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THE SPACE STRUCTURE NEAR A BLOW-UP POINT FOR SEMILINEAR HEAT EQUATIONS: A FORMAL APPROACH

Рассматриваются положительные решения полулинейных параболических уравнений $u_t-u_{xx}=u^p$, $p>1$, и $u_t-u_{xx}=e^u$ при $-\infty<x<+\infty$, $t>0$, которые обостряются в единственной точке $x=0$ в конечный момент времени $t=T>0$. Посредством формальных методов дано описание типов возможного асимптотического поведения решений при $(x, t)\to(0, T)$.

§ 1. Introduction

In this paper we shall consider positive solutions of the semilinear equations

(1.1a) $\quad u_t-u_{xx}=u^p, \quad -\infty<x<+\infty, \quad t>0, \quad p>1,$

(1.1b) $\quad u_t-u_{xx}=e^u, \quad -\infty<x<+\infty, \quad t>0.$

Since the arguments which follow are very similar for both equations, these will be devoloped in parallel. Formulae and statements corresponding to (1.1a) (resp. to (1.1b)) will be labeled with the letter a (resp. with letter b).

We shall assume that there is a single-point blow-up at $x=0$ and $t=T$, in the sense that

$$\limsup_{t\to T} u(0,t)=+\infty.$$

Moreover, we shall also suppose that the following assumptions hold

(1.2a) $\quad \lim_{t\to T}(T-t)^{1/(p-1)}u(x,t)=(p-1)^{-1/(p-1)}$

uniformly for $|x|\leqslant C(T-t)^{1/2}$ and any $C>0$;

(1.2b) $\quad \lim_{t\to T}[u(x,t)+\ln(T-t)]=0$

uniformly for $|x|\leqslant C(T-t)^{1/2}$ and any $C>0$.

Conditions under which (1.2a) holds can be found in [1] — [6]. As to (1.2b), see [7] — [9]. In some of these papers equations (1.1a) or (1.1b) are considered in bounded domains rather than in the whole line as here. However, we shall keep to this last framework for convenience.

We now proceed to describe our results. These are of a formal nature, since they are derived (without proof) by the method of matched asymptotic expansions. We need to introduce some notations, and to this end, the following changes of variables are performed

(1.3a) $\quad y=x(T-t)^{-1/2}, \quad \tau=-\ln(T-t),$

(1.4a) $\quad u(x, t)=(T-t)^{-1/(p-1)}\Phi(y, \tau),$

$$(1.5a) \qquad \Phi=W^{-1/(p-1)};$$
$$(1.3b) \qquad y=x(T-t)^{-1/2}, \quad \tau=-\ln(T-t),$$
$$(1.4b) \qquad u(x,t)=-\ln(T-t)+\Phi(y,\tau),$$
$$(1.5b) \qquad \Phi=-\ln W.$$

Notice that we are denoting by $W=W(y, \tau)$ the two auxiliary functions in (1.5a), (1.5b). We then obtain that W should solve

$$(1.6a) \qquad W_\tau=W_{yy}-\frac{1}{2}yW_y+W-\frac{p}{p-1}\frac{W_y^2}{W}-(p-1),$$

$$(1.6b) \qquad W_\tau=W_{yy}-\frac{1}{2}yW_y+W-\frac{W_y^2}{W}-1.$$

Assimptions (1.2) read now

$$(1.7a) \qquad \lim_{\tau\to\infty} W(y,\tau)=p-1, \text{ uniformly on sets } |y|\leqslant C,$$

$$(1.7b) \qquad \lim_{\tau\to\infty} W(y,\tau)=1 \text{ uniformly on sets } |y|\leqslant C.$$

Taking into account (1.7), we define $\psi(y, \tau)$ as follows

$$(1.8a) \qquad W(y,\tau)=(p-1)+\psi(y,\tau),$$
$$(1.8b) \qquad W(y,\tau)=1+\psi(y,\tau).$$

By (1.7), $\psi(y, \tau)\to 0$ as $\tau\to\infty$. We shall see that three kinds of possible behaviours are expected for $\psi(y, \tau)$ as $\tau\to\infty$. To this end, let $H_n(y)$ be given by

$$(1.9) \qquad H_n(y)=c_n\tilde{H}_n(y/2),$$

where $c_n=(2^{n/2}(\pi)^{1/4}(n!)^{1/2})^{-1}$ and $\tilde{H}_n(s)$ is the standard n-th Hermite polynomial. Then we obtain that, for (1.1a), $\psi(y, \tau)$ may behave to the first order in one of the following manners

$$(1.10a) \qquad \psi_0(y,\tau)=0 \text{ as } \tau\to\infty,$$

$$(1.11a) \qquad \psi_2(y,\tau)\approx\frac{(4\pi)^{1/4}(p-1)^2}{2^{1/2}p}\frac{H_2(y)}{\tau} \text{ as } \tau\to\infty,$$

$$(1.12a) \qquad \psi_n(y,\tau)\approx k\exp[(1-n/2)\tau]H_n(y) \text{ as } \tau\to\infty, \quad n=3,4,\ldots.$$

Here k is a free constant, depending on the data corresponding to any solution. As to (1.1b), we have

$$(1.10b) \qquad \psi_0(y,\tau)=0 \text{ as } \tau\to\infty,$$

$$(1.11b) \qquad \psi_2(y,\tau)\approx\frac{(4\pi)^{1/4}}{2^{1/2}}\frac{H_2(y)}{\tau} \text{ as } \tau\to\infty,$$

$$(1.12b) \qquad \psi_n(y,\tau)\approx k\exp[(1-n/2)\tau]H_n(y) \text{ as } \tau\to\infty, \quad n=3,4,\ldots;$$

k being a free constant as before. As a matter of fact, we shall derive higher-order expansions corresponding to (1.10)–(1.12) (cf. (4.6), (4.15) below).

Let us briefly remark the previous results. To begin with, (1.10) is satisfied by the explicit solutions

$$u(x,t)=[(p-1)(T-t)]^{-1/(p-1)} \quad (\text{for } (1.1a)),$$
$$u(x,t)=-\ln(T-t) \quad (\text{for } (1.1b)),$$

and we conject that these are the only solutions behaving in this way. The asymptotic behaviour of type (1.11a) was obtained by a different formal method in [10] (for $p=3$) and in [1], [3] for the case $p>1$, and that of type (1.11b) was obtained in [11], cf. references in [5], [6], [12]. In several papers [5], [10], [13] asymptotic behaviour of the type (1.11a) was pointed out by numerical methods. For (1.11b) numerical results are given in [11], [5]. Related rigorous upper bounds can be found in [5] for (1.1a) and in [14], [5] for (1.1b). Recently, the existence of solutions behaving as in (1.11b) has been proved in [8]. In our opinion, the main contribution of this note consists in pointing out the possibility of the behaviours described in (1.12). Notice that our method here is quite different from techniques previously used to obtain the discrete set of non-monotone, self-similar blowing-up solutions of quasilinear heat equations as

$$u_t=\operatorname{div}(u^\sigma Du)+u^\beta \quad \text{and} \quad u_t=\operatorname{div}(|Du|^\sigma Du)+u^\beta,$$

where $\sigma>0$, $\beta>\sigma+1$ are fixed (cf. [15], [16] for the first equation and [17] — for the second one; see also a full list of references in [12, Chapt, IV]). We finally point out that our approach can be applied in a variety of situations. For instance, in [18] we use it to study how nonnegative solutions approach an extinction time in the **semilinear equation**

$$u_t=u_{xx}-u^p, \qquad -\infty<x<+\infty, \quad t>0, \quad 0<p<1.$$

§ 2. First-order expansions

Substituting (1.8) into (1.6), we obtain for ψ

$$\text{(2.1a)} \qquad \psi_\tau=\psi_{yy}-\frac{1}{2}y\psi_y+\psi-\frac{p}{p-1}\frac{(\psi_y)^2}{(p-1)+\psi},$$

$$\text{(2.1b)} \qquad \psi_\tau=\psi_{yy}-\frac{1}{2}y\psi_y+\psi-\frac{(\psi_y)^2}{1+\psi}.$$

We now assume that $\psi(y,\tau)$ can be written as

$$\text{(2.2)} \qquad \psi(y,\tau)=\sum_{n=0}^{\infty}a_n(\tau)H_n(y).$$

We need to introduce some notation. Set

$$L_\omega^2(\mathbb{R})=\left\{f\in L^2_{\text{loc}}(\mathbb{R}):\ \int_{-\infty}^{+\infty}|f(y)|^2e^{-y^2/4}\,dy<+\infty\right\},$$

$$\langle f,g\rangle=\int_{-\infty}^{+\infty}f(y)g(y)e^{-y^2/4}\,dy, \qquad \|f\|^2=\langle f,f\rangle.$$

We also consider the linear operator

$$A\varphi(y)=\varphi''(y)-\frac{1}{2}y\varphi'(y)+\varphi(y)$$

with domain $D(A)=\{\varphi\in H^2_{\text{loc}}(\mathbb{R}):\ \varphi,\ \varphi'$ and φ'' belong to $L_\omega^2(\mathbb{R})\}$. This operator is self-adjoint in $L_\omega^2(\mathbb{R})$, and its spectrum consists of the eigenvalues $\{1-n/2:\ n=0,1,2,\dots\}$. The eigenfunction corresponding to the n-th eigenvalue is the modified n-th Hermite polynomial $H_n(y)$ given in (1.9). Notice that $\|H_n\|=1$ for any n. Therefore, substituting (2.2) in (2.1), multiplying by $H_n(y)e^{-y^2/4}$ and integrating over the whole line, we obtain for $n=$ $=0, 1, 2, \ldots$

$$\text{(2.3a)}\qquad \dot{a}_n(\tau)=\left(1-\frac{n}{2}\right)a_n(\tau)-\frac{p}{p-1}\left\langle H_n(y), \left[\sum_{k=0}^{\infty} a_k(\tau)H_k'(y)\right]^2\times\right.$$

$$\left.\times\left[(p-1)+\sum_{k=0}^{\infty} a_k(\tau)H_k(y)\right]^{-1}\right\rangle,$$

$$\text{(2.3b)}\qquad \dot{a}_n(\tau)=\left(1-\frac{n}{2}\right)a_n(\tau)-\left\langle H_n(y), \left[\sum_{k=0}^{\infty} a_k(\tau)H_k'(y)\right]^2\times\right.$$

$$\left.\times\left[1+\sum_{k=0}^{\infty} a_k(\tau)H_k(y)\right]^{-1}\right\rangle.$$

For any nonnegative integer n, m and l, set now

$$A_{nml}=\int_{-\infty}^{+\infty} H_n(y)H_m(y)H_l(y)e^{-y^2/4}\,dy.$$

We shall show at the end of this paper (cf. Appendix) that

$$\text{(2.4)}\qquad \begin{aligned} &A_{nml}\neq 0 \text{ if and only if } n+m+l \text{ is even},\quad n<m+l,\quad m<n+l,\\ &n<m+l,\\ &A_{nml}=(4\pi)^{-1/4}(n!m!l!)^{1/2}\left(\frac{n+m-l}{2}!\ \frac{n+l-m}{2}!\ \frac{m+l-n}{2}!\right)^{-1}. \end{aligned}$$

Since, by standart results

$$\text{(2.5)}\qquad H_k'(y)=(k/2)^{1/2}H_{k-1}(y),\qquad k=1, 2, \ldots,$$

assuming that all coefficients $a_k(\tau)$ (and hence $\psi(y, \tau)$) are small enough for large τ, we then have that

$$\left\langle H_n(y), \left[\sum_{k=0}^{\infty} a_k(\tau)H_k'(y)\right]^2\left[(p-1)+\sum_{k=0}^{\infty} a_k(\tau)H_k(y)\right]^{-1}\right\rangle=$$

$$=\frac{1}{p-1}\left\langle H_n(y), \left[\sum_{k=0}^{\infty} a_k(\tau)H_k'(y)\right]^2\right\rangle+\ldots=$$

$$=\frac{1}{2(p-1)}\left\langle H_n(y), \left[\sum_{k=1}^{\infty} k^{1/2}a_k(\tau)H_k(y)\right]^2\right\rangle+\ldots=$$

$$=\frac{1}{2(p-1)}\sum_{k,m=1}^{\infty}(km)^{1/2}a_ka_mA_{n,k-1,m-1}+\ldots,$$

so that (2.3) yields, for $n=0, 1, 2, \ldots$

$$\text{(2.6a)}\qquad \dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\frac{p}{2(p-1)^2}\sum_{k,m=1}^{\infty}(km)^{1/2}a_ka_mA_{n,k-1,m-1}+\ldots,$$

$$\text{(2.6b)}\qquad \dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\frac{1}{2}\sum_{k,m=1}^{\infty}(km)^{1/2}a_ka_mA_{n,k-1,m-1}+\ldots.$$

By analogy with classical ODE theory, we expect that one of the modes in (2.2) will eventually dominate, i. e. $\psi(y,\tau)\approx a_j(\tau)H_j(y)$ as $\tau\to\infty$ for some $j=0, 1, 2,\ldots$, so that the behaviour of the a_n's is given in the first approximation by $a_n(\tau)\approx\delta_{j,n}\tilde{a}_n(\tau)$, where $\delta_{j,n}=1$ if $j=n$ and $\delta_{j,n}=0$ otherwise. It then follows from (2.6) that, if $j=3, 4,\ldots$, the linear part dominates there, and (1.12) follows. For $j=0$, 1 this linear approximation yields unstable terms so that, in view of (1.7), (1.8) this case must be excluded. Finally, when $j=2$, we have

(2.7a) $$\dot{a}_2(\tau)=-\frac{p}{(p-1)^2}a_2^2(\tau)A_{211}+\ldots=-\frac{2^{1/2}p}{(4\pi)^{1/4}(p-1)^2}a_2^2(\tau)+\ldots,$$

(2.7b) $$\dot{a}_2(\tau)=-a_2^2(\tau)A_{211}+\ldots=-\frac{2^{1/2}}{(4\pi)^{1/4}}a_2^2(\tau)+\ldots.$$

Integrating (2.7), (1.11) follows. We shall denote henceforth ψ_2 and ψ_n in (1.11), (1.12) as the second and the third type solutions respectively.

§ 3. Higher-order asymptotics: the inner region

Having obtained (1.10)–(1.12), we next proceed to derive higher-order expansions for the second and the third type solutions. To this end, we notice that equations (2.1) yield, for ψ small enough,

$$\psi_\tau=\psi_{yy}-\frac{1}{2}y\psi_y+\psi-\frac{p}{p-1}\frac{(\psi_y)^2}{(p-1)+\psi}=$$

$$=\psi_{yy}-\frac{1}{2}y\psi_y+\psi-\frac{p}{(p-1)^2}(\psi_y)^2\left(1-\frac{\psi}{p-1}+\ldots\right),$$

$$\psi_\tau=\psi_{yy}-\frac{1}{2}y\psi_y+\psi-(\psi_y)^2(1-\psi+\ldots),$$

whence, using (2.2)

(3.1a) $$\dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\frac{p}{(p-1)^2}\langle H_n,(\psi_y)^2\rangle+\frac{p}{(p-1)^3}\langle H_n,\psi(\psi_y)^2\rangle+\ldots$$

(3.1b) $$\dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\langle H_n,(\psi_y)^2\rangle+\langle H_n,\psi(\psi_y)^2\rangle+\ldots.$$

When $n\neq2$, the main contribution in (3.1) comes from the quadratic terms, which yields

(3.2a) $$\dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\frac{\theta_1^2p}{(p-1)^2}A_{n11}\frac{1}{\tau^2}+\ldots\text{ as }\tau\to\infty,$$

(3.2b) $$\dot{a}_n=\left(1-\frac{n}{2}\right)a_n-\theta_2^2A_{n11}\frac{1}{\tau^2}+\ldots\ as\ \tau\to\infty,$$

where

(3.3a) $\theta_1=(2^{1/2}p)^{-1}(4\pi)^{1/4}(p-1)^2$,

(3.3b) $\theta_2=2^{-1/2}(4\pi)^{1/4}$.

Notice that by (2.4), $A_{n11}\neq0$ if and only if $n=0,2$. To lower order terms, we have to retain only the case $n=0$, and since $A_{011}=(4\pi)^{-1/4}$, we arrive at

$$\dot{a}_0=a_0-\frac{\theta_1^2p}{(4\pi)^{1/4}(p-1)^2}\frac{1}{\tau^2}+\ldots\text{ as }\tau\to\infty,$$

$$\dot{a}_0=a_0-\frac{\theta_2^2}{(4\pi)^{1/4}}\frac{1}{\tau^2}+\ldots\text{ as }\tau\to\infty.$$

As in the first approximation $a_0=0$, we then obtain

$$a_0(\tau)\approx\frac{\theta_1^2 p}{(4\pi)^{1/4}(p-1)^2}\int_\tau^\infty e^{(\tau-s)}s^{-2}\,ds\approx$$

$$\approx\frac{\theta_1^2 p}{(4\pi)^{1/4}(p-1)^2}\frac{1}{\tau^2}\quad\text{as } \tau\to\infty,$$

$$a_0(\tau)\approx\frac{\theta_2^2}{(4\pi)^{1/4}}\int_\tau^\infty e^{(\tau-s)}s^{-2}\,ds\approx\frac{\theta_2^2}{(4\pi)^{1/4}}\frac{1}{\tau^2}\quad\text{as } \tau\to\infty,$$

whence

(3.4a) $$a_0(\tau)\approx\frac{(4\pi)^{1/4}(p-1)^2}{2p}\frac{1}{\tau^2}\quad\text{as } \tau\to\infty,$$

(3.4b) $$a_0(\tau)\approx\frac{(4\pi)^{1/4}}{2}\frac{1}{\tau^2}\quad\text{as } \tau\to\infty.$$

To estimate $a_2(\tau)$, we set

(3.5) $$a_n(\tau)=\xi(\tau)\delta_{n2}+\omega_n(\tau),$$

where $|\omega_n(\tau)|\ll 1/\tau$ as $\tau\to\infty$, $\delta_{ij}=1$ if $i=j$ and zero otherwise, and $\xi(\tau)=\theta/\tau$, where $\theta=\theta_1$ or θ_2 (cf. (3.3)) according to the context. On the other hand, using (2.2), (2.5), we have

$$\langle H_2,\psi(\psi_y)^2\rangle\approx\frac{\theta^3}{\tau^3}\langle H_2,H_2(H_1)^2\rangle=\frac{\theta^3}{\tau^3}\langle H_2H_1,H_2H_1\rangle=$$

$$=\frac{\theta^3}{\tau^3}\sum_{l=0}^{\infty}\langle H_2H_1,H_l\rangle\langle H_l,H_2H_1\rangle=\frac{\theta^3}{\tau^3}\sum_{l=0}^{\infty}A^2_{21l}=\frac{\theta^3}{\tau^3}(A^2_{211}+A^2_{213}).$$

Taking into account that by (3.1) $\dot\omega_4=-\omega_4+O(\omega_2/\tau)$, it then follows that ω_2 satisfies

$$\dot\omega_2=-\frac{2\omega_2}{\tau}+\frac{5(4\pi)^{1/4}(p-1)^3}{2^{3/2}p^2}\frac{1}{\tau^3}+\dots\quad\text{as } \tau\to\infty,$$

$$\dot\omega_2=-\frac{2\omega_2}{\tau}+\frac{5(4\pi)^{1/4}}{2^{3/2}}\frac{1}{\tau^3}+\dots\quad as\ \tau\to\infty,$$

whence, after integration

(3.6a) $$\omega_2(\tau)=\frac{\alpha}{\tau^2}+\frac{5(4\pi)^{1/4}(p-1)^3}{2^{3/2}p^2}\frac{\ln\tau}{\tau^2}+\dots\quad\text{as } \tau\to\infty,$$

(3.6b) $$\omega_2(\tau)=\frac{\alpha}{\tau^2}+\frac{5(4\pi)^{1/4}}{2^{3/2}}\frac{\ln\tau}{\tau^2}+\dots\quad\text{as } \tau\to\infty.$$

We thus have obtained for solutions of the second type

(3.7a) $$\psi_2(y,\tau)=\frac{(4\pi)^{1/4}(p-1)^2}{2^{1/2}p}\frac{H_2(y)}{\tau}+\frac{\alpha}{\tau^2}H_2(y)+$$

$$+\frac{5(4\pi)^{1/4}}{2^{3/2}}\frac{(p-1)^3}{p^2}\frac{\ln\tau}{\tau^2}H_2(y)+\frac{(4\pi)^{1/4}(p-1)^2}{2p}\frac{H_0(y)}{\tau^2}+\dots$$

as $\tau\to\infty$,

(3.7b) $$\psi_2(y,\tau)=\frac{(4\pi)^{1/4}}{2^{3/2}}\frac{H_2(y)}{\tau}+\frac{\alpha}{\tau^2}H_2(y)+\frac{5(4\pi)^{1/4}}{2^{3/2}}\frac{\ln\tau}{\tau^2}H_2(y)+$$

$$+\frac{(4\pi)^{1/4}}{2}\frac{H_0(y)}{\tau^2}+\dots\text{as } \tau\to\infty,$$

where α is a free constant, and

(3.8) $$H_0(y)=(4\pi)^{-1/4}, \quad H_2(y)=[2^{3/2}(4\pi)^{1/4}]^{-1}(y^2-2).$$

We next obtain corresponding expansions for the third type solutions. We then set

$$a_l(\tau)=Ce^{(1-n/2)\tau}\delta_{ln}+\omega_l(\tau), \qquad l=3, 4, \dots,$$

where C is arbitrary. Substituting in (2.6), we obtain respectively

$$\dot\omega_l(\tau)=\left(1-\frac{l}{2}\right)\omega_l(\tau)-\frac{pn}{2(p-1)^2}A_{l,n-1,n-1}C^2e^{(2-n)\tau}+\dots,$$

$$\dot\omega_l(\tau)=\left(1-\frac{l}{2}\right)\omega_l(\tau)-\frac{n}{2}A_{l,n-1,n-1}C^2e^{(2-n)\tau}+\dots.$$

By (2.4), $A_{l,n-1,n-1}=0$ if $l>2(n-1)$. Integrating the expressions above, we deduce

(3.9a) $$\omega_l(\tau)=\alpha_l e^{(1-l/2)\tau}-\frac{pn}{(p-1)^2}\frac{A_{l,n-1,n-1}}{l-2(n-1)}C^2e^{(2-n)\tau}+\dots$$

if $l=0, 1, \dots, 2(n-1)-1$,

$$\omega_{2(n-1)}(\tau)=\alpha_{2(n-1)}e^{(2-n)\tau}-\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}\tau e^{(2-n)\tau}+\dots,$$

(3.9b) $$\omega_l(\tau)=\alpha_l e^{(1-l/2)\tau}-n\frac{A_{l,n-1,n-1}}{l-2(n-1)}C^2e^{(2-n)\tau}+\dots$$

if $l=0, 1, \dots, 2(n-1)-1$,

$$\omega_{2(n-1)}(\tau)=\alpha_{2(n-1)}e^{(2-n)\tau}-\frac{nC^2}{2}A_{2(n-1),n-1,n-1}\tau e^{(2-n)\tau}+\dots,$$

where $\alpha_l=0$ if $l=0, 1, \dots, n$, since $e^{(1-n/2)\tau}$ must be the dominant term as $\tau\to\infty$ (cf. (3.1)). We have then obtained

(3.10a) $$\psi_n(y,\tau)=Ce^{(1-n/2)\tau}H_n(y)-\frac{pnC^2}{(p-1)^2}e^{(2-n)\tau}\sum_{l=0}^{n}\frac{A_{l,n-1,n-1}}{l-2(n-1)}H_l(y)+$$
$$+\sum_{l=n+1}^{2(n-1)-1}\left[\alpha_lH_l(y)e^{(1-l/2)\tau}-\frac{pnC^2}{(p-1)^2}\frac{A_{l,n-1,n-1}}{l-2(n-1)}H_l(y)e^{(2-n)\tau}\right]+$$
$$+\left[\alpha_{2(n-1)}-\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}\tau\right]e^{(2-n)\tau}H_{2(n-1)}(y)+\dots$$

as $\tau\to\infty$,

(3.10b) $$\psi_n(y,\tau)=Ce^{(1-n/2)\tau}H_n(y)-nC^2e^{(2-n)\tau}\sum_{l=0}^{n}\frac{A_{l,n-1,n-1}}{l-2(n-1)}H_l(y)+$$
$$+\sum_{l=n+1}^{2(n-1)-1}\left[\alpha_lH_l(y)e^{(1-l/2)\tau}-nC^2\frac{A_{l,n-1,n-1}}{l-2(n-1)}H_l(y)e^{(2-n)\tau}\right]+$$
$$+\left[\alpha_{2(n-1)}-\frac{n}{2}C^2A_{2(n-1),n-1,n-1}\tau\right]e^{(2-n)\tau}H_{2(n-1)}(y)+\dots \text{ as } \tau\to\infty.$$

§ 4. Higher-order expansions: th intermediate region

Consider first the case of the second type solutions. Taking into account (3.8), one readily see that (3.7) should lose its validity when $|y|^2\approx\tau$. To analyze the situation which then arises, we set

$$\xi = y\tau^{-1/2} \equiv x(T-t)^{-1/2}[-\ln(T-t)]^{-1/2}.$$

Equations (1.6) become then

$$W_\tau = \frac{1}{\tau} W_{\xi\xi} + \frac{\xi}{2}\left(\frac{1}{\tau} - 1\right) W_\xi + W - \frac{p}{(p-1)\tau} \frac{(W_\xi)^2}{W} - (p-1), \tag{4.1a}$$

$$W_\tau = \frac{1}{\tau} W_{\xi\xi} + \frac{\xi}{2}\left(\frac{1}{\tau} - 1\right) W_\xi + W - \frac{1}{\tau} \frac{(W_\xi)^2}{W} - 1. \tag{4.1b}$$

Our plan is now to match solutions of (4.1) with the functions ψ_2 in (3.7). Using the definition of W and (3.8), we first write the expansions already obtained for W in the inner region in terms of the new variable ξ. This gives

$$W(\xi,\tau) = (p-1) + \frac{(p-1)^2}{4p}\xi^2 - \frac{(p-1)^2}{2p}\frac{1}{\tau} + \frac{\alpha}{\tau}\xi^2 + \tag{4.2a}$$
$$+ \frac{5(p-1)^3}{8p^2}\frac{\ln\tau}{\tau}\xi^2 - \frac{2\alpha}{\tau^2} - \frac{5(p-1)^3}{4p^2}\frac{\ln\tau}{\tau^2} + \frac{(p-1)^2}{2p}\frac{1}{\tau^2} + \dots,$$

$$W(\xi,\tau) = 1 + \frac{1}{4}\xi^2 - \frac{1}{2\tau} + \frac{\alpha}{\tau}\xi^2 + \frac{5}{8}\frac{\ln\tau}{\tau}\xi^2 - \frac{2\alpha}{\tau^2} - \frac{5}{4}\frac{\ln\tau}{\tau^2} + \tag{4.2b}$$
$$+ \frac{1}{2\tau^2} + \dots,$$

where α is a free constant. It is then natural to try in (4.1) an expansion

$$W(\xi,\tau) = W_0(\xi) + \frac{\ln\tau}{\tau} W_1(\xi) + \frac{1}{\tau} W_2(\xi) + \dots. \tag{4.3}$$

Substituting (4.3) in (4.1), and matching with (4.2), we obtain

$$W_0(\xi) = (p-1) + c_{0,1}\xi^2, \qquad W_1(\xi) = c_{1,1}\xi^2, \tag{4.4a}$$

$$W_0(\xi) = 1 + c_{0,2}\xi^2, \qquad W_1(\xi) = c_{1,2}\xi^2, \tag{4.4b}$$

where

$$c_{0,1} = \frac{(p-1)^2}{4p}, \qquad c_{1,1} = \frac{5(p-1)^3}{8p^2}, \qquad c_{0,2} = \frac{1}{4}, \qquad c_{1,2} = \frac{5}{8}.$$

As to $W_2(\xi)$, we obtain

$$W_2(\xi) = -\frac{(p-1)^2}{2p} + \frac{(p-1)^2}{4p}\xi^2 \ln\left(1 + \frac{p-1}{4p}\xi^2\right) + c_{3,1}\xi^2, \tag{4.5a}$$

$$W_2(\xi) = -\frac{1}{2} + \frac{1}{4}\xi^2 \ln\left(1 + \frac{1}{4}\xi^2\right) + c_{3,2}\xi^2 \tag{4.5b}$$

for some free real constants $c_{3,1}$ and $c_{3,2}$. As a conclusion, we obtain the following behaviour for solutions of the second type in the region where $|\xi|$ is bounded:

$$W(\xi,\tau) = (p-1) + \frac{(p-1)^2}{4p}\xi^2 + \frac{5(p-1)^3}{8p^2}\frac{\ln\tau}{\tau}\xi^2 + \tag{4.6a}$$
$$+ \frac{1}{\tau}\left[c_{3,1}\xi^2 - \frac{(p-1)^2}{2p} + \frac{(p-1)^2}{4p}\xi^2 \ln\left(1 + \frac{p-1}{4p}\xi^2\right)\right] + \dots$$
as $\tau \to \infty$,

$$W(\xi,\tau) = 1 + \frac{1}{4}\xi^2 + \frac{5}{8}\frac{\ln\tau}{\tau}\xi^2 + \frac{1}{\tau}\left[c_{3,2}\xi^2 - \frac{1}{2} + \tag{4.6b}\right.$$
$$\left. + \frac{1}{4}\xi^2 \ln\left(1 + \frac{1}{4}\xi^2\right)\right] + \dots \text{ as } \tau \to \infty.$$

We conclude by performing the corresponding expansion for the third type solutions. It is then natural to use, as a new variable

$$\xi=y\exp\left[\left(\frac{1}{n}-\frac{1}{2}\right)\tau\right]\equiv x(T-t)^{-1/n}$$

in the region where $|\xi|\approx 1$. Equations (1.4) are then transformed into

$$\text{(4.7a)}\qquad W_\tau=e^{-(1-2/n)\tau}W_{\xi\xi}-\frac{\xi}{n}W_\xi+W-\frac{p}{(p-1)}e^{-(1-2/n)\tau}\frac{(W_\xi)^2}{W}-(p-1),$$

$$\text{(4.7b)}\qquad W_\tau=e^{-(1-2/n)\tau}W_{\xi\xi}-\frac{\xi}{n}W_\xi+W-e^{-(1-2/n)\tau}\frac{(W_\xi)^2}{W}-1.$$

As in the previous case, we first write the expansion already obtained for W in the inner region in terms of the new variable. This yields

$$\text{(4.8a)}\qquad W(\xi,\tau)=(p-1)+Cc_n\xi^n-Cc_n\frac{n(n-1)}{2}\xi^{n-2}e^{-(1-2/n)\tau}+$$

$$+\sum_{l=n+1}^{2(n-1)}a_lc_l\xi^le^{(1-l/n)\tau}-$$

$$-\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\tau\xi^{2(n-1)}e^{-(1-2/n)\tau}+\ldots,$$

$$\text{(4.8b)}\qquad W(\xi,\tau)=1+Cc_n\xi^n-Cc_n\frac{n(n-1)}{2}\xi^{n-2}e^{-(1-2/n)\tau}+$$

$$+\sum_{l=n+1}^{2(n-1)}a_lc_l\xi^le^{(1-l/n)\tau}-$$

$$-\frac{nC^2}{2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\tau\xi^{2(n-1)}e^{-(1-2/n)\tau}+\ldots$$

as $\tau\to\infty$, where $H_n(\xi)=c_n\xi^n-2^{-1}n(n-1)c_n\xi^{n-2}+\ldots$ as $\xi\to\infty$. In view of (4.8), we now try in (4.7) the following expansion

$$W(\xi,\tau)=W_0(\xi)+\sum_{l=n+1}^{2(n-1)}W_l(\xi)e^{(1-l/n)\tau}+\tau e^{-(1-2/n)\tau}Q(\xi)+\ldots.$$

Integrating the differential equations obtained for terms of similar order, and matching with the corresponding quantities in (4.8), we get

$$\text{(4.9a)}\qquad W_0(\xi)=(p-1)+A_0\xi^n,$$

$$\text{(4.9b)}\qquad W_0(\xi)=1+A_0\xi^n,$$

$$\text{(4.10)}\qquad W_l(\xi)=A_l\xi^l,\qquad Q(\xi)=R\xi^{2(n-1)},$$

where

$$\text{(4.11)}\qquad A_0=Cc_n,\qquad A_l=a_lc_l,\qquad l=n+1,\ldots,2(n-1)-1,$$

$$\text{(4.12a)}\qquad R=-\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}c_{2(n-1)},$$

$$\text{(4.12b)}\qquad R=-\frac{nC^2}{2}A_{2(n-1),n-1,n-1}c_{2(n-1)}.$$

Concerning $W_{2(n-1)}(\xi)$, we obtain

$$\text{(4.13a)}\qquad \left(\frac{2}{n}-1\right)W_{2(n-1)}+\frac{1}{n}\xi(W_{2(n-1)})_\xi-W_{2(n-1)}=$$

$$=(W_0)_{\xi\xi}-Q-\frac{p}{p-1}\frac{[(W_0)_\xi]^2}{W},$$

(4.13b) $$\left(\frac{2}{n}-1\right)W_{2(n-1)}+\frac{1}{n}\xi(W_{2(n-1)})_\xi-W_{2(n-1)}=$$

$$=(W_0)_{\xi\xi}-Q-\frac{[(W_0)_\xi]^2}{W}.$$

Taking into account (4.9)–(4.12), integrating the differential equation obtained for $W_{2(n-1)}(\xi)$ yields

(4.14a) $$W_{2(n-1)}(\xi)=S\xi^{2(n-1)}-n(n-1)Cc_n\xi^{n-2}+ +|R|\xi^{2(n-1)}\ln[(p-1)+A_0\xi^n]+\ldots,$$

(4.14b) $$W_{2(n-1)}(\xi)=S\xi^{2(n-1)}-n(n-1)Cc_n\xi^{n-2}+ +|R|\xi^{2(n-1)}\ln(1+A_0\xi^n)+\ldots$$

ar $\tau\to\infty$, where S, C are free constants. Summing these results up, we have obtained the expansions

(4.15a) $$W(\xi,\tau)=[(p-1)+Cc_n\xi^n]+\sum_{l=n+1}^{2(n-1)-1}a_lc_l\xi^le^{(1-l/n)\tau}-$$
$$-\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\tau\xi^{2(n-1)}e^{-(1-2/n)\tau}+$$
$$+e^{-(1-2/n)\tau}\Big\{S\xi^{2(n-1)}-n(n-1)Cc_n\xi^{n-2}+$$
$$+\frac{pnC^2}{2(p-1)^2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\xi^{2(n-1)}\ln[(p-1)+Cc_n\xi^n]\Big\}+\ldots.$$

(4.15b) $$W(\xi,\tau)=(1+Cc_n\xi^n)+\sum_{l=n+1}^{2(n-1)-1}a_lc_l\xi^le^{(1-l/n)\tau}-$$
$$-\frac{nC^2}{2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\tau\xi^{2(n-1)}e^{-(1-2/n)\tau}+$$
$$+e^{-(1-2/n)\tau}\Big\{S\xi^{2(n-1)}-n(n-1)Cc_n\xi^{n-2}+$$
$$+\frac{nC^2}{2}A_{2(n-1),n-1,n-1}c_{2(n-1)}\xi^{2(n-1)}\ln(1+Cc_n\xi^n)\Big\}+\ldots$$

as $\tau\to\infty$, where the expansions above are to be understood as consisting of $[(p-1)+Cc_n\xi^n]$ (resp. $(1+Cc_n\xi^n)$) and the first nonzero term afterwards. Notice that to avoid singularities, it seems reasonable to reduce ourselves the cases $C>0$ and n even.

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APPENDIX

We compute here the terms A_{nml} defined in (2.4). To this end, we first consider the integrals

$$I_{nml}=\int_{-\infty}^{+\infty}H_n(x)H_m(x)H_l(x)e^{-x^2}dx,$$

where, changing slightly our previous notation, $H_n(x)$ represents now the n-th Hermite polynomial. By a well-known generation formula

$$\text{(A1)} \qquad \exp(2tx-t^2+2sx-s^2+2rx-r^2)=\sum_{n,m,l=0}^{\infty}\frac{H_n(x)H_m(x)H_l(x)}{n!m!l!}t^n s^m r^l.$$

We now recall Cauchy integral formula in polydiscs for functions $f(z)$ analytical in $\mathbb{C}^n$:

$$\text{(A2)} \qquad f(z)=\frac{1}{(2\pi i)^n}\int_{\partial D_n}\int_{\partial D_{n-1}}\cdots\int_{\partial D_1}\frac{f(\xi_1,\ldots,\xi_n)}{(\xi_1-z_1)\ldots(\xi_n-z_n)}d\xi_1\ldots d\xi_n=$$

$$=\frac{1}{(2\pi i)^n}\sum_{j_1,\ldots,j_n=0}^{\infty}z_1^{j_1}\ldots z_n^{j_n}\int_{|\xi_n-w_n|=r_n}\cdots$$

$$\cdots\int_{|\xi_1-w_1|=r_1}\frac{f(\xi_1,\ldots,\xi_n)}{\xi_1^{j_1+1}\ldots\xi_n^{j_n+1}}d\xi_1\ldots d\xi_n,$$

where $z=(z_1,\ldots,z_z)\in D_1\times\ldots\times D_n=D(w_1,r_1)\times\ldots\times D(w_n,r_n)$, $D(w_k,r_k)$ being a disc in $\mathbb{C}$ centered at w_k with radius r_k. Set now

$$\Phi(t,s,r)=\exp[-(t^2+s^2+r^2)]\int_{-\infty}^{+\infty}\exp[-x^2+2(t+s+r)x]dx.$$

We now multiply in (A1) by $\exp(-x^2)$ and integrate over $\mathbb{R}$, to get

$$\text{(A3)} \qquad \Phi(t,s,r)=\sum_{n,m,l=0}^{\infty}\frac{I_{nml}}{n!m!l!}t^n s^m r^l.$$

From (A2) and (A3), we deduce that for some $\rho>0$

$$\frac{I_{nml}}{n!m!l!}=\frac{1}{(2\pi i)^3}\int_{|t|=\rho}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\Phi(t,s,r)}{t^{1+n}s^{1+m}r^{1+l}}dt\,ds\,dr.$$

Notice that, since

$$\int_{-\infty}^{+\infty}\exp(-x^2+2\alpha x)\,dx=\exp(\alpha^2)\int_{-\infty}^{+\infty}\exp[-(x-\alpha)^2]\,dx=\pi^{1/2}\exp(\alpha^2),$$

we have that

$$\Phi(t,s,r)=\pi^{1/2}\exp[2(ts+tr+sr)],$$

whence

$$I_{nml}=\pi^{1/2}\frac{n!m!l!}{(2\pi i)^3}\int_{|t|=\rho}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp[2(ts+tr+sr)]}{t^{1+n}s^{1+m}r^{1+l}}dt\,dt\,dr.$$

Moreover, setting $a=2(s+r)$, we obtain

$$\text{(A4)} \qquad \frac{n!}{2\pi i}\int_{|t|=\rho}\frac{\exp(2(ts+tr))}{t^{1+n}}dt=\frac{n!}{2\pi i}\int_{|t|=\rho}\frac{\exp(at)}{t^{1+n}}dt=\frac{d^n}{dt^n}(e^{ta})\big|_{t=0}=a^n.$$

Therefore

$$I_{nml}=\pi^{1/2}\frac{m!l!}{(2\pi i)^2}\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp(2sr)[2(s+r)]^n}{s^{1+m}r^{1+l}}ds\,dr=$$

$$=2^n\pi^{1/2}\frac{m!l!}{(2\pi i)^2}\sum_{k=0}^{n}\left\|\begin{matrix}n\\k\end{matrix}\right\|\int_{|s|=\rho}\int_{|r|=\rho}\frac{\exp(2sr)}{s^{1+m-k}r^{1+l-n+k}}ds\,dr.$$

Assume for instance that $m \geqslant n$. Then, arguing as in (A4)

$$\int\limits_{|s|=\rho} \frac{\exp(2sr)}{s^{1+m-k}}\, ds = (2r)^{m-k} \frac{2\pi i}{(m-k)!},$$

so that

$$I_{nml} = \frac{2^n \pi^{1/2} m!\, l!}{2\pi i} \sum_{k=0}^{n} \left\| \begin{matrix} n \\ k \end{matrix} \right\| 2^{m-k} [(m-k)!]^{-1} \int\limits_{|r|=\rho} r^{-(1+l-n-m+2k)}\, dr,$$

and since

$$\int\limits_{|r|=\rho} r^{-(1+l-n-m+2k)}\, dr = 2\pi i \delta_{2k,m+n-l}$$

we arrive at

$$I_{nml} = 2^n \pi^{1/2} m!\, l! \sum_{k=0}^{n} \left\| \begin{matrix} n \\ k \end{matrix} \right\| 2^{m-k} [(m-k)!]^{-1} \delta_{2k,m+n-l}.$$

Therefore $I_{nml} \neq 0$ if $m+n-l$ is an even integer, $0 \leqslant m+n-l \leqslant 2n$, whence $m+n+l$ has to be even and $l \leqslant m+n$, $m \leqslant l+n$. Under such assumptions

$$\text{(A5)} \qquad I_{nml} = \frac{2^n \pi^{1/2} m!\, l!\, 2^{m-(m+n-l)/2}}{(m-(m+n-l)/2)!} \left\| \begin{matrix} n \\ \dfrac{m+n-l}{2} \end{matrix} \right\| =$$

$$= 2^{(n+m+l)/2} \pi^{1/2} \frac{n!\, m!\, l!}{[(m+l-n)/2]!\, [(m+n-l)/2]!\, [(n+l-m)/2]!}.$$

Having obtained (A5), we turn our attention to the integral

$$A_{nml} = \int\limits_{-\infty}^{+\infty} \tilde{H}_n(x) \tilde{H}_m(x) \tilde{H}_l(x) e^{-x^2/4}\, dx.$$

where $\tilde{H}_n(x) = c_n H_n(x/2)$, $c_n = [2^{n/2} (4\pi)^{1/4} (n!)^{1/2}]^{-1}$, and we just notice that

$$A_{nml} = 2 c_n c_m c_l I_{nml},$$

whence the result.

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