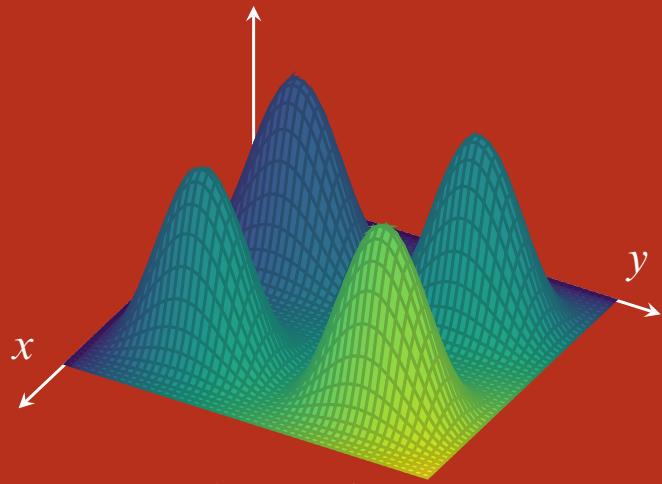




Mathematical Methods II

Lecture Notes

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1. LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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THIS course starts with a brief introduction to Partial Differential Equations (PDE) in its simplest form, linear PDEs. We recall some elementary definitions of complex numbers arithmetic and complex functions. We will also present some of the most paradigmatic linear PDEs of Mathematical Physics, such as the wave, heat, Laplace, Poisson, and Schrödinger equations. We will discuss changes of variable in a PDE and the notion of general solution as well. Boundary and/or initial value conditions will be also considered. We discuss then linear problems that are formulated naturally in terms of linear differential operators. The idea of eigenvalue and eigenfunction is introduced at this point. Finally, uniqueness for problems in Electrostatics, Geophysics and Fluid Mechanics is studied.

§1.1. PDE Definition. Linear PDE



Basic ideas and concepts for the study of linear PDEs, such as complex numbers, partial derivatives and their notations, dependent and independent variables, and order of a PDE are introduced here. Also, we introduce the most relevant PDEs of Mathematical Physics that we will discuss along this course, the Poisson equation, the wave equation, the Schrödinger equation and the heat equation. We will study changes of variables and conclude the section with the concept of general solution.

1.1.1. PDEs The functions that appear in our discussions will take complex values.

Complex Valued Functions

We consider functions that take complex values and that depend on a certain number of real variables $(t, x, \dots)$ and write $u = u(t, x, \dots) = u_1(t, x, \dots) + i u_2(t, x, \dots)$ to denote a function that depends on the real variables $(t, x, \dots)$, that takes complex values with real parts and imaginary given by $\operatorname{Re} u = u_1 = u_1(t, x, \dots)$ and $\operatorname{Im} u = u_2 = u_2(t, x, \dots)$. As complex numbers, the function u can be conjugated $\bar{u} = u_1 - i u_2$, and have a modulus and an argument: $|u| = +\sqrt{u_1^2 + u_2^2} = +\sqrt{u \bar{u}}$ and $\arg u = \arctan \frac{u_2}{u_1}$. Hence, if we denote $r = |u|$ and $\theta = \arg u$, we can write $u_1 = r \cos \theta$ and $u_2 = r \sin \theta$.

The complex exponential function Recall that in the arithmetic of complex numbers, given $a, b \in \mathbb{R}$, exponentials with complex exponents are defined by the Euler formulas

$$e^{ib} = \cos b + i \sin b, \quad e^{a+ib} = e^a e^{ib} = e^a (\cos b + i \sin b) \quad a, b \in \mathbb{R}.$$

Euler's formula allows us to express functions with complex values in the form

$$u = r \cos \theta + i r \sin \theta = r e^{i\theta}, \quad r = |u|, \theta = \arg u.$$

This expression is very useful for applications in wave phenomena and Quantum Physics.



Examples:

1. Let $u(x, y) = xy + i e^{x^2+y^2}$. In this case $u_1(x, y) = xy$ and $u_2(x, y) = e^{x^2+y^2}$. Consequently, $\bar{u} = xy - i e^{x^2+y^2}$ and $|u| = \sqrt{(xy)^2 + e^{2(x^2+y^2)}}$.
2. Let $u(x, y) = e^{xy+i(x^2+y^2)}$. Using Euler formulas $u = e^{xy}(\cos(x^2 + y^2) + i \sin(x^2 + y^2))$, hence $u_1 = e^{xy} \cos(x^2 + y^2)$ and $u_2 = e^{xy} \sin(x^2 + y^2)$, with modulus and argument given by $|u| = e^{xy}$ and $\arg u = x^2 + y^2$.

The complex logarithm

Sometimes we will also use the multi-valued **complex logarithm**

$$\log z = \log |z| + i(\arg z + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots,$$

where $\log |z|$ denotes natural logarithm (also known as Naperian logarithm) of the positive real number $|z|$ and $\arg z$ is any of the possible arguments of z .



Examples:

1. Solve the equation

$$e^u = 1.$$

Taking $\arg 1 = 0$ we get

$$u = \log 1 = i 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

2. Solve the equation $e^u = -2i$. Taking $\arg(-2i) = -\pi/2$ we get $u = \log(-2i) = \log 2 + i\left(-\frac{\pi}{2} + 2n\pi\right)$, $n = 0, \pm 1, \pm 2, \dots$

Extended Notation for Partial Derivatives

On many occasions the partial derivatives of u will be denoted as shown by following examples:

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}.$$

We will always assume that the functions we handle admit derivatives to the order required by the operations that we make. The most comfortable situation appears when we assume that all derivatives of all orders do exist. In that case, according to the Schwarz–Clairaut theorem, the result of a multiple derivation is independent of the order in which we make the individual derivations.

As an example, we will have $u_{xxyxzy} = u_{xxxyyz} = u_{zxyxyx}$. The basic rule to derive and integrate functions u with complex values is to think of the imaginary unit i as a constant. So, for example

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} = \frac{\partial u_1}{\partial t} + i \frac{\partial u_2}{\partial t}, & u_x &= \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + i \frac{\partial u_2}{\partial x}, \\ u_{xx} &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u_1}{\partial x^2} + i \frac{\partial^2 u_2}{\partial x^2}, & u_{xy} &= \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u_1}{\partial x \partial y} + i \frac{\partial^2 u_2}{\partial x \partial y}, \\ \int u \, dx &= \int (u_1 + i u_2) \, dx = \int u_1 \, dx + i \int u_2 \, dx. \end{aligned}$$

In particular, the exponential function e^u of a function u with complex values fulfill the same derivation rule as if u took real values. For example

$$\frac{\partial e^u}{\partial x} = e^u u_x.$$

To prove it, just proceed as follows

$$\begin{aligned} \frac{\partial e^u}{\partial x} &= \frac{\partial}{\partial x} \left(e^{u_1} e^{i u_2} \right) = \frac{\partial}{\partial x} \left(e^{u_1} \cos u_2 + i e^{u_1} \sin u_2 \right) = \frac{\partial}{\partial x} \left(e^{u_1} \cos u_2 \right) + i \frac{\partial}{\partial x} \left(e^{u_1} \sin u_2 \right) \\ &= e^{u_1} \frac{\partial u_1}{\partial x} \cos u_2 - e^{u_1} \sin u_2 \frac{\partial u_2}{\partial x} + i \left(e^{u_1} \frac{\partial u_1}{\partial x} \sin u_2 + e^{u_1} \cos u_2 \frac{\partial u_2}{\partial x} \right) \\ &= \left(e^{u_1} \cos u_2 + i e^{u_1} \sin u_2 \right) \left(\frac{\partial u_1}{\partial x} + i \frac{\partial u_2}{\partial x} \right) = e^u u_x. \end{aligned}$$



Examples:

1. Given $u(x, y) = xy + i e^{x^2+y^2}$. we have $u_x = y + 2ix e^{x^2+y^2}$, $u_{xx} = i(2 + 4x^2) e^{x^2+y^2}$.
2. For $u(x, y) = e^{xy+i(x^2+y^2)}$ the first order derivatives are

$$\begin{aligned} u_x &= (y + i2x) e^{xy} (\cos(x^2 + y^2) + i \sin(x^2 + y^2)), \\ u_y &= (x + i2y) e^{xy} (\cos(x^2 + y^2) + i \sin(x^2 + y^2)). \end{aligned}$$

3. The indefinite integral of the exponential function $u(x) = e^{(a+ib)x}$, $a + ib \neq 0$, with exponent depending linearly on x , is done as if $c = a + ib$ was a real number. That is to say, $\int e^{(a+ib)x} dx = \frac{e^{(a+ib)x}}{a+ib}$. In this manner, one can compute the indefinite integral as follows

$$\int_0^\pi e^{(1+i)x} dx = \frac{e^{(1+i)x}}{1+i} \Big|_0^\pi = \frac{1}{1+i} (e^{(1+i)\pi} - e^0) = -\frac{1}{1+i} (e^\pi + 1) = \frac{1+e^\pi}{2} (-1+i)$$

Condensed Notation for Partial Derivatives

Let's consider complex functions $u = u(x) = u_1(x) + i u_2(x)$ that depend on

$$x = (x_0, x_1, \dots, x_{n-1}),$$

denotes a vector in $\mathbb{R}^n$. Frequently, but not always, the variable x_0 will be identified with a time variable t . For derivatives we will write

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}}}, \quad |\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_{n-1},$$

where vector indexes are

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}_+^n \subset \mathbb{R}^n,$$

with n non-negative integer components. Observe that $|\alpha|$ is the order of the derivative $D^\alpha u$. By definition if $\alpha = (0, 0, \dots, 0)$ then $D^\alpha u \equiv u$.

The relationship between the two types of notations is easy to establish. For example, if $(x_0, x_1, x_2, x_3) = (t, x, y, z)$:

$$u_{xxzyz} = D^\alpha u, \quad \alpha = (0, 2, 1, 2).$$

When we have a single independent variable x , we will use the notation

$$D^n u := \frac{d^n u}{dx^n}.$$

To define the concept of PDE it is convenient to use the condensate notation.

PDE Definition

A PDE is a relation of the form

$$(1) \quad F(x, D^\alpha u) = 0,$$

being F a function depending upon $x = (x_0, x_1, \dots, x_{n-1})$ ($n > 1$), and on a finite number of derivatives $D^\alpha u$. The nomenclature is as follows:

1. The variables x_i , $i = 0, \dots, n-1$, are named as **independent variables** of the PDE.
2. The unknown function u of the PDE is called the **dependent variable** of the PDE.
3. If r is the maximum order of derivatives $D^\alpha u$ upon which the F function depends, then r is by definition the **order** of the PDE.

The most frequent situation is that the PDE is defined by a function F that is a polynomial in the variables $D^\alpha u$. However, there are physical situations in which more general equations appear. For example, the so-called equation of sine-Gordon, $u_{tt} - u_{xx} = \sin u$. In particular, if F is a polynomial of degree one in the variables $D^\alpha u$ it is said that the PDE is a **linear PDE**. In this case the PDE is of the form

$$(1) \quad \boxed{\sum_{\alpha}' a_{\alpha}(x) D^{\alpha} u - f(x) = 0,}$$

where $\sum_{\alpha}'$ means that the sum extends to a finite set of multi-indices α with $|\alpha| \geq 0$. The functions $a_{\alpha}(x)$ and $f(x)$ are supposed to be given. Normally, when we deal with a PDE as in (1) we clean $f(x)$ in the LHS and put it on the RHS and write

$$\sum_{\alpha}' a_{\alpha}(x) D^{\alpha} u = f(x).$$

We refer to $f(x)$ as the **inhomogeneous term** of the equation. If the function $f(x) \equiv 0$ we will say that the linear PDE is linear is **homogeneous**. Otherwise, $f \neq 0$, we say that the linear PDE is **inhomogeneous**.

In general, nonlinear PDEs are much more difficult to treat than the linear ones. We will not study them in this course. To consider a specific PDE the extended notation is more convenient.

Examples:

1. The PDE

$$u_x + e^{x+y} u_y - u = x^2 y^2,$$

is linear and has order 1. It is inhomogeneous with an independent term given by $x^2 y^2$.

2. The PDE

$$u_{xx}u + u_y + xy = 0,$$

is nonlinear, because of the appearance of the term $u_{xx}u$, and has order 2.



1.1.2. Some linear PDEs relevant in Physics Physics is plenty of linear and nonlinear PDEs. Both in Electromagnetism and in Quantum Mechanics the basic equations are linear, but in other areas, such as Continuous Media Dynamics or General Relativity, the fundamental equations are nonlinear.

Linear PDEs in Physics

There are four linear PDE examples, all of them second order, to which we will devote a particular attention in this course. Its main versions, with three spatial variables (x, y, z) , are:

1. **Poisson equation**

$$u_{xx} + u_{yy} + u_{zz} = f$$

being $f = f(x, y, z)$ a given function. If $f \equiv 0$ the PDE is named Laplace equation. Both PDEs appear often in Electrostatics and in Fluid Mechanics.

2. **Wave equation**

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$$

with c a positive real number which represents the propagation speed of the wave.

3. **Schrödinger equation**

$$i\hbar u_t = -\frac{\hbar^2}{2m}(u_{xx} + u_{yy} + u_{zz}) + V(x, y, z)u$$

which describes the dynamics of a particle of mass m in a field of forces with potential function $V = V(x, y, z)$. The symbol $\hbar$ represents the normalized Planck constant. Notice the presence of the imaginary number i in the coefficient of u_t . This fact is the main reason why in this course we consider functions with complex values.

4. **Heat equation**

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz})$$

It is relevant in thermal diffusion and fluid diffusion processes in general. The symbol a^2 represents the diffusion coefficient.

To write the above equations in abbreviated form is convenient to use the notation of the Laplacian operator:

$$\Delta u := u_{xx} + u_{yy} + u_{zz},$$

which is a fundamental example of a concept, that of **differential operator**, to which we will devote great attention in this course. In terms of the Laplacian the previous equations are expressed as follows:

1. Poisson equation::

$$\Delta u = f$$

2. Wave equation:

$$u_{tt} = c^2 \Delta u$$

3. Schrödinger equation:

$$i \hbar u_t = -\frac{\hbar^2}{2m} \Delta u + V u$$

4. Heat equation:

$$u_t = a^2 \Delta u$$

Sometimes we will consider simplified versions of the previous equations in which u does not depend on some of the variables (x, y, z) . Thus, a version in $1 + 2$ dimensions of the wave, Schrödinger or heat equations is a PDE in which we assume that u depends on (t, x, y) only.

1.1.3. Change of independent variables Given a PDE (1), one of the most frequent manipulations that we must do is determine the form it acquires when we perform a change of independent variables $x \mapsto x' = x'(x)$ with transformation equations:

$$x'_i = x'_i(x_0, x_1, \dots, x_{n-1}), \quad i = 0, 1, \dots, n-1,$$

which we always assume to be invertible $x' \mapsto x = x(x')$, being the inverse transformation given by the equations

$$x_i = x_i(x'_0, x'_1, \dots, x'_{n-1}), \quad i = 0, 1, \dots, n-1.$$

For the sake of simplicity will not use a new function symbol to the compound function $u(x(x'))$, which we will simply denote $u(x')$.

The form that takes (1) in the new variables is determined by substituting the variables x in $F(x, D^a u)$ by $x(x')$, and the derivatives with respect to x ($D^a u$) by their expressions in terms of derivatives with respect to x' . For the latter we must use the chain rule. The expressions derived from orders one and two are

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \sum_k \frac{\partial u}{\partial x'_k} \frac{\partial x'_k}{\partial x_j}, \\ \frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i} \left(\sum_k \frac{\partial x'_k}{\partial x_j} \frac{\partial u}{\partial x'_k} \right) \\ &= \sum_k \frac{\partial^2 x'_k}{\partial x_i \partial x_j} \frac{\partial u}{\partial x'_k} + \sum_l \sum_k \frac{\partial x'_l}{\partial x_i} \frac{\partial x'_k}{\partial x_j} \frac{\partial^2 u}{\partial x'_l \partial x'_k} \end{aligned}$$

Sometimes a change of variables can convert an PDE into another simpler one. The typical example is the $1 + 1$ wave equation dimensions.



Examples:

1. Consider the 1D wave equation, $u_{tt} - u_{xx} = 0$ and let us perform the change of variables $y_1 = t + x$, $y_2 = t - x$, with inverse change given by $t = \frac{1}{2}(y_1 + y_2)$, $x = \frac{1}{2}(y_1 - y_2)$. We immediately obtain:

$$\begin{aligned} u_t &= u_{y_1} \frac{\partial y_1}{\partial t} + u_{y_2} \frac{\partial y_2}{\partial t} = u_{y_1} + u_{y_2}, \\ u_x &= u_{y_1} \frac{\partial y_1}{\partial x} + u_{y_2} \frac{\partial y_2}{\partial x} = u_{y_1} - u_{y_2}, \\ u_{tt} &= \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right) (u_{y_1} + u_{y_2}) = u_{y_1 y_1} + u_{y_2 y_2} + 2u_{y_1 y_2}, \\ u_{xx} &= \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right) (u_{y_1} - u_{y_2}) = u_{y_1 y_1} + u_{y_2 y_2} - 2u_{y_1 y_2}. \end{aligned}$$

As a consequence the PDE is transforms into

$$4u_{y_1 y_2} = 0.$$

In this new form the PDE can be integrated immediately and the solution is

$$u = f(y_1) + g(y_2) = f(x + t) + g(t - x),$$

with f and g being arbitrary functions of one variable only.

- ii) Let us write the Laplace equation

$$u_{xx} + u_{yy} = 0,$$

in polar coordinates:

$$\begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta, \\ r &= \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}. \end{aligned}$$

Applying the chain rule we get:

$$\begin{aligned} u_x &= \frac{x}{\sqrt{x^2 + y^2}} u_r - \frac{y}{x^2 + y^2} u_\theta = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta, \\ u_y &= \frac{y}{\sqrt{x^2 + y^2}} u_r + \frac{x}{x^2 + y^2} u_\theta = \sin \theta u_r + \frac{\cos \theta}{r} u_\theta, \\ u_{xx} &= \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right) \\ &= \cos^2 \theta u_{rr} + 2 \frac{\cos \theta \sin \theta}{r^2} u_\theta - 2 \frac{\cos \theta \sin \theta}{r} u_{\theta r} + \frac{\sin^2 \theta}{r} u_r + \frac{\sin^2 \theta}{r^2} u_{\theta \theta}, \\ u_{yy} &= \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right) \\ &= \sin^2 \theta u_{rr} - 2 \frac{\cos \theta \sin \theta}{r^2} u_\theta + 2 \frac{\cos \theta \sin \theta}{r} u_{\theta r} + \frac{\cos^2 \theta}{r} u_r + \frac{\cos^2 \theta}{r^2} u_{\theta \theta}. \end{aligned}$$

Therefore, the 2D Laplace equation in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

1.1.4. General solution of a PDE The idea of **general solution** of an ordinary differential equation (ODE) or a PDE arises when trying to solve the equation by applying integration operations. In principle, in an ODE of order r , indefinite integrations are required, and since each one introduces an arbitrary integration constant, the solution obtained will be an expression with n arbitrary constants called **general solution** of the ODE. But life is hard. First, because only for simple cases is it possible to obtain explicit solutions by applying indefinite integration operations. On the other hand, even in cases where it is possible, it is not guaranteed that we can generate all the solutions of an ODE of order n through a single dependent expression of arbitrary constant r . The following examples illustrate this situation.

Examples:



1. The equation

$$\frac{d^2 u}{dx^2} \left(\frac{du}{dx} - u \right) = 0,$$

has order 2. However, we cannot characterize all of its solutions through a single expression with two arbitrary constants since u is a solution if and only if at least one of the two factors is canceled, $\frac{d^2 u}{dx^2}$ or $\left(\frac{du}{dx} - u \right)$. In other words, all its solutions are given by two expressions

$$u = c_1 x + c_2 \text{ or } u = c_3 e^x$$

where (c_1, c_2, c_3) are arbitrary constants. Note that the expression $c_1 x + c_2$ is a general solution, but is not the expression $c_3 e^x$ although it is a solution to the equation.

2. The equation

$$\frac{du}{dx} = \sqrt{10 - u},$$

is order 1. All its solutions are given by

$$u = 10 - \frac{(x - c_1)^2}{4},$$

which is a general solution, and the constant solution

$$u = 10,$$

not included in the previous general solution.

In the PDE context the situation is even more complicated. In principle, in an r -th order PDE with n independent variables, when an indefinite integration with respect to one the independent variables is performed, an arbitrary function of the other $(n - 1)$ variables is introduced instead of an arbitrary constant. If the integration process can be carried out until the end, r indefinite integrations will be required, and since each one introduces an arbitrary function of $(n - 1)$ variables, the solution obtained will be an expression with r arbitrary functions of $(n - 1)$

variables called **general solution** of the PDE. Despite the limitations of this notion, its use is useful when calibrating whether the methods used to obtain solutions of PDE characterize a sound solution space. An easy example of general solution is obtained for the PDE

$$u_{xy} = 0.$$

Integrating the equation with respect to y we get

$$u_x + c(x) = 0$$

and performing integration x we arrive to

$$u = c_1(x) + c_2(y), \quad c_1(x) := \int^x c(x) \, dx.$$

We must note that the arbitrary functions, which appear in the construction methods of general solutions of PDEs, do not always depend on the independent variables of the PDE. For example, for the PDE

$$u_x + u_y = 0,$$

with $r = 1$ and $n = 2$ we can proceed as follows to build a general solution. First we observe that

$$u = e^{i\lambda(x-y)}$$

is a solution for any possible value $\lambda \in \mathbb{R}$. Also, as the PDE is linear, we conclude that

$$u = \int_{\mathbb{R}} c(\lambda) e^{i\lambda(x-y)} \, d\lambda,$$

with $c(\lambda)$ an arbitrary function of a variable, it is a general solution of the PDE. These types of general solutions are the ones that will appear in these lectures.

§1.2. Boundary and/or Initial Conditions



WE move forward presenting the boundary conditions and/or initial conditions that usually accompany the PDEs which are required to be solved. First, we present the notions of domain and boundary, and then we discuss the idea of boundary value problems for PDEs. In particular, we introduce the problems of Dirichlet and Neumann, the concept of normal derivative to the surface of the boundary being necessary for the latter. Periodic boundary conditions and what is understood by initial condition will also be treated. Finally, we briefly discuss the type of solutions to be considered: the smooth functions.

1.2.1. Domains. Boundaries

Domains and Boundaries for PDEs

In general, when we consider a PDE the unknown function $u = u(x)$ is assumed to be defined over a given set Ω of $\mathbb{R}^n$. We always assume that Ω satisfies the following two conditions:

1. Ω is an open set. That is, for all $a \in \Omega$ there exists an $r > 0$ such that every point $x \in \mathbb{R}^n$ with a distance to a less than r ($d(x, a) < r$) belongs to Ω .
2. Ω is connected. That is, it is not possible to find two non empty open sets Ω_i , ($i = 1, 2$) such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$.

In such a case we say that Ω is a **domain** of $\mathbb{R}^n$. The **boundary** $S(\Omega)$ of Ω is the set of those points $a \in \mathbb{R}^n$ such that for any $r > 0$, there exists points x inside $x \in \Omega$ and exists points x outside $x \notin \Omega$, with $d(x, a) < r$.

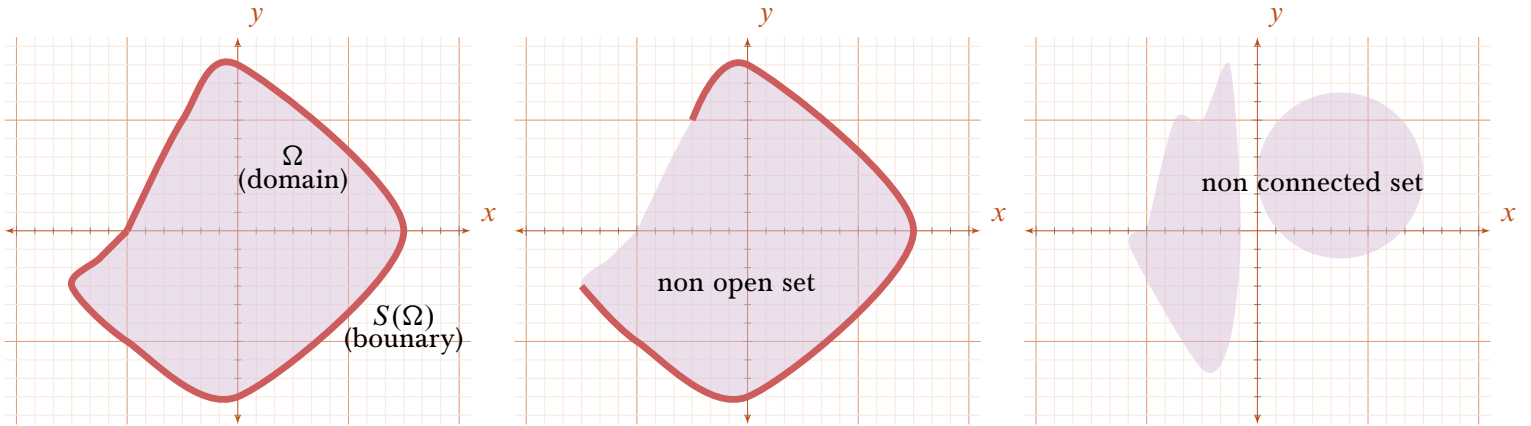
Obviously, property 1) means that Ω has no points in common with its boundary $S(\Omega)$. As for property 2), we can interpret it as the prohibition that Ω can not be divided in two separate sectors. The union set

$$\overline{\Omega} = \Omega \cup S(\Omega),$$

is called *the closure of Ω* .

In the 1D situation, $n = 1$, the domain is necessarily an open interval of the real line $\Omega = (a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ and, consequently, if the interval is finite, $S(\Omega) = \{a, b\}$.

Next we show diagrams of a domain and of sets that are not domains since they violate the property 1) or property 2).



1.2.2. Boundary conditions

Boundary Conditions for PDEs

Given a PDE defined in a domain $\Omega \subset \mathbb{R}^n$:

$$(2) \quad F(x, D^\alpha u) = 0, \quad x \in \Omega,$$

normally we are asked not only to find a function $u = u(x)$ that satisfies the PDE at all points of Ω , but also that such a function satisfies a number of conditions

$$f_i(x, D^\alpha u) = 0, \quad x \in S_i, \quad i = 1, \dots, m,$$

where the symbols S_i denote parts of the boundary $S(\Omega)$ of Ω , and f_i are functions depending on the variables x_i , and on a finite number of derivatives $D^\alpha u$ with $|\alpha| \geq 0$. Conditions of this class are called **boundary conditions**. We will only consider boundary conditions in which the functions f_i are degree one polynomials in the variables $D^\alpha u$ (linear boundary conditions). That is, in the form

$$\sum_{\alpha} b_{i,\alpha}(x) D^\alpha u - g_i(x) = 0, \quad x \in S_i,$$

or $\sum_{\alpha} b_{i,\alpha} D^\alpha u|_{S_i} = g_i$. A problem consisting in solving a PDE over a domain Ω and a set of boundary conditions is called **boundary value problem**.

A PDE does not always admit the imposition of certain boundary conditions. As an example we can consider the equation

$$u_{xy} = 0$$

in the square $\Omega = \{(x, y) \in (0, 1) \times (0, 1)\}$. In this case the boundary is the union of the four sides of the square

$$S(\Omega) = S_1 \cup S_2 \cup S_3 \cup S_4,$$

with

$$\begin{aligned} S_1 &:= \{(x, 0) \in \mathbb{R}^2 : 0 < x \leq 1\}, & S_2 &:= \{(1, y) : 0 < y \leq 1\}, \\ S_3 &:= \{(x, 1) \in \mathbb{R}^2 : 0 \leq x < 1\} & S_4 &:= \{(0, y) : 0 \leq y < 1\}. \end{aligned}$$

Let us request the following boundary conditions

$$u|_{S_1} = f_1(x), \quad u|_{S_2} = f_2(y), \quad u|_{S_3} = f_3(x), \quad u|_{S_4} = f_4(y).$$

Let's see now that this problem may have no solution. As is satisfied the PDE, $u_{xy} = 0$, the function u_x do not depends on y , $u_x|_{S_1} = u_x|_{S_3}$, and therefore we must have

$$f_1'(x) = f_3'(x),$$

and analogous argument leads to

$$f_2'(y) = f_4'(y).$$

Therefore, for the boundary value problem to have a solution, it is necessary additional conditions on the boundary data.

Dirichlet and Neumann Boundary Conditions

In problems on a domain Ω in the space $\mathbb{R}^3$ the following nomenclature is used for the simplest boundary conditions on a S surface contained in $S(\Omega)$:

1. **Dirichlet condition:**

$$u|_S = g$$

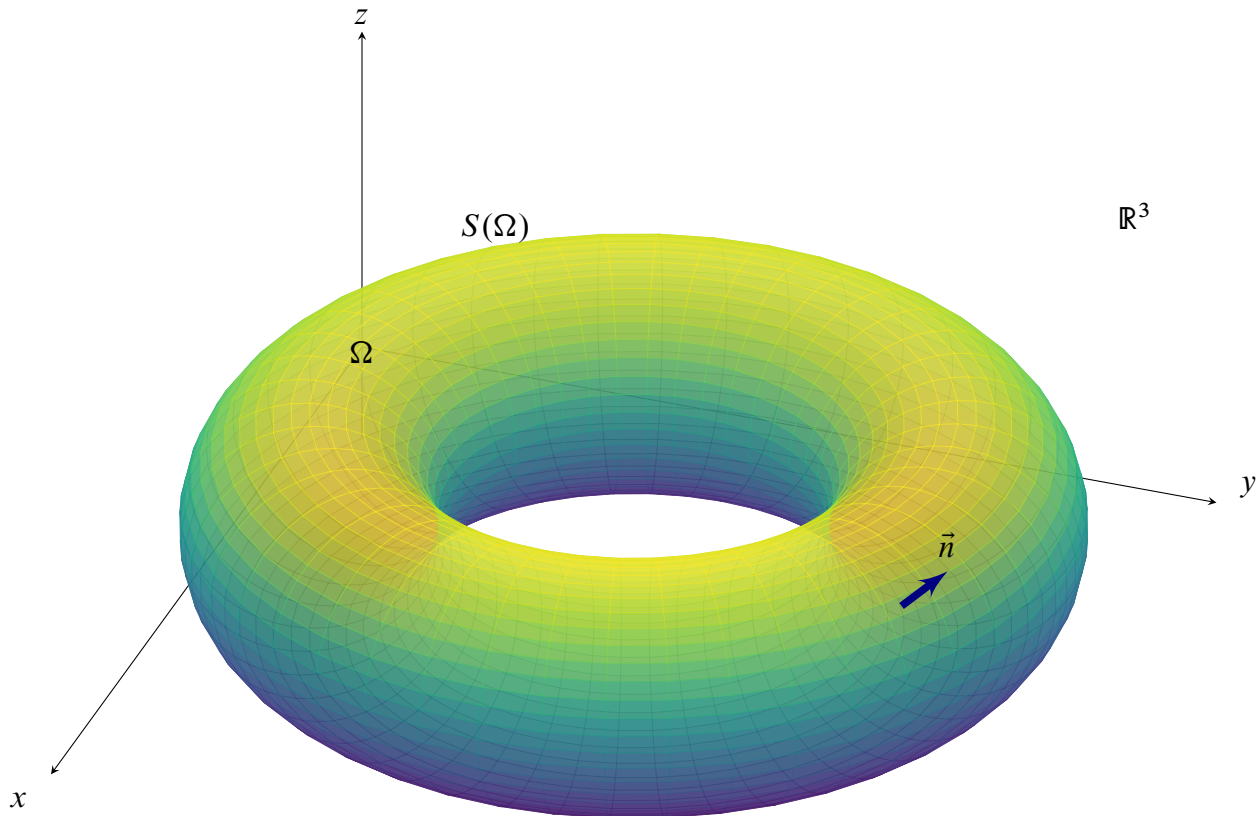
2. **Neumann condition:**

$$\left. \frac{\partial u}{\partial \mathbf{n}} \right|_S = g$$

with the normal derivative to the surface given by

$$\frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u = n_1 u_x + n_2 u_y + n_3 u_z,$$

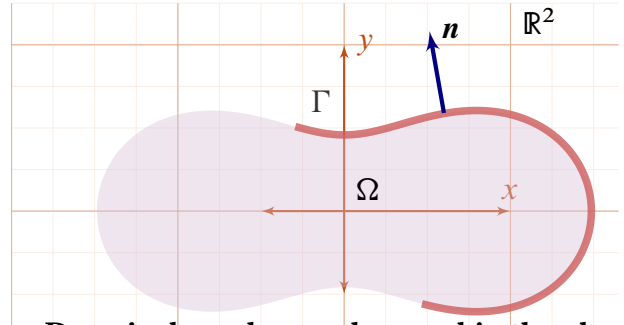
with $\mathbf{n} = (n_1, n_2, n_3)$ a vector field of unit normals to the surface S .



Domain, boundary and normal in the space: Torus

Sometimes we will consider $\mathbb{R}^2$ versions of the previous boundary conditions. In such cases Ω will be a region in the plane, S a curve contained in $S(\Omega)$ and instead of the notation $S, S(\Omega)$ we would prefer to use $\Gamma, \Gamma(\Omega)$, respectively.

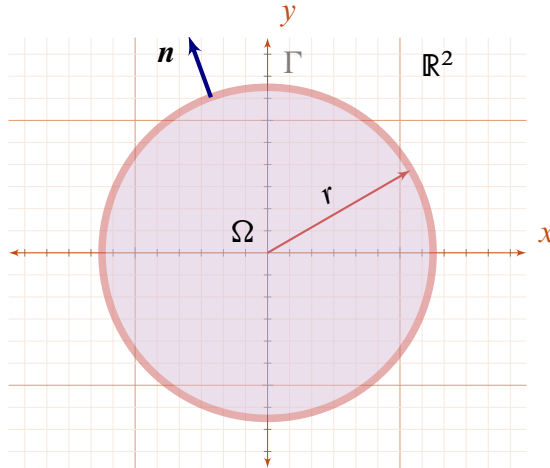
In any case, whenever we work with a boundary condition on a part S of the border of $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, we will assume that S can be described by implicit equation, $f_S(x) = 0$, such that $\nabla f_S(x) \neq \mathbf{0}$, $x \in S$. We can define a unit normal vector field according to $\mathbf{n}(x) := \frac{\nabla f_S(x)}{\|\nabla f_S(x)\|}$.



**Domain, boundary and normal in the plane:
Cassini oval**



Examples: The following examples show the normal vector fields and the corresponding derivation operations in the direction of the normal vector.



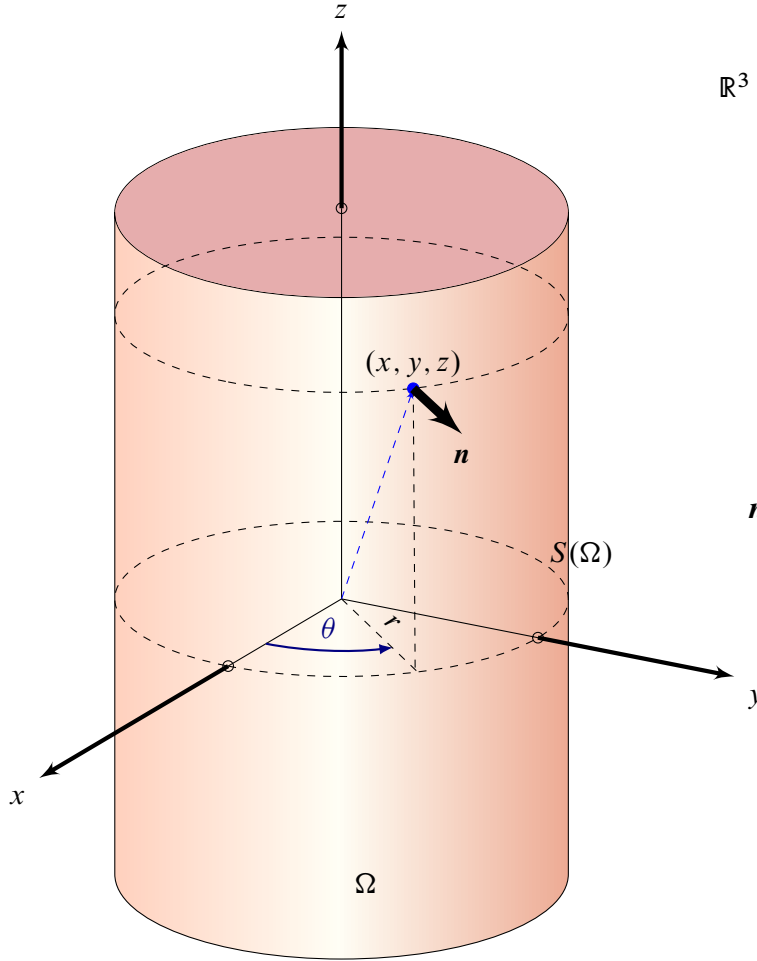
Domain, boundary and normal

1. In this first example Ω is a disk centered at the origin with radius r in $\mathbb{R}^2$ and its boundary is a circle $\Gamma = \Gamma(\Omega)$ of radius r , we have

$$f_\Gamma(x, y) = x^2 + y^2 - r^2,$$

$$\mathbf{n} = \frac{(2x, 2y)}{\|(2x, 2y)\|} \Big|_\Gamma = \frac{1}{r}(x, y),$$

$$\frac{\partial u}{\partial \mathbf{n}} \Big|_\Gamma = \frac{1}{r}(xu_x + yu_y).$$



2. Secondly, we consider that Ω is the inside of the cylinder with axis OZ and radius r in $\mathbb{R}^3$, $S = S(\Omega)$. Therefore, the boundary is the cylindrical surface, that is, the points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation

$$f_S(x, y, z) = x^2 + y^2 - r^2,$$

the unit normal to the surface is

$$\mathbf{n} = \frac{(2x, 2y, 0)}{\|(2x, 2y, 0)\|_{\Gamma}} = \frac{1}{r}(x, y, 0),$$

and the normal derivative over surface is computed according to

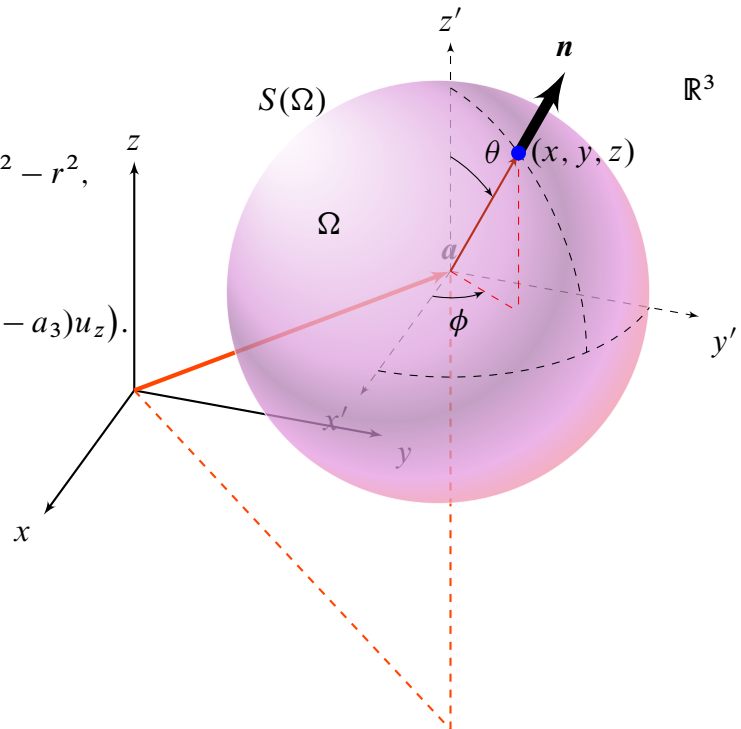
$$\frac{\partial u}{\partial \mathbf{n}} \Big|_S = \frac{1}{r}(xu_x + yu_y).$$

3. Finally, let us consider Ω as the interior of the sphere centered at $\mathbf{a} = (a_1, a_2, a_3)$ and radius r in $\mathbb{R}^3$,

$$f_S(x, y, z) = (x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 - r^2,$$

$$\mathbf{n} = \frac{1}{r}(x - a_1, y - a_2, z - a_3),$$

$$\frac{\partial u}{\partial \mathbf{n}} \Big|_S = \frac{1}{r}((x - a_1)u_x + (y - a_2)u_y + (z - a_3)u_z).$$



There are other types of boundary conditions associated with appropriate pairs of hypersurfaces of the boundary of Ω .

Periodic Conditions

Let us suppose that S_i , $i = 1, 2$ of $\mathbb{R}^n$ which belong to $S(\Omega)$, are such that there exists a bijective map between them

$$\begin{aligned}\sigma : S_1 &\longrightarrow S_2, \\ x &\mapsto \sigma(x),\end{aligned}$$

such that both σ and its inverse, when expressed in local coordinates of its domain surfaces, are functions that admit all derivatives. A **periodic boundary condition** is then expressed as an equation of the form

$$f(x, D^\alpha u(x)) = f(\sigma(x), D^\alpha u(\sigma(x))), \quad x \in S_1.$$



Example: When $n = 1$, we may consider $S_1 = \{0\} \subset \mathbb{R}$ and σ as the translation map $x \mapsto x + a$, with $\sigma(0) = a$, the boundary conditions could be $u|_{x=0} = u_{x=a}$, $u_x|_{x=0} = u_x|_{x=a}$.

1.2.3. Initial conditions

Initial Conditions

Another type of conditions that are usually required for the solutions of a PDE are the so-called *initial conditions* of one of the independent variables that we will denote t or x_0 . They are usually a set of conditions of the form:

$$u|_{t=t_0} = \Phi_0, \frac{\partial u}{\partial t}|_{t=t_0} = \Phi_1, \dots, \frac{\partial^{r-1} u}{\partial t^{r-1}}|_{t=t_0} = \Phi_{r-1},$$

where $r \geq 0$ and Φ_i are functions depending on the rest of independent variables. In general, the initial conditions are not boundary conditions as situations are also considered in which the set determined by the equation $t = t_0$ can be in the *inside* of Ω .

In physical problems in which the evolution of a system is analyzed, both boundary and initial conditions usually coexist.



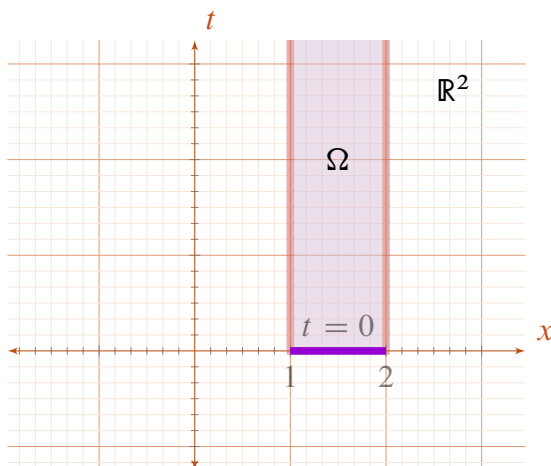
Examples: The following examples show a couple of different generic situations.

1. Let us consider the heat equation in 1+1 dimensions:

$$u_t = u_{xx},$$

over the domain

$$\Omega = \{(t, x) \in \mathbb{R}^2 \mid t > 0, 1 < x < 2\}.$$



Domain, boundary and initial condition

We can impose the conditions

$$u|_{t=0} = (x-1)(x-2), \quad u|_{x=1} = 0, \quad u|_{x=2} = 0.$$

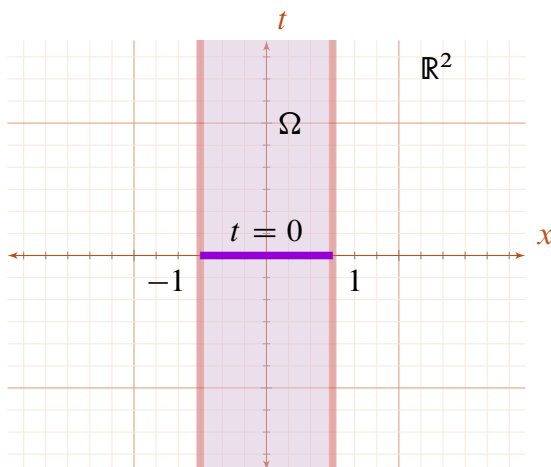
In this case the initial condition is a boundary condition, as well.

2. Let us discuss the Schrödinger equation in 1+1 dimensions:

$$i u_t = -u_{xx},$$

over the domain

$$\Omega = \{(t, x) \in \mathbb{R}^2 \mid -\infty < t < \infty, -1 < x < 1\}.$$



Domain, boundary and initial condition

We can impose the conditions

$$u|_{t=0} = \sin(\pi x) e^{-x^2}, \quad u|_{x=-1} = 0, \quad u|_{x=1} = 0.$$

In this case the initial condition is not a boundary condition.

1.2.4. Smooth functions When looking for solutions to a boundary value problem, different types of functional spaces are considered in which to investigate the existence of such solutions. For boundary value problems, the space $C^\infty(\overline{\Omega})$ of functions of class C^∞ or smooth in the closure $\overline{\Omega}$ of Ω .

Smooth Functions

By definition $u \in C^\infty(\overline{\Omega})$ is a smooth function if and only if all its derivatives do exist for all orders $D^\alpha u(x)$ in every point x of some open set $\Omega_0 \supset \overline{\Omega}$. The most interesting properties C^∞ are

1. If $u, v \in C^\infty(\overline{\Omega})$ then the functions

$$\lambda u(x) + \mu v(x), \quad \forall \lambda, \mu \in \mathbb{C}$$

$$u(x) \cdot v(x),$$

$$\frac{u(x)}{v(x)}, \quad (\text{if } v(x) \neq 0 \text{ for all } x \in \overline{\Omega}),$$

also belong to $C^\infty(\overline{\Omega})$.

2. If u is in $C^\infty(\overline{\Omega})$ then all its derivatives $D^\alpha u(x)$ also belong to $C^\infty(\overline{\Omega})$.

§1.3. Linear Problems. Linear Differential Operators



LINEAR boundary value and/or initial conditions problems, which are the subject of study of this course, are discussed. Appropriate notations to deal with such problems are provided. Linear differential operators and some of its properties are briefly discussed. The notion of eigenvalues and eigenfunction for such operators is considered.

1.3.1. Differential Operators

Differential Operators

Let be $C^\infty(\overline{\Omega})$ the space of functions of class C^∞ in $\overline{\Omega}$. A differential operator L on $C^\infty(\overline{\Omega})$ is an application $L : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\overline{\Omega})$, $u \mapsto Lu$, as follows

$$Lu := \sum_{\alpha}' a_{\alpha}(x) D^{\alpha} u,$$

where $\sum_{\alpha}'$ means that the sum extends to a finite set of multi-indexes α with $|\alpha| \geq 0$ and the coefficients $a_{\alpha}(x)$ are supposed to be given functions in $C^\infty(\overline{\Omega})$.

Every differential operator is a linear application. That is, it satisfies

$$L(\lambda u + \mu v) = \lambda Lu + \mu Lv, \quad \forall u, v \in C^\infty(\overline{\Omega}), \lambda, \mu \in \mathbb{C}.$$

A common notation we will use to refer to a differential operator is

$$L := \sum_{\alpha}' a_{\alpha}(x) D^{\alpha}.$$

Examples:



1. Let us consider the following differential operator acting on $C^{\infty}(\mathbb{R})$

$$L = x^2 D + x.$$

Its action on $u = \cos x$ is

$$Lu = x^2 Du + xu = -x^2 \sin x + x \cos x.$$

2. The Laplacian operator

$$Lu := u_{xx} + u_{yy} + u_{zz},$$

can be written as follows

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Is a differential operator over $C^{\infty}(\overline{\Omega})$, being Ω any domain in $\mathbb{R}^3$. Its action over the function

$$u := e^{x^2+y^2+z^2},$$

is

$$Lu = \frac{\partial^2}{\partial x^2} e^{x^2+y^2+z^2} + \frac{\partial^2}{\partial y^2} e^{x^2+y^2+z^2} + \frac{\partial^2}{\partial z^2} e^{x^2+y^2+z^2} = (4(x^2 + y^2 + z^2) + 6) e^{x^2+y^2+z^2}.$$

Differential operators are the appropriate mathematical objects to handle linear PDEs. Thus, a linear PDE of the form

$$\sum_{\alpha}' a_{\alpha}(x) D^{\alpha} u = f(x),$$

is written in condensed form as

$$Lu = f,$$

being L the differential operator

$$Lu := \sum_{\alpha}' a_{\alpha}(x) D^{\alpha} u.$$

Therefore, the problem with linear PDE consists of find functions $u \in C^{\infty}(\overline{\Omega})$ whose image by linear application L matches the function f .

Relevant cases of these PDEs are the following:

1. **Poisson equation** in 3 dimensions:

$$Lu = f, \quad Lu := \Delta u.$$

2. **Wave equation** in 1+3 dimensions:

$$Lu = 0, \quad Lu := u_{tt} - c^2 \Delta u.$$

3. **Schrödinger equation** in 1+3 dimensions:

$$Lu = 0, \quad Lu := i\hbar u_t + \frac{\hbar^2}{2m} \Delta u + Vu.$$

4. **Heat equation** in 1+3 dimensions:

$$Lu = 0, \quad Lu := u_t - a^2 \Delta u.$$

Differential Operators

Differential operators have very important algebraic properties. The following natural operations are relevant

1. Addition of operators

$$(L + M)u := Lu + Mu.$$

2. Product of complex numbers and operators

$$(\lambda L)u := \lambda(Lu).$$

3. Product of operators

$$(LM)u := L(Mu).$$

4. Commutator of operators

$$[L, M]u := L(Mu) - M(Lu).$$

Two operators commute if and only if their commutator is the zero operator $[L, M] = 0$.

With respect to the addition and the product of complex numbers the differential operators form a linear space. The zero operator, defined as the differential operator with all coefficients equal to zero, is denoted $L = 0$ (obviously in such a case $Lu = 0$ for all u). Regarding the operation of operators' product, which coincides with the composition of operators as applications, the most outstanding peculiarity is that it is not a commutative operation.



Examples:

1. If we take

$$L = D, \quad M = D + x,$$

we have that

$$L(Mu) = D^2u + xDu + u, \quad M(Lu) = D^2u + xDu.$$

Hence $LM \neq ML$.

2. Let us now calculate an example in three dimensions with $L_x = y\partial_z - z\partial_y$ and $L_z = x\partial_y - y\partial_x$, the commutator is

$$\begin{aligned} [L_x, L_z] &= [y\partial_z - z\partial_y, x\partial_y - y\partial_x] = [y\partial_z, x\partial_y] + [y\partial_z, -y\partial_x] + [-z\partial_y, x\partial_y] + [-z\partial_y, -y\partial_x] \\ &= [y\partial_z, x\partial_y] + [z\partial_y, y\partial_x] = (-x\partial_z + z\partial_x)[\partial_y, y] \\ &= z\partial_x - x\partial_z \\ &=: L_y. \end{aligned}$$

This set of three operators (L_x, L_y, L_z) is essentially the **orbital angular momentum operator** that appears in **Quantum Physics** (a $-\mathrm{i}\hbar$ factor is missing).

1.3.2. Initial condition and boundary operators

Boundary Operators

Let $C^\infty(\overline{\Omega})$ be the space of differentiable functions over $\overline{\Omega}$. A boundary operator l on $C^\infty(\overline{\Omega})$ is a map $l : C^\infty(\overline{\Omega}) \rightarrow C(S)$ with $u \mapsto l(u)$, being $C(S)$ the set of continuous functions on a hypersurface $S \subset S(\Omega)$ that belongs to the boundary of Ω , as follows

$$l(u) := \sum_{\alpha}' b_{\alpha}(x) D^{\alpha} u|_S,$$

where $\sum_{\alpha}'$ means that the sum extends to a finite set of multi-indices α with $|\alpha| \geq 0$ and the coefficients $b_{\alpha}(x)$ are given functions in $C(S)$.

Every boundary operator is a linear map. That is, it satisfies

$$l(\lambda u + \mu v) = \lambda l(u) + \mu l(v), \quad \forall u, v \in C^\infty(\overline{\Omega}), \lambda, \mu \in \mathbb{C}.$$

Initial Condition Operators

When one of the independent variables is time

$$x = (t, \mathbf{x}), \quad \mathbf{x} := (x_1, \dots, x_{n-1}),$$

and S denotes the subset intersection of a hyperplane $t = t_0$ with $\overline{\Omega}$ we define an initial condition operator as an application of the form $l : C^\infty(\overline{\Omega}) \rightarrow C(S)$ with $u \mapsto l(u)$, given by

$$l(u) := \left. \frac{\partial^r u}{\partial t^r} \right|_{t=t_0}, \quad r \geq 0.$$

It is clear that these initial condition operators are also linear.

Boundary and initial condition operators are the appropriate mathematical objects to handle linear boundary conditions and initial conditions. Thus, a linear boundary condition

$$\sum_{\alpha}' b_{\alpha}(x) D^{\alpha} u|_S = g,$$

can be written in a condensate manner as follows

$$l(u) = g,$$

being l the boundary operator

$$l(u) := \sum_{\alpha}' b_{\alpha}(x) D^{\alpha} u|_S,$$

Relevant cases of boundary operators associated with boundary conditions $l(u) = g$ are

1. **Dirichlet condition:**

$$l(u) := u|_S.$$

2. **Neumann condition:**

$$l(u) := \frac{\partial u}{\partial \mathbf{n}} \Big|_S.$$

Problems with Linear PDEs

The paradigmatic problem that we will consider in this course is that of characterize functions $u \in C^{\infty}(\overline{\Omega})$ which are the solution of a system of equations of the form

$$(2) \quad \begin{cases} Lu = f, \\ l_i(u) = g_i, \quad i = 1, \dots, m, \end{cases}$$

where L is a differential operator and l_i a series of boundary or initial conditions operators.

1.3.3. Eigenvalues and eigenfunctions of differential operators One of the basic ingredients of our methods for solving linear problems such as (2) will be to use eigenvalues and eigenfunctions of differential operators.

Associated Subspace to Boundary Value Conditions

Given a set of boundary operators $\{l_i\}_{i=1}^m$ acting on the space $C^{\infty}(\overline{\Omega})$ we call **associated subspace** $\mathcal{D}$ to the set of functions such that

$$l_i(u) = 0 \text{ for all } i = 1, \dots, m.$$

Domain of a Linear Differential Operator

Let L be a differential operator over $C^\infty(\overline{\Omega})$ and $\mathcal{D}$ the associated subspace to the boundary operators $\{l_i\}_{i=1}^m$ on $C^\infty(\overline{\Omega})$. If we limit ourselves to consider the action of L on the elements of $\mathcal{D}$ we can consider L as a linear map that takes us from $\mathcal{D}$ to $C^\infty(\overline{\Omega})$, $L : \mathcal{D} \rightarrow C^\infty(\overline{\Omega})$ with $u \mapsto Lu$. In that case we will say that we consider the operator L defined on the **domain** $\mathcal{D}$ of $C^\infty(\overline{\Omega})$

Eigenvalues of a Linear Differential Operator

Let L be a differential operator defined on a **domain** $\mathcal{D}$ of $C^\infty(\overline{\Omega})$. We'll say that a complex number λ is an **eigenvalue** of L over $\mathcal{D}$ if

$$\text{there exists some function } u \neq 0 \text{ in } \mathcal{D} \text{ such that } Lu = \lambda u$$

In that case we will say that such function u is an **eigenfunction** of L corresponding to the eigenvalue λ . The set $\sigma(L)$ of all eigenvalues of L is called the **spectrum** of L . For each eigenvalue $\lambda \in \sigma(L)$, we define the corresponding **eigensubspace** as the following linear subspace

$$\mathcal{D}_\lambda := \{u \in \mathcal{D} : Lu = \lambda u\}$$

When $\dim \mathcal{D}_\lambda = 1$, we say that the eigenvalue is **simple** and if $\dim \mathcal{D}_\lambda \geq 2$ that is **degenerate**.

In general, solving an eigenvalue and eigenfunction problem $Lu = \lambda u$, $u \in \mathcal{D}$ in $\mathbb{R}^n$ requires to find the solutions of the homogeneous linear PDE, $Lu - \lambda u = 0$, in n independent variables. Until the next chapter we will not have a method to treat this type of PDE with $n \geq 2$. So, for now, we'll limit ourselves to problems with a single variable.

Examples:

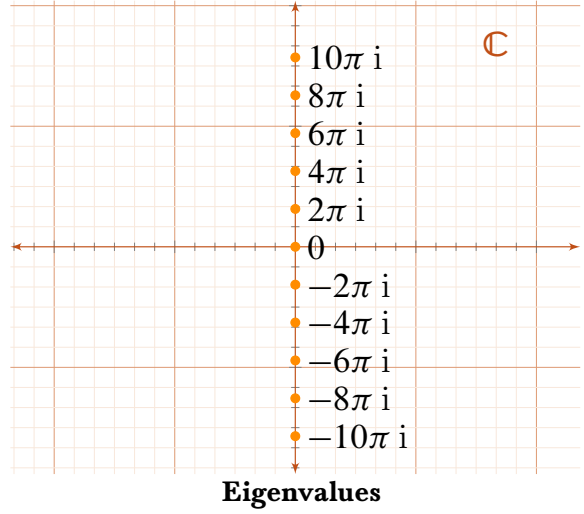


1. Let $Lu = u_x$, in the domain $\mathcal{D} = \{u \in C^\infty[0, 1] \mid u(0) = 0\}$. The eigenvalue problem is

$$u_x = \lambda u, \quad u(0) = 0.$$

The solution to the differential equation is $u(x) = c e^{\lambda x}$ and the boundary condition implies that $c = 0$ and therefore $u = 0$. Then there is no eigenvalue and $\sigma(L) = \emptyset$.

2. Let be the previous operator but now in a different domain associated with periodic boundary conditions. That is, $Lu = u_x$ with $\mathcal{D} = \{u \in C^\infty[0, 1] \mid u(0) = u(1)\}$. The solution to the differential equation is $u(x) = c e^{\lambda x}$ and the boundary condition implies $e^\lambda = 1$. Therefore, $\lambda = \log 1 = 2n\pi i$, $\sigma(L) = \{2n\pi i\}_{n \in \mathbb{Z}}$, with eigensubspaces $\mathcal{D}_{2n\pi i} = \mathbb{C} e^{2n\pi i x}$ and simple eigenvalues ($\dim \mathcal{D}_{2n\pi i} = 1$). The attached figure shows the distribution of the spectrum in the complex plane.
3. Let be the operator $Lu = -u_{xx}$ en el dominio $\mathcal{D} = \{u \in C^\infty[0, 1] \mid u(0) = 0, u(1) = 0\}$. The eigenvalue problem is



$$-u_{xx} = \lambda u, \quad u(0) = 0, u(1) = 0.$$

For $\lambda = 0$ the solution to the differential equation is $u(x) = c_1 x + c_2$ and the boundary conditions imply $c_2 = 0$, $c_1 + c_2 = 0$. That is $c_1 = c_2 = 0$, so that $u \equiv 0$. Then $\lambda = 0$ is not an eigenvalue. For $\lambda \neq 0$ the solution to the differential equation is $u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$. Boundary conditions are now reduced to

$$\begin{cases} c_1 = 0, \\ c_1 \cos \sqrt{\lambda} + c_2 \sin \sqrt{\lambda} = 0. \end{cases}$$

For there to be a non-trivial solution $(c_1, c_2) \neq (0, 0)$ of this system it is necessary and sufficient that the determinant of the matrix of coefficients of the system is annulled

$$\begin{vmatrix} 1 & 0 \\ \cos \sqrt{\lambda} & \sin \sqrt{\lambda} \end{vmatrix} = \sin \sqrt{\lambda} = 0.$$

Therefore, the eigenvalues are

$$\lambda_n = n^2 \pi^2, \quad n = 1, 2, \dots$$

For each of these eigenvalues the solutions of the system are $c_1 = 0$ and c_2 arbitrary. So the eigensubspaces are

$$\mathcal{D}_{n^2 \pi^2} = \mathbb{C} \{\sin n \pi x\}.$$

The eigenvalues and eigenfunctions of a differential operator depend on its domain $\mathcal{D}$.

§1.4. 1D Eigenvalue Problems



STRATEGIES for solving eigenvalue problems for differential operators in one variable in suitable domains, i.e. smooth functions that fulfill as many boundary conditions as the order of the differential operator, are discussed now. We will give some examples for ordinary differential operators with constant coefficients.

Eigenvalues of a 1D Linear Differential Operator

Let $Lu = \sum_{n=0}^N a_n(x) D^n u(x)$ be a N -th order 1D linear differential operator in a domain $\mathcal{D}$ determined by a series of linear boundary conditions on the closed interval $[a, b]$, $l_j(u) = 0$ for $j = 1, \dots, N$. To solve the spectral problem $Lu = \lambda u$ with $u \in \mathcal{D}$ we may proceed as follows:

1. Solve the linear ODE, i.e. $Lu - \lambda u = 0$, considering λ as a parameter. The general solution will be of the form

$$(2) \quad u(\lambda, x) = c_1 u_1(\lambda, x) + \dots + c_N u_N(\lambda, x),$$

being $\{u_i\}_{i=1, \dots, N}$ a maximal set of linearly independent solutions, i. e., a fundamental set.

2. Impose the boundary conditions to the previous general solution

$$l_j(u) = 0, \quad j = 1, \dots, N.$$

Which, by substituting in each equation $l_j(u) = 0$ and due to the linearity of the boundary operators, provides a system of N homogeneous linear equations

$$(3) \quad l_j(u_1)c_1 + \dots + l_j(u_N)c_N = 0, \quad j = 1, \dots, N,$$

for the N coefficients $c_1, c_2, \dots, c_N$ of the general solution. The elements λ of the spectrum of L are characterized by the property of admitting nontrivial solutions $u \in \mathcal{D}$, of $Lu = \lambda u$. But this happens if and only if the equations (3) for $(c_1, \dots, c_N)$ admits non-trivial solutions $(c_1, \dots, c_N) \neq (0, \dots, 0)$. This, in turn, takes place if and only if the determinant of the system matrix of coefficients vanish.

$$\begin{vmatrix} l_1(u_1) & \dots & l_1(u_N) \\ \vdots & & \vdots \\ l_N(u_1) & \dots & l_N(u_N) \end{vmatrix} = 0.$$

This condition constitutes an equation of the type

$$f(\lambda) = 0,$$

and their solutions form the spectrum $\sigma(L)$.

3. For every $\lambda \in \sigma(L)$ we solve the linear system (3) for $(c_1, \dots, c_N)$ and determine the general solution $u = u(\lambda, x)$ in (2) which will constitute the associated eigensubspace $\mathcal{D}_{\lambda_n}$.

1.4.1. Constant coefficients operators A typical case we will find is a problem with a L differential operator with constant coefficients. We describe below how to proceed in the two simplest cases.

1. First order operators

$Lu = \alpha u_x + \beta u$, $\mathcal{D} = \{u \in C^\infty([a, b]), \text{ such that } \gamma u(a) - \nu u(b) = 0\}$,
where, to avoid trivial cases, we assume that the constants $\alpha, \beta, \gamma, \nu \in \mathbb{C}$ satisfy

$$\alpha \neq 0, \gamma \neq 0.$$

The eigenvalue equation, $Lu = \lambda u$, is

$$\alpha u_x + (\beta - \lambda) u = 0,$$

whose general solution is given by

$$u(x) = c \exp\left(\frac{\lambda - \beta}{\alpha} x\right), \text{ with } c \in \mathbb{C} \text{ arbitrary.}$$

By imposing the boundary condition we obtain the equation that characterizes the eigenvalues

$$c \left(\gamma \exp\left(\frac{\lambda - \beta}{\alpha} a\right) - \nu \exp\left(\frac{\lambda - \beta}{\alpha} b\right) \right) = 0.$$

Simplifying the constant factor c leaves us with

$$\gamma \exp\left(\frac{\lambda - \beta}{\alpha} (a - b)\right) = \nu.$$

As $\gamma \neq 0$ we have that

$$\exp\left(\frac{\lambda - \beta}{\alpha} (a - b)\right) = \frac{\nu}{\gamma} \implies \lambda = \beta + \frac{\alpha}{a - b} \log \frac{\nu}{\gamma},$$

then eigenvalues are

$$\lambda_n = \beta + \frac{\alpha}{a - b} \left(\log \left| \frac{\nu}{\gamma} \right| + i \left(\arg \left(\frac{\nu}{\gamma} \right) + 2n\pi \right) \right), \quad n \in \mathbb{Z}.$$

with corresponding eigenfunctions given by

$$u_n(x) = c \exp\left(\frac{\lambda_n - \beta}{\alpha} x\right),$$

with $c \in \mathbb{C}$ arbitrary.

2. Second order operators

$Lu = \alpha u_{xx} + \beta u_x + \gamma u$, $\mathcal{D} = \{u \in C^\infty([a, b]), \text{ such that } l_i(u) = 0, i = 1, 2\}$,

being the boundary operators of the form

$$l_i(u) = v_i u(a) + \mu_i u_x(a) + v'_i u(b) + \mu'_i u_x(b), \quad i = 1, 2.$$

The general solution of the differential equation

$$\alpha u_{xx} + \beta u_x + \gamma u = \lambda u,$$

is of the form

$$u = c_1 e^{r_1(\lambda)x} + c_2 e^{r_2(\lambda)x},$$

where $r_1(\lambda)$ and $r_2(\lambda)$ are two different solutions of the characteristic equation

$$\alpha r^2 + \beta r + \gamma = \lambda.$$

The boundary conditions require

$$\begin{cases} c_1 l_1(e^{r_1(\lambda)x}) + c_2 l_1(e^{r_2(\lambda)x}) = 0, \\ c_1 l_2(e^{r_1(\lambda)x}) + c_2 l_2(e^{r_2(\lambda)x}) = 0. \end{cases}$$

For there to be a non-trivial solution $(c_1, c_2) \neq (0, 0)$ of this system it is necessary and sufficient that the determinant of the matrix of coefficients of the system is zero.

$$\begin{vmatrix} l_1(e^{r_1(\lambda)x}) & l_1(e^{r_2(\lambda)x}) \\ l_2(e^{r_1(\lambda)x}) & l_2(e^{r_2(\lambda)x}) \end{vmatrix} = 0.$$

This is a condition of the form $f(\lambda) = 0$, whose solutions are the operator's eigenvalues. Once these are determined, the system is solved for c_1 and c_2 and the respective eigenfunctions are determined.

§1.5. Boundary Value Problems in Electrostatics, Geophysics and Fluid Mechanics

Electrostatics/Geophysics Equations

Given the domain $\Omega \subset \mathbb{R}^3$, if ρ represents the distribution of electric charge (mass density) and a_i are boundary operators. Then, the electric potential (gravitational) u satisfies the boundary value problem.

$$\begin{aligned} \Delta u &= \rho, \\ a_i(u) &= g_i, \quad i = 1, \dots, m. \end{aligned}$$

Perfect Fluid Equations

Given the domain $\Omega \subset \mathbb{R}^3$, in perfect fluid mechanics the potential for velocities is determined by the same type of problem but with $\rho \equiv 0$.

$$\begin{aligned} \Delta u &= 0, \\ a_i(u) &= g_i, \quad i = 1, \dots, m. \end{aligned}$$

1.5.1. Uniqueness In this course we will study methods that allow us, in particular, to determine solutions to this type of boundary value problems. The important question that arises from the point of view of Physics is whether these solutions are the only possible ones (question of uniqueness). Let us see that it is possible to analyze in a simple way such a question for the Dirichlet boundary value problem

$$\begin{cases} \Delta u = \rho, \\ u|_{S(\Omega)} = g, \end{cases}$$

and Neumann boundary value problem

$$\begin{cases} \Delta u = \rho, \\ \frac{\partial u}{\partial \mathbf{n}} \Big|_{S(\Omega)} = g, \end{cases}$$

for the Poisson equation. We are looking for $u \in C^\infty(\overline{\Omega})$. However, there may be two cases:

- **Inner Problem:** Ω is a bounded set of $\mathbb{R}^3$.
- **Exterior Problem:** Ω is an unbounded set and its complement is bounded.

We must remember the **theorem of divergence** that assures that for a smooth vector field A we have

$$\int_{\Omega} \nabla \cdot A \, d^3 x = \int_{S(\Omega)} A \cdot dS.$$

Applying it to $A = u \nabla u$, with u a smooth function, and bearing in mind that

$$\nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u \Delta u,$$

we get Green's identity

$$\int_{\Omega} (|\nabla u|^2 + u \Delta u) \, d^3 x = \int_{S(\Omega)} u \frac{\partial u}{\partial \mathbf{n}} \, dS.$$

Uniqueness for the inner problem: Suppose now that two solutions u_1 and u_2 of the Dirichlet or of Neumann boundary value problems are given. The difference $u = u_1 - u_2$ satisfies

$$\begin{cases} \Delta u = 0, \\ u|_{S(\Omega)} = 0 \end{cases}$$

or

$$\begin{cases} \Delta u = 0, \\ \frac{\partial u}{\partial \mathbf{n}} \Big|_{S(\Omega)} = 0, \end{cases}$$

respectively. Applying Green's identity to this function we have

$$\int_{\Omega} |\nabla u|^2 \, d^3 x = 0$$

so that $\nabla u = \mathbf{0}$ in the connected set Ω and, consequently, u is constant in Ω . For the problem of Dirichlet boundary value problem as u is zero at the boundary such constant is zero. Therefore, we conclude

Uniqueness for the Inner Problem

- The inner Dirichlet problem admits at most one solution.
- In the inner Neumann problem the difference of two different solutions is a constant.

Uniqueness of the exterior problem: Now Ω is not bounded and we cannot apply carelessly the Green's identity to the difference function $u = u_1 - u_2$ of two solutions. For that reason, we change to a domain that we will denote by Ω_R as described below. The complementary set $\Omega^c := \mathbb{R}^3 \setminus \Omega$ of Ω is bounded, thus there exists a $R_0 > 0$ such that if $R > R_0 \Rightarrow \Omega^c \subset B(0, R)$, where $B(0, R) := \{x \in \mathbb{R}^3 : |x| < R\}$ is a ball of radius R centered at the origin. We denote by $\Omega_R := B(0, R) \setminus \Omega^c$, with boundary given by $S(\Omega_R) = S_1(\Omega) \cup S_2(\Omega) \cup \dots \cup S_m(\Omega) \cup S(0, R)$ with $S(0, R) := \{x \in \mathbb{R}^3 : \|x\| = R\}$ the sphere of radius R centered at the origin and $S(\Omega) = S_1(\Omega) \cup S_2(\Omega) \cup \dots \cup S_m(\Omega)$.

Applying Green's identity to u in Ω_R we get

$$\int_{\Omega_R} |\nabla u|^2 \, d^3 x = \int_{S(0, R)} u \frac{\partial u}{\partial \mathbf{n}} \, dS$$

and remembering that the normal derivative on the sphere is u_r , we are lead to

$$\int_{\Omega_R} |\nabla u|^2 \, d^3 x = \int_{S(0, R)} u u_r \, dS.$$

The surface integral can be bounded as follows

$$\left| \int_{S(0, R)} u u_r \, dS \right| \leq 4\pi R^2 \max_{x \in S(0, R)} |u u_r|,$$

and assuming that¹ $u = O\left(\frac{1}{r}\right)$ and $u_r = O\left(\frac{1}{r^2}\right)$ for $r \rightarrow \infty$, we get

$$\int_{\Omega} |\nabla u|^2 \, d^3 x = \lim_{R \rightarrow \infty} \int_{\Omega_R} |\nabla u|^2 \, d^3 x \leq C \lim_{R \rightarrow \infty} \frac{1}{R} = 0.$$

A harmonic function u is said to be regular at infinity if $u = O\left(\frac{1}{r}\right)$ and $|\nabla u| = O\left(\frac{1}{r^2}\right)$ for $r \rightarrow \infty$. Hence, a harmonic function u regular at infinity satisfies

$$\lim_{R \rightarrow \infty} \int_{S(0, R)} u u_r \, dS = 0.$$

It can be proven that² that a harmonic function that goes to zero uniformly at infinity is a regular at infinity.

¹Recall that given to two functions f and g we say that $f = O(|g|)$ if one can find $M, r_0 > 0$ such that $|f| \leq M|g| \, \forall r \geq r_0$. This O notation is called asymptotic or Landau notation.

²See for example A. Tijonov & A. Samarski, *Equations of Mathematical Physics*, Dover

Uniqueness in the External Problem

To ensure uniqueness in the Dirichlet's problem and uniqueness up to constant in the problem of Neumann it is necessary to demand that at infinity the solutions satisfy a condition of the form

$$u = f + u_0, \quad r \rightarrow \infty,$$

where f is a given fixed function and u_0 is a harmonic function that tends to zero uniformly at infinity.

§1.6. Exercises

1.6.1. Exercises with solutions

1. Determine the commutator $[L, M]$ of the operators

a)

$$L := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad M := x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + z.$$

b)

$$L := \frac{\partial^2}{\partial x \partial y}, \quad M := x^2 + y^2 + z^2.$$

Solution:

- a) To evaluate the commutator we use the Leibnitz property, the commutator is a derivation, which is called inner, i.e., $[A, BC] = [A, B]C + B[A, C]$, and also that

$$\left[\frac{\partial}{\partial x_i}, x_j \right] = \delta_{i,j}. \text{ Hence, we have}$$

$$\begin{aligned} [L, M] &= \left[\frac{\partial^2}{\partial x^2}, x \frac{\partial}{\partial y} \right] - \left[\frac{\partial^2}{\partial y^2}, y \frac{\partial}{\partial x} \right] + \left[\frac{\partial^2}{\partial z^2}, z \right] \\ &= 2 \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x}, x \right] \frac{\partial}{\partial y} - 2 \frac{\partial}{\partial y} \left[\frac{\partial}{\partial y}, y \right] \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial z} \left[\frac{\partial}{\partial z}, z \right] \\ &= 2 \frac{\partial^2}{\partial x \partial y} - 2 \frac{\partial^2}{\partial y \partial x} + 2 \frac{\partial}{\partial z}. \end{aligned}$$

Consequently,,

$$[L, M] = 2 \frac{\partial}{\partial z}.$$

- b) Now we have

$$\begin{aligned} [L, M] &= \left[\frac{\partial^2}{\partial x \partial y}, x^2 + y^2 + z^2 \right] = \left[\frac{\partial^2}{\partial x \partial y}, x^2 \right] + \left[\frac{\partial^2}{\partial x \partial y}, y^2 \right] \\ &= \left[\frac{\partial}{\partial x}, x^2 \right] \frac{\partial}{\partial y} + \left[\frac{\partial}{\partial y}, y^2 \right] \frac{\partial}{\partial x} = 2x \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}. \end{aligned}$$

1.6.2. Exercises

1. Deduce from Euler formulas the following properties

- a) $\overline{e^u} = e^{\bar{u}}, \quad u \in \mathbb{C}.$
b) $e^{u_1} e^{u_2} = e^{u_1 + u_2}, \quad u_1, u_2 \in \mathbb{C}.$
c) $\frac{1}{e^u} = e^{-u}, \quad u \in \mathbb{C}.$

2. To prove that the natural logarithm operation $\log$ allows us to express the solutions of an equation for a complex number u

$$e^u = z_0,$$

in the form

$$u = \log z_0 = \log |z_0| + i(\arg z_0 + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

3. a) Find complex numbers z that satisfy

$$1) \quad e^z = 1, \quad 2) \quad e^z = i, \quad 3) \quad e^{z^2} = 1.$$

- b) Determine the real and imaginary parts, the module and first-order partial derivatives of the following functions:

$$f(x, y, z) = e^{i(k_1x + k_2y + k_3z)}, \quad (k_1, k_2, k_3) \in \mathbb{R}^3,$$

$$f(x, y) = (x + y) e^{x^2 + iy^2},$$

$$f(x, y, z) = e^{-r^2}, \quad f(x, y, z) = \frac{1}{r}, \quad \text{with } r = \sqrt{x^2 + y^2 + z^2}.$$

4. Prove that the associated subspaces $\mathcal{D}$ to sets $\{l_i\}_{i=1}^m$ of linear boundary operators are linear subspaces of linear space $C^\infty(\overline{\Omega})$. That is

$$\forall u, v \in \mathcal{D}, \forall \lambda, \mu \in \mathbb{C} \implies \lambda u + \mu v \in \mathcal{D}.$$

5. Determine the order of the following PDE. Indicate which are linear and which are homogeneous linear.

$$u_x - x u_y = 0; \quad u + u_x u_y = 0,$$

$$\sqrt{1 + x^2} (\cos y) u_x + u_{xy} - e^{x/y} u = x^2,$$

$$u_t + u_{xxx} + u u_x = 0; \quad i u_t + u_{xx} + |u|^2 u = 0,$$

$$u_x + e^y u_y = x^2; \quad u_{xy} = e^u, \quad u_{tt} - \Delta u = xyz.$$

6. Consider the heat equation in one spatial dimension

$$u_t = a^2 u_{xx}$$

- a) Show that the function

$$u = \frac{M}{2\sqrt{\pi a^2 t}} e^{-\frac{x^2}{4a^2 t}},$$

where M is an arbitrary constant, is a solution to the equation.

- b) For each fixed value of $t \neq 0$ represent u as a function of x . Determine its maximum and inflection points as functions of t .
c) Determine the value of the integral

$$\int_{-\infty}^{\infty} u(t, x) dx.$$

Hint: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

- d) Describe how the shape of the u graph changes when it grows t .

- e) If $u(t, x)$ represents the concentration of a substance per unit length, interpret what this solution describes.

7. Considering the Schrödinger equation in a spatial dimension

$$i u_t = -C u_{xx}, C > 0.$$

- a) Prove that the function

$$u = \frac{M}{2\sqrt{\pi C t}} e^{i \frac{x^2}{4Ct}},$$

where M is an arbitrary constant, is a solution to the equation.

- b) For each fixed value of $t \neq 0$ represent the real and imaginary parts of u as functions of x .

- c) For each fixed value of $t \neq 0$ represent $|u|$ as a function of x .

8. Determine a general solution of the equations:

$$\begin{aligned} u_{xy} &= 0; & u_{xxyy} &= 0, \\ 3u_y + u_{xy} &= 0; & u_{xy} &= x \\ u_x - u_y &= 0; & u_x + u_y &= 0 \\ u_{tt} - c^2 u_{xx} &= 0; & u_{tt} - c^2 u_{xx} &= t. \end{aligned}$$

9. Determine unit normal outgoing vectors $\mathbf{n}$ for the following curves and surfaces:

the circle: $(x - a)^2 + (y - b)^2 = r^2$,

the sphere: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$,

the cylinder: $(x - a)^2 + (y - b)^2 = r^2, 0 \leq z \leq h$,

the inner rectangle: $a \leq x \leq b, c \leq y \leq d$,

the interior parallelepiped: $a \leq x \leq b, c \leq y \leq d, e \leq z \leq f$.

10. Let the following differential operators be

$$L = \frac{\partial^2}{\partial x \partial y}, \quad M = \sin(xy) \frac{\partial}{\partial y}.$$

Calculate the result $LM u$ of applying the product operator LM to the function $u = x e^y$.

11. Be $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$ the Laplacian operator in three dimensions. Compute the result Δu of applying the operator Δ to the function

$$u = \cos r, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

12. Show that the commutator of two differential operators satisfies the properties

$$[A, BC] = [A, B]C + B[A, C], \quad [AB, C] = A[B, C] + [A, C]B.$$

Determine the following commutators:

$$[H, Q], \quad [H, P],$$

where

$$H = -\partial_{xx}, \quad Q = x, \quad P = -i \partial_x.$$

13. Consider the differential operator $Lu = -i u_x$ over the domain $\mathcal{D}$ of the functions u of infinite class in the interval $[-1, 1]$ that satisfy the boundary condition $u(1) = i u(-1)$. Determine the eigenvalues and eigenfunction of L .
14. Find operator eigenvalues and eigenfunctions, $Lu = -u_{xx}$, over domains $\mathcal{D}$ of functions in $C^\infty[-1, 1]$ that meet boundary conditions

$$a) \quad u(0) = u(1), \quad u_x(0) = u_x(1).$$

$$b) \quad u_x(0) = 0, \quad u_x(1) = 0.$$

2. SEPARATION OF VARIABLES

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SEPARATION of variables is one of the oldest methods of generating solutions of linear PDEs. Basically it allows to reduce the problem of seeking solutions to certain types of PDEs to problems solving ODEs. On the other hand, it constitutes the basic element for eigenfunction expansion method to determine solutions of boundary value and/or initial conditions problems.

§2.1. Separable Linear Homogeneous PDEs



WE now discuss what is meant by separable differential operator, and how to build solutions with the so-called method of separation of variables for homogeneous PDEs. We will see how the method leads us to a general solution of the equation.

2.1.1. Separable Differential Operators Let L be a linear differential operator acting on $C^\infty(\overline{\Omega})$ where Ω is a domain in $\mathbb{R}^n$

$$Lu := \sum_{\alpha} a_{\alpha}(x) D^{\alpha} u.$$

For the following discussion it is convenient to write L in the form

$$L = L\left(x_0, x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right),$$

which indicates that L takes derivatives with respect to the variables $(x_0, \dots, x_{n-1})$ and that their coefficients $a_{\alpha} = a_{\alpha}(x)$ depend on those variables.

Separable Operators

It is said that L is a **separable** operator with respect to the variable x_0 if it can be split down into the sum of two operators

$$L = A + B,$$

of the form

$$A = A\left(x_0; \frac{\partial}{\partial x_0}\right), \quad B = B\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right).$$

In other words, A only takes derivations with respect to x_0 and its coefficients can only depend on x_0 , while B takes derivatives with respect to the variables $(x_1, \dots, x_{n-1})$ and their coefficients can only depend on $(x_1, \dots, x_{n-1})$.



Examples:

1. The operator

$$Lu = xu_x + yu_{yy} + yzu_z + x^2u,$$

is separable with respect to x since $L = A + B$ being

$$A\left(x; \frac{\partial}{\partial x}\right) = x \frac{\partial}{\partial x} + x^2, \quad B\left(y, z; \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = y \frac{\partial^2}{\partial y^2} + yz \frac{\partial}{\partial z}.$$

2. The Laplacian operator

$$Lu := \Delta u = u_{xx} + u_{yy} + u_{zz},$$

is separable from any of its three variables (x, y, z) .

3. The d'Alembertian operator that appears in the 1+3 dimensional wave equation:.

$$Lu := u_{tt} - c^2 \Delta u,$$

is separable with respect to any of its four variables (t, x, y, z) .

4. The operator who appears in the Schrödinger equation in 1+3 dimensions:

$$Lu := i\hbar u_t + \frac{\hbar^2}{2m} \Delta u + V(x, y, z)u,$$

is separable in the variable t . It will be separable with respect to the variable x only when the function $V = V(x, y, z)$ is $V = u(x) + v(y, z)$. Analogously for variables y and z .

2.1.2. Solutions to Separable Homogeneous Linear PDEs Separation of Variables Method (SVM) applies to homogeneous linear PDEs where the operator L is separable..

Separation of Variables Method

Given a linear homogeneous PDE of the form

$$A\left(x_0; \frac{\partial}{\partial x_0}\right)u + B\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)u = 0,$$

any pair of functions

$$v = v(x_0), \quad w = w(x_1, \dots, x_{n-1}),$$

which satisfy the system of equations

$$Av = \lambda v,$$

$$Bw = -\lambda w,$$

with arbitrary $\lambda \in \mathbb{C}$, determine a solution given by

$$u = v(x_0)w(x_1, \dots, x_{n-1}).$$

PROOF. Given two non-zero autofunctions $v = v(x_0)$ and $w = w(x_1, \dots, x_{n-1})$, by the form of the operators A and B it is clear that $u = vw$ satisfies

$$\begin{cases} Au = A(vw) = wAv = \lambda vw = \lambda u, \\ Bu = B(vw) = vBw = -\lambda vw = -\lambda u, \end{cases}$$

therefore

$$Au + Bu = \lambda u - \lambda u = 0.$$

□

Observations:



1. The utility of the SVM lies in the fact that it reduces the problem of the search for solutions of a PDE

$$Au + Bu = 0,$$

with n variables independent $(x_0, x_1, \dots, x_{n-1})$ to a system which consists of an ODE

$$Av = \lambda v,$$

and a PDE

$$Bw + \lambda w = 0,$$

with $(n-1)$ independent variables $(x_1, \dots, x_{n-1})$, which is in turn another homogeneous linear PDE. If to this PDE we can apply the SVM, that is to say if it is separable with respect to some of its independent variables, we will reduce it to a ODE and a homogeneous linear PDE with $n-2$ independent variables. Iterating the process, we conclude that if it is possible apply the SVM $n-1$ times we will have managed to reduce the PDE

to a system of n ordinary differential equations. When this happens we say that the PDE is **resoluble through the SVM**.

2. The functions v and w that appear in the SVM are eigenfunctions of the operators A and B with eigenvalues λ and $-\lambda$, respectively.
3. When we apply SVM we assume that λ is an arbitrary complex parameter. Then, the obtained solutions u will depend on of that parameter λ . In a PDE with n independent variables, soluble through SVM we must apply SVM $n - 1$ times, then the solutions obtained will depend on the $n - 1$ parameters $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ introduced by SVM. In other words, the SVM provides a family $(n - 1)$ -parametric solutions

$$u = u(\lambda, x).$$

4. As the PDE

$$Au + Bu = 0,$$

is a linear and homogeneous, given any family of solutions,

$$\{u = u(\lambda, x), \lambda \in \Lambda\},$$

any linear combination of elements in the family

$$\sum_{k=1}^N c_k u(\lambda_k, x),$$

is also a solution of the PDE. Under appropriate conditions we will also be able to build solutions through generalized linear combinations of the infinite series type

$$\sum_{k=1}^{\infty} c_k u(\lambda_k, x),$$

or an integral expression.

$$\int_{\Lambda_0} c(\lambda) u(\lambda, x) d^{n-1} \lambda, \quad \Lambda_0 \subset \Lambda.$$

Even also a superposition of both forms

$$\sum_{k=1}^{\infty} c_k u(\lambda_k, x) + \int_{\Lambda_0} c(\lambda) u(\lambda, x) d^{n-1} \lambda.$$

Although we will not use the concept of general solution, it can be proved in concrete examples that this scheme to generate solutions leads, under appropriate condition, to general solutions of the problem.

§2.2. Separable Homogeneous Linear Boundary Value Problems



SEPARATION of variables can also be applied in situations where a linear equation in partial derivatives must be satisfied in a domain and, in addition, at the boundary must meet m boundary conditions

$$\begin{cases} Lu = 0, \\ l_i(u) = 0, \quad i = 1, \dots, m, \end{cases}$$

Boundary Operators in One Variable

It is said that a boundary operator acts only on the variable x_0 if for all pair of functions $v = v(x_0)$ and $w = w(x_1, \dots, x_{n-1})$ is satisfied

$$l(vw) = w(x_1, \dots, x_{n-1})l(v).$$

Similarly, it is said that l acts only on variables $(x_1, \dots, x_{n-1})$ when for all pair of functions $v = v(x_0)$ and $w = w(x_1, \dots, x_{n-1})$ is satisfied that

$$l(vw) = v(x_0)l(w).$$

Examples:



1. Let $\Omega \subset \mathbb{R}^3$, the functionals

$$l(u) = u|_{x=1}, \quad l(u) = u_x|_{x=1},$$

only acts on the variable x . However,

$$l(u) = (u + u_y)|_{z=1}, \quad l(u) = u|_{y+z=3},$$

only acts on variables (y, z) .

2. Let $\Omega \subset \mathbb{R}^2$ and Γ the unit circle, the functional

$$l(u) = u|_{\Gamma},$$

is not separable either with respect to x neither to y . However, if we take polar coordinates (r, θ) , the functional is separable with respect to the radial variable r since

$$l(u) = u|_{r=1}.$$

SVM can be applied to homogeneous boundary value problems when the PDE is separable and the boundary conditions are appropriate. We can summarize the propitious situation in the following statement.

Separation of Variables for Homogeneous Boundary Value Problems

Let us consider a boundary value problem of the form

$$\begin{aligned} A\left(x_0; \frac{\partial}{\partial x_0}\right)u + B\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)u &= 0, \\ a_i(u) &= 0, \quad i = 1, \dots, r, \\ b_j(u) &= 0, \quad j = 1, \dots, s, \end{aligned}$$

such that

1. Boundary operators a_i acts only on the variable x_0 .
2. Boundary operators b_j acts only on the variables $(x_1, \dots, x_{n-1})$.

Then, every couple of functions

$$v = v(x_0), \quad w = w(x_1, \dots, x_{n-1}),$$

that satisfy the systems of equations

$$\begin{cases} Av = \lambda v, \\ a_i(v) = 0, \quad i = 1, \dots, r, \\ Bw = -\lambda w, \\ b_j(w) = 0, \quad j = 1, \dots, s, \end{cases}$$

with $\lambda \in \mathbb{C}$, determine a solution given by

$$u = v(x_0)w(x_1, \dots, x_{n-1}).$$

PROOF. The only thing left to prove is that if $v = v(x_0)$ satisfies the first r boundary conditions and $w = w(x_1, \dots, x_{n-1})$ satisfies the s second boundary conditions, then $u = vw$ is solution to our problem. But this property is an immediate consequence of the properties of boundary operators a_i and b_j , with $a_i(u) = a_i(vw) = wa_i(v) = 0$ and $b_j(u) = b_j(vw) = vb_j(w) = 0$. $\square$

Examples:

1. Let $v(x, z)$ be the velocity field of a stationary fluid moving in the plane xz above an impenetrable background at depth $z = -h$.

The fluid velocity potential $u = u(x, z)$ is defined by the relationship

$$\mathbf{v} = \nabla u,$$

and satisfies the boundary value problem for the planar Laplace equation

$$\begin{cases} u_{xx} + u_{zz} = 0, \\ u_z|_{z=-h} = 0. \end{cases}$$

We can apply the SVM, looking for solutions of the form $u = v(z)w(x)$ where

$$v_{zz} = \lambda v, \quad v_z|_{z=-h} = 0,$$

and

$$w_{xx} = -\lambda w.$$

Introducing the change $\lambda = k^2$, it is obtained that the solution of the ODE for v is

$$v(z) = A e^{kz} + B e^{-kz},$$

imposing the boundary value condition to v one obtains

$$k(A e^{-kh} - B e^{kh}) = 0, \quad B = A e^{-2kh}.$$

That is

$$v(z) = A(e^{kz} + e^{-2kh} e^{-kz}) = 2A e^{-kh} \cosh k(z + h).$$

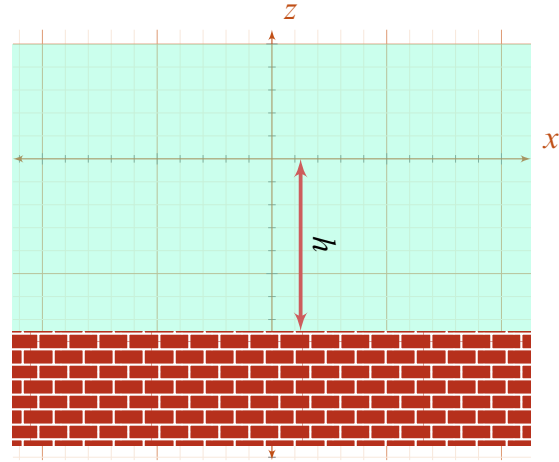
On the other hand, solving the ODE for w

$$w(x) = C e^{ikx} + D e^{-ikx}.$$

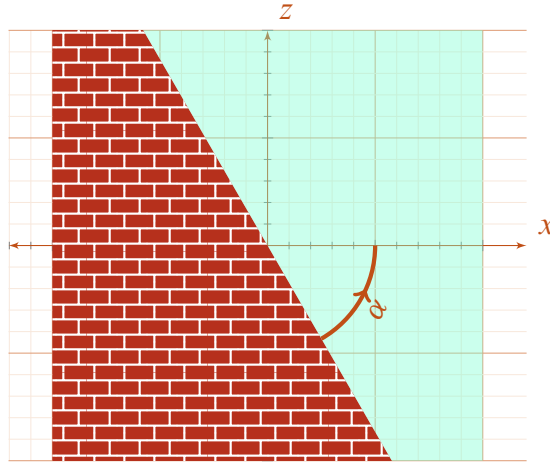
In this way we have determined the following family of solutions

$$u(x, z) = 2\tilde{A}(C e^{ikx} + D e^{-ikx}) \cosh k(z + h), \quad \tilde{A} \in \mathbb{C}.$$

2. It is very frequent to find boundary value problems in which the SVM is not initially applicable, but it is when making an appropriate change of independent variables. For example, let's consider again the problem of the velocity field of a stationary perfect fluid moving in the xz plane over an impenetrable bottom, but let's suppose now that this bottom is not horizontal but is described by a line of equation $z = -mx$, $m = \tan \alpha$.



Perfect fluid on horizontal plane



Perfect fluid on an inclined plane

The velocity potential of the fluid $u = u(x, z)$ then satisfies the boundary value problem

$$\begin{cases} u_{xx} + u_{zz} = 0, \\ \frac{\partial u}{\partial \mathbf{n}} \Big|_{z=-mx} = 0. \end{cases}$$

Bearing in mind that a unit normal at the bottom is

$$\mathbf{n} = \sin \alpha \mathbf{i} + \cos \alpha \mathbf{k}$$

the boundary value problem becomes

$$\begin{cases} u_{xx} + u_{zz} = 0, \\ (\sin \alpha u_x + \cos \alpha u_z) \Big|_{z=-mx} = 0. \end{cases}$$

It is clear that the boundary value condition does not allow for a direct application of SVM. However, if we introduce the change of coordinates

$$x' = \cos \alpha x - \sin \alpha z, \quad z' = \sin \alpha x + \cos \alpha z,$$

the problem is reduced to

$$\begin{cases} u_{x'x'} + u_{z'z'} = 0, \\ u_z \Big|_{z'=0} = 0, \end{cases}$$

to which we can apply SVM, and as we saw in the previous example

$$u(x', z') = 2A(C e^{ikx'} + D e^{-ikx'}) \cosh kz'.$$

To write the solution in terms of the variables (x, z) it is enough to introduce in u the expressions of (x', z') in function of (x, z) .

§2.3. Application to the Equations of Mathematical Physics



MATHEMATICAL-PHYSICS problems solved by the SVM frequently require the consideration of a linear equation in partial derivatives of evolutionary type, in which a variable of type *temporal* is clearly separated from the rest of the variables, let's say of spatial type. The variable *time* belongs to an interval and the spatial variables to a domain in multidimensional space, therefore the domain for the problem is the Cartesian product of the time interval with the spatial region. In addition, initial conditions for the temporal variable and boundary conditions

at the spatial boundary will be satisfied

$$(4) \quad \left\{ \begin{array}{l} \alpha \frac{\partial^N u}{\partial t^N} - Lu = f, \quad t \in (a, b), \quad \mathbf{x} \in \Omega_0 \subseteq \mathbb{R}^{n-1}, \\ \left\{ \frac{\partial^i u}{\partial t^i} \right|_{t=t_0} = f_i(\mathbf{x}), \quad i = 1, \dots, N-1, \\ l_j(u) = g_j(t, \mathbf{x}), \quad j = 1, \dots, s, \end{array} \right.$$

where $\alpha \in \mathbb{C}$, L is an operator in spatial variables $\mathbf{x} = (x_1, \dots, x_{n-1})$

$$L = L\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)$$

and the l_j are boundary operators in the spatial variables. These problems arise naturally in Electromagnetism, Quantum Mechanics and Mechanics of Continuous Media. It should be noted that in general the problem is not homogeneous, although the first member of the PDE

$$\alpha \frac{\partial^N u}{\partial t^N} - Lu$$

is a separable operator with respect to the variable t .

As we will see later, the method to solve these problems is the *eigenfunction expansion method* that is based on the construction of the solution through families of solutions of the **associated spectral problem**

$$(5) \quad \left\{ \begin{array}{l} Lw = \lambda w, \\ l_j(w) = 0, \quad j = 1, \dots, s. \end{array} \right.$$

The latter is a homogeneous problem, which when the appropriate separability conditions are met, can be solved by the SVM. The solutions $w(\mathbf{x})$ of (5) determine the so-called **standing wave series** of the physical problem described by (4). These standing waves are functions of the form

$$u = e^{-i\omega t} w(\mathbf{x}), \quad \omega \in \mathbb{R},$$

where $w(\mathbf{x})$ is a solution of (5) and the time factor $e^{-i\omega t}$ is found demanding that u satisfies the PDE in (4). This condition determines the *frequency* ω of the wave, since when replacing the u function in the PDE you get

$$\alpha(-i\omega)^N e^{-i\omega t} w = \lambda e^{-i\omega t} w,$$

by simplifying you get the *dispersion relation*

$$\alpha(-i\omega)^N = \lambda,$$

which links ω to λ . Only if there are real solutions ω to this relationship can we talk about standing waves in the corresponding physical phenomenon.

In many interesting physical problems you have $L = \Delta$, so the corresponding eigenvalue equation is

$$-\Delta w = \lambda w.$$

Denoting $\lambda = k^2$ this equation becomes the **Helmholtz equation**

$$\Delta w + k^2 w = 0,$$

that will be fundamental in what follows.



Examples:

1. Let's consider a typical problem for the wave equation in 1 + 1 dimensions

$$\left\{ \begin{array}{l} c^{-2}u_{tt} = u_{xx}, \quad t \in (a, b), \quad x \in (0, l), \\ \left\{ \begin{array}{l} u|_{t=t_0} = f_1(x), \\ u_t|_{t=t_0} = f_2(x); \end{array} \right. \\ \left\{ \begin{array}{l} u|_{x=0} = g_1(t), \\ u|_{x=l} = g_2(t). \end{array} \right. \end{array} \right.$$

The associated spectral problem is

$$\begin{aligned} w_{xx} &= \lambda w, \\ \left\{ \begin{array}{l} w(0) = 0, \\ w(l) = 0. \end{array} \right. \end{aligned}$$

The solutions are

$$w_n(x) = \sin(k_n x), \quad k_n = \frac{\pi}{l}n, \quad n = 1, 2, \dots,$$

with $\lambda_n = -k_n^2$. The dispersion relation is

$$c^{-2}(-i\omega)^2 = -k^2,$$

then the standing waves of the problem are

$$u_n(t, x) = e^{-i\omega_n t} \sin(k_n x), \quad \omega_n = \pm c k_n.$$

2. Let us now consider a problem with the free Schrödinger equation in 1 + 1 dimensions

$$\left\{ \begin{array}{l} i\hbar u_t = -\frac{\hbar^2}{2m}u_{xx}, \quad t \in (a, b), \quad x \in (0, l), \\ u|_{t=t_0} = f_1(x); \\ \left\{ \begin{array}{l} u|_{x=0} = g_1(t), \\ u|_{x=l} = g_2(t). \end{array} \right. \end{array} \right.$$

The associated spectral problem is

$$\begin{aligned} -\frac{\hbar^2}{2m}w_{xx} &= \lambda w, \\ \left\{ \begin{array}{l} w(0) = 0, \\ w(l) = 0. \end{array} \right. \end{aligned}$$

The solutions are

$$w_n(x) = \sin(k_n x), \quad k_n = \frac{\pi}{l} n, \quad n \geq 1,$$

where

$$\lambda_n = \frac{\hbar^2 k_n^2}{2m}.$$

The dispersion relation is

$$\omega = \frac{\hbar k^2}{2m},$$

and the standing waves of the problem are the following *stationary states*

$$u_n(t, x) = e^{-i\omega_n t} \sin(k_n x), \quad \omega_n = \frac{\hbar k_n^2}{2m}.$$

§2.4. Helmholtz Equation in Cartesian Coordinates



ELMHOLTZ equation in Cartesian coordinates, that can be considered as a spectral problem in three dimensions and, moreover, being a homogeneous linear PDE separable with respect to the three Cartesian coordinates to which the SVM can be applied to find solutions, reads

$$u_{xx} + u_{yy} + u_{zz} + k^2 u = 0, \quad k \in \mathbb{C}.$$

In this way we start looking for a solution of the form

$$u(x, y, z) = X(x)w(y, z),$$

so that

$$\begin{cases} \frac{d^2 X}{dx^2} = \lambda X, \\ w_{yy} + w_{zz} + k^2 w = -\lambda w, \end{cases}$$

where $\lambda = -k_1^2 \in \mathbb{C}$. Separating variables again, we seek for a solution to the second equation of the form

$$w(y, z) = Y(y)Z(z)$$

such that

$$\begin{cases} \frac{d^2 Y}{dy^2} = \lambda' Y, \\ \frac{d^2 Z}{dz^2} + (k^2 - k_1^2) Z = -\lambda' Z \end{cases}$$

where $\lambda' = -k_2^2 \in \mathbb{C}$. Writing $k_3^2 = k^2 - k_1^2 - k_2^2$ we see that the solution sought has the form

$$(6) \quad \begin{cases} u_{k_1, k_2, k_3}(x, y, z) = X_{k_1}(x)Y_{k_2}(y)Z_{k_3}(z), \\ X_{k_1}(x) = \begin{cases} a_1x + b_1, & k_1 = 0, \\ a_1 e^{ik_1x} + b_1 e^{-ik_1x}, & k_1 \neq 0, \end{cases} \\ Y_{k_2}(y) = \begin{cases} a_2y + b_2, & k_2 = 0, \\ a_2 e^{ik_2y} + b_2 e^{-ik_2y}, & k_2 \neq 0, \end{cases} \\ Z_{k_3}(z) = \begin{cases} a_3z + b_3, & k_3 = 0 \\ a_3 e^{ik_3z} + b_3 e^{-ik_3z}, & k_3 \neq 0. \end{cases} \end{cases}$$

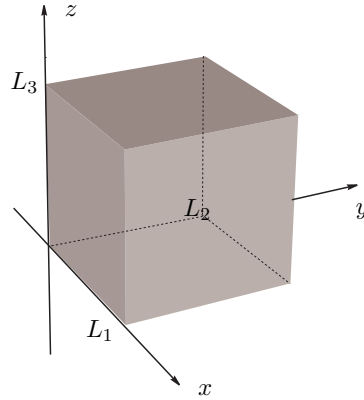
with

$$k^2 = k_1^2 + k_2^2 + k_3^2.$$

The boundary conditions to which these solutions are suited are those in which the boundary is formed by planes parallel to the coordinate planes. Let's analyze three examples of this type.

Quantum particle in an impenetrable box Stationary states of energy E of a quantum particle in a impenetrable box of sides L_1 , L_2 and L_3 is described by the next Dirichlet's problem

$$\begin{cases} -\frac{\hbar^2}{2m} \Delta u = Eu, \\ u|_{\text{walls}} = 0. \end{cases}$$



The PDE is a Helmholtz equation with $k^2 = \frac{2mE}{\hbar^2}$ and the boundary value condition breakdown into the following conditions

$$\begin{aligned} u(0, y, z) &= 0, & u(L_1, y, z) &= 0; \\ u(x, 0, z) &= 0, & u(x, L_2, z) &= 0; \\ u(x, y, 0) &= 0, & u(x, y, L_3) &= 0. \end{aligned}$$

Let us impose these boundary conditions on $u_{k_1,k_2,k_3}(x,y,z) = X_{k_1}(x)Y_{k_2}(y)Z_{k_3}(z)$ in (6). One can check that when any k_1, k_2 or k_3 cancels the solution is trivial. Hence we require $k_1, k_2, k_3 \in \mathbb{N}$. For X_{k_1} we have

$$\begin{aligned} a_1 + b_1 &= 0, \\ a_1 e^{ik_1 L_1} + b_1 e^{-ik_1 L_1} &= 0 \end{aligned}$$

which implies

$$a_1 = -b_1, \quad k_1 = \frac{\pi}{L_1} n_1, \quad n_1 \in \mathbb{N}$$

and, consequently,

$$X_{k_1}(x) = A_1 \sin \frac{\pi n_1}{L_1} x.$$

An analogous procedure in the variables y, z allows us to conclude that

$$u_{k_1,k_2,k_3}(x,y,z) = a \sin\left(\frac{\pi n_1}{L_1} x\right) \sin\left(\frac{\pi n_2}{L_2} y\right) \sin\left(\frac{\pi n_3}{L_3} z\right), \quad n_1, n_2, n_3 \in \mathbb{N}.$$

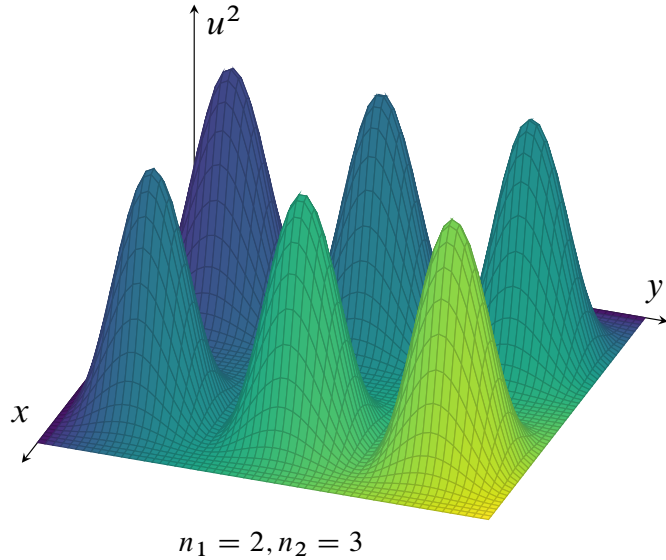
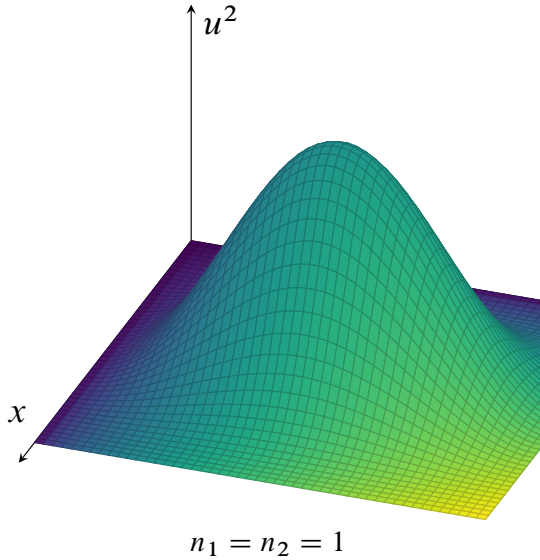
The parameter $k^2 = k_1^2 + k_2^2 + k_3^2$ is, therefore, of the form

$$k^2 = \pi^2 \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right)$$

and the following energy quantification is obtained

$$E_{n_1,n_2,n_3} = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right),$$

with $n_1, n_2, n_3 \in \mathbb{N}$. We can consider a version of the same problem in two dimensions making $z = 0$. We now show the graphic of $|u(x,y)|^2$ which represents the probability density of presence of the quantum particle at the point (x,y) of the rectangle $[0, 2] \times [0, 4]$



Quantum particle in a box with periodic conditions We now impose periodic boundary conditions, that is:

$$\begin{aligned} u(0, y, z) &= u(L_1, y, z), & u_x(0, y, z) &= u_x(L_1, y, z); \\ u(x, 0, z) &= u(x, L_2, z), & u_y(x, 0, z) &= u_y(x, L_2, z); \\ u(x, y, 0) &= u(x, y, L_3), & u_z(x, y, 0) &= u_z(x, y, L_3). \end{aligned}$$

These conditions applied to eigenfunctions u_{k_1, k_2, k_3} of (6) lead us for $k_i = 0$ to $a_i = 0$ and when $k_i \neq 0$ to

$$\begin{aligned} a_i + b_i &= a_i e^{ik_i L_i} + b_i e^{-ik_i L_i}, \\ a_i - b_i &= a_i e^{ik_i L_i} - b_i e^{-ik_i L_i}, \end{aligned}$$

with $i = 1, 2$ and 3 . For these linear systems to have non-trivial solutions $(a_i, b_i) \neq (0, 0)$ we must demand that

$$\begin{vmatrix} 1 - e^{ik_i L_i} & 1 - e^{-ik_i L_i} \\ 1 - e^{ik_i L_i} & -(1 - e^{-ik_i L_i}) \end{vmatrix} = 0;$$

this is

$$(e^{ik_i L_i/2} - e^{-ik_i L_i/2})^2 = 0.$$

Condition that is identically satisfied when

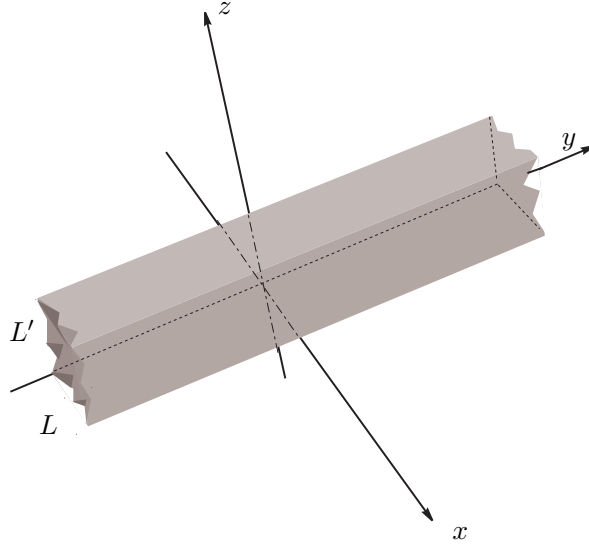
$$k_i = \frac{2\pi}{L_i} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3.$$

Therefore, the possible energies are

$$E_{n_1, n_2, n_3} = \frac{\hbar^2}{2m} 4\pi^2 \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right).$$

So solutions u are those of (6) with these discretized $k_i = \frac{2\pi}{L_i} n_i$.

Fluid in a parallelepipedic pipe The fluid in a pipe, modeled as the set $[0, L] \times \mathbb{R} \times [0, L'] \subset \mathbb{R}^3$, as shown in the following picture,



has a velocity potential u determined by the following Neumann boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0, \\ \begin{cases} u_x(0, y, z) = 0, \\ u_x(L, y, z) = 0, \\ u_z(x, y, 0) = 0, \\ u_z(x, y, L') = 0. \end{cases} \end{cases}$$

Let us, therefore, impose upon $u_{k_1, k_2, k_3} = X_{k_1}(x)Y_{k_2}(y)Z_{k_3}(z)$ these conditions. For $k_1 = 0$ we have $X_0(x) = a_1x + b_1$, with a_1, b_1 constants to be determined by the boundary conditions. From $X'_0(0) = X'_0(L) = 0$ we deduce that $a_1 = 0$ but b_1 is free. When $k_1 \neq 0$, for the function $X_{k_1} = a_1 e^{ik_1x} + b_1 e^{-ik_1x}$ we have

$$X'(0) = X'(L) = 0,$$

that leads to the system

$$\begin{aligned} a_1 - b_1 &= 0, \\ a_1 e^{ik_1L} - b_1 e^{-ik_1L} &= 0. \end{aligned}$$

The existence of non-trivial solutions $(a_1, b_1) \neq (0, 0)$ requires

$$\begin{vmatrix} 1 & -1 \\ e^{ik_1L} & -e^{-ik_1L} \end{vmatrix} = 0,$$

this is $\sin 2k_1L = 0$ and, consequently,

$$k_1 = \frac{\pi}{L}n, \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

As $a_1 = b_1$ the eigenfunctions are

$$X_n(x) = \cos \frac{\pi}{L} n x.$$

A similar analysis in the z variable leads to

$$k_3 = \frac{\pi}{L'} n', \quad n' \in \mathbb{Z}_+,$$

with associated eigenfunctions

$$Z_{n'}(z) = \cos \frac{\pi}{L'} n' z.$$

For the variable y we don't have boundary conditions to impose, however, we are now solving Laplace's equation; this is the Helmholtz's equation with

$$k^2 = 0.$$

So, as $k_1^2 + k_2^2 + k_3^2 = k^2 = 0$, we deduce that

$$k_2 = \pm i \pi \sqrt{\frac{n^2}{L^2} + \frac{n'^2}{L'^2}},$$

and, for $k_2 \neq 0$, we obtain the following solutions

$$u_{n,n'}(x, y, z) = \cos \frac{n \pi x}{L} \cos \frac{n' \pi z}{L'} e^{\pm \pi \sqrt{\frac{n^2}{L^2} + \frac{n'^2}{L'^2}} y},$$

while for $n = n' = 0$ the solution is

$$u_{n,n'}(x, y, z) = Ay + B,$$

with A, B arbitrary constants.

§2.5. Exercises

2.5.1. Exercises with solutions

1. Apply the MSV to the following linear and first-order PDEs

$$\begin{aligned}u_x + u_y &= 0, \\xu_x + u_y &= 0, \\xu_x - yu_y &= 0.\end{aligned}$$

Provide a general solution candidate by linear superposition.

Solution: For the first equation we have the operators

$$A = \frac{d}{dx}, \quad B = \frac{d}{dy},$$

and we look for factorized solutions in the form

$$u(x, y) = v(x)w(y),$$

where $v(x)$ and $w(y)$ are eigenfunctions

$$Av = \lambda v, \quad Bw = -\lambda w.$$

That is,

$$v' = \lambda v, \quad w' = -\lambda w.$$

Therefore, we can write

$$v(x) = c_1 e^{\lambda x}, \quad w(y) = c_2 e^{-\lambda y},$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants and the eigenvalues $\lambda \in \mathbb{C}$ are not constrained in any way. This leads to the factored solution

$$u(x, y) = c e^{\lambda(x-y)},$$

where we have absorbed the two constants c_1, c_2 in the constant $c = c_1 c_2$. Therefore, the SVM gives us the following solution

$$u(x, y) = \int c(\lambda) e^{\lambda(x-y)} d\lambda,$$

that can be considered a general solution.

We observe that the general solution to this equation is in the form of $u(x, y) = f(x - y)$, for any derivable function f .

In the second case we have

$$A = x \frac{d}{dx}, \quad B = \frac{d}{dy},$$

and the factors of

$$u(x, y) = v(x)w(y),$$

will be eigenfunctions

$$Av = \lambda v,$$

$$Bw = -\lambda w.$$

and, therefore, will satisfy,

$$xv' = \lambda v,$$

$$w' = -\lambda w.$$

In this way, we conclude

$$v(x) = c_1 x^\lambda,$$

$$w(y) = c_2 e^{-\lambda y},$$

that leads to

$$u(x, y) = c e^{\lambda(\log x - y)}.$$

and given a function $c(\lambda)$ to the general solution

$$u(x, y) = \int c(\lambda) e^{\lambda(\log x - y)} d\lambda.$$

In particular, given a derivable function $f(x)$ and $x > 0$ the general solution of the equation is $u(x, y) = f(\log x - y)$.

Finally, for the third equation, we have

$$A = x \frac{d}{dx},$$

$$B = -y \frac{d}{dy},$$

and the factorized solutions

$$u(x, y) = v(x)w(y),$$

require their factors to be solutions of

$$xv' = \lambda v,$$

$$yw' = \lambda w.$$

Therefore, we can write

$$v(x) = c_1 x^\lambda,$$

$$w(y) = c_2 y^\lambda.$$

So,

$$u(x, y) = c x^\lambda y^\lambda.$$

The general solution is

$$\int c(\lambda) (xy)^\lambda d\lambda.$$

Given any derivable function $f(x)$ the function $u(x, y) = f(xy)$ is a general solution.

2. Apply the SVM to the Laplace equation in two dimensions

$$u_{xx} + u_{yy} = 0.$$

Solution: We seek for solutions in factorized form $u(x, y) = v(x)w(y)$, which will be solutions to the following eigenvalue problems

$$v'' = k^2 v, \quad w'' = -k^2 w.$$

Therefore, they can be expressed as follows

$$v(x) = a e^{kx} + b e^{-kx}, \quad w(y) = c e^{iky} + d e^{-iky},$$

for arbitrary complex constants $a, b, c, d \in \mathbb{C}$. Therefore, using the complex variable $z = x + iy$ and we can write

$$u(x, y) = A e^{kz} + B e^{-kz} + C e^{k\bar{z}} + D e^{-k\bar{z}},$$

with $A, B, C, D \in \mathbb{C}$ arbitrary constants. This leads us to the following general solution

$$u(x, y) = \int A(k) e^{kz} dk + \int C(k) e^{k\bar{z}} dk.$$

In fact, the most general harmonic function is the sum of a holomorphic function $f(z)$, i.e. does not depend on $\bar{z}$, and an antiholomorphic $g(\bar{z})$, i.e. it does not depend on z , that is to say $u(x, y) = f(z) + g(\bar{z})$ is a general solution.

3. Apply the MSV to the 2D Laplace equation

$$u_{xx} + u_{yy} = 0,$$

with Dirichlet boundary conditions

$$\begin{cases} u(0, y) = 0, \\ u(1, y) = 0. \end{cases}$$

Solution: We use the solutions discussed in the previous problem and impose the corresponding boundary conditions:

$$\begin{cases} v(0) = 0, \\ v(1) = 0. \end{cases}$$

This is, on the one hand

$$\begin{pmatrix} 1 & 1 \\ e^k & e^{-k} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which possesses non-trivial solutions when

$$0 = \begin{vmatrix} 1 & 1 \\ e^k & e^{-k} \end{vmatrix} = e^k - e^{-k},$$

and hence $e^{2k} = 1$, whose solutions are

$$k = n\pi i, \quad n = 1, 2, \dots$$

In addition, you must have $a + b = 0$. So

$$v(x) = A \sin n\pi x, \quad A \in \mathbb{C}, \quad n = 1, 2, \dots$$

The solution to the Laplace equation will be

$$u(x, y) = \sin n\pi x (c e^{-n\pi y} + d e^{n\pi y}), \quad c, d \in \mathbb{C}.$$

4. Apply the MSV to the 2D Laplace equation

$$u_{xx} + u_{yy} = 0,$$

with boundary conditions

$$\begin{cases} u(0, y) = 0, \\ u(1, y) = 0, \\ u(x, 0) = 0. \end{cases}$$

Solution: We use the solutions discussed in the previous problem and impose the corresponding additional boundary value condition

$$w(0) = 0.$$

So, we get $c + d = 0$, and therefore

$$w(y) = B \sinh n\pi y.$$

We have concluded that

$$u(x, y) = A \sin n\pi x \sinh n\pi y,$$

is the solution to our problem. The most general solution that the SVM provides will take the following form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi y.$$

For some adequate sequence $\{c_n\}_{n=1}^{\infty}$.

5. Apply the method of separation of variables to the 1D heat equation, with $D > 0$,

$$u_t = Du_{xx}.$$

Solution: The separation of variables is done with the operators

$$A = \frac{d}{dt}, \quad B = -D \frac{d^2}{dx^2},$$

looking for solution in the form $u(t, x) = v(t)w(x)$, each factor being an eigenfunction of the corresponding differential operator, i.e.,

$$Av = -k^2 v, \quad Bw = k^2 w,$$

where we've written the eigenvalue as $\lambda = k^2$ with $k \in \mathbb{C}$. Therefore, the following equations must be fulfilled

$$v' = -k^2 v, \quad Dw'' = -k^2 w,$$

whose solutions are

$$v(t) = c e^{-k^2 t}, \quad w(x) = a e^{\frac{ikx}{\sqrt{D}}} + b e^{-\frac{ikx}{\sqrt{D}}}.$$

Therefore, the solution of the heat equation is

$$u(t, x) = a e^{-k^2 t + \frac{ikx}{\sqrt{D}}} + b e^{-k^2 t - \frac{ikx}{\sqrt{D}}};$$

and the principle of linear superposition leads to the following general solution

$$u(t, x) = \int \left(a(k) e^{-k^2 t + \frac{ikx}{\sqrt{D}}} + b(k) e^{-k^2 t - \frac{ikx}{\sqrt{D}}} \right) dk.$$

6. Apply the SVM to the 1D free Schrödinger equation

$$i \hbar u_t = -\frac{\hbar^2}{2m} u_{xx}.$$

Solution: To work with the SVM we consider the operators

$$A = i \hbar \frac{d}{dt}, \quad B = \frac{\hbar^2}{2m} \frac{d^2}{dx^2},$$

and we look for solutions in the form of $u(t, x) = v(t)w(x)$, where

$$Av = k^2 v, \quad Bw = -k^2 w,$$

where $\lambda = k^2$ with $k \in \mathbb{C}$. Hence, we obtain the following ODEs

$$i \hbar v' = k^2 v, \quad \frac{\hbar^2}{2m} w'' = -k^2 w,$$

whose solutions are

$$v(t) = c e^{-i \frac{k^2 t}{\hbar}}, \quad w(x) = a e^{\frac{i \sqrt{2m} k x}{\hbar}} + b e^{-\frac{i \sqrt{2m} k x}{\hbar}}.$$

The corresponding solution of the Schrödinger equation is

$$u(t, x) = a e^{-i \frac{k^2 t}{\hbar} + \frac{i \sqrt{2m} k x}{\hbar}} + b e^{-i \frac{k^2 t}{\hbar} - \frac{i \sqrt{2m} k x}{\hbar}},$$

which leads to the following general solution

$$u(t, x) = \int \left(a(k) e^{-i \frac{k^2 t}{\hbar} + \frac{i \sqrt{2m} k x}{\hbar}} + b(k) e^{-i \frac{k^2 t}{\hbar} - \frac{i \sqrt{2m} k x}{\hbar}} \right) dk.$$

7. Apply the SVM to the 1D wave equation

$$u_{tt} - c^2 u_{xx} = 0.$$

Solution: SVM can be applied with the operators

$$A = \frac{d^2}{dt^2}, \quad B = -c^2 \frac{d^2}{dx^2},$$

and the *ansatz* $u(t, x) = v(t)w(x)$. The following eigenvalue problems appear (we take eigenvalues in the form $\lambda = -k^2$)

$$Av = -k^2v, \quad Bw = k^2w,$$

which are the following ODEs

$$v'' = -k^2v, \quad -c^2w'' = k^2w.$$

The solutions are given by

$$v(t) = P e^{ikt} + Q e^{-ikt}, \quad w(x) = R e^{ik\frac{x}{c}} + S e^{-ik\frac{x}{c}}, \quad P, Q, R, S \in \mathbb{C},$$

and therefore, the solution of the wave equation given by the SVM is

$$u(t, x) = \alpha e^{ik(t-\frac{x}{c})} + \beta e^{-ik(t-\frac{x}{c})} + \gamma e^{ik(t+\frac{x}{c})} + \delta e^{-ik(t+\frac{x}{c})}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

Therefore, the overall solution will be

$$u(t, x) = \int \left(\alpha(k) e^{ik(t-\frac{x}{c})} + \gamma(k) e^{ik(t+\frac{x}{c})} \right) dk.$$

That is, the general solution of the wave equation is of the form $f(x - ct) + g(x + ct)$.

8. A 1D quantum particle enclosed in a box of length L is described by a type boundary value problem:

$$\begin{cases} i u_t = -u_{xx}, & t > 0, 0 < x < L; \\ \begin{cases} u|_{x=0} = 0, \\ u|_{x=L} = 0. \end{cases} \end{cases}$$

Determine the solutions provided by the SVM.

Solution: To apply the SVM we write

$$A = i \frac{d}{dt}, \quad B = \frac{d^2}{dx^2},$$

and we look for solutions in the form $u(t, x) = v(t)w(x)$, each factor being an eigenfunction

$$Av = k^2v, \quad Bw = -k^2w,$$

where we've written the eigenvalue in the form of $\lambda = k^2$. Therefore, the following ODEs must be met

$$i v' = k^2v, \quad w'' = -k^2w,$$

which has as its solution

$$v(t) = c e^{-ik^2t}, \quad w(x) = a e^{ikx} + b e^{-ikx},$$

for arbitrary complex constants a, b, c . The boundary conditions act only on the x variable, so we must impose that

$$\begin{cases} w(0) = 0, \\ w(1) = 0. \end{cases}$$

that leads to the system

$$\begin{pmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The existence of non-trivial solutions requires

$$\begin{vmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{vmatrix} = 0,$$

i.e., $k = \frac{n\pi}{L}$, with $n = 1, 2, \dots$ and the eigenfunctions are

$$w_n(x) = \sin \frac{n\pi x}{L}.$$

Hence, the solutions that the SVM provides for this problem are of the following form

$$u(t, x) = \sum_{n=1}^{\infty} c_n e^{-i \frac{n^2 \pi^2 t}{L^2}} \sin \frac{n\pi x}{L},$$

for appropriate sequences of numbers $\{c_n\}_{n=1}^{\infty}$.

9. Determine the solutions provided by the SVM for the boundary value problem:

$$\begin{cases} tu_t = u_{xx} + 2u, & t > 0, 0 < x < \pi, \\ \begin{cases} u|_{x=0} = 0, \\ u|_{x=\pi} = 0. \end{cases} \end{cases}$$

Solution: The separation of variables is considered in this case with the following differential operators

$$A = t \frac{d}{dt} - 2, \quad B = -\frac{d^2}{dx^2},$$

that for factored solutions in the form $u(t, x) = v(t)w(x)$ leads to the following eigenvalue problems, with $\lambda = -k^2$,

$$Av = -k^2 v, \quad Bw = k^2 w.$$

Thus, we must solve the following ordinary differential equations

$$tv' = (2 - k^2)v, \quad w'' = -k^2 w,$$

the solutions to which are

$$v(t) = ct^{2-k^2}, \quad w(x) = a \sin kx + b \cos kx, \quad a, b, c, \in \mathbb{C}.$$

The boundary conditions acts only on the variable x , then they will have to be demanded to $w(x)$, that is

$$\begin{cases} w(0) = 0, \\ w(\pi) = 0, \end{cases}$$

which leads to the system

$$\begin{pmatrix} 0 & 1 \\ \sin k & \cos k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But for non-trivial solutions to exist, we will have to ask the system determinant to cancel, that is $\sin k = 0$, then

$$k = n\pi, \quad n = 1, 2, \dots$$

That's it, the eigenvalues are

$$\lambda_n = -n^2\pi^2, \quad n = 1, 2, \dots,$$

and the corresponding eigenfunctions are given by

$$w_n(x) = \sin n\pi x, \quad n = 1, 2, \dots$$

Therefore, we come to the following solution

$$u(t, x) = ct^{2-n^2\pi^2} \sin n\pi x, \quad c \in \mathbb{C}, \quad n = 1, 2, \dots$$

and their linear combinations link to the solution

$$u(t, x) = \sum_{n=1}^{\infty} c_n t^{2-n^2\pi^2} \sin n\pi x,$$

for suitable number sequences $\{c_n\}_{n=1}^{\infty} \subset \mathbb{C}$.

10. Determine the solutions provided by the SVM for the boundary value problem:

$$\begin{cases} u_t = u_{xx}, & t > 0, \quad 0 < x < 1; \\ \begin{cases} u|_{x=0} = 0, \\ (u_x + \alpha u)|_{x=1} = 0, \end{cases} \end{cases}$$

Solution: Before solving the problem we give its physical interpretation. We have a problem of heat diffusion in a thread of unit length, with a given zero temperature at one edge and in the other edge there is a mechanism that cools/warms in proportion to the temperature at which it is found.

To apply SVM we consider eigenvalue problems for operators

$$A = \frac{d}{dt}, \quad B = -\frac{d^2}{dx^2},$$

that follow

$$Av = -k^2v, \quad Bw = k^2w,$$

which enable solutions to be found in the following factored form $u(t, x) = v(t)w(x)$. Therefore, the following ODEs need to be solved

$$v' = -k^2 v, \quad w'' = -k^2 w,$$

whose solutions are

$$v(t) = c e^{-k^2 t}, \quad w(x) = a \sin kx + b \cos kx, \quad a, b, c \in \mathbb{C}.$$

Boundary conditions acts only on the x variable, so we impose them on $w(x)$, i.e.,

$$\begin{cases} w(0) = 0, \\ w'(1) + \alpha w(1) = 0, \end{cases}$$

which, bearing in mind that

$$w'(x) = ka \cos kx - kb \sin kx,$$

and that because of this

$$w'(x) + \alpha w(x) = a(\alpha \sin kx + k \cos kx) + b(\alpha \cos kx - k \sin kx)$$

are written as the following linear system

$$\begin{pmatrix} 0 & 1 \\ \alpha \sin k + k \cos k & \alpha \cos k - k \sin k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The existence of non-trivial solutions to it leads to

$$\begin{vmatrix} 0 & 1 \\ \alpha \sin k + k \cos k & \alpha \cos k - k \sin k \end{vmatrix} = 0,$$

and, therefore, that the eigenvalues are in the form $\lambda_n = -k_n^2$ where k_n is one of the infinite solutions of the following transcendent equation

$$\tan k = -\frac{k}{\alpha}.$$

The eigenfunctions will be

$$w_n(x) = c \sin k_n x,$$

and the solutions of the boundary value problem for the heat equation are

$$u(t, x) = c e^{-k_n^2 t} \sin k_n x, \quad n = 1, 2, \dots$$

Finally, the most general solution provided by SVM will be in the form

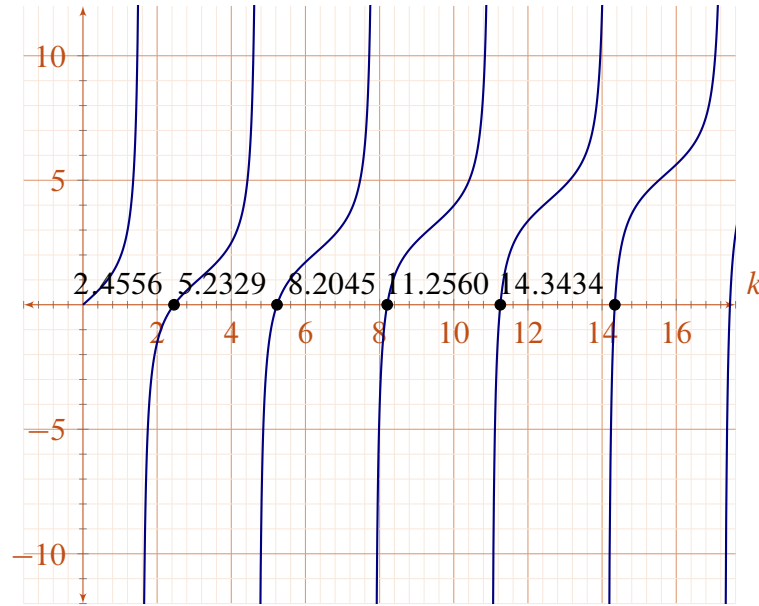
$$u(t, x) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} \sin k_n x,$$

for a conveniently chosen sequence of numbers $\{c_n\}_{n=1}^{\infty} \subset \mathbb{C}$.

The table of the first k_n is

First zeros of $\tan k + \frac{k}{3}$						
2.4556	5.2329	8.2045	11.2560	14.3434	17.4490	20.5652

For $\alpha = 3$ we represent below the transcendent function and its zeros



Graph of $f(k) = \tan(k) + \frac{k}{3}$ and the first five zeros

2.5.2. Exercises

1. An infinite string held by one of its ends is described by the boundary value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < 0, t > 0; \\ u|_{x=0} = 0. \end{cases}$$

Determine the standing waves of the problem and interpret them dynamically.

2. Under certain conditions some components of the electromagnetic field inside a rectangular section waveguide are described by the boundary value problem:

$$\begin{cases} u_{tt} = c^2 \Delta u, & 0 < x < L_1, 0 < y < L_2, -\infty < z < +\infty; \\ u|_{\text{walls}} = 0. \end{cases}$$

Determine the standing waves of the model.

3. Under certain conditions the components of an electromagnetic field confined inside a parallelepipedic box are described by the boundary value problem:

$$\begin{cases} u_{tt} = c^2 \Delta u, & 0 < x < L_1, 0 < y < L_2, 0 < z < L_3; \\ u|_{\text{walls}} = 0. \end{cases}$$

Determine the standing waves of the model and interpret them dynamically.

4. The height above equilibrium position of the membrane of a rectangular drum obeys the boundary value problem:

$$\begin{cases} u_{tt} = c^2 \Delta u, & 0 < x < L_1, 0 < y < L_2, \\ u|_{\text{walls}} = 0. \end{cases}$$

Determine the standing waves of the model and interpret them dynamically.

3. SYMMETRIC OPERATORS IN HILBERT SPACE

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HILBERT spaces and orthogonal sets, which allow for eigenfunctions series expansions of solutions of boundary value problems, are introduced. We discuss symmetric differential operators in these functional spaces. We also present a very important class of second order differential operators, the so called Sturm–Liouville operators. These operators are important in the resolution of separable linear PDEs.

§3.1. Hilbert Spaces

3.1.1. Scalar Product of Functions

Scalar Product in Vector Spaces

Suppose we have a complex vector space V . A scalar product in V is an application $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ such that

1. $(u, v) = \overline{(v, u)}, \forall u, v \in V$.
2. $(u, \alpha v_1 + \beta v_2) = \alpha(u, v_1) + \beta(u, v_2), \forall u, v_1, v_2 \in V$ and $\forall \alpha, \beta \in \mathbb{C}$.
3. $(u, u) \geq 0, \forall u \in V; (u, u) = 0 \Leftrightarrow u = 0$.

From properties I) and II) we deduce that $(\alpha v_1 + \beta v_2, u) = \bar{\alpha}(v_1, u) + \bar{\beta}(v_2, u)$. We see that $(\cdot, \cdot)$ is almost a bilinear form that is sometimes called a sesquilinear form. Property III) tells us that the scalar product of a vector with itself is always a real non-negative number and that it is zero only for the vector 0. The scalar product allows us to endow vectors with length, which we will call norm:

$$\|u\| := +\sqrt{(u, u)}.$$

A basic inequality (Cauchy–Schwarz inequality) that relates the norm and the scalar product is

$$|(u, v)| \leq \|u\| \|v\|.$$

Three fundamental properties are fulfilled for the norm

1. $\|u\| = 0 \Leftrightarrow u = 0$.
2. $\|\alpha u\| = |\alpha| \|u\|$, $\forall \alpha \in \mathbb{C}$ and $\forall u \in V$.
3. $\|u + v\| \leq \|u\| + \|v\|$ (triangular inequality), $\forall u, v \in V$.

These properties allow us to define a distance in V

$$d(u, v) := \|u - v\| = \sqrt{(u - v, u - v)}.$$

The notion of scalar product as outlined here can be applied to *functional spaces* as we shall now see.

Scalar Product in Functional Spaces

Let's consider an open set $\Omega \subset \mathbb{R}^n$ and a function $\rho : \Omega \rightarrow \mathbb{R}$ differentiable and positive ($\rho(x) > 0 \forall x \in \Omega$). Given two functions u, v defined on Ω their scalar product corresponding to the weight function ρ is defined by

$$(u, v) = \int_{\Omega} \bar{u}(x) v(x) \rho(x) d^n x.$$

We must remember that the integral of functions on $\bar{\Omega}$ with complex values is understood as:

$$\int_{\bar{\Omega}} u(x) \rho(x) d^n x := \int_{\bar{\Omega}} \operatorname{Re}(u(x)) \rho(x) d^n x + i \int_{\bar{\Omega}} \operatorname{Im}(u(x)) \rho(x) d^n x.$$

Obviously, for this scalar product to make sense it is necessary that the corresponding integral¹ exists. This condition is guaranteed if we work in the space of functions

$$\mathcal{L}_{\rho}^2(\bar{\Omega}) := \left\{ u : \bar{\Omega} \rightarrow \mathbb{C} \mid \|u\|^2 = \int_{\bar{\Omega}} |u(x)|^2 \rho(x) d^n x < \infty \right\}.$$

Actually the $(\cdot, \cdot)$ operation we just defined is not strictly a scalar product, what fails is that there are functions $u(x) \neq 0$ in Ω that have a zero norm $\|u\| = 0$.² To solve this problem we must consider a function to be the zero function if its norm is zero. Likewise, two functions will be considered to be the same if and only if their difference has a zero norm. The set obtained from $\mathcal{L}_{\rho}^2(\bar{\Omega})$ by means of these identifications will be denoted by $L_{\rho}^2(\bar{\Omega})$ and is a vector space of infinite dimension with scalar product. The scalar product in $L_{\rho}^2(\bar{\Omega})$ endows to this space of functions of a mathematical structure that is known by the name de **Hilbert space**.³ The elements of $L_{\rho}^2(\bar{\Omega})$ are known as **integrable square functions** with weight ρ over $\bar{\Omega}$. If $\bar{\Omega}$ is a

¹The reader is already familiar with Riemann's integral; however, this is not entirely adequate in this context. In fact, it is necessary to use a more sophisticated integration theory due to Lebesgue.

²For example, the function of Dirichlet that on rationals is 1 and on irrationals 0, is Lebesgue integrable and with norm 0.

³Strictly speaking, what we are doing here is nothing more than considering function equivalence classes: $u \sim v$ provided that $\|u - v\| = 0$. The quotient space $\mathcal{L}_{\rho}^2(\bar{\Omega}) / \sim$ is what we denote by $L_{\rho}^2(\bar{\Omega})$.

compact set (which is equivalent to Ω being a bounded set) then it is clear that any differentiable function in $\overline{\Omega}$ is square integrable in $\overline{\Omega}$. That is to say

$$\overline{\Omega} \text{ compact} \Rightarrow C^\infty(\overline{\Omega}) \subset L^2_\rho(\overline{\Omega}).$$

If $\overline{\Omega}$ is not compact such inclusion is not true and only those differentiable functions that are decaying adequately at infinity will be of integrable square.

Examples:



1. Let $\Omega = (0, +\infty) \subset \mathbb{R}$; $u(x) = x$ and $v(x) = 1$, if the weight function is $\rho(x) = 1$ then the scalar product of u and v do not exist due to

$$(u, v) = \int_0^\infty x \, dx = \infty.$$

If we now take the weight $\rho(x) = e^{-x^2}$, the scalar product of u and v exist

$$(u, v) = \int_0^\infty x e^{-x^2} \, dx = \frac{1}{2}.$$

2. If we take the domain $\Omega = (0, 1)$ and weight $\rho(x) = 1$, the scalar product of the functions $u(x) = \frac{1}{x}$ and $v(x) = 1$ is not defined due to the singularity of the integrand at $x = 0$.

$$(u, v) = \int_0^1 \frac{1}{x} \, dx = \infty.$$

However, for the weight $\rho(x) = x$ we get the following scalar product

$$\int_0^1 x \frac{1}{x} \, dx = 1.$$

3. We take now $\Omega = (0, 1)$, $u(x) = x + ix^2$ and $v(x) = 1 - ix$ with $\rho(x) = 1$. Then

$$(u, v) = \int_0^1 (x - ix^2)(1 - ix) \, dx = \int_0^1 (x - x^3 - 2ix^2) \, dx = \frac{1}{2} - \frac{2i}{3}$$

and the norm of u is

$$\|u\|^2 = (u, u) = \int_0^1 |x + ix^2|^2 \, dx = \int_0^1 (x^2 + x^4) \, dx = \frac{8}{15}.$$

4. If $\Omega = (0, 2\pi)$, $u(x) = e^{ix}$ and $v(x) = e^{2ix}$ with $\rho(x) = 1$ we obtain

$$(u, v) = \int_0^{2\pi} e^{-ix} e^{2ix} \, dx = 0$$

and the square of the length of u will be

$$\|u\|^2 = \int_0^{2\pi} |e^{ix}|^2 \, dx = \int_0^{2\pi} 1 \, dx = 2\pi.$$

3.1.2. Coordinate changes We have defined the scalar product of functions on domains in $\mathbb{R}^n$ using Cartesian coordinates. However, due to the theorem of the change of variables of the Integral Calculus, these scalar products can be expressed in other types of coordinates. Let us consider two important cases:

- If the definition domain is the plane $\mathbb{R}^2$ we can consider **polar coordinates** and get to

$$(u, v) = \int_{\mathbb{R}^2} \bar{u}(x, y)v(x, y)\rho(x, y) \, dx \, dy = \int_0^\infty \int_0^{2\pi} \bar{u}(r, \theta)v(r, \theta)\rho(r, \theta)r \, dr \, d\theta.$$

So, when we switch to polar coordinates we have $\rho(x, y) \rightarrow r\rho(r, \theta)$.

- In space $\mathbb{R}^3$ we have the **spherical coordinates** and now

$$\begin{aligned} (u, v) &= \int_{\mathbb{R}^3} \bar{u}(x, y, z)v(x, y, z)\rho(x, y, z) \, dx \, dy \, dz \\ &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \bar{u}(r, \theta, \phi)v(r, \theta, \phi)\rho(r, \theta, \phi)r^2 \sin \theta \, dr \, d\theta \, d\phi. \end{aligned}$$

and $\rho(x, y, z) \rightarrow r^2 \sin \theta \rho(r, \theta, \phi)$.



Example: Be the weight $\rho = 1$ and the functions $u(r, \theta, \phi) = \frac{e^{\frac{i\phi}{2}}}{r^2+1}$ y $v(r, \theta, \phi) = \frac{\sin \theta}{r^2}$; then, their scalar product is

$$\begin{aligned} (u, v) &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{-\frac{i\phi}{2}}}{(r^2+1)} \frac{\sin \theta}{r^2} r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \left[\int_0^\infty \frac{dr}{1+r^2} \right] \left[\int_0^\pi \sin^2 \theta \, d\theta \right] \left[\int_0^{2\pi} e^{-\frac{i\phi}{2}} \, d\phi \right] \\ &= \left[\arctan r \right]_0^\infty \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^\pi \left[2i e^{-\frac{i\phi}{2}} \right]_0^{2\pi} = \left[\frac{\pi}{2} \right] \left[\frac{\pi}{2} \right] \left[2i(-2) \right] = -i\pi^2. \end{aligned}$$

§3.2. Orthogonal Sets of Functions



ORTHOGONALITY, complete set of functions and the orthogonal bases of a functional space, that as we will see are fundamental elements for this course, are discussed in this section. Later on we will show that they are instrumental objects in the resolution of some important PDEs in Mathematical-Physics.

Orthogonal Set

Given a set of functions $\{u_n(x)\}_{n \in J}$ in $L^2_\rho(\bar{\Omega})$, where $J \subset \mathbb{Z}$ is a set of finite or infinite indices, we say it is **orthogonal** if $(u_n, u_m) = 0, \forall n \neq m; n, m \in J$.

Examples:

1. The set $\left\{ \sin \frac{n\pi x}{\ell} \right\}_{n \geq 1}$ is orthogonal on the interval $[0, \ell]$ with weight function $\rho(x) = 1$. To realize this fact it suffice to note that

$$\int_0^\ell \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx = \int_0^\ell \frac{1}{2} \left(\cos \frac{(n-m)\pi x}{\ell} - \cos \frac{(n+m)\pi x}{\ell} \right) dx = 0,$$

for $n \neq m$.⁴

2. The set $\left\{ \cos \frac{n\pi x}{\ell} \right\}_{n \geq 0}$ is orthogonal in the interval $[0, \ell]$ with weight $\rho(x) = 1$. As before we have that for $n \neq m$

$$\int_0^\ell \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} dx = \int_0^\ell \frac{1}{2} \left(\cos \frac{(n-m)\pi x}{\ell} + \cos \frac{(n+m)\pi x}{\ell} \right) dx = 0.$$

3.2.1. Orthogonal function series expansions Given an orthogonal set

$$\{u_n(x)\}_{n \in J}$$

an important question is to know what kind of functions can be expanded in the form $v = \sum_{n \in J} c_n u_n$. When the set of indices J that label the orthogonal set is infinite, we must be very careful with the meaning of the sum extended to the indices in J . If $J = \mathbb{N}$ the expression

$u(x) = \sum_{n=1}^{\infty} c_n u_n(x)$ is understood as the limit $\lim_{N \rightarrow \infty} \sum_{n=1}^N c_n u_n(x)$. In the same way when $J = \mathbb{Z}$

$$u(x) = \sum_{n=-\infty}^{\infty} c_n u_n(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N c_n u_n(x) + \lim_{M \rightarrow \infty} \sum_{m=1}^M c_{-m} u_{-m}(x).$$

However, when we deal with functions there are several different notions of limit.

From this moment on, unless we say otherwise, we will only refer to series of functions that converge strongly, i.e. in L^2 mean. The following properties on orthogonal function series expansions are of great importance.

⁴In the integral calculation we have reduced quadratic expressions in trigonometric functions in linear forms thanks to sine and cosine formulas:

$$\begin{aligned} \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B, \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B, \end{aligned}$$

which give rise to the following addition formulas

$$\begin{aligned} \sin A \sin B &= \frac{1}{2}(\cos(A - B) - \cos(A + B)), \\ \sin A \cos B &= \frac{1}{2}(\sin(A - B) + \sin(A + B)), \\ \cos A \cos B &= \frac{1}{2}(\cos(A - B) + \cos(A + B)). \end{aligned}$$

Types of Functional Limits

In this course there are three notions of limit that we will use:

1. Pointwise limit: $\forall x_0 \in \overline{\Omega}$ we have $u(x_0) = \lim_{N \rightarrow \infty} \sum_{n=1}^N c_n u_n(x_0)$.
2. Uniform limit: $\lim_{N \rightarrow \infty} \sup_{x \in \Omega} \left| u(x) - \sum_{n=1}^N c_n u_n(x) \right| = 0$.
3. Strong limit (L^2 mean limit): $\lim_{N \rightarrow \infty} \left\| u - \sum_{n=1}^N c_n u_n \right\| = 0$.

Uniform convergence implies pointwise convergence, however in general the inverse is not true. On the other hand, when the closure of Ω is compact, uniform convergence implies strong convergence and again the reciprocal is not satisfied in general.

Expansions in Orthogonal Functions

The following properties are fulfilled:

1. If u admits an expansion $u = \sum_{n \in J} c_n u_n$, then this expansion is unique and its coefficients c_n are determined by

$$(5) \quad c_n = \frac{(u_n, u)}{\|u_n\|^2}.$$

2. If $u = \sum_{n \in J} c_n u_n$ and $v = \sum_{n \in J} c'_n u_n$ their scalar product is

$$(6) \quad (u, v) = \sum_{n \in J} \bar{c}_n c'_n \|u_n\|^2.$$

3. **Parseval identity** If $u = \sum_{n \in J} c_n u_n$ its norm fulfills the following

$$(7) \quad \|u\|^2 = \sum_{n \in J} \frac{|(u_n, u)|^2}{\|u_n\|^2}.$$

PROOF.⁵

1. As $u = \sum_{m \in J} c_m u_m$ and $\{u_n\}_{n \in J}$ is a orthogonal set we find

$$(u_n, u) = (u_n, \sum_{m \in J} c_m u_m) = \sum_{m \in J} c_m (u_n, u_m) = c_n (u_n, u_n).$$

2. As $u = \sum_{n \in J} c_n u_n$ and $v = \sum_{m \in J} c'_m u_m$ the corresponding scalar product is

$$(u, v) = \sum_{n, m \in J} \bar{c}_n c'_m (u_n, u_m) = \sum_n \bar{c}_n c'_n \|u_n\|^2.$$

3. It is a consequence of the two previous properties taking $v = u$.

□

A relevant property that allows to find orthogonal sets of functions in several variables is the following

Orthogonal Sets in Product Spaces

$\{u_n(x_1)\}_{n \in J_1}$ is orthogonal in the domain Ω_1 with respect to the weight $\rho_1(x)$ and $\{v_m(x_2)\}_{m \in J_2}$ is orthogonal in the domain Ω_2 with respect to the weight $\rho_2(x)$, then $\{u_n(x_1)v_m(x_2)\}_{(n,m) \in J_1 \times J_2}$ is orthogonal in the domain $\Omega_1 \times \Omega_2$ with weight $\rho_1(x_1)\rho_2(x_2)$.

PROOF. According to Fubini's theorem we factor the integral

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} \bar{u}_n(x_1) \bar{v}_m(x_2) u_{n'}(x_1) v_{m'}(x_2) \rho(x_1) \rho(x_2) dx_1 dx_2 \\ = \left[\int_{\Omega_1} \bar{u}_n(x_1) u_{n'}(x_1) \rho_1(x_1) dx_1 \right] \left[\int_{\Omega_2} \bar{v}_m(x_2) v_{m'}(x_2) \rho_2(x_2) dx_2 \right] = 0 \end{aligned}$$

if $n \neq n'$ or $m \neq m'$.

□

Complete Orthogonal Sets

Given an orthogonal set $\{u_n\}_{n \in J}$ in $L^2_\rho(\bar{\Omega})$, we say that it is **complete** when all function $u \in L^2_\rho(\bar{\Omega})$ can be expanded in series $u = \sum_{n \in J} c_n u_n$. In that case it is said that $\{u_n\}_{n \in J}$ is a **orthogonal basis** of the space $L^2_\rho(\bar{\Omega})$.

If an orthogonal set is not complete it can always be extended by adding new elements to form an orthogonal basis.

⁵In this proof, the sums within the scalar product are taken out. This manipulation is obviously licit as long as the sums are finite. However, it also turns out to be so in the case of strong convergent series.

§3.3. Symmetric Differential Operators



SYMMETRIC differential operators, a quite important notion for these lectures, are introduced in this section. It should be stressed here that the concept of symmetric operator is closely linked to the concept of fundamental self-adjoint operator in Quantum Mechanics, which happen to be the physical observables of the system.

Consider linear differential operators L

$$L = \sum_{\alpha} ' a_{\alpha}(x) D^{\alpha},$$

defined over a linear subspace of functions $\mathcal{D}$ in $L^2_{\rho}(\overline{\Omega})$ which we'll call the operator's domain.

Symmetric Differential Operators

We will say that a differential operator L is **symmetric** over a domain $\mathcal{D}$ provided that

$$(v, Lw) = (Lv, w), \quad \forall v, w \in \mathcal{D}.$$

The natural way to obtain domains of functions in which an operator is symmetric is to characterize the domain by imposing appropriate boundary conditions on its elements.



Examples:

1. Let us consider the operator

$$L = -\frac{d^2}{dx^2}, \quad \Omega = (a, b) \text{ bounded set of } \mathbb{R}.$$

To determine domains $\mathcal{D} \subset L^2[a, b]$ in where this operator is symmetric, let's observe that for all pair of differentiable functions v and w in $[a, b]$ the following Lagrange's identity is fulfilled

$$\begin{aligned} (v, Lw) &= \int_a^b \bar{v}(x)(-w''(x)) dx = -(\bar{v}(x)w'(x))\Big|_a^b + \int_a^b \bar{v}'(x)w'(x) dx \\ &= (-\bar{v}(x)w'(x) + \bar{v}'(x)w(x))\Big|_a^b + \int_a^b (-\bar{v}''(x)w(x)) dx \\ &= (-\bar{v}(x)w'(x) + \bar{v}'(x)w(x))\Big|_a^b + (Lv, w); \end{aligned}$$

therefore a domain for a symmetric operator L must satisfy that

$$(7) \quad \boxed{(-\bar{v}(x)w'(x) + \bar{v}'(x)w(x))\Big|_a^b = 0, \quad \forall v, w \in \mathcal{D}.}$$

Using the condition (7) it is immediate to see that for this operator there are at least three types of boundary conditions that determine domains where the operator is symmetric.

- a) Homogeneous Dirichlet

$$\boxed{\mathcal{D} := \{u \in C^{\infty}[a, b] : u(a) = u(b) = 0\}.$$

b) Homogeneous Neumann

$$\mathcal{D} := \{u \in C^\infty[a, b] : u'(a) = u'(b) = 0\}.$$

c) Periodic

$$\mathcal{D} := \{u \in C^\infty[a, b] : u(a) = u(b), u'(a) = u'(b)\}.$$

2. Let us now look at the following example in three dimensions

$$Lu = -\Delta u = -\operatorname{div}(\nabla u).$$

To determine domains where this operator is symmetric consider the space $L^2(\overline{\Omega})$, for a bounded domain $\Omega \subset \mathbb{R}^3$. Using the scalar product in $L^2(\overline{\Omega})$ and applying the divergence theorem, we obtain

$$\begin{aligned} (Lv, w) - (v, Lw) &= \int_{\overline{\Omega}} \left[\left(-\operatorname{div}(\nabla \bar{v}(x)) \right) w(x) + \bar{v}(x) \left(\operatorname{div}(\nabla w(x)) \right) \right] d^3 x \\ &= \int_{\overline{\Omega}} \operatorname{div} \left[\bar{v}(\nabla w) - (\nabla \bar{v})w \right] d^3 x \\ &= \int_{S(\Omega)} \left[\bar{v}(\nabla w) - (\nabla \bar{v})w \right] \cdot d\mathbf{S} \\ &= \int_{S(\Omega)} \left[\bar{v} \frac{\partial w}{\partial \mathbf{n}} - \frac{\partial \bar{v}}{\partial \mathbf{n}} w \right] dS. \end{aligned}$$

It is then immediate to deduce from this identity that L is symmetric in the following domains

a) Homogeneous Dirichlet

$$\mathcal{D} := \{u \in C^\infty(\overline{\Omega}) : u|_{S(\Omega)} = 0\}.$$

b) Homogeneous Neumann

$$\mathcal{D} := \left\{ u \in C^\infty(\overline{\Omega}) : \frac{\partial u}{\partial \mathbf{n}} \Big|_{S(\Omega)} = 0 \right\}.$$

3.3.1. Eigenvalues and Eigenfunctions of Symmetric Operators Symmetrical operators play a key role in PDE's resolution methods in this course. This is due to the properties of their eigenfunctions.

Spectrum of Symmetric Operators

If L is a symmetric differential operator on a domain $\mathcal{D}$ in a L^2 Hilbert space, then the following properties are satisfied:

1. Eigenvalues are real numbers

$$\sigma(L) \subset \mathbb{R}.$$

2. If $v, w \in \mathcal{D}$ are eigenfunctions corresponding to different eigenvalues then they are orthogonal

$$(v, w) = 0.$$

PROOF. 1. If $\lambda \in \sigma(L)$ then there's a non-null function $u \in \mathcal{D}$ such that $Lu = \lambda u$. For this reason

$$(Lu, u) = (\lambda u, u) = \bar{\lambda} \|u\|^2, \quad (u, Lu) = (u, \lambda u) = \lambda \|u\|^2.$$

As $(Lu, u) = (u, Lu)$ and $\|u\| \neq 0$ we must have $\bar{\lambda} = \lambda$. Then λ is a real number.

2. Given two different elements λ, μ of the spectrum of L , which we already know will be real numbers, there are eigenfunctions v and w such that $Lv = \lambda v$ and $Lw = \mu w$; hence,

$$(Lv, w) = \lambda(v, w), \quad (v, Lw) = \mu(v, w).$$

Since L is symmetric, $(Lv, w) = (v, Lw)$ and thus $(\lambda - \mu)(v, w) = 0$ and so, as $\lambda \neq \mu$, it follows $(v, w) = 0$.

□

§3.4. Sturm–Liouville Operators



TURM–LIOUVILLE operators are a special type of differential operators that play a fundamental role in the equations of Mathematical-Physics. We devote this section to give a brief view of what they are and some of their properties.

3.4.1. 1D case

1D Sturm–Liouville Operators

A second order differential operator defined on a domain $\mathcal{D} \subset L^2_\rho[a, b]$ is of the Sturm–Liouville type if it is of the form

$$Lu = \frac{1}{\rho} \left[-\frac{d}{dx} \left(p \frac{du}{dx} \right) + qu \right],$$

with ρ, p, q real functions in $C^\infty(a, b)$ such that $\rho, p > 0$ in the open interval (a, b) .

Note that in general the functions ρ, p, q can be singular in $x = a, b$ and that the functions ρ, p can be zero in $x = a, b$.

Regular Sturm–Liouville Operators

A Sturm–Liouville operator on a domain

$$(6) \quad \mathcal{D} := \{u \in C^\infty[a, b] : l_i(u) = 0\},$$

with boundary operators of the form

$$\begin{aligned} l_1(u) &= \alpha_1 u(a) + \tilde{\alpha}_1 u(b) + \beta_1 \frac{d u}{d x}(a) + \tilde{\beta}_1 \frac{d u}{d x}(b) = 0, \\ l_2(u) &= \alpha_2 u(a) + \tilde{\alpha}_2 u(b) + \beta_2 \frac{d u}{d x}(a) + \tilde{\beta}_2 \frac{d u}{d x}(b) = 0, \end{aligned}$$

where $\tilde{\alpha}_1, \tilde{\beta}_1', \tilde{\beta}_1, \tilde{\alpha}_2, \alpha_1, \beta_2, \beta_1, \alpha_2 \in \mathbb{R}$ and (linearly independent boundary conditions)

$$\text{rank} \begin{pmatrix} \alpha_1 & \tilde{\alpha}_1 & \beta_1 & \tilde{\beta}_1 \\ \alpha_2 & \tilde{\alpha}_2 & \beta_2 & \tilde{\beta}_2 \end{pmatrix} = 2,$$

is regular if and only if

- i) The interval (a, b) is bounded ($a \neq -\infty$ and $b \neq \infty$).
- ii) The functions ρ, p, q belong to the space $C^\infty[a, b]$.
- iii) The ρ, p are strictly positive functions in the closed interval $[a, b]$.

We are now faced to an important question: For what domains does a regular Sturm–Liouville operator turn out to be symmetric?

Domains for Symmetric Operators

A regular Sturm–Liouville operator L on a domain $\mathcal{D} \subset L^2_\rho[a, b]$ is symmetric if and only if

$$(6) \quad p(a) \begin{vmatrix} \bar{v}(a) & w(a) \\ \bar{v}_x(a) & w_x(a) \end{vmatrix} = p(b) \begin{vmatrix} \bar{v}(b) & w(b) \\ \bar{v}_x(b) & w_x(b) \end{vmatrix},$$

$\forall v, w \in \mathcal{D}$.

PROOF. To prove it, it is enough to show that the difference $(Lv, w) - (v, Lw)$ vanish for all $v, w \in \mathcal{D}$ when the identity (6) is satisfied. This is true given that if $v, w \in \mathcal{D}$

$$\begin{aligned} (Lv, w) - (v, Lw) &= \int_a^b dx \left[- \left(\frac{d}{dx} \left(p(x) \frac{d \bar{v}}{dx}(x) \right) \right) w(x) + \bar{v}(x) \left(\frac{d}{dx} \left(p(x) \frac{d w}{dx}(x) \right) \right) \right] \\ &= - \int_a^b dx \frac{d}{dx} \left[p(x) \left(\frac{d \bar{v}}{dx} w - \bar{v} \frac{d w}{dx} \right) \right] \\ &= - \left[p(x) \left(\bar{v}_x w - \bar{v} w_x \right) \right] \Big|_a^b \\ &= -p(b) \begin{vmatrix} \bar{v}(b) & w(b) \\ \bar{v}_x(b) & w_x(b) \end{vmatrix} + p(a) \begin{vmatrix} \bar{v}(a) & w(a) \\ \bar{v}_x(a) & w_x(a) \end{vmatrix} = 0. \end{aligned}$$

Note that the contribution of q is canceled. Therefore, the symmetric character of L :

$$(Lv, w) = (v, Lw), \quad \forall v, w \in \mathcal{D}$$

is equivalent to the given criterion. □

Below are some basic examples of boundary conditions that guarantee (7) and therefore determine domains over which regular Sturm–Liouville operators are symmetric.

1. **Separate boundary conditions:** One at point a and another at the other point b :

$$\alpha u(a) + \tilde{\alpha} u_x(a) = 0,$$

$$\beta u(b) + \tilde{\beta} u_x(b) = 0.$$

Two particularly relevant cases of these are the conditions from **Dirichlet**

$$u(a) = 0, \quad u(b) = 0,$$

and the **Neumann** ones

$$u_x(a) = 0, \quad u_x(b) = 0.$$

2. **Periodic boundary conditions:**

$$u(a) = u(b), \quad p(a)u_x(a) = p(b)u_x(b).$$

The relevance of Sturm–Liouville operators lies in the fact that under appropriate conditions their eigenfunction sets provide complete **orthogonal sets**.

Orthogonal Bases for Regular & Symmetric Sturm–Liouville operators

If L is a symmetric regular Sturm–Liouville operator in a domain $\mathcal{D} \subset L^2_\rho[a, b]$, then there is a complete orthogonal set $\{u_n\}_{n=1}^\infty$ of $L^2_\rho[a, b]$ built up with L^2 eigenfunctions. In addition, the corresponding expansion of any $u \in \mathcal{D}$

$$u = \sum_{n=1}^{\infty} c_n u_n,$$

converges uniformly to u in the closed interval $[a, b]$.

Regular Sturm–Liouville Operators with Separate Boundary Conditions

If L is a regular Sturm–Liouville operator with separate boundary conditions in (a, b) . Then,

1. Eigenvalues are simple.
2. Eigenfunction u_n corresponding to the n -th eigenvalue λ_n has $n - 1$ zeros in the open interval (a, b) .
3. Whenever $p(x) > 0$ in $[a, b]$ the eigenvalues can be ordered as follows

$$\lambda_1 < \lambda_2 < \dots,$$

with $\lim_{n \rightarrow \infty} \lambda_n = \infty$. If $p(x) < 0$ en $[a, b]$

$$\lambda_1 > \lambda_2 > \dots,$$

and $\lim_{n \rightarrow \infty} \lambda_n = -\infty$.

For symmetric Sturm–Liouville operators of singular type there are generalizations of this result, although its description requires to introduce new concepts such as continuous spectrum and generalized eigenfunctions. When we study the Fourier transform we will comment on some of these situations.

3.4.2. 3D situation

3D Sturm–Liouville Operators

A second order differential operator defined on a domain $\mathcal{D} \subset L^2_\rho(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^3$ is of the Sturm–Liouville type if it is of the form

$$Lu = \frac{1}{\rho} \left(-\operatorname{div} (p \nabla u) + qu \right),$$

with ρ, p, q real functions in $C^\infty(\Omega)$ such that $\rho, p > 0$ on Ω .

To determine domains where these operators are symmetric let's suppose that Ω is a bounded set of $\mathbb{R}^3$ and that ρ, p, q are functions in $C^\infty(\overline{\Omega})$. Then, using the expression of the scalar product in $L^2_\rho(\overline{\Omega})$ and applying the divergence theorem, we get the Lagrange's identity

$$\begin{aligned} (Lv, w) - (v, Lw) &= \int_{\overline{\Omega}} \left[\bar{v}(x) \left(\nabla \cdot (p(x) \nabla w(x)) \right) - \left(\nabla \cdot (p(x) \nabla \bar{v}(x)) \right) w(x) \right] d^3 x \\ &= - \int_{\overline{\Omega}} \nabla \cdot \left[p(x) \left((\nabla \bar{v}) w - \bar{v} (\nabla w) \right) \right] d^3 x \\ &= - \int_{S(\Omega)} \left[p(x) \left((\nabla \bar{v}) w - \bar{v} (\nabla w) \right) \right] \cdot dS \\ &= - \int_{S(\Omega)} \left[p(x) \left(\frac{\partial \bar{v}}{\partial \mathbf{n}} w - \bar{v} \frac{\partial w}{\partial \mathbf{n}} \right) \right] dS. \end{aligned}$$

Therefore, we can state

3D Domains for Symmetric Sturm–Liouville Operators

Assume that Ω is a bounded domain in $\mathbb{R}^3$ and that ρ, p, q are functions in $C^\infty(\overline{\Omega})$. The Sturm–Liouville operator L is symmetric on a domain $\mathcal{D}$ in $L^2_\rho(\overline{\Omega})$ if and only if $\forall v, w \in \mathcal{D}$

$$(6) \quad \int_{S(\Omega)} p(x) \left(\frac{\partial \bar{v}}{\partial \mathbf{n}} w - \bar{v} \frac{\partial w}{\partial \mathbf{n}} \right) dS = 0.$$

As an immediate consequence we deduce the following domains on which the Sturm–Liouville operator is symmetric

1. Homogeneous Dirichlet

$$\mathcal{D} := \{u \in C^\infty(\overline{\Omega}) : u|_{S(\Omega)} = 0\}.$$

2. Homogeneous Neumann

$$\mathcal{D} := \left\{ u \in C^\infty(\overline{\Omega}) : \frac{\partial u}{\partial \mathbf{n}} \Big|_{S(\Omega)} = 0 \right\}.$$

Relevant examples of Sturm–Liouville operators in three dimensions are as follows

1. **Laplacian operator.** If we take $\rho(x) = p(x) \equiv 1$ we get the Laplacian operator with negative sign

$$L = -\Delta$$

2. **Schrödinger's Hamiltonian.** If we take $\rho = 1$ and $p = \frac{\hbar^2}{2m}$ the corresponding operator is the familiar Schrödinger's Hamiltonian operator from Quantum Mechanics of a particle in a force field with a potential $q = V$.

$$Lu = -\frac{\hbar^2}{2m} \Delta u + V(x, y, z)u,$$

§3.5. Exercises

3.5.1. Exercises with solutions

1. Let's consider the operator $L = -D^2$ on the set of differentiable functions u in the interval $[a, b]$ with weight $\rho(x) = 1$. Determine which of the following boundary conditions define domains where the operator is symmetric

- a) $u(a) = 0, u'(a) = 0$
- b) $2u(a) - u'(a) = 0, 3u(a) + 2u(b) = 0$
- c) $u(a) + u'(a) = 0, 2iu(a) + u(b) = 0$
- d) $u(a) - 2u'(b) = 0, u(b) + 2u'(a) = 0$
- e) $u(a) + u(b) = 0, u'(a) - u'(b) = 1$

Solution: The boundary conditions must be homogeneous and with real coefficients, so options c), e) are discarded. In order to discern the correct answer among the three remaining options, let's remember that the operator $-D^2$ is symmetric if for all pair of functions u, v of the domain $\bar{u}(a)v'(a) - \bar{u}'(a)v(a) = \bar{u}(b)v'(b) - \bar{u}'(b)v(b)$ is fulfilled. Therefore, option a) is immediately discarded ($0 \neq \bar{u}(b)v'(b) - \bar{u}'(b)v(b)$). In option b) we can express $u'(a) = 2u(a)$ and $u(b) = -3/2u(a)$ in terms of $u(a)$, so we have $\bar{u}(a)v'(a) - \bar{u}'(a)v(a) = 0$ and $\bar{u}(b)v'(b) - \bar{u}'(b)v(b) = -3/2(\bar{u}(a)v'(b) - v(a)\bar{u}(a))$; therefore, it is not correct. Let's see that option d) is correct, now we have $u'(b) = u(a)/2$ and $u'(a) = -u(b)/2$, so $\bar{u}(a)v'(a) - \bar{u}'(a)v(a) = -(\bar{u}(a)v(b) - \bar{u}(b)v(a))/2$ and $\bar{u}(b)v'(b) - \bar{u}'(b)v(b) = (\bar{u}(b)v(a) - \bar{u}(a)v(b))/2$ and equality is satisfied.

2. Determine the eigenvalues of the differential operator

$$Lu = -D^2u,$$

acting on the domain of differentiable functions in $[0, 1]$ that meet the boundary conditions

$$u_x(0) = 0, \quad u(1) = 0.$$

Solution: First we observe that $\lambda = 0$ is not an eigenvalue because if it was the case then $u(x) = A + Bx$, with A, B constants and the boundary conditions give $A = B = 0$. Solutions to the eigenvalue problem $-u_{xx} = \lambda u$, $\lambda \neq 0$, are of the form $u = A e^{ikx} + B e^{-ikx}$, with $\lambda = k^2$. As $u_x|_{x=0} = ik(A - B)$ and $u|_{x=1} = e^{ik}A + e^{-ik}B$ when imposing the boundary conditions we obtain the following linear system

$$\begin{cases} ik(A - B) = 0, \\ e^{ik}A + e^{-ik}B = 0. \end{cases} \quad \text{This system has non-zero solutions, e.g. } (A, B) \neq (0, 0), \text{ as long as}$$

$\begin{vmatrix} ik & -ik \\ e^{ik} & e^{-ik} \end{vmatrix} = 0$, what leads to $\cos k = 0 \Leftrightarrow k = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \dots$, so that

eigenvalues are $\lambda_n = (n + \frac{1}{2})^2 \pi^2$, $n = 0, 1, 2, \dots$

3. Given the differential operator

$$Lu := -D(e^x Du),$$

on the differentiable functions in $[0, 1]$, determine for which value of c the following boundary conditions

$$\begin{cases} u(0) + cu(1) = 0, \\ u'(0) + cu'(1) = 0 \end{cases}$$

define a domain of $L^2[0, 1]$ in which the operator is symmetric.

Solution: The operator L is of the Sturm–Liouville type with $p(x) = e^x$ and $q(x) = 0$ over the closed interval $[0, 1]$. Therefore, the condition for it to be symmetric is

$$\begin{vmatrix} \bar{v}(0) & w(0) \\ \bar{v}_x(0) & w_x(0) \end{vmatrix} = e \begin{vmatrix} \bar{v}(1) & w(1) \\ \bar{v}_x(1) & w_x(1) \end{vmatrix},$$

for all pair of functions $v(x)$ and $w(x)$ in the domain. Using the boundary conditions we obtain that the first determinant is

$$\begin{vmatrix} \bar{v}(0) & w(0) \\ \bar{v}_x(0) & w_x(0) \end{vmatrix} = c^2 \begin{vmatrix} \bar{v}(1) & w(1) \\ \bar{v}_x(1) & w_x(1) \end{vmatrix},$$

from which it is concluded that one must have $c^2 = e$.

4. Let us consider the differential operator

$$Lu = -D^2u,$$

acting on the domain of differentiable functions in $[0, 1]$ that fulfill the boundary conditions

$$\begin{cases} u'(0) = 0, \\ u(1) + u'(1) = 0. \end{cases}$$

Find the relationship that determines the eigenvalues $\lambda = k^2$ of L .

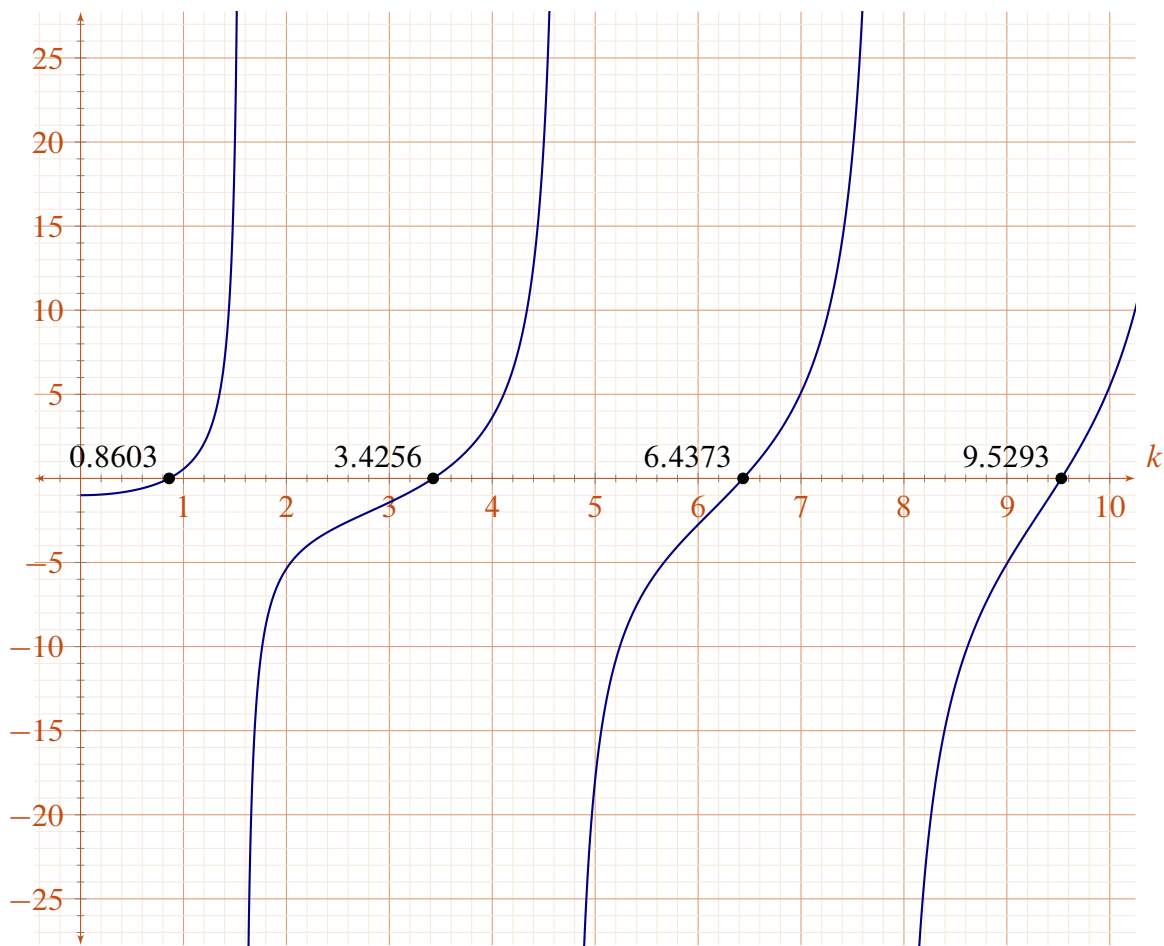
Solution: The possible eigenfunctions of L must be in the form of $u(x) = a \cos kx + b \sin kx$, $a, b \in \mathbb{R}$ and with nonzero eigenvalue $\lambda = k^2$ (for $\lambda = 0$ sought solutions are $u(x) = a + bx$, which once the given boundary conditions are applied leads to $u = 0$). The derivative is $u'(x) = -ak \sin kx + bk \cos kx$. Therefore, the existence of non-trivial solutions leads to

$$\begin{vmatrix} 0 & k \\ \cos k - k \sin k & \sin k + k \cos k \end{vmatrix} = 0$$

implying

$$k(\cos k - k \sin k) = 0.$$

and therefore the spectral relationship is $\boxed{k \tan k = 1}$.



Graph of $f(k) = k \tan(k) - 1$ and the first four zeros

First zeros of $k \tan k = 1$								
0.8603	3.4256	6.4373	9.5293	12.6453	15.7713	18.9024	22.0365	25.1724

5. Compute the eigenvalues of the boundary value problem

$$-(1+x) \frac{d}{dx} \left((1+x) \frac{du}{dx} \right) = \lambda u, \quad 0 < x < 1,$$

$$\begin{cases} u(0) = 0, \\ u(1) = 0. \end{cases}$$

Solution: Using the variable $y := x + 1$ the eigenvalue problem with $\lambda = k^2$, $k \in \mathbb{C}$, is a Euler's equation

$$y^2 u_{yy} + y u_y + k^2 u = 0,$$

whose general solution is

$$u(y) = a e^{ik \log y} + b e^{-ik \log y}, \quad a, b \in \mathbb{C}.$$

Note that $y \in (1, 2)$ and therefore you can consider the function $\log y$ as a real function and thus avoid a discussion on determinations. In terms of the variable x is written as

follows

$$u(x) = a e^{ik \log(x+1)} + b e^{-ik \log(x+1)}, \quad a, b \in \mathbb{C}.$$

Now we impose the boundary conditions, i.e.

$$\begin{aligned} a + b &= 0, \\ a e^{ik \log 2} + b e^{-ik \log 2} &= 0. \end{aligned}$$

Therefore, the existence of nontrivial solutions requires

$$e^{i2k \log 2} = 1.$$

This is, $k_n = \frac{\pi}{\log 2} n$, $n = 1, 2, \dots$ and the eigenvalues are

$$\lambda_n = \frac{\pi^2}{(\log 2)^2} n^2$$

for $n = 1, 2, \dots$.

6. The differential operator

$$L = -x \frac{d^2}{dx^2}$$

acts on the set of square integrable functions with weight $\rho(x) = \frac{1}{x}$ defined in the interval $[1, e]$ that satisfy the following boundary conditions

$$\begin{cases} u(1) = 0, \\ u(e) + cu'(e) = 0. \end{cases}$$

Determine which of the following values of the constant c defines a domain over which the operator L is symmetric.

- a) c arbitrary real.
- b) c arbitrary pure imaginary.
- c) $c = 1 + i$.
- d) There is no c for which the operator is symmetric.
- e) $c = 1 - i$.

Solution: The operator is of the Sturm–Liouville type. Also, if $c \in \mathbb{R}$ then L is regular with separate boundary conditions and therefore symmetric. One can also check that it is a necessary condition. The correct answer is a).

7. Given $h, l > 0$ find the spectral equation that determines the eigenvalues $\lambda = k^2$ of the problem

$$\begin{aligned} -u_{xx} &= \lambda u, & 0 < x < l, \\ \begin{cases} hu(0) - u_x(0) = 0, \\ hu(l) + u_x(l) = 0. \end{cases} \end{aligned}$$

Solution: We are faced with separate boundary conditions and a regular Sturm-Liouville operator, so the operator is symmetric, with simple and increasing eigenvalues. The value $\lambda = 0$ would have as solution $u = c_1 + c_2x$ with $u_x = c_2$ and the boundary conditions would impose

$$\begin{pmatrix} h & -1 \\ h & hl + 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

that has non-trivial solutions as long as $\begin{vmatrix} h & -1 \\ h & hl + 1 \end{vmatrix} = 0$, i.e. $h(hl + 1) = -h$, this is $h = 0$ or $hl = -2$, both impossible, since $h, l > 0$. Therefore, 0 is not an eigenvalue. If $\lambda \neq 0$, we impose the boundary conditions to

$$u(x) = c_1 \cos kx + c_2 \sin kx.$$

As $u_x(x) = -kc_1 \sin kx + kc_2 \cos kx$, we get

$$\begin{pmatrix} h & -k \\ h \cos kl - k \sin kl & h \sin kl + k \cos kl \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will have non-trivial solutions as long as $\begin{vmatrix} h & -k \\ h \cos kl - k \sin kl & h \sin kl + k \cos kl \end{vmatrix} = 0$, i.e.,

$$h(h \sin kl + k \cos kl) + k(h \cos kl - k \sin kl) = 0$$

that after simplifying leads to

$$(h^2 - k^2) \sin kl + 2kh \cos kl = 0,$$

and the spectral relationship is

$$\tan kl = \frac{2hk}{k^2 - h^2}.$$

8. Let us consider the spectral problem

$$\begin{aligned} -u_{xx} &= \lambda u, & 0 < x < 1, \\ \begin{cases} u_x(0) + u(0) = 0, \\ u(1) = 0. \end{cases} \end{aligned}$$

- Find the equation that determines eigenvalues and prove it has infinite solutions.
- Calculate the corresponding eigenfunctions. Do they form a complete orthogonal set in $L^2[0, 1]$?
- Expand the function $u(x) = x \sin \pi x$ in a series of such eigenfunctions.

Hints:

$$\begin{aligned}
\int_0^1 x \sin \pi x \sin k_n x \, dx &= \pi \frac{\pi^2 \sin k - 2k_n \cos k_n - k_n^2 \sin k_n - 2k_n}{(\pi^2 - k_n^2)^2}, \\
\int_0^1 x \sin \pi x \cos k_n x \, dx &= \pi \frac{\pi^2 \cos k - 2k_n \sin k_n - k_n^2 \cos k_n}{(\pi^2 - k_n^2)^2}, \\
\int_0^1 \sin^2 k_n x \, dx &= \frac{1}{2} \left(1 - \frac{\sin k_n \cos k_n}{k_n} \right), \\
\int_0^1 x \sin k_n x \cos k_n x \, dx &= \frac{\sin^2 k_n}{2k_n}, \\
\int_0^1 \cos^2 k_n x \, dx &= \frac{1}{2} \left(1 + \frac{\sin k_n \cos k_n}{2k_n} \right).
\end{aligned}$$

Solution: It is a regular Sturm–Liouville problem with separate boundary conditions, therefore its eigenvalues are real, simple and form an increasing sequence.

$$\lambda_0 < \lambda_1 < \dots$$

with $\lambda_n \xrightarrow{n \rightarrow \infty} \infty$. In addition, the corresponding set of eigenfunctions $\{u_n\}_{n=1}^\infty$ is an orthogonal base of $L^2[0, 1]$. Let's write $\lambda = k^2$ so either $k \in \mathbb{R}$ or $k \in i\mathbb{R}$. Also, the function $u(x) = x \sin \pi x$ fulfills that

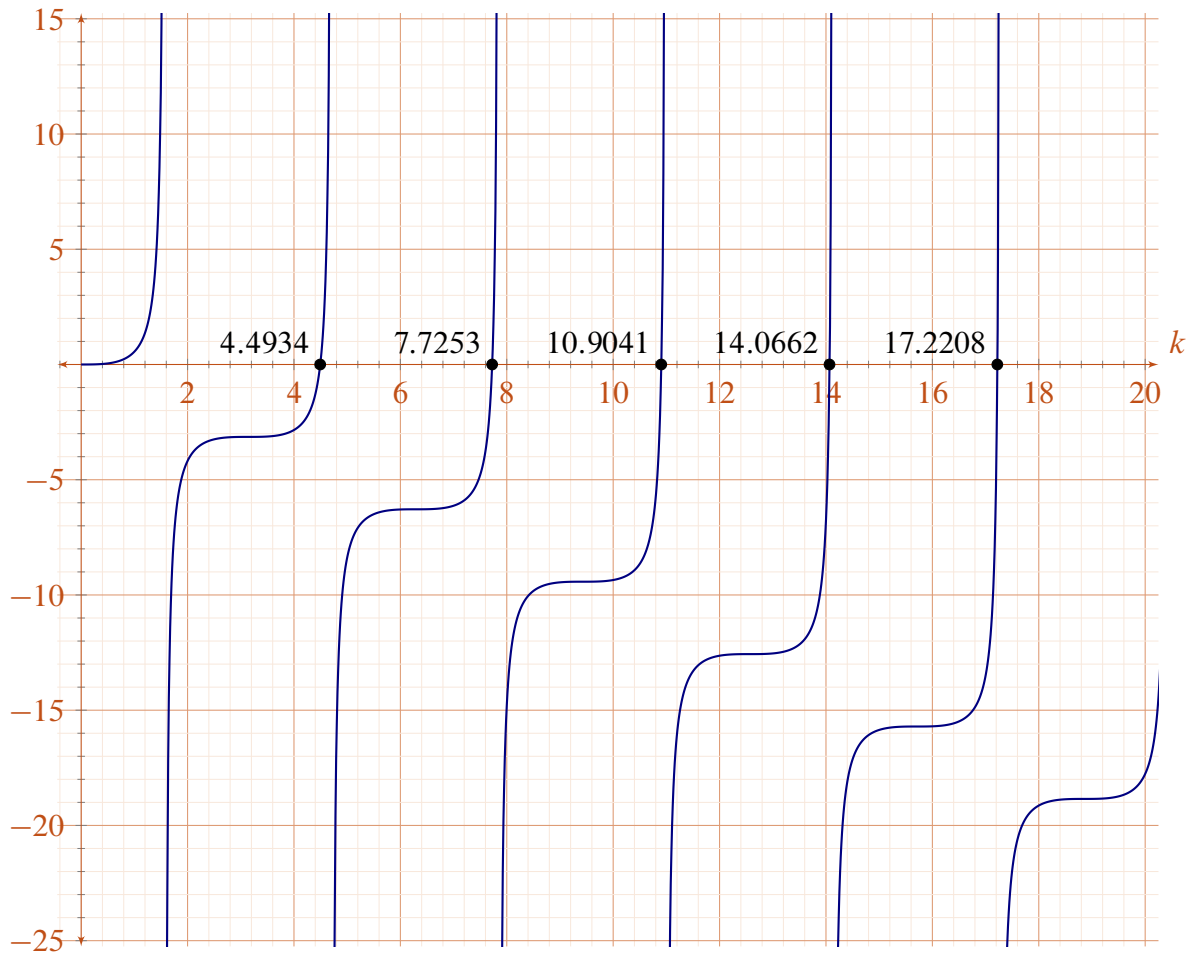
$$\begin{aligned}
u(1) &= (x \sin \pi x)|_{x=1} = \sin \pi = 0, \\
u'(0) + u(0) &= (\sin \pi x + x \pi \cos \pi x)|_{x=0} + (x \sin \pi x)|_{x=0} = 0.
\end{aligned}$$

Thus, the solution satisfies the boundary conditions and is therefore in the domain of the differential operator. So, we have the uniform convergence of this eigenfunction series.

If $k = 0$ the eigenfunction has the form $u(x) = a + bx$ and the boundary conditions are satisfied if and only if $a + b = 0$ leading to 0 being an eigenvalue with corresponding eigenfunction given by $u_0(x) = 1 - x$. On the other hand, if $k \neq 0$, the boundary conditions on the possible solution $a \sin kx + b \cos kx$ imply $\begin{vmatrix} k & 1 \\ \sin k & \cos k \end{vmatrix} = 0$: therefore, the spectral equation that determines the eigenvalues is

$$\boxed{\tan k = k.}$$

Therefore, given the parity of the spectral equation we only deal with the solutions with $k > 0$.



Graph of $f(k) = \tan(k) - k$ and the first five zeros

First zeros of $\tan k = k$					
4.4934	7.7253	10.9041	14.0662	17.2208	20.4204

An asymptotic analysis of the tangent near its poles leads to zeros k_n satisfying $k_n = (n + \frac{1}{2})\pi - \frac{1}{(n + \frac{1}{2})\pi} + \mathcal{O}(\frac{1}{n^2})$ for $n \rightarrow \infty$. Indeed, from the addition relations for trigonometric functions we know that $\tan((n + \frac{1}{2})\pi + \epsilon) = -\cot \epsilon$; so the relationship $\tan k = k$ for $k = (n + \frac{1}{2})\pi + \epsilon$, with $\epsilon > 0$ small, leads to $-\frac{\cos \epsilon}{\sin \epsilon} = (n + \frac{1}{2})\pi + \epsilon$. Recalling that, for $\epsilon \sim 0$ small, we have $\cos \epsilon \sim 1$ and $\sin \epsilon \sim \epsilon$ we conclude $-\frac{1}{\epsilon} \sim (n + \frac{1}{2})\pi$ for $n \rightarrow \infty$.

Once we have discussed the eigenvalues $\lambda_n = k_n^2$ we go on to consider the eigenfunctions. Given the spectral equation the eigenfunctions for $n = 1, 2, \dots$ can be chosen in the form

$$u_n(x) = \sin k_n x - k_n \cos k_n x.$$

Then, you have an orthogonal basis given by $\{u_n(x)\}_{n=0}^{\infty}$, which allows you to write

$$x \sin \pi x = \sum_{n=0}^{\infty} c_n u_n(x),$$

where the series coefficients are given by the following formulas

$$c_n = \begin{cases} \frac{\int_0^1 (1-x)x \sin \pi x \, dx}{\int_0^1 (1-x)^2 \, dx}, & n = 0 \\ \frac{\int_0^1 (\sin k_n x - k_n \cos k_n x)x \sin \pi x \, dx}{\int_0^1 (\sin k_n x - k_n \cos k_n x)^2 \, dx}, & n = 1, 2, \dots \end{cases}$$

So, for $n = 0$, we get $c_0 = \frac{12}{\pi^3}$. On the other hand, using the hint, we can write

$$\begin{aligned} H_n &:= \int_0^1 (\sin k_n x - k_n \cos k_n x)^2 \, dx \\ &= \int_0^1 \sin^2 k_n x \, dx + k_n^2 \int_0^1 \cos^2 k_n x \, dx - 2k_n \int_0^1 \sin k_n x \cos k_n x \, dx \\ &= \frac{1}{2} \left(1 - \frac{\sin k_n \cos k_n}{k_n} \right) + k_n^2 \frac{1}{2} \left(1 + \frac{\sin k_n \cos k_n}{k_n} \right) - 2k_n \frac{\sin^2 k_n}{2k_n} \\ &= \frac{1 + k_n^2}{2} - \frac{1 - k_n^2}{2} \frac{\sin k_n \cos k_n}{k_n} - \sin^2 k_n, \end{aligned}$$

and if we now remember that $\sin k_n = k_n \cos k_n$ we have

$$\begin{aligned} H_n &= \frac{1 + k_n^2}{2} - \frac{1 - k_n^2}{2} \cos^2 k_n - k_n^2 \cos^2 k_n = \frac{1 + k_n^2}{2} - \frac{1 + k_n^2}{2} \cos^2 k_n \\ &= \frac{1 + k_n^2}{2} \sin^2 k_n. \end{aligned}$$

On the other hand, from the hints we deduce that

$$I_n := \int_0^1 (\sin k_n x - k_n \cos k_n x)x \sin \pi x \, dx = \frac{\pi}{(\pi^2 - k_n^2)^2} (A_n - k_n B_n)$$

where

$$\begin{aligned} A_n &:= \pi^2 \sin k_n - 2k_n \cos k_n - k_n^2 \sin k_n - 2k_n, \\ B_n &:= \pi^2 \cos k_n + 2k_n \sin k_n - k_n^2 \cos k_n. \end{aligned}$$

But, from the spectral relation $\sin k_n = k_n \cos k_n$ it follows that

$$\begin{aligned} A_n - k_n B_n &= \pi^2 \sin k_n - 2k_n \cos k_n - k_n^2 \sin k_n - 2k_n - k_n(\pi^2 \cos k_n + 2k_n \sin k_n - k_n^2 \cos k_n) \\ &= -2(1 + k_n^2) \sin k_n - 2k_n. \end{aligned}$$

Finally, remembering the spectral relation we can write,

$$\begin{aligned} c_n &= -\frac{4\pi}{(\pi^2 - k_n^2)^2} \frac{k_n + (1 + k_n^2) \sin k_n}{(1 + k_n^2) \sin^2 k_n} = -\frac{4\pi}{(\pi^2 - k_n^2)^2} \left(\frac{\frac{\sin k_n}{\cos k_n}}{1 + \frac{\sin^2 k_n}{\cos^2 k_n}} \frac{1}{\sin^2 k_n} + \frac{1}{\sin k_n} \right) \\ &= -\frac{4\pi}{(\pi^2 - k_n^2)^2} \left(\frac{\cos k_n}{\sin k_n} + \frac{1}{\sin k_n} \right), \end{aligned}$$

that leads to

$$c_n = -\frac{4\pi}{(\pi^2 - k_n^2)^2} \left(\frac{1}{\sin k_n} + \frac{1}{k_n} \right).$$

From where we get the table

First coefficients c_n					
0.3870	0.0946	-0.0058	9.646×10^{-4}	-3.8177×10^{-4}	-7.9528×10^{-5}

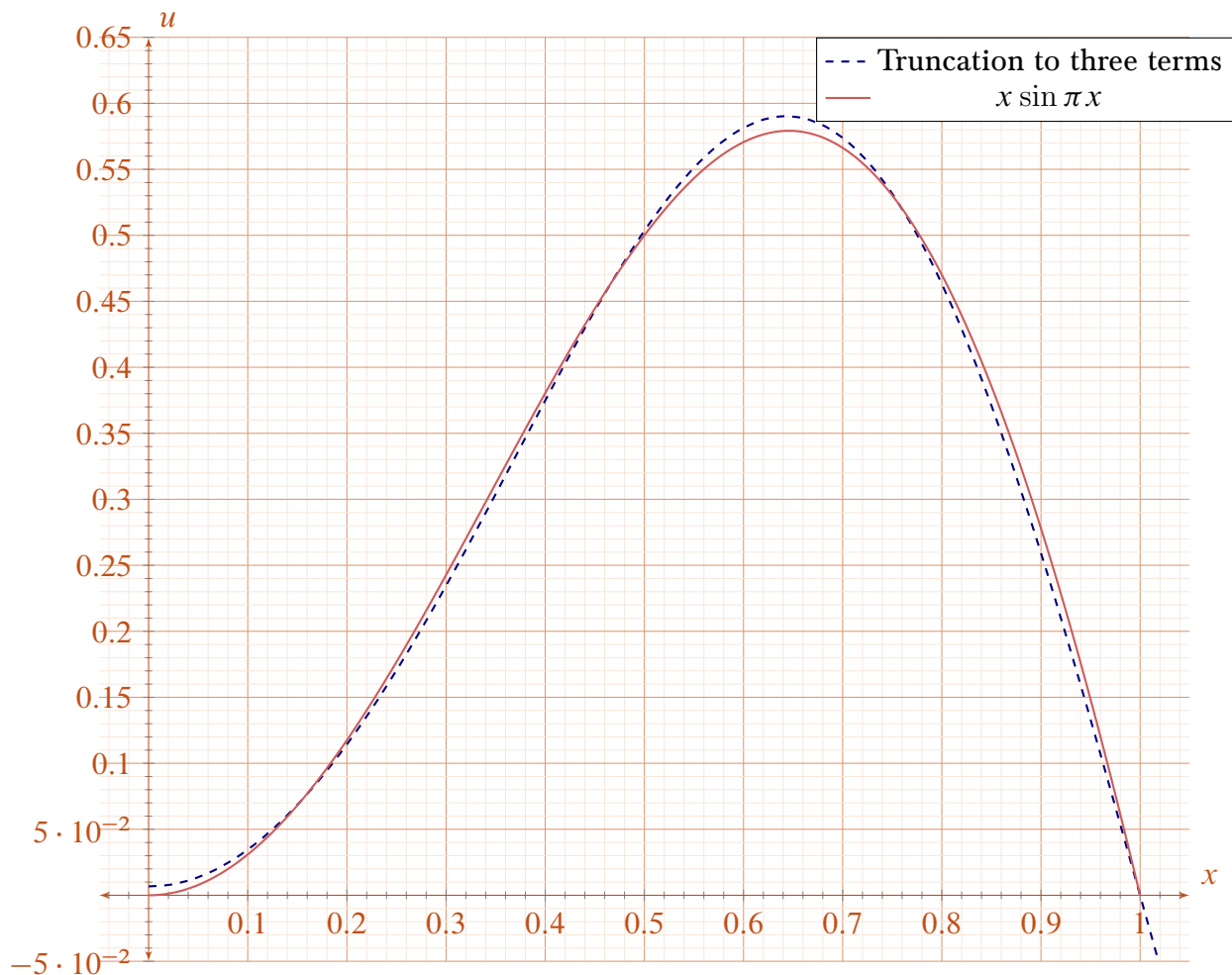
Therefore, the series expansion requested in $[0, 1]$ will be

$$x \sin \pi x = \frac{12}{\pi^3} (1 - x) - \sum_{n=1}^{\infty} \frac{4\pi}{(\pi^2 - k_n^2)^2} \left(\frac{1}{\sin k_n} + \frac{1}{k_n} \right) (\sin k_n x - k_n \cos k_n x).$$

In particular, the truncation to three terms is

$$\begin{aligned} x \sin \pi x &\sim 0.3870(1 - x) + 0.0946 \sin 4.4934x - 0.4251 \cos 4.4934x \\ &\quad - 0.0058 \sin 7.7253x + 0.0448 \cos 7.7253x, \end{aligned}$$

whose graph we compare with that of $u(x) = x \sin \pi x$ in the following figure



3.5.2. Exercises

1. Calculate the scalar products:

$$(u, v) = \int_{\bar{\Omega}} \bar{u}(x)v(x)\rho(x) d^n x,$$

corresponding to the following data:

- a) $u(x, y) = xy + i x^2 y^2$, $v(x, y) = 1 - ixy$, $\bar{\Omega} = [0, 1] \times [0, 1]$, $\rho(x, y) = x^2 y^2$.
 - b) $u(x, y) = \exp(-(x^2 + y^2))$, $v(x, y) = 1$, $\bar{\Omega} = \mathbb{R}^2$, $\rho(x, y) \equiv 1$.
2. Prove that if L_1 and L_2 are symmetric operators on a certain domain $\mathcal{D}$, then all the linear combinations $\lambda_1 L_1 + \lambda_2 L_2$, with real coefficients λ_1 and λ_2 , are also symmetric operators on that domain.
 3. Prove that the operator $L = -\frac{d^2}{dx^2}$ is symmetric over the domain $\mathcal{D}$ of class C^∞ functions in $[a, b]$ such that they satisfy one of the following types of boundary conditions:

$$u(b) = \alpha u(a) + \beta u'(a), \quad u'(b) = \gamma u(a) + \delta u'(a),$$

being $\alpha, \beta, \gamma, \delta$ real numbers such that:

$$\alpha\delta - \beta\gamma = 1.$$

4. Let us consider the operator $L = -i \frac{d}{dx}$ on the domain $\mathcal{D}$ of functions of class C^{infy} in $[a, b]$ such that they satisfy the boundary value condition

$$\alpha u(a) + \beta u(b) = 0,$$

being α and β given complex numbers. Prove that L is symmetric if and only if

$$|\alpha| = |\beta|.$$

Determine in such a case the spectrum and eigenfunctions of L .

5. Let us consider the operator $Lu = \frac{d^2 u}{dx^2} - 2u$ on the domain $\mathcal{D}$ of functions of class C^2 in $[0, 1]$ such that

$$u(0) = 0, u(1) = 0.$$

Determine the spectrum and the eigenfunctions.

6. Prove that an operator of Sturm–Liouville regular on an interval $[a, b]$ of the form

$$Lu = \frac{1}{\rho} (-D(pDu) + qu),$$

is symmetric over domains corresponding to the boundary conditions

a) Separate:

$$\begin{cases} \alpha u(a) + \alpha' u'(a) = 0, \\ \beta u(b) + \beta' u'(b) = 0, \end{cases}$$

with $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$.

b) Periodic

$$\begin{cases} u(a) = u(b), \\ p(a) u'(a) = p(b) u'(b). \end{cases}$$

4. FOURIER SERIES AND FOURIER TRANSFORM

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FOURIER analysis, series and transformations, is the the topic that we are about to introduce in this lecture. We will divide it into two parts. First, we study Fourier series expansions, either with imaginary exponentials or in sine and cosines, either with sine-only expansions or cosines-only expansions. Next, we will deal with the continuous version of these expansions, which appears when the intervals where the functions are expanded are not bounded. This *corpus* of knowledge, Fourier's analysis, is one of the key pieces for understanding natural phenomena and was introduced some two hundred years ago, by Joseph Fourier, in the study of heat transmission.

§4.1. Fourier Trigonometric Bases



EIGENVALUE problems associated with two particularly simple differential operators will be consider. The corresponding developments in eigenfunctions constitute the so-called **Fourier trigonometric series**. The operator (which in Quantum Mechanics is the linear moment operator)

$$L = -i \frac{d}{dx}.$$

is symmetric in a domain built up with the differentiable functions u in a bounded interval $[a, b]$ such that $u(a) = u(b)$, i.e., functions that satisfy periodic conditions.

The problem of eigenvalues

$$\begin{cases} -i \frac{d}{dx} u = \lambda u, \\ u(a) = u(b), \end{cases}$$

has as a solution

$$u(x) = c e^{i\lambda x}$$

the boundary value condition being satisfied if and only if

$$e^{i\lambda(b-a)} = 1.$$

This condition determines the possible elements of the spectrum

$$\lambda_n = n\omega, \quad n \in \mathbb{Z}, \quad \omega := \frac{2\pi}{b-a}$$

with the following associated eigenfunctions

$$u_n(x) = e^{in\omega x}.$$

The second operator we consider is

$$L = -\frac{d^2}{dx^2}.$$

We will study domains associated with three types of boundary conditions

1. **Periodic.** The problem of eigenvalues is

$$\begin{cases} -\frac{d^2 u}{dx^2} = \lambda u, \\ u(a) = u(b), \\ \frac{du}{dx}(a) = \frac{du}{dx}(b). \end{cases}$$

When $\lambda \neq 0$, the solution is

$$u(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$$

verifying the conditions at the boundary if and only if the following homogeneous linear system for (c_1, c_2) is fulfilled

$$(8) \quad \begin{aligned} [\cos \sqrt{\lambda}a - \cos \sqrt{\lambda}b]c_1 + [\sin \sqrt{\lambda}a - \sin \sqrt{\lambda}b]c_2 &= 0, \\ [\sin \sqrt{\lambda}a - \sin \sqrt{\lambda}b]c_1 - [\cos \sqrt{\lambda}a - \cos \sqrt{\lambda}b]c_2 &= 0. \end{aligned}$$

Non-trivial solutions $(c_1, c_2) \neq (0, 0)$ of this system exist if and only if the matrix of coefficients has zero determinant

$$\begin{vmatrix} \cos \sqrt{\lambda}a - \cos \sqrt{\lambda}b & \sin \sqrt{\lambda}a - \sin \sqrt{\lambda}b \\ -\sin \sqrt{\lambda}a + \sin \sqrt{\lambda}b & \cos \sqrt{\lambda}a - \cos \sqrt{\lambda}b \end{vmatrix} = 0,$$

this is

$$[\cos \sqrt{\lambda}a - \cos \sqrt{\lambda}b]^2 + [\sin \sqrt{\lambda}a - \sin \sqrt{\lambda}b]^2 = 0.$$

Expanding the squares and using the trigonometric formulas we get

$$1 - \cos \sqrt{\lambda}(b-a) = 0$$

and that's why non-null eigenvalues are

$$\lambda_n = \omega^2 n^2, \quad n = 1, 2, \dots \quad \omega := \frac{2\pi}{b-a}.$$

These eigenvalues are not simple (note that this is a homogeneous Sturm–Liouville problem with periodic boundary conditions), since for any of them the linear system (8) is reduced to a system in which the matrix of coefficients is the zero matrix; therefore, it admits as solutions all the values of (c_1, c_2) . In this way

$$\mathcal{D}_{\lambda_n} = \mathbb{C}\{\sin n\omega x, \cos n\omega x\}.$$

On the other hand, $\lambda = 0$ is also an eigenvalue and an eigenfunction can be taken as the constant 1 (the solutions of $-u'' = 0$ are $u(x) = c_1 + c_2x$, but the boundary conditions impose $c_2 = 0$). Therefore, the set of eigenvalues is

$$\lambda_n = \omega^2 n^2, \quad n = 0, 1, \dots, \quad \omega := \frac{2\pi}{b-a}$$

and the set of eigenfunctions is

$$\{1, \cos n\omega x, \sin n\omega x\}_{n=1}^{\infty},$$

and constitutes a complete orthogonal set in $L^2[a, b]$.

2. **Homogeneous Dirichlet.** The problem of eigenvalues is

$$\begin{cases} -\frac{d^2 u}{dx^2} = \lambda u. \\ u(a) = 0, \\ u(b) = 0. \end{cases}$$

The value $\lambda = 0$ is not in the spectrum because if it were, the corresponding eigenfunction would be as follows $u(x) = c_1 + c_2x$ and the boundary conditions would imply that $c_1 + c_2a = c_1 + c_2b = 0$, but as $b - a \neq 0$ we have $c_1 = c_2 = 0$. The solutions to the EDO for $\lambda \neq 0$ have the form

$$u(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x,$$

and the boundary conditions are satisfied if and only if

$$c_1 \cos \sqrt{\lambda}a + c_2 \sin \sqrt{\lambda}a = 0,$$

$$c_1 \cos \sqrt{\lambda}b + c_2 \sin \sqrt{\lambda}b = 0.$$

There will be non-trivial solutions $(c_1, c_2) \neq (0, 0)$ if and only if

$$\begin{vmatrix} \cos \sqrt{\lambda}a & \sin \sqrt{\lambda}a \\ \cos \sqrt{\lambda}b & \sin \sqrt{\lambda}b \end{vmatrix} = 0,$$

this is

$$\sin(\sqrt{\lambda}(b-a)) = 0.$$

So, eigenvalues are

$$\lambda_n = \frac{\omega^2}{4}n^2, \quad n = 1, 2, \dots, \quad \omega := \frac{2\pi}{b-a},$$

and the corresponding eigenfunctions turn out to be

$$\begin{aligned} u_n(x) &= -\sin n \frac{\omega}{2} a \cos n \frac{\omega}{2} x + \cos n \frac{\omega}{2} a \sin n \frac{\omega}{2} x \\ &= \sin n \frac{\omega}{2} (x - a). \end{aligned}$$

3. **Homogeneous Neumann.** Now, we have

$$\left\{ \begin{array}{l} -\frac{d^2 u}{dx^2} = \lambda u, \\ \frac{du}{dx}(a) = 0, \\ \frac{du}{dx}(b) = 0. \end{array} \right.$$

In this case $\lambda = 0$ is in the spectrum: the corresponding solution is $u(x) = c_1 + c_2 x$, the boundary conditions imply $c_2 = 0$ and the corresponding eigenfunction can be taken as $u_0(x) = 1$. When $\lambda \neq 0$ the solutions have the form

$$u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

and the boundary conditions are satisfied if and only if

$$\begin{aligned} -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} a + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} a &= 0, \\ -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} b + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} b &= 0. \end{aligned}$$

There are nontrivial solutions $(c_1, c_2) \neq (0, 0)$ if and only if

$$\sqrt{\lambda} \begin{vmatrix} -\sin \sqrt{\lambda} a & \cos \sqrt{\lambda} a \\ -\sin \sqrt{\lambda} b & \cos \sqrt{\lambda} b \end{vmatrix} = 0$$

this is

$$\sin \left(\sqrt{\lambda} (b - a) \right) = 0.$$

Thus, the eigenvalues are

$$\lambda_n = \frac{\omega^2}{4} n^2, \quad n = 0, 1, \dots \quad \omega := \frac{2\pi}{b - a}$$

and the eigenfunctions are

$$\begin{aligned} u_n(x) &= \cos n \frac{\omega}{2} a \cos n \frac{\omega}{2} x + \sin n \frac{\omega}{2} a \sin n \frac{\omega}{2} x \\ &= \cos n \frac{\omega}{2} (x - a). \end{aligned}$$

§4.2. Fourier Series



EXPANSIONS in the eigenfunctions of the symmetric differential operators discussed in the previous section constitute the so-called Fourier trigonometric series expansions. The operator $L = -i \frac{d}{dx}$ is symmetric over the domain $\mathcal{D}$ of differentiable functions $u = u(x)$ in the interval $[a, b]$ such that $u(a) = u(b)$.

Fourier Exponential Series Expansions

Any function $u \in L^2[a, b]$ can be expanded in the form

$$u = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}, \quad \omega := \frac{2\pi}{b-a},$$

with coefficients given by

$$c_n = \frac{1}{b-a} \int_a^b e^{-in\omega x} u(x) dx.$$

In general the series converges in square mean to the function u , however if $u \in \mathcal{D}$ the series converges uniformly.

Note that in the previous calculation of the coefficients c_n it has been taken into account that $\|u_n\|^2 = \int_a^b dx = b-a$.

The operator $L = -\frac{d^2}{dx^2}$ is of the Sturm–Liouville regular and symmetric type provided that $[a, b]$ is a finite interval and any of the three boundary conditions studied occur: periodic, Dirichlet and Neumann. Therefore, the associated sets of eigenfunctions are orthogonal bases in $L^2[a, b]$. Thus all the series that we are going to deal with converge in L^2 mean in to the function expanded and if, in addition, this function belongs to the corresponding domain $\mathcal{D}$ the convergence is uniform.

Fourier Sine and Cosine Series (Periodic Conditions)

All functions $u \in L^2[a, b]$ can be expanded in the form

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x), \quad \omega := \frac{2\pi}{b-a},$$

where the coefficients are determined by the so-called **Euler formulas**

$$a_n = \frac{2}{b-a} \int_a^b \cos n\omega x u(x) dx, \quad n \geq 0,$$

$$b_n = \frac{2}{b-a} \int_a^b \sin n\omega x u(x) dx, \quad n \geq 1.$$

The eigenfunction basis used is

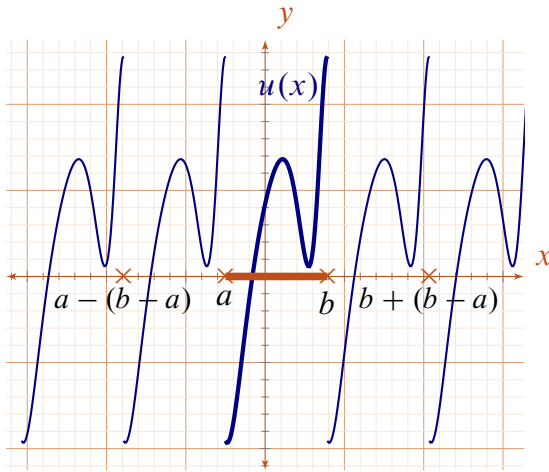
$$\left\{ u_0 = \frac{1}{2}, u_n = \cos n\omega x, v_n = \sin n\omega x \right\}_{n=1}^{\infty}$$

and it has been taken into account that

$$\begin{aligned} \|u_0\|^2 &= \frac{1}{4} \int_a^b dx = \frac{b-a}{4}, \\ \|u_n\|^2 &= \int_a^b \cos^2 n\omega x dx = \frac{1}{2} \int_a^b (1 + \cos 2n\omega x) dx = \frac{b-a}{2}, \\ \|v_n\|^2 &= \int_a^b \sin^2 n\omega x dx = \int_a^b (1 - \cos^2 n\omega x) dx = \frac{b-a}{2}. \end{aligned}$$

It is clear that there is a close relationship between the Fourier series of exponentials and those of sines and cosines. We can go from one to another by using the formulas

$$\begin{aligned} e^{in\omega x} &= \cos n\omega x + i \sin n\omega x, \\ \cos n\omega x &= \frac{1}{2} (e^{in\omega x} + e^{-in\omega x}), \\ \sin n\omega x &= \frac{1}{2i} (e^{in\omega x} - e^{-in\omega x}). \end{aligned}$$



Periodic extension $u_{\text{per}}(x)$ of $u(x)$

Although the function u is, in principle, defined only on the interval $[a, b]$, the two types of Fourier expansions that we have discussed provide series of periodic functions with period $(b-a)$. Thus, the functions defined by these series when they are considered on the domain $\mathbb{R}$ are periodic functions with period $(b-a)$. Therefore, these Fourier series lead to the extension of u defined as follows:

Given $x \in \mathbb{R}$, there is only one integer $m \in \mathbb{Z}$ such that $x \in [a + m(b-a), b + m(b-a)]$ and the periodic extension is

$$u_{\text{per}}(x) := u(x - m(b-a)).$$

That is, the $u_{\text{per}}(x)$ graph is constructed by simply pasting copies, one after the other, of

the u graph over $[a, b]$. Expansions in exponential series or cosines and sines can be applied either to u or to the corresponding periodic extension.

Fourier Sine Series (Dirichlet Conditions)

All $u \in L^2[a, b]$ can be expanded as

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\omega}{2}(x-a), \quad \omega := \frac{2\pi}{b-a},$$

with the coefficients of the series given by

$$b_n = \frac{2}{b-a} \int_a^b \sin \frac{n\omega}{2}(x-a) u(x) dx, \quad n \geq 1.$$

The orthogonal basis of eigenfunctions is

$$\left\{ v_n = \sin \frac{n\omega}{2}(x-a) \right\}_{n \geq 1},$$

whose squared norms are

$$\|v_n\|^2 = \int_a^b \sin^2 \frac{n\omega}{2}(x-a) dx = \int_a^b \frac{1}{2} \left(1 - \cos n \frac{2\pi}{b-a}(x-a) \right) dx = \frac{b-a}{2}.$$

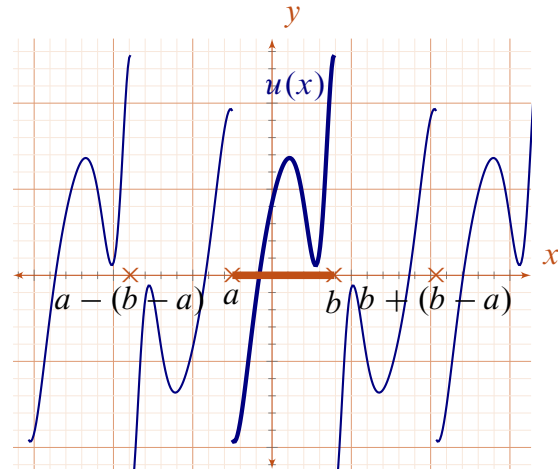
The Fourier sine series of makes sense not only in the interval $[a, b]$ but throughout the whole real line $\mathbb{R}$. The result is the so-called odd periodic extension of the function u . This odd extension of u to the entire real line is determined in two stages: first extends the function to the interval $[a - (b-a), b] = [2a - b, a] \cup [a, b]$ in odd form, that is, if $x \in [a, b]$ we define

$$u_{\text{odd}}(a - (x-a)) = -u(x),$$

and then a periodic extension is made from the interval $[2a - b, a]$ to the entire real line. This construction is justified if we observe that the functions $\frac{\omega}{2}(x-a)$ that appear in the series considered as functions in $\mathbb{R}$ fulfill that

$$\sin \frac{\omega}{2}(X-a) \Big|_{X=2a-x} = -\sin \frac{\omega}{2}(x-a),$$

which leads to the odd extension, which is also periodic with a $2(b-a)$ -period.



Odd extension $u_{\text{odd}}(x)$ of $u(x)$

Fourier Cosine Series (Neumann conditions)

All $u \in L^2[a, b]$ can be expanded as

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \frac{\omega}{2} (x - a), \quad \omega := \frac{2\pi}{b-a},$$

where the coefficients of the series are given by

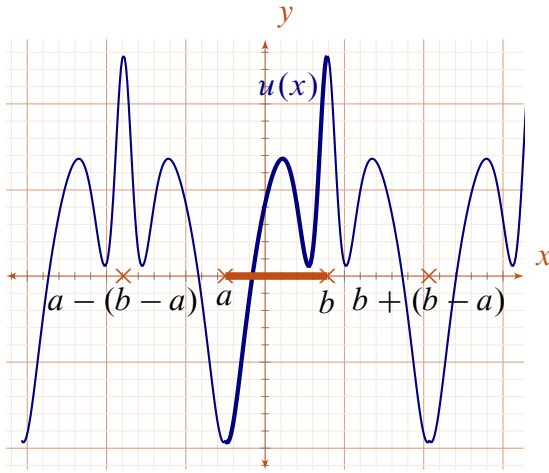
$$a_n = \frac{2}{b-a} \int_a^b \cos n \frac{\omega}{2} (x - a) u(x) dx, \quad n \geq 0.$$

The eigenfunction basis used is

$$\left\{ u_0 = \frac{1}{2}, u_n = \cos \left(n \frac{\omega}{2} (x - a) \right) \right\}_{n \geq 1},$$

with squared norms

$$\begin{aligned} \|u_0\|^2 &= \frac{1}{4} \int_a^b dx = \frac{b-a}{4}, \\ \|u_n\|^2 &= \int_a^b \cos^2 n \frac{\omega}{2} (x - a) dx = \frac{1}{2} \int_a^b \left(1 + \cos n \frac{2\pi}{b-a} (x - a) \right) dx = \frac{b-a}{2}. \end{aligned}$$



Even extension $u_{\text{even}}(x)$ of $u(x)$

The Fourier cosine series is a function defined on $\mathbb{R}$ which is the even extension of u defined as follows: we extend the function u to the interval $[a - (b - a), b] = [2a - b, a] \cup [a, b]$ evenly, that is, if $x \in [a, b]$ we define

$$u_{\text{par}}(a - (x - a)) = u(x),$$

and then the periodic extension is made from $[2a - b, b]$ to the whole real line. The justification for this is analogous to the sine Fourier series. We observe that the functions $\cos \frac{\omega}{2} (x - a)$ considered as functions in $\mathbb{R}$ fulfill that $\cos \frac{n\omega}{2} (X - a) \Big|_{X=2a-x} = \cos \frac{n\omega}{2} (x - a)$, which leads to even extension, and is also periodic with period $2(b - a)$.



Examples: Consider the function $u(x) = e^x$ in the interval $[0, 1]$. Let's analyze the three different Fourier trigonometric expansions. First we consider the sine and cosines series expansion and their periodic extension. Then, we analyze series expansions in cosine only and in sines only and their even and odd extensions, respectively.

1. The Fourier sine and cosine series of

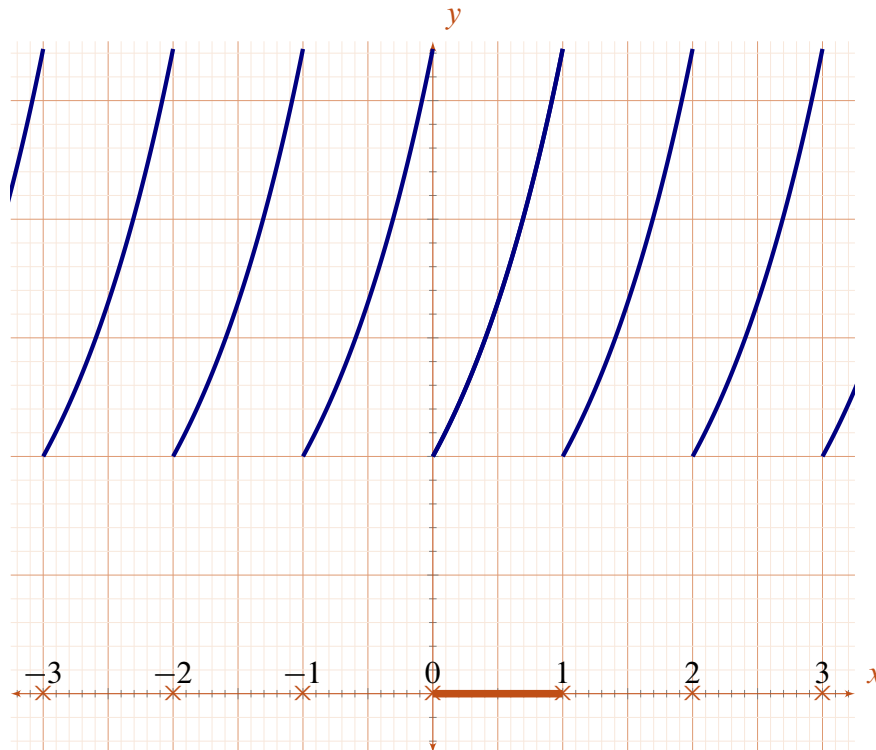
$$e^x = \frac{a_0}{2} + \sum_{n \geq 1} [a_n \cos 2\pi n x + b_n \sin 2\pi n x]$$

has for coefficients

$$\begin{aligned} a_n &= 2 \int_0^1 e^x \cos 2\pi n x \, dx = 2 \operatorname{Re} \left[\int_0^1 e^x e^{2i\pi n x} \, dx \right] = 2 \operatorname{Re} \left[\int_0^1 e^{(1+2i\pi n)x} \, dx \right] \\ &= 2 \operatorname{Re} \left[\frac{e^{(1+2i\pi n)x}}{1+2i\pi n} \right]_0^1 \\ &= \frac{2[e - 1]}{1 + 4\pi^2 n^2}, \end{aligned}$$

$$\begin{aligned} b_n &= 2 \int_0^1 e^x \sin 2\pi n x \, dx = 2 \operatorname{Im} \left[\int_0^1 e^x e^{2i\pi n x} \, dx \right] = 2 \operatorname{Im} \left[\frac{e^{(1+2i\pi n)x}}{1+2i\pi n} \right]_0^1 \\ &= -2\pi n a_n. \end{aligned}$$

The periodic extension of e^x has as a graph

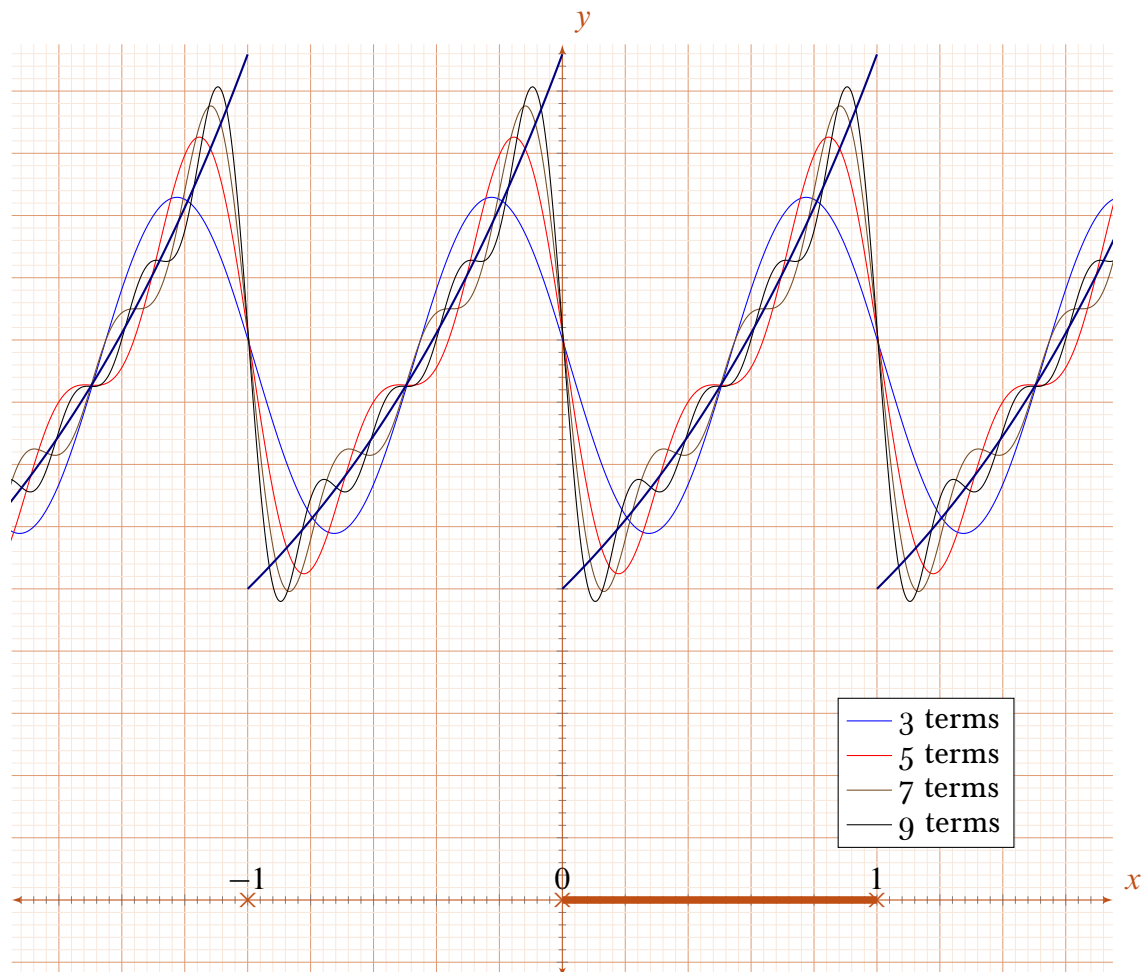


Periodic extension $[e^x]_{\text{per}}$ of e^x

and the first terms of the series in sines and cosines are

$$\begin{aligned}
 [e^x]_{\text{per}} \sim & 2(e-1) \left[\frac{1}{2} + \frac{1}{1+4\pi^2} \cos 2\pi x + \frac{1}{1+16\pi^2} \cos 4\pi x \right. \\
 & \left. + \frac{1}{1+36\pi^2} \cos 6\pi x + \frac{1}{1+64\pi^2} \cos 8\pi x + \dots \right] \\
 & - 4\pi(e-1) \left[\frac{1}{1+4\pi^2} \sin 2\pi x + \frac{2}{1+16\pi^2} \sin 4\pi x \right. \\
 & \left. + \frac{3}{1+36\pi^2} \sin 6\pi x + \frac{4}{1+64\pi^2} \sin 8\pi x + \dots \right].
 \end{aligned}$$

We represent below the graph of the periodic extension of the exponential and those of its Fourier trigonometric series truncated for 3, 5, 7 and 9 terms (with a different color each)



Periodic extension $[e^x]_{\text{per}}$ and truncations of the Fourier sine and cosine series in

Observe that in the discontinuities of the function the series tends to the mean value of the left and right values of the given function. In addition we see that near these discontinuities the truncated series oscillates, the presence of these oscillations is the so-called Gibbs phenomenon, and it remains even if we take more and more terms in the series.

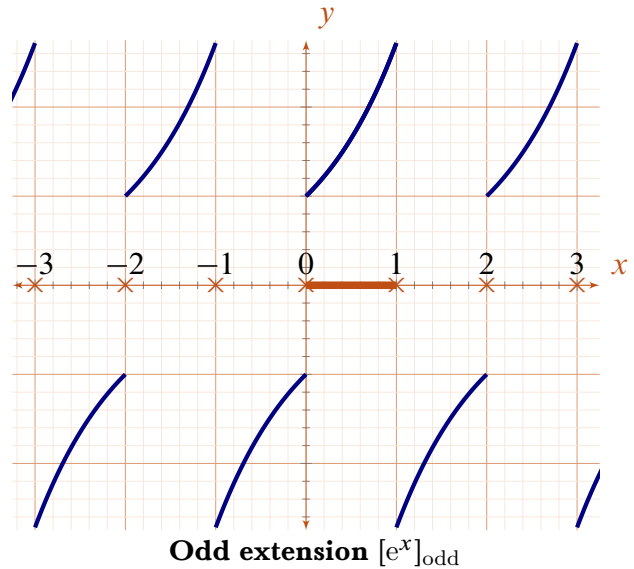
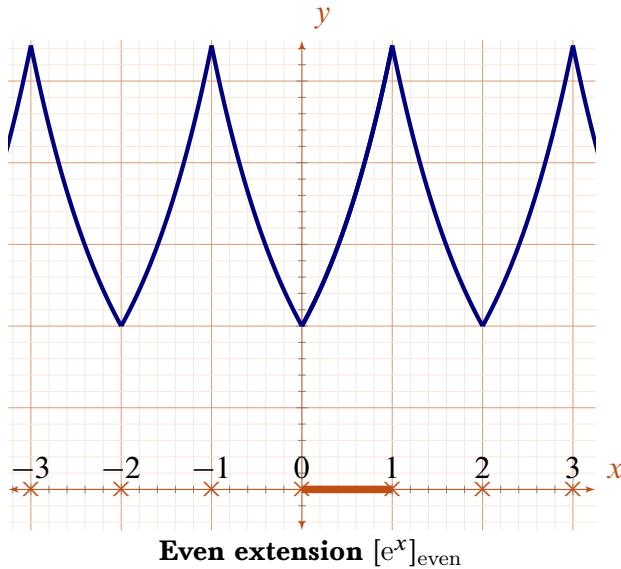
2. We now consider the series of only cosines and only sines,

$$e^x = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos \pi n x, \quad e^x = \sum_{n \geq 1} b_n \sin \pi n x.$$

It is satisfied that

$$\begin{aligned} a_n &= 2 \int_0^1 e^x \cos \pi n x \, dx = 2 \operatorname{Re} \left[\int_0^1 e^x e^{i \pi n x} \, dx \right] = 2 \operatorname{Re} \left[\int_0^1 e^{(1+i \pi n)x} \, dx \right] \\ &= 2 \operatorname{Re} \left[\frac{e^{(1+i \pi n)x}}{1+i \pi n} \right]_0^1 \\ &= \frac{2[(-1)^n e - 1]}{1 + \pi^2 n^2}, \\ b_n &= 2 \int_0^1 e^x \sin \pi n x \, dx = 2 \operatorname{Im} \left[\int_0^1 e^x e^{i \pi n x} \, dx \right] = 2 \operatorname{Im} \left[\frac{e^{(1+i \pi n)x}}{1+i \pi n} \right]_0^1 \\ &= -n \pi a_n \end{aligned}$$

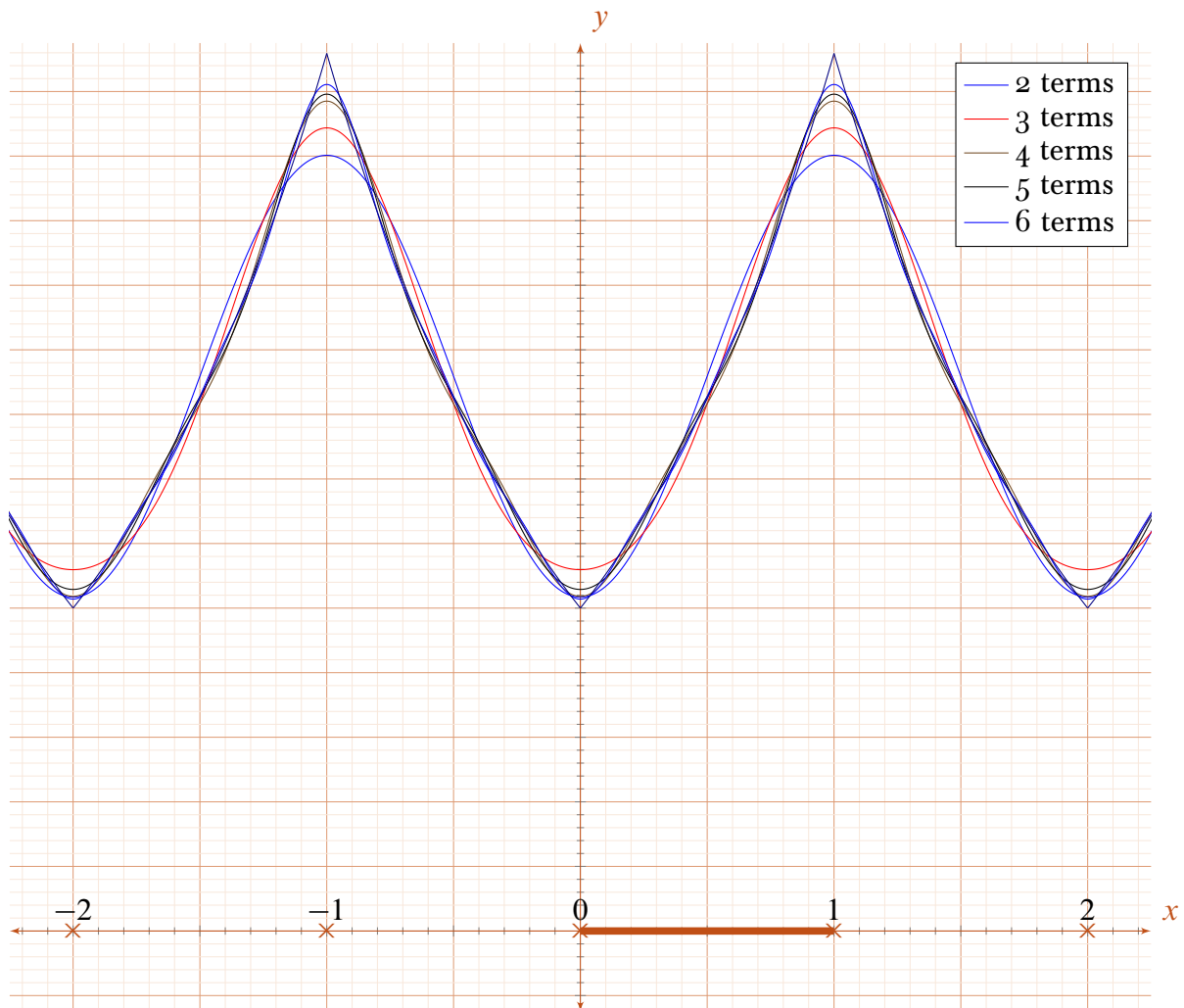
The even extension $[e^x]_{\text{even}}$ and odd extension $[e^x]_{\text{odd}}$ is represented graphically as follows



The corresponding series are

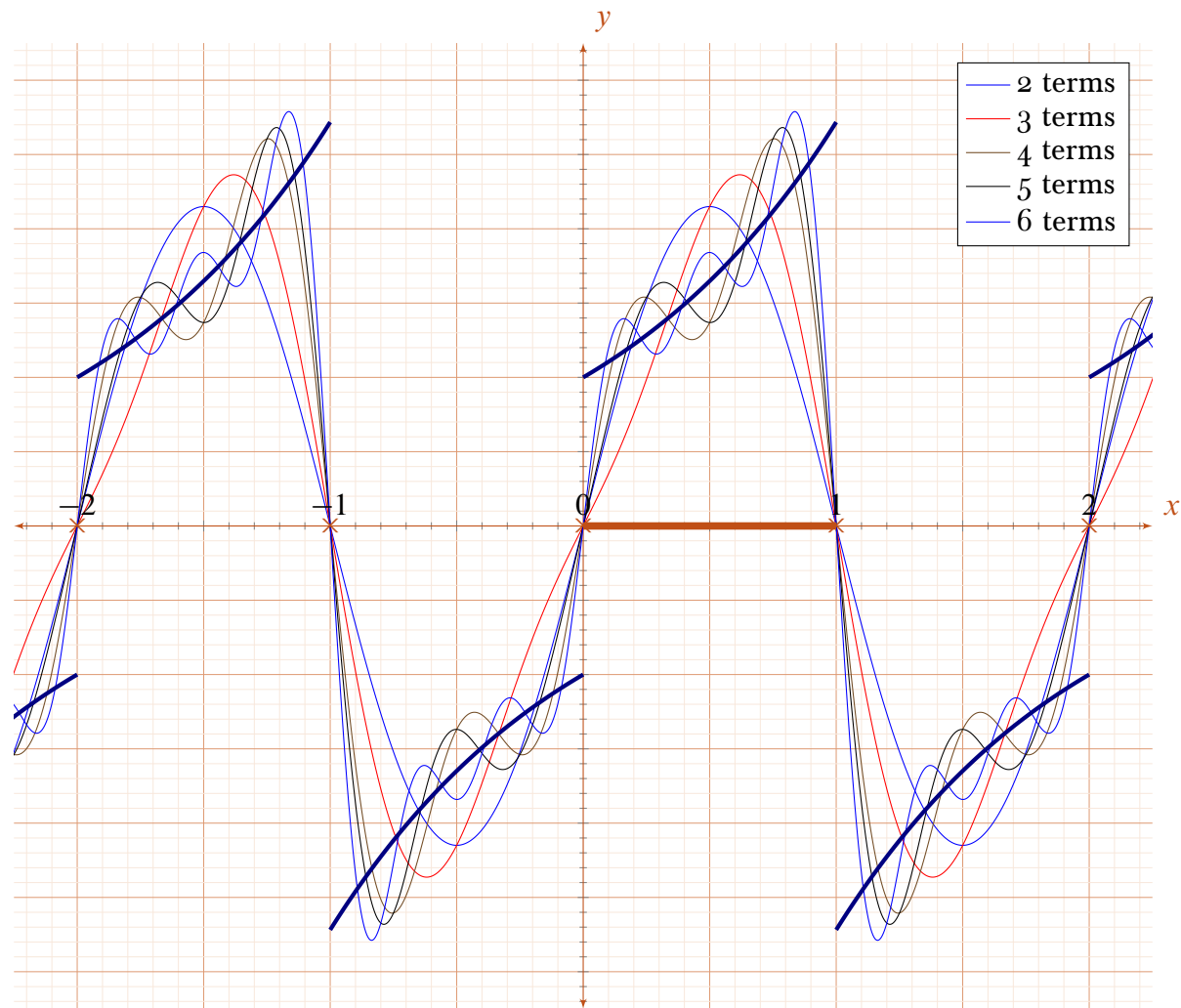
$$\begin{aligned}
 [e^x]_{\text{even}} &\sim e - 1 + 2 \left[-\frac{e+1}{1+\pi^2} \cos \pi x + \frac{e-1}{1+4\pi^2} \cos 2\pi x \right. \\
 &\quad \left. - \frac{e+1}{1+9\pi^2} \cos 3\pi x + \frac{e-1}{1+16\pi^2} \cos 4\pi x + \dots \right], \\
 [e^x]_{\text{odd}} &\sim 2\pi \left[\frac{e+1}{1+\pi^2} \sin \pi x - 2\frac{e-1}{1+4\pi^2} \sin 2\pi x \right. \\
 &\quad \left. + 3\frac{e+1}{1+9\pi^2} \sin 3\pi x - 4\frac{e-1}{1+16\pi^2} \sin 4\pi x + \dots \right]
 \end{aligned}$$

Next, we show the graphs of the extended functions and their truncated series. For $[e^x]_{\text{even}}$ we have



Even extension of $[e^x]_{\text{par}}$ and Fourier cosine series truncations

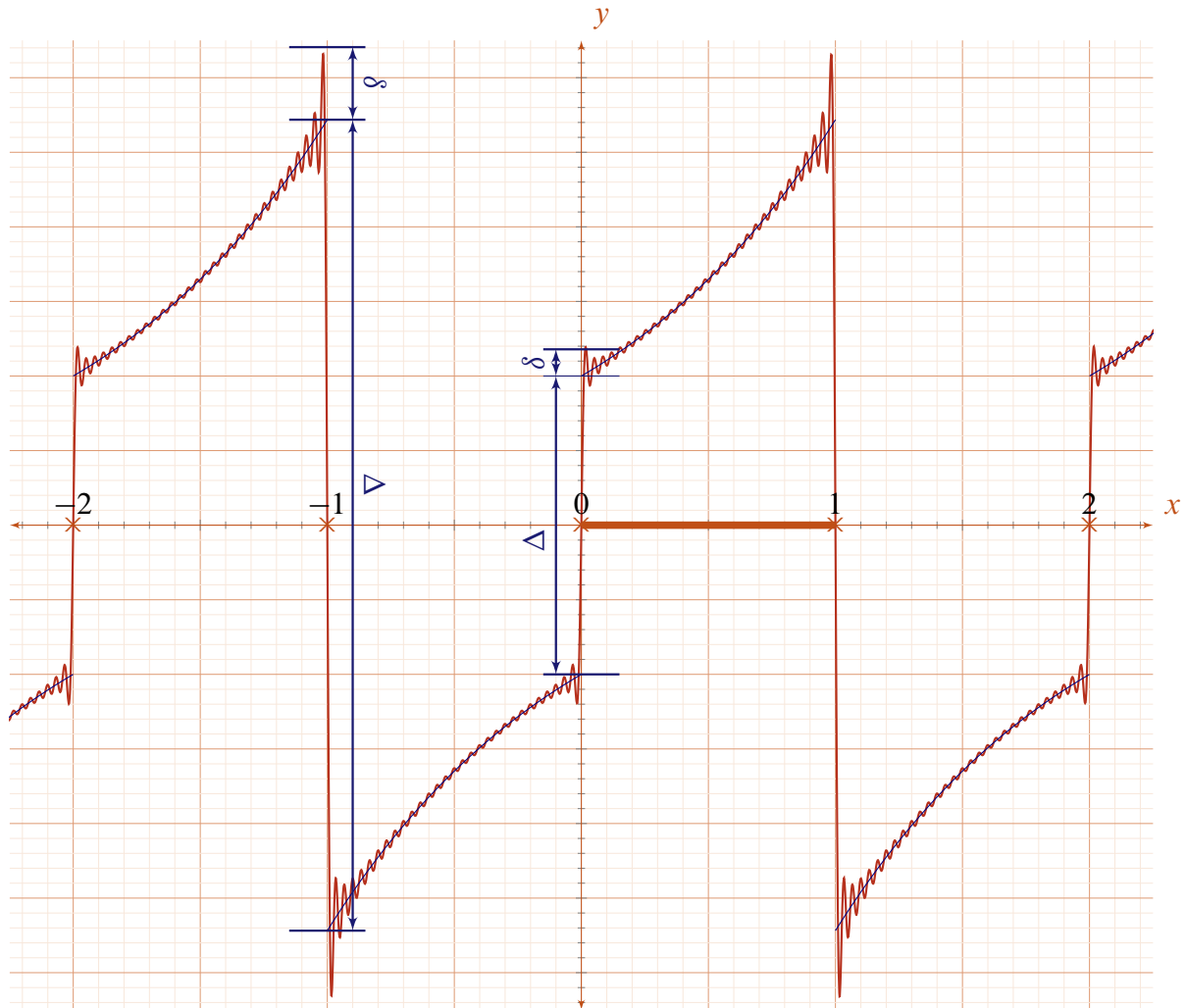
We observe that it is a very good approximation. Furthermore, we see that and by other that in the points $x = 0, \pm 1, \pm 2, \dots$ the tangent to the truncated series has null slope, this is the the derivative is zero (as it should be since the derivative only has sines that vanish at those points) as opposed to the original function where the derivative jumps from -1 to 1 . Whereas for $[e^x]_{\text{odd}}$ we have



**Odd extension $[e^x]_{\text{impar}}$ and truncations of
Fourier sine series**

The oscillating line represents the Fourier series of sines truncated to sixth order. We should note that in the problematic points $0, \pm 1, \pm 2, \dots$ the discrepancy between function and truncated series is remarkable. For example, in $x = 0$ the original function has a discontinuous jump of 2 units while the series at that point is zero, as it should be. These phenomena are due to the fact that the chosen function does not comply with the boundary requirements but, nevertheless, in the interior the Fourier series converges to the given extension.

Below we show the approximation to 31 terms of the Fourier sine series, it begins to approximate quite well the original graph. To compare with the previous expansions that approximated even better with a much smaller number of terms. The **Gibbs phenomenon** is clearly visualized. The partial sum near the discontinuity point, of jump Δ , deviates from the expected jump by approximately 9% of the total jump of the function approaching at that point, $\delta = 0,089489872236\Delta$. The lower the jump, the lower the rebound due to the Gibbs phenomenon.



Gibbs phenomenon: truncation to 61 terms of the Fourier series in sines of $[e^x]_{\text{odd}}$

§4.3. Convergence of Fourier Series



CONVERGENCE of Fourier series of a function $u(x)$ is the subject to which we now turn. As we already know for any function $u \in L^2[a, b]$ their exponential Fourier series, in sines and cosines, in only sines and in only cosines converge strongly. Moreover, if the function is differentiable and satisfies the suitable boundary conditions (periodic, homogeneous Dirichlet

or homogeneous Neumann) the corresponding series converge uniformly (and therefore also pointwise) to the function u . A fundamental question is to know when the series of a general function $u \in L^2[a, b]$ converge pointwise and what is the relationship between the limit function and the function u . A kind of functions for which we can give very precise answers to these questions is as follows:

Piecewise Smooth Functions

A $u = u(x)$ function is **smooth piecewise** (or C^1 piecewise) at an interval $[a, b]$ when there is a partition $a = c_1 < c_2 < \dots < c_M = b$ of $[a, b]$ such that $\forall i = 1, \dots, M-1$, both u and its first derivative u' are continuous in the subintervals (c_i, c_{i+1}) and have finite lateral limits at the edges c_i and c_{i+1} .

For example, the pointwise convergence of the series of sines and cosines of a function $u \in L^2[a, b]$ requires to find out for which points $x \in \mathbb{R}$ there exists the limit

$$\lim_{N \rightarrow \infty} S_N(u, x),$$

where $S_N(u, x)$ denotes the partial sum N -th of the series of sines and cosines

$$S_N(u, x) := \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos n\omega x + b_n \sin n\omega x).$$

and determine its relationship to the value of u in x . One answer of great practical interest is the following theorem

Dirichlet Theorem (1829)

If $u(x)$ is a C^1 **piecewise function** in $[a, b]$ then its **Fourier series converge pointwise** at all points $x \in \mathbb{R}$ and its limit is equal to

$$\frac{u_{\text{ext}}(x + 0^+) + u_{\text{ext}}(x + 0^-)}{2},$$

where u_{ext} denotes the corresponding extension (periodic, even or odd) from u to all the real line $\mathbb{R}$ and

$$u_{\text{ext}}(x + 0^\pm) := \lim_{\varepsilon \rightarrow 0} u_{\text{ext}}(x \pm |\varepsilon|).$$

This result completely solves the problem of the pointwise convergence of the Fourier series discussed above. Basically, we just need to know the extension to $\mathbb{R}$ of the function $u = u(x)$ and apply the following consequences of Dirichlet's theorem

1. If x is a point of continuity of u_{ext} then both limits $u_{\text{ext}}(x + 0^\pm)$ match $u_{\text{ext}}(x)$, so the corresponding Fourier series pointwise converges to u_{ext} .

2. If x is a discontinuity point of u_{ext} the two limits $u_{\text{ext}}(x + 0^\pm)$ are different and constitute two equally attractive *candidates* for the limit of the Fourier series. In this case, the series makes a *Solomonic* choice and decides to converge to the semi-sum of these two limits.



Example: Let the function

$$u(x) = \begin{cases} \frac{\pi}{4}, & 0 \leq x \leq \pi, \\ -\frac{\pi}{4}, & -\pi \leq x < 0. \end{cases}$$

Let's consider its corresponding Fourier expansions in exponentials. Bearing in mind that in this case $b - a = 2\pi$, $\omega = 1$, we have

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-inx} dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{1}{2in}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$u(x) = \frac{1}{2i} \left[e^{ix} + \frac{1}{3} e^{i3x} + \dots \right] + \frac{1}{2i} \left[-e^{-ix} - \frac{1}{3} e^{-i3x} - \dots \right],$$

that by grouping terms takes the form

$$u(x) = \sin x + \frac{1}{3} \sin 3x + \dots = \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}.$$

This is the Fourier sine and cosine series of for v in the interval $[-\pi, \pi]$ (the terms in the cosines have zero coefficient).

The u function is piecewise C^1 in $[-\pi, \pi]$ thus Fourier series must satisfy the properties required by Dirichlet's theorem. We directly check some of such properties. Let's take the points $x = 0, \pm\pi$. At them all the sine terms of the series cancel, then the sum of the series converges pointwise into $x = 0, \pm\pi$ to zero, which is precisely the value that takes the periodic extension

$$\frac{u_{\text{ext}}(x + 0^+) + u_{\text{ext}}(x + 0^-)}{2},$$

at such points.

Note that pointwise convergence for $x = \frac{\pi}{2}$ implies

$$\boxed{\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}.$$

Bounded Variation Functions

Given a function $u(x)$ in the closed interval $[a, b]$ and a partition $\mathcal{P} = \{x_i\}_{i=0}^n$, with $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, the **variation** of u associated with the partition $\mathcal{P}$ is given by

$$\text{var}(u, \mathcal{P}) := \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|.$$

the **total variation** of $u(x)$ is defined as

$$\text{var } u := \sup_{\mathcal{P}} \text{var}(u, \mathcal{P}).$$

Bounded monotonic functions are of bounded variation with

$$\text{var } u = u(b) - u(a),$$

if a function $u(x)$ is derivable in $[a, b]$ with integrable derivative $u'(x)$ in $[a, b]$ its variation is

$$\text{var } u = \int_a^b |u'(x)| \, dx.$$

Jordan's decomposition of a function assures us that a function is of bounded variation if and only if it can be written as the difference of two non-decreasing derivable functions $u(x) = u_1(x) - u_2(x)$. A function $u(x)$ belongs to the bounded variation class $BV[a, b]$ if $\text{var } u < +\infty$.

For example, the function

$$u(x) = \begin{cases} x^2, & x \in [-1, 0], \\ -x^2, & x \in (0, 1] \end{cases}$$

has derivative given by

$$u'(x) = \begin{cases} 2x, & x \in [-1, 0], \\ -2x, & x \in (0, 1] \end{cases}$$

and, consequently, its variation is

$$\text{var } u = \int_{-1}^1 |u'(x)| \, dx = \int_{-1}^0 |-2x| \, dx + \int_0^1 |2x| \, dx = -x^2 \Big|_{-1}^0 + x^2 \Big|_0^1 = 2 = u(b) - u(a).$$

Continuity Module

A function $u(x)$ admits the function $\omega : \mathbb{R} \rightarrow \mathbb{R}_+$ as a **continuity module** in $[a, b]$ if

$$|u(x) - u(y)| \leq \omega(|x - y|).$$

Hölder Continuity

It is said that $u(x)$ is of Hölder class, or **Hölder continuous**, provided that

$$|u(x) - u(y)| \leq C |x - y|^\alpha,$$

where $\alpha, C \geq 0$ are non-negative constants. If $\alpha = 1$ we say that the function is in the **Lipschitz class**, when $\alpha > 0$ the continuity is assured, and when $\alpha = 0$, it is equivalent to the boundness of the function.

Behavior of the Coefficients

Given a function $u : [a, b] \rightarrow \mathbb{C}$ the coefficients of its Fourier series expansion in exponentials satisfy the following bounds:

1. If $u(x)$ is of bounded variation we have

$$|c_n| \leq \frac{\text{var } u}{b - a} \frac{1}{|n|}.$$

2. If $u \in C^p[a, b]$ then

$$|c_n| \leq \frac{\int_a^b |u^{(p)}(x)| \, dx}{|n|^p}.$$

3. If $u \in C^p[a, b]$ and $\omega(x)$ is a continuity module you have

$$|c_n| \leq \omega\left(\frac{b - a}{n}\right) \frac{1}{|n|^p}.$$

4. If $u(x)$ is α -Hölder continuous then there is $K > 0$ such that

$$|c_n| \leq \frac{K}{|n|^\alpha}.$$

Uniform Convergence of Fourier Series

1. The **Fourier series of continuous Hölder functions converge uniformly** and

$$|u(x) - S_N(u, x)| \leq K \frac{\log N}{N^\alpha}.$$

for some $K > 0$.

2. The **Fourier series of continuous functions of bounded variation converge uniformly**.
3. If $u \in C^p[a, b]$ and $u^{(p)}(x)$ has as continuity module $\omega(x)$ then the Fourier series converges uniformly and there is $K > 0$ such that

$$|u(x) - S_N(u, x)| \leq K \frac{\log N}{N^p} \omega\left(\frac{b - a}{N}\right).$$

§4.4. Fourier Transform

FOURIER transform of a function $u = u(x)$ can be described as a expansion of u in eigenfunctions of the operator $Lu = -Du$, when the domain of L is $\mathcal{D} := \{u \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) : Lu \in L^2(\mathbb{R})\}$. Note that the definition of $\mathcal{D}$ is motivated by the fact that when considering functions u defined over the whole real line, it is not assured that $u \in L^2(\mathbb{R})$ when $u \in C^\infty$, so we have to demand that u belongs to both spaces. On the other hand, we must also demand that $Lu \in L^2(\mathbb{R})$ for L to be an application $\mathcal{D} \rightarrow L^2(\mathbb{R})$. The special thing about this domain is that, even though L is symmetric in it, there are no eigenfunctions of L in $\mathcal{D}$. This is clear since

$$Lu = \lambda u,$$

implies

$$u = c e^{i\lambda x}, \quad c \neq 0,$$

but such functions are not in $L^2(\mathbb{R})$ since

$$\int_{-\infty}^{+\infty} |u|^2 dx = |c|^2 \int_{-\infty}^{+\infty} e^{-2\operatorname{Im}(\lambda)x} dx = \infty.$$

However, we are going to show that a subset of these eigenfunctions

$$B := \{e_k(x) := e^{ikx} / \sqrt{2\pi} : k \in \mathbb{R}\},$$

allows to expand any function $u \in L^2(\mathbb{R})$. Now, the expansion will not be a series, but an integral. In fact the basis B of functions in the expansion is not a discrete set, since it has as many elements e_k , $k \in \mathbb{R}$, as the cardinality of the set of real numbers.

To introduce this new expansion we start from the Fourier series of exponentials of a function $u \in L^2[a, b]$

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c_n e^{ik_n x}, \quad c_n = \frac{\sqrt{2\pi}}{b-a} \int_a^b e^{-ik_n x} u(x) dx,$$

where we have changed the expression that we used of the coefficients c_n in a factor 2π and we denote

$$k_n := \omega n, \quad \omega := \frac{2\pi}{b-a}.$$

Our idea is to write this sum in the form of Riemann's sum of an integral with respect to the variable k . In this sense we have

$$(9) \quad u(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c(k_n) e^{ik_n x} \Delta k$$

with

$$(10) \quad \begin{aligned} \Delta k &:= k_{n+1} - k_n = \omega, \\ c(k_n) &:= \frac{c_n}{\Delta k} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ik_n x} u(x) dx. \end{aligned}$$

If we take the limit

$$a \rightarrow -\infty, \quad b \rightarrow +\infty,$$

observe that $\Delta k \rightarrow 0$ and that we have

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c(k) e^{ikx} dk,$$

$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x) dx.$$

This is the expansion of u in the $e_k(x)$ eigenfunctions. The function $c = c(k)$ plays the role of *expansion coefficients*. All the information of the source function $u = u(x)$ is encoded in the new function $c = c(k)$. The usual nomenclature and notation for this expansion are

$$c(k) =: \mathcal{F}(u), \quad \text{Fourier transform of } u.$$

$$u(x) =: \mathcal{F}^{-1}(c), \quad \text{inverse Fourier transform of } c.$$

The Fourier transform thus appears as a continuous limit of the Fourier series expansion concept.

We now consider the n -dimensional extension of the Fourier transform. We will denote

$$x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n, \quad k = (k_0, k_1, \dots, k_{n-1}) \in \mathbb{R}^n,$$

$$x \cdot k = x_0 k_0 + \dots + x_{n-1} k_{n-1}.$$

Fourier Transform

The Fourier transform of a function $u(x)$ is defined by

$$c(k) = \mathcal{F}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ik \cdot x} u(x) d^n x.$$

A basic problem is knowing when the Fourier transform of a function do in fact exists.

Fourier Transform in $L^1(\mathbb{R}^n)$

If the absolute value of $u(x)$ is integrable, this is if

$$\int_{\mathbb{R}^n} |u(x)| d^n x < \infty,$$

then the Fourier transform $\mathcal{F}(u)$ **exists** and it's a continuous function in all $\mathbb{R}^n$.

However, the space of functions with integrable absolute value does not remain invariant under the Fourier transform. That is, **there are functions with an integrable absolute value whose Fourier transform has a non-integrable absolute value**. In this sense the space $L^2(\mathbb{R}^n)$ is more appropriate since it is satisfied

Fourier Transform in $L^2(\mathbb{R}^n)$

If $u(x)$ is square integrable function, $u \in L^2(\mathbb{R}^n)$, this is if

$$\int_{\mathbb{R}^n} |u(x)|^2 d^n x < \infty,$$

then the Fourier transform $\mathcal{F}(u)$ exists and is also an square integrable function. In addition, the Fourier transform defines a bijective linear application

$$\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

that preserves the scalar product

$$(\mathcal{F}u, \mathcal{F}v) = (u, v), \quad \forall u, v \in L^2(\mathbb{R}^n).$$

The inverse of the Fourier transform is what we have defined as the inverse Fourier transformation.

It is necessary to warn that the improper integral that accompanies the Fourier transform operation on elements of $L^2(\mathbb{R}^n)$ must be carried out in a different sense from the usual one. Specifically, it is defined as

$$c(k) = \lim_{R \rightarrow \infty} c_R(k), \quad c_R(k) := \frac{1}{(2\pi)^{n/2}} \int_{|x| \leq R} e^{-ikx} u(x) d^n x,$$

where the limit operation is that associated with convergence in L^2 mean

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^n} |c(k) - c_R(k)|^2 d^n k = 0.$$

Another function space in which the Fourier transform has important properties is the *Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ u \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (x^\alpha D^\beta u) < \infty, \forall \alpha, \beta \in \mathbb{Z}_+^n \right\},$$

of **test functions** of fast decay. We are using the notation

$$x^\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}}.$$

This space is contained in $L^2(\mathbb{R}^n)$.

Fourier Transform in $\mathcal{S}(\mathbb{R}^n)$

If $u = u(x)$ belongs to $\mathcal{S}(\mathbb{R}^n)$, then **the Fourier transform exists** and belongs to $\mathcal{S}(\mathbb{R}^n)$. In addition, the Fourier transform defines a bijective linear application

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n).$$

Let us have a look at some examples of Fourier transforms of single-variable functions.



Examples:

1. Given the characteristic function of the interval $[-a, a]$

$$u(x) = \begin{cases} 1 & x \in [-a, a], \\ 0 & x \in \mathbb{R} \setminus [-a, a], \end{cases}$$

its Fourier transform is easily calculated

$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{-ikx} dx = -\frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{ik} \Big|_{-a}^a dx = \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k}.$$

2. Let us consider the function

$$u(x) = \begin{cases} e^{-ax} & x > 0, \\ 0 & \text{en el resto,} \end{cases}$$

with $a > 0$. The transform is

$$\begin{aligned} c(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ikx} e^{-ax} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a+ik)x} dx = -\frac{1}{\sqrt{2\pi}} \frac{e^{-(a+ik)x}}{(a+ik)} \Big|_{x=0}^{\infty} \\ &= \frac{1}{\sqrt{2\pi}} \frac{a-ik}{a^2+k^2} \end{aligned}$$

3. The Lorentzian function

$$u(x) = \frac{1}{x^2 + a^2}, \quad a > 0$$

has as its Fourier transform

$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x^2 + a^2} dx,$$

that can be calculated using complex variable techniques. Let $f(z) = \frac{e^{-ikz}}{z^2 + a^2}$ and γ be the positive oriented real line. We are going to compute $\int_{\gamma} f(z) dz$ using the Cauchy's Residue Theorem. The singularities of f are two simple poles in $z = \pm ia$, there the residues of $f(z)$ are: $\text{Res}(f, \pm ia) = \pm e^{\pm ka} / (2ia)$. When $k > 0$ the integral is the limit when $R \rightarrow \infty$ of the integral over the lower semi-circle centered on the radius origin R . Whereas if $k < 0$ the upper semi-circle must be chosen instead. Therefore, in each situation contributes only one pole in the formula $\int_{\gamma} f(z) dz = 2\pi i \sum_{p \in \text{poles}} \text{Res}(f, p)$, and

the result is

$$c(k) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{a}.$$

4.4.1. Fourier transform properties Under appropriate conditions (for example, if they belong to the Schwartz space) the Fourier transform has a number of basic properties we describe now. In this first table we collect the most immediate properties

Fourier Transform: Properties I

1. Linearity

$$\begin{aligned} u_i(x) &\xrightarrow{\mathcal{F}} c_i(k), \quad i = 1, 2 \\ &\Downarrow \\ \lambda_1 u_1(x) + \lambda_2 u_2(x) &\xrightarrow{\mathcal{F}} \lambda_1 c_1(k) + \lambda_2 c_2(k), \quad \forall \lambda_1, \lambda_2 \in \mathbb{C} \end{aligned}$$

2. Translations

$$u(x) \xrightarrow{\mathcal{F}} c(k) \implies u(x + a) \xrightarrow{\mathcal{F}} e^{ik \cdot a} c(k)$$

3. Scaling

$$u(x) \xrightarrow{\mathcal{F}} c(k) \implies e^{i\ell \cdot x} u(x) \xrightarrow{\mathcal{F}} c(k - \ell)$$

4. If $A \in M_N(\mathbb{R})$ is an invertible matrix then

$$u(x) \xrightarrow{\mathcal{F}} c(k) \implies u(Ax) \xrightarrow{\mathcal{F}} \frac{1}{\det A} c((A^\top)^{-1}k)$$

5. Complex conjugation

$$u(x) \xrightarrow{\mathcal{F}} c(k) \implies \bar{u}(x) \xrightarrow{\mathcal{F}} \bar{c}(-k)$$

The second set of properties, which require further analysis, is set out below. We will use the notation D_x^α, D_k^α to denote the multiple derivation operators D^α with respect to the variables x or k , respectively. We also introduce the operation of **convolution of two functions** u and v as follows

$$(u * v)(x) := \int_{\mathbb{R}^n} u(x - y)v(y) \, d^n y = \int_{\mathbb{R}^n} u(y)v(x - y) \, d^n y.$$

Fourier transform: Properties II

1. Fourier transform of derivatives:

$$\mathcal{F}(D_x^\alpha(u)) = (ik)^\alpha \mathcal{F}(u).$$

2. Fourier transform and multiplication by polynomials:

$$\mathcal{F}(x^\alpha(u)) = (i D_k)^\alpha \mathcal{F}(u).$$

3. Fourier transform of the convolution:

$$\mathcal{F}(u * v) = (2\pi)^{n/2} \mathcal{F}(u) \mathcal{F}(v).$$

4. Parseval identity:

$$\int_{\mathbb{R}^n} |u(x)|^2 d^n x = \int_{\mathbb{R}^n} |c(k)|^2 d^n k.$$

PROOF. 1. In the first place we have

$$\begin{aligned} \mathcal{F}\left(\frac{\partial u}{\partial x_i}\right) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik \cdot x} \frac{\partial u}{\partial x_i} d^n x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left[e^{-ik \cdot x} u(x) \Big|_{x_i=-\infty}^{\infty} \right. \\ &\quad \left. - \int_{-\infty}^{\infty} (-ik_i) e^{-ik \cdot x} u(x) dx_i \right] dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n, \end{aligned}$$

where we've integrated by parts. If $u \in \mathcal{S}(\mathbb{R}^n)$ is fulfilled that

$$u(x) \Big|_{x_i=-\infty}^{\infty} = 0,$$

and we obtain

$$\mathcal{F}\left(\frac{\partial u}{\partial x_i}\right) = ik_i \mathcal{F}(u).$$

Consequently,

$$\mathcal{F}\left(\frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} \dots \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} u\right) = (ik_0)^{\alpha_0} \dots (ik_{n-1})^{\alpha_{n-1}} \mathcal{F}(u).$$

2. We began by observing that

$$\mathcal{F}(x_i u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik \cdot x} x_i u(x) d^n x = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} i \frac{\partial(e^{-ik \cdot x} u(x))}{\partial k_i} d^n x.$$

If $u \in \mathcal{S}(\mathbb{R}^n)$ we can extract out of the integral the derivative with respect to the parameter k_i and write

$$\mathcal{F}(x_i u) = i \frac{\partial}{\partial k_i} \mathcal{F}(u).$$

Thus,

$$\mathcal{F}(x_0^{\alpha_0} \dots x_{n-1}^{\alpha_{n-1}} u) = \left(i \frac{\partial}{\partial k_0}\right)^{\alpha_0} \dots \left(i \frac{\partial}{\partial k_{n-1}}\right)^{\alpha_{n-1}} \mathcal{F}(u)$$

3. Let us write the transformation of a convolution

$$\mathcal{F}(u * v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik \cdot x} \left[\int_{\mathbb{R}^n} u(y) v(x - y) d^n y \right] d^n x.$$

Since $u, v \in \mathcal{S}(\mathbb{R}^n)$ the multiple integral is independent of the order in which we make the integrals on each variable. So we can write

$$\mathcal{F}(u * v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-ik \cdot x} u(x - y) v(y) d^n y d^n x,$$

that with the change of variables

$$x = \xi + \eta, \quad y = \eta,$$

becomes

$$\mathcal{F}(u * v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-ik \cdot (\xi + \eta)} u(\xi) v(\eta) d^n \xi d^n \eta = (2\pi)^{n/2} \mathcal{F}(u) \mathcal{F}(v),$$

as we wanted to show.

4. The scalar product in $L^2(\mathbb{R}^n)$ of $u, v \in \mathcal{S}(\mathbb{R}^n)$ can be written as

$$(u, v) = \int_{\mathbb{R}^n} \bar{u}(y) v(y) d^n y = (\overline{Pu} * v)(0),$$

where $Pu(x) = u(-x)$. Therefore, using the result of 3) we have

$$(u, v) = (2\pi)^{n/2} [\mathcal{F}^{-1}(\mathcal{F}(\overline{Pu}) \mathcal{F}(v))](0),$$

this is to say

$$(u, v) = \int_{\mathbb{R}^n} e^{ik \cdot x} \bar{c}(k) d(k) d^n k \Big|_{x=0}$$

where c, d are the Fourier transforms of u, v , respectively. Therefore,

$$\int_{\mathbb{R}^n} \bar{u}(x) v(x) d^n x = \int_{\mathbb{R}^n} \bar{c}(k) d(k) d^n k,$$

and in particular, for $u = v$ we get the identity of Parseval.

□

Although in previous proofs we have confined ourselves to fast decaying functions, these properties are valid in more general situations.



Examples:

1. First we are going to calculate the Fourier transform of the 1D Gaussian function

$$u(x) = e^{-a^2 x^2}, \quad a > 0.$$

Its the derivative is

$$D_x u(x) = -2a^2 x u(x).$$

Applying the Fourier transform to both members of this equation, and taking into account the properties i) and ii) that we have proved, we deduce that the transform $c(k)$ of the function $u(x)$ satisfies

$$D_k c(k) = -\frac{1}{2a^2} k c(k).$$

Integrating this equation we get

$$c(k) = c(0) e^{-\frac{k^2}{4a^2}}.$$

On the other hand, from the Fourier transform definition we know that

$$\begin{aligned} c(0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-a^2 x^2} dx \\ &= \frac{1}{\sqrt{2\pi} a} \int_{-\infty}^{+\infty} e^{-x^2} dx = \frac{1}{a\sqrt{2}}. \end{aligned}$$

Therefore

$$c(k) = \frac{1}{a\sqrt{2}} e^{-\frac{k^2}{4a^2}}.$$

In other words, the transformation of a Gaussian is a Gaussian.

2. Consider now the Fourier transform of the Gaussian function in n variables

$$u(x) = e^{-\frac{1}{2} \sum_{i,j=0}^{n-1} A_{ij} x_i x_j}$$

being $A = (A_{ij})$ a symmetric positive definite matrix ($A = A^t$ and $(x, Ax) > 0, \forall x \neq 0$).

Taking partial derivatives with respect to x_i we get

$$\frac{\partial u}{\partial x_i} = -u(x) \left(\sum_{j=0}^{n-1} A_{ij} x_j \right).$$

Therefore, using properties (i) and (ii), we find that

$$i k_i c(k) = - \sum_{j=0}^{n-1} A_{ij} i \frac{\partial c}{\partial k_j},$$

and so

$$\frac{\partial c}{\partial k_j} = - \left[\sum_{i=0}^{n-1} (A^{-1})_{ji} k_i \right] c.$$

This first-order PDE has the following solution

$$c(k) = c(0) e^{-\frac{1}{2} \sum_{i,j=0}^{n-1} (A^{-1})_{ji} k_i k_j}.$$

Let us note that

$$c(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{i,j=0}^{n-1} A_{ij} x_i x_j} d^n x.$$

Being A symmetric and positive definite, it can be factor as $A = O\Lambda O^t$ where O is orthogonal and $\Lambda = \text{diag}(\lambda_0, \dots, \lambda_{n-1})$ is the diagonal matrix of A , with $\lambda_i > 0$, $i = 0, \dots, n-1$. That's why, after the change of coordinates $x \rightarrow \tilde{x} = O^{-1}x$, with unit Jacobian (since O is an orthogonal matrix you have $|\det O| = 1$), $c(0)$ is expressed as

$$c(0) = \frac{1}{(2\pi)^{n/2}} \prod_{i=0}^{n-1} \int_{\mathbb{R}} e^{-\lambda_i \tilde{x}_i^2/2} d\tilde{x}_i = \frac{1}{(2\pi)^{n/2}} \prod_{i=0}^{n-1} \sqrt{\frac{2}{\lambda_i}} \int_{\mathbb{R}} e^{-\hat{x}_i^2} d\hat{x}_i.$$

Hence, as $\det A = \lambda_1 \cdots \lambda_n$ we get

$$c(0) = \frac{1}{\sqrt{\det A}},$$

so that

$$c(k) = \frac{1}{\sqrt{(2\pi)^n \det A}} e^{-\frac{1}{2} \sum_{i,j=0}^{n-1} (A^{-1})_{ji} k_i k_j}.$$

For example, the Fourier transform of

$$u(x_1, x_2) = e^{-(x_1^2 + x_2^2 + x_1 x_2)}$$

is

$$c(k_1, k_2) = \frac{1}{\sqrt{3}} e^{-\frac{1}{3}(k_1^2 + k_2^2 - k_1 k_2)}.$$

Now $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\det A = 3$ and $A^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$ and eigenvalues $\lambda_1 = 1, \lambda_2 = 3$.

3. We now calculate the Fourier transform of

$$u(x_1, x_2, x_3) = \begin{cases} x_1 x_2 x_3 & x_1, x_2, x_3 \in [-a, a], \\ 0 & \text{in the rest.} \end{cases}$$

First of all, let us note that the Fourier transform in $\mathbb{R}^3$ of

$$v(x_1, x_2, x_3) = \begin{cases} 1 & x_1, x_2, x_3 \in [-a, a], \\ 0 & \text{in the rest.} \end{cases}$$

is

$$\prod_{i=1}^3 \frac{1}{(2\pi)^{1/2}} \int_{-a}^a e^{-ik_i x_i} dx_i = \sqrt{\frac{8}{\pi^3}} \frac{\sin ak_1}{k_1} \frac{\sin ak_2}{k_2} \frac{\sin ak_3}{k_3}.$$

and as $u(x_1, x_2, x_3) = x_1 x_2 x_3 v(x_1, x_2, x_3)$ we must have

$$c(k_1, k_2, k_3) = i \frac{\partial}{\partial k_1} i \frac{\partial}{\partial k_2} i \frac{\partial}{\partial k_3} \sqrt{\frac{8}{\pi^3}} \frac{\sin ak_1}{k_1} \frac{\sin ak_2}{k_2} \frac{\sin ak_3}{k_3}.$$

Thus,

$$\begin{aligned} c(k_1, k_2, k_3) &= -i \sqrt{\frac{8}{\pi^3}} \frac{\partial[\sin ak_1/k_1]}{\partial k_1} \frac{\partial[\sin ak_2/k_2]}{\partial k_2} \frac{\partial[\sin ak_3/k_3]}{\partial k_3} \\ &= -i \sqrt{\frac{8}{\pi^3}} \left[\frac{ak_1 \cos ak_1 - \sin ak_1}{k_1^2} \right] \left[\frac{ak_2 \cos ak_2 - \sin ak_2}{k_2^2} \right] \\ &\quad \times \left[\frac{ak_3 \cos ak_3 - \sin ak_3}{k_3^2} \right] \end{aligned}$$

4. To calculate the Fourier transform of

$$u(x) = (x-1)^2 e^{-(x+1)^2}.$$

we expand the factor $(x-1)^2$:

$$u(x) = (x^2 - 2x + 1) e^{-(x+1)^2},$$

and since we know that

$$e^{-(x+1)^2} \xrightarrow{\mathcal{F}} e^{ik} \frac{e^{-k^2/4}}{\sqrt{2}},$$

we obtain

$$\begin{aligned} c(k) &= \left[\left(i \frac{d}{dk} \right)^2 - 2i \frac{d}{dk} + 1 \right] \frac{e^{-k^2/4+ik}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \left[- \left(-\frac{1}{2} + \left(-\frac{k}{2} + i \right)^2 \right) - 2i \left(-\frac{k}{2} + i \right) + 1 \right] e^{-k^2/4+ik} \\ &= \frac{-k^2 + 8ik + 18}{4\sqrt{2}} e^{-k^2/4+ik}. \end{aligned}$$

§4.5. Exercises

4.5.1. Exercises with solutions

1. Determine the value in $x = 0$ of the Fourier series in sines and cosines of

$$u(x) = \frac{\sin x}{|x|}, \quad -1 < x < 1.$$

Solution: The function $u(x)$ is not continuous at the origin, its lateral limits are $\lim_{x \rightarrow \pm} u(x) = \pm 1$ and therefore its semisum is $\frac{u_+ + u_-}{2} = 0$. Recalling Dirichlet's theorem we conclude that the series converges pointwise to 0 in 0.

2. Consider the Fourier series expansion in sines and cosines

$$x^{10} + 2x^8 + 4 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad -\pi < x < \pi.$$

Find out which of the following statements is correct

- a) $b_n = \frac{1}{n^3}$.
 b) $a_n = \begin{cases} \frac{1}{n^3}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$
 c) $a_n = 0$.
 d) $b_n = 0$.
 e) $a_n = b_n$.

Solution: As the function $u(x) := x^{10} + 2x^8 + 4$ is even with respect to the midpoint of the interval, $x = 0$, we have that $b_n = 0$. Therefore, (a), (c) and (e) are false and (d) is true. The answer (b) is false because $u(-\frac{\pi}{4}) \neq u(\frac{3\pi}{4})$.

3. Given the function

$$u(x) = |x| \cot x, \quad -1 < x < 1,$$

determine the value at $x = 0$ of the Fourier series in sines.

Solution: The function

$$u(x) = |x| \frac{\cos x}{\sin x}.$$

is continuous throughout the interval except for $x = 0$ where the lateral limits of the function exist: $u_- = -1$ and $u_+ = 1$. Therefore, Dirichlet's theorem implies the value of the sum of the corresponding Fourier series in $x = 0$ is the average value $\frac{u_- + u_+}{2} = 0$.

4. Determine how many points in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ the following Fourier series

$$\sum_{n \in \mathbb{Z}} c_n e^{i2nx}$$

of the function

$$u(x) = \frac{\sin x}{|x|}$$

is canceled.

Solution: Being a Fourier series of exponentials e^{i2nx} we deduce that $\omega = \frac{2\pi}{b-a} = 2$ and therefore the length of the interval is $b - a = \pi$. So, we can take $[a, b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$. The function belongs to the piecewise C^1 class in this interval with a single discontinuity at $x = 0$. Here we have the following lateral limits

$$\lim_{x \rightarrow 0^\pm} u(x) = \pm 1,$$

whose mean is zero. Dirichlet's theorem implies that the Fourier series vanishes at $x = 0$. On the edges of the interval we have $u(\pm\frac{\pi}{2}) = \pm\frac{2}{\pi}$, and as the Fourier series in exponentials will recover the periodic extension of

5. In the next Fourier series expansion

$$x^2 = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad 0 < x < 2\pi,$$

determine the value of the coefficients b_n .

Solution: The coefficients b_n are determined by the formula

$$b_n = \frac{2}{b-a} \int_a^b u(x) \sin n\omega x \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx.$$

This integral, which is immediate, can be calculated by integrating by parts as follows

$$\begin{aligned} x^2 \sin x &= \left(-x^2 \frac{\cos nx}{n} \right)' + 2x \frac{\cos nx}{n} \\ &= \left(-x^2 \frac{\cos nx}{n} + 2x \frac{\sin nx}{n^2} \right)' - 2 \frac{\sin nx}{n^2} \\ &= \left(-x^2 \frac{\cos nx}{n} + 2x \frac{\sin nx}{n^2} + 2 \frac{\cos nx}{n^3} \right)' \end{aligned}$$

so that

$$b_n = \frac{1}{\pi} \left(-x^2 \frac{\cos nx}{n} + 2x \frac{\sin nx}{n^2} + 2 \frac{\cos nx}{n^3} \right) \Big|_{x=0}^{x=2\pi},$$

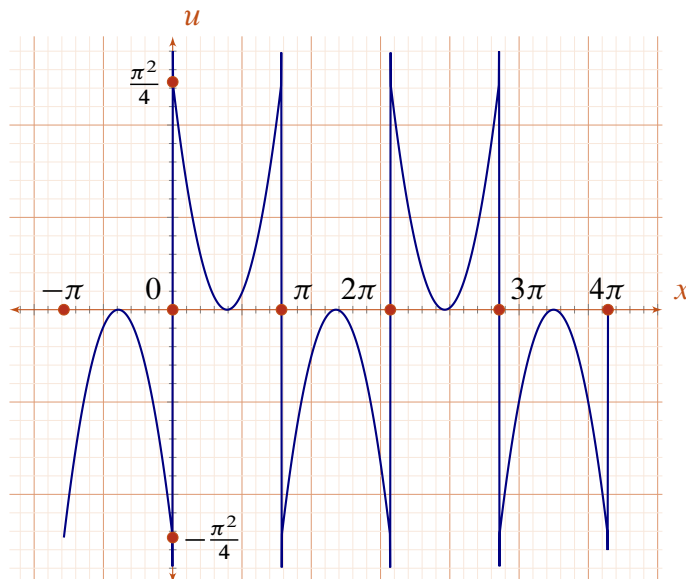
this is,

$$b_n = -\frac{4\pi}{n}.$$

6. Determine how many points in the interval $[0, 3\pi]$ the Fourier series $\sum_{n=1}^{\infty} b_n \sin nx$ of $u(x) = (x - \frac{\pi}{2})^2$ is zero.

Solution: As we are dealing with sine Fourier series with the functions $\sin nx$, $n = 1, 2, \dots$, we see $\omega = 2$ and $a = 0$ (remember that in the series the sines appear in the form $\sin n\omega \frac{x-a}{2}$, and therefore the interval where the function $u(x) = (x - \frac{\pi}{2})^2$ is defined is $[0, \pi]$).

According to Dirichlet's theorem, the Fourier series of sines, will converge pointwise to the periodic extension to the real line of the odd extension of u to the interval $[-\pi, \pi]$. In the attached graph of this odd periodic extension we can see that the function is discontinuous at the points $\{0, \pi, 2\pi, 3\pi\}$ and, according to Dirichlet's theorem, the Fourier series will converge to the semisum of the lateral limits which in this case is 0. On the other hand, we have the following roots of the series $\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\}$. In short, in the interval $[0, 3\pi]$ the Fourier series is cancelled 7 times.



Odd extension of $u(x) = (x - \frac{\pi}{2})^2$

7. Let $u(x)$ be a solution of the Poisson equation

$$\Delta u = \rho, \quad x \in \mathbb{R}^3,$$

with

$$\hat{\rho}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \rho(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x};$$

the Fourier transform of $\rho(x)$. Show that the Fourier transform $\hat{u}$ of u fulfills $|\mathbf{k}|^2 \hat{u}(\mathbf{k}) + \hat{\rho}(\mathbf{k}) = 0$

Solution: Performing the Fourier transform of the Poisson equation we have

$$\mathcal{F}(\Delta u) = \hat{\rho},$$

recalling that $i k_i \mathcal{F}(u) = \mathcal{F}(D_{x_i} u)$ we get

$$\mathcal{F}(\Delta u) = ((i k_1)^2 + (i k_2)^2 + (i k_3)^2) \hat{u} = -|\mathbf{k}|^2 \hat{u}.$$

8. Calculate the Fourier transform of the function

$$u(x) = \begin{cases} x e^x, & -\infty < x < 1, \\ 0, & 1 \leq x < \infty. \end{cases}$$

Solution: Calling

$$v(x) := \begin{cases} e^x, & -\infty < x < 1, \\ 0, & 1 \leq x < \infty \end{cases}$$

We have the Fourier transform $\mathcal{F}(u) = \mathcal{F}(xv) = i D_k \mathcal{F}(v)$. But,

$$\mathcal{F}(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^1 e^{(1-ik)x} dx = \frac{e^{1-ik}}{\sqrt{2\pi}(1-ik)}.$$

Therefore, we conclude that

$$c(k) = -\frac{1}{\sqrt{2\pi}} \frac{ik e^{(1-ik)}}{(1-ik)^2}.$$

9. Determine the 2D Fourier transform

$$c(k_1, k_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x, y) e^{-i(k_1 x + k_2 y)} dx dy,$$

of the function

$$u(x, y) = (x + y) e^{-x^2 - |y|}.$$

Hint: $\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx = \sqrt{\pi} e^{-k^2/4}.$

Solution: In the first place we must bear in mind that

$$\mathcal{F}(u) = i(D_{k_1} + D_{k_2}) \mathcal{F}(e^{-x^2 - |y|}).$$

Secondly that

$$\begin{aligned} \mathcal{F}(e^{-x^2 - |y|}) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-x^2 - |y|} e^{-i(k_1 x + k_2 y)} dx dy = \left[\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-x^2} e^{-ik_1 x} dx \right] \left[\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-|y|} e^{-ik_2 y} dy \right] \\ &= \frac{1}{2\sqrt{\pi}} e^{-k_1^2/4} \int_{\mathbb{R}} e^{-|y|} e^{-ik_2 y} dy. \end{aligned}$$

Finally,

$$\int_{\mathbb{R}} e^{-|y|} e^{-ik_2 y} dy = \int_{-\infty}^0 e^{(1-ik_2)y} dy + \int_0^{\infty} e^{-(1+ik_2)y} dy = \frac{1}{1-ik_2} + \frac{1}{1+ik_2} = \frac{2}{1+k_2^2}.$$

Thus,

$$\mathcal{F}(e^{-x^2 - |y|}) = \frac{1}{\sqrt{\pi}} e^{-k_1^2/4} \frac{1}{1+k_2^2}$$

and

$$\mathcal{F}(u) = -\frac{i}{\sqrt{\pi}} \frac{e^{-k_1^2/4}}{1+k_2^2} \left(\frac{k_1}{2} + \frac{2k_2}{1+k_2^2} \right).$$

10. Determine the Fourier transform $c(k_1, k_2, k_3)$ of the following function

$$u(x, y, z) := \begin{cases} \frac{e^{-x-y}}{z^2+4}, & \text{si } x > 0 \text{ and } y > 0, \\ 0, & \text{in the rest of space} \end{cases}$$

Hint: Whenever $a > 0$ is satisfied that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2 + a^2} dx = \frac{e^{-a|k|}}{2a}.$$

Solution: As the function factors $u = u_1(x)u_2(y)u_3(z)$, we have

$$\mathcal{F}^{(3)}(u) = \mathcal{F}^{(1)}(u_1)\mathcal{F}^{(1)}(u_2)\mathcal{F}^{(1)}(u_3).$$

But, we know that

$$\begin{aligned}\mathcal{F}^{(1)}(u_1) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ik_1x} e^{-x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{1+ik_1}, \\ \mathcal{F}^{(1)}(u_2) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ik_2y} e^{-y} dy = \frac{1}{\sqrt{2\pi}} \frac{1}{1+ik_2}, \\ \mathcal{F}^{(1)}(u_3) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ik_3z} \frac{1}{z^2+4} dz = \sqrt{\frac{\pi}{2}} \frac{e^{-2|k_3|}}{4},\end{aligned}$$

and, therefore,

$$c(k_1, k_2, k_3) = \frac{1}{2\sqrt{2\pi}} \frac{e^{-2|k_3|}}{(1+ik_1)(1+ik_2)}.$$

11. Find the Fourier transform $c(k_1, k_2)$ of the following function

$$u(x, y) := \begin{cases} xy e^{-x-y}, & \text{if } x > 0 \text{ and } y > 0, \\ 0, & \text{in the remaining plane.} \end{cases}$$

Solution: Let us denote

$$G(x, y) := \begin{cases} e^{-x-y}, & \text{if } x > 0 \text{ and } y > 0, \\ 0, & \text{in the remaining plane.} \end{cases}$$

We know that

$$\mathcal{F}^{(2)}(G) = \frac{1}{2\pi} \frac{1}{(1+ik_1)(1+ik_2)},$$

as well as

$$\mathcal{F}^{(2)}(xyG) = iD_{k_1} iD_{k_2} \mathcal{F}^{(2)}(G).$$

Thus, as $u(x, y) = xyG(x, y)$, we deduce

$$\mathcal{F}^{(2)}(u) = \frac{1}{2\pi} \frac{1}{(1+ik_1)^2(1+ik_2)^2}.$$

12. Find the Fourier transform $c(k)$ of the following function

$$u(x) = x e^{-\frac{|x+2|}{4}}.$$

Solution: First, we calculate

$$\begin{aligned}\mathcal{F}(e^{-\frac{|x|}{4}}) &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 e^{-ikx + \frac{x}{4}} dx + \int_0^{\infty} e^{-ikx - \frac{x}{4}} dx \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{\frac{1}{4} - ik} + \frac{1}{\frac{1}{4} + ik} \right) \\ &= \frac{1}{\sqrt{\pi}} \frac{8}{1 + 16k^2}.\end{aligned}$$

Secondly, we remind that $\mathcal{F}(u(x+a)) = e^{ika} \mathcal{F}(u(x))$ and, consequently,

$$\mathcal{F}(e^{-\frac{|x+2|}{4}}) = \frac{8}{\sqrt{\pi}} \frac{1}{1 + 16k^2} e^{i2k}.$$

We also observe that

$$\mathcal{F}(x e^{-\frac{|x+2|}{4}}) = i D_k \mathcal{F}(e^{-\frac{|x+2|}{4}}) = i D_k \frac{4}{\pi} \frac{1}{1 + 16k^2} e^{i2k}.$$

Therefore, we arrive at the final result

$$c(k) = -\frac{16}{\sqrt{\pi}} \frac{16k^2 + 16ik + 1}{(16k^2 + 1)^2} e^{i2k}.$$

13. Consider the function defined by

$$u(x) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2}}{1 + y^2} dy, \quad -\infty < x < \infty.$$

a) Determine functions $f(x)$ and $g(x)$ such that $u(x)$ is expressed as the convolution

$$u(x) = (f * g)(x).$$

b) Calculate the Fourier transform of $u(x)$.

Solution:

a) If we pick the functions $f(x)$ and $g(x)$ with the following Gaussian and Lorentzian functions

$$f(x) = e^{-x^2}, \quad g(x) = \frac{1}{1 + x^2},$$

we have that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy = u(x).$$

b) Therefore, bearing in mind that

$$\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g)$$

and recalling that

$$\mathcal{F}(f) = \frac{e^{-\frac{k^2}{4}}}{2\sqrt{\pi}}, \quad \mathcal{F}(g) = \frac{e^{-|k|}}{2},$$

we obtain

$$\hat{u}(k) = \frac{1}{2} e^{-\frac{k^2}{4} - |k|}.$$

4.5.2. Exercises

1. Expand the function $u(x) = 1$ in sine Fourier series in the interval $[0, l]$. Analyze whether you can derive the expansion term by term. Also determine the expansion in cosine Fourier series in of this function.
2. Let be the function $u(x) = x$.
 - a) Find its expansion in Fourier series in sines over the interval $[0, l]$.
 - b) Find its Fourier series expansion in cosines over the interval $[0, l]$.
 - c) Find the Fourier series expansion in sines and cosines over the interval $[-l, l]$.

3. Determine the Fourier series of sines and cosines of the function $u(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0, \\ x & \text{if } 0 \leq x \leq 1. \end{cases}$

What value does the series take in $x = 1$?

4. Determine the Fourier series of sines and cosines of the function $u(x) = e^{ix}$, over the interval $[0, \pi]$.
5. Determine the Fourier series of sines and cosines of the function

$$u(x) := \begin{cases} 1, & \text{if } -2 \leq x \leq 0, \\ x, & \text{if } 0 \leq x \leq 2 \end{cases},$$

and the corresponding sum of the series at $x = 0$.

6. Determine Fourier transforms of

- a) $u(x) = \frac{1}{x + \alpha}$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$,
- b) $u(x) = \begin{cases} x & \text{if } -a < x < a, \\ 0 & \text{in the rest.} \end{cases}$
- c) $u(x) = \frac{\sin ax}{x}$, $a \in \mathbb{R}$.
- d) $u(x) = (x^2 + 1) e^{-a^2 x^2}$, $a > 0$.
- e) $u(x) = \frac{x}{x^2 + a^2}$, $a \in \mathbb{R}$.

7. Determine the Fourier transform of

$$u(x, y) = \begin{cases} 1, & \text{if } -a < x < a \text{ and } -b < y < b, \\ 0, & \text{in the rest.} \end{cases}$$

8. Consider the Fourier Transform

$$c(\mathbf{k}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\mathbf{k} \cdot \mathbf{x}} u(\mathbf{x}) d^2 \mathbf{x}.$$

of a function $u = u(\mathbf{x})$ defined on the plane $\mathbf{x} \in \mathbb{R}^2$.

- a) Write the integral that defines $c(\mathbf{k})$ in terms of polar coordinates.

b) Determine the Fourier transform of the function

$$u(x, y) = \begin{cases} y, & \text{if } \mathbf{x} \text{ is in the first quadrant and } |\mathbf{x}| < 1, \\ 0, & \text{in the remaining of } \mathbb{R}^2. \end{cases}$$

Hint: Take the x axis in the direction and orientation of the vector $\mathbf{k}$

g. Determine the Fourier transform

$$c(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}} u(\mathbf{x}) d^3 \mathbf{x}.$$

of the function

$$u(\mathbf{x}) = \frac{e^{-a|\mathbf{x}|}}{|\mathbf{x}|} \quad a > 0.$$

Hint: Use spherical coordinates and take the z axis in the direction and orientation of the vector $\mathbf{k}$.

5. EIGENFUNCTION EXPANSION METHOD

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EIGENFUNCTION expansions are used to extend variable separation techniques to situations that go beyond homogeneous problems. Hence, we will treat the separation of variables as a problem of eigenvalues and eigenfunctions. The associated differential operator must be symmetric, and therefore the set of its eigenfunctions must be an orthogonal set, which leads to the consideration of expansions with respect to this set of eigenfunctions. This will allow us to apply the SVM in inhomogeneous problems using expansions in the associated eigenfunctions. The idea is to present a variety of examples that cover a wide spectrum of situations but do not require the use of coordinates other than Cartesian coordinates, the latter problems that we will deal with these techniques in the last chapter uses special functions that we will introduce in the next chapter.

§5.1. Eigenfunction Expansion Method (EEM)



EIGENFUNCTION expansion method is a classical tool in solving boundary and/or initial value problems of linear type. It is applied, as we have explained, in situations where the equations are inhomogeneous and is based on the use of expansions in **complete sets of eigenfunctions** of one of the operators appearing in the corresponding PDE.

5.1.1. EEM and inhomogeneous problems EEM can be applied to inhomogeneous problems of the type

$$\begin{cases} Au + Bu = f, & x \in \Omega, \\ a_i(u) = g_i, & i = 1, \dots, r \\ b_j(u) = 0, & j = 1, \dots, s. \end{cases}$$

where $x = (x_0, \mathbf{x})$ with $\mathbf{x} = (x_1, \dots, x_{n-1})$ belong to a domain $\Omega = I \times \Omega_0$

$$x_0 \in I, \quad \mathbf{x} \in \Omega_0,$$

being I an open interval of $\mathbb{R}$ and Ω_0 an open set of $\mathbb{R}^{n-1}$. The PDE and the boundary and/or initial conditions must satisfy:

EEM: Prepared Problems

1. The PDE has the form

$$(11) \quad A\left(x_0; \frac{\partial}{\partial x_0}\right)u + B\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)u = f(x).$$

2. The system of boundary and/or initial conditions consists of two subsystems. One of them is formed by conditions of inhomogeneous type

$$(12) \quad a_i(u) = g_i(x), \quad i = 1, \dots, r,$$

which contains **operators** a_i **that act only on the variable** x_0 , while the other is made up of homogeneous conditions

$$b_j(u) = 0, \quad j = 1, \dots, s,$$

such that **operators** b_j **act only on the variables** $(x_1, \dots, x_{n-1})$.

3. The inhomogeneous terms f and g_i of the problem admit expansions of the form

$$\begin{cases} f(x) = \sum_m f_m(x_0)w_m(x), \\ g_i(x) = \sum_m c_{im}w_m(x), \quad i = 1, \dots, r, \end{cases}$$

in a set of eigenfunctions of the operator B

$$Bw_m = \lambda_m w_m,$$

that satisfy homogeneous boundary conditions

$$b_j(w_m) = 0, \quad j = 1, \dots, s.$$

The third of the above conditions is the most demanding. The typical case in which it is fulfilled is when the **operator** B **symmetric on the domain**

$$(13) \quad \mathcal{D} = \{w \in C^\infty(\overline{\Omega_0}) : b_j(w) = 0, \quad j = 1, \dots, s\},$$

in the space $L^2_\rho(\overline{\Omega_0})$. This is the reason for the importance of symmetric operators in linear PDE theory. Assuming that all three conditions are met, the EEM is implemented through the following process:

EEM I

A solution of the problem is sought in the form of a series expansion in eigenfunctions

$$u(x) = \sum_m v_m(x_0)w_m(x).$$

The unknowns are then the coefficients $v_m(x_0)$ of the series.

EEM II

Since the A operator only acts on the x_0 variable and is a linear operator, it is to be expected under favorable regularity conditions that

$$Au(x) = A\left(\sum_m v_m(x_0)w_m(x)\right) = \sum_m (Av_m)(x_0)w_m(x).$$

In the same manner, the linear operator B only acts on the variables $x_1, \dots, x_{n-1}$ and it is to be expected under favorable regularity conditions that

$$Bu(x) = B\left(\sum_m v_m(x_0)w_m(x)\right) = \sum_m v_m(x_0)(Bw_m)(x) = \sum_m v_m(x_0)\lambda_m w_m(x),$$

where we have used that $Bw_m = \lambda_m w_m$. If we also use the expansion

$$f(x) = \sum_m f_m(x_0)w_m(x)$$

the PDE is reduced to

$$\sum_m (Av_m)(x_0)w_m(x) + \sum_m \lambda_m v_m(x_0)w_m(x) = \sum_m f_m(x_0)w_m(x).$$

Identifying now the coefficients of the functions $w_m(x)$ you get

$$(14) \quad \boxed{Av_m + \lambda_m v_m = f_m.}$$

Each coefficient v_m will then be a solution to this ODE.

EEM III

To sought for a solution $u(x)$ fulfilling the conditions

$$\begin{cases} a_i(u) = g_i, & i = 1, \dots, r, \\ b_j(u) = 0, & j = 1, \dots, s, \end{cases}$$

we proceed in a similar way. Firstly, as the boundary operators a_i only act on the variable x_0

$$a_i(u) = a_i\left(\sum_m v_m(x_0)w_m(x)\right) = \sum_m a_i(v_m)w_m(x),$$

so using the expansions of the functions g_i

$$g_i(x) = \sum_m c_{im}w_m(x),$$

conditions $a_i(u) = g_i$ are expressed as

$$\sum_m a_i(v_m)w_m(x) = \sum_m c_{im}w_m(x).$$

Identifying the coefficients of the functions $w_{bm}(x)$ we obtain that each coefficient v_m , besides being a solution of the ODE(14) must satisfy the conditions.

$$(15) \quad \boxed{a_i(v_m) = c_{im} \quad i = 1, \dots, r.}$$



Observations:

1. It should be noted that the conditions $b_j(u) = 0$ are fulfilled automatically since

$$b_j(u) = b_j\left(\sum_m v_m(x_0)w_m(x)\right) = \sum_m v_m(x_0)b_j(w_m(x)) = 0,$$

where we have used the fact that, by the definition of eigenfunctions $w_{bm}(x)$, we have $b_j(w_m) = 0$ for $j = 1, \dots, s$.

2. The method that we have just seen admits a natural generalization to the case in which the expansions in eigenfunctions of the data f and g_i are generalized linear combinations of eigenfunctions of the operator B , using integration operations such as Fourier transform. In such situations a u solution is sought in the form of a expansion of the same type, replaced in (11) and (12) and the coefficients in the expansion obtained are identified to obtain equations analogous to (14) and (15).
3. The hypotheses about the existence of expansions of the functions f and g_i in eigenfunctions of B are always fulfilled when the set eigenfunctions of B is complete, which is true for broad classes of symmetric operators seen in the previous chapter.



Example: Let us consider the following problem with the Laplace's equation on a rectangle

$$(16) \quad \begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & 0 < x < a, \ 0 < y < b, \\ \begin{cases} u(0, y) = 0, \\ u(a, y) = 0, \\ u(x, 0) = \sin \frac{\pi x}{a}, \\ u(x, b) = 0. \end{cases} \end{cases}$$

This problem describes the stationary distribution of temperature on a plate $[0, a] \times [0, b]$, with all sides at zero temperature except one that has a sine type temperature distribution. The spectral problem

$$\begin{cases} -w_{xx} = \lambda^2 w, \\ \begin{cases} w|_{x=0} = 0, \\ w|_{x=a} = 0, \end{cases} \end{cases}$$

has as solution

$$\lambda_n = \frac{n^2 \pi^2}{a^2}, \quad w_n(x) = \sin \frac{n\pi x}{a}, \quad n = 1, 2, \dots$$

Applying the EEM we expand the solution as the sine Fourier series

$$u(x, y) = \sum_{n=1}^{\infty} v_n(y) \sin \frac{n\pi x}{a}.$$

For inhomogeneities we only have the term $\sin \frac{\pi x}{a}$ and, consequently, we seek for our solution in the form

$$u(x, y) = v(y) \sin \frac{\pi x}{a}.$$

The function v is determined by

$$\begin{cases} v'' = \frac{\pi^2}{a^2}v, \\ \begin{cases} v|_{y=0} = 1, \\ v|_{y=b} = 0. \end{cases} \end{cases}$$

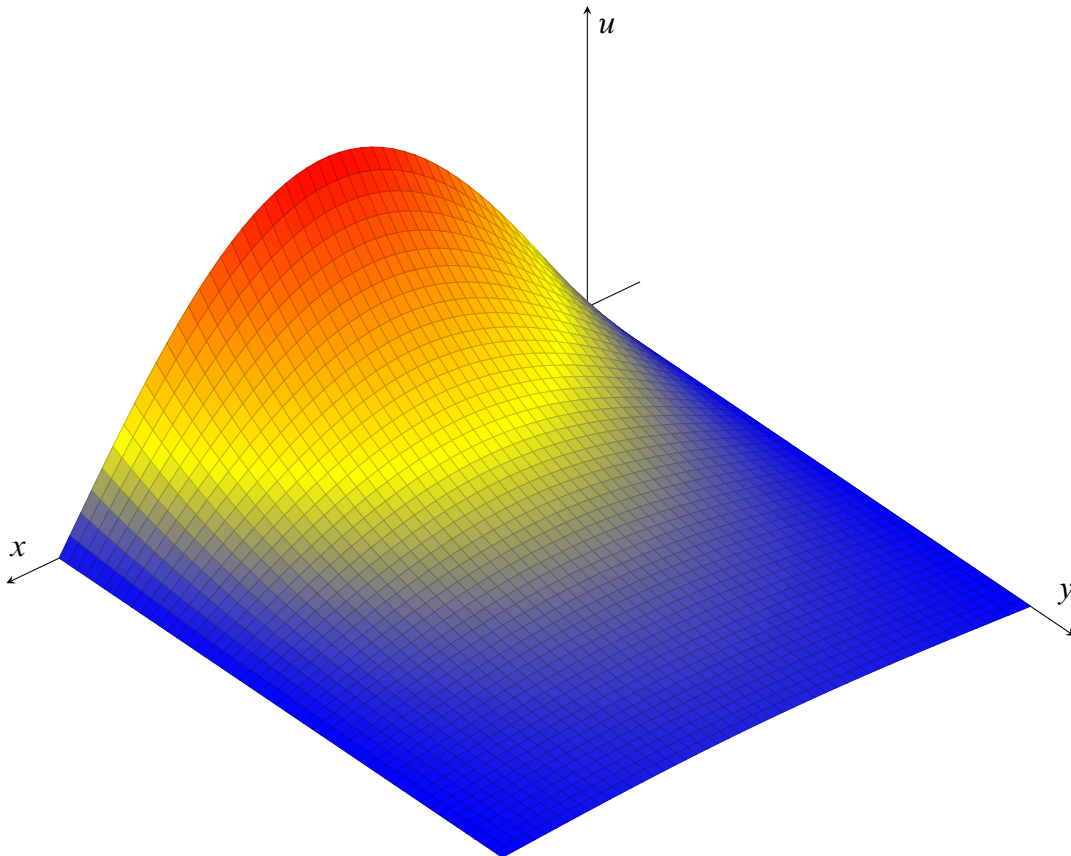
Hence, $v(y) = A e^{\frac{\pi}{a}y} + B e^{-\frac{\pi}{a}y}$ where

$$\begin{cases} A + B = 1, \\ A e^{\frac{\pi}{a}b} + B e^{-\frac{\pi}{a}b} = 0. \end{cases}$$

The solution to this system is $A = -\frac{e^{-\pi \frac{b}{a}}}{2 \sinh \pi \frac{b}{a}}$, $B = \frac{e^{\pi \frac{b}{a}}}{2 \sinh \pi \frac{b}{a}}$ so that the solution reads

$$u(x, y) = \frac{\sinh \frac{\pi}{a}(b - y)}{\sinh \frac{\pi b}{a}} \sin \frac{\pi}{a}x.$$

The graph for this temperature distribution on a plate with $a = \pi$, $b = 2\pi$ is



Temperature $u(x, y)$ in the plate $[0, \pi] \times [0, 2\pi]$

§5.2. EEM in Action



ONE of the most frequent situations in which EEM is used is in problems where evolution equations appear together with a number of initial conditions, as many as the order of the derivative in t , and a few boundary conditions.

1. The PDE is an evolution equation of the form

$$a \frac{\partial^r u}{\partial t^r} + B\left(x_1, \dots, x_{n-1}; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right)u = f(x).$$

con $a \in \mathbb{C}$.

2. The solution must meet r initial conditions

$$\frac{\partial^i u}{\partial t^i}(t_0, \mathbf{x}) = g_i(\mathbf{x}), \quad i = 0, \dots, r-1.$$

as well as a set of s boundary conditions

$$b_j(u) = 0, \quad j = 1, \dots, s,$$

with operators b_j that act only on variables $(x_1, \dots, x_{n-1})$.

In this case the operator A is

$$Au = a \frac{\partial^r u}{\partial t^r},$$

and the conditions $a_i(u) = g_i$ are

$$\frac{\partial^i u}{\partial t^i}(t_0, \mathbf{x}) = g_i(\mathbf{x}), \quad i = 1, \dots, r.$$



Examples:

1. We now consider a problem of initial and boundary conditions for the Schrödinger equation

$$\left\{ \begin{array}{l} i\hbar u_t = -\frac{\hbar^2}{2m} u_{xx} + V(x)u, \quad x \in (a, b), \quad t > 0, \\ \left\{ \begin{array}{l} u|_{t=0} = g(x), \\ \left\{ \begin{array}{l} u|_{x=a} = 0, \\ u|_{x=b} = 0, \end{array} \right. \end{array} \right. \end{array} \right.$$

which describes the evolution of a quantum particle enclosed in the segment (a, b) subject to the potential $V(x)$. Introducing the differential operator, known as the quantum Hamiltonian,

$$Hu := -\frac{\hbar^2}{2m} u_{xx} + V(x)u$$

the Schrödinger equation is

$$-i\hbar u_t + Hu = 0.$$

The EEM is based in this case on the problem of eigenvalues

$$\begin{aligned} Hw &= \lambda w, \\ \begin{cases} w|_{x=a} = 0, \\ w|_{x=b} = 0. \end{cases} \end{aligned}$$

Assuming that $\{\lambda_n\}_{n=1,2,\dots}$ is the set of eigenvalues with associated eigenfunctions $\{w_n(x)\}_{n=1,2,\dots}$ we seek for solutions

$$u(x) = \sum_{n=1}^{\infty} v_n(t) w_n(x)$$

that when introduced in the Schrödinger equation

$$\sum_{n=1}^{\infty} \left(-i\hbar \frac{d}{dt} v_n(t) + \lambda_n v_n(t) \right) w_n(x) = 0,$$

leads to

$$-i\hbar \frac{d}{dt} v_n(t) + \lambda_n v_n(t) = 0;$$

whose solution is

$$v_n(t) = a_n e^{-i \frac{\lambda_n}{\hbar} t}.$$

If we now assume that the initial condition has the expansion

$$g(x) = \sum_{n=1}^{\infty} c_n w_n(x),$$

we conclude that $a_n = c_n$ and therefore the solution to our problem is

$$u(t, x) = \sum_{n=1}^{\infty} c_n e^{-i \frac{\lambda_n}{\hbar} t} w_n(x).$$

We are going to specify this scheme with the following two examples:

- Let us study now

$$\begin{cases} i\hbar u_t = -\frac{\hbar^2}{2m} u_{xx}, & x \in (0, l), t > 0, \\ \begin{cases} u|_{t=0} = 2 \sin \frac{\pi}{l} x - \sin 3 \frac{\pi}{l} x, \\ \begin{cases} u|_{x=0} = 0, \\ u|_{x=l} = 0. \end{cases} \end{cases} \end{cases}$$

That is, a free quantum particle confined in the $(0, l)$ segment. The eigenvalue problem is

$$-\frac{\hbar^2}{2m} w_{xx} = \lambda w,$$

$$\begin{cases} w|_{x=0} = 0, \\ w|_{x=l} = 0, \end{cases}$$

with eigenvalues

$$\lambda_n = \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, \dots$$

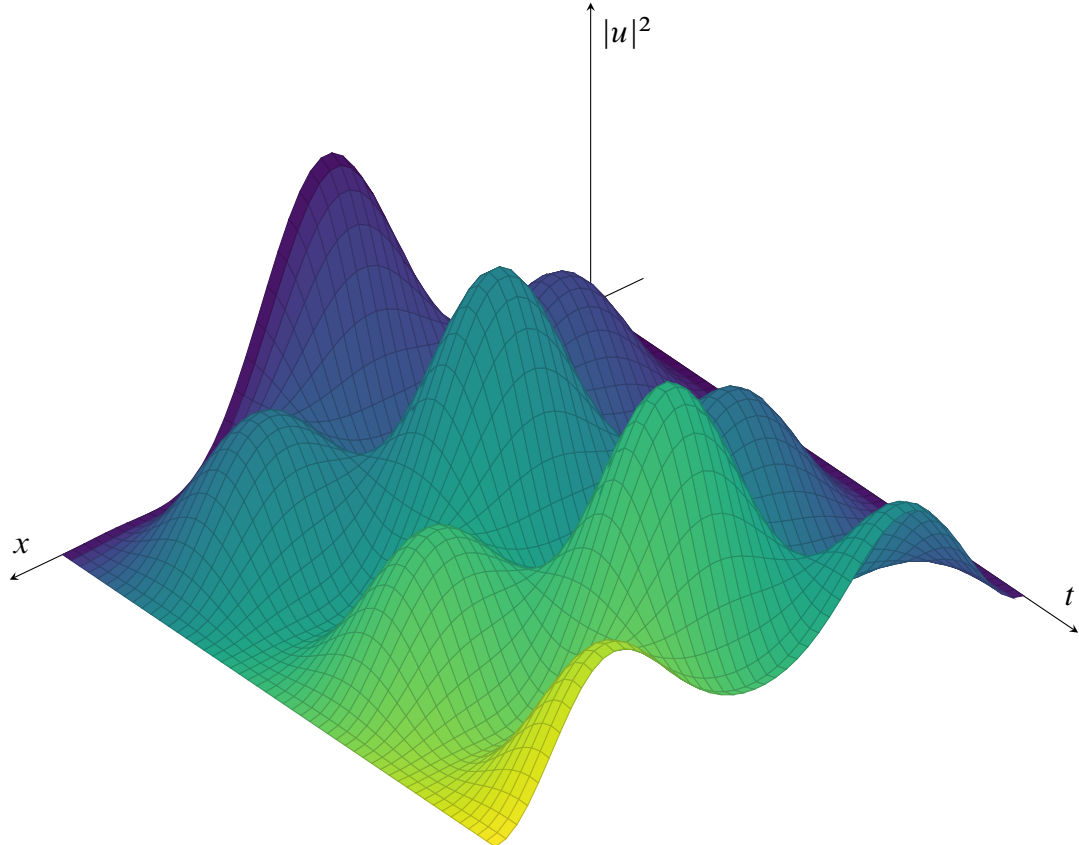
and corresponding eigenfunctions given by

$$w_n(x) = \sin \frac{n\pi}{l} x, \quad n = 1, 2, \dots$$

Therefore, the initial condition is expanded in eigenfunctions and thus we obtain for the state of the particle

$$u(t, x) = 2 \sin \frac{\pi}{l} x e^{-i \frac{\hbar}{2m} \frac{\pi^2}{l^2} t} - \sin 3 \frac{\pi}{l} x e^{-i \frac{\hbar}{2m} \frac{9\pi^2}{l^2} t}$$

Below we show the space-time graph for probability density



Probability density $|u|^2$ for $l = \pi$

- We continue with the free case although we change the Dirichlet boundary conditions by periodic conditions. So the initial and boundary value problem to solve is

$$\begin{cases} i\hbar u_t = -\frac{\hbar^2}{2m}u_{xx}, & x \in (0, l), t > 0, \\ u|_{t=0} = 2i - 2\cos\frac{2\pi}{l}x + 7\sin\frac{6\pi}{l}x, \\ \begin{cases} u|_{x=0} = u|_{x=l}, \\ u_x|_{x=0} = u_x|_{x=l}. \end{cases} \end{cases}$$

The problem of eigenvalues

$$\begin{cases} -\frac{\hbar^2}{2m}w_{xx} = \lambda w, \\ \begin{cases} w|_{x=0} = w|_{x=l}, \\ w_x|_{x=0} = w_x|_{x=l}. \end{cases} \end{cases}$$

with eigenvalues (all double but for zero which is simple)

$$\lambda_n = \frac{\hbar^2}{2m} \frac{4n^2\pi^2}{l^2}, \quad n = 0, 1, 2, \dots$$

and corresponding eigenfunctions given by

$$\left\{ w_0(x) = 1, w_n^{(+)}(x) = \cos\frac{2\pi}{l}nx, w_n^{(-)}(x) = \sin\frac{2\pi}{l}nx \right\}_{n=1,2,\dots}.$$

In the initial condition the eigenfunctions *only* w_0 , $w_1^{(+)}$ and $w_3^{(-)}$ appear. Consequently,

$$u(t, x) = 2i - 2\cos\frac{2\pi}{l}x e^{-i\frac{\hbar}{2m}\frac{4\pi^2}{l^2}t} + 7\sin\frac{6\pi}{l}x e^{-i\frac{\hbar}{2m}\frac{36\pi^2}{l^2}t}.$$

2. We now analyze a problem of initial conditions for the heat equation with periodic boundary conditions. The problem is

$$\begin{cases} u_t = u_{xx}, & x \in (0, l), t > 0, \\ u|_{t=0} = \frac{2}{l^2}x^3 - \frac{3}{l}x^2 + x, \\ \begin{cases} u|_{x=0} = u|_{x=l}, \\ u_x|_{x=0} = u_x|_{x=l}. \end{cases} \end{cases}$$

We stop for a moment to discuss the consistency between the initial condition and boundary conditions. If $g(x) = ax^3 + bx^2 + x$ and we demand that $g(0) = g(l)$ and $g'(0) = g'(l)$ lead us to the conditions

$$\begin{aligned} al^2 + bl + c &= 0, \\ 3al + 2b &= 0, \end{aligned}$$

which sets a uniparametric family of solutions a, b, c . A possible choice is given by our initial condition. The problem of eigenvalues is

$$\begin{aligned} -w_{xx} &= \lambda w, \\ \begin{cases} w|_{x=0} = w|_{x=l}, \\ w_x|_{x=0} = w_x|_{x=l}. \end{cases} \end{aligned}$$

which has as eigenvalues

$$\lambda_n = n^2 \omega^2, \quad \omega = \frac{2\pi}{l}$$

and eigenfunctions

$$w_n(x) = e^{in\omega x}.$$

To apply the EEM we have to determine the Fourier series

$$g(x) = \sum_{n \in \mathbb{Z}} c_n e^{in\omega x}, \quad c_n = \frac{1}{l} \int_0^l g(x) e^{-in\omega x} dx.$$

Since our g is a polynomial, we'll use the integrals

$$I_{m,n} := \int_0^l x^m e^{-in\omega x} dx = \begin{cases} i \frac{1}{n\omega} (l^m - m I_{m-1,n}) & n \neq 0, \\ \frac{l^{m+1}}{m+1} & n = 0. \end{cases}$$

In the first identity we have integrated by parts. It is easy to conclude that

$$I_{0,n} = \begin{cases} 0 & n = 0, \\ l & n \neq 0, \end{cases} \quad I_{1,n} = \begin{cases} \frac{l^2}{2} & n = 0, \\ i \frac{l}{n\omega} & n \neq 0, \end{cases} \quad I_{2,n} = \begin{cases} \frac{l^3}{3} & n = 0, \\ i \frac{l^2}{n\omega} + \frac{2l}{n^2 \omega^2} & n \neq 0, \end{cases}$$

and

$$I_{3,n} = \begin{cases} \frac{l^4}{4} & n = 0, \\ i \frac{l^3}{n\omega} + \frac{3l^2}{n^2 \omega^2} - i \frac{6l}{n^3 \omega^3} & n \neq 0. \end{cases}$$

Therefore, the Fourier series of the initial condition is

$$\begin{aligned} \frac{2}{l^2} x^3 - \frac{3}{l} x^2 + x &= \frac{1}{l} \left(\frac{2}{l^2} \frac{l^4}{4} - \frac{3}{l} \frac{l^3}{3} + \frac{l^2}{2} \right. \\ &\quad \left. + \sum_{n \neq 0} \left[\frac{2}{l^2} \left(i \frac{l^3}{n\omega} + \frac{3l^2}{n^2 \omega^2} - i \frac{6l}{n^3 \omega^3} \right) - \frac{3}{l} \left(i \frac{l^2}{n\omega} + \frac{2l}{n^2 \omega^2} \right) + i \frac{l}{n\omega} \right] e^{in\omega x} \right) \\ &= \frac{24}{l^2 \omega^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin n\omega x. \end{aligned}$$

We then look for a solution of the form $u(t, x) = \sum_{n=1}^{\infty} v_n(t) \sin n\omega x$ implying

$$\begin{aligned} v'_n &= \lambda_n v_n, \\ v_n(0) &= \frac{24}{l^2 \omega^3} \frac{1}{n^3} \end{aligned}$$

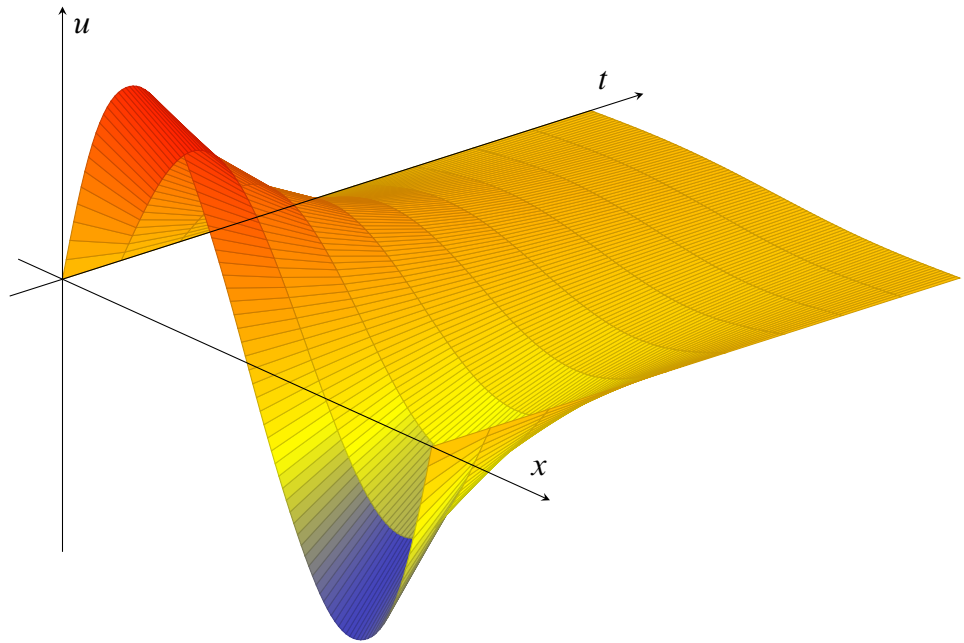
and for this reason

$$v_n(t) = \frac{24}{l^2 \omega^3} \frac{1}{n^3} e^{-n^2 \omega^2 t}.$$

Finally, the solution is

$$u(t, x) = \frac{24}{l^2 \omega^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 \omega^2 t} \sin n\omega x.$$

Given the fast decay of the coefficients v_n in n and t the graph is obtained with a good degree of approximation with a partial sum of only few terms, next we show the space-time representation of this evolution for $l = 1$ and a partial sum to 5 terms.



Temperature $u(x, t)$ in a trend $[0, 1]$ with times $0 < t < 0, 1$

3. We now study the wave equation and the problem

$$\left\{ \begin{array}{l} u_{tt} = u_{xx}, \\ u|_{t=0} = x(x^2 - 1), \\ u_t|_{t=0} = 0, \\ u|_{x=-1} = u|_{x=1}, \\ u_x|_{x=-1} = u_x|_{x=1}. \end{array} \right. \quad x \in (-1, 1), \quad t > 0,$$

The problem of finding eigenvalues is as in the previous cases. We went on to expand $x(x^2 - 1)$ in Fourier series of exponentials $x(x^2 - 1) = \sum_{n \in \mathbb{Z}} c_n e^{in\pi x}$ with

$$c_n = \frac{1}{2} \int_{-1}^1 x(x^2 - 1) e^{-in\pi x} dx$$

that after repeated integrations by parts leads us to

$$c_n = \begin{cases} 0, & n = 0, \\ -(-1)^n \frac{6i}{n^3 \pi^3}, & n \neq 0. \end{cases}$$

Therefore, the series is

$$g(x) := x(x^2 - 1) = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi^3} \sin n\pi x.$$

Now, the eigenfunction expansion of the solution

$$u(t, x) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

leads to

$$\begin{cases} v_n'' = -n^2 \pi^2 v_n, \\ v_n(0) = 12 \frac{(-1)^n}{n^3 \pi^3}, \\ v_n'(0) = 0, \end{cases}$$

so that

$$v_n(t) = 12 \frac{(-1)^n}{n^3 \pi^3} \cos n\pi t$$

and the solution is

$$u(t, x) = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi^3} \cos n\pi t \sin n\pi x.$$

Recalling that $\cos n\pi t \sin n\pi x = \frac{1}{2} [\sin n\pi(x + t) + \sin n\pi(x - t)]$ we deduce

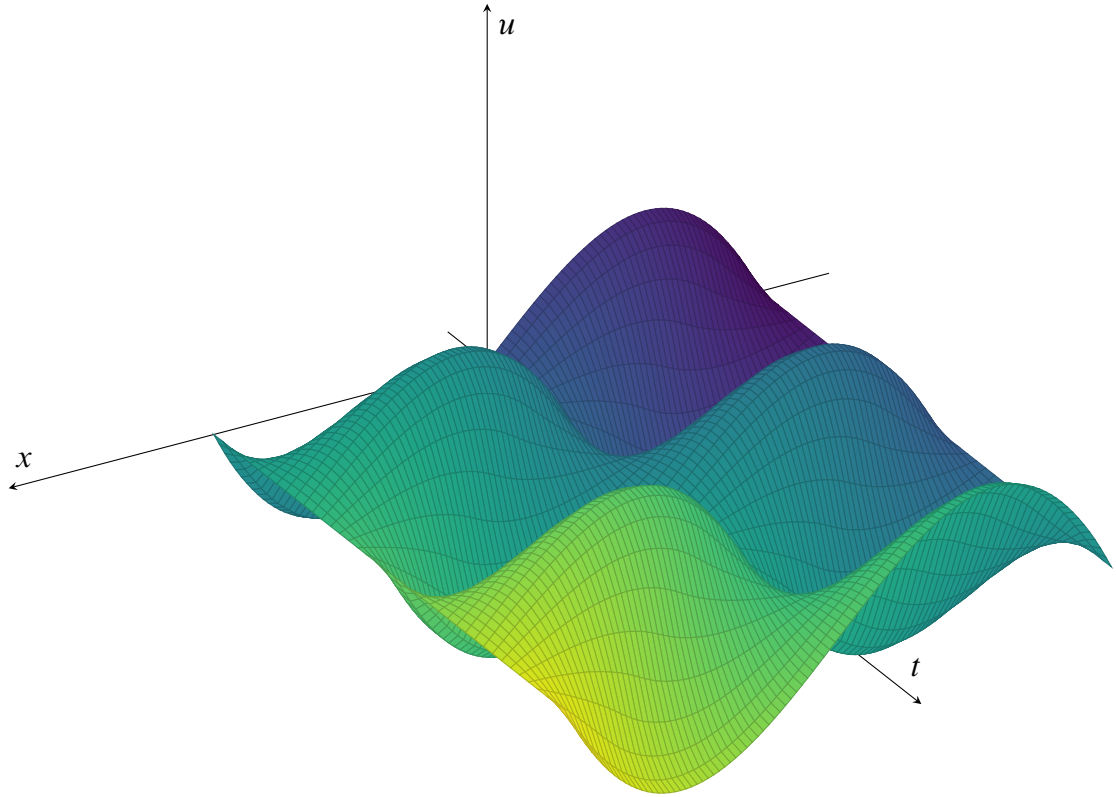
$$u(t, x) = \frac{1}{2} \left[12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi^3} \sin n\pi(x + t) + 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi^3} \sin n\pi(x - t) \right].$$

Thus, we can write

$$u(t, x) = \frac{g_{\text{per}}(x + t) + g_{\text{per}}(x - t)}{2}$$

where g_{per} is the periodic extension of $g(x) = x^3 - x$ from $[-1, 1]$ to $\mathbb{R}$. So we have the semi-sum of two traveling waves with opposite velocities $v_{\pm} = \pm 1$ and with the form of the periodic extension of the initial condition of each one of them.

The space-time graph of these waves is



Semi-sum of traveling waves on $[-1, 1]$ with periodic boundary conditions

5.2.1. Bonus: more general inhomogeneities Now, instead of a prepared problem we confront with a more general problem in where again the system of boundary and/or initial conditions consists of two subsystems. One of them, as before, is built up by conditions of inhomogeneous type

$$a_i(u) = g_i(x), \quad i = 1, \dots, r,$$

which contains *operators* a_i *that act only on the variable* x_0 , while the other, and this is the difference with the cases discussed above, is also made up of inhomogeneous conditions

$$(17) \quad b_j(u) = h_j(x), \quad x \in I \times S(\Omega_0) \quad j = 1, \dots, s,$$

such that *operators* b_j *act only on the variables* $(x_1, \dots, x_{n-1})$.

To be able to apply the EEM we require that these last boundary conditions must be equivalent to

$$(18) \quad (w_m, Bu) = (Bw_m, u) + I_m(h),$$

where the terms $I_m(h)$ must have the form

$$I_m(h) = \int_{S(\Omega_0)} \sum_j C_{m,j} h_j \, dS,$$

with $C_{m,j}$ being linear operators acting of the boundary data h_j that must be found for each case. Then, (14) modifies to

$$(19) \quad \boxed{Av_m + \lambda_m v_m + \frac{I_m(h)}{\|w_m\|^2} = f_m.}$$

To illustrate this procedure let's assume that $B = L$ with

$$Lu = \frac{1}{\rho}(-\operatorname{div}(p\nabla u) + qu),$$

is a $(n-1)$ -dimensional Sturm–Liouville operator, where ρ, p, q are real smooth functions in Ω_0 and $\rho, p > 0$ in Ω_0 . Such an Sturm–Liouville type operator satisfies the multidimensional Lagrange's identity (remember that we proved it for the 3D case)

$$(w, Lu) = (Lw, u) - \int_{S(\Omega_0)} \left[p(x) \left(\frac{\partial u}{\partial \mathbf{n}} \bar{w} - u \frac{\partial \bar{w}}{\partial \mathbf{n}} \right) \right] dS.$$

This operator will be symmetric in domains where the boundary terms in the Lagrange's identity cancel. For example, mixed conditions of Robin type

$$(20) \quad b(u) = \left(p \frac{\partial u}{\partial \mathbf{n}} + \sigma u \right) \Big|_{S(\Omega_0)} = 0 \quad \text{or} \quad b(u) = \left(\sigma p \frac{\partial u}{\partial \mathbf{n}} + u \right) \Big|_{S(\Omega_0)} = 0$$

with σ a smooth real function on the boundary, define a domain $\mathcal{D} = \{w \in C^\infty(\Omega_0) : b(w) = 0\}$ such that the Sturm–Liouville operator is symmetric in $\mathcal{D}$ (prove it). The first, for $\sigma = 0$, delivers the Neumann condition while the second condition for $\sigma = 0$ reduces to the Dirichlet case. The Lagrange's identity is not only useful for the identification of domains of symmetricity of this Sturm–Liouville operator but also to treat with the more general inhomogeneities. Indeed, if (17) is determined by (20)

$$b(u) = h$$

with h being a given function defined over $I \times S(\Omega_0)$, then we get

$$(w_m, Lu) = (Lw_m, u) + I_m(h)$$

with I_m , in the first case, given by

$$\begin{aligned} I_m(h) &= \int_{S(\Omega_0)} \left[p \left(\frac{\partial u}{\partial \mathbf{n}} \bar{w}_m - u \frac{\partial \bar{w}_m}{\partial \mathbf{n}} \right) \right] dS \\ &= \int_{S(\Omega_0)} \left[\left(p \frac{\partial u}{\partial \mathbf{n}} + \sigma u \right) \bar{w}_m \right] dS \\ &= \int_{S(\Omega_0)} h \bar{w}_m dS \end{aligned}$$

and by

$$\begin{aligned}
 I_{\mathbf{m}}(h) &= \int_{S(\Omega_0)} \left[p(x) \left(\frac{\partial u}{\partial \mathbf{n}} \bar{w}_{\mathbf{m}} - u \frac{\partial \bar{w}_{\mathbf{m}}}{\partial \mathbf{n}} \right) \right] dS \\
 &= - \int_{S(\Omega_0)} \left[\sigma p \left(\frac{\partial u}{\partial \mathbf{n}} + u \right) p \frac{\partial \bar{w}_{\mathbf{m}}}{\partial \mathbf{n}} \right] dS \\
 &= - \int_{S(\Omega_0)} h p \frac{\partial \bar{w}_{\mathbf{m}}}{\partial \mathbf{n}} dS,
 \end{aligned}$$

for the second case.

Example: Let us study the temperature $u(x, y, t)$ of a square plate $[0, 1] \times [0, 1]$ that fulfills 

$$\left\{ \begin{array}{l} u_t = u_{xx} + u_{yy}, \quad t > 0, \quad (x, y) \in [0, 1] \times [0, 1], \\ \left\{ \begin{array}{l} u|_{t=0} = xy, \\ \left\{ \begin{array}{l} u|_{x=0} = \alpha y t, \\ u|_{x=1} = 0, \\ u|_{y=0} = \beta x t, \\ u|_{y=1} = 0, \end{array} \right. \end{array} \right. \end{array} \right.$$

with $\alpha, \beta \in \mathbb{R}$. Here $L = -\Delta$ and the associated homogeneous boundary value problem is of Dirichlet type. Therefore, we have a complete set of eigenfunctions

$$w_{\mathbf{m}} = \sin(m_1 \pi x) \sin(m_2 \pi y), \quad \mathbf{m} = (m_1, m_2) \in \mathbb{N}^2$$

with corresponding eigenvalues

$$\lambda_{\mathbf{m}} = \pi^2(m_1^2 + m_2^2), \quad m_1, m_2 = 1, 2, \dots$$

We notice that $\|w_{\mathbf{m}}\| = 2$. The initial condition has the following expansion, a 2D sine Fourier series, indeed,

$$xy = \sum_{m_1, m_2=1}^{\infty} \frac{(-1)^{m_1+m_2}}{4\pi^2 m_1 m_2} \sin(m_1 \pi x) \sin(m_2 \pi y),$$

where we have used the Euler's formula for the Fourier coefficients

$$xy = \sum_{m_1, m_2=1}^{\infty} c_{\mathbf{m}} w_{\mathbf{m}}, \quad c_{\mathbf{m}} = \int_{[0,1]^2} xy \frac{w_{\mathbf{m}}(x, y)}{\|w_{\mathbf{m}}\|^2} dx dy.$$

Let us deal now with the new inhomogeneities, that appear in the two boundary conditions of Dirichlet type if the inhomogeneity is removed. Hence, we are required to compute

$$I_{\mathbf{m}}(h) = - \int_{\Gamma} h \frac{\partial w_{\mathbf{m}}}{\partial \mathbf{n}} dl$$

where Γ is the boundary of the unit square, that is transversed anticlockwise. The inhomogeneities are not zero only in two segments of this border, Γ_1 , that corresponds to $y = 0$ and Γ_4

that corresponds to $x = 0$. The unit normal vectors are $-\mathbf{j}$ in Γ_1 and $-\mathbf{i}$ in Γ_4 . Consequently, as

$$\nabla w_{\mathbf{m}} = (m_1 \pi \cos(m_1 \pi x) \sin(m_2 \pi y), m_2 \pi \sin(m_1 \pi x) \cos(m_2 \pi y))$$

we have

$$\frac{\partial w_{\mathbf{m}}}{\partial \mathbf{n}} = \begin{cases} -m_2 \pi \sin(m_1 \pi x), & \text{in } \Gamma_1, \\ -m_1 \pi \sin(m_2 \pi y), & \text{in } \Gamma_4. \end{cases}$$

Thus,

$$\begin{aligned} I_{\mathbf{m}} &= \beta \int_0^1 m_2 \pi \sin(m_1 \pi x) x t \, dx + \alpha \int_0^1 m_1 \pi \sin(m_2 \pi y) y t \, dy \\ &= -\left((-1)^{m_1} \frac{m_2}{m_1} \beta + (-1)^{m_2} \frac{m_1}{m_2} \alpha\right) t. \end{aligned}$$

The ODE satisfied by $v_{\mathbf{m}}$ is

$$\begin{cases} \frac{d v_{\mathbf{m}}}{d t} + \pi^2 (m_1^2 + m_2^2) v_{\mathbf{m}} - \left((-1)^{m_1} \frac{m_2}{m_1} \beta + (-1)^{m_2} \frac{m_1}{m_2} \alpha\right) \frac{t}{4} = 0, \\ v_{\mathbf{m}}(0) = \frac{(-1)^{m_1+m_2}}{4\pi^2 m_1 m_2}, \end{cases}$$

whose solution is

$$\begin{aligned} v_{\mathbf{m}}(t) &= \left(\frac{(-1)^{m_1+m_2}}{4\pi^2 m_1 m_2} + \left((-1)^{m_1} \frac{m_2}{m_1} \beta + (-1)^{m_2} \frac{m_1}{m_2} \alpha\right) \frac{1}{\pi^4 (m_1^2 + m_2^2)^2} \right) e^{-\pi^2 (m_1^2 + m_2^2) t} \\ &\quad - \left((-1)^{m_1} \frac{m_2}{m_1} \beta + (-1)^{m_2} \frac{m_1}{m_2} \alpha\right) \frac{1 - \pi^2 (m_1^2 + m_2^2) t}{\pi^4 (m_1^2 + m_2^2)^2}. \end{aligned}$$

Finally, the temperature $u(x, y, t)$ of the plate will be $u = \sum_{\mathbf{m} \in \mathbb{N}^2} v_{\mathbf{m}}(t) w_{\mathbf{m}}(x, y)$.

§5.3. Fourier Transform and EEM



UNBOUNDED domains problems are incorporated within EEM using the Fourier transform (FT). For example, when we have two independent variables (x_0, x_1) the FT is used with respect to the variable x_1 whenever the following circumstances occur

1. The PDE is of constant coefficients.
2. The variable x_1 varies over the whole line $-\infty < x_1 < \infty$.
3. There are no boundary conditions with respect to the variable x_1 or they are of the type $\lim_{x_1 \rightarrow \pm\infty} u = 0$.

In such cases, the inhomogeneous terms of the problem are expanded through its Fourier transformation and a solution expansion in the same way is sought for the unknown solution. A situation of this type appears for problems of initial conditions with a time variable and a spatial variable $(x_0, x_1) = (t, x)$, which belong to unbounded domains.

Unbounded Problem. I: Fourier Transform Strategy

To solve an initial value problem of the form

$$(21) \quad \begin{cases} \frac{\partial^N u}{\partial t^N} + \sum_{n=0}^M a_n \frac{\partial^n u}{\partial x^n} = f, & t > 0, -\infty < x < \infty, \\ \begin{cases} u|_{t=0} = g_1(x), \\ \frac{\partial u}{\partial t}|_{t=0} = g_2(x), \\ \vdots \\ \frac{\partial^{N-1} u}{\partial t^{N-1}}|_{t=0} = g_N(x) \end{cases} \end{cases}$$

we *expand* the inhomogeneous terms in the form

$$(22) \quad g_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_n(k) e^{ikx} dk, \quad f(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(t, k) e^{ikx} dk$$

where c_n and c are the Fourier transforms of g_n and f , respectively,

$$c_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_n(x) e^{-ikx} dx = \mathcal{F}(g_n), \quad c(t, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t, x) e^{-ikx} dx = \mathcal{F}(f(t, \cdot)).$$

We then seek for a solution with the same form

$$(23) \quad u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t, k) e^{ikx} dk,$$

where v is the Fourier transform of u with respect to the variable x

$$v(t, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t, x) e^{-ikx} dx = \mathcal{F}(u(t, \cdot)).$$

Unbounded Problem. II: The Problem for the FT

Substituting this expression in the equations of the problem (21) and identifying coefficients in the exponentials e^{ikx} we find the following N -th order ODE for $v(t, k)$ with respect to t , with N initial conditions

$$(24) \quad \begin{cases} \frac{\partial^N v(t, k)}{\partial t^N} + \sum_{n=0}^M a_n (ik)^n v(t, k) = c(t, k), & t > 0, \\ \begin{cases} v|_{t=0} = c_1(k), \\ \frac{\partial v}{\partial t}|_{t=0} = c_2(k), \\ \vdots \\ \frac{\partial^{N-1} v}{\partial t^{N-1}}|_{t=0} = c_N(k). \end{cases} \end{cases}$$

Unbounded Problem. III: The FT Solution

The solution to this problem is of the form

$$v(t, k) = v_0(t, k) + \sum_{n=1}^N e_n(t, k) c_n(k),$$

where $v_0(t, k)$ is solution of the PDE (24) with homogeneous initial conditions, and $e_n(t, k)$ is a solution of the PDE resulting from the elimination of the inhomogeneous terms in (24), and assuming that its n -th t derivative evaluated $t = 0$ equals 1 and the remaining derivatives are taken as 0. In this way the solution found is

$$\begin{aligned} u(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t, k) e^{ikx} dk \\ &= u_0(t, x) + \sum_{n=1}^N \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e_n(t, k) c_n(k) e^{ikx} dk, \end{aligned}$$

where

$$u_0(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_0(t, k) e^{ikx} dk = \mathcal{F}^{-1}(v_0(t, \cdot)),$$

is the inverse FT of $v_0(t, k)$. Thus, if we know the inverse FT of the functions $e_n(t, k)$

$$G_n(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e_n(t, k) e^{ikx} dk = \mathcal{F}^{-1}(e_n(t, \cdot)),$$

then

$$u(t, x) = u_0(t, x) + \sum_{n=1}^N \mathcal{F}^{-1}(\mathcal{F}(G_n) \cdot \mathcal{F}(g_n)) = u_0(t, x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^N G_n(t, x) * g_n(x),$$

and we can write the solution (21) in the following form

$$(25) \quad u(t, x) = u_0(t, x) + \sum_{n=1}^N \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_n(t, x - y) g_n(y) dy.$$



Observation: The method that we have just described really follows the EEM strategy because what we do basically is to *expand* with respect to a set of eigenfunctions. $\{w_k(x) = e^{ikx}\}_{k \in \mathbb{R}}$, now a continuous set, of the operator with constant coefficients $B = \sum_{n=1}^M a_n \frac{\partial^n u}{\partial x^n}$.



Examples:

1. We apply these ideas to the **heat equation**

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u|_{t=0} = f(x), \end{cases}$$

with $t > 0$ and $-\infty < x < \infty$. We expand

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{ikx} dk, \quad c(k) = \mathcal{F}(f),$$

and look for a solution in the form

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t, k) e^{ikx} dk, \quad v(t, k) = \mathcal{F}(u(t, \cdot)).$$

Substituting in the equation and identifying coefficients in the exponentials e^{ikx} we get

$$\begin{aligned} \frac{\partial v(t, k)}{\partial t} &= -a^2 k^2 v(t, k), \\ v|_{t=0} &= c(k). \end{aligned}$$

The solution to this problem is

$$v(t, k) = e(t, k) c(k), \quad \text{siendo } e(t, k) = e^{-a^2 t k^2}$$

and, since we know the inverse FT of $e(t, k)$, i.e.,

$$\mathcal{F}^{-1}(e^{-a^2 t k^2}) = \sqrt{\frac{1}{2t a^2}} e^{-\frac{x^2}{4a^2 t}},$$

we can write the solution in the following way

$$u(t, x) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2 t}} f(y) dy.$$

It is possible to generalize this result to the multidimensional case

$$\begin{cases} u_t = a^2 \Delta u, & t > 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \\ u|_{t=0} = f(x). \end{cases}$$

The EEM leads to the solution

$$u(t, x) = \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4a^2 t}} f(y) d^n y.$$

2. We apply the FT to the free Schrödinger equation. The above results on the heat equation can be applied to deduce analogous formulas for free Schrödinger equation solutions in any dimension

$$\begin{cases} i\hbar u_t = -\frac{\hbar^2}{2m} \Delta u, & t > 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \\ u|_{t=0} = f(x). \end{cases}$$

Note that this initial condition problem for the Schrödinger equation is obtained from the heat equation by taking

$$a^2 = i \frac{\hbar}{2m}.$$

Therefore, making that identification of parameter a we deduce the following expression of the solution

$$u(t, x) = \left(\frac{m}{2\pi i \hbar t} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i \frac{m|x-y|^2}{2\hbar t}} f(y) \, d^n y.$$

§5.4. Exercises

5.4.1. Exercises with solutions

1. For the quantum harmonic oscillator in one dimension solve the following initial and boundary value problem

$$\begin{cases} i\hbar u_t = -\frac{\hbar^2}{2m}u_{xx} + \frac{1}{2}kx^2u, & -\infty < x < \infty, t > 0, \\ u|_{t=0} = g(x), \\ \lim_{x \rightarrow \pm\infty} u = 0. \end{cases}$$

Solution: The eigenvalue equation is

$$-\frac{\hbar^2}{2m}w_{xx} + \frac{1}{2}kx^2w = \lambda w, \quad -\infty < x < \infty, \\ \lim_{x \rightarrow \pm\infty} w = 0.$$

That bears the name of Weber, who was the first to study it in 1869. Using the variables

$$K := \frac{mk}{\hbar}, \quad X := \sqrt[4]{K}x, \quad \Lambda := \frac{2m}{\hbar\sqrt{K}}\lambda$$

we can write the eigenvalue problem as

$$-w_{XX} + X^2w = \Lambda w,$$

which is the well-known Hermite equation that we will deal with in the next chapter. The conditions at infinity lead to the eigenvalues

$$\Lambda_n = 2n + 1, \quad n = 0, 1, 2, \dots,$$

and corresponding eigenfunctions

$$w_n(X) = H_n(X) e^{-\frac{X^2}{2}},$$

with $H_n(X)$, $n = 0, 1, \dots$ the Hermite polynomials that we will treat in detail in the following chapter conform an orthogonal basis $\{H_n(x)\}_{n=0}^\infty$ en $L^2(\mathbb{R})$. We can write the eigenvalues as

$$\lambda_n = \hbar \sqrt{\frac{k}{2m}}(2n + 1)$$

with eigenfunctions, known as Hermite or Hermite–Weber functions

$$w_n(x) = H_n\left(\sqrt{\frac{\sqrt{mk}}{\hbar}}x\right) e^{-\frac{\sqrt{mk}}{2\hbar}x^2},$$

for $n = 0, 1, \dots$. Hence, if $g(x) \in L^2(\mathbb{R})$ we have the expansion

$$g(x) = \sum_{n=0}^{\infty} c_n w_n(x)$$

and the solution will be

$$u(t, x) = \sum_{n=0}^{\infty} c_n e^{-i \frac{\lambda_n}{\hbar} t} w_n(x).$$

If we put $\sqrt{mk\hbar} = 1$ the functions known as Weber or cylindrical-parabolic functions are obtained

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x),$$

that make up an orthonormal basis of Hilbert space $L^2(\mathbb{R})$.

2. Using the FT solve for the wave equation on the real straight line the following initial value problem

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, -\infty < x < \infty, \\ \begin{cases} u|_{t=0} = f(x), \\ u_t|_{t=0} = g(x). \end{cases} \end{cases}$$

How is this method linked to the d'Alembert formula?

Solution: We can apply the EEM using the eigenfunction $\{e^{ikx}\}_{k \in \mathbb{R}}$ to the spectral problem

$$w_{xx} = -k^2 w, \quad x \in \mathbb{R}.$$

Therefore, we write for the solution sought

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(t, k) e^{ikx} dk.$$

Introducing this expression in the wave equation we obtain

$$\int_{\mathbb{R}} (v_{tt}(k, t) + k^2 v(k, t)) e^{ikx} dk = 0.$$

For this reason, we impose

$$v_{tt}(k, t) + k^2 v(k, t) = 0$$

whose general solution is

$$v(k, t) = A(k) e^{ikt} + B(k) e^{-ikt}.$$

We conclude that the EEM leads to a solution of the form

$$\begin{aligned} u(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (A(k) e^{ik(x+t)} + B(k) e^{ik(x-t)}) dk, \\ u_t(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ik (A(k) e^{ik(x+t)} - B(k) e^{ik(x-t)}) dk, \end{aligned}$$

that at $t = 0$ reduces to

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (A(k) + B(k)) e^{ikx} dk, \quad g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ik (A(k) - B(k)) e^{ikx} dk,$$

Therefore, for the FT of the initial data

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx, \quad \hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-ikx} dx,$$

we obtain

$$\hat{f}(k) = A(k) + B(k), \quad \hat{g}(k) = ik(A(k) - B(k)),$$

which is solved immediately

$$A(k) = \frac{1}{2} \left(\hat{f}(k) + \frac{\hat{g}(k)}{ik} \right), \quad B(k) = \frac{1}{2} \left(\hat{f}(k) - \frac{\hat{g}(k)}{ik} \right).$$

So, finally, we can write the solution in terms of the Fourier transforms of the initial data as follows

$$u(t, x) = \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(k) e^{ik(x+t)} dk + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(k) e^{ik(x-t)} dk \right) + \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(k) \frac{e^{ik(x+t)}}{ik} dk - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(k) \frac{e^{ik(x-t)}}{ik} dk \right).$$

The formula

$$\frac{e^{ik(x \pm t)}}{ik} = \frac{1}{ik} + \int_0^{x \pm t} e^{iks} ds$$

allows us to express this as follows

$$u(t, x) = \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(k) e^{ik(x+t)} dk + \int_{\mathbb{R}} \hat{f}(k) e^{ik(x-t)} dk \right) + \frac{1}{2} \int_{x-t}^{x+t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{g}(k) e^{iks} dk ds.$$

where we immediately recognize, using the inverse FT, the d'Alembert formula

$$u(t, x) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds.$$

3. Using the EEM, solve the problem of initial and boundary conditions for the heat equation in an infinite plate of width L as follows

$$\begin{cases} u_t = u_{xx} + u_{yy}, & (x, y) \in \mathbb{R} \times (0, L), t > 0. \\ u|_{t=0}(x, y) = g(x, y) = \begin{cases} \sin \frac{\pi y}{L}, & x \in [-1, 1], \\ 0, & x \notin [-1, 1], \end{cases} \\ \begin{cases} u|_{y=0} = 0, \\ u|_{y=L} = 0. \end{cases} \end{cases}$$

Solution: The associated spectral problem is

$$\begin{aligned} -(w_{xx} + w_{yy}) &= \lambda w, & \lambda &= k_1^2 + k_2^2, \\ \begin{cases} w|_{y=0} = 0, \\ w|_{y=L} = 0. \end{cases} \end{aligned}$$

Which, in turn, is a homogeneous boundary value problem, for the two-dimensional Helmholtz equation, in which we can separate variables into Cartesian coordinates. Thus, the solutions are written as

$$w(x, y) = (a_1 e^{ik_1 x} + b_1 e^{-ik_1 x})(a_2 e^{ik_2 y} + b_2 e^{-ik_2 y}).$$

The boundary conditions give

$$\begin{pmatrix} 1 & 1 \\ e^{ik_2 L} & e^{-ik_2 L} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Non-zero solutions of this system exist as long as $k_2 = n\frac{\pi}{L}$, $n = 1, 2, \dots$. Therefore, the eigenvalues will be

$$\left\{ k^2 + n^2 \frac{\pi^2}{L^2} \right\}_{k \in \mathbb{R}, n \in \mathbb{N}}$$

with corresponding eigenfunction given by

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \sin n\pi y \right\}_{k \in \mathbb{R}, n \in \mathbb{N}}$$

Note that there is a continuous set of eigenvalues, since we have no boundary conditions in the variable x . The solution will be expressed according to the EEM as follows

$$u(t, x, y) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v_n(k, t) e^{ikx} \sin \frac{n\pi y}{L} dk$$

Inserting this expression in the PDE delivers the following ODE for $v_n(k, t)$

$$\frac{\partial v_n}{\partial t}(t, k) + \left(k^2 + \frac{n^2 \pi^2}{L^2}\right) v_n(t, k) = 0$$

whose solution is

$$v_n(k, t) = a_n(k) e^{-(k^2 + \frac{n^2 \pi^2}{L^2})t}.$$

To find out which eigenfunction are involved in this expansion, we analyze the corresponding expansions of the inhomogeneities. In this case it is only the function $g(x, y)$, whose form indicates that can be expressed in terms of $\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \sin n\pi y \right\}_{k \in \mathbb{R}}$ only. Hence

$$g(x, y) = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c(k) e^{ikx} dk \right) \sin \frac{\pi y}{L}, \quad c(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-ikx} dx = \sqrt{\frac{2}{\pi}} \frac{\sin k}{k}.$$

This is,

$$g(x, y) = \left(\int_{\mathbb{R}} \frac{\sin k}{\pi k} e^{ikx} dk \right) \sin \frac{\pi y}{L},$$

that leads to

$$u(t, x, y) = e^{-\frac{\pi^2}{L^2}t} I(t, x) \sin \frac{\pi}{L} y, \quad I(t, x) := \int_{\mathbb{R}} e^{-k^2 t + i k x} \frac{\sin k}{\pi k} dk.$$

We now evaluate $I(t, x)$ that we write in the form

$$I(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-k^2 t} \left(\frac{e^{i k(x+1)} - e^{i k(x-1)}}{i k} \right) dk,$$

and recalling that

$$\frac{e^{i k(x \pm 1)}}{i k} = \frac{1}{i k} + \int_0^{x \pm 1} e^{i k s} ds$$

it is expressed as follows

$$I(t, x) = \frac{1}{\sqrt{2\pi}} \int_{x-1}^{x+1} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-k^2 t} e^{i k s} dk \right) ds.$$

Now, as we have already calculated, the inverse FT of a Gaussian is another Gaussian, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-k^2 t} e^{i k s} dk = \frac{1}{\sqrt{2t}} e^{-\frac{s^2}{4t}}$$

which implies

$$\begin{aligned} I(t, x) &= \frac{1}{\sqrt{\pi}} \int_{x-1}^{x+1} \frac{1}{2\sqrt{t}} e^{-\frac{s^2}{4t}} ds && \text{change of variable } \frac{s}{2\sqrt{t}} = \zeta \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-\zeta^2} d\zeta. \end{aligned}$$

In terms of the error function

$$(26) \quad \operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta,$$

we can express it as follows

$$I(t, x) = \frac{1}{2} \left(\operatorname{erf} \left(\frac{x+1}{2\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-1}{2\sqrt{t}} \right) \right).$$

In short, we have proven that the solution to the problem is

$$u(t, x, y) = \frac{1}{2} e^{-\frac{\pi^2}{L^2}t} \left(\operatorname{erf} \left(\frac{x+1}{2\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-1}{2\sqrt{t}} \right) \right) \sin \frac{\pi}{L} y.$$

Notice that for $x > 0$ the error function has the following asymptotic behavior

$$\operatorname{erf} x = 1 + \frac{1}{\sqrt{\pi} x} e^{-x^2} \left(1 - \mathcal{O}\left(\frac{1}{x^2}\right) \right), \quad x \rightarrow \infty,$$

and also that is an odd function

$$\operatorname{erf} x = -\operatorname{erf}(-x).$$

Therefore,

$$\lim_{t \rightarrow 0^+} \frac{1}{2} \left(\operatorname{erf} \left(\frac{x+1}{2\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x-1}{2\sqrt{t}} \right) \right) = \begin{cases} 1, & x \in [-1, 1], \\ 0, & x \notin [-1, 1]. \end{cases}$$

and the initial condition is recovered $\lim_{t \rightarrow 0^+} u(t, x, y) = g(x, y)$.

4. Using the Fourier transform, find the solution to the following initial value problem for the heat equation on the straight line

$$\begin{cases} u_t = u_{xx}, & t > 0, -\infty < x < \infty, \\ u|_{t=0} = g(x) := x e^{-|x|}. \end{cases}$$

Solution: Using the FT, as EEM, we obtain

$$u(t, x) = \int_{\mathbb{R}} c(k) e^{-k^2 t + i k x} dk,$$

where we have used the FT $c(k)$ of the initial data $g(x)$

$$\begin{aligned} c(k) &= \mathcal{F}(g) = i D_k \mathcal{F} = \frac{1}{2\pi} i D_k \left(\int_{-\infty}^0 e^{(1-ik)x} dx + \int_0^{\infty} e^{-(1+ik)x} dx \right) \\ &= \frac{1}{2\pi} i D_k \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right) = -\frac{2ik}{1+k^2}. \end{aligned}$$

Therefore, the requested solution is

$$u(t, x) = -2i \int_{\mathbb{R}} \frac{k}{1+k^2} e^{-k^2 t + i k x} dk.$$

Since the integral is difficult to calculate, we approach its computation with another strategy, also based on the Fourier transform, which allows you to express the solution in the following way

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} s e^{-|s|} e^{-\frac{(x-s)^2}{4t}} ds.$$

To analyze this integral we write it as follows

$$u(t, x) = F(t, x) - F(t, -x), \quad F(t, x) := \frac{1}{2\sqrt{\pi t}} \int_0^{\infty} s e^{-s - \frac{(x-s)^2}{4t}} ds.$$

Relationships

$$(x-s) e^{-\frac{(x-s)^2}{4t}} = 2t \frac{d}{ds} e^{-\frac{(x-s)^2}{4t}} = 2t \left(1 + e^s \frac{d}{ds} e^{-s} \right) e^{-\frac{(x-s)^2}{4t}},$$

imply

$$\left(x - 2t - 2t \frac{d}{ds} \right) e^{-s - \frac{(x-s)^2}{4t}} = s e^{-s - \frac{(x-s)^2}{4t}},$$

and we deduce

$$\begin{aligned}
 F(t, x) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty \left(x - 2t - 2t \frac{d}{ds} \right) e^{-s - \frac{(x-s)^2}{4t}} ds \\
 &= \frac{1}{2\sqrt{\pi t}} \left((x - 2t) \int_0^\infty e^{-s - \frac{(x-s)^2}{4t}} ds - 2t e^{-s - \frac{(x-s)^2}{4t}} \Big|_{s=0}^{s=\infty} \right) \\
 &= \frac{x - 2t}{2\sqrt{\pi t}} \int_0^\infty e^{-s - \frac{(x-s)^2}{4t}} ds + \sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4t}}.
 \end{aligned}$$

Performing the variable change

$$\zeta := -\frac{x-s}{2\sqrt{t}} + \sqrt{t}, \quad s = 2\sqrt{t}\zeta + x - 2t, \quad ds = 2\sqrt{t} d\zeta, \quad \zeta^2 - (t-x) = \frac{(x-s)^2}{4t} + s,$$

we get, in terms of the error function and its complementary $\operatorname{erfc}(z) := 1 - \operatorname{erf}(z)$, the following

$$\begin{aligned}
 \int_0^\infty e^{-s - \frac{(x-s)^2}{4t}} ds &= 2\sqrt{t} e^{t-x} \int_{-\frac{x-2t}{2\sqrt{t}}}^\infty e^{-\zeta^2} d\zeta \\
 &= \sqrt{\pi t} e^{t-x} \operatorname{erfc}\left(-\frac{x-2t}{2\sqrt{t}}\right) = \sqrt{\pi t} e^{t-x} \left(1 - \operatorname{erf}\left(-\frac{x-2t}{2\sqrt{t}}\right)\right).
 \end{aligned}$$

So,

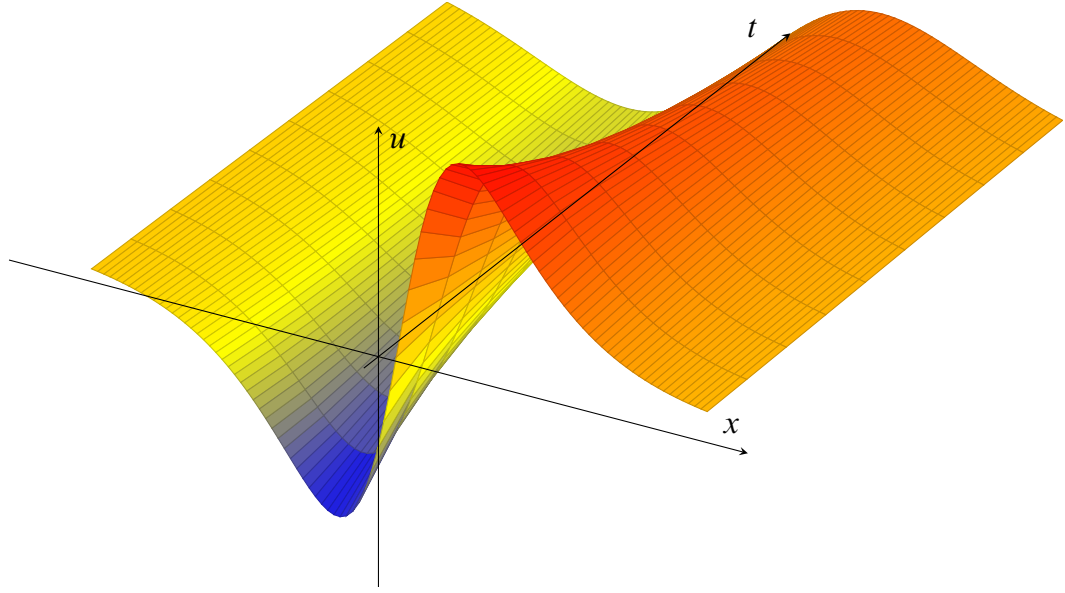
$$F(t, x) = \frac{x - 2t}{2} e^{t-x} \operatorname{erfc}\left(-\frac{x - 2t}{2\sqrt{t}}\right)$$

and, finally, we can express the solution as follows

$$u(t, x) = \frac{x - 2t}{2} e^{t-x} \operatorname{erfc}\left(-\frac{x - 2t}{2\sqrt{t}}\right) + \frac{x + 2t}{2} e^{t+x} \operatorname{erfc}\left(\frac{x + 2t}{2\sqrt{t}}\right).$$

Given that $\lim_{x \rightarrow \infty} \operatorname{erfc}(x) = 0$ and $\lim_{x \rightarrow -\infty} \operatorname{erfc}(x) = 2$ we have $\lim_{t \rightarrow 0^+} u(t, x) = \begin{cases} x e^{-x}, & x > 0 \\ x e^x, & x < 0, \end{cases}$

and we recover the initial temperature data.



5. Solve, by EEM, for the vibrating string the following boundary and initial value problem

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, -1 < x < 1, \\ \begin{cases} u|_{t=0} = \sin \pi x, \\ u_t|_{t=0} = |x|, \end{cases} \\ \begin{cases} u|_{x=-1} = u|_{x=1}, \\ u_x|_{x=-1} = u_x|_{x=1}. \end{cases} \end{cases}$$

Solution: In order to apply the EEM we must solve the following eigenvalue problem

$$\begin{aligned} -w_{xx} &= \lambda w, & x &\in [-1, 1] \\ \begin{cases} w|_{x=-1} &= w|_{x=1}, \\ w_x|_{x=-1} &= w_x|_{x=1}. \end{cases} \end{aligned}$$

The corresponding eigenvalues are $\lambda_n = n^2\pi^2$ with eigenfunctions given by

$$\{1, \cos n\pi x, \sin n\pi x\}_{n=1}^{\infty}.$$

The eigenfunction expansion of the solution is $u(t, x)$ is

$$u(t, x) = v_0(t) + \sum_{n=1}^{\infty} (v_n(t) \cos n\pi x + \tilde{v}_n(t) \sin n\pi x).$$

By introducing this expression into the wave equation and decoupling the different modes we get

$$\begin{aligned} v_{0,tt} &= 0, \\ v_{n,tt} + n^2\pi^2 v_n &= 0, \\ \tilde{v}_{n,tt} + n^2\pi^2 \tilde{v}_n &= 0, \end{aligned}$$

that leads to

$$\begin{cases} v_0(t) = A_0 + B_0 t, \\ v_n(t) = A_n \cos n\pi t + B_n \sin n\pi t, & n = 1, 2, \dots, \\ \tilde{v}_n(t) = \tilde{A}_n \cos n\pi t + \tilde{B}_n \sin n\pi t, & n = 1, 2, \dots \end{cases}$$

We must now expand the initial conditions. As $u|_{t=0} = f(x) := \sin \pi x$ is already expanded it is enough to calculate the Fourier series of $u_t|_{t=0} = g(x) := |x|$. Being the function $g(x)$ even in $[-1, 1]$ its expansion is reduced to a Fourier series in cosines. The corresponding coefficients will be

$$\begin{cases} a_0 = 2 \int_0^1 x \, dx = 1, \\ a_n = 2 \int_0^1 x \cos n\pi x \, dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\frac{4}{\pi^2 n^2}, & \text{if } n \text{ is odd.} \end{cases} \end{cases}$$

Then, in the eigenfunction expansion of $u(t, x)$ should only appear

$$1, \sin \pi x, \cos \pi x, \cos 3\pi x, \cos 5\pi x, \dots$$

and, therefore, the only non-zero coefficients are $v_0, v_1, v_3, v_5, \dots$ and $\tilde{v}_1$. Because all these we have

$$\begin{cases} A_0 = v_0(0) = 0, \\ A_{2n+1} = v_{2n+1} = 0, \\ \tilde{A}_1 = \tilde{v}_1(0) = 1 \end{cases}, \quad \begin{cases} B_0 = v'_0(0) = \frac{1}{2}, \\ (2n+1)\pi B_{2n+1} = v'_{2n+1}(0) = -\frac{4}{\pi(2n+1)^2}, \\ \pi \tilde{B}_1 = \tilde{v}'_1 = 0, \end{cases}$$

with $n = 0, 1, \dots$. Hence, we conclude that the Fourier series for the sought solution is

$$u(t, x) = \frac{t}{2} + \cos \pi t \sin \pi x - \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin((2n+1)\pi t) \cos((2n+1)\pi x).$$

This solution can be interpreted as follows. First of all, using the addition formulas of the trigonometric functions we write it as follows

$$\begin{aligned} u(t, x) = \frac{1}{2} \Big(\sin \pi(x-t) + \sin \pi(x+t) + t - \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin((2n+1)\pi(x+t)) \\ + \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin((2n+1)\pi(x-t)) \Big) \end{aligned}$$

According to Dirichlet's theorem, the Fourier series

$$\frac{1}{2} - \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)\pi x$$

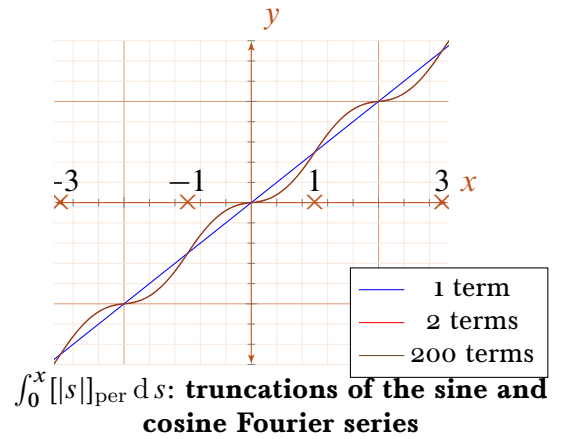
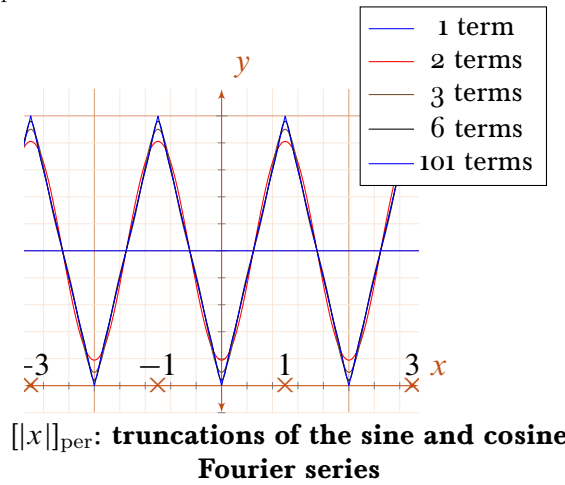
converge pointwise, for all $x \in \mathbb{R}$ to the periodic extension $g_{\text{per}}(x)$ of $g(x) = |x|$, $x \in [-1, 1]$. As the $|x|$ function is continuous and of bounded variation¹ this Fourier series on sines and cosines converge uniformly. Therefore, it is possible to derive and integrate term by term

$$\int_0^x g_{\text{per}}(s) \, ds = \frac{x}{2} - \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin(2n+1)\pi x.$$

Therefore, the Fourier expansion obtained for the solution corresponds to d'Alembert's formula

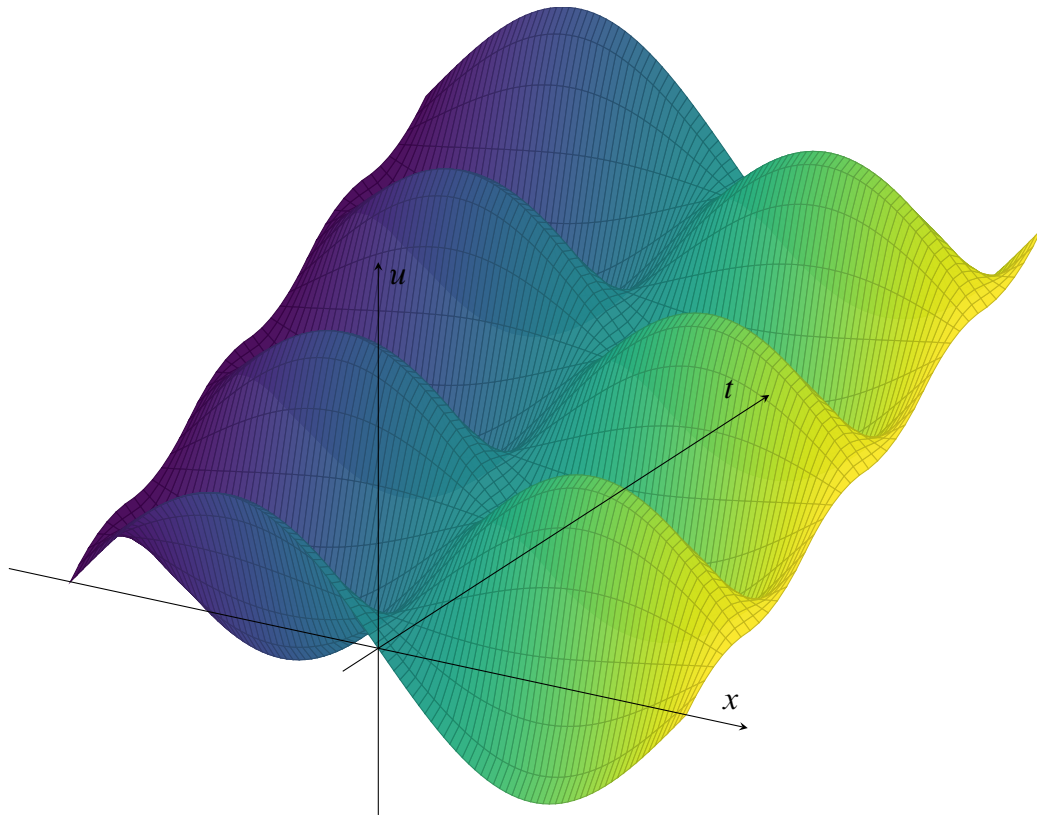
$$u(t, x) = \frac{1}{2} \left(f_{\text{per}}(x+t) + f_{\text{per}}(x-t) + \int_{x-t}^{x+t} g_{\text{per}}(s) \, ds \right).$$

Next we show the graphs of the truncations of the series of Fourier of $g_{\text{per}}(x)$ and of $\int_0^x g_{\text{per}}(s) \, ds$



We also show a space-time graph of the solution of the d'Alembert solution and another of the truncation to ten terms of the corresponding Fourier series.

¹It is the difference of two non-decreasing functions $|x| = u_2(x) - u_1(x)$, where $u_2(x) = \begin{cases} x, & 0 < x < 1, \\ 0, & -1 < x < 0, \end{cases}$ and $u_1(x) = \begin{cases} 0, & 0 < x < 1, \\ x, & -1 < x < 0, \end{cases}$



6. Applying the EEM solve the following initial and boundary value problem for the vibrating string

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, 0 < x < 2\pi, \\ \begin{cases} u|_{t=0} = x(x - 2\pi), \\ u_t|_{t=0} = 0, \end{cases} \\ \begin{cases} u|_{x=0} = 0, \\ u|_{x=2\pi} = 0. \end{cases} \end{cases}$$

Solution: To apply the EEM we must find the solution of

$$\begin{cases} -w_{xx} = \lambda w, \\ \begin{cases} w|_{x=0} = 0, \\ w|_{x=2\pi} = 0, \end{cases} \end{cases}$$

which, as we know, has as its eigenvalues $\lambda_n = \frac{n^2}{4}$ with eigenfunctions $w_n(x) = \sin \frac{nx}{2}$, $n = 1, 2, \dots$. We must now expand the initial conditions $u|_{t=0} = f(x) := x(x - 2\pi)$ and $u_t|_{t=0} = g(x) = 0$ in terms of these eigenfunctions. Therefore, it is necessary to compute the sine Fourier series of $f(x)$ and $g(x)$. For the Fourier expansion of $f(x)$,

which is the only non-trivial case, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{nx}{2}, \quad b_n := \frac{1}{\pi} \int_0^{2\pi} x(x-2\pi) \sin \frac{nx}{2} dx = \begin{cases} 0, & n \text{ is even,} \\ -\frac{32}{n^2\pi^3}, & n \text{ is odd.} \end{cases}$$

The eigenfunction expansion of the solution is

$$u(t, x) := \sum_{n=1}^{\infty} v_n(t) \sin \frac{nx}{2},$$

where $v_n(t)$ solves

$$v_n''(t) + \frac{n^2}{4} v_n(t) = 0, \quad \begin{cases} v_n(0) = b_n, \\ v_n'(0) = 0. \end{cases}$$

So that

$$v_n(t) = b_n \cos \frac{nt}{2},$$

and we conclude that

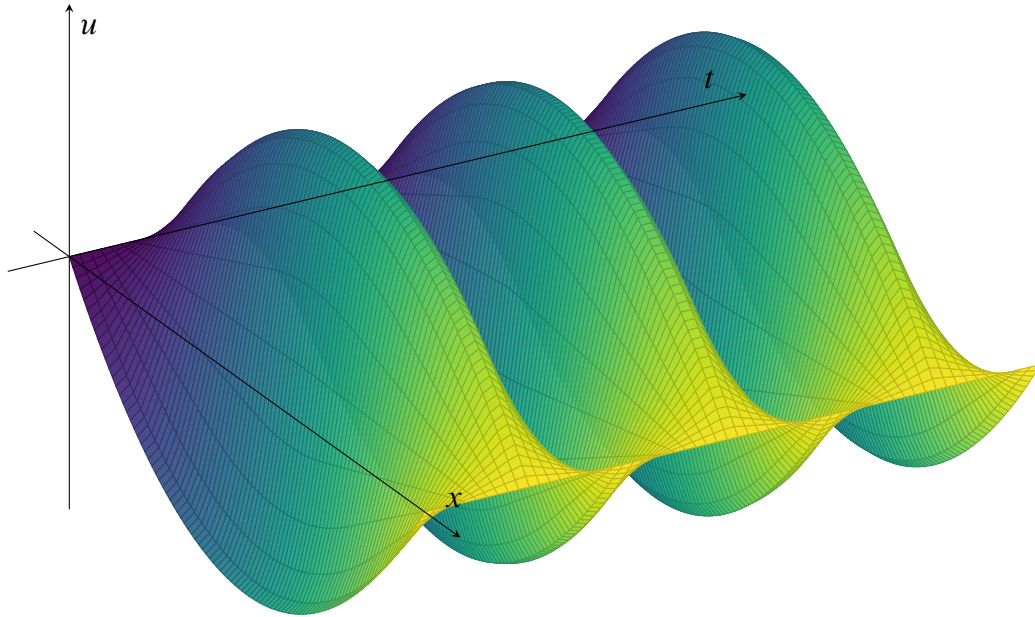
$$u(t, x) = -\frac{32}{\pi} \sum_{n \text{ is odd}} \frac{1}{n^3} \cos \frac{nt}{2} \sin \frac{nx}{2}.$$

As in previous exercises, the use of trigonometric addition formulas leads to

$$u(t, x) = -\frac{1}{2} \left(\frac{32}{\pi} \sum_{n \text{ is odd}} \frac{1}{n^3} \sin \frac{n(x-t)}{2} + \frac{32}{\pi} \sum_{n \text{ is odd}} \frac{1}{n^3} \sin \frac{n(x+t)}{2} \right).$$

Hence, given the Fourier series expansion of $f_{\text{odd}}(x)$, we get the d'Alembert formula

$$u(t, x) = \frac{f_{\text{odd}}(x-t) + f_{\text{odd}}(x+t)}{2}.$$



Waves in $[0, 2\pi]$ with Dirichlet boundary type conditions

7. Solve by Fourier Transform the initial value problem

$$\begin{cases} u_t + u_x - u_{xx} = 0, & t > 0, -\infty < x < \infty, \\ u(x, 0) = e^{-x^2}, \end{cases}$$

Hint: $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx = \frac{1}{2\sqrt{\pi}} e^{-\frac{k^2}{4}}.$

Solution: The functions e^{ikx} are eigenfunctions of $B = \frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}$ with eigenvalues $\lambda = ik + k^2$. Hence, we seek for a solution in the form

$$u(x, t) = \int_{\mathbb{R}} c(k) e^{-(ik+k^2)t} e^{ikx} dk,$$

where

$$c(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-x^2} e^{-ikx} dx = \frac{1}{2\sqrt{\pi}} e^{-\frac{k^2}{4}}.$$

Thus

$$u(t, x) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-k^2(t+\frac{1}{4})} e^{-ik(t-x)} dk,$$

that with the change of variables

$$X = k, \quad K = t - x, \quad a = \sqrt{t + \frac{1}{4}},$$

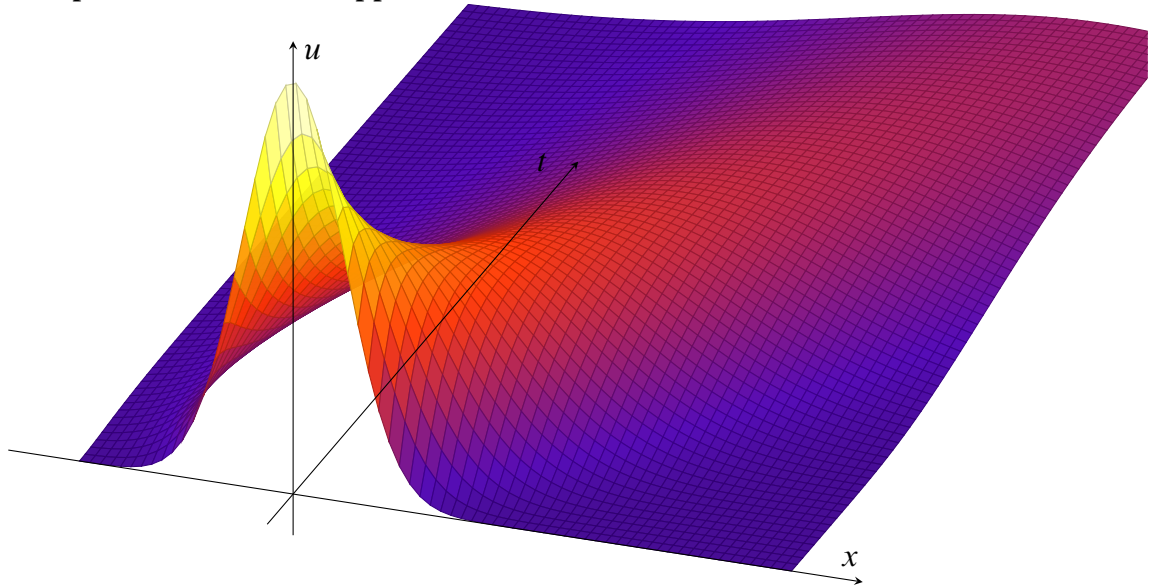
gives

$$u(t, x) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-a^2 X^2} e^{-i K X} dX,$$

which implies that

$$u(t, x) = \frac{1}{\sqrt{1+4t}} e^{-\frac{(t-x)^2}{1+4t}}.$$

This equation is the heat equation to which a convectional or drag term has been added, which accounts for a forced drag, for some reason, of the heat flow. The space-time graph is shown below, where the drag-out together with the typical dissipation of calorific phenomena can be appreciated.



Temperature space-time graph

8. Using the EEM determine the solution of the following boundary and initial value problem for the heat equation

$$\begin{cases} u_t = u_{xx}, & t > 0, -1 < x < 1, \\ u|_{t=0} = x^2 \sin \pi x, \\ \begin{cases} u|_{x=-1} = 0, \\ u|_{x=1} = 0. \end{cases} \end{cases}$$

Hint:

$$\int \sin ax \sin bx \, dx = \begin{cases} \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, & a \neq b \\ \frac{x}{4} - \frac{\sin 2ax}{2a}, & a = b. \end{cases}$$

Solution: The application of the EEM in this case requires the resolution of the following eigenvalue problem: $-\frac{d^2 u}{dx^2} = \lambda u$ with $u(-1) = u(1) = 0$. This Dirichlet problem has as eigenvalues $\lambda_n = \frac{n^2 \pi^2}{4}$ and corresponding eigenfunctions $w_n(x) = \sin \frac{n\pi(x+1)}{2}$. Therefore, we look for the solution $u(x, t)$ in the form of a sine Fourier series

$$u(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi(x+1)}{2}.$$

Hence, $v_n(t)$ must fulfill $v' = -\lambda v$ so that $v(t) = c e^{-\frac{n^2 \pi^2 t}{4}}$ and the expansion is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2 \pi^2 t}{4}} \sin \frac{n\pi(x+1)}{2}.$$

The coefficients c_n are the Fourier coefficients of the initial condition. Thus, we have

$$c_n = \int_{-1}^1 x \sin \pi x \sin \frac{n\pi(x+1)}{2} dx.$$

Expanding the last factor of the integrand and analyzing the parity of the resulting integrands we conclude that

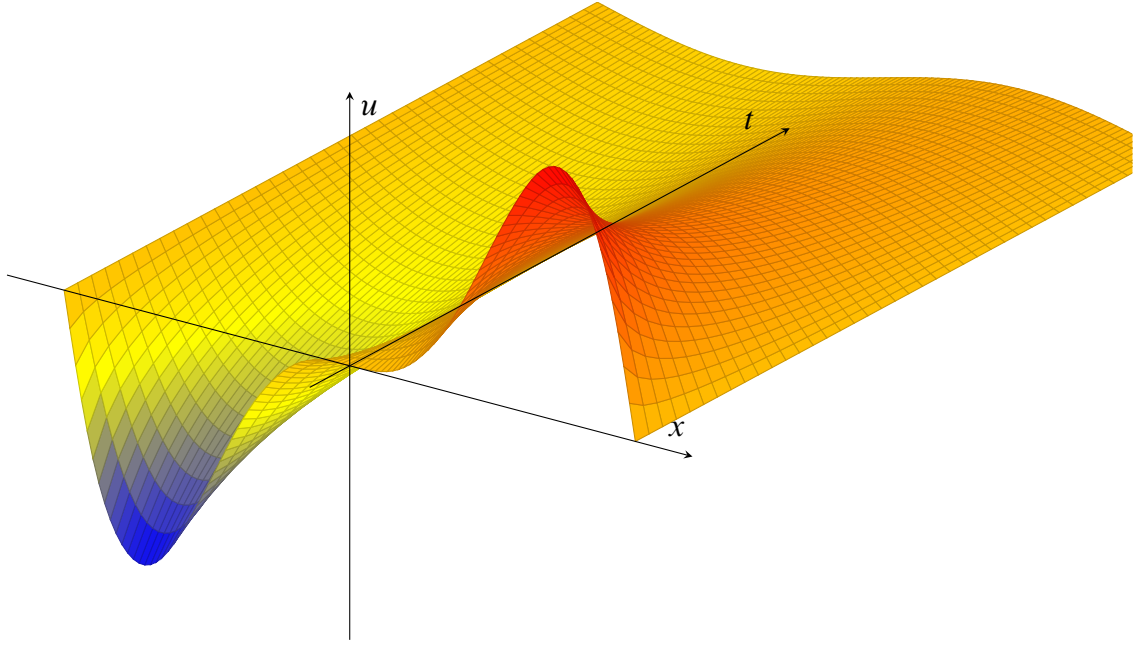
$$c_n = \begin{cases} 0, & n \text{ is odd,} \\ (-1)^m \int_{-1}^1 x^2 \sin \pi x \sin m\pi x dx, & n = 2m, \end{cases}$$

Integrating twice in parts, with the help of the hint provided, we conclude that

$$c_n = \begin{cases} 0, & n \text{ is odd,} \\ -\frac{2\pi^2-3}{6\pi^2}, & n = 2, \\ -\frac{8}{(m^2-1)^2\pi^2}, & n = 2m, m > 1. \end{cases}$$

This is how we get the final result

$$u(t, x) = -\frac{2\pi^2-3}{6\pi^2} e^{-\pi^2 t} \sin \pi(x+1) - \sum_{m=2}^{\infty} \frac{8}{(m^2-1)^2\pi^2} e^{-m^2\pi^2 t} \sin m\pi(x+1).$$



Temperature given by the 10-terms truncated Fourier series

9. Solve, using Fourier Transform, the following initial value problem for the heat equation in the plane

$$\begin{cases} u_t = u_{xx} + u_{yy}, & t > 0, (x, y) \in \mathbb{R}^2, \\ u|_{t=0} = x^2 e^{-x^2-y^2}. \end{cases}$$

Hint: $\int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx = \sqrt{\pi} e^{-\frac{k^2}{4}}.$

Solution: The solution is expressed in terms of the Fourier transform as follows

$$u(t, x, y) = \int_{\mathbb{R}^2} v(t, k, q) e^{i(kx+qy)} dk dq,$$

where $\frac{\partial v}{\partial t} = -(k^2 + q^2)v$ and, consequently, $v(t, k, q) = c(k, q) e^{-(k^2+q^2)t}$. Hence,

$$u(t, x, y) = \int_{\mathbb{R}^2} c(k, q) e^{-(k^2+q^2)t} e^{i(kx+qy)} dk dq.$$

where $c(k, q)$ satisfies

$$c(k, q) = -\frac{d^2}{dk^2} \mathcal{F}(e^{-x^2-y^2}) = -\frac{k^2 - 2}{16\pi} e^{-\frac{k^2}{4}}.$$

Therefore, we conclude that

$$u(t, x, y) = - \int_{\mathbb{R}^2} \frac{k^2 - 2}{16\pi} e^{-\frac{k^2}{4}} e^{-(k^2+q^2)t} e^{i(kx+qy)} dk dq,$$

this is

$$u(t, x, y) = -\frac{\pi}{4} \frac{1}{2\pi} \int_{\mathbb{R}} (k^2 - 2) e^{-k^2(t+\frac{1}{4})} e^{ikx} dk \frac{1}{2\pi} \int_{\mathbb{R}} e^{-q^2(t+\frac{1}{4})} e^{iqy} dq,$$

from where, using the basic properties of the Fourier transform, we obtain

$$u(t, x, y) = \frac{2t(1 + 4t) + 4x^2}{(1 + 4t)^2} e^{-\frac{x^2+y^2}{1+4t}}.$$

10. Solve the following initial and boundary value problem for the heat equation in a 1D bar

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < 1, \\ u|_{t=0} = \sin \pi x, \\ \begin{cases} u_x|_{x=0} = 0, \\ u|_{x=1} = 0. \end{cases} \end{cases}$$

Note that the boundary condition in 0, of Neumann type, models, according to Fourier's law, the thermal insulation of that border. In fact, the Neumann type conditions for the heat equation represent conditions on the heat flow over the boundary, in this case the thermal insulation of the boundary. Therefore, in this problem we study the thermal evolution of a bar with one thermally insulated end and another end at zero temperature.

Solution: In order to apply the EEM we first consider the eigenvalue problem for the differential operator $B = -\frac{d^2}{dx^2}$, $Bw = \lambda w$, $\lambda = k^2$, in the space $\{w \in C^\infty[0, 1], w_x|_{x=0} = 0, w|_{x=1} = 0\}$. It is easy to deduce that the eigenvalues are $\lambda_n = k_n^2$ con $k_n := \frac{(2n+1)\pi}{2}$ and the eigenfunctions $w_n(x) := \cos k_n x$. As the boundary conditions are separated for a regular Sturm–Liouville operator, the spectrum is not degenerated and the eigenfunction set forms an orthogonal basis. Therefore, we can expand the inhomogeneity $\sin \pi x$ on this basis

$$\sin \pi x = \sum_{n=0}^{\infty} c_n \cos \frac{(2n+1)\pi}{2} x,$$

with

$$\begin{aligned} c_n &= \frac{(w_n, \sin \pi x)}{\|w_n\|^2} = 2 \int_0^1 \sin \pi x \cos \frac{(2n+1)\pi}{2} x dx \\ &= -\frac{8}{\pi} \frac{1}{4n^2 + 4n - 3}. \end{aligned}$$

The expansion in eigenfunctions will be

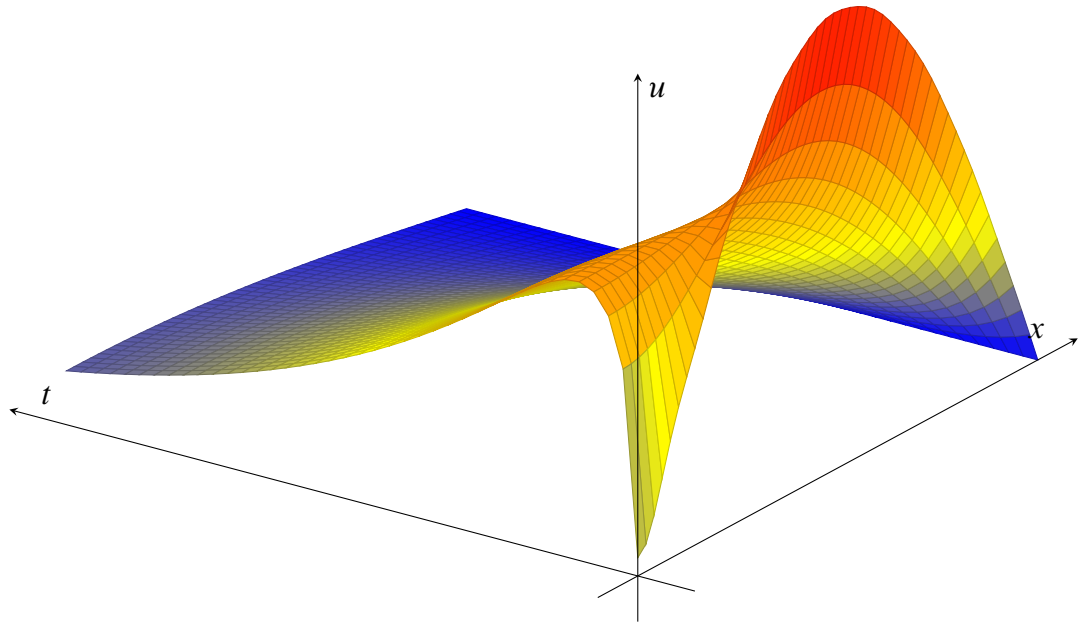
$$u(t, x) = \sum_{n=0}^{\infty} v_n(t) \cos \frac{(2n+1)\pi}{2} x,$$

where $v_n(t)$ solves

$$\begin{aligned} \frac{dv_n}{dt} + \frac{(2n+1)^2 \pi^2}{4} v_n &= 0, \\ v_n(0) &= c_n. \end{aligned}$$

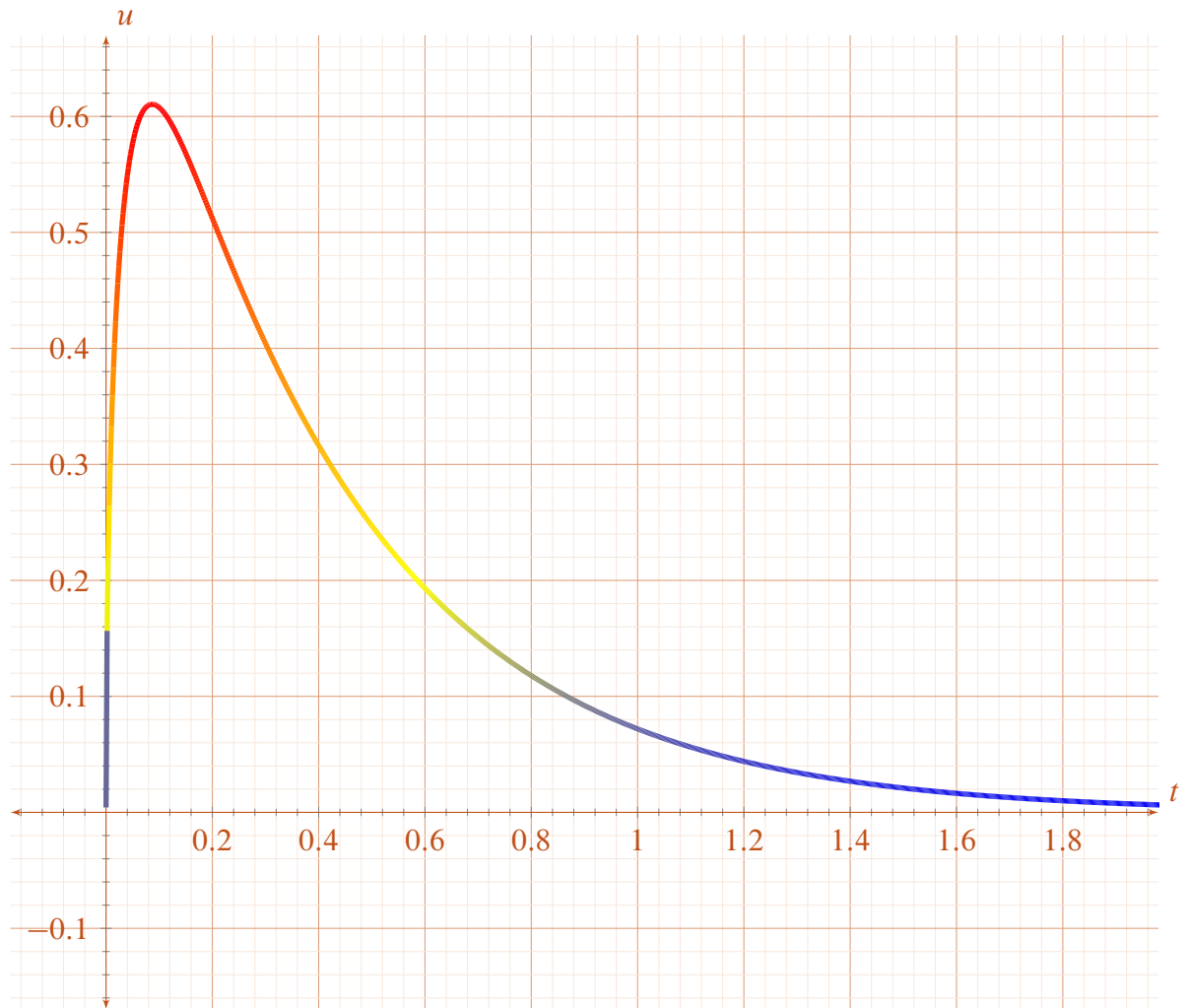
In short, the expansion for the solution is given by

$$u(t, x) = -\frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{4n^2 + 4n - 3} e^{-\frac{(2n+1)^2 \pi^2 t}{4}} \cos \frac{(2n+1)\pi}{2} x.$$



Temperature $u(x, t)$ with Fourier series truncated to 16 terms

The following graph shows the edge temperature with thermal insulation as a function of time.



Fourier Series, truncated to 150 terms, of the temperature $u(t, 0)$ with thermally insulated edge

5.4.2. Exercises

1. Apply EEM to solve the initial and boundary value problem for the vibrating string

$$\left\{ \begin{array}{l} u_{tt} = u_{xx}, \quad 0 < x < 1; \\ \left\{ \begin{array}{l} u|_{x=0} = 0, \\ u|_{x=l} = 0; \end{array} \right. \\ \left\{ \begin{array}{l} u|_{t=0} = g_1(x), \\ u_t|_{t=0} = g_2(x). \end{array} \right. \end{array} \right.$$

Calculate solution expansion for the following initial data

$$g_1(x) = x(x-1), \quad g_2(x) = 0.$$

2. The Fourier transform method applied to the heat equation initial value problem

$$(27) \quad \begin{cases} u_t = a^2 u_{xx}, & -\infty < x < +\infty; \\ u|_{t=0} = f(x), \end{cases}$$

provides the following expression for the solution

$$u(t, x) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2 t}} f(y) \, dy.$$

a) Prove that the following alternative expression holds

$$u(t, x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} f(x + 2a\sqrt{t}s) \, ds.$$

b) Determine the solution of (27) in the following two cases

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{in the remaining,} \end{cases} \quad f(x) = \begin{cases} x, & 0 < x < 1, \\ 0, & \text{in the remaining.} \end{cases}$$

Hint: Use error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds.$$

6. SPECIAL FUNCTIONS

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FOURIER series and Fourier transform are key elements in the resolution of linear PDE with homogeneous boundary conditions, described naturally in Cartesian coordinates, via the EEM. The fundamental reason for this to happen relies in the fact that the corresponding eigenvalue problems are associated with constant coefficients differential operators. But when the boundaries of the regions are not expressed in a simple way in Cartesian coordinates and we must change of variables to curvilinear coordinate systems in which the boundaries are described in a natural manner, then the EEM leads us to use eigenvalue problems associated with differential operators of the Sturm–Liouville type with variable coefficients. This happens, in particular, with problems for the Laplacian with boundaries naturally expressed in cylindrical or spherical coordinates. We are going to study in this chapter the functions that are required in these cases, in particular we will focus on how they are expanded in terms of series around certain points.

§6.1. Frobenius Series. Regular and Singular Regular Points



EIGENVALUE problems for these boundary value problems, which are naturally formulated in the framework of cylindrical or spherical coordinates, requires to solve a certain type of second-order linear ODE of non constant coefficients

$$(28) \quad u'' + a(x)u' + b(x)u = 0,$$

where $a = a(x)$ and $b = b(x)$ are functions of the independent variable x . As we know from the theory of linear ODE. The general solution of (28) is an arbitrary linear combination

$$(29) \quad u(x) = C_1u_1(x) + C_2u_2(x),$$

of any two of their linearly independent solutions.

In this chapter we seek for particular solutions of (28) starting from expansions around a given fixed point x_0 given by

$$(30) \quad u(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 (x - x_0)^\alpha + c_1 (x - x_0)^{\alpha+1} + \dots,$$

where **the exponent** α that appears in the factor $(x - x_0)^\alpha$ **is generally not an integer**, and the coefficients c_n need to be found. These type of expressions are called **Frobenius series**. To study such series expansions it becomes necessary to consider (28) in the complex plane $\mathbb{C}$. Thus, the sought solutions $u = u(z)$ and the coefficients $a = a(z)$ and $b = b(z)$ will be, in our discussion, functions of the complex variable $z \in \mathbb{C}$, while $u' = \frac{d u}{d z}$ denotes the derivative with respect to the complex variable z .

When looking for solutions $u = u(z)$ expressed in Frobenius series we must determine around which type of point (that we will now denote z_0) is expanded in series.

Regular Point

A point $z_0 \in \mathbb{C}$, $z_0 \neq \infty$, is said to be a **regular point** of (28) if the functions $a(z)$ and $b(z)$ are analytical in z_0 . That is to say, if there are expansions in power series

$$\begin{aligned} a(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots, \\ b(z) &= b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \dots, \end{aligned}$$

convergent on some disk centered on z_0 .

Regular Singular Point

A point $z_0 \in \mathbb{C}$, $z_0 \neq \infty$, is said to be a **regular singular point** of (28) if any of the functions $a(z)$ or $b(z)$ is singular in $z = z_0$ and yet the modified functions $(z - z_0)a(z)$ and $(z - z_0)^2 b(z)$ are analytical in z_0 . That is to say, if there are expanded in power series

$$\begin{aligned} (z - z_0)a(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots, \\ (z - z_0)^2 b(z) &= b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \dots, \end{aligned}$$

convergent on some disk centered on z_0 . Observe that we have to demand that some of the coefficients a_0, b_0, b_1 are nonzero, because if all vanish we would be faced to removable singularities, and therefore the functions $a(z)$ and $b(z)$ would be both analytical in $z = z_0$.

We will only discuss ODE in which only these two types of points appear, leaving out of the study the irregular singular points that are those that do not fall into the two previous types.

To **classify** the types of points that arise in a particular equation the only thing we need to look at are the possible singularities of the functions $a(z)$ (coefficient of u') and $b(z)$ (coefficient of u).

Regular points are those where both functions $a(z)$ and $b(z)$ are analytical. Points where at least one of the functions $a(z)$ or $b(z)$ has a pole, of maximum order equal to one for $a(z)$ and of maximum order equal to two for $b(z)$, are **regular singular points**.

Discussion for Relevant Equations

1. Hermite equation

$$u'' - 2zu' + 2vu = 0, \quad v \in \mathbb{Z}.$$

has $a(z) = -2z$ and $b(z) = 2v$, which are analytical functions throughout $\mathbb{C}$, then all points are regular.

2. Airy Equation

$$u'' - zu = 0.$$

In this case $a(z) \equiv 0$ and $b(z) = -z$, which are analytical functions throughout $\mathbb{C}$, then all points are regular.

3. Legendre Equation

$$((1 - z^2)u')' + \lambda u = 0, \quad \lambda \in \mathbb{C}.$$

Solving for the second derivative, it can be written in the form

$$u'' - \frac{2z}{1 - z^2}u' + \frac{\lambda}{1 - z^2}u = 0.$$

Hence

$$a(z) = -\frac{2z}{1 - z^2}, \quad b(z) = \frac{\lambda}{1 - z^2}.$$

The points $z_0 = 1$ and $z_0 = -1$ are simple poles of $a(z)$ and $b(z)$, then they are regular single points. The remaining points are regular.

4. Bessel equation

$$z^2u'' + zu' + (z^2 - v^2)u = 0, \quad v \in \mathbb{C}.$$

Clearing the second derivative leaves us with

$$u'' + \frac{1}{z}u' + \left(1 - \frac{v^2}{z^2}\right)u = 0,$$

so that

$$a(z) = \frac{1}{z}, \quad b(z) = 1 - \frac{v^2}{z^2}.$$

Hence, $z_0 = 0$ is a simple pole of $a(z)$ and a double pole of $b(z)$, then it is the only regular singular point and the remaining points of the complex plane are regular points.

5. Laguerre Equation

$$zu'' + (1 - z)u' + ku = 0, \quad k \in \mathbb{C}.$$

Clearing the second derivative leaves us with

$$u'' + \left(\frac{1}{z} - 1\right)u' + \frac{k}{z}u = 0,$$

so that

$$a(z) = \frac{1}{z} - 1, \quad b(z) = \frac{k}{z}.$$

In this case $z_0 = 0$ is a simple pole of $a(z)$ and $b(z)$, it is the only regular singular point and the rest of points of the complex plane are regular points.

§6.2. Expansions Around Regular Points

By definition, if z_0 is a regular point for the ODE

$$(31) \quad u'' + a(z)u' + b(z)u = 0,$$

then the functions $a(z)$ and $b(z)$ have power series expansions

$$(32) \quad \begin{aligned} a(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \\ b(z) &= b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots. \end{aligned}$$

The following theorem summarizes everything we need to know about the properties of the solutions of (31) about z_0 .

Regular Points

If z_0 is a regular point of (31) then:

1. Any solution u of (31) is analytical in $z = z_0$
2. The radius of convergence of the corresponding power series is greater than or equal to the distance of z_0 to the nearest singularity of the functions $a(z), b(z)$.
3. The coefficients of the power series expansion of a solution of (31)

$$(33) \quad u = \sum_{n \geq 0} c_n (z - z_0)^n = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

can be determined from the expansion coefficients of $a(z)$ and $b(z)$ by substituting the series (32) and (33) in the ODE (31), and identifying coefficients of the powers $(z - z_0)^n$.

We should note that the first part of the theorem implies that the only singularities that a solution u of (31) can have are the singularities of the functions $a(z)$ and $b(z)$. Therefore, if $a(z)$ and $b(z)$ do not have singular points neither will have the solutions of (31). It should also be noted that the theorem provides us with a method for calculating solutions of (28) in the form of a power series around regular points. The following example illustrates that method.

6.2.1. Hermite Equation

The quantum version of the harmonic oscillator, after simplifying the corresponding Schrödinger equation, leads to the following ODE¹.

$$v'' - (z^2 - 2v - 1)v = 0$$

that once performed the change $u = e^{z^2/2} v$ becomes the Hermite's ODE²

$$(34) \quad u'' - 2zu' + 2vu = 0,$$

¹View solved problems

²That can be written in the form of eigenvalue problem for an operator of Sturm–Liouville (not regular for having a domain not delimited) in the following way: $-\frac{1}{e^{-z^2}}(e^{-z^2}u')' = 2vu$, with $\rho(x) = p(x) = e^{-x^2}$ and $q(x) = 0$.

This equation fits in our discussion with the identification $a(z) = -2z$ and $b(z) = 2v$. Let's take $z_0 = 0$. From the previous theorem we deduce that the power series of the solutions converge in the whole complex plane, because the functions $a(z), b(z)$ do not have singularities. We sought for a solution in the form of a power series expansion centered at $z_0 = 0$

$$u = \sum_{n \geq 0} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots,$$

that once introduced in (34) and given that the functions $a(z)$ and $b(z)$ are already expanded in powers of z , leads to

$$\sum_{n=0}^{\infty} ((n+2)(n+1)c_{n+2} - 2(n-v)c_n)z^n = 0.$$

Identifying coefficients of the powers z^n we find

$$(n+2)(n+1)c_{n+2} - 2(n-v)c_n = 0, \quad n \geq 0,$$

from which we deduce the recurrence relation for the unknown coefficients of u :

$$(35) \quad c_{n+2} = -\frac{2(v-n)}{(n+2)(n+1)}c_n.$$

Thus, each coefficient c_n determines the coefficient c_{n+2} and, therefore, by iterating this process we obtain that

$$\begin{aligned} c_{2n} &= \frac{(-1)^n 2^n}{(2n)!} v(v-2) \cdots (v-2n+2) c_0, \\ c_{2n+1} &= \frac{(-1)^n 2^n}{(2n+1)!} (v-1)(v-3) \cdots (v-2n+1) c_1. \end{aligned}$$

In conclusion, taking on one side $(c_0, c_1) = (1, 0)$ and on the other $(c_0, c_1) = (0, 1)$ we get two linearly independent solutions of the Hermite equation given by

$$\begin{aligned} u_0(z) &= \sum_{n \geq 0} c_{2n} z^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} v(v-2) \cdots (v-2n+2) z^{2n}, \\ u_1(z) &= \sum_{n \geq 0} c_{2n+1} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} (v-1)(v-3) \cdots (v-2n+1) z^{2n+1}. \end{aligned}$$

Hermite Polynomials It follows from (35) that if a coefficient c_n of the solution series is zero then all c_{n+2k} with $k \geq 0$ are also zero. For n an even number, we would get that $u_0(x)$ is a polynomial and if n is odd it would imply that $u_1(x)$ is a polynomial. In view of this observation and of (35) it is clear that for $v = 2k$ and $v = 2k+1$ the solutions $u_0(z)$ and $u_1(z)$ reduce to polynomial functions. These polynomials are proportional to the so-called Hermite polynomials.

Hermite Polynomials

The Hermite polynomials can be determined by the *Rodrigues' formula*

$$H_n = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$

The first Hermite polynomials are

$$\begin{aligned} H_0(z) &= 1, & H_1(z) &= 2z, & H_2(z) &= 4z^2 - 2, \\ H_3(z) &= 8z^3 - 12z, & H_4(z) &= 16z^4 - 48z^2 + 12, & H_5(z) &= 32z^5 - 160z^3 + 120z. \end{aligned}$$

Hermite Polynomials Properties

1. **Recurrence Formula:** $H_n(z) = 2zH_{n-1}(z) - 2(n-1)H_{n-2}(z)$.
2. **Generating Function:** $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$.
3. **ODE:** $H_n'' - 2zH_n' + 2nH_n = 0$.
4. **Orthogonality:** $\int_{\mathbb{R}} H_n(x)H_m(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{n,m}$.
5. **Completeness** The Hermite polynomial set $\{H_n(x)\}_{n=0}^{\infty}$ is a **complete orthogonal set** in $L^2_{e^{-x^2} dx}(\mathbb{R})$.

What happens to the solutions $u_0(z)$ and $u_1(z)$ when $v \notin \{0, 1, 2, \dots\}$? From Fuchs' theorem, we know that both solutions are analytical in $\mathbb{C}$, therefore perfectly *reasonable*, but as we well know, from Liouville's theorem, at $z = \infty$ a singularity will appear. For applications, particularly in the context of Quantum Physics, we need solutions $u_0(x)$ and $u_1(x)$ that belong to $L_{e^{-x^2} dx}(\mathbb{R}) = \{f(x) : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx < \infty\}$, that is satisfied if the series end (are truncated) and are simply polynomials. Let's see that this is not the case when series do not truncate and have an infinite number of non-vanishing terms, in which case $u_0(x), u_1(x) \notin L_{e^{-x^2} dx}(\mathbb{R})$. Although we are not going to prove it, we are going to give a plausibility argument, which can be made mathematically rigorous. Let's compare the coefficients c_n of $u_0(z)$ with those of e^{z^2} when $n \rightarrow \infty$. For $a > 0$, we have the following power series

$$e^{a^2 z^2} = \sum_{n=0}^{\infty} C_n z^n, \quad C_n = \begin{cases} \frac{a^2}{\frac{n}{2}!}, & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

whose even coefficients C_n have the following quotient *à la* d'Alambert (for odd n the quotient does not make sense)

$$\frac{C_{n+2}}{C_n} = \frac{a^2}{\frac{n}{2} + 1}$$

which behaves as follows for n large and even

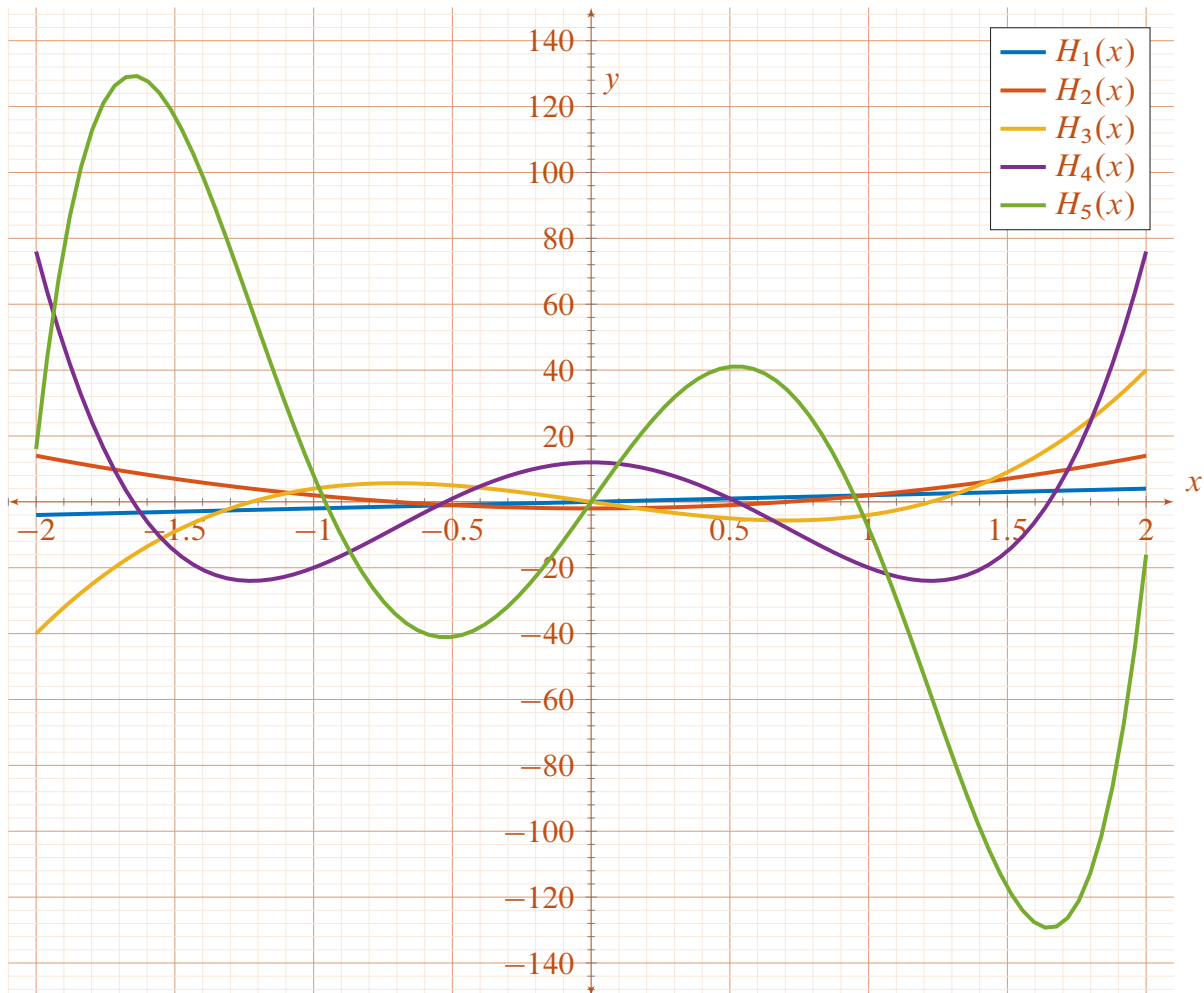
$$\frac{C_{n+2}}{C_n} \sim \frac{2a^2}{n}, \quad n \rightarrow \infty.$$

But, for the even coefficients c_n (the only nonzero ones) of the power series $u_0(x)$ we have

$$\begin{aligned} \frac{c_{n+2}}{c_n} &= \frac{2(\nu - n)}{(n+2)(n+1)} \\ &\sim \frac{2}{n}, \quad n \rightarrow \infty. \end{aligned}$$

Therefore, it is reasonable to suppose that the power series of $u_0(x)$ behaves like the power series for e^{x^2} , except, of course, that ν is an even number. So, when ν is not an even number, on the real line for x large we have $|u_0(x)|^2 \sim e^{2x^2}$, so that $u_0(x) \notin L_{e^{-x^2}dx}(\mathbb{R})$. A similar argument leads that, when ν is not an odd number, that $|u_1(x)|^2 \sim x^2 e^{2x^2}$ and $u_1(x) \notin L_{e^{-x^2}dx}(\mathbb{R})$.

Hermite Polynomials



The Hermite polynomials not only appear in Quantum Physics when the quantum harmonic oscillator is solved, as we have seen in one of the solved problems of the previous unit, but also

in other fields of Mathematics such as: probability theory, combinatorics, numerical analysis and system theory.

6.2.2. Legendre Equation We have previously seen that the Legendre equation³

$$u'' - \frac{2z}{1-z^2}u' + \frac{\lambda}{1-z^2}u = 0,$$

has as regular singular points $z_0 = \pm 1$, all other points being regular. Let's take the origin $z_0 = 0$ as the point where the expansions are centered. Every solution admits a power series expansion

$$u = \sum_{n \geq 0} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots,$$

whose convergence radius will be greater than or equal to 1. To determine the coefficients let us write the Legendre equation in the form

$$(1-z^2)u'' - 2zu' + \lambda u = 0.$$

Substituting now the series of u in the Legendre equation and grouping coefficients in powers z^n we get

$$\sum_{n \geq 0} ((n+2)(n+1)c_{n+2} - n(n-1)c_n - 2nc_n + \lambda c_n)z^n = 0.$$

Which gives us the following recurrence

$$(36) \quad c_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} c_n, \quad n \geq 0.$$

Taking the two options $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, we obtain two linearly independent solutions of the Legendre equation

$$u_0(z) = \sum_{n \geq 0} c_{2n} z^{2n}, \quad u_1(z) = \sum_{n \geq 0} c_{2n+1} z^{2n+1}.$$

Legendre Polynomials Based on (36) we can now make a reasoning similar to what we have done with the Hermite functions and conclude that if we take

$$\lambda = n(n+1),$$

for a certain fixed n , then it is verified:

1. If n is even ($n = 2k$) all coefficients c_{2m} ($m > k$) are cancelled and therefore $u_0(z)$ is a polynomial.
2. If n is odd ($n = 2k + 1$) all coefficients c_{2m+1} ($m > k$) are cancelled and therefore $u_1(z)$ is a polynomial.

The polynomials obtained in this way are the **Legendre polynomials**.

³As an eigenvalue problem for a Sturm–Liouville type operator it is written in the form $-((1-x^2)u')' = \lambda u$, with $\rho = 1$, $p(x) = 1-x^2$ and $q(x) = 0$, that is not regular, given the behavior in $x = \pm 1$.

Legendre Polynomials

Legendre polynomials can be determined using the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n, \quad n \geq 0.$$

Legendre polynomials are calculated immediately and the first turn out to be

$$\begin{aligned} P_0(z) &= 1, & P_1(z) &= z, & P_2(z) &= \frac{1}{2}(3z^2 - 1), \\ P_3(z) &= \frac{1}{2}(5z^3 - 3z), & P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), & P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z) \end{aligned}$$

Properties of Legendre Polynomials

1. **Bonnet Recurrence Formula:** $(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z)$.
2. **Generating function:** $\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ for $|t| < 1$ and $|x| \leq 1$.
3. **Second order ODE:** $((1-z^2)P'_n)' + n(n+1)P_n = 0$.
4. **Orthogonality:** $\int_{-1}^1 P_n(x)P_m(x) dx = \frac{1}{n + \frac{1}{2}} \delta_{n,m}$.
5. **Completeness:** The set $\{P_n(x)\}_{n=0}^{\infty}$ is a complete orthogonal set in $L^2[-1, 1]$.

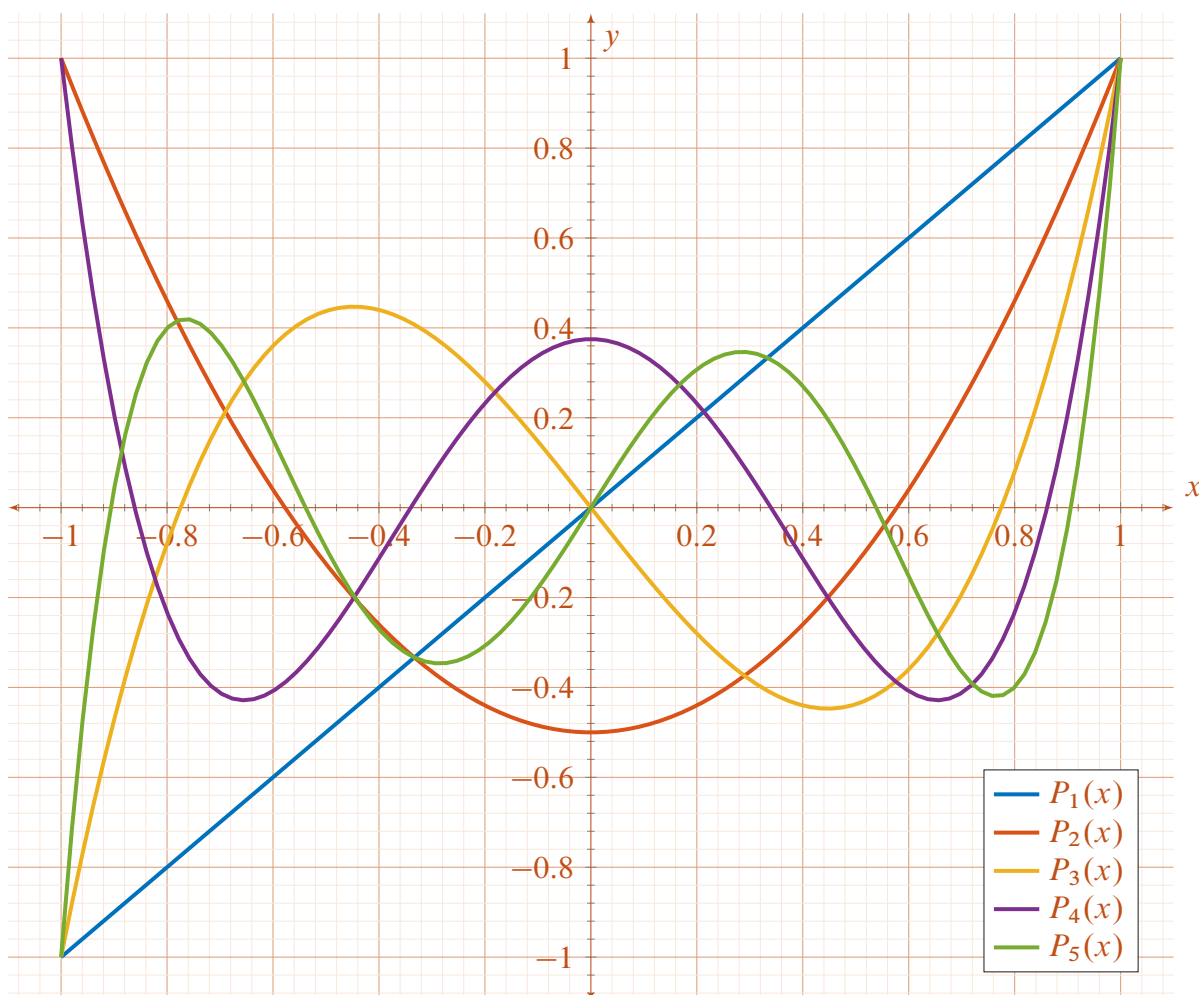
What about the solutions $u_0(z)$ y $u_1(z)$ when $\lambda \neq n(n+1)$, $n \in \{0, 1, 2, \dots\}$? In this case the discussion is simpler than the case of Hermite. We know that our solutions $u_0(z)$ and $u_1(z)$ converge uniformly inside the unit disk. However, when n is even the $u_0(z)$ series ends and is a polynomial, this is $u_0(z)$ is defined in all $\mathbb{C}$, and when n is odd the $u_1(z)$ series ends and is also a polynomial and, therefore, is again defined in all $\mathbb{C}$. When the series do not end it is easy to realize that the convergence radius of $u_0(z)$ and $u_1(z)$ is 1, indeed.

Applications Legendre polynomials appear when Newtonian or Coulombian potential is expanded in the form

$$\frac{1}{r - r_0} = \frac{1}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta}} = \begin{cases} \frac{1}{r} \sum_{n=0}^{\infty} \frac{r_0^n}{r^n} P_n(\cos \theta), & |r| > |r_0|, \\ \frac{1}{r_0} \sum_{n=0}^{\infty} \frac{r^n}{r_0^n} P_n(\cos \theta), & |r| < |r_0|, \end{cases}$$

here $\theta = \widehat{rr_0}$ is the latitude of the point r with respect to the axis determined by the vector $r - r_0$. This series applies when considering the multi-polar expansion of a point load outside the origin.

Legendre Polynomials



§6.3. Expansions Around Regular Singular Points

6.3.1. Euler equation The Euler equation serves as a toy model to understand what we can expect from solutions around regular singular points. The ODE is

$$(37) \quad u'' + \frac{q_1}{z}u' + \frac{q_0}{z^2}u = 0$$

where $q_1, q_0 \in \mathbb{C}$ are two nonzero complex numbers. Clearly $z_0 = 0$ is a regular singular point. Multiplying by z^2 the Euler equation becomes

$$z^2 u'' + q_1 z u' + q_0 u = 0,$$

or equivalently

$$Lu = 0,$$

where L is the differential operator

$$L = z^2 \frac{d^2}{dz^2} + q_1 z \frac{d}{dz} + q_0.$$

We should note that

$$Lz^\alpha = P(\alpha)z^\alpha, \quad P(\alpha) := \alpha(\alpha - 1) + q_1\alpha + q_0,$$

This equation tells us that z^α is always an eigenfunction of L with eigenvalue $\lambda = P(\alpha)$, and that $u = z^\alpha$ is a solution to Euler's equation when $P(\alpha) = 0$. The function $P(\alpha)$ is a polynomial of order 2 and will therefore have two roots counting multiplicities. If the two roots α_1, α_2 are different, then z^{α_1} and z^{α_2} are linearly independent solutions and a general solution of the Euler equation is

$$u = c_1 z^{\alpha_1} + c_2 z^{\alpha_2}.$$

The case gets involved if the indicial polynomial has a double root λ_1 , and a solution is $u = z^{\alpha_1}$ so we need to find another independent solution. A method to find another linearly independent solution is as follows. Since the root is double we must have $P(\alpha) = (\alpha - \alpha_1)^2$ and, consequently, $\left. \frac{dP}{d\alpha} \right|_{\alpha=\alpha_1} = 0$. Now, if we derive with respect to α the equation

$$Lz^\alpha = P(\alpha)z^\alpha,$$

we get

$$(38) \quad \frac{\partial L(z^\alpha)}{\partial \alpha} = L\left(\frac{dz^\alpha}{d\alpha}\right) = L(z^\alpha \log z) = P'(\alpha)z^\alpha + P(\alpha)z^\alpha \log z.$$

Since α_1 is a double root we have

$$P(\alpha_1) = P'(\alpha_1) = 0,$$

then, (38) for $\alpha = \alpha_1$ reduces to

$$L(z^{\alpha_1} \log z) = 0,$$

that is, $u = z^{\alpha_1} \log z$ is another solution to the equation. Summarizing, $\{z^{\alpha_1}, z^{\alpha_1} \log z\}$ is a system of two linearly independent solutions and a general solution is

$$u = c_1 z^{\alpha_1} + c_2 z^{\alpha_1} \log z.$$

We have to observe the appearance of possible powers of non-integer exponent and of the logarithm function in the expression of the solutions. This fact illustrates what we can expect in this type of situations.

6.3.2. Frobenius Series Given a regular singular point $z_0 \in \mathbb{C}$, a solution u of (28) is not always analytic on a disk centered on z_0 , because it can have a singularity, a pole or logarithmic branch. In this case the solution does not have a power series expansion around z_0 but a more complicated form. Fuchs found that there is always at least one solution in the form of a factor $(z - z_0)^\alpha$, where α is generally a certain complex number not always integer, multiplying a power series centered on z_0 . That is to say of the form

$$(39) \quad u = (z - z_0)^\alpha \sum_{n \geq 0} c_n (z - z_0)^n = c_0 (z - z_0)^\alpha + c_1 (z - z_0)^{\alpha+1} + \cdots, \quad c_0 \neq 0.$$

This type of expansion is called **Frobenius series**. On the other hand, it also shows that there is always another solution whose form is either of Frobenius type (39) or else in the form

$$(40) \quad u = (z - z_0)^{\alpha_1} \log(z - z_0) \sum_{n \geq 0} b_n (z - z_0)^n + (z - z_0)^{\alpha_2} \sum_{n \geq 0} c_n (z - z_0)^n,$$

where the power series converge on a disk with a radius greater than or equal to the distance of z_0 to the nearest singularity of $a(z), b(z)$.



Example: Consider Euler's equation

$$u'' + \frac{1}{4z^2}u = 0,$$

that has a regular singular point in $z_0 = 0$. If we look for a solution in the form of a power series around $z_0 = 0$

$$u = c_0 + c_1 z + \cdots,$$

by replacing the series in the equation we get

$$(n - 1/2)^2 c_n = 0, \quad n = 1, 2, \dots$$

which implies that $c_n = 0$ and therefore $u \equiv 0$, the trivial solution. Being this an Euler equation we know that a general solution is

$$u = c_1 \sqrt{z} + c_2 \sqrt{z} \log z.$$

The solution u_1 as a Frobenius series Suppose that z_0 is a regular singular point of

$$(41) \quad u'' + a(z)u' + b(z)u = 0.$$

According to the definition of regular singular point, there are in this case expansions of the form

$$(42) \quad \begin{aligned} A(z) &:= (z - z_0)a(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \\ B(z) &:= (z - z_0)^2 b(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots. \end{aligned}$$

To try to determine a solution in the Frobenius series form of (39) with $c_0 = 1$, we start by multiplying the equation (41) by $(z - z_0)^2$ and writing it in the equivalent form.

$$(z - z_0)^2 u'' + (z - z_0)A(z)u' + B(z)u = 0.$$

Substituting now in this equation the Frobenius series (39), with unknown exponent and coefficients, we find when grouping the coefficients of the powers in $z - z_0$ that is written in the form

$$(43) \quad (z - z_0)^\alpha P(\alpha) + \sum_{n \geq 1} (z - z_0)^{\alpha+n} Q_n(\alpha) = 0,$$

being

$$(44) \quad P(\alpha) = \alpha^2 + (a_0 - 1)\alpha + b_0,$$

the **indicial polynomial**, and

$$\begin{aligned} Q_n(\alpha) &= (\alpha(\alpha - 1) + 2\alpha n + n(n - 1) + \alpha a_0 + n a_0 + b_0)c_n + \sum_{k=0}^{n-1} ((\alpha + k)a_{n-k} + b_{n-k})c_k \\ &= P(\alpha + n)c_n + \sum_{k=0}^{n-1} ((\alpha + k)a_{n-k} + b_{n-k})c_k \end{aligned}$$

Now let's identify power coefficients in (43). By canceling the coefficient of $(z - z_0)^\alpha$ we see that α must be the root of the indicial polynomial

$$\boxed{P(\alpha) = 0},$$

which is called **indicial equation**, which being a second-degree polynomial will have two roots $\alpha = \alpha_1, \alpha_2$, generally complex, and that we can order assuming

$$\operatorname{Re} \alpha_1 \geq \operatorname{Re} \alpha_2.$$

Non-resonant case. Suppose the indicial polynomial (44) has two roots such that its difference is not integer,

$$(45) \quad \alpha_1 - \alpha_2 \notin \mathbb{Z},$$

that is, let's analyze the non-resonant case. Let's take $\alpha = \alpha_1$ in our calculation, we can equate to zero in (43) the coefficients of $(z - z_0)^{\alpha+n}$ ($n \geq 1$) and find the relationships

$$(46) \quad P(\alpha_1 + n)c_n = - \sum_{k=0}^{n-1} ((\alpha_1 + k)a_{n-k} + b_{n-k})c_k, \quad n \geq 1.$$

But $P(\alpha_1 + n) \neq 0$ for all $n = 1, 2, \dots$. Indeed, let us assume the contrary. If there was a $n_0 = 1, 2, \dots$ such that $P(\alpha_1 + n_0) = 0$, since $P(\alpha)$ only has two roots, then $\alpha_2 = \alpha_1 + n_0$ which would be in contradiction with (45). In this manner (46) gives us each coefficient c_n in terms of the previous ones $c_0 = 1, c_2, \dots, c_{n-1}$. In other words, we completely determine a solution in the form of a Frobenius series.

If we now take $\alpha = \alpha_2$ in our calculation, we come back to (46) with α_2 instead of α_1 . It is therefore clear that if $P(\alpha_2 + n) \neq 0$ for all $n = 1, 2, \dots$ (which is equivalent to saying that $\alpha_1 - \alpha_2 \neq n, n = 1, 2, \dots$), then the method provides us with another solution in the form of Frobenius series, now with exponent $\alpha = \alpha_2$.

Resonant Case. We'll discuss only the case where $\alpha_1 = \alpha_2$. Let's proceed as in the study of the Euler equation by introducing the differential operator

$$L = (z - z_0)^2 \frac{d^2}{dz^2} + (z - z_0)A(z) \frac{d}{dz} + B(z).$$

Let's make L act on a Frobenius series (39) with variable exponent α and whose coefficients $c_n = c_n(\alpha)$ depend on α except $c_0 = 1$, so we get

$$(47) \quad Lu = (z - z_0)^\alpha P(\alpha) + \sum_{n \geq 1} (z - z_0)^{\alpha+n} Q_n(\alpha),$$

where $P(\alpha)$ and $Q_n(\alpha)$ are the same functions that appear in (43). That is, the **indicial polynomial** is

$$(48) \quad \boxed{P(\alpha) = \alpha^2 + (a_0 - 1)\alpha + b_0},$$

and

$$\begin{aligned} Q_n(\alpha) &= (\alpha(\alpha - 1) + 2\alpha n + n(n - 1) + \alpha a_0 + n a_0 + b_0)c_n + \sum_{k=0}^{n-1} ((\alpha + k)a_{n-k} + b_{n-k})c_k \\ &= P(\alpha + n)c_n + \sum_{k=0}^{n-1} ((\alpha + k)a_{n-k} + b_{n-k})c_k \end{aligned}$$

For $\alpha = \alpha_1$, as there are no more roots we can build a solution in a recurrent form imposing $Q_n(\alpha_1) = 0$, which will give a solution $u_0(z)$.

For any $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha > \operatorname{Re} \alpha_1 - 1$ we have $P(\alpha + n) \neq 0$ for all $n = 1, 2, \dots$ and we can determine all coefficients $c_n(\alpha)$ through the recurrence relation

$$P(\alpha + n)c_n = - \sum_{k=0}^{n-1} ((\alpha + k)a_{n-k} + b_{n-k})c_k, \quad n = 1, 2, \dots,$$

and the corresponding series fulfills

$$Lu = (z - z_0)^\alpha P(\alpha).$$

Taking the derivative with respect to α in this last equation we obtain

$$L\left(\frac{\partial u}{\partial \alpha}\right) = (z - z_0)^\alpha P'(\alpha) + (z - z_0)^\alpha \log(z - z_0)P(\alpha),$$

and, given that $P(\alpha) = (\alpha - \alpha_1)^2$ and $P'(\alpha) = 2(\alpha - \alpha_1)$ vanish at $\alpha = \alpha_1$, we get

$$L\left(\frac{\partial u}{\partial \alpha}\right)\bigg|_{\alpha=\alpha_1} = 0.$$

We conclude that $\frac{\partial u}{\partial \alpha}\bigg|_{\alpha=\alpha_1}$ is a second solution of the form (28)

$$u_1(z) = u_0(z) \log(z - z_0) + (z - z_0)^{\alpha_1} \sum_{n=1} c'_n (z - z_0)^n,$$

where $u_0(z)$ is the Frobenius series solution corresponding to α_1 .

Fuchs Theorem

Let α_1 and α_2 be the roots of the indicial polynomial ordered so that $\operatorname{Re} \alpha_1 \geq \operatorname{Re} \alpha_2$, then we have the following alternatives for the solutions of (41):

1. If $\alpha_1 - \alpha_2 \neq 0, 1, 2, \dots$, there are two linearly independent solutions in the form of Frobenius series

$$u_1(z) = (z - z_0)^{\alpha_1} \sum_{n \geq 0} c_n (z - z_0)^n, \quad u_2(z) = (z - z_0)^{\alpha_2} \sum_{n \geq 0} c'_n (z - z_0)^n,$$

with $c_0 = c'_0 = 1$.

2. If $\alpha_1 - \alpha_2 = 0$, that is to say if $\alpha_1 = \alpha_2$, there are two linearly independent solutions of the form

$$u_1(z) = (z - z_0)^\alpha \sum_{n \geq 0} c_n (z - z_0)^n, \quad u_2(z) = u_1(z) \log(z - z_0) + (z - z_0)^\alpha \sum_{n \geq 0} c'_n (z - z_0)^n,$$

where $\alpha := \alpha_1 = \alpha_2$ and $c_0 = 1, c'_0 = 0$.

3. If $\alpha_1 - \alpha_2 = 1, 2, 3, \dots$, there are two linearly independent solutions of the form

$$u_1(z) = (z - z_0)^{\alpha_1} \sum_{n \geq 0} c_n (z - z_0)^n, \quad u_2(z) = c u_1(z) \log(z - z_0) + (z - z_0)^{\alpha_2} \sum_{n \geq 0} c'_n (z - z_0)^n,$$

where $c_0 = 1, c'_0 = 1$ and c is a constant that can be zero.

The radii of convergence of the power series involved in the solutions u_1 and u_2 is at least the distance to the nearest singularity.

§6.4. Bessel Equation

Let's consider the Bessel equation

$$(49) \quad z^2 u'' + z u' + (z^2 - \nu^2) u = 0, \quad \nu \in \mathbb{R}, \quad \nu \geq 0.$$

In this case we have $a(z) = \frac{1}{z}$ y $b(z) = 1 - \frac{\nu^2}{z^2}$ and thus $z_0 = 0$ is a regular single point. The indicial polynomial is reduced to

$$P(\alpha) = \alpha^2 - \nu^2 = (\alpha - \nu)(\alpha + \nu).$$

So, since we assume that $\nu > 0$, the two solutions α_1, α_2 of the indicial equation with $\operatorname{Re} \alpha_1 \geq \operatorname{Re} \alpha_2$ are $\alpha_1 = \nu$ and $\alpha_2 = -\nu$.

6.4.1. The solution u_1 and its Frobenius series The Frobenius series corresponding to $\alpha_1 = \nu$ will be of the form

$$u_1 = c_0 z^\nu + c_1 z^{\nu+1} + \dots$$

which introduced in (49) leads to

$$P(\nu + 1)c_1 = 0 \quad \Rightarrow \quad ((\nu + 1)^2 - \nu^2)c_1 = 0, \quad \Rightarrow \quad c_1 = 0$$

and

$$P(\nu + n)c_n + c_{n-2} = 0 \quad \Rightarrow \quad c_n = -\frac{c_{n-2}}{n(n + 2\nu)}.$$

For the corresponding Frobenius series all the coefficients of odd order vanish

$$c_1 = c_3 = \cdots = c_{2n+1} = \cdots = 0,$$

while those of even order satisfy the recurrence relation

$$c_{2k} = -\frac{c_{2(k-1)}}{4k(k+\nu)},$$

that leads to

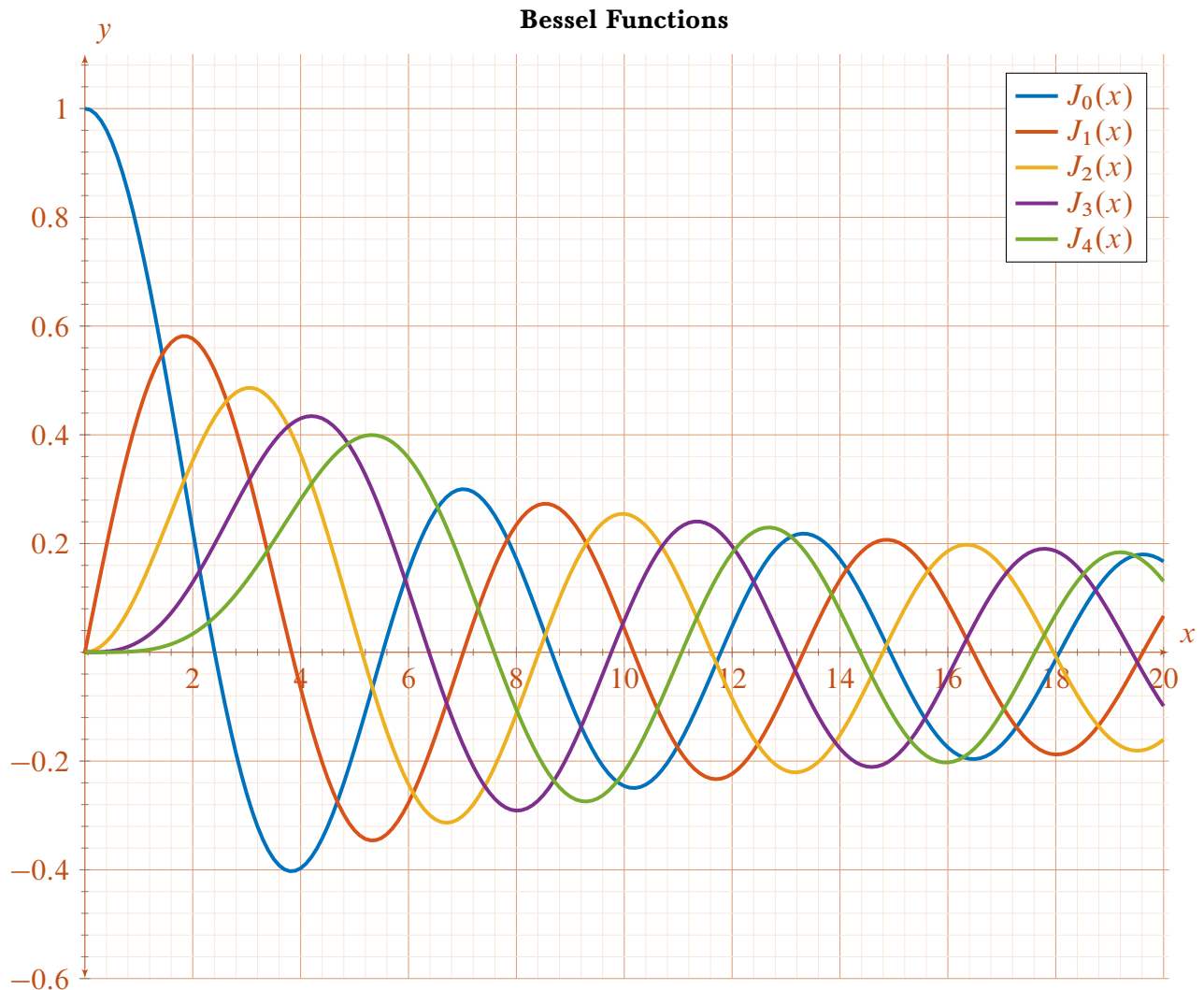
$$c_{2k} = \frac{(-1)^k c_0}{4^k k! (\nu + k) \cdots (\nu + 2)(\nu + 1)}.$$

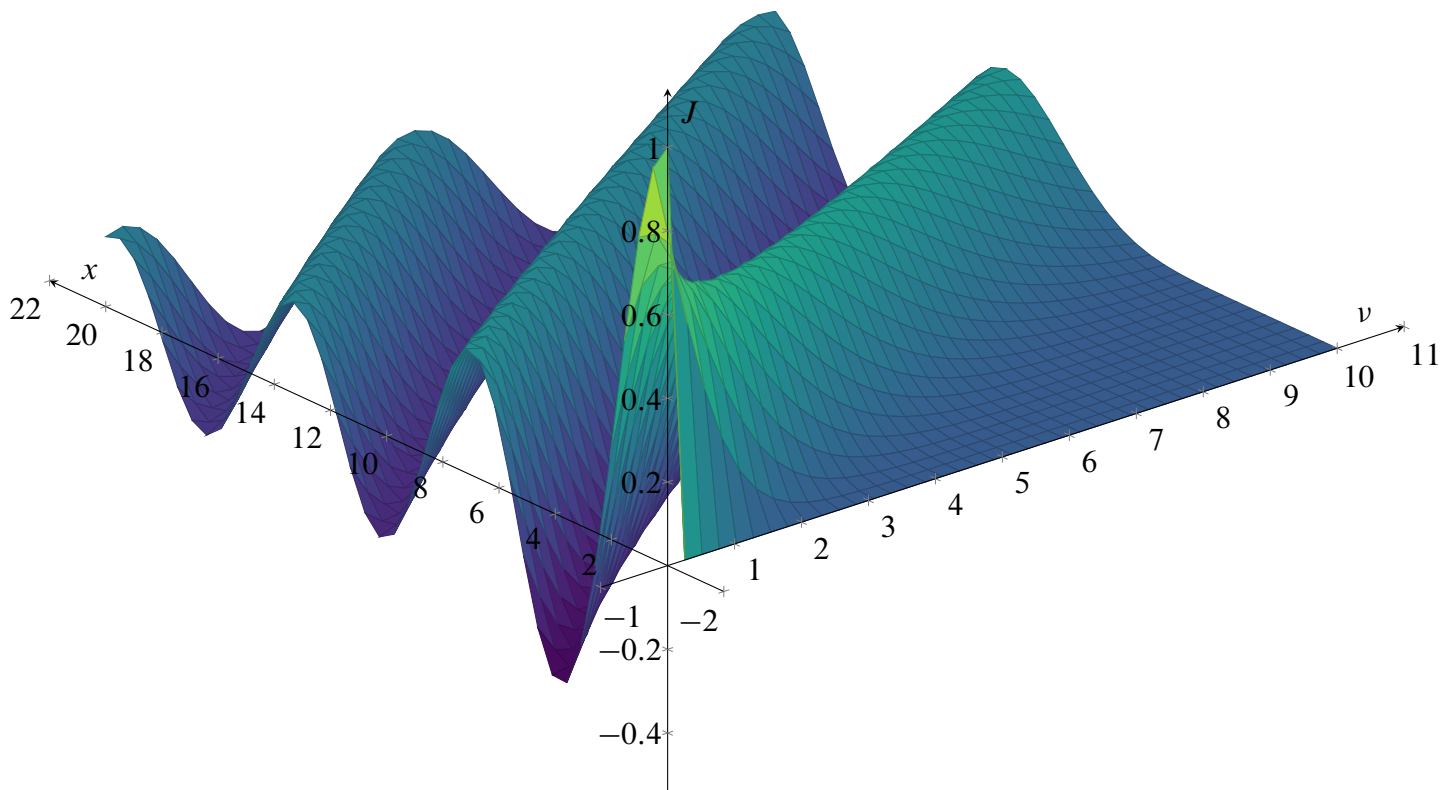
Taking $c_0 = 2^{-\nu}$ then we find the following solution in series form

(50)

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + 1)(\nu + n) \cdots (\nu + 2)(\nu + 1)} \left(\frac{z}{2}\right)^{2n+\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left(\frac{z}{2}\right)^{2n+\nu}$$

which determines the so-called **first kind Bessel's function of order ν** . Note that we have used the function Γ (see the next section of this lecture for a summary of the theory of this function). The series that defines $J_\nu(z)$ converges in the whole complex plane due to the absence of singularities outside the origin of the coefficients $a(z)$ and $b(z)$,





Graph of $F(x, v) = J_v(x)$ as function of (x, v)

6.4.2. The solution u_2 According to the statement of Fuchs' Theorem, to determine the solution u_2 associated with the other root $\alpha_2 = -\nu$ of the indicial polynomial we have to distinguish the following cases according to the values it takes $\alpha_1 - \alpha_2 = 2\nu$

1. $2\nu \neq 0, 1, 2, \dots$
2. $2\nu = 0, 1, 2, \dots$ In this case there are two possible variants
 - a) $\nu = 0, 1, 2, \dots$
 - b) $\nu = 1/2, 3/2, 5/2, \dots$

Case 1: $2\nu \neq 0, 1, 2, \dots$ If $\alpha_1 - \alpha_2 = 2\nu \neq 0, 1, 2, \dots$ alternative 1) of the Fuchs Theorem assures us that the second solution u_2 will be a Frobenius series with exponent $\alpha_2 = -\nu$. In fact, if we repeat the previous calculation of the Frobenius series for $\alpha = \alpha_1 = \nu$ but now with $\alpha = \alpha_2 = -\nu$ we get as a solution u_2 the Bessel function of order $-\nu$

$$J_{-\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n - \nu + 1)} \left(\frac{z}{2}\right)^{2n-\nu}.$$

A general solution of the Bessel equation will then be

$$u(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x),$$

where C_1 and C_2 are two arbitrary constants.

Case 2a: $\nu = 0, 1, 2, \dots$ Now $\alpha_1 - \alpha_2 = 2\nu$ is a nonnegative integer number $0, 2, 4, \dots$ and we have two possibilities

- If $\nu = 0$ then $\alpha_1 = \alpha_2 = 0$, we fall into alternative 2) of the Fuchs Theorem and therefore there is a solution u_2 of the form

$$(51) \quad u_2(z) = J_0(z) \log z + \sum_{n \geq 0} c_n z^n,$$

- If $\nu = m$ con $m = 1, 2, \dots$ then $\alpha_1 = m, \alpha_2 = -m$, we fall into alternative 3) of the Fuchs Theorem and therefore there is a solution u_2 of the type

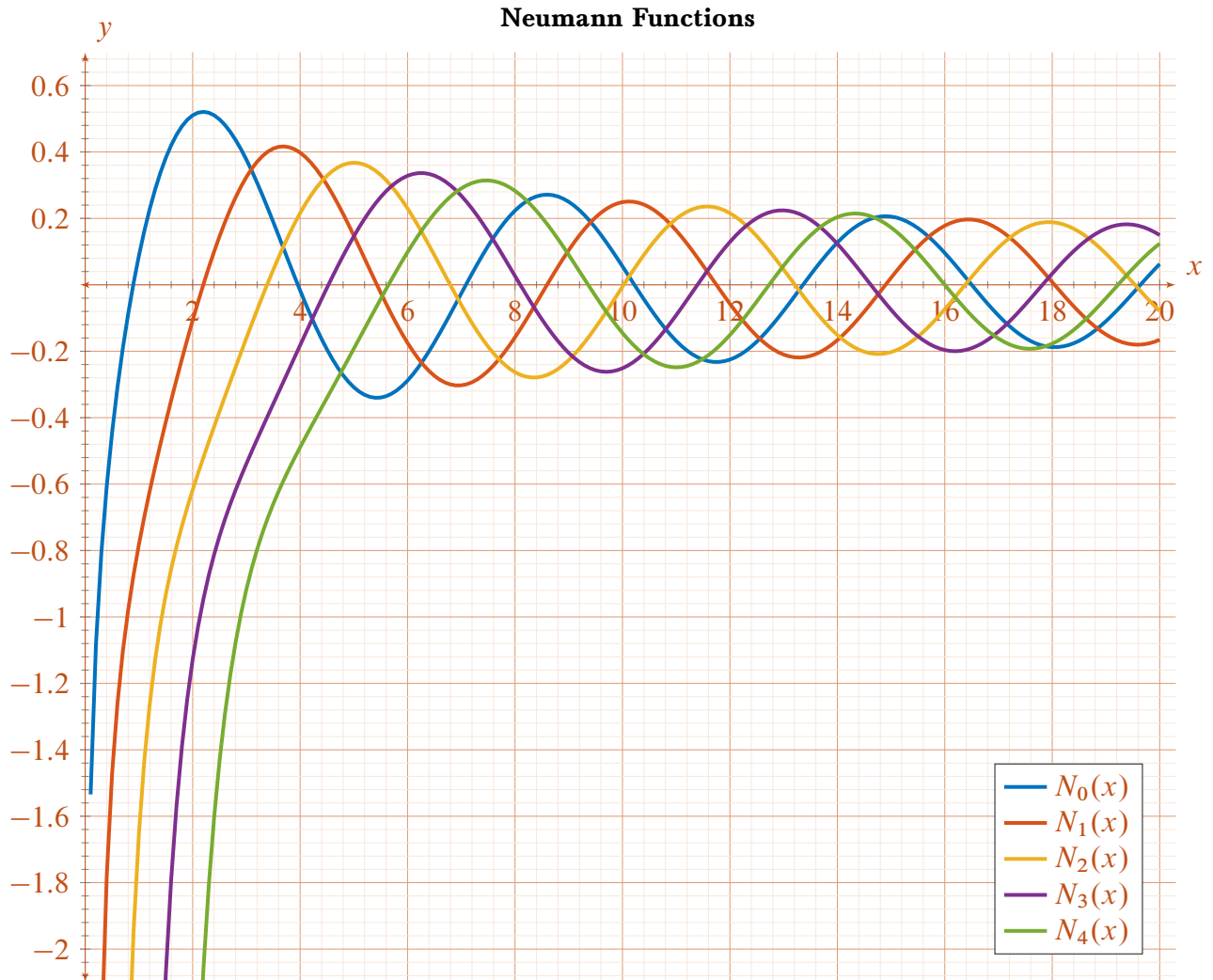
$$(52) \quad u_2(z) = c J_m(z) \log z + z^{-m} \sum_{n \geq 0} c_n z^n,$$

where c could be zero.

The second solution $u_2(x)$ in both cases can be expressed in terms of the so-called Neumann functions

$$(53) \quad N_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin \nu\pi}.$$

Also known as Bessel functions of second kind and denoted alternatively as $Y_\nu(x)$.



For $\nu = 0, 1, 2, \dots$ the denominator in (53) vanishes and the definition (53) is not valid, nevertheless it happens that also the numerator is zero due to the following identity, whose

proof we omit, that satisfies the functions of Bessel of integer order

$$J_{-m}(z) = (-1)^m J_m(z), \quad m = 0, 1, 2, \dots$$

We can then apply the l'Hôpital rule to define the Neumann functions $N_m(z)$ by means of the limit

$$N_m(z) = \lim_{\nu \rightarrow m} N_\nu(z), \quad m = 0, 1, 2, \dots,$$

and thus obtain the desired second solution of the Bessel equation. It can be shown that these functions can be expanded in the form

(54)

$$N_m(z) = \frac{2}{\pi} \left[\log \left(\frac{z}{2} \right) + \gamma \right] J_m(z) - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{\Gamma(m-k)}{\Gamma(k+1)} \left(\frac{z}{2} \right)^{-m+2k} \\ - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^m}{\Gamma(m+k-1)\Gamma(k+1)} \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m+k} \right] \left(\frac{z}{2} \right)^{m+2k},$$

where

$$\gamma = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right] \approx 0.5772156649$$

is the so-called **Euler-Mascheroni constant**. Note that this formula shows that the constant c of the expression (52) is not zero.

In summary, the general solution of the Bessel equation when ν is an integer $m = 0, 1, 2, \dots$ can be written in the form

$$u(x) = C_1 J_m(x) + C_2 N_m(x),$$

where C_1 and C_2 are two arbitrary constants.

The expressions (50) and (54) for functions $J_m(x)$ and $N_m(x)$ allow us to see that their asymptotic behavior when $x \rightarrow 0$ is given by

$$(55) \quad J_m(x) \sim \frac{1}{m!} \frac{x^m}{2^m}, \quad N_m(x) \sim \begin{cases} \frac{2}{\pi} \log \frac{x}{2}, & \text{if } m = 0, \\ -\frac{(m-1)!}{\pi} \frac{2^m}{x^m}, & \text{if } m = 1, 2, \dots \end{cases}$$

The asymptotic behavior when $x \rightarrow \infty$ is deduced from other characterizations of these functions and turns out to be

$$(56) \quad \begin{cases} J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{m\pi}{2} - \frac{\pi}{4} \right), \\ N_m(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{m\pi}{2} - \frac{\pi}{4} \right). \end{cases}$$

Case 2b: $\nu = 1/2, 3/2, 5/2, \dots$ **Spherical Bessel and Neumann Functions** If $\nu = l + \frac{1}{2}, l = 0, 1, 2, \dots$, the difference between roots $\alpha_1 = \nu$ and $\alpha_2 = -\nu$ is a positive integer $\alpha_1 - \alpha_2 = 2l + 1$. We are again within the alternative 3) of the Fuchs Theorem and therefore

$$(57) \quad u_2(z) = c J_{l+\frac{1}{2}}(x) \log z + \frac{1}{z^{l+\frac{1}{2}}} \sum_{n \geq 0} c_n z^n,$$

where c could be zero. In fact, in this case it is verified that $c = 0$, so the solution u_2 does not have a logarithmic term and is reduced to a series of Frobenius that coincides with $J_{-(l+\frac{1}{2})}(x)$. Indeed, $J_{-(l+\frac{1}{2})}(x)$ is the solution of the Bessel equation and is not proportional to $J_{l+\frac{1}{2}}(x)$ because $J_{l+\frac{1}{2}}(0) = 0$ while $J_{-(l+\frac{1}{2})}(x)$ diverge at $x = 0$.

In summary, a general solution of the Bessel equation when $\nu = l + \frac{1}{2}, l = 0, 1, 2, \dots$ can be written in the form $u(x) = C_1 J_{l+\frac{1}{2}}(x) + C_2 J_{-(l+\frac{1}{2})}(x)$, where C_1 and C_2 are two arbitrary constants. An alternative form, using Neumann functions, is

$$u(x) = C_1 J_{l+\frac{1}{2}}(x) + C_2 N_{l+\frac{1}{2}}(x).$$

A modified form of the functions of Bessel and Neumann of semi-integer order are their spherical versions

$$(58) \quad \boxed{j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x), \quad n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x),}$$

that appear in important applications. It should be noted that, using the negative index Bessel functions we have $n_l(x) = (-1)^{n+1} j_{-l}(x)$.

It is immediate to verify that if u is solution of the equation of Bessel of order ν

$$x^2 u'' + x u' + (x^2 - \nu^2) u = 0,$$

then $v = \frac{u}{\sqrt{x}}$ fulfills

$$x^2 v'' + 2x v' + \left(x^2 - \nu^2 + \frac{1}{4}\right) v = 0,$$

that for $\nu = l + \frac{1}{2}$ reduces to

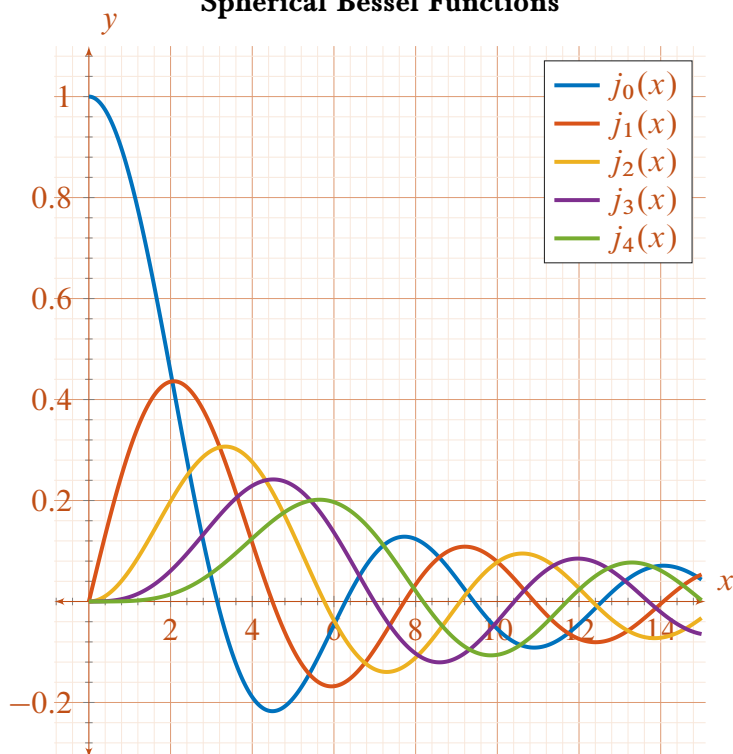
$$(59) \quad x^2 v'' + 2x v' + (x^2 - l(l+1)) u = 0.$$

Therefore, this is the differential equation that satisfies the spherical functions of Bessel and Neumann, so a general solution is $v(x) = C_1 j_l(x) + C_2 n_l(x)$, where C_1 and C_2 are two arbitrary constants. To end our discussion on these spherical functions we will only add that their explicit form turns out to be very simple since they can be determined by means of the following **Rayleigh's formulas**

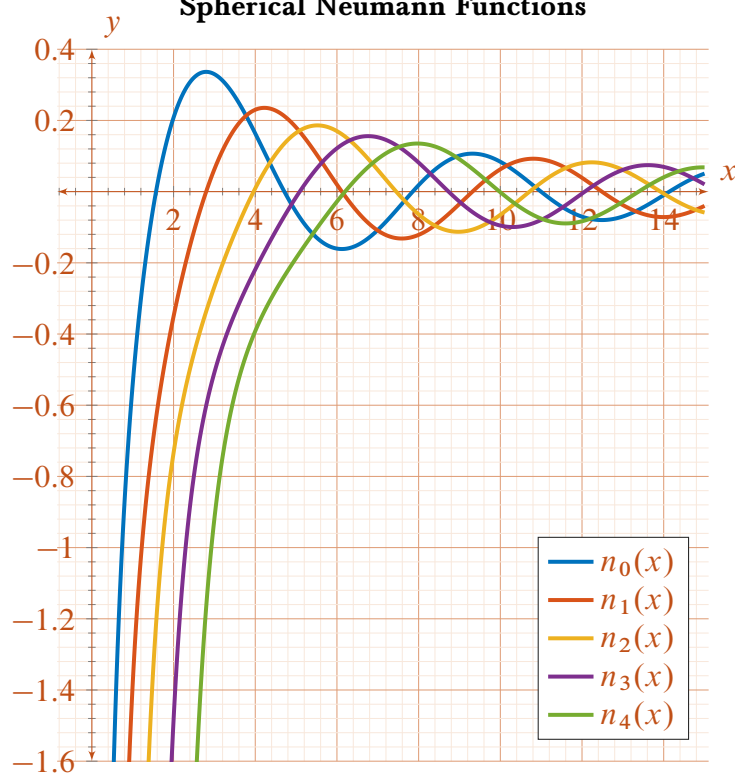
$$(60) \quad \boxed{\begin{cases} j_l(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx}\right)^l \frac{\sin x}{x}, \\ n_l(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx}\right)^l \frac{\cos x}{x}. \end{cases}}$$

The graph of the first spherical functions is shown below

Spherical Bessel Functions



Spherical Neumann Functions



Spherical Bessel Functions

The first spherical Bessel functions are

$$j_0(x) = \frac{\sin x}{x},$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_2(x) = \left(\frac{3}{x^2} - 1\right) \frac{\sin x}{x} - \frac{3 \cos x}{x^2},$$

$$j_3(x) = \left(\frac{15}{x^3} - \frac{6}{x}\right) \frac{\sin x}{x} - \left(\frac{15}{x^2} - 1\right) \frac{\cos x}{x},$$

$$j_4(x) = \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right) \frac{\sin x}{x} - \left(\frac{105}{x^3} - \frac{10}{x}\right) \frac{\cos x}{x}.$$

Spherical Neumann Functions

The first spherical Neumann functions are

$$\begin{aligned} n_0(x) &= -\frac{\cos x}{x}, \\ n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \\ n_2(x) &= -\left(\frac{3}{x^2} - 1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}, \\ n_3(x) &= -\left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\cos x}{x} - \left(\frac{15}{x^2} - 1\right)\frac{\sin x}{x}, \\ n_4(x) &= -\left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\frac{\cos x}{x} - \left(\frac{105}{x^3} - \frac{10}{x}\right)\frac{\sin x}{x}. \end{aligned}$$

Fourier–Bessel Expansions

For $\nu > -1$, let $\{c_{\nu,n}\}_{n=1}^{\infty}$ be the set of infinite zeros of $J_{\nu}(x) = 0$, ordered in a strictly increasing way, $c_{\nu,1} < c_{\nu,2} < \dots$. Then,

$$\{J_{\nu}(c_{\nu,n}x)\}_{n=1}^{\infty}$$

is a complete orthogonal set in $L^2_{x \, dx}[0, 1]$. For any function $f(x) \in L^2_{x \, dx}[0, 1]$, the expansion

$$f(x) = \sum_{n=1}^{\infty} c_n J_{\nu}(c_{\nu,n}x), \quad c_n = \frac{2}{(J_{\nu+1}(c_{\nu,n}))^2} \int_0^1 J_{\nu}(c_{\nu,n}x) f(x) x \, dx,$$

that converges strongly in $L^2_{x \, dx}[0, 1]$, it's called Fourier–Bessel expansion. Note that $(J_{\nu+1}(c_{\nu,n}))^2 = (J'_{\nu}(c_{\nu,n}))^2$.

Fourier–Dini Expansions

Similar to the Fourier–Bessel series, the Fourier–Dini series are built, now let $\{c_{\nu,n}\}_{n=1}^{\infty}$ be the set of infinite zeros of $aJ_{\nu}(x) + xJ'_{\nu}(x)$, ordered in a strictly increasing way, $c_{\nu,1} < c_{\nu,2} < \dots$. Then,

$$\{J_{\nu}(c_{\nu,n}x)\}_{n=1}^{\infty}$$

is a complete orthogonal set in $L^2_{x \, dx}[0, 1]$. Any function in $L^2_{x \, dx}[0, 1]$ admits the following expansion

$$f(x) = \sum_{n=1}^{\infty} c_n J_{\nu}(c_{\nu,n}x), \quad c_n = \frac{2c_{\nu,n}^2}{(c_{\nu,n}^2 - \nu^2 + a^2)(J_{\nu}(c_{\nu,n}))^2} \int_0^1 J_{\nu}(c_{\nu,n}x) f(x) x \, dx,$$

that converges strongly in $L^2_{x \, dx}[0, 1]$, it's called Fourier–Dini expansion

Dirichlet's theorem for Fourier–Bessel and Fourier–Dini series

For every function piecewise C^1 in $[0, 1]$ the corresponding Fourier–Bessel series and Fourier–Dini series converge pointwise in $x \in [0, 1]$ to the average $\frac{1}{2}(f(x_+) + f(x_-))$.

Recursion Relation for Bessel and Neumann Functions

Let's assume that $\operatorname{Re} \nu > -\frac{1}{2}$, then the Bessel and Neumann functions satisfy the following differential relations

$$(x^{\pm \nu} J_\nu(x))' = \pm x^{\pm \nu} J_{\nu \mp 1}(x), \quad (x^{\pm \nu} N_\nu(x))' = \pm x^{\pm \nu} N_{\nu \mp 1}(x),$$

this is

$$J'_\nu(x) = \mp J_{\nu \pm 1}(x) \pm \frac{\nu}{x} J_\nu(x), \quad N'_\nu(x) = \mp N_{\nu \pm 1}(x) \pm \frac{\nu}{x} N_\nu(x),$$

that imply

$$\begin{aligned} \frac{2\nu}{x} J_\nu(x) &= J_{\nu-1}(x) + J_{\nu+1}(x), & \frac{2\nu}{x} N_\nu(x) &= N_{\nu-1}(x) + N_{\nu+1}(x), \\ 2J'_\nu(x) &= J_{\nu-1}(x) - J_{\nu+1}(x), & 2N'_\nu(x) &= N_{\nu-1}(x) - N_{\nu+1}(x), \end{aligned}$$

Generating Functions for the Bessel Functions

1. Generating Function:

$$e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x).$$

2. Jacobi–Anger Expansion:

$$e^{iz \cos(\phi)} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi},$$

$$e^{\pm iz \sin(\phi)} = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) \cos 2n\phi \pm 2i \sum_{n=0}^{\infty} J_{2n+1}(z) \sin (2n+1)\phi.$$

§6.5. The Euler Gamma Function



ULER gamma function is an extension to the complex plane of the factorial of a positive integer, $n! := n(n-1) \cdots 1$ and there are several equivalent ways to define it. In these notes we will present three of them. Firstly, the original Euler definition, due to Legendre, as an infinite limit, then we discuss the the main definition of the Euler gamma function as an integral and finally the Weierstrass definition as an infinite product

■ **Euler's definition as an infinite product.**

$$(61) \quad \Gamma(z) := \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1) \cdots (z+n)} n^z, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \quad n^z = e^{z \log n},$$

we see that

$$(62) \quad \begin{aligned} \Gamma(z+1) &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{(z+1) \cdots (z+1+n)} n^{z+1} = \lim_{n \rightarrow \infty} \frac{zn}{z+1+n} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1) \cdots (z+n)} n^z \\ &= z\Gamma(z). \end{aligned}$$

This difference equation, $\Gamma(z+1) = z\Gamma(z)$, is the basic relation that characterizes Euler's Γ function. It is curious that this Γ function does not satisfy any differential equation as it occurs with other relevant functions of Mathematical-Physics. In fact, as was proven by Otto Hölder in 1887, for every $n \in \mathbb{N}_0$ there is no non-zero polynomial $P \in \mathbb{C}[X; Y_0, Y_1, \dots, Y_n]$ such that

$$P\left(z; \Gamma(z), \Gamma'(z), \dots, \Gamma^{(n)}(z)\right) = 0,$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. That is, the Euler Gamma function does not satisfies an algebraic ODE.

Also, from the definition (61) you can get the value of the function in $z = 1$;

$$\Gamma(1) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{2 \cdots (1+n)} n = 1$$

and from (62) we get

$$\Gamma(2) = 1, \quad \Gamma(3) = 2\Gamma(2) = 2, \quad \dots \quad \Gamma(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!.$$

Thus, Euler's Γ function extends the factorial function to the complex plane and sometimes the notation $z! = \Gamma(z+1)$ is used.

■ **Legendre's definition as a definite integral.**

$$(63) \quad \Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re} z > 0.$$

The restriction on the real part of z is to avoid divergences of the integral at the origin $t = 0$. An integration by parts leads immediately to (62)

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = [t^z (-e^{-t})]_{t=0}^\infty + \int_0^\infty z t^{z-1} e^{-t} dt = z\Gamma(z),$$

Making different changes of variable we conclude that for $\operatorname{Re} z > 0$

$$\begin{aligned}\Gamma(z) &= 2 \int_0^\infty e^{-t^2} t^{2z-1} dt, \\ &= \int_0^1 \left[\log \left(\frac{1}{t} \right) \right]^{z-1} dt,\end{aligned}$$

from which, for example, we deduce

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

To check that both definitions of Euler's Γ function are equivalent we introduce the following function

$$F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt, \quad \operatorname{Re} z > 0,$$

for $n = 1, 2, \dots$ a positive integer. We immediately conclude that $\lim_{n \rightarrow \infty} F(z, n) = \Gamma(z)$. Using the variable $u = \frac{t}{n}$ we get the alternative expression

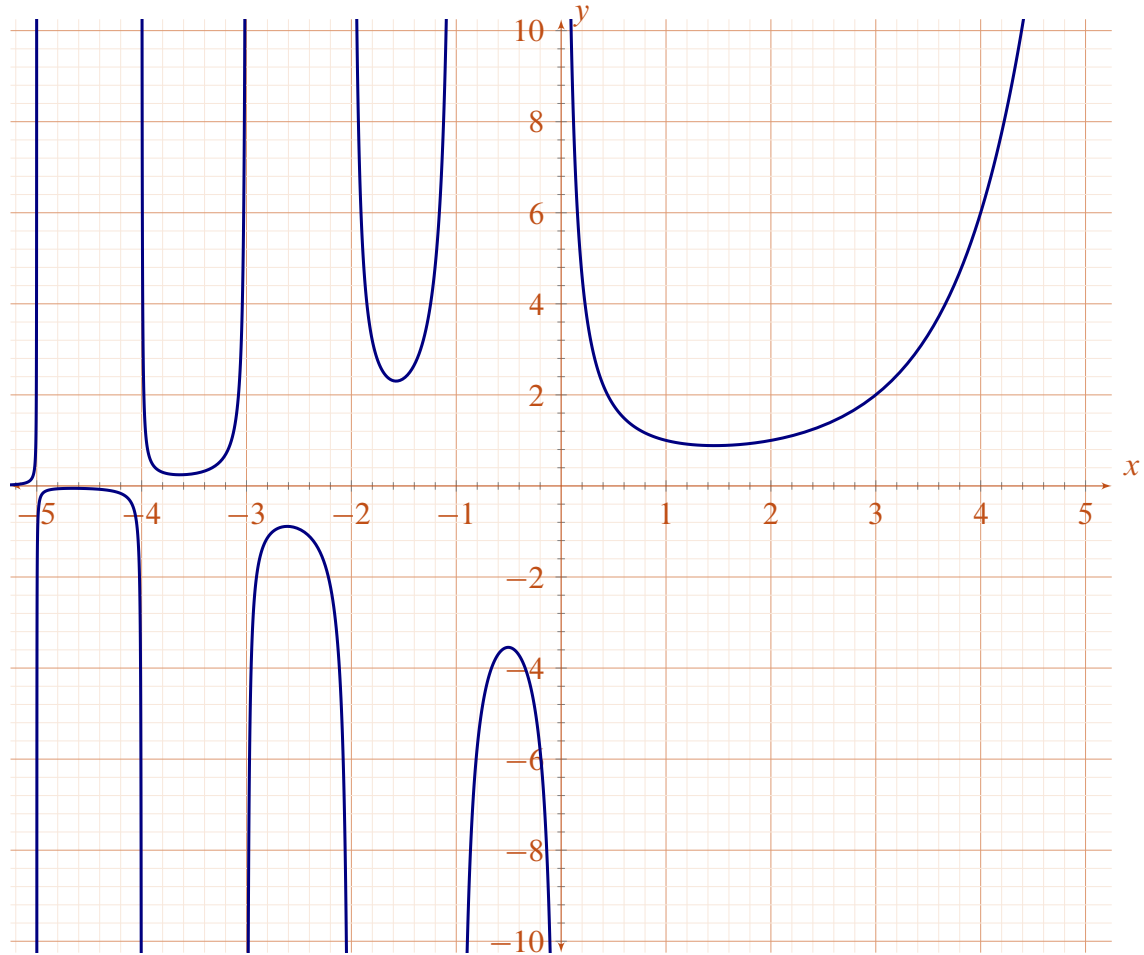
$$F(z, n) = n^z \int_0^1 (1-u)^n u^{z-1} du,$$

that after successive integrations by parts becomes

$$F(z, n) = \frac{1 \cdot 2 \cdot 3 \cdots n}{z \cdots (z+n)} n^z,$$

and so we get the announced result.

Although the definition (63) is true only for $\operatorname{Re} z > 0$ using the relationship (62) you can obtain the values for negative real parts. For example, $(-1/2)\Gamma(-1/2) = \Gamma(1 - 1/2) = \sqrt{\pi}$ so that $\Gamma(-1/2) = -2\sqrt{\pi}$. Relation (62) implies $\lim_{z \rightarrow 0} z\Gamma(z) = \lim_{z \rightarrow 0} \Gamma(z+1) = \Gamma(1) = 1$, so $z = 0$ is a simple pole of the Euler gamma function. A similar argument applies to the other poles $z = -n$, the negative integers, of the Γ function. Therefore, we have a meromorphic extension in the complex plane with poles, which are simple, of the factorial function.

Euler gamma function

- **Weierstrass infinite product** Another possible definition of Euler's Γ function is due to Weierstrass

$$\frac{1}{\Gamma(z)} := z e^{yz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where we've used **Euler–Mascheroni constant**.

Finally we comment on two very important properties of Euler's Γ function

- **Euler's Reflection Formula**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

- **Stirling Formula**

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}, \quad |z| \rightarrow \infty, \quad -\pi < \arg z < \pi.$$

In Statistical Physics this formula is usually written as

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad n \rightarrow \infty.$$

§6.6. Exercises

6.6.1. Exercises with solutions

1. To solve by means of the Frobenius method the initial values problem of the classical harmonic oscillator

$$(64) \quad \begin{aligned} u'' + \omega^2 u &= 0, \\ \begin{cases} u(0) = a_0, \\ u'(0) = a_1. \end{cases} \end{aligned}$$

Solution: Given that $a(z) \equiv 0$ and $b(z) = \omega^2$ are entire functions, analytic in $\mathbb{C}$, so will be the solution. The power series expansion (33) is

$$u'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}z^n$$

which, once introduced in (64), gives

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} + \omega^2 c_n \right) z^n = 0$$

which necessarily requires that

$$\begin{aligned} (n+2)(n+1)c_{n+2} + \omega^2 c_n &= 0, & n &= 0, 1, \dots \\ c_0 &= a_0, \\ c_1 &= a_1. \end{aligned}$$

Therefore

$$c_{2n} = a_0 \frac{(-1)^n}{(2n)!} \omega^{2n}, \quad c_{2n+1} = a_1 \frac{(-1)^n}{(2n+1)!} \omega^{2n+1}$$

and the power series expansion is

$$\begin{aligned} u &= a_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \omega^{2n} z^{2n} \right) + \frac{a_1}{\omega} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \omega^{2n+1} z^{2n+1} \right) \\ &= a_0 \cos \omega z + \frac{a_1}{\omega} \sin \omega z. \end{aligned}$$

Therefore, the solution is a combination of trigonometric functions defined in the complex plane $\mathbb{C}$ and therefore it is an entire function.

2. Solve by the Frobenius method the Airy equation initial value problem

$$(65) \quad \begin{aligned} u'' &= zu, \\ \begin{cases} u(0) = a_0, \\ u'(0) = a_1. \end{cases} \end{aligned}$$

Solution: In this case $a(z) \equiv 0$ and $b(z) = -z$ so that $z_0 = 0$ is a regular point and the solution will be an entire function that admits a power series expansion (33) at that point

$$u(z) = \sum_{n \geq 0} c_n z^n,$$

which is convergent in the whole complex plane. Substituting this series in the differential equation we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}z^n = \sum_{n=1}^{\infty} c_{n-1}z^n$$

which is rewritten as

$$2c_2 + \sum_{n=0}^{\infty} \left((n+3)(n+2)c_{n+3} - c_n \right) z^{n+1} = 0$$

and, consequently, we have

$$c_2 = 0,$$

$$c_{n+3} = \frac{1}{(n+3)(n+2)} c_n, \quad n = 0, 1, \dots$$

$$c_0 = a_0,$$

$$c_1 = a_1.$$

So, in terms of Euler gamma function, we get

$$c_{3n} = \frac{a_0}{3n(3n-1)(3n-3)(3n-4) \cdots 6 \cdot 5 \cdot 3 \cdot 2} = a_0 \Gamma(2/3) \frac{1}{n! 9^n \Gamma(n+2/3)},$$

$$c_{3n+1} = \frac{a_1}{(3n+1)3n(3n-2)(3n-3) \cdots 7 \cdot 6 \cdot 4 \cdot 3} = a_1 \Gamma(4/3) \frac{1}{n! 9^n \Gamma(n+4/3)},$$

$$c_{3n+2} = 0.$$

and hence the solution⁴ is

$$u = a_0 \Gamma(2/3) \sum_{n=0}^{\infty} \frac{z^{3n}}{n! 9^n \Gamma(n+2/3)} + a_1 \Gamma(4/3) \sum_{n=0}^{\infty} \frac{z^{3n+1}}{n! 9^n \Gamma(n+4/3)}.$$

3. Consider the following ODE in complex plane

$$\frac{d^2 u}{dz^2} + \frac{1+z}{6z} \frac{du}{dz} + \frac{1+2z}{6z^2} u = 0$$

⁴The so-called Airy functions are often used in literature

$$\begin{aligned} \text{Ai}(z) &= 3^{-2/3} \sum_{n=0}^{\infty} \frac{z^{3n}}{n! 9^n \Gamma(n+2/3)} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{n! 9^n \Gamma(n+4/3)}, \\ \text{Bi}(z) &= 3^{-1/6} \sum_{n=0}^{\infty} \frac{z^{3n}}{n! 9^n \Gamma(n+2/3)} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{n! 9^n \Gamma(n+4/3)} \end{aligned}$$

and answer the following questions:

- To classify the points of the plane according to whether they are regular, regular singular or irregular.
- For the origin, find the indicial polynomial and its two roots.
- Find the difference between the two roots and conclude how the two linearly independent solutions corresponding to each root of the indicial polynomial are.
- To find, for the first solution its Frobenius series and the recurrence relation. Determine the first three non-zero coefficients.
- To find, for the second solution, the recurrence relation between its coefficients.

Solution:

- In this case, $a(z) = \frac{1+z}{6z}$ and $b(z) = \frac{1+2z}{6z^2}$, so $a(z)$ and $b(z)$ are meromorphic functions with a single singularity in $z = 0$, a simple pole and a double pole, respectively. Therefore, all the points of the plane except the origin are regular points of the ODE. However, $z = 0$ is a regular singular point.
- We have $a_0 = \frac{1}{6}$ and $b_0 = \frac{1}{6}$, so the indicial polynomial is

$$P(\alpha) = \alpha(\alpha - 1) + a_0\alpha + b_0 = \alpha^2 - \frac{5}{6}\alpha + \frac{1}{6}.$$

The roots of the indicial polynomial are

$$\alpha = \frac{\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{4}{6}}}{2} = \begin{cases} \alpha_1 = \frac{1}{2}, \\ \alpha_2 = \frac{1}{3}. \end{cases}$$

- Since $\alpha_1 - \alpha_2 = \frac{1}{6}$ is not integer, there are no resonances. Then two ODE linearly independent solutions are written in the form of Frobenius series as follows

$$u_1(z) = \sum_{n=0}^{\infty} c_{1,n} z^{n+\frac{1}{2}}, \quad u_2(z) = \sum_{n=0}^{\infty} c_{2,n} z^{n+\frac{1}{3}},$$

with coefficients $c_{1,n}$ and $c_{2,n}$ to be determined recursively by ODE itself.

- We write the ODE in the following way

$$6z^2 \frac{d^2 u}{dz^2} + (z + z^2) \frac{du}{dz} + (1 + 2z)u = 0$$

and calculate the first two derivatives, deriving term by term,

$$\frac{du_1}{dz} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) c_{1,n} z^{n-\frac{1}{2}}, \quad \frac{d^2 u_1}{dz^2} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \left(n - \frac{1}{2}\right) c_{1,n} z^{n-\frac{3}{2}}.$$

Hence,

$$\begin{aligned}
 6z^2 \frac{d^2 u_1}{dz^2} &= \sum_{n=0}^{\infty} 6\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)c_{1,n}z^{n+\frac{1}{2}} \\
 &= -\frac{3}{2}c_{1,0}z^{\frac{1}{2}} + \sum_{n=0}^{\infty} 6\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right)c_{1,n+1}z^{n+\frac{3}{2}}, \\
 (z + z^2) \frac{du_1}{dz} &= \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)c_{1,n}z^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)c_{1,n}z^{n+\frac{3}{2}} \\
 &= \frac{1}{2}c_{1,0}z^{\frac{1}{2}} + \sum_{n=0}^{\infty} \left(\left(n + \frac{3}{2}\right)c_{1,n+1} + \left(n + \frac{1}{2}\right)c_{1,n}\right)z^{n+\frac{3}{2}}, \\
 (1 + 2z)u_1 &= \sum_{n=0}^{\infty} c_{1,n}z^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} 2c_{1,n}z^{n+\frac{3}{2}} = c_{1,0}z^{\frac{1}{2}} + \sum_{n=0}^{\infty} (c_{1,n+1} + 2c_{1,n})z^{n+\frac{3}{2}}
 \end{aligned}$$

Therefore, the ODE is written in terms of Frobenius series as follows

$$\sum_{n=0}^{\infty} \left(6\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right)c_{1,n+1} + \left(n + \frac{3}{2}\right)c_{1,n+1} + \left(n + \frac{1}{2}\right)c_{1,n} + c_{1,n+1} + 2c_{1,n}\right)z^{n+\frac{3}{2}} = 0.$$

That implies

$$6\left(n + \frac{3}{2}\right)\left(n + \frac{1}{2}\right)c_{1,n+1} + \left(n + \frac{3}{2}\right)c_{1,n+1} + \left(n + \frac{1}{2}\right)c_{1,n} + c_{1,n+1} + 2c_{1,n} = 0,$$

so

$$2((2n + 3)(3n + 2) + 1)c_{1,n+1} + (2n + 5)c_{1,n} = 0$$

and, finally, we can write the recurrence as

$$c_{1,n+1} = -\frac{2n + 5}{2((2n + 3)(3n + 2) + 1)}c_{1,n},$$

with $n \in \{1, 2, \dots\}$. Thus, $c_{1,0}$ is arbitrary and $c_{1,1} = -\frac{5}{14}c_{1,0}$, $c_{1,2} = -\frac{7}{82}c_{1,1} = \frac{35}{1148}c_{1,0}$.

e) Proceed as above

$$\frac{du_2}{dz} = \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right)c_{2,n}z^{n-\frac{2}{3}}, \quad \frac{d^2 u_2}{dz^2} = \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right)c_{2,n}z^{n-\frac{5}{3}}.$$

Hence,

$$\begin{aligned}
 6z^2 \frac{d^2 u_2}{dz^2} &= \sum_{n=0}^{\infty} 6\left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right)c_{2,n}z^{n+\frac{1}{3}} \\
 &= -\frac{4}{3}c_{2,0}z^{\frac{1}{3}} + \sum_{n=0}^{\infty} 6\left(n + \frac{4}{3}\right)\left(n + \frac{1}{3}\right)c_{2,n+1}z^{n+\frac{4}{3}}, \\
 (z + z^2) \frac{du_2}{dz} &= \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right)c_{2,n}z^{n+\frac{1}{3}} + \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right)c_{2,n}z^{n+\frac{4}{3}} \\
 &= \frac{1}{3}c_{2,0}z^{\frac{1}{3}} + \sum_{n=0}^{\infty} \left(\left(n + \frac{4}{3}\right)c_{2,n+1} + \left(n + \frac{1}{3}\right)c_{2,n}\right)z^{n+\frac{4}{3}}, \\
 (1 + 2z)u_2 &= \sum_{n=0}^{\infty} c_{2,n}z^{n+\frac{1}{3}} + \sum_{n=0}^{\infty} 2c_{2,n}z^{n+\frac{4}{3}} = c_{2,0}z^{\frac{1}{3}} + \sum_{n=0}^{\infty} (c_{2,n+1} + 2c_{2,n})z^{n+\frac{4}{3}}
 \end{aligned}$$

Therefore, the ODE is written in terms of Frobenius series as follows

$$\sum_{n=0}^{\infty} \left(6\left(n + \frac{4}{3}\right)\left(n + \frac{1}{3}\right)c_{2,n+1} + \left(n + \frac{4}{3}\right)c_{2,n+1} + \left(n + \frac{1}{3}\right)c_{2,n} + c_{2,n+1} + 2c_{2,n}\right)z^{n+\frac{4}{3}} = 0.$$

Hence,

$$6\left(n + \frac{4}{3}\right)\left(n + \frac{1}{3}\right)c_{2,n+1} + \left(n + \frac{4}{3}\right)c_{2,n+1} + \left(n + \frac{1}{3}\right)c_{2,n} + c_{2,n+1} + 2c_{2,n} = 0.$$

So that,

$$c_{2,n+1} = -\frac{3n+7}{2(3n+4)(3n+1)+3}c_{2,n},$$

for $n \in \{0, 1, 2, \dots\}$.

4. For the ODE in the complex plane

$$3z^2 \frac{d^2 u}{dz^2} + (2z^2 - z) \frac{du}{dz} + u = 0.$$

- Classify the points of the complex plane, according to whether they are ordinary, regular singular or irregular singular. What type of point is the origin $z = 0$?
- Find the indicial polynomial in $z = 0$, find its two roots α_1 and α_2 , $\alpha_1 \geq \alpha_2$.
- Find the Frobenius series at the origin $u_1(z)$ for the largest root α_1 .
- Find the Frobenius series at the origin $u_2(z)$ for the smaller root α_2 , express this solution in terms of exponentials and radicals.

Solution:

a) The ODE can be written as follows

$$\frac{d^2 u}{dz^2} + a(z) \frac{du}{dz} + b(z)u = 0, \quad a(z) := -\frac{1}{3z} + \frac{2}{3}, \quad b(z) := \frac{1}{3z^2}.$$

The functions $a(z)$ and $b(z)$ are holomorphic in $\mathbb{C} \setminus \{0\}$, so **all points of the complex plane are regular but for the origin**. In the origin, $a(z)$ has a single pole and $b(z)$ a double pole, so **the origin is a regular singular point**.

b) As we know, if

$$a(z) = \frac{a_0}{z} + a_1 + a_2 z + \cdots, \quad b(z) = \frac{b_0}{z^2} + \frac{b_1}{z} + b_0 + \cdots,$$

the indicial polynomial in the origin is

$$P(\alpha) = \alpha(\alpha - 1) + a_0\alpha + b_0.$$

In this case $a_0 = -\frac{1}{3}$ and $b_0 = \frac{1}{3}$, and hence $P(\alpha) = \alpha(\alpha - 1) - \frac{\alpha}{3} + \frac{1}{3}$, i.e.

$$P(\alpha) = \alpha^2 - \frac{4}{3}\alpha + \frac{1}{3}.$$

The corresponding roots are

$$\frac{\frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{3}}}{2} = \begin{cases} 1, \\ \frac{1}{3}. \end{cases}$$

So the roots are $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{3}$.

c) We are now looking for a solution in the form

$$u_1(z) = z \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^{n+1}.$$

We calculate the first two derivatives

$$\frac{d u_1}{d z} = \sum_{n=0}^{\infty} (n+1) c_n z^n, \quad \frac{d^2 u_1}{d z^2} = \sum_{n=0}^{\infty} (n+1) n c_n z^{n-1},$$

so that

$$\begin{aligned} 3z^2 \frac{d^2 u_1}{d z^2} &= \sum_{n=0}^{\infty} 3(n+1) n c_n z^{n+1}, \\ (2z^2 - z) \frac{d u_1}{d z} &= \sum_{n=0}^{\infty} 2(n+1) c_n z^{n+2} - \sum_{n=0}^{\infty} (n+1) c_n z^{n+1} \\ &= \sum_{n=0}^{\infty} (2n c_{n-1} - (n+1) c_n) z^{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} 3z^2 \frac{d^2 u_1}{dz^2} + (2z^2 - z) \frac{du_1}{dz} + u_1 &= \sum_{n=0}^{\infty} (3(n+1)nc_n + 2nc_{n-1} - (n+1)c_n + c_n)z^{n+1} \\ &= \sum_{n=0}^{\infty} n((3n+2)c_n + 2c_{n-1})z^{n+1}. \end{aligned}$$

Consequently, we must require

$$c_n = -\frac{2}{3n+2}c_{n-1}, \quad n \in \{1, 2, \dots\}.$$

In particular, $c_1 = -\frac{2}{5}c_0$, $c_2 = \frac{2^2}{8 \times 5}c_0$, $c_3 = -\frac{2^3}{11 \times 8 \times 5}c_0$, and

$$c_n = (-1)^n \frac{2^n}{(3n+2)!!!}$$

in terms of the triple factorial $n!!! = n(n-3)(n-6)\dots$, and the Frobenius series (power series in this case) is

$$u_1(z) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(3n+2)!!!} z^{n+1}.$$

d) The requested solution is

$$u_2(z) = z^{\frac{1}{3}} \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^{n+\frac{1}{3}}.$$

We calculate the first two derivatives

$$\frac{du_2}{dz} = \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right) c_n z^{n-\frac{2}{3}}, \quad \frac{d^2 u_2}{dz^2} = \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right) \left(n - \frac{2}{3}\right) c_n z^{n-\frac{5}{3}},$$

hence

$$\begin{aligned} 3z^2 \frac{d^2 u_2}{dz^2} &= \sum_{n=0}^{\infty} 3 \left(n + \frac{1}{3}\right) \left(n - \frac{2}{3}\right) c_n z^{n+\frac{1}{3}}, \\ (2z^2 - z) \frac{du_2}{dz} &= \sum_{n=0}^{\infty} 2 \left(n + \frac{1}{3}\right) c_n z^{n+\frac{4}{3}} - \sum_{n=0}^{\infty} \left(n + \frac{1}{3}\right) c_n z^{n+\frac{1}{3}} \\ &= -\frac{c_0}{3} z^{\frac{1}{3}} + \sum_{n=1}^{\infty} \left(2 \left(n - \frac{2}{3}\right) c_{n-1} - \left(n + \frac{1}{3}\right) c_n\right) z^{n+\frac{1}{3}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 3z^2 \frac{d^2 u}{dz^2} + (2z^2 - z) \frac{du}{dz} + u \\
 &= \left(-3 \times \frac{1}{3} \times \frac{2}{3} - \frac{1}{3} + 1 \right) c_0 z^{\frac{1}{3}} \\
 &\quad + \sum_{n=1}^{\infty} \left[\left(3 \left(n + \frac{1}{3} \right) \left(n - \frac{2}{3} \right) - \left(n + \frac{1}{3} \right) + 1 \right) c_n + 2 \left(n - \frac{2}{3} \right) c_{n-1} \right] z^{n+\frac{1}{3}} \\
 &= \sum_{n=1}^{\infty} (3n-2) \left(n c_n + \frac{2}{3} c_{n-1} \right) z^{n+\frac{1}{3}}.
 \end{aligned}$$

so that

$$c_n = -\frac{2}{3n} c_{n-1}, \quad n \in \{1, 2, \dots\},$$

$$c_n = \left(-\frac{2}{3} \right)^n \frac{1}{n!} c_0, \quad n \in \{1, 2, \dots\}.$$

and the solution is

$$u_2(x) = c_0 z^{\frac{1}{3}} \sum_{n=0}^{\infty} \left(-\frac{2z}{3} \right)^n \frac{1}{n!} = c_0 z^{\frac{1}{3}} e^{-\frac{2z}{3}}.$$

5. For the ODE

$$z(z^2 + 2) \frac{d^2 u}{dz^2} + 2 \frac{du}{dz} - 2zu = 0.$$

- Classify the points of the complex plane according to its role with respect the ODE: regular, regular singular and irregular singular.
- For $z_0 = 0$, find the indicial polynomial $P(\alpha)$, of the ODE and its roots. Discuss the application of Fuchs theorem Will terms with roots appear in the first series solution u_1 ? and what about logarithms in the second series solution $u_2(z)$?
- For the first solution $u_1(z)$, as a Frobenius series with respect to $z = 0$, find c_1 and the recurrence relationship between the coefficients.
- Discuss the recurrence. Find the value of odd coefficients c_{2n+1} and show that the series ends. Find the polynomial $u_1(z)$. What about the radius of convergence, is it bigger than the distance to the nearest singularity?
- Find the second solution and discuss the convergence radius of the corresponding power series.

Solution.

- We write the ODE as follows

$$\frac{d^2 u}{dz^2} + a(z) \frac{du}{dz} + b(z)u = 0, \quad a(z) := \frac{2}{z(z^2 + 2)}, \quad b(z) := -\frac{2}{z^2 + 2}.$$

We see that the functions $a(z)$ and $b(z)$ are meromorphic for $\mathbb{C}$ with simple poles located at $\{0, i\sqrt{2}, -i\sqrt{2}\}$. Therefore, all points in $\mathbb{C} \setminus \{0, i\sqrt{2}, -i\sqrt{2}\}$ are regular points. On the other hand, the regular single points make up the set $\{0, i\sqrt{2}, -i\sqrt{2}\}$, and there are no irregular singular points.

b) To analyze local aspects around $z = 0$, we wrote the ODE in the form

$$z^2 \frac{d^2 u}{dz^2} + zA(z) \frac{du}{dz} + B(z)u = 0,$$

$$A(z) := za(z) = \frac{2}{z^2 + 2}, \quad B(z) := z^2 b(z) = -\frac{2z^2}{z^2 + 2}.$$

Where $A(z)$ and $B(z)$ present the following power series expansions of converging powers on the disk $D(0, \sqrt{2}) := \{z \in \mathbb{C} : |z| < \sqrt{2}\}$

$$A(z) = 1 - \frac{z^2}{2} + \frac{z^4}{4} - \frac{z^6}{8} \cdots, \quad B(z) = -z^2 + \frac{z^4}{2} - \frac{z^6}{4} + \frac{z^8}{8} \cdots,$$

Therefore, as $a_0 = 1$ and $b_0 = 0$ we have for the indicial polynomial

$$P(\alpha) = \alpha^2 + (a_0 - 1)\alpha + b_0$$

that

$$P(\alpha) = \alpha^2.$$

There's only one root in $\boxed{\alpha_1 = \alpha_2 = 0}$ that's double. **We deduce that, on the one hand, $u_1(z)$ is a power series, and in the other hand, that logarithms will appear for the second series solution $u_2(z)$.**

c) We write the Frobenius series for the root $\alpha_1 = 0$, which is a power series, and the other terms that appear in the ODE

$$\begin{aligned} -2zu_1(z) &= -2z \sum_{n=0}^{\infty} c_n z^n = -\sum_{n=1}^{\infty} 2c_{n-1} z^n, \\ 2u_1'(z) &= \sum_{n=0}^{\infty} 2nc_n z^{n-1} = \sum_{n=0}^{\infty} 2(n+1)c_{n+1} z^n, \\ z(z^2 + 2)u_1''(z) &= (z^3 + 2z) \sum_{n=0}^{\infty} n(n-1)c_n z^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1)c_n z^{n+1} + 2 \sum_{n=0}^{\infty} n(n-1)c_n z^{n-1} \\ &= \sum_{n=1}^{\infty} (n-1)(n-2)c_{n-1} z^n + 2 \sum_{n=0}^{\infty} (n+1)nc_{n+1} z^n, \end{aligned}$$

and we obtain:

$$2c_1 + \sum_{n=1}^{\infty} ((n-3)nc_{n-1} + 2(n+1)(n+1)c_{n+1})z^n = 0$$

that leads to $\boxed{c_1 = 0}$ and to the recursion law

$$\boxed{c_{n+2} = -\frac{(n-2)(n+1)}{2(n+2)^2}c_n,}$$

for $n \in \{0, 1, 2, \dots\}$.

- d) From the recurrence, since $c_1 = 0$, it follows that **all odd coefficients are zero**, $c_{2n+1} = 0$. On the other hand, it also follows from the recurrence that $c_4 = 0$ and therefore that $0 = c_6 = c_8 = c_{10} = \dots$. Effectively, the series ends and the only coefficients not necessarily zero are c_0 and c_2 , the latter, according to the recurrence is $c_2 = \frac{1}{4}$ and, therefore, choosing $c_0 = 1$, we can

$$u_1(z) = 1 + \frac{z^2}{4}.$$

Obviously, as the power series collapses to a polynomial the radius of convergence becomes infinity and is bigger than $\sqrt{2}$, the distance to the closet singularity.

- e) According to Fuch's Theorem, the second solution can be written in the form

$$u_2 = u_1 \log z + \sum_{n=1}^{\infty} c'_n z^n$$

Using the notation

$$L = z(z^2 + 2) \frac{d^2}{dz^2} + 2 \frac{d}{dz} - 2z = 0,$$

we find

$$L(u_2) = L(u_1 \log z) + L\left(\sum_{n=1}^{\infty} c'_n z^n\right) = L(u_1) \log z + \frac{3}{4} z^3 + z + L\left(\sum_{n=1}^{\infty} c'_n z^n\right).$$

Thus, as $L(u_1) = 0$ we find the coefficients c'_n from

$$L\left(\sum_{n=1}^{\infty} c'_n z^n\right) = -\frac{3}{4} z^3 - z.$$

As we have previously accomplished,

$$L\left(\sum_{n=1}^{\infty} c'_n z^n\right) = 2c'_1 + \sum_{n=1}^{\infty} ((n-3)nc'_{n-1} + 2(n+1)^2 c'_{n+1}) z^n,$$

with $c'_0 = 0$. Therefore, equating coefficients power by power

$$c'_1 = 0, \quad c'_2 = -\frac{1}{8}, \quad c'_3 = 0, \quad c'_4 = -\frac{3}{128}$$

and, all the odd coefficients are zero

$$c'_{2n+1} = 0,$$

and, for $n \geq 2$, we find the recurrence

$$c'_{2n+2} = -\frac{(2n+1)(2n-2)}{2(2n+2)^2} c'_{2n}.$$

Hence, as

$$\lim_{n \rightarrow \infty} \frac{c'_{2n+2}}{c'_{2n}} = \frac{1}{2}$$

according to d'Alembert's ratio convergence criterion the convergence radius is $\sqrt{2}$, and the second solution is

$$u_2 = \left(\frac{z^2}{4} + 1\right) \log z - \frac{1}{8}z^2 - \frac{3}{128}z^4 + \frac{5}{1536}z^6 + \cdots.$$

6.6.2. Exercises

1. The Legendre equation for the polynomials of the same name $P_n(x)$ can be written as

$$LP_n = \lambda_n P_n, \quad \lambda_n = n(n+1).$$

where L is the operator of Sturm–Liouville

$$Lu := -D((1-x^2)Du) = -(1-x^2)D^2u + 2xDu.$$

Prove that L is symmetric over the domain

$$\mathcal{D} := \{u \in C^\infty(-1, 1) : \lim_{x \rightarrow \pm 1} (1-x^2)u'(x) = 0\} \subset L^2([-1, 1]).$$

As a consequence demonstrate that the polynomials of Legendre $\{P_n(x)\}_{n \geq 0}$ form an orthogonal set in $L^2[-1, 1]$.

2. Rodrigues' formula for Legendre's polynomials is:

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2 - 1)^n, \quad n \geq 0.$$

By using this expression:

- a) Determine the first three polynomials.
 - b) To represent graphically in the interval $[-1, 1]$ the polynomials P_2 and P_3 .
 - c) Show that $P_n(-x) = (-1)^n P_n(x)$, $n = 0, 1, \dots$.
3. Through Rodrigues' formula for Legendre's polynomials:
 - a) Show that $P_n(x)$ is a polynomial of degree n .
 - b) Using the identity

$$(x^2 - 1)^n = (x + 1)^n (x - 1)^n,$$

in Rodrigues' formula and Leibniz's rule to differentiate, you get

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} [D^k (x+1)^n] [D^{n-k} (x-1)^n].$$

Use this formula to prove that

$$P_n(1) = 1, \quad \forall n \geq 0,$$

and deduce then that

$$P_n(-1) = (-1)^n, \quad \forall n \geq 0.$$

4. Developing by means of Newton's binomial formula we have to

$$(x^2 - 1)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^{n-k} x^{2k}.$$

Making use of this identity in Rodrigues' formula proves that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}, \quad P_{2n+1}(0) = 0, \quad \forall n \geq 0.$$

5. Making use of the identity

$$DP_{n+1} - DP_{n-1} = (2n+1)P_n, \quad n \geq 0,$$

determine the series expansion of Legendre polynomials

$$u(x) = \sum_{n \geq 0} c_n P_n(x),$$

of the function

$$u(x) = \begin{cases} 1 & \text{si } 0 < x < 1 \\ 0 & \text{si } -1 < x < 0; \end{cases}$$

6. Let L be the following Sturm–Liouville operator

$$L_m u = -\frac{1}{x} D(xDu) + \frac{m^2}{x^2} u, \quad m = 0, 1, 2, \dots$$

a) Show that L_m is symmetric in the domain

$$\mathcal{D} := \{u \in C^2((0, b)) : \exists \lim_{x \rightarrow 0} u(x), \lim_{x \rightarrow 0} (xu'(x)) = 0, u(b) = 0\} \subset L^2_\rho([0, b]),$$

siendo $\rho(x) = x$

b) If c_{mn} ($n \geq 1$) are the zeros of the Bessel function $J_m(x)$ then, if

$$\lambda_n = (c_{mn}/b)^2,$$

the functions $u_n(x) = J_m(\sqrt{\lambda_n}x)$, $n \geq 1$, belong to $\mathcal{D}$ and satisfy

$$L_m u_n = \lambda_n u_n.$$

c) Based on the asymptotic behavior of Bessel functions at large coordinate values x ,

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right),$$

graphically sketch the eigenfunctions $u_n(x)$ in the interval $[0, b]$.

d) Using the same asymptotic expression of $J_m(x)$ provide an approximate formula for eigenvalues λ_n .

7. Consider the Bessel equation

$$\frac{1}{x}(xu')' + \left(1 - \frac{m^2}{x^2}\right)u = 0.$$

Multiplying by $2x^2u'$ show that

$$2xu^2 = [(xu')^2 + (x^2 - m^2)u^2]'$$

From here prove that

$$\int_0^b x J_m(\sqrt{\lambda_n} x)^2 dx = \frac{b^2}{2} \left(J_m'(\sqrt{\lambda_n} b) \right)^2,$$

with c_{mn} ($n \geq 1$) being the zeros of the function $J_m(x)$, y

$$\lambda_n = (c_{mn}/b)^2.$$

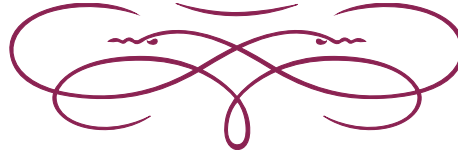
8. Show that the spherical Bessel function $j_l(x), n_l(x)$ fulfill

$$x^2 v'' + 2xu' + (x^2 - l(l+1))u = 0.$$

7. HELMHOLTZ EQUATION IN CURVILINEAR COORDINATES

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ELMHOLTZ equation for functions satisfying certain boundary value problems, with boundaries described naturally in given curvilinear coordinates are discussed in this lecture. Equipped with the knowledge and tools that we developed in previous lectures we will consider this topic for the case in which cylindrical or spherical coordinates can be used. In the case of cylindrical coordinates, the functions of Bessel and Neumann of integer index will appear, in the case of spherical coordinates we will need the Legendre polynomials and their associated polynomials, which will allow us to construct spherical harmonics, and their real versions, the tesseral spherical harmonics.

§7.1. Cylindrical Coordinates



LAPLACIAN operator and its eigenvalues, i.e. the analysis of boundary value problems for the Helmholtz equation, is now addressed but in situations where the boundary are described in a particularly simple way in cylindrical coordinates. The introduction of these coordinates into the Helmholtz and MSV equation leads to the appearance of the Bessel and Neumann functions that will be key in this section. We will deal with the study of examples from Quantum Physics and Fluid Mechanics.

When using cylindrical coordinates as illustrated in the figure below, this is

$$\begin{aligned}x &= r \cos \theta, \\y &= r \sin \theta, \\z &= z,\end{aligned}$$

the Helmholtz equation is written as follows

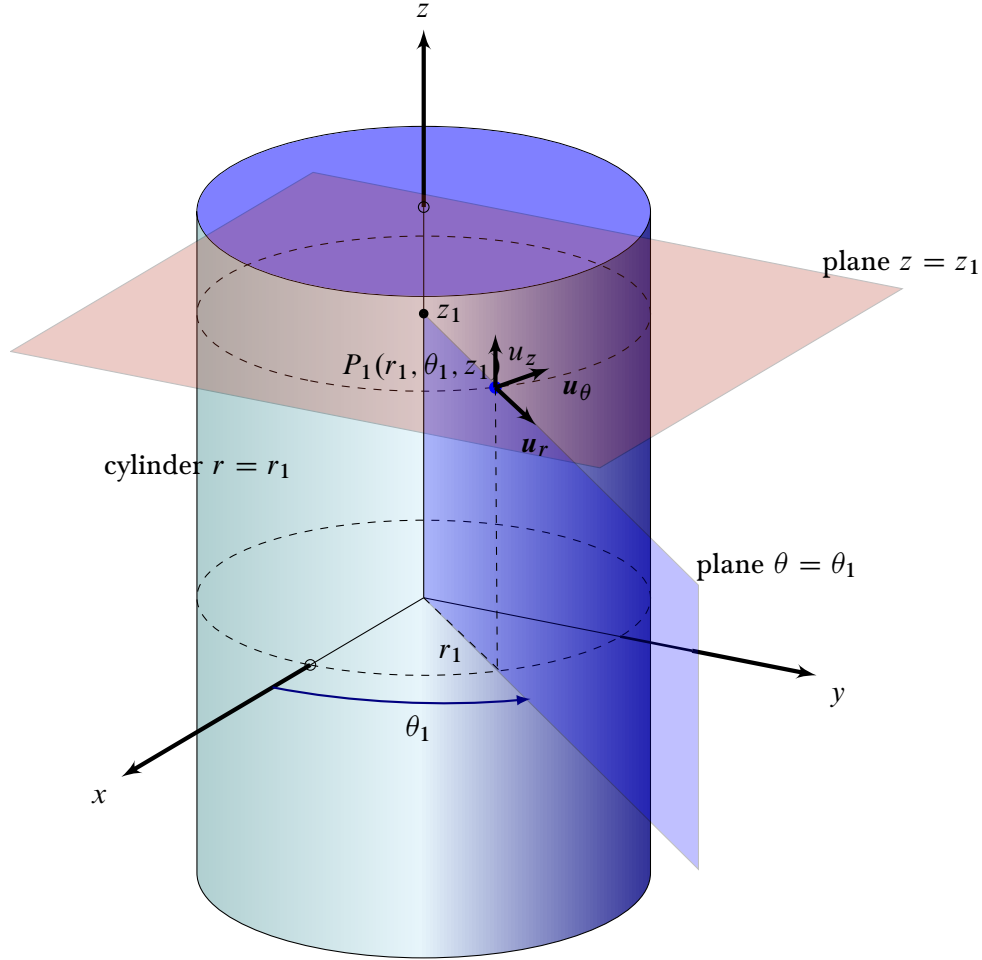
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} + k^2u = 0.$$

We separate variables through factorization

$$u(r, \theta, z) = V(r, \theta)Z(z)$$

which transforms the Helmholtz equation into

$$\begin{aligned} Z'' + k^2 Z &= \alpha^2 Z, \\ r^2 V_{rr} + r V_r + V_{\theta\theta} &= -\alpha^2 r^2 V. \end{aligned}$$



The solution of the first ODE is

$$(66) \quad Z_\alpha(z) = \begin{cases} Az + B, & k^2 = \alpha^2, \\ A e^{i\sqrt{k^2 - \alpha^2}z} + B e^{-i\sqrt{k^2 - \alpha^2}z}, & k^2 \neq \alpha^2. \end{cases}$$

For the second equation we can also separate variables

$$V(r, \theta) = R(r)\Theta(\theta)$$

to get the equations

$$\begin{aligned} r^2 R'' + r R' + \alpha^2 r^2 R &= m^2 R, \\ \Theta'' &= -m^2 \Theta. \end{aligned}$$

The general solution of the second equation is

$$(67) \quad \Theta_m(\theta) = \begin{cases} C\theta + D, & m = 0 \\ C e^{im\theta} + D e^{-im\theta}, & m \neq 0. \end{cases}$$

It remains to analyze the ODE in R , known as **radial equation**. There are two different cases for this one. First we consider $\alpha = 0$ and the corresponding equation is

$$r^2 R'' + rR' - m^2 R = 0$$

whose solution is

$$(68) \quad R_{\alpha=0,m}(r) = \begin{cases} c_1 \log r + c_2 & m = 0, \\ c_1 r^m + c_2 r^{-m} & m \neq 0. \end{cases}$$

When $\alpha \neq 0$, by changing the variable $\rho = \alpha r$, the corresponding ODE is reduced to

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\rho^2 - m^2)R = 0,$$

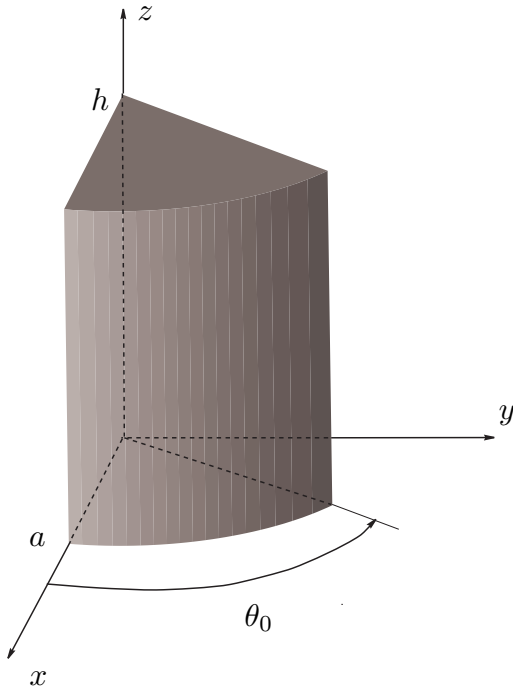
which is the well-known Bessel equation (49) in the variable ρ . Therefore, the solution is

$$(69) \quad R_{\alpha,m}(r) = E J_m(\alpha r) + F N_m(\alpha r).$$

7.1.1. Polar Coordinates The Helmholtz equation in two dimensions, written in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + k^2 u = 0.$$

It is the same equation that appears when in the Helmholtz equation in cylindrical coordinates we seek for solutions that do not depend on the independent variable z . Therefore, it is enough to make $\alpha^2 = k^2$ in the above to obtain the corresponding solutions..



7.1.2. Quantum particle in an impenetrable cylindrical wedge We now consider a quantum particle enclosed in a cylindrical wedge with radius a , height h and with aperture angle $0 < \theta_0 < 2\pi$ as shown in the figure. The stationary states are described by the square integrable solutions of the following Dirichlet boundary problem

$$\begin{cases} -\frac{\hbar^2}{2M} \Delta u = Eu, \\ u|_{\text{walls}} = 0 \end{cases}$$

that calling

$$E = \frac{\hbar^2 k^2}{2M}$$

we write as the following boundary value problem for the Helmholtz equation in cylindrical coordinates

$$\begin{cases} \Delta u + k^2 u = 0, \\ \begin{cases} u|_{r=a} = 0, \\ u|_{\theta=0} = 0, \\ u|_{\theta=\theta_0} = 0, \\ u|_{z=0} = 0, \\ u|_{z=L} = 0. \end{cases} \end{cases}$$

Applying the MSV solutions are found in the form

$$R_{\alpha,m}(r)\Theta_m(\theta)Z_\alpha(z),$$

which must satisfy the boundary value conditions and regularity. First we focus on the angular function Θ_m (67); the boundary conditions in the variable θ son

$$\Theta_m(0) = \Theta_m(\theta_0) = 0,$$

that lead, for $m \neq 0$, to the next linear system for C and D

$$\begin{cases} C + D = 0, \\ e^{im\theta_0} C + e^{-im\theta_0} D = 0. \end{cases}$$

For $m = 0$ it implies that $\Theta = 0$. This linear system has a non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 \\ e^{im\theta_0} & e^{-im\theta_0} \end{vmatrix} = 0,$$

which is tantamount to

$$\sin m\theta_0 = 0.$$

Therefore, we infer that

$$\boxed{m = j \frac{\pi}{\theta_0}, \quad j \in \mathbb{N}.}$$

Moreover $D = -C$ and

$$\Theta_m(\theta) = 2C \sin(m\theta).$$

Secondly, we deal with the variable z and impose the corresponding boundary conditions in (66) to Z_α . If $k^2 = \alpha^2$ one easily concludes that $Z = 0$. For $k^2 \neq \alpha^2$ we obtain the following linear system for A and B

$$\begin{cases} A + B = 0, \\ e^{i\sqrt{k^2-\alpha^2}L} A + e^{-i\sqrt{k^2-\alpha^2}L} B = 0, \end{cases}$$

which admits a nontrivial solution if and only if

$$\sin \sqrt{k^2 - \alpha^2} L = 0.$$

And so

$$k^2 = n^2 \frac{\pi^2}{L^2} + \alpha^2, \quad n \in \mathbb{N}.$$

Also $B = -A$ and

$$Z_\alpha(z) = 2A \sin \frac{\pi}{L} n z.$$

Finally, we analyzed the radial variable r . The radial function $R(r)$ must not have singularities for $r = 0$ and must also satisfy $R(a) = 0$. Therefore, and taking into account (68), it follows that there are no non-trivial solutions for $\alpha = 0$ and this case is discarded. It only remains to be seen what happens for $\alpha \neq 0$, now from (69) we deduce $F = 0$ (regularity at the origin) and the boundary condition imposes that

$$J_m(\alpha a) = 0.$$

Hence, if $\{c_{m,\ell}\}_{\ell=1}^\infty$ are the zeros⁴ of the Bessel function $J_m(x)$ we come to the conclusion that

$$\begin{aligned} m &= j \frac{\pi}{\theta_0}, \quad j \in \mathbb{N}, \\ \alpha &= \frac{c_{m,\ell}}{a}, \quad \ell \in \mathbb{N}. \end{aligned}$$

The admissible energies are therefore

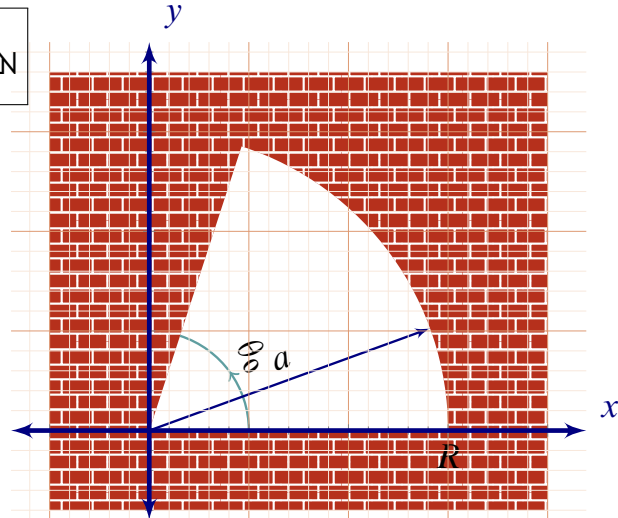
$$E_{j,\ell,n} = \frac{\hbar^2}{2M} \left[\frac{\pi^2 n^2}{L^2} + \frac{(c_{j \frac{\pi}{\theta_0}, \ell})^2}{a^2} \right], \quad j, \ell, n \in \mathbb{N}$$

and the corresponding eigenfunctions will be

$$J_{j \frac{\pi}{\theta_0}} \left(c_{j \frac{\pi}{\theta_0}, \ell} \frac{r}{a} \right) \sin \left(j \pi \frac{\theta}{\theta_0} \right) \sin \left(n \pi \frac{z}{h} \right).$$

The problem of a two-dimensional quantum particle enclosed in a wedge, as the figure indicates, is resolved (in polar coordinates) by making $n = 0$. For example, if $a = 1$ and $\theta_0 = \pi/2$ the energies and the stationary states are

$$\frac{\hbar^2}{2M} (c_{2j,\ell})^2, \quad J_{2j}(c_{2j,\ell} r) \sin(2j\theta), \quad j, \ell \in \mathbb{N},$$



⁴Given the asymptotic behavior

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x - (2m+1) \frac{\pi}{4} \right)$$

we observe for zeros that

$$c_{m,\ell} \sim (2\ell+1) \frac{\pi}{2} + (2m+1) \frac{\pi}{4}.$$

respectively. For example, for $j = 1$, the first two zeros of the Bessel function J_2 are

$$c_{2,1} \cong 5.135622302, c_{2,2} \cong 8.417244140,$$

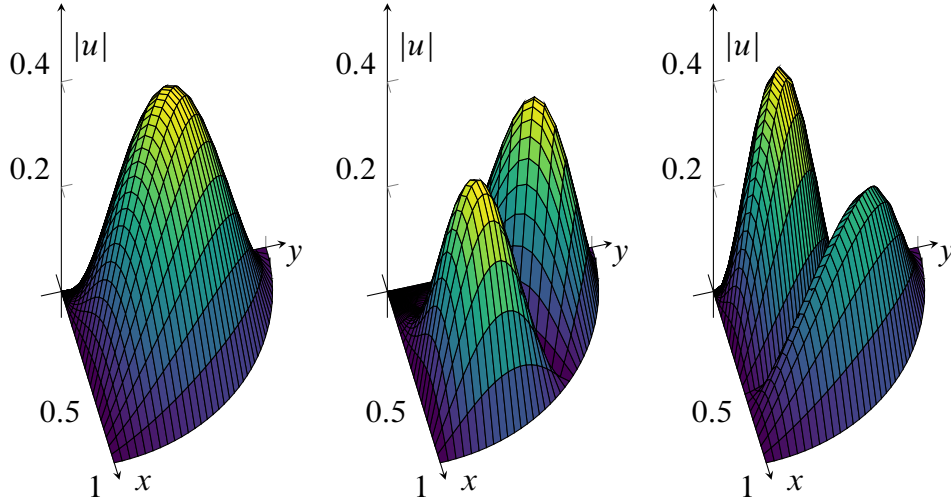
for $j = 2$ the first zero of the function J_4 is

$$c_{4,1} \cong 7.588342435.$$

It's easy to see that these are the first three zeros to appear. Therefore, the fundamental state, the first and second excited are represented by

$$J_2(c_{2,1}r) \sin(2\theta), \quad J_4(c_{4,1}r) \sin(4\theta), \quad J_2(c_{2,2}r) \sin(2\theta).$$

Next we show the sequence formed by the squares of these functions



7.1.3. Quantum particle in an impenetrable cylinder Let's look at the case of a quantum particle enclosed in a cylinder ($\theta_0 = 2\pi$) of radius a . In this case the conditions in θ become

$$\begin{cases} \Delta u + k^2 u = 0, \\ \begin{cases} u|_{r=a} = 0, \\ u|_{\theta=0} = u|_{\theta=2\pi}, \\ u|_{z=0} = 0, \\ u|_{z=L} = 0. \end{cases} \end{cases}$$

The periodic condition $u|_{\theta=0} = u|_{\theta=2\pi}$ implies that $m \in \mathbb{Z}$ and $\Theta_m(\theta) = C e^{im\theta}$. The admissible energies are therefore

$$E_{m,\ell,n} = \frac{\hbar^2}{2M} \left[\frac{\pi^2 n^2}{L^2} + \frac{(c_{m,\ell})^2}{a^2} \right], \quad m \in \mathbb{Z}_+, \ell, n \in \mathbb{N}$$

and the corresponding eigenfunctions will be

$$J_m\left(c_{m,\ell} \frac{r}{a}\right) e^{\pm im\theta} \sin\left(n\pi \frac{z}{h}\right).$$

Where we see a degeneration for the states with $|m| > 0$.

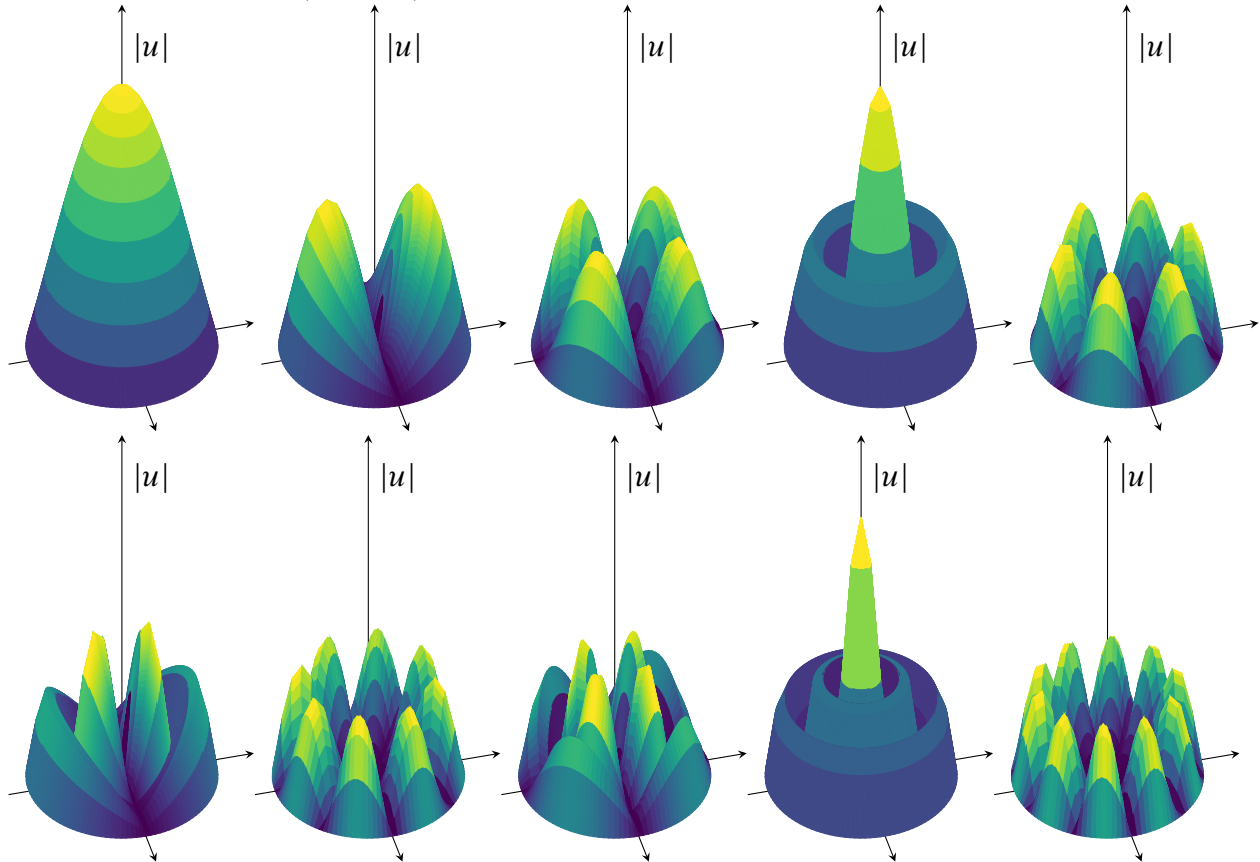
7.1.4. Quantum particle trapped in a quantum corral As before we put $n = 0$ so that $E_{m,\ell} = \frac{\hbar^2}{2M} \frac{(c_{m,\ell})^2}{a^2}$ and the corresponding eigenfunctions will be $J_m\left(c_{m,\ell} \frac{r}{a}\right) e^{\pm im\theta}$ for $m \in \mathbb{Z}_+, \ell \in \mathbb{N}$.

ℓ	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$	$J_6(x)$	$J_7(x)$	$J_8(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715	9.9361	11.0864	12.2251
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386	13.5893	14.8213	16.0378
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002	17.0038	18.2876	19.5545
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208	21.6415	22.9452
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178	23.5861	24.9349	26.2668

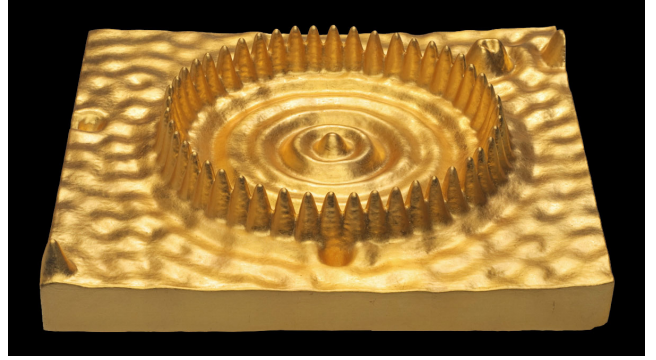
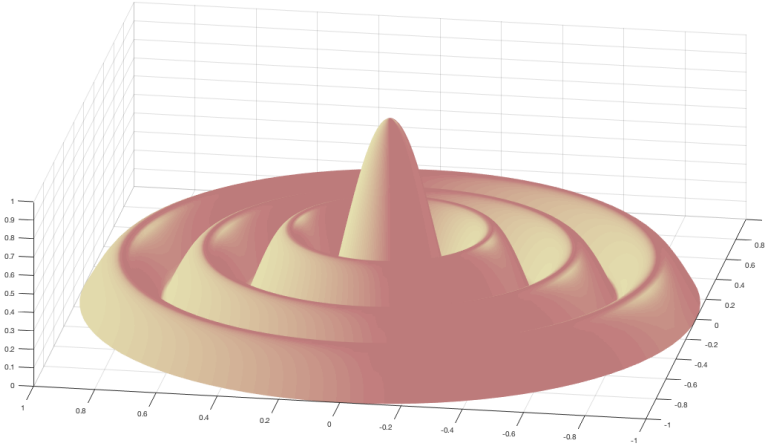
We see that states are organized according to their energy as

$$\begin{aligned}
 E_{0,1} &< E_{1,1} < E_{2,1} \\
 &< E_{0,2} < E_{3,1} < E_{1,2} < E_{4,1} < E_{2,2} \\
 &< E_{0,3} < E_{5,0} < E_{3,2} < E_{6,0} < E_{1,3} < E_{4,2} < E_{7,1} < E_{2,3} \\
 &< E_{0,4} \cdots
 \end{aligned}$$

For $a = 1$ fundamental state is $J_0(2.4048r)$, the first excited state, which is degenerated, $J_1(3.8317r) \cos(\theta - \beta)$, the second excited, which is degenerate, $J_2(5.1356r) \cos 2(\theta - \beta)$ and the third excited state $J_0(5.1356r)$.



We now show the excited state 16, which corresponds to $J_0(c_{0,4}r)$ with energy $\frac{\hbar^2}{2M} \frac{(c_{0,4})^2}{a^2}$ and it can be compared to a famous art creation of a solid state surface quantum corral



7.1.5. Quantum particle trapped in an annulus Let's analyze the case of a quantum particle enclosed between two cylinders with the same axis and radii $0 < R_1 < R_2 < \infty$. In this case the conditions in θ become

$$\begin{cases} \Delta u + k^2 u = 0, \\ \begin{cases} u|_{r=R_1} = 0, \\ u|_{r=R_2} = 0, \\ u|_{\theta=0} = u|_{\theta=2\pi}, \\ u|_{z=0} = 0, \\ u|_{z=L} = 0. \end{cases} \end{cases}$$

The periodic condition $u|_{\theta=0} = u|_{\theta=2\pi}$ implies that $m \in \mathbb{Z}$ and $\Theta_m(\theta) = C e^{im\theta}$. The discussion in z is the same, but when we go to the radial variable the situation is completely different. If $m = 0$ we can have $\alpha = 0$ because the boundary values lead to

$$\begin{pmatrix} \log R_1 & 1 \\ \log R_2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, you must have $\begin{vmatrix} \log R_1 & 1 \\ \log R_2 & 1 \end{vmatrix} = 0$, but $\begin{vmatrix} \log R_1 & 1 \\ \log R_2 & 1 \end{vmatrix} = \log R_1 - \log R_2 = \log \frac{R_1}{R_2}$, so that from $\log \frac{R_1}{R_2} = 0$ we deduce $R_1 = R_2$, which is absurd. So far so good, and for the next case we have

$$\begin{pmatrix} R_1^m & R_1^{-m} \\ R_2^m & R_2^{-m} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is, we require that $\begin{vmatrix} R_1^m & R_1^{-m} \\ R_2^m & R_2^{-m} \end{vmatrix} = 0$, or $\frac{R_1^m}{R_2^m} = \frac{R_2^m}{R_1^m}$ which again implies that $R_1 = R_2$, which is impossible. It is therefore ruled out that $\alpha = 0$. If $\alpha \neq 0$ we will have

$$\begin{pmatrix} J_m(\alpha R_1) & N_m(\alpha R_1) \\ J_m(\alpha R_2) & N_m(\alpha R_2) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $\begin{vmatrix} J_m(\alpha R_1) & N_m(\alpha R_1) \\ J_m(\alpha R_2) & N_m(\alpha R_2) \end{vmatrix} = 0$, i.e., α must be one of the zeros $\{\alpha_{n,\ell}\}$ of

$$J_m(\alpha R_1)N_m(\alpha R_2) - N_m(\alpha R_1)J_m(\alpha R_2) = 0.$$

Calling $c = \alpha R_1$ and $\rho = \frac{R_2}{R_1}$ you can write

$$J_m(c)N_m(\rho c) - N_m(c)J_m(\rho c) = 0.$$

The admissible energies are therefore

$$E_{m,\ell,n} = \frac{\hbar^2}{2M} \left[\frac{\pi^2 n^2}{L^2} + \frac{(c_{m,\ell})^2}{R_1^2} \right], \quad m \in \mathbb{Z}_+, \ell, n \in \mathbb{N}$$

and the corresponding eigenfunctions are

$$\left(N_m(c_{m,\ell})J_m\left(c_{m,\ell}\frac{r}{R_1}\right) - J_m(c_{m,\ell})N_m\left(c_{m,\ell}\frac{r}{R_1}\right) \right) e^{\pm im\theta} \sin n\pi \frac{z}{L}.$$

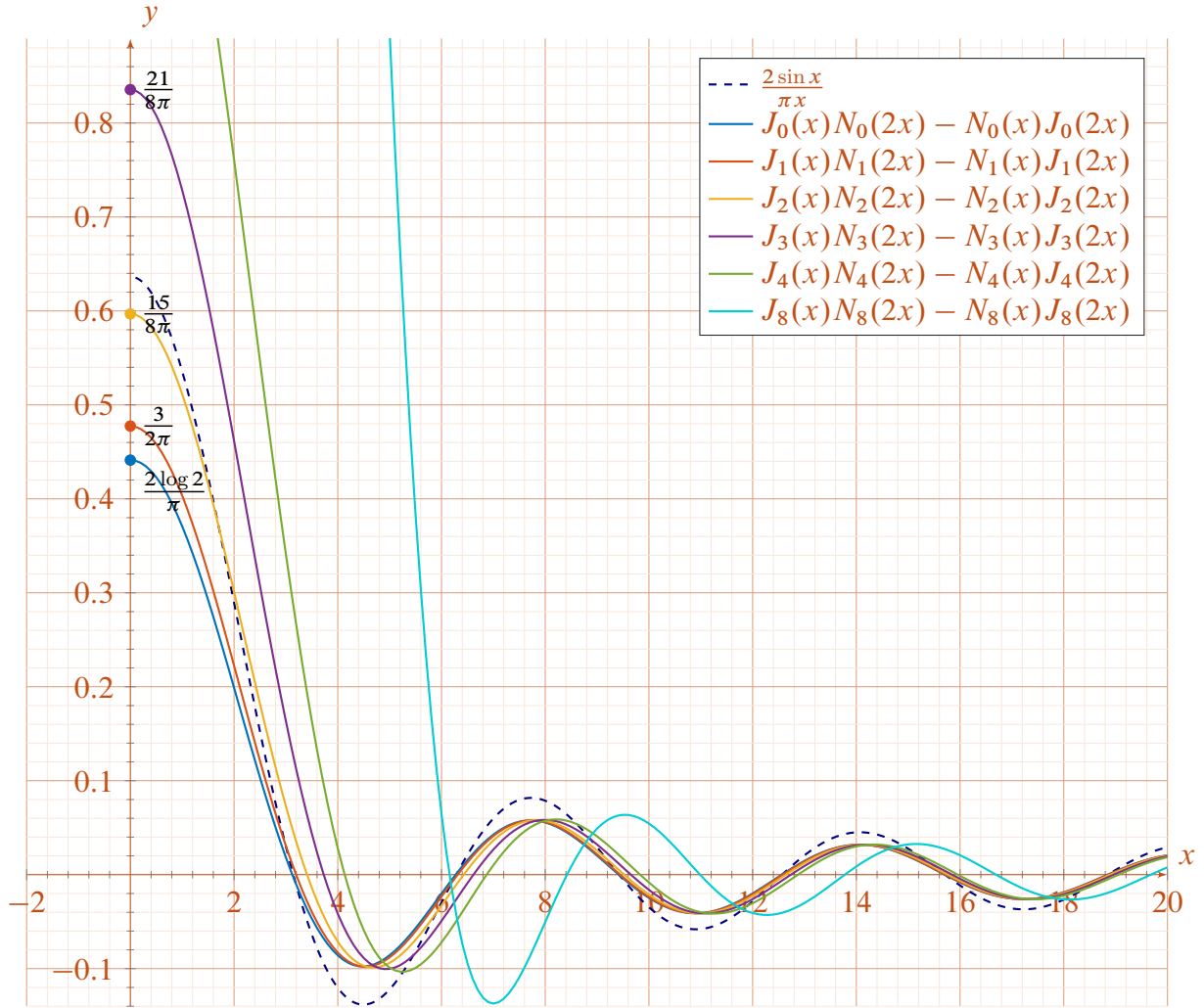
In the planar reduction to the annulus one obtains the energies

$$E_{m,\ell} = \frac{\hbar^2}{2M} \frac{(c_{m,\ell})^2}{R_1^2}, \quad m \in \mathbb{Z}_+, \ell \in \mathbb{N}$$

and the eigenfunctions

$$\left(N_m(c_{m,\ell})J_m\left(c_{m,\ell}\frac{r}{R_1}\right) - J_m(c_{m,\ell})N_m\left(c_{m,\ell}\frac{r}{R_1}\right) \right) e^{\pm im\theta}.$$

Let us put $R_1 = 1$ y $R_2 = 2$, with $\rho = 2$, we will have to find the collection of zeros of $f_m(x) := J_m(x)N_m(2x) - N_m(x)J_m(2x)$. We represent below the functions $f_m(x)$



From the asymptotic behaviors of the Bessel and Neumann functions to large x , and of the trigonometric relation $\sin(a - b) = \sin a \cos b - \sin b \cos a$, we deduce

$$\begin{aligned}
 f_m(x) &\sim \frac{2\pi}{x} \left(\cos \left(x - (2m+1)\frac{\pi}{4} \right) \sin \left(2x - (2m+1)\frac{\pi}{4} \right) \right. \\
 &\quad \left. - \sin \left(x - (2m+1)\frac{\pi}{4} \right) \cos \left(2x - (2m+1)\frac{\pi}{4} \right) \right) \\
 &= \frac{2 \sin x}{\pi x}.
 \end{aligned}$$

Then, for a given m and x large enough, *much larger* than $(2m+1)\frac{\pi}{4}$ the graph of $f_m(x)$ will practically overlap with that of $\frac{2 \sin x}{\pi x}$. On the other hand, the asymptotic behavior for small x is $f_0(x) \sim \frac{2 \log 2}{\pi}$ y $f_m(x) \sim \frac{4^m - 1}{m\pi 2^m}$.

The table of zeros $\{c_{m,\ell}\}$ is

ℓ	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$
1	3.1230	3.1966	3.4069	3.7289	4.1334	4.5950	5.0945	5.6179	6.1557	6.7015
2	6.2734	6.3123	6.4278	6.6159	6.8712	7.1866	7.5550	7.9687	8.4206	8.9038
3	9.4182	9.4445	9.5229	9.6522	9.8309	10.0564	10.3261	10.6371	10.9862	11.3703
4	12.5614	12.5812	12.6404	12.7385	12.8747	13.0481	13.2574	13.5012	13.7779	14.0858
5	15.7040	15.7199	15.7673	15.8462	15.9561	16.0964	16.2665	16.4656	16.6929	16.9474

Which leads to the following ordering

$$E_{0,1} < E_{1,1} < E_{2,1} < E_{3,1} < E_{4,1} < E_{5,1} < E_{6,1} < E_{7,1} < E_{8,1} \\ < E_{0,2} < E_{1,2} < E_{2,2} < E_{3,2} < E_{9,1} < \dots$$

The wave functions are, $m = 0, 1, 2, \dots$,

$$u_{m,\ell} = (N_m(c_{m,\ell})J_m(c_{m,\ell}r) - J_m(c_{m,\ell})N_m(c_{m,\ell}r))e^{\pm im\theta}.$$

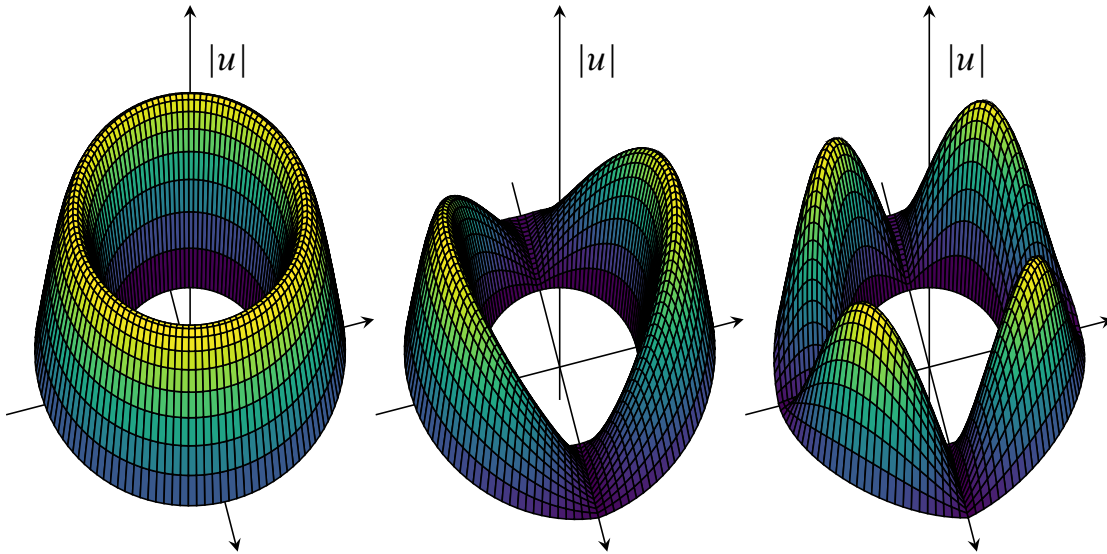
The fundamental eigenstate is

$$N_0(c_{0,1})J_0(c_{0,1}r) - J_0(c_{0,1})N_0(c_{0,1}r)$$

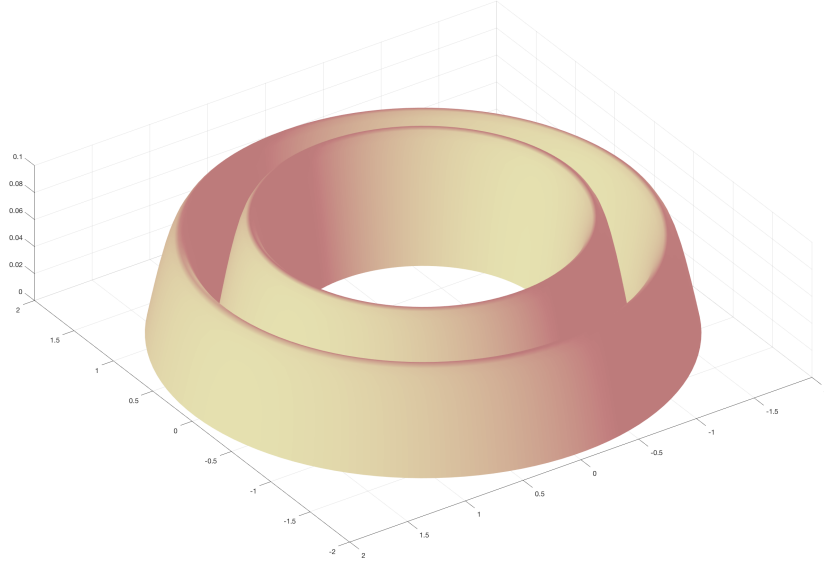
and the first two excited states, both degenerated, with a two-dimensional eigensubspace are expanded by

$$(N_1(c_{1,1})J_1(c_{1,1}r) - J_0(c_{0,1})N_1(c_{1,1}r))\sin(\theta - \beta), \\ (N_2(c_{2,1})J_1(c_{2,1}r) - J_0(c_{2,1})N_1(c_{2,1}r))\sin(2(\theta - \beta)),$$

Next we draw the probability densities corresponding to these three states



We now show the excited state g , which corresponds to $u = J_0(c_{0,2}r)$ with energy $\frac{\hbar^2}{2M}(c_{0,2})^2$



7.1.6. Fluid in a cylindrical pipe The u velocity potential of a stationary fluid in a circular section pipe like the one shown in the figure is characterized by the following Neumann boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0, \\ \frac{\partial u}{\partial r} \Big|_{r=a} = 0. \end{cases}$$

We have taken into account that

$$\nabla u = u_r \mathbf{u}_r + \frac{1}{r} u_\theta \mathbf{u}_\theta + u_z \mathbf{u}_z,$$

where the orthonormal trihedron $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ is constructed in terms of the Cartesian trihedron $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as follows

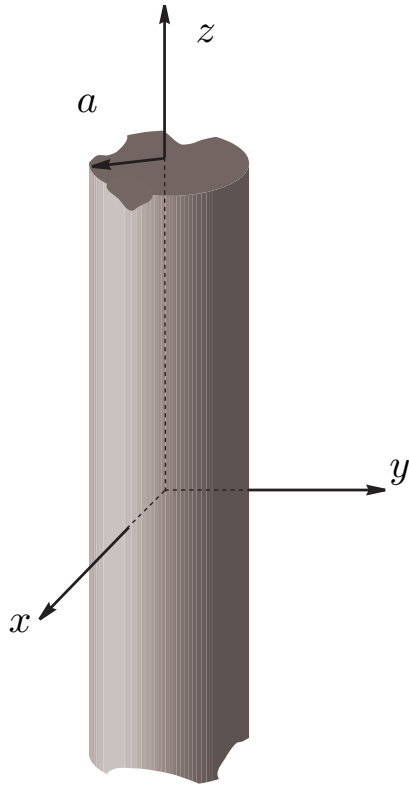
$$\mathbf{u}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad \mathbf{u}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad \mathbf{u}_z = \mathbf{k}.$$

Therefore, the unitary normal vector to the surface of the cylindrical pipe is precisely $\mathbf{n} = \mathbf{u}_r$ from which it follows that the normal derivative on that surface is

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u = u_r.$$

Let's impose these boundary conditions on

$$R_{\alpha,m}(r) \Theta_m(\theta) Z_\alpha(z)$$



where we take $k = 0$ (the PDE is the Laplace equation). The functions must be regular at the origin $r = 0$ as well as 2π -periodic in θ , then $m \in \mathbb{Z}_+$, and satisfy the homogeneous Neumann condition on the surface of the cylinder $r = a$. That is,

$$R'_{\alpha,m}(a) = 0.$$

For $\alpha = 0$ only the trivial solution, u constant, with the fluid at rest $\mathbf{v} = \mathbf{0}$ is possible. If $\alpha \neq 0$ the radial functions will be, by regularity in $r = 0$,

$$J_m(\alpha r)$$

and therefore the boundary condition is

$$J'_m(\alpha a) = 0.$$

If we denote $\{c'_{m,\ell}\}_{\ell \in \mathbb{N}}$ to zeros⁴ of J'_m then

$$\alpha_{m,\ell} = \frac{c'_{m,\ell}}{a}, \quad m \in \mathbb{Z}_+, \ell \in \mathbb{N}.$$

Finally, the solutions are

$$J_m\left(c'_{m,\ell} \frac{r}{a}\right) (C e^{im\theta} + D e^{-im\theta}) (A e^{\alpha_{m,\ell} z} + B e^{-\alpha_{m,\ell} z}).$$

§7.2. Spherical coordinates



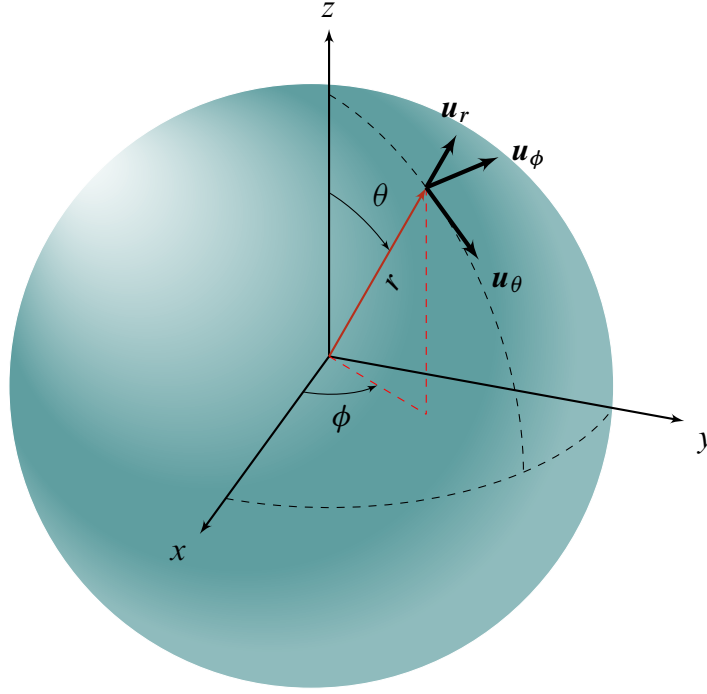
ELMHOLTZ equation in situations where boundary conditions to be satisfied are described simply in spherical coordinates is considered in this section. For its study it will be necessary the introduction of the spherical harmonics, eigenfunctions of the angular momentum and their real or tesseral versions. We will illustrate these developments with different examples of Quantum Physics, Fluid Mechanics and Electrostatics.

⁴Given the asymptotic behavior

$$J'_m(x) \sim -\sqrt{\frac{2}{\pi x}} \sin\left(x - (2m+1)\frac{\pi}{4}\right)$$

we observe for the zeros

$$c'_{m,\ell} \sim \ell\pi + (2m+1)\frac{\pi}{4}.$$



In spherical coordinates

$$\begin{cases} x = r \sin \theta \cos \phi, \\ y = r \sin \theta \sin \phi, \\ z = r \cos \theta, \end{cases}$$

the Helmholtz equation is written

$$r(ru)_{rr} + k^2 r^2 u + \frac{1}{\sin \theta} (\sin \theta u_\theta)_\theta + \frac{1}{\sin^2 \theta} u_{\phi\phi} = 0.$$

A first separation of variables:

$$u(r, \theta, \phi) = R(r)Y(\theta, \phi)$$

leads us to decouple the equation in radial and angular part¹ as follows

$$\begin{aligned} r(rR)'' + k^2 r^2 R &= \lambda R, \\ \frac{1}{\sin \theta} (\sin \theta Y_\theta)_\theta + \frac{1}{\sin^2 \theta} Y_{\phi\phi} &= -\lambda Y. \end{aligned}$$

The angular equation written in the form

$$\sin \theta (\sin \theta Y_\theta)_\theta + \lambda Y \sin^2 \theta + Y_{\phi\phi} = 0,$$

¹Note that

$$\frac{1}{\sin \theta} (\sin \theta Y_\theta)_\theta + \frac{1}{\sin^2 \theta} Y_{\phi\phi} = -\frac{L^2 Y}{\hbar^2}$$

where L is the angular momentum operator in Quantum Mechanics.

is separable. Thus, the factorization

$$Y(\theta, \phi) = P(\theta)\Phi(\phi),$$

leads to the following pair of ODEs

$$\begin{aligned}\sin \theta (\sin \theta P_\theta)_\theta + \lambda \sin^2 \theta P &= m^2 P, \\ \Phi_{\phi\phi} &= -m^2 \Phi.\end{aligned}$$

Consequently,

$$\boxed{\Phi(\phi) = C e^{im\phi} .}$$

We will assume that m belongs to $\mathbb{Z}$ in order to ensure continuity in the xz plane. On the other hand, using the variable

$$\xi := \cos \theta$$

the equation for P takes the form

$$(70) \quad \boxed{\frac{d}{d\xi} \left[(1 - \xi^2) \frac{dP}{d\xi} \right] + \left(\lambda - \frac{m^2}{1 - \xi^2} \right) P = 0.}$$

This is the **associated Legendre** equation that for $m = 0$ reduces to the Legendre ODE. One can check that if $\mathcal{P}(\xi)$ solves the Legendre ODE

$$(71) \quad \frac{d}{d\xi} \left[(1 - \xi^2) \frac{d\mathcal{P}}{d\xi} \right] + \lambda \mathcal{P} = 0,$$

then

$$P(\xi) =: (1 - \xi^2)^{|m|/2} \frac{d^{|m|} \mathcal{P}}{d\xi^{|m|}},$$

is a solution to the associated Legendre equation. The Legendre equation (71) has regular solutions at $\xi = \pm 1$ if and only if

$$(72) \quad \lambda = \ell(\ell + 1), \quad \ell = 0, 1, 2, \dots,$$

which are given by the Legendre polynomials of whose expression (without normalizing) is

$$\mathcal{P}_\ell(\xi) := \frac{d^\ell (\xi^2 - 1)^\ell}{d\xi^\ell}, \quad \ell = 0, 1, \dots$$

Therefore⁴ (70) will have regular solutions at $\xi = \pm 1$ if and only if the condition (72) is met, and the solutions of (70) are

$$(73) \quad P_\ell^{|m|}(\xi) := \frac{(-1)^m}{2^\ell \ell!} (1 - \xi^2)^{|m|/2} \frac{d^{|m|+\ell} (\xi^2 - 1)^\ell}{d\xi^{|m|+\ell}}, \quad |m| \leq \ell, \ell = 0, 1, \dots$$

⁴The associated Legendre equation is a problem of Sturm–Liouville in the interval $[-1, 1]$ with $\rho(\xi) = 1$, $p(\xi) = 1 - \xi^2$ and $q(x) = m^2/(1 - \xi^2)$. In spite of being singular, it has a complete orthogonal set of eigenfunctions:

$$\int_{-1}^1 P_\ell^{|m|}(\xi) P_{\ell'}^{|m|}(\xi) d\xi = 0, \quad \text{si } \ell \neq \ell'.$$

where the factor $(-1)^m$ is the so called Condon–Shortley phase. It should also be noted that these solutions are not trivial if and only if $|m| \leq \ell$, and that for $m = 0$ we recover the Legendre polynomials $P_\ell(\xi)$. Inspired by

$$\frac{d^{\ell-m}}{d\xi^{\ell-m}}(\xi^2 - 1)^\ell = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} (1 - \xi^2)^m \frac{d^{\ell+m}}{d\xi^{\ell+m}}(\xi^2 - 1)^\ell,$$

we define the associated Legendre functions $P_\ell^m(\xi)$ for any $m \in \mathbb{Z}$ as follows

$$(74) \quad P_\ell^{-|m|}(\xi) := (-1)^m \frac{(\ell - |m|)!}{(\ell + |m|)!} P_\ell^{|m|}(\xi), \quad |m| \leq \ell, \ell = 0, 1, \dots$$

Associated Legendre Functions

For $\ell = 0, 1, 2, \dots$ and $m = 0, \pm 1, \dots, \pm \ell$, the associated Legendre functions are given by

$$P_\ell^m(\xi) = \frac{(-1)^m}{2^\ell \ell!} (1 - \xi^2)^{m/2} \frac{d^{\ell+m}}{d\xi^{\ell+m}}(\xi^2 - 1)^\ell.$$

Note that for m even the associated Legendre functions are polynomials and they are not polynomials for m odd, in that case appears a factor $\sqrt{1 - \xi^2}$ multiplying a polynomial. Nevertheless, $P_\ell^m(\cos \theta)$ is a trigonometric polynomial in $\sin \theta$ and $\cos \theta$. This is why these associated Legendre functions are sometimes called Legendre associated “polynomials”.

Associated Legendre Functions

The first associated Legendre functions are

$$\begin{aligned} P_0^0(x) &= 1, \\ P_1^1(x) &= -\sqrt{1-x^2}, \quad P_1^0(x) = x, \quad P_1^{-1}(x) = \frac{1}{2}\sqrt{1-x^2}, \\ P_2^2(x) &= 3(1-x^2), \quad P_2^1(x) = -3x\sqrt{1-x^2}, \quad P_2^0(x) = \frac{1}{2}(3x^2-1), \\ &\quad P_2^{-1}(x) = \frac{1}{2}x\sqrt{1-x^2}, \quad P_2^{-2}(x) = \frac{1}{8}(1-x^2), \\ P_3^3(x) &= -15(1-x^2)\sqrt{1-x^2}, \quad P_3^2(x) = 15x(1-x^2), \\ P_3^1(x) &= -\frac{3}{2}(5x^2-1)\sqrt{1-x^2}, \quad P_3^0(x) = \frac{1}{2}(5x^3-3x), \quad P_3^{-1}(x) = \frac{1}{8}(5x^2-1)\sqrt{1-x^2}, \\ &\quad P_3^{-2}(x) = \frac{1}{8}x(1-x^2), \quad P_3^{-3}(x) = \frac{1}{48}(1-x^2)\sqrt{1-x^2}. \end{aligned}$$

Spherical Harmonics

The solutions of the angular equation, which are known as *spherical harmonics*, for $\ell = 0, 1, 2, \dots$ and $m = 0, \pm 1, \dots, \pm \ell$, can be written in following the form

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}.$$

We must mention that there are other ways to introduce spherical harmonics. This is the form used in Quantum Mechanics. In other fields of Physics other normalizations are used, for example in Seismology: $Y_\ell^m(\theta, \phi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$, in Geodesy and Geophysics: $Y_\ell^m(\theta, \phi) = \sqrt{(2\ell+1) \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$ in Magnetism $Y_\ell^m(\theta, \phi) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$. The spherical harmonics fulfill

$$\bar{Y}_\ell^m(\theta, \phi) = (-1)^m Y_\ell^{-m}(\theta, \phi),$$

where $\bar{Y}_\ell^m$ is the complex conjugate of the spherical harmonic.

Spherical Harmonics Expansions

The spherical harmonics constitute a complete orthonormal set in

$$L^2(S^2) := \left\{ f = f(\theta, \phi) : \int_{S^2} |f(\theta, \phi)|^2 dS < \infty \right\},$$

where $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ unitary sphere $\mathbb{R}^3$, and $dS = \sin \theta d\theta d\phi$ is the area element in the sphere. The scalar product is

$$(f, g) := \int_{S^2} \bar{f}(\theta, \phi) g(\theta, \phi) dS = \int_0^{2\pi} \int_0^\pi \bar{f}(\theta, \phi) g(\theta, \phi) \sin \theta d\theta d\phi.$$

So, we have the orthonormal basis $\{Y_\ell^m\}_{\ell=0,1,\dots, m=-\ell,\dots,\ell}$. That is, they form an orthonormal set

$$(Y_\ell^m, Y_{\ell'}^{m'}) = \delta_{\ell,\ell'} \delta_{m,m'}$$

and any function $f(\theta, \phi)$ on $L^2(S^2)$ has an expansion

$$f(\theta, \phi) = \sum_{\ell,m} c_\ell^m Y_\ell^m(\theta, \phi)$$

where

$$c_\ell^m = (Y_\ell^m, f) = \int_0^{2\pi} \int_0^\pi \bar{Y}_\ell^m(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi.$$

Spherical harmonics give the atomic orbitals of Quantum Physics and Quantum Chemistry, which are called orbitals or s waves (*sharp*, $\ell = 0$), p (*main*, $\ell = 1$), d (*diffuse*, $\ell = 2$) and f (*fundamental*, $\ell = 3$), for greater angular moments $\ell = 4, 5, 6, 7, \dots$ the letters g, h, i, k, ... are used, in alphabetical order and omitting the j, since some languages do not distinguish them. Next we give a table of the first spherical harmonics, where we have used that $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$,

$$\begin{aligned}
 \text{s wave, } \ell = 0 : \quad Y_0^0(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{1}{\pi}}, \\
 \text{p wave, } \ell = 1 : \quad \left\{ \begin{aligned} Y_1^{-1}(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi} = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \frac{(x - iy)}{r}, \\ Y_1^0(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r}, \\ Y_1^1(\theta, \phi) &= -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi} = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \frac{(x + iy)}{r}, \end{aligned} \right. \\
 \text{d wave, } \ell = 2 : \quad \left\{ \begin{aligned} Y_2^{-2}(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{-2i\phi} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x - iy)^2}{r^2}, \\ Y_2^{-1}(\theta, \phi) &= \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{-i\phi} = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x - iy)z}{r^2}, \\ Y_2^0(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1) = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{(2z^2 - x^2 - y^2)}{r^2}, \\ Y_2^1(\theta, \phi) &= -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\phi} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)z}{r^2}, \\ Y_2^2(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)^2}{r^2} \end{aligned} \right. \\
 \text{f wave, } \ell = 3 : \quad \left\{ \begin{aligned} Y_3^{-3}(\theta, \phi) &= \frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3 \theta e^{-3i\phi} = \frac{1}{8} \sqrt{\frac{35}{\pi}} \frac{(x - iy)^3}{r^3} \\ Y_3^{-2}(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{-2i\phi} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \frac{(x - iy)^2 z}{r^3} \\ Y_3^{-1}(\theta, \phi) &= \frac{1}{8} \sqrt{\frac{21}{\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{-i\phi} = \frac{1}{8} \sqrt{\frac{21}{\pi}} \frac{(x - iy)(4z^2 - x^2 - y^2)}{r^3} \\ Y_3^0(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{7}{\pi}} (5 \cos^3 \theta - 3 \cos \theta) = \frac{1}{4} \sqrt{\frac{7}{\pi}} \frac{z(2z^2 - 3x^2 - 3y^2)}{r^3} \\ Y_3^1(\theta, \phi) &= -\frac{1}{8} \sqrt{\frac{21}{\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi} = -\frac{1}{8} \sqrt{\frac{21}{\pi}} \frac{(x + iy)(4z^2 - x^2 - y^2)}{r^3} \\ Y_3^2(\theta, \phi) &= \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\phi} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \frac{(x + iy)^2 z}{r^3} \\ Y_3^3(\theta, \phi) &= -\frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3 \theta e^{3i\phi} = -\frac{1}{8} \sqrt{\frac{35}{\pi}} \frac{(x + iy)^3}{r^3} \end{aligned} \right.
 \end{aligned}$$

Real or Tesseral Spherical Harmonics

You can build a base of real spherical harmonics $Y_{\ell,m}(\theta, \phi)$, also known as tesseral spherical harmonics, with

$$Y_{\ell,m}(\theta, \phi) = \begin{cases} \frac{i}{\sqrt{2}}(Y_{\ell}^m(\theta, \phi) - (-1)^m Y_{\ell}^{-m}(\theta, \phi)), & m < 0, \\ Y_{\ell}^0(\theta, \phi), & m = 0, \\ \frac{i}{\sqrt{2}}(Y_{\ell}^m(\theta, \phi) + (-1)^m Y_{\ell}^{-m}(\theta, \phi)), & m > 0, \end{cases}$$

which are the real and imaginary parts

$$Y_{\ell,m}(\theta, \phi) = \begin{cases} \sqrt{2}(-1)^m \operatorname{Im} Y_{\ell}^{|m|}(\theta, \phi), & m < 0, \\ Y_{\ell}^0(\theta, \phi), & m = 0, \\ \sqrt{2}(-1)^m \operatorname{Re} Y_{\ell}^m(\theta, \phi), & m > 0. \end{cases}$$

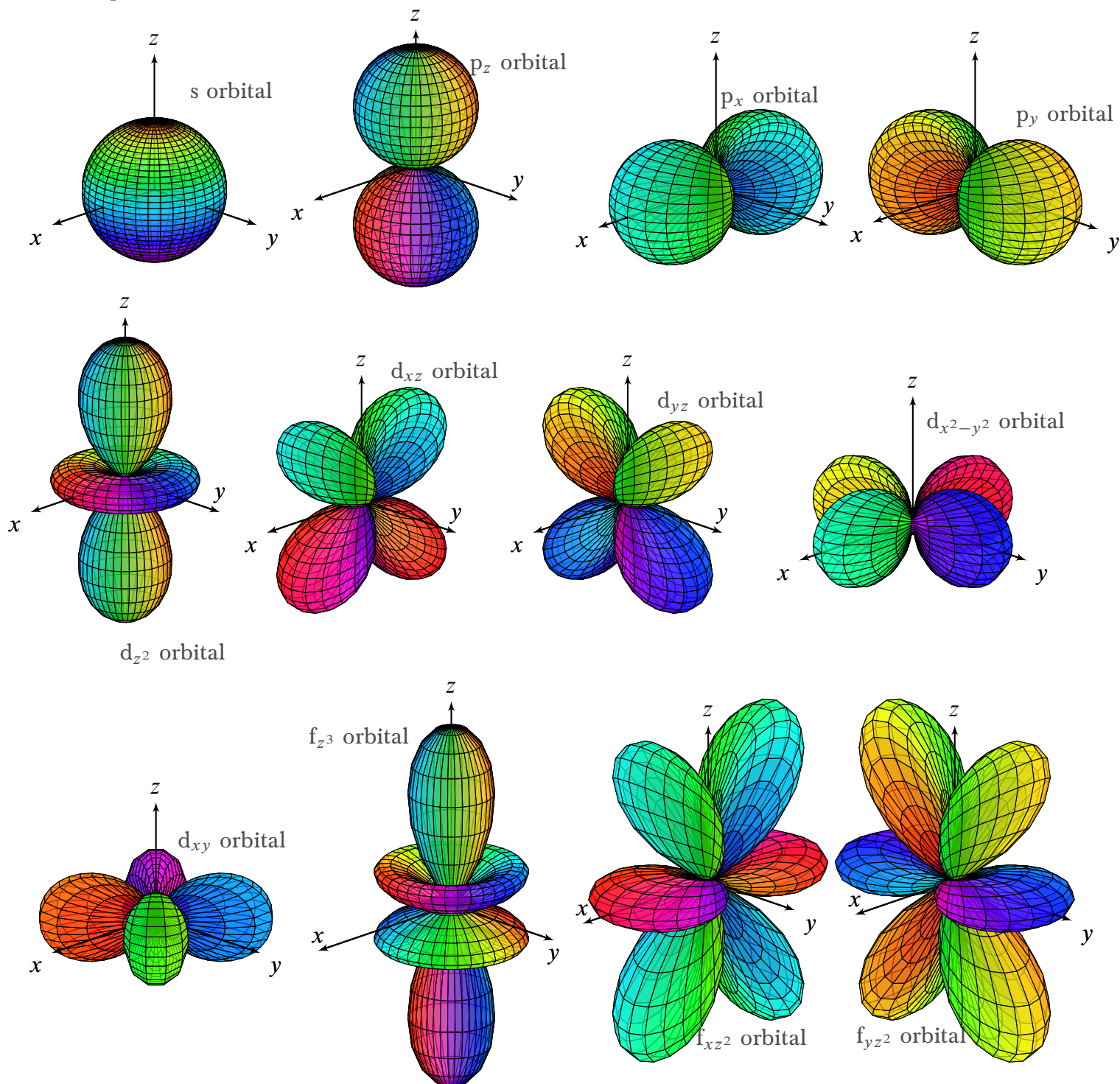
These tesseral harmonics form an orthonormal basis of $L^2(S^2)$, said cosine for $m > 0$, and sine for $m < 0$

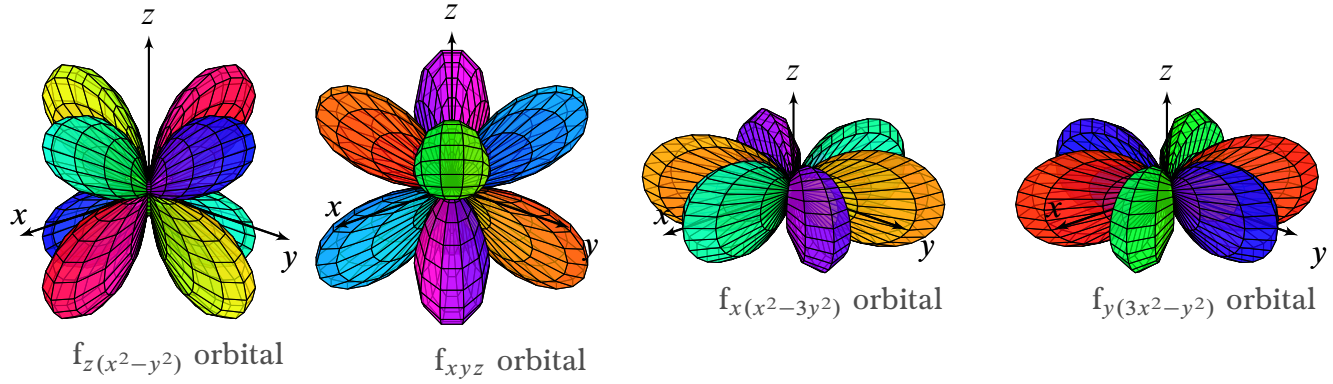
$$Y_{\ell,m}(\theta, \phi) = \begin{cases} \sqrt{2} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_{\ell}^{|m|}(\cos \theta) \sin |m|\phi, & m < 0, \\ \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}^m(\cos \theta), & m = 0, \\ \sqrt{2} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^m(\cos \theta) \cos m\phi & m > 0. \end{cases}$$

Below is a table of the different tesseral harmonics, using s, p, d, f wave notation and presenting the different multipolar components of the orbitals.

$$\begin{aligned}
 &\text{wave s, } l = 0 : Y_{0,0}(\theta, \phi) = s(\theta, \phi) = \frac{1}{2} \sqrt{\frac{1}{\pi}}, \\
 &\text{wave p, } l = 1 : \left\{ \begin{aligned} Y_{1,-1}(\theta, \phi) &= p_y(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \frac{y}{r} = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi, \\ Y_{1,0}(\theta, \phi) &= p_z(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \frac{z}{r} = \sqrt{\frac{3}{4\pi}} \cos \theta, \\ Y_{1,1}(\theta, \phi) &= p_x(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \frac{x}{r} = \sqrt{\frac{3}{4\pi}} \sin \theta \cos \phi, \end{aligned} \right. \\
 &\text{wave d, } l = 2 : \left\{ \begin{aligned} Y_{2,-2}(\theta, \phi) &= d_{xy}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{xy}{r^2} = \frac{1}{2} \sqrt{\frac{15}{\pi}} \sin^2 \theta \cos \phi \sin \phi, \\ Y_{2,-1}(\theta, \phi) &= d_{yz}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{yz}{r^2} = \frac{1}{2} \sqrt{\frac{15}{\pi}} \cos \theta \sin \theta \sin \phi, \\ Y_{2,0}(\theta, \phi) &= d_{z^2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} \frac{-x^2 - y^2 + 2z^2}{r^2} = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1) \\ Y_{2,1}(\theta, \phi) &= d_{xz}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{zx}{r^2} = \frac{1}{2} \sqrt{\frac{15}{\pi}} \cos \theta \sin \theta \cos \phi, \\ Y_{2,2}(\theta, \phi) &= d_{x^2-y^2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{15}{\pi}} \frac{x^2 - y^2}{r^2} = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) \end{aligned} \right. \\
 &\text{wave f, } l = 3 : \left\{ \begin{aligned} Y_{3,-3}(\theta, \phi) &= f_{y(3x^2-y^2)}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{35}{2\pi}} \frac{y(3x^2 - y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{35}{2\pi}} \sin^3 \theta \sin \phi (4 \cos^2 \phi - 1), \\ Y_{3,-2}(\theta, \phi) &= f_{xyz}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{105}{\pi}} \frac{xyz}{r^3} = \frac{1}{2} \sqrt{\frac{105}{\pi}} \cos \theta \sin^2 \theta \sin \phi \cos \phi, \\ Y_{3,-1}(\theta, \phi) &= f_{yz^2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{21}{2\pi}} \frac{y(4z^2 - x^2 - y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{21}{2\pi}} (5 \cos^2 \theta - 1) \sin \theta \sin \phi, \\ Y_{3,0}(\theta, \phi) &= f_{z^3}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{7}{\pi}} \frac{z(2z^2 - 3x^2 - 3y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{7}{\pi}} (5 \cos^2 \theta - 3) \cos \theta, \\ Y_{3,1}(\theta, \phi) &= f_{xz^2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{21}{2\pi}} \frac{x(4z^2 - x^2 - y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{21}{2\pi}} (5 \cos^2 \theta - 1) \sin \theta \cos \phi, \\ Y_{3,2}(\theta, \phi) &= f_{z(x^2-y^2)}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{105}{\pi}} \frac{z(x^2 - y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{105}{\pi}} \cos \theta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi), \\ Y_{3,3}(\theta, \phi) &= f_{x(x^2-3y^2)}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{35}{2\pi}} \frac{x(x^2 - 3y^2)}{r^3} = \frac{1}{4} \sqrt{\frac{35}{2\pi}} \sin^3 \theta \cos \phi (1 - 4 \sin^2 \phi). \end{aligned} \right.
 \end{aligned}$$

We are going to represent the orbitals, or tesseral waves, as the surface determined by the polar equation $r = \sqrt{4\pi}|Y_{\ell,m}(\theta, \phi)|$, which are therefore not level surfaces





7.2.1. Radial equation We distinguish two cases according to whether k is null or not.

- If $k = 0$ the radial equation reads

$$r^2 R'' + 2rR' - \ell(\ell + 1)R = 0$$

which is an Euler ODE. Trying solutions of the form r^α we immediately conclude that a general solution that is of the form

$$(75) \quad R(r) = Ar^\ell + B\frac{1}{r^{\ell+1}}.$$

- For $k \neq 0$, the radial ODE for R is

$$r^2 R'' + 2rR' + (k^2 r^2 - \ell(\ell + 1))R = 0.$$

Bearing in mind that with the change of variable $x = kr$ this radial equation becomes the differential equation which satisfy the spherical functions of Bessel and Neumann

$$j_\ell(x) := (-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\sin x}{x}, \quad n_\ell(x) := -(-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\cos x}{x},$$

it is clear that the general solution of the radial equation for $k \neq 0$ is

$$(76) \quad R_\ell(r) = Aj_\ell(kr) + Bn_\ell(kr).$$

The first spherical functions are:

	$\ell = 0$	$\ell = 1$	$\ell = 2$
$j_\ell(r)$	$\frac{\sin r}{r}$	$\frac{\sin r}{r^2} - \frac{\cos r}{r}$	$3\frac{\sin r}{r^3} - 3\frac{\cos r}{r^2} - \frac{\sin r}{r}$
$n_\ell(r)$	$-\frac{\cos r}{r}$	$-\frac{\cos r}{r^2} - \frac{\sin r}{r}$	$-3\frac{\cos r}{r^3} - 3\frac{\sin r}{r^2} + \frac{\cos r}{r}$

The behavior of these functions in the origin⁶ is

$$(77) \quad \begin{aligned} j_\ell(r) &\sim \frac{r^\ell}{(2\ell+1)!!}, \\ n_\ell(r) &\sim -\frac{(2\ell-1)!!}{r^{\ell+1}}, \end{aligned} \quad r \rightarrow 0.$$

7.2.2. Quantum particle in a spherical box The stationary states of energy E of a free particle inside a sphere of radius a with impenetrable walls are described by the solutions of the following boundary value problem:

$$\begin{cases} \Delta u + k^2 u = 0, & E = \frac{\hbar^2 k^2}{2M}, \\ u|_{r=a} = 0. \end{cases}$$

We must impose on the solutions of the Helmholtz equation

$$u(r, \theta, \phi) = R_{k,\ell}(r) Y_{\ell,m}(\theta, \phi)$$

both the regularity at the origin and the boundary value conditions. The regularity at the origin imposes that $B = 0$ in (75) and (76). On the other hand, the boundary value conditions for the case (75) leads to $A = 0$ and the trivial solution, while for (76) it leads to

$$j_\ell(ka) = 0.$$

Hence, if $\{c_{\ell,n}\}_{n=1}^\infty$ are the zeros, which form an increasing sequence, of the spherical function of Bessel j_ℓ the admissible values of energy are

$$E_{\ell,n} = \frac{\hbar^2}{2M} \frac{c_{\ell,n}^2}{a^2}$$

and the corresponding stationary states

$$u_{\ell,m,n}(r, \theta, \phi) = j_\ell\left(\frac{c_{\ell,n}}{a}r\right) Y_{\ell,m}(\theta, \phi).$$

The probability density $|u_{\ell,m,n}|^2$ does not depend on the variable ϕ . Therefore, we can represent, using polar coordinates (r, θ) , the probability in the planes that contain the axis z , $\phi = \text{constant}$, and this representation will be the same for all these planes.

The first three zeros are

$$c_{0,1} = \pi, \quad c_{1,1} \cong 4.493409458, \quad c_{2,1} \cong 5.763459197,$$

⁶The functions of Bessel and spherical Neumann have the following behavior in infinity

$$\begin{aligned} j_\ell(r) &= \frac{1}{r} \cos\left(r - (2\ell+1)\frac{\pi}{2}\right) \\ n_\ell(r) &= \frac{1}{r} \sin\left(r - (2\ell+1)\frac{\pi}{2}\right) \end{aligned} \quad r \rightarrow \infty.$$

and therefore the energy of the fundamental state and of the first excited state is approximately given by

$$\frac{\hbar^2}{2Ma^2}9.869604404, \quad \frac{\hbar^2}{2Ma^2}20.190728557, \quad \frac{\hbar^2}{2Ma^2}33.217461915.$$

7.2.3. Fluid inside a sphere The velocity potential u for a stationary fluid in a sphere of radio a is characterized by the following Neumann problem

$$\begin{cases} \Delta u = 0, \\ u_r|_{r=a} = 0. \end{cases}$$

Where we have taken into account that

$$\nabla u = u_r \mathbf{u}_r + \frac{1}{r} u_\theta \mathbf{u}_\theta + \frac{1}{r \sin \theta} u_\phi \mathbf{u}_\phi$$

and the orthonormal basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi\}$ is

$$\begin{cases} \mathbf{u}_r = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}, \\ \mathbf{u}_\theta = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k}, \\ \mathbf{u}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}. \end{cases}$$

Therefore, the unitary normal to the spherical surface is $\mathbf{n} = \mathbf{u}_r$ so that

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u = u_r.$$

We have a Helmholtz equation with $k = 0$ (Laplace equation). The regularity at the origin implies that $B = 0$ so the possible eigenfunctions are of the form

$$r^\ell Y_{\ell,m}(\theta, \phi).$$

The boundary condition implies that $\ell = 0$ and therefore the potential is constant. This means that the fluid is at rest.

7.2.4. EEM in problems of Electrostatics and Fluid Mechanics with spherical symmetry In Electrostatics it is common to find boundary value problems such as

$$\begin{cases} \Delta u = \rho, \\ a_i(u) = g_i, \quad i = 1, \dots, n \end{cases}$$

where ρ is the charge density, u the electric potential, and therefore the electric field is $\mathbf{E} = -\nabla u$, and boundary operators a_i act only on r and $g_i = g_i(\theta, \phi)$. A similar problem also appears in Fluid Mechanics

$$\begin{cases} \Delta u = 0, \\ a_i(u) = g_i, \quad i = 1, \dots, n \end{cases}$$

where u is the potential for fluid velocities, and therefore the velocity is $\mathbf{v} = \nabla u$. The Poisson equation in spherical coordinates is written as follows

$$\frac{1}{r}(ru)_{rr} + \frac{1}{r^2 \sin \theta}(\sin \theta u_\theta)_\theta + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi} = \rho(r, \theta, \phi).$$

To apply the EEM we write the Poisson equation as

$$(A + B)u = r^2 \rho$$

where

$$\begin{aligned} Au &:= r^2 u_{rr} + 2ru_r, \\ Bu &:= \frac{1}{\sin \theta} (\sin \theta u_\theta)_\theta + \frac{1}{\sin^2 \theta} u_{\phi\phi}. \end{aligned}$$

The B operator is symmetric in $C^\infty(S^2)$ and the corresponding eigenvalue problem is solved as follows

$$BY_{l,m} = -l(l+1)Y_{l,m}, \quad l = 0, 1, 2, \dots, m = -l, \dots, l.$$

As we know the spherical harmonics form a complete set in $L^2(S^2)$ and that's why we can always expand

$$\begin{aligned} r^2 \rho(r, \theta, \phi) &= \sum_{l,m} \rho_{l,m}(r) Y_{l,m}(\theta, \phi), \\ g_i(\theta, \phi) &= \sum_{l,m} c_{i,l,m} Y_{l,m}(\theta, \phi). \end{aligned}$$

Therefore, the EEM seeks solutions in the form of

$$u(r, \theta, \phi) = \sum_{l,m} v_{l,m}(r) Y_{l,m}(\theta, \phi),$$

where the functions $v_{l,m}(r)$ are subject to

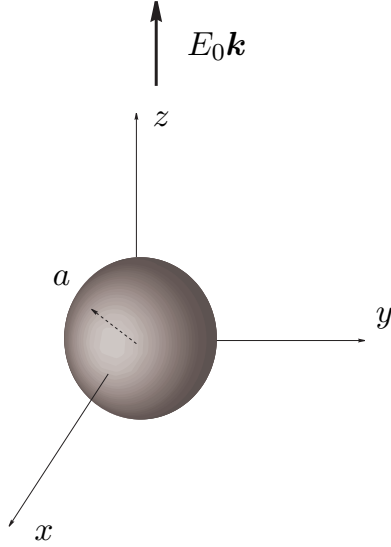
$$\begin{cases} r^2 v''_{l,m} + 2r v'_{l,m} - l(l+1)v_{l,m} = \rho_{l,m}, \\ a_i(v_{l,m}) = c_{ilm} \end{cases}$$

In the homogeneous case, $\rho = 0$, corresponds to Laplace equation, and it has as a solution

$$v_{l,m}(r) = A_{l,m} r^l + \frac{B_{l,m}}{r^{l+1}}.$$

7.2.5. Charged conducting sphere in electrostatic equilibrium in a constant electric field Let us consider a conducting sphere, with radius a and centered at the origin, with charge Q and that has reach the equilibrium immerse in a constant electric field $E_0 \mathbf{k}$. As we know, equilibrium is reached if the potential on the sphere is constant, say u_0 . The electrostatic potential u is the solution of the following boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0, \\ \begin{cases} u|_{r=a} = u_0, \\ u \sim -E_0 z + V_0 + \mathcal{O}\left(\frac{1}{r}\right), \quad r \rightarrow \infty. \end{cases} \end{cases}$$



Applying the EEM, we look for a solution of the form

$$u(r, \theta, \phi) = \sum_{\ell, m} \left(A_{\ell, m} r^\ell + B_{\ell, m} \frac{1}{r^{\ell+1}} \right) Y_{\ell, m}(\theta, \phi).$$

The boundary value condition in $r = a$ can be written

$$u|_{r=a} = u_0 \sqrt{4\pi} Y_{0,0},$$

whereas, like $z = r \cos \theta$, the condition in infinity is

$$u \sim -E_0 r \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \phi) + V_0 \sqrt{4\pi} Y_{0,0} + \mathcal{O}\left(\frac{1}{r}\right).$$

Thus, we can restrict ourselves to an expansion

$$u(r, \theta, \phi) = \left(A_{0,0} + B_{0,0} \frac{1}{r} \right) Y_{0,0}(\theta, \phi) + \left(A_{1,0} r + B_{1,0} \frac{1}{r^2} \right) Y_{1,0}(\theta, \phi).$$

We now impose the boundary value conditions to obtain

$$\begin{cases} A_{0,0} + B_{0,0} \frac{1}{a} = \sqrt{4\pi} u_0, \\ A_{1,0} a + B_{1,0} \frac{1}{a^2} = 0, \end{cases} \quad \begin{cases} A_{0,0} = \sqrt{4\pi} V_0, \\ A_{1,0} = -\sqrt{\frac{4\pi}{3}} E_0. \end{cases}$$

Therefore,

$$(78) \quad u(r, \theta, \phi) = V_0 + \frac{(u_0 - V_0)a}{r} - E_0 \left(r - \frac{a^3}{r^2} \right) \cos \theta.$$

The potential is composed of a monopolar term

$$\frac{(u_0 - V_0)a}{r}$$

and a dipolar term

$$-E_0 \left(r - \frac{a^3}{r^2} \right) \cos \theta.$$

The monopolar moment is $4\pi\epsilon_0(u_0 - V_0)a$ and the net charge is $Q = (u_0 - V_0)a$. Thus, the potential is

$$u(r, \theta, \phi) = V_0 + \frac{Q}{r} - E_0 \left(r - \frac{a^3}{r^2} \right) \cos \theta.$$

The electric field $\mathbf{E} = -\nabla u$ is

$$\mathbf{E} := \left[\frac{Q}{r^2} + E_0 \left(1 + 2\frac{a^3}{r^3} \right) \cos \theta \right] \mathbf{u}_r - E_0 \left(1 - \frac{a^3}{r^3} \right) \sin \theta \mathbf{u}_\theta.$$

7.2.6. Fluid in uniform motion deformed by a spherical ball Let us now consider a fluid in uniform motion in which we submerge a solid sphere of radius a . The velocity potential must satisfy the following boundary problem

$$\begin{cases} \Delta u = 0, \\ u_r|_{r=a} = 0, \\ u \sim v_0 z + \mathcal{O}\left(\frac{1}{r}\right), \quad r \rightarrow \infty. \end{cases}$$

As in the previous problem we seek for a solution of the form

$$u(r, \theta, \phi) = \sum_{\ell, m} \left(A_{\ell, m} r^\ell + B_{\ell, m} \frac{1}{r^{\ell+1}} \right) Y_{\ell, m}(\theta, \phi).$$

In addition the boundary value conditions can be written as follows

$$\begin{aligned} u_r|_{r=a} &= 0, \\ u(r, \theta, \phi) &\sim v_0 r \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \phi) + \mathcal{O}\left(\frac{1}{r}\right), \quad r \rightarrow \infty. \end{aligned}$$

Thus, we restrict ourselves to solutions of the form

$$u(r, \theta, \phi) = \left(A_{1,0} r + B_{1,0} \frac{1}{r^2} \right) Y_{1,0}(\theta, \phi).$$

Imposing boundary conditions

$$\begin{cases} A_{1,0} = \sqrt{\frac{4\pi}{3}} v_0, \\ A_{1,0} - 2B_{1,0} \frac{1}{a^3} = 0. \end{cases}$$

Therefore

$$B_{1,0} = \frac{1}{2} \sqrt{\frac{4\pi}{3}} v_0 a^3.$$

So

$$u(r, \theta, \phi) = v_0 \left(r + \frac{a^3}{2r^2} \right) \cos \theta$$

and the velocity is

$$\mathbf{v} = v_0 \left(\left[1 - \frac{a^3}{r^3} \right] \cos \theta \mathbf{u}_r - \left[1 + \frac{a^3}{2r^3} \right] \sin \theta \mathbf{u}_\theta \right).$$

§7.3. Exercises

7.3.1. Exercises with solutions

1. Determine the standing waves in spherical coordinates, $u(t, r, \theta, \phi) = e^{-i\omega t} w(r, \theta, \phi)$ of the following wave equation

$$\begin{cases} u_{tt} = \Delta u, & r < a, \\ u|_{r=a} = 0. \end{cases}$$

Solution: Standing waves are obtained by considering the separation of variables $u(t, r, \theta, \phi) = e^{-i\omega t} w(r, \theta, \phi)$, where the following dispersion relation and Helmholtz equation are satisfied

$$\omega = \pm k, \quad \Delta w + k^2 w = 0.$$

By imposing the condition of regularity at $r = 0$ it is clear that admissible $w(r, \theta, \phi)$ are $j_l(kr)Y_{l,m}(\theta, \phi)$ with $j_l(ka) = 0$ and $\omega = \pm k$.

2. Determine the regular solutions, in cylindrical coordinates, $u(r, \theta, z)$ of the following Neumann problem

$$\begin{cases} \Delta u = 0, & r < a, \\ \frac{\partial u}{\partial r}\bigg|_{r=a} = 0. \end{cases}$$

Solution: The functions that satisfy all requirements are those of the form $J_m(\alpha r) e^{im\theta + \alpha z}$ with $m \in \mathbb{Z}$, $J'_m(\alpha a) = 0$.

3. Find the value of the constant a for which there is a solution to the following boundary value problem in spherical coordinates

$$\begin{cases} \Delta u = 1, & r < 1, \\ \frac{\partial u}{\partial r}\bigg|_{r=1} = a. \end{cases}$$

Solution: Applying the divergence theorem in the ball $B(0, 1)$, we have

$$\int_{B(0,1)} \Delta u d^3x = \int_{S(0,1)} \nabla u \cdot d\mathbf{S}$$

as $\Delta u = 1$ and $u_r dS = \nabla u \cdot d\mathbf{S}$ we conclude

$$\frac{4}{3}\pi = a4\pi$$

so that $a = 1/3$.

4. Solve, by the method of separating variables in cylindrical coordinates, the following boundary value problem

$$\begin{cases} \Delta u = 0, & 1 < r < 2, \\ u|_{r=1} = u|_{r=2}. \end{cases}$$

Solution: As we know the resolution by separation of variables in cylindrical coordinates of the Laplace equation leads to consider

$$u(r, \theta, z) = R_{\alpha, m}(r) \Theta_m(\theta) Z_\alpha(z),$$

with

$$\begin{cases} Z_\alpha(z) = \begin{cases} Az + B, & \alpha = 0, \\ A e^{\alpha z} + B e^{-\alpha z}, & \alpha \neq 0, \end{cases} \\ \Theta_m(\theta) = \begin{cases} C\theta + D, & m = 0, \\ C e^{im\theta} + D e^{-im\theta}, & m \neq 0 \end{cases} \\ R_{\alpha, m}(r) = \begin{cases} c_1 \log r + c_2, & \alpha = m = 0, \\ c_1 r^m + c_2 r^{-m}, & \alpha = 0, m \neq 0, \\ E J_m(\alpha r) + F N_m(\alpha r), & \alpha \neq 0. \end{cases} \end{cases}$$

Not all of these functions are solutions to our problem. **As we want the solution to be single valued** $\Rightarrow m = 0, 1, 2, \dots$. The boundary condition, which is imposed on the $R_{\alpha, m}$, determine the following possibilities

a) $\alpha = m = 0 \Rightarrow c_2 = c_1 \log 2 + c_2$ and so $c_1 = 0$. Hence,

$$R_{0,0} = 1.$$

b) $\alpha = 0, m \neq 0 \Rightarrow c_1 + c_2 = 2^m c_1 + 2^{-m} c_2$. Therefore,

$$R_{0,m}(r) = (1 - 2^{-m}) r^m - (1 - 2^m) r^{-m}.$$

c) $\alpha \neq 0 \Rightarrow E J_m(\alpha) + F N_m(\alpha) = E J_m(2\alpha) + F N_m(2\alpha)$. Thus,

$$R_{\alpha, m}(r) = (N_m(\alpha) - N_m(2\alpha)) J_m(\alpha r) - (J_m(\alpha) - J_m(2\alpha)) N_m(\alpha r).$$

5. The solutions of the Helmholtz equation in spherical coordinates are of the form

$$[A_{lm} j_l(kr) + B_{lm} n_l(kr)] Y_{l,m}(\theta, \phi).$$

Determine the values of k for which there are non-trivial solutions with $l = 0$ of the boundary problem

$$\begin{cases} \Delta u + k^2 u = 0, & 1 < r < 2, \\ \begin{cases} u|_{r=1} = 0, \\ \frac{\partial u}{\partial r}|_{r=2} = 0. \end{cases} \end{cases}$$

$$\text{Hint: } j_0(r) = \frac{\sin r}{r}, \quad n_0(r) = -\frac{\cos r}{r}.$$

Solution: The solution has the form

$$u(r, \theta, \phi) = (A j_0(kr) + B n_0(kr)) Y_{0,0},$$

and the boundary conditions impose

$$\begin{cases} A j_0(k) + B n_0(k) = 0, \\ A j'_0(2k) + B n'_0(2k) = 0. \end{cases}$$

For a non-trivial solution to exist, it is necessary that

$$\begin{vmatrix} j_0(k) & n_0(k) \\ j'_0(2k) & n'_0(2k) \end{vmatrix} = 0.$$

This is,

$$\begin{vmatrix} \sin k & -\cos k \\ 2k \cos 2k - \sin 2k & 2k \sin 2k + \cos 2k \end{vmatrix} = 0$$

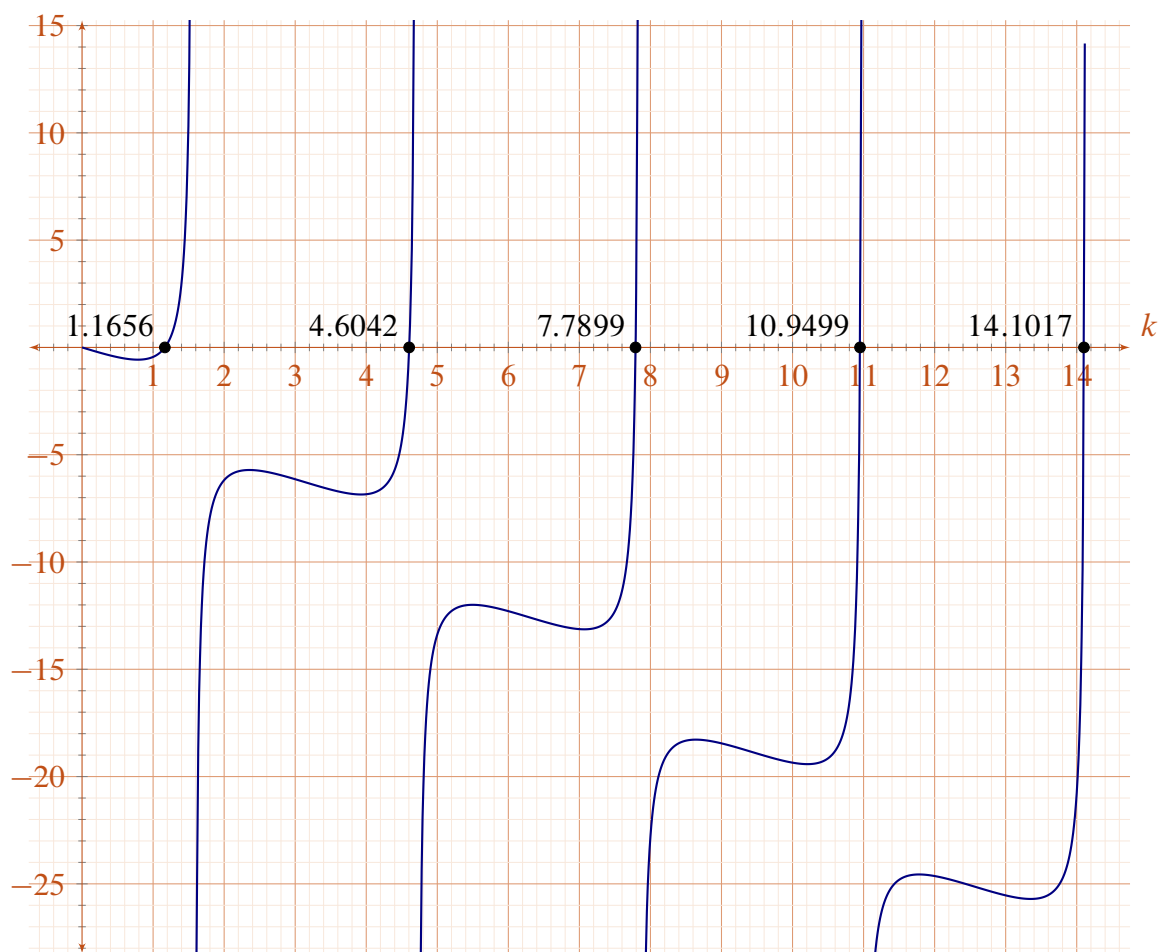
which is written as

$$\tan k = 2k.$$

The table of the first five zeros is

First zeros of $\tan k = 2k$				
1.1656	4.6042	7.7899	10.9499	14.1017

and the corresponding graph



Graph $f(k) = \tan(k) - 2k$ **and the first five zeros**

6. Determine standing waves using the variable separation method

$$u(t, x) = e^{-i\omega t} w(x), \quad \omega \geq 0,$$

of the next boundary problem in spherical coordinates

$$\begin{cases} u_{tt} = \Delta u, & r > 1, \\ \begin{cases} u|_{r=1} = 0, \\ \lim_{r \rightarrow \infty} u = 0. \end{cases} \end{cases}$$

Solution: If we introduce the standing wave ansatz in the boundary value problem for the wave equation we get the following boundary problem for the Helmholtz equation

$$\begin{cases} \Delta w + \omega^2 w = 0, \\ \begin{cases} w|_{r=1} = 0, \\ \lim_{r \rightarrow \infty} w = 0. \end{cases} \end{cases}$$

As we know the separation of variables into spherical coordinates leads to the following solutions of the Helmholtz equation

$$R_l(r)Y_{l,m}(\theta, \phi),$$

where Y_{lm} are the spherical harmonics and R_l is a function that takes one form or another depending on whether ω is null or not. When $\omega = 0$ we have $R_l(r) = ar^l + \frac{b}{r^{l+1}}$, but boundary conditions impose $a = 0$ and $a + b = 0$ then the solution is trivial. If $\omega \neq 0$ the radial function is $R_l(r) = aj_l(\omega r) + bn_l(\omega r)$, with $j_l(r)$ and $n_l(r)$ are the spherical Bessel and spherical Neumann functions respectively, which automatically satisfy the boundary condition at $r \rightarrow \infty$. The boundary condition in $r = 1$ imposes that the radial function is proportional to $n_l(\omega)j_l(\omega r) - j_l(\omega)n_l(\omega r)$. For all of it, the requested standing waves are

$$a e^{-i\omega t} (n_l(\omega)j_l(\omega r) - j_l(\omega)n_l(\omega r))Y_{l,m}(\theta, \phi)$$

7. Solve the following boundary problem for the Laplace equation by means of the eigenfunction expansion method in polar coordinates

$$\begin{cases} \Delta u = 0, & 0 \leq r < 1, \quad 0 \leq \theta < 2\pi, \\ u|_{r=1} = \sin^2 \theta. \end{cases}$$

Solution: In polar coordinates we must solve

$$\begin{cases} r^2 u_{rr} + r u_r + u_{\theta\theta} = 0, & 0 \leq r < 1, \quad 0 \leq \theta < 2\pi, \\ u|_{r=1} = \sin^2 \theta. \end{cases}$$

To apply the EEM we first solve the eigenvalue problem for $\frac{d^2}{d\theta^2}$ with periodic boundary conditions: $u(0) = u(2\pi)$. The eigenfunctions are

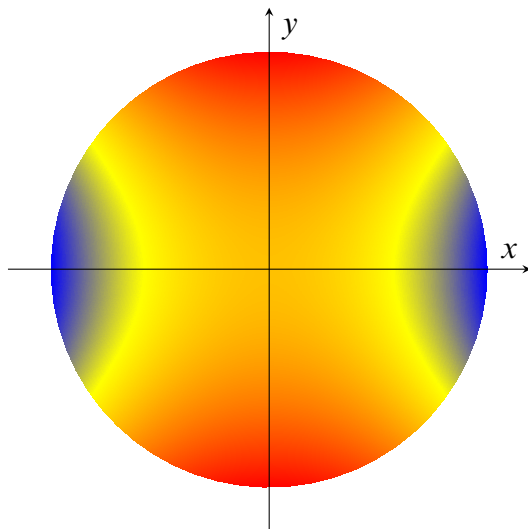
$$\{1, \sin m\theta, \cos m\theta\}_{m \in \mathbb{N}}$$

and the corresponding eigenvalues λ_m are $\{1, m^2\}_{m \in \mathbb{N}}$. Therefore, we expand the solution as

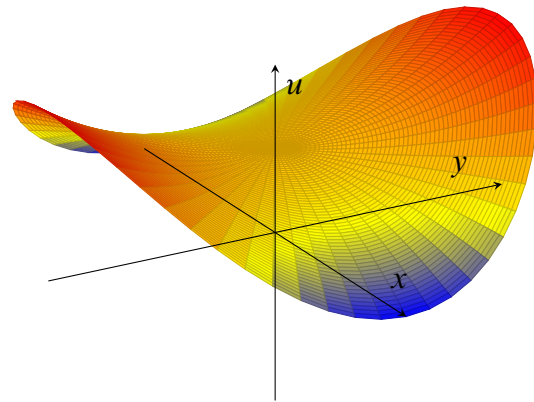
$$u(r, \theta) = v_0(r) + \sum_{m=1}^{\infty} (v_m(r) \cos m\theta + w_m(r) \sin m\theta).$$

The functions $v(r), w(r)$ are characterized for being solutions of $r^2 v'' + r v' + \lambda_m v_m = 0$. Considering now the regularity of the solution for $r = 0$ we conclude that $v_0 = c_0$ and that $v_m(r) = c_m r^m$ and $w_m(r) = d_m r^m$, where the coefficients c_m, d_m are determined by the boundary condition $\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$. Therefore, the expansion of the solution is

$$u(r, \theta) = \frac{1 - r^2 \cos 2\theta}{2}.$$



Color temperature
on the circular plate



Temperature surface $u(r, \theta)$
on the circular plate

8. Consider the Laplace equation in a cylindrical layer with Dirichlet boundary conditions

$$\begin{cases} \Delta u = 0, & 1 < r < 2, \\ \begin{cases} u|_{r=1} = 0, \\ u|_{r=2} = 0. \end{cases} \end{cases}$$

Determine, by means of the method of separation of variables the solutions $u(r, z)$ independent of θ .

Solution: We are solving the Laplace equation, so the solutions obtained by the method of separation of variables will be linear combinations of

$$u = R(r)\Theta(\theta)Z(z),$$

where

$$Z(z) = A e^{\alpha z} + B e^{-\alpha z},$$

$$\Theta(\theta) = C e^{im\theta} + D e^{-im\theta},$$

$$R(r) = \begin{cases} c_1 \log r + c_2, & \alpha = 0, m = 0, \\ c_1 r^m + \frac{c_2}{r^m}, & \alpha = 0, m \neq 0, \\ E J_m(\alpha r) + F N_m(\alpha r), & \alpha \neq 0. \end{cases}$$

As we don't want to have contributions dependent on θ we put $m = 0$ and $\Theta = 1$. Therefore, the solutions are $u(r, z) = R(r)Z(z)$ con

$$Z(z) = A e^{\alpha z} + B e^{-\alpha z},$$

$$R(r) = \begin{cases} c_1 \log r + c_2, & \alpha = 0, m = 0, \\ E J_0(\alpha r) + F N_0(\alpha r), & \alpha \neq 0. \end{cases}$$

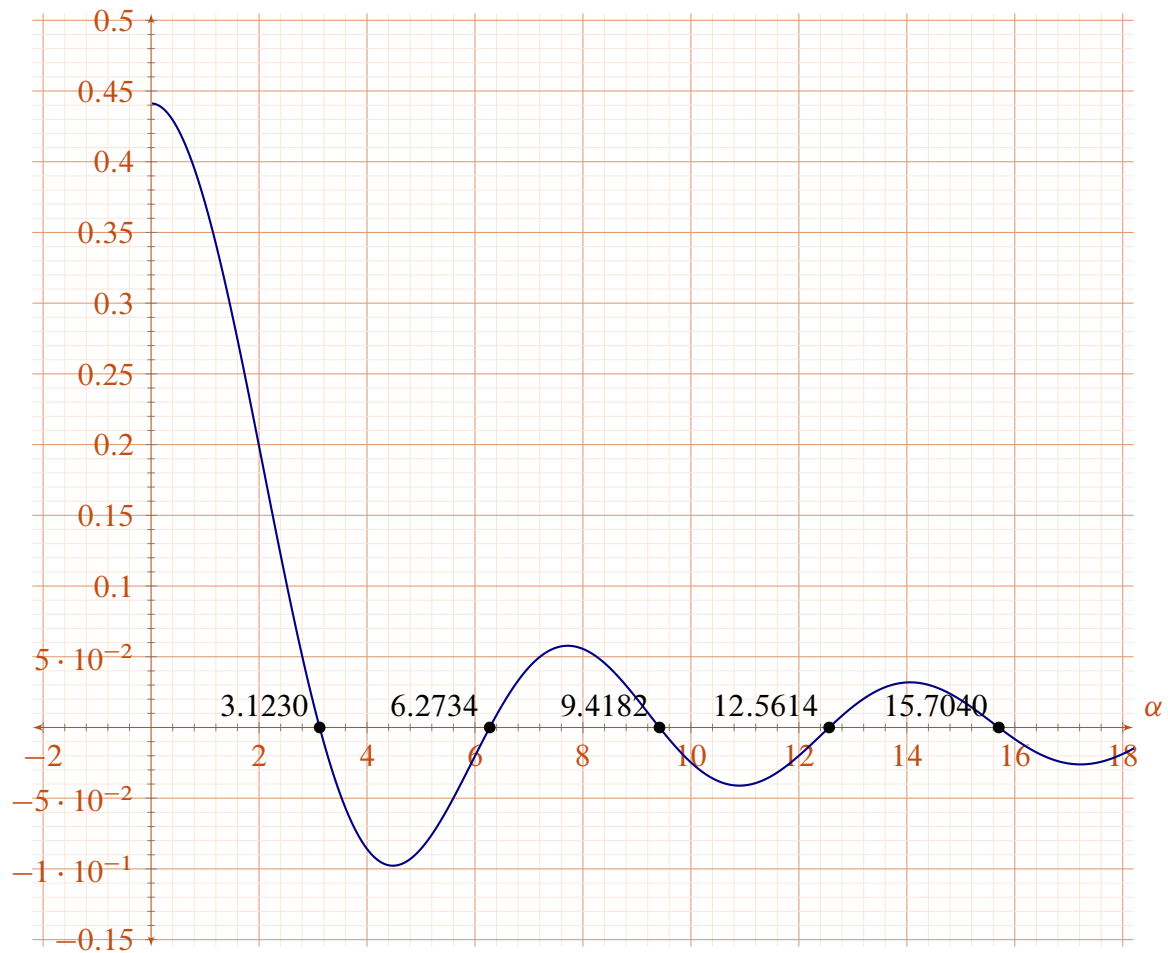
Imposing the boundary conditions $R(r)|_{r=1} = R(r)|_{r=2} = 0$ we see that $\alpha \neq 0$ and that we must have

$$J_0(\alpha)N_0(2\alpha) = J_0(2\alpha)N_0(\alpha).$$

The table of the first five zeros is

First zeros of $J_0(\alpha)N_0(2\alpha) = J_0(2\alpha)N_0(\alpha)$					
3.1230	6.2734	9.4182	12.5614	15.7040	18.8462

and the corresponding graph is



Graph $f(\alpha) = J_0(\alpha)N_0(2\alpha) = J_0(2\alpha)N_0(\alpha)$ and the first five zeros

This transcendental equation determines the possible values of α . The solution sought is

$$u = e^{\pm\alpha z} (N_0(\alpha)J_0(\alpha r) - J_0(\alpha)N_0(\alpha r)).$$

Radial functions

$$R_1(r) := J_0(3.1230)N_0(3.1230r) - N_0(3.1230)J_0(3.1230r),$$

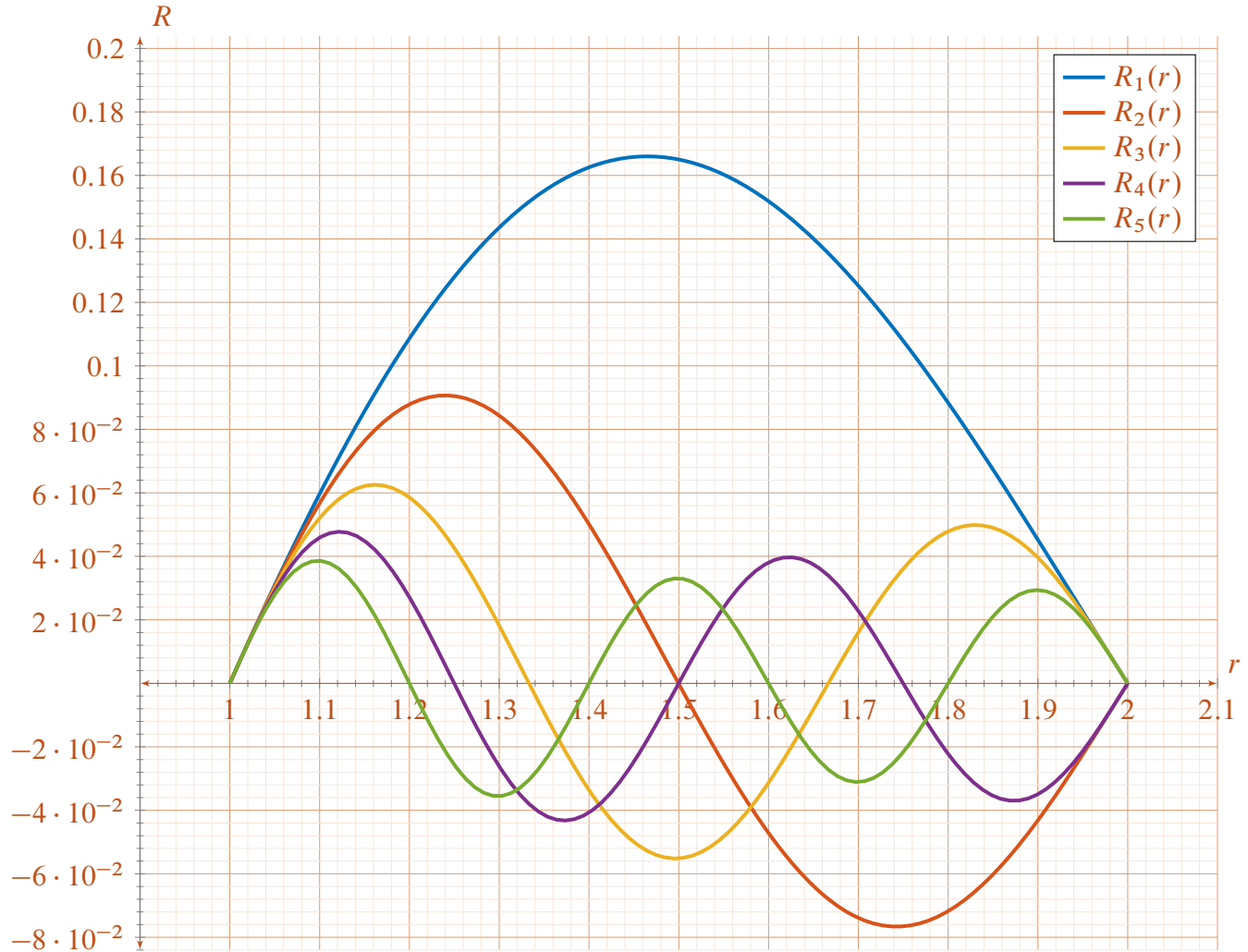
$$R_2(r) := J_0(6.2734)N_0(6.2734r) - N_0(6.2734)J_0(6.2734r),$$

$$R_3(r) := J_0(9.4182)N_0(9.4182r) - N_0(9.4182)J_0(9.4182r),$$

$$R_4(r) := J_0(12.5614)N_0(12.5614r) - N_0(12.5614)J_0(12.5614r),$$

$$R_5(r) := J_0(15.7040)N_0(15.7040r) - N_0(15.7040)J_0(15.7040r)$$

have the following graphs, for $1 < r < 2$,



9. Find the standing waves of the form $u(t, x) = \sin t w(r, \theta, \phi)$ for the wave equation in a spherical layer with the following Dirichlet boundary conditions

$$\begin{cases} u_{tt} = \Delta u, & \frac{\pi}{2} < r < \pi, \\ \begin{cases} u|_{r=\frac{\pi}{2}} = \cos \theta, \\ u|_{r=\pi} = 0. \end{cases} \end{cases}$$

Hint: $Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta$.

Solution: The conditions for $w(r, \theta, \phi)$ are

$$\begin{cases} \Delta w + w = 0, & \frac{\pi}{2} < r < \pi, \\ \left\{ \begin{array}{l} w|_{r=\frac{\pi}{2}} = \cos \theta, \\ w|_{r=\pi} = 0. \end{array} \right. \end{cases}$$

Which is a boundary value for the Helmholtz equation with $k^2 = 1$. To apply the EEM we write

$$\Delta + 1 = A + B,$$

$$A = r \frac{d}{dr} r + r^2,$$

$$B = \sin \theta \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2}{d\phi^2},$$

and analyze the eigenvalue problem

$$BY = -\lambda Y$$

in $C^\infty(S^2)$. Let us remember that, in this case, the eigenfunctions are the spherical harmonics. $\{Y_{l,m}(\theta, \phi)\}_{\substack{l=0,1,2,\dots \\ m=-l,\dots,l}}$ that form an orthogonal basis, with eigenvalues $\lambda = -l(l+1)$. We only have one inhomogeneous term in a boundary condition, $g_1(\theta) = \cos \theta$, for which we have

$$g_1(\theta) = \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \phi).$$

The eigenfunction expansion of the solution is

$$w(r, \theta, \phi) = v(r) Y_{1,0}(\theta, \phi),$$

where $v(r)$ satisfies

$$Av + \lambda v = 0$$

so that

$$v = c_1 j_1(r) + c_2 n_1(r),$$

where the spherical functions of Bessel of order 1 are

$$j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r}, \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r}.$$

The boundary conditions at $r = \frac{\pi}{2}$ and $r = \pi$ lead to the following system for the coefficients c_1 and c_2 ,

$$\begin{pmatrix} \frac{2}{\pi} & -1 \\ \pi & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\pi}{3}} \pi \\ 0 \end{pmatrix}$$

that implies

$$c_1 = \frac{2\sqrt{\pi}}{3\sqrt{3}}, \quad c_2 = -\frac{2\pi\sqrt{\pi}}{3\sqrt{3}},$$

and the solution is

$$u(t, r, \theta) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \frac{\sin t \cos \theta}{r^2} ((1 + \pi r) \sin r + (\pi - r) \cos r).$$

7.3.2. Exercises

1. Determine the stationary states of a quantum particle enclosed in a cylindrical box of radius a and height h .
2. The velocity potential of a stationary fluid inside a circular section pipe of radius a is described in cylindrical coordinates by the boundary problem:

$$\begin{cases} \Delta u = 0, & 0 \leq r < a, 0 \leq \theta < 2\pi, z \in \mathbb{R}; \\ \frac{\partial u}{\partial n} \Big|_{\text{wall}} = 0. \end{cases}$$

Determine the solutions provided by the variable separation method.

3. The velocity potential of a stationary fluid inside a h deep circular pond is described in cylindrical coordinates by the boundary value problem:

$$\begin{cases} \Delta u = 0, & 0 \leq r < a, 0 \leq \theta < 2\pi, -h < z < 0; \\ \frac{\partial u}{\partial n} \Big|_{\text{pared}} = 0. \end{cases}$$

Determine the solutions provided by the variable separation method.

4. Under certain conditions certain components of the field confined between two concentric spheres are described by the boundary problem in spherical coordinates:

$$\begin{cases} u_{tt} = c^2 \Delta u, & a_1 < r < a_2, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi \\ u|_{\text{paredes}} = 0. \end{cases}$$

Determine the standing waves of the model.

5. Prove that in Cartesian coordinates on the unit sphere:

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} z, \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} (x + iy), \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} (x - iy),$$

Determine the development of the functions $u_1 = x$, $u_2 = y$, $u_3 = z$ on the unit sphere in spherical harmonics.

6. Solve the initial value problem

$$\begin{cases} u_t = \Delta u, & 1 < r < 2, t > 0, \\ u|_{t=0} = \frac{1}{r} \sin\left(\frac{\pi}{2}r\right), \\ \left(u + \frac{\partial u}{\partial n}\right) \Big|_{r=1} = u|_{r=2} = 0. \end{cases}$$

7. Solve the initial value problem

$$\begin{cases} u_t = \Delta u + \frac{\sin 2r}{r} e^{-t}, & t > 0, \\ u|_{t=0} = \frac{\sin r}{r} \end{cases}$$

8. Solve the boundary value problem

$$\begin{cases} \Delta u = \frac{C}{r}, & r > R, \\ u|_{r=R} = z, \\ \frac{\partial u}{\partial r}|_{r=R} = 0. \end{cases}$$

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