3, \\ \mu^1(X_7, X_8) = (2 + \alpha) X_5 + X_4, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \mu^2(X_1, X_i) = X_{i-1}, \quad 3 \leq i \leq 8, \\ \mu^2(X_4, X_7) = X_2, \\ \mu^2(X_4, X_8) = X_3 + X_2, \\ \mu^2(X_5, X_6) = -X_2, \\ \mu^2(X_5, X_7) = (-2/5)X_2, \\ \mu^2(X_5, X_8) = X_4 + (3/5)X_3, \\ \mu^2(X_6, X_7) = (-2/5)X_3, \\ \mu^2(X_6, X_8) = X_5 + (1/5)X_4 + X_3 + \alpha X_2, \\ \mu^2(X_7, X_8) = X_6 + (1/5)X_5 + \alpha X_3, \end{array} \right.$$

We cannot perturb any law of type  $\mu^1$  in a law of type  $\mu^2$  and conversely. Then the closure of the orbits of these two one-parameter families are irreducible components. As we have shown that the open of filiforms contains no more than two components, we have proved the theorem.  $\square$

**Theorem.** *The variety  $N^8$  has at least two irreducible components of dimension 55 and so it is reducible.*

**Proof.** This is a direct consequence of the previous theorem. The computation of the dimensions of the components have been done on a micro computer.  $\square$

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