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Decision Support

Values of games with weighted graphs

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ABSTRACT

In this paper we deal with TU games in which cooperation is restricted by means of a weighted network. We admit several interpretations for the weight of a link: capacity of the communication channel, flow across it, intimacy or intensity in the relation, distance between both incident nodes/players, cost of building or maintaining the communication link or even probability of the relation (as in Calvo, Lasaga, and van den Noweland, 1999). Then, according to the different interpretations, we introduce several point solutions for these restricted games in a way parallel to the familiar environment of Myerson. Finally, we characterize these values in terms of the (adapted) component efficiency, fairness and balanced contributions properties and we analyze the extent to which they satisfy a link/weight monotonicity property.

1. Introduction

A cooperative TU game describes a situation in which several actors can obtain certain transferable payoffs by means of the cooperation. Mathematically a TU game consists of a set of players and a characteristic function that assigns to each subset of players (coalition) a real number representing the worth of the coalition.

In this paper we deal with TU games in which cooperation is restricted by means of a weighted network. The seminal work on games in which restrictions in the cooperation are given by a graph is due to Myerson (1977). He assumed that the nodes in the graph are the players in the game, each link representing a direct bilateral communication channel. From this starting point, he defined the graph-restricted game and proposed as a point solution for players in this environ the so called Myerson value, i.e., the Shapley value of the restricted game. Moreover he characterized this value in terms of component efficiency and fairness. Later, Myerson (1980) gave another characterization of his value replacing fairness by balanced contributions.

In the Myerson model, bilateral restrictions in communication are dichotomous: they exist or they do not. Nevertheless, as has long been appreciated, each direct connection can be only partially (not totally) limited. We use in this paper a weighted graph as a model of these partial restrictions in the communications. A weighted graph consists of a set of nodes and a set of links, each link having an associated weight, a non-negative real number that can be interpreted in different ways: the capacity or the capability of the communication channel, the flow across it, the degree of intimacy, intensity or frequency if the link represents a social relation, the distance between both incident nodes, or even the cost of building or maintaining the communication link. Calvo, Lasaga, and van den Noweland (1999) introduced a probabilistic model in which the weight of a link is the probability of establishing the relation, these probabilities being independent. Later, Gómez, González-Arangüena, Manuel, and Owen (2008) generalized this model omitting the independence hypothesis. In a related context, Jiménez-Losada, Fernández, Ordoñez, and Grabisch (2010) consider cooperative TU games with Choquet players in which the restrictions in the communications are modeled using an (undirected) fuzzy graph.[1]

Other authors have considered values for games with node-weighted communication graphs. For example, in a game theoretical centrality analysis of terrorist networks, Lindelauf, Hamers, and Husslage (2013), Husslage, Borm, Burg, Hamers, and Lindelauf (2014) and Michalak et al. (2013) introduce a weighted connectivity game and propose its Shapley value as a centrality measure. This game can depend on information about relationship between terrorists in the network (link weights) and/or about individuals (node weights).

In this paper we introduce point solutions for games with restrictions in communications modeled by a weighted graph, following a parallel way to the familiar territory of Myerson.

In a TU game, each set of players is able to obtain its total dividend Harsanyi (1959). The basic idea underlying the definition of the Myerson game is that, under restrictions in the communications, every coalition that can be connected in the graph (possibly using intermediaries) obtains all its dividend in the game. Otherwise, i.e., in absence of connectedness, its dividend vanishes. And thus, the binary

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[1] For details on fuzzy graphs, readers can see Mordeson and Nair (2000).

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framework of communication is projected in an all or nothing possibility of obtaining the dividend. Nevertheless, the weighted graph introduces a non dichotomous but fuzzy scheme of relations. In accordance with this, we propose to consider a weighted graph game in which players partially connected obtain a part but not all their dividend and the lack of connectedness leading, of course, to the loss of the dividend. We introduce several forms of calculating this fraction of the dividend taking into account various alternative interpretations of the weights. Then, we use the Shapley value of the corresponding weighted graph restricted game as an appropriate solution concept for this situation and we characterize it in terms of the (adapted to this framework) component efficiency, fairness and balanced contributions properties. Finally, we explore the extent to which the defined values satisfy a link/weight monotonicity property.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and preliminaries; in Section 3 we define the weighted graph restricted games and the different extensions of the Myerson value in accordance with the various interpretations of the weights. Section 4 is devoted to characterization of the defined values. In Section 5 we deal with the link/weight monotonicity property and the paper ends with a section of final remarks and conclusions.

2. Preliminaries

A cooperative n-person game or a *TU-game* is a pair (N, v), $N = \{1, \ldots, n\}$ being the set of players and $v : 2^N \to \mathbb{R}$, *the characteristic function*, a map satisfying $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ represents the transferable utility that S can obtain whenever its members cooperate.

We will use G^N to denote the set of all TU-games with players set N. It is easy to see that G^N is a vector space. The game (N, u_S), $\emptyset \neq S \subseteq N$, with characteristic function given by:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise} \end{cases}$$

is known as the *unanimity game* corresponding to S. The family of all games $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$ is a basis for G^N. As a consequence, the characteristic function v of every game in G^N can be written:

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S.$$

The coefficients (coordinates) of v in such a basis are known as *Harsanyi dividends* (Harsanyi, 1959). The worth of every coalition S can be written in terms of its Harsanyi dividends. For each $S \subseteq N$, $S \neq \emptyset$:

$$v(S) = \sum_{\emptyset \neq T \subseteq S} \Delta_v(T).$$

A very popular point solution for TU-games is *the Shapley value* (Shapley, 1953), which assigns to every player the following convex linear combination of his marginal contributions to different coalitions:

$$\mathrm{Sh}_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} (v(S \cup \{i\}) - v(S)), \quad i \in N.$$

An alternative expression for this value in terms of the dividends is:

$$\mathrm{Sh}_i(N, v) = \sum_{i \in S \subseteq N} \frac{\Delta_v(S)}{s}, \quad i \in N.$$

A graph or a network is a pair (N, γ), $N = \{1, 2, \ldots, n\}$ being the set of nodes and γ a subset of $\gamma_N = \{\{i, j\}, i, j \in N, i \neq j\}$. Each *link* $\{i, j\} \in \gamma$ represents a direct relation or a communication channel between i and j. Γ^N denotes the set of all graphs with nodes set N.

Given $(N, \gamma) \in \Gamma^N$, we will say that two nodes i and j are *directly connected in* γ if $\{i, j\} \in \gamma$. And we will say they are *connected* in γ if there exists a sequence of nodes $i_1, i_2, \ldots, i_k$ with $i_1 = i$, $i_k = j$ such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \ldots, k-1$. A set $S \subseteq N$ is connected in γ if every pair of nodes in S is connected. *A connected component*, C, of the graph (N, γ) is a maximal connected subset. That is, C is connected and, for all $C' \subseteq N$, if $C \subsetneq C'$ then, C' is not connected. A graph (N, γ) induces a partition N/γ of the set N in connected components.

Given a set $S \subseteq N$ and a graph (N, γ), the restriction of the graph γ to the set S is the graph $(S, \gamma_{|S})$. Let S/γ be the set of the connected components of S in $(S, \gamma_{|S})$. A *subgraph* of a graph (N, γ) is a graph (N, γ') with $\gamma' \subseteq \gamma$.

Given a graph (N, γ) and a link $l \in \gamma$, $(N, \gamma \setminus \{l\})$ is the subgraph obtained when the relation l is severed and (N, γ_{-i}) is the resulting subgraph when all the incident in i links are broken and then i becomes an isolated node in the resulting graph.

Suppose that $(N, \gamma) \in \Gamma^N$ and let $(R, \gamma_{|R})$ denote the restriction to $R \subseteq N$, $R \neq \emptyset$. Suppose that $S \subseteq R$, $S \neq \emptyset$. If $S = \{i\}$ then the (unique) *connection subgraph of* S in $(R, \gamma_{|R})$ is defined as $(\{i\}, \emptyset)$. If $|S| > 1$, then a (not necessarily unique) *connection subgraph of* S in $(R, \gamma_{|R})$ is a graph $(D(\eta), \eta)$ satisfying:

(i) $\eta \subseteq \gamma_{|R}$ and S is connected in η, and
(ii) $D(\eta) = \{i \in N$ such that there exists $j \in N$ with $\{i, j\} \in \eta\}$.

A connection subgraph of $S \subseteq R$, $S \neq \emptyset$, in $(R, \gamma_{|R})$ is *minimal* [2] if there is no other connection subgraph $(D(\eta'), \eta')$ of S in $(R, \gamma_{|R})$ with $\eta' \subsetneq \eta$. Given a graph (N, γ) and $\emptyset \neq S \subseteq R \subseteq N$, $\mathcal{CG}(S, R, \gamma)$ will denote the family of all connection graphs of S in $(R, \gamma_{|R})$. Of course, it can occur that, for some S, R and (N, γ), $\mathcal{CG}(S, R, \gamma) = \emptyset$. Nevertheless, when we make use of it, this set will always be non empty. $\mathcal{MCG}(S, R, \gamma)$ will denote the family of all minimal connections graphs of $S \subseteq R$, $S \neq \emptyset$, in $(R, \gamma_{|R})$.

An *unweighted communication situation* or simply a *communication situation* is a triple (N, v, γ), (N, v) being a TU game and (N, γ) a graph. We will use $\mathcal{CS}^N$ to denote the set of all communication situations with players-nodes set N. An *allocation rule* ψ on $\mathcal{CS}^N$ is a map $\psi : \mathcal{CS}^N \to \mathbb{R}^n$, $\psi_i(N, v, \gamma)$ representing the outcome for player i in game (N, v) given the restrictions in the communication imposed by the graph (N, γ).

The *Myerson value* (Myerson, 1977) is the allocation rule μ on $\mathcal{CS}^N$ defined:

$$\mu(N, v, \gamma) = \mathrm{Sh}(N, v^\gamma), \quad \text{where } v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \text{ for all } S \subseteq N.$$

Myerson (1977) characterized this allocation rule in terms of *component efficiency* (for all $C \in N/\gamma$, $\sum_{i \in C} \mu_i(N, v, \gamma) = v(C)$) and *fairness* (for each $l = \{i, j\} \in \gamma$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma \setminus \{l\}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma \setminus \{l\})$). He also characterized it (Myerson, 1980) in terms of component efficiency and *balanced contributions* (given $i, j \in N$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma_{-j}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma_{-i})$).

3. Weighted Myerson values

In this section we admit several interpretations for weights in a graph and we accordingly define several weighted graph restricted games for players involved in a TU game with restrictions in the connections given by such a weighted network. Finally, we propose some values that generalize the classical Myerson one.

3.1. Weighted graphs

Definition 3.1. A weighted graph or a weighted network is a pair (N, γ_w), $N = \{1, \ldots, n\}$ being a set of nodes and $\gamma_w = \{\gamma, \{w_l\}_{l \in \gamma}\}$,

[2] Let us observe that in the case $S = R = N$ a minimal connection subgraph of N in $(N, \gamma_{|N} = \gamma)$ is a spanning tree of (N, γ).

with $\gamma \in \Gamma^N$ and the weights $w_l \geq 0$, for all $l \in \gamma$. We will use Γ_w^N to denote the set of all weighted graphs with nodes set N.

The weights w_l in a weighted network can admit many different interpretations. If the network is viewed as a model of communications or transport among its nodes, weights can be interpreted as the capacities of the communication channels and then, we would assume $w_l \in (0, 1]$ for all $l \in \gamma$. Alternatively, weights can be viewed as flow values in the network and so $w_l \in [0, \infty)$ (or $w_l \in [0, k_l]$, if we suppose that each channel has a bounded flow). Sometimes, the network will describe the social relations among actors and, in this case, a reasonable meaning of weights is that they measure intensity, intimacy or frequency of bilateral relations. So, for those situations, we will assume $w_l \in (0, 1]$. Weights can also represent distances between nodes or even costs of creating or maintaining links. In this latter case, $w_l \in [0, \infty)$ for all $l \in \gamma$. Calvo et al. (1999) propose to interpret each weight as the probability of the corresponding relation and to assume the links independence.

We will say that two nodes $i, j \in N$ are *directly connected (connected)* in the weighted graph (N, γ_w) if they are directly connected (connected) in the graph (N, γ). As a consequence, the set of connected components in (N, γ_w), that we will call N/γ_w, coincides with N/γ.

We will use $(N, \gamma_w \backslash \{l\})$ to denote the *weighted subgraph* of (N, γ_w) given by $(N, (\gamma \backslash \{l\})_{w \backslash \{w_l\}})$ in which all the links $l' \neq l$ have the corresponding weight in (N, γ_w). $(N, \gamma_w{}^{-k})$ will be the weighted subgraph obtained from (N, γ_w) when deleting all the incident in k links and, obviously, its associated weights, i.e., $(N, \gamma_w{}^{-k}) = (N, (\gamma_{-k})_{\{w_l\}_{l \in \gamma_{-k}}})$.

3.2. Weighted graph restricted games

Definition 3.2. An α-weighted communication situation is a tuple $(N, v, \gamma_w, \alpha(\gamma_w))$ in which (N, v) is a TU-game, (N, γ_w) a weighted network, the nodes in the network being the players in the game, and α is a function defined on Γ_w^N that assigns to each (N, γ_w) the family of real numbers $\alpha(N, \gamma_w)$ (or $\alpha(\gamma_w)$ for short):

$$\alpha(\gamma_w) = \left\{\alpha_T^R(\{w_l\}_{l \in \gamma}) \in [0, 1], \emptyset \neq T \subseteq R, R \in S/\gamma, S \subseteq N\right\}.$$

Definition 3.3. Given $(N, v, \gamma_w, \alpha(\gamma_w))$ an α-weighted communication situation, we define the α-weighted graph restricted game as the TU-game in G^N with characteristic function:

$$v^{\gamma_w, \alpha}(S) = \sum_{R \in S/\gamma} v^{\gamma_w, \alpha}(R),$$

where, for $R \in S/\gamma$,

$$v^{\gamma_w, \alpha}(R) = \sum_{\emptyset \neq T \subseteq R} \Delta_v(T) \alpha_T^R(\{w_l\}), \tag{1}$$

$\alpha_T^R(\{w_l\}) \in [0, 1]$ being the proportion of the dividend $\Delta_v(T)$ that the coalition $T \subseteq R$ retains as a consequence of the restrictions in the communication imposed by the weighted graph. When there is no ambiguity with respect to the weights $\{w_l\}_{l \in \gamma}$ we will simplify the notation writing α_T^R instead of $\alpha_T^R(\{w_l\})$.

Let $\mathcal{WCS}^{N,\alpha}$ be the class of all α-weighted communication situations with players set N. In accordance with the different meanings of weights considered in Section 3.1, we can distinguish four particular classes: $\mathcal{WCS}^{N,\alpha_c}$, $\mathcal{WCS}^{N,\alpha_d}$, $\mathcal{WCS}^{N,\alpha_f}$ and $\mathcal{WCS}^{N,\alpha_p}$, respectively including the α_c-weighted communication situations (c for capacity) in which weights represent capacities of channels or intimacy in the relations, the α_d-weighted communication situations (d for distance), the α_f-weighted communication situations (f for flow) and the α_p-weighted communication situations (p for probability).

To define formally these classes, consider $\emptyset \neq T \subseteq R$, $R \in S/\gamma$, $S \subseteq N$ and let $\{\eta_1^{T,R}, \eta_2^{T,R}, \ldots, \eta_{t(R)}^{T,R}\}$ be $\mathcal{MCG}(T, R, \gamma)$,[3,4] the family of all minimal connection graphs of $T \subseteq R$ in $(R, \gamma_{|R})$. Then:

- $\mathcal{WCS}^{N,\alpha_c}$ is the set of $(N, v, \gamma_w, \alpha_c(\gamma_w))$ for which $w_l \in (0, 1]$ for all $l \in \gamma$ and

  $$(\alpha_c)_T^R = \max_{i=1,\ldots,t(R)} \min_{l \in \eta_i^{T,R}} \{w_l\}, \quad \text{for } |T| \geq 2, \text{ and } (\alpha_c)_T^R = 1, \text{ if } |T| = 1.$$

  In the previous expression it is assumed that the possibilities of communication or transport are bounded by the capacity of the channel (the minimal capacity of its links). If several alternatives are available for players, then they will prefer the one in which the minimal capacity is maximal. Similarly it seems natural to assume that the intimacy or confidence among several actors is at most the minimal of the bilateral ones. When a set of actors want to establish relations they choose, among the possible intermediaries, those such that the minimal total intimacy (or confidence) is maximal.
- $\mathcal{WCS}^{N,\alpha_d}$ is the set of $(N, v, \gamma_w, \alpha_d(\gamma_w))$ for which $w_l \in [0, \infty)$ represents the distance between the two incident nodes in the link l or the cost of creating or maintaining this link and

  $$(\alpha_d)_T^R = \max_{i=1,\ldots,t(R)} \left\{\frac{1}{1 + \sum_{l \in \eta_i^{T,R}} w_l}\right\},$$
  $$\text{for } |T| \geq 2, \text{ and } (\alpha_d)_T^R = 1, \text{ if } |T| = 1.$$

  Here, we assume that the communication possibilities decrease with the distance or the cost. If several minimal connection graphs are available, then the geodesic (minimal total distance or minimal cost) ones are preferred.
- $\mathcal{WCS}^{N,\alpha_f}$ is the set of $(N, v, \gamma_w, \alpha_f(\gamma_w))$ for which $w_l \in [0, \infty)$ represents the flow between the two incident nodes in the link l and

  $$(\alpha_f)_T^R = \max_{i=1,\ldots,t(R)} \left\{\frac{1}{1 + \max_{l \in \eta_i^{T,R}} w_l}\right\},$$
  $$\text{for } |T| \geq 2, \text{ and } (\alpha_f)_T^R = 1, \text{ if } |T| = 1.$$

  Now, we assume that the communication possibilities decrease with the flow (or the traffic intensity). If several alternatives are possible then the one in which the traffic jam is minimal will be preferred.
- $\mathcal{WCS}^{N,\alpha_p}$ is in fact the set of probabilistic communication situations defined in Calvo et al. (1999), and thus:

  $$(\alpha_p)_T^R = \sum_{i=1}^{t(R)} \prod_{l \in \eta_i^{T,R}} w_l - \sum_{i<j} \prod_{l \in \eta_i^{T,R} \cup \eta_j^{T,R}} w_l + \cdots + (-1)^{t(R)+1}$$
  $$\times \prod_{l \in \bigcup_{j=1}^{t(R)} \eta_j^{T,R}} w_l, \quad \text{for } |T| \geq 2, \text{ and } (\alpha_p)_T^R = 1, \text{ if } |T| = 1.$$

  In this case, $(\alpha_p)_T^R$ can be interpreted as the probability of connecting T using links between members of R in a randomly selected graph with independent links whose probabilities are the weights.

Remark 3.1. The Myerson approach can be considered as a particular case of each one of the previous situations. In fact, if for $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$, $w_l = 1$ for all $l \in \gamma$, so that the capacities are not restricted, then we find the unweighted graph case of Myerson. Analogously, for $(N, v, \gamma_w, \alpha_d(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_d}$, the case of null distances or costs leads to the Myerson model. Similarly, for

[3] The same definitions will be obtained if, instead of $\mathcal{MCG}(T, R, \gamma)$, we use $\mathcal{CG}(T, R, \gamma)$ or even any subset of $\mathcal{CG}(T, R, \gamma)$ containing $\mathcal{MCG}(T, R, \gamma)$. For details, see Gómez, González-Arangüena, Manuel, Owen, and del Pozo (2004).

[4] Let us observe that, as R is a connected component in (N, γ), every $T \subseteq R$, $T \neq \emptyset$, is connected in $(R, \gamma_{|R})$ and thus $\mathcal{MCG}(T, R, \gamma) \neq \emptyset$.

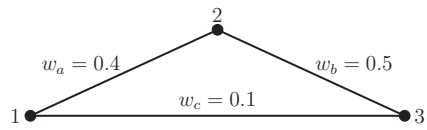

Fig. 1. A representation of the weighted network in Example 3.1.

$(N, v, \gamma_w, \alpha_f(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_f}$ the case of null flow or traffic intensity gives the Myerson model. Finally, it is known that probabilistic communication situations generalize the case of Myerson in which links are dichotomous. As a consequence, $\mathcal{CS}^N$, the set of all unweighted communication situations can be viewed as a subset of $\mathcal{WCS}^{N,\alpha_c}$, $\mathcal{WCS}^{N,\alpha_d}$, $\mathcal{WCS}^{N,\alpha_f}$ or $\mathcal{WCS}^{N,\alpha_p}$. In all of these cases, given an α-weighted communication situation $(N, v, \gamma_w, \alpha(\gamma_w))$, for all $T \subseteq R \subseteq N$ with R connected in γ, $\alpha_T^R = 1$ holds, and then:

$$v^{\gamma_w,\alpha}(S) = \sum_{R\in S/\gamma} v^{\gamma_w}(R) = \sum_{R\in S/\gamma}\sum_{T\subseteq R} \Delta_v(T)\alpha_T^R(\{w_l\})$$
$$= \sum_{R\in S/\gamma}\sum_{T\subseteq R} \Delta_v(T) = \sum_{R\in S/\gamma} v(R) = v^{\gamma}(S)$$

for all $S \subseteq N$ and thus, the α-weighted graph restricted game coincides with the classical graph restricted game (Myerson, 1977).

Remark 3.2. The definition of α in each one of the four given classes is consistent also with the idea of continuity in the following sense. Given $(N, v, \gamma_w, \alpha_c(\gamma_w))$ and $(N, v, (\gamma_w \setminus \{l^*\}), \alpha_c(\gamma_w \setminus \{l^*\})) \in \mathcal{WCS}^{N,\alpha_c}$, then for all $T \subseteq R \subseteq N$, $(\alpha_c)_T^R(\{w_l\}) \to (\alpha_c)_T^R(w \setminus \{w_{l^*}\})$ if $w_{l^*} \to 0$ (analogously in $\mathcal{WCS}^{N,\alpha_p}$). Similarly, given $(N, v, \gamma_w, \alpha_d(\gamma_w))$ and $(N, v, (\gamma_w \setminus \{l^*\}), \alpha_d(\gamma_w \setminus \{l^*\})) \in \mathcal{WCS}^{N,\alpha_d}$ then, for all $T \subseteq R \subseteq N$, $(\alpha_d)_T^R(\{w_l\}) \to (\alpha_d)_T^R(w \setminus \{w_{l^*}\})$ if $w_{l^*} \to \infty$ (analogously in $\mathcal{WCS}^{N,\alpha_f}$).

In order to clarify previous ideas, let us consider the following example.

Example 3.1. Consider the α_c-weighted communication situation $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ in which $N = \{1, 2, 3\}$, $v = u_{\{1,3\}}$ and γ_w is given in Fig. 1. From definition, $\alpha_c(\gamma_w)$ will be:

$$\Big\{(\alpha_c)_{\{i\}}^R = 1, \quad \text{if } i \in R \subseteq N;\ (\alpha_c)_{\{1,2\}}^{\{1,2\}} = (\alpha_c)_{\{1,2\}}^N = (\alpha_c)_{\{1,3\}}^N = (\alpha_c)_N^N$$
$$= 0.4,\ (\alpha_c)_{\{1,3\}}^{\{1,3\}} = 0.1,\ (\alpha_c)_{\{2,3\}}^{\{2,3\}} = (\alpha_c)_{\{2,3\}}^N = 0.5\Big\}.$$

If link weights represent intimacies, we can interpret this α_c-weighted communication situation in the following way: $v = u_{\{1,3\}}$ is the game in which actors 1 and 3 want to begin a love relationship. Nevertheless the intimacy level of 1 and 3 is low, say 0.1, and also they are very timid. On the other hand, 1 is best related with 2 and also 3 has more confidence in 2. These intimacies are valued as 0.4 and 0.5, respectively. Then, $v^{\gamma_w,\alpha}$ models the game of obtaining the highest level of intimacy or confidence when establishing the relation between 1 and 3 given the strength of respective relations among all three players and thus

$$v^{\gamma_w,\alpha}(R) = \begin{cases} 0, & \text{if } R \not\supseteq \{1,3\} \\ 0.1, & \text{if } R = \{1,3\} \\ 0.4, & \text{if } R = \{1,2,3\}. \end{cases} \tag{2}$$

As we can see, the possibility that players 1 and 3 will begin a relation increases from 0.1 to 0.4 if player 2 acts as intermediary. In this situation, player 2 is a catalyst[5] as he is better related with 1 and 3 (and so they probably better face the question if 2 introduces them). On the other hand, if in this example the degree of intimacy between 1 and 3 increases to reach a level $\beta \geq 0.4$, player 2 becomes a superfluous intermediary and thus $v^{\gamma_w,\alpha} = \beta u_{\{1,3\}}$.

Using (1) we can also obtain (2). For example, let us calculate $v^{\gamma_w,\alpha}(R)$ for $R = N = \{1, 2, 3\}$. As $v = u_{\{1,3\}}$, only $T = \{1, 3\}$ is such that $\Delta_v(T) \neq 0$. Then, $v^{\gamma_w,\alpha}(N) = \Delta_v(\{1, 3\})(\alpha_c)_T^N = 0.4$.

The functions $\alpha_c, \alpha_d, \alpha_f$ and α_p exhibit some common features that will permit us to unify several results in the sequel. With this purpose we introduce the following definition.

Definition 3.4. Given $\mathcal{WCS}^{N,\alpha}$, the class of α-weighted communication situations, we will say that α is decomposable on unanimity games (decomposable for short) if, for all $(N, \gamma_w) \in \Gamma_w^N$ and all $\emptyset \neq T \subseteq N$ such that $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \ldots, \eta_{t(N)}^T\}$[6] it exists for each $\{\eta_{i_1}^T, \ldots, \eta_{i_s}^T\} \subseteq \mathcal{MCG}(T, N, \gamma)$ a number in $[0, 1]$, $\beta(\{w_l\}_{l\in\eta_{i_1}^T}, \ldots, \{w_l\}_{l\in\eta_{i_s}^T})$, or simply $\beta_{\eta_{i_1}^T\ldots\eta_{i_s}^T}$, verifying:

(i) $$u_T^{\gamma_w,\alpha} = \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i<j} \beta_{\eta_i^T\eta_j^T} u_{D(\eta_i^T\cup\eta_j^T)} + \cdots$$
$$+ (-1)^{t(N)-1} \beta_{\eta_1^T\cdots\eta_{t(N)}^T} u_{D(\cup_{i=1}^{t(N)}\eta_i^T)},$$
$$\text{if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) \neq \emptyset,$$

(ii) $u_T^{\gamma_w,\alpha} = 0$, if $|T| > 1$ and $\mathcal{MCG}(T, N, \gamma) = \emptyset$, and

(iii) $u_T^{\gamma_w,\alpha} = u_T$ if $|T| = 1$.

Remark 3.3. Note that α is not necessarily decomposable. As an example, consider $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$, $\gamma = \{\{1, 2\}, \{2, 3\}\}$ and $w_l > 0$ for all $l \in \gamma$. If α is such that $\alpha_{\{1,2\}}^{\{1,2,3\}} \neq \alpha_{\{1,2\}}^{\{1,2\}}$, then α is not decomposable.

In the following propositions, we obtain the corresponding α_c-weighted graph restricted game for α_c-weighted communication situations (respectively α_d, α_f and α_p) in which the game is a unanimity one. A direct consequence of these propositions is that $\alpha_c, \alpha_d, \alpha_f$ and α_p are decomposable.

Proposition 3.1. *Let* $(N, u_T, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$, $\emptyset \neq T \subseteq N$. *If* $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \ldots, \eta_{t(N)}^T\}$ *and* $\beta_{\eta_i^T} = \min_{l\in\eta_i^T}\{w_l\}$ *for* $i = 1, \ldots, t(N)$, *then:*

(i) $$u_T^{\gamma_w,\alpha_c} = \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i<j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T\cup\eta_j^T)} + \cdots$$
$$+ (-1)^{t(N)-1} \min_{i=1,\ldots,t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)}\eta_i^T)},$$
$$\textit{if } |T| > 1 \textit{ and } \mathcal{MCG}(T, N, \gamma) \neq \emptyset, \tag{3}$$

(ii) $u_T^{\gamma_w,\alpha_c} = 0$, *if* $|T| > 1$ *and* $\mathcal{MCG}(T, N, \gamma) = \emptyset$, *and*

(iii) $u_T^{\gamma_w,\alpha_c} = u_T$ *if* $|T| = 1$.

Proof. To prove (i), consider $S \subseteq N$. If $T \not\subseteq S$ then, by the definition of $u_T^{\gamma_w,\alpha_c}$, $u_T^{\gamma_w,\alpha_c}(S) = 0$ as no coalition with the dividend different from zero is contained in S. On the other hand, the set of nodes of each minimal connection graph of T in N contains by definition all nodes in T and thus the characteristic function in the right hand term of (3) evaluated in S also gives 0.

Let us then consider the case in which $S \subseteq N$ is such that $T \subseteq S$ and S is connectable in γ, i.e., S is a subset of a connected component of N in γ. If S is not connectable in γ, we would consider the intersection R of S with the connected component of N in γ containing T and we

[5] The difficulties to build a relationship, as well as the (comic and tragic) situations arising in these cases, have ensured the regular appearance of these themes in best-selling books and films such as John Ford's *The quiet man*. In this movie Barry Fitzgerald in the role of Michaeleen Oge Flynn had an intermediary part in the relationship between John Wayne (in the role of Sean Thornton) and Maureen O'Hara (acting as Mary Kate Danaher).

[6] As we consider the minimal connection graphs of T in (N, γ), we simplify the notation using η_i^T instead of $\eta_i^{T,N}$.

would suppose that this intersection still contains T. Otherwise the previous reasoning applies and both members in (3) applied to R give zero. So, let us suppose that $\mathcal{MCG}(T,S,\gamma)=\{\eta_1^{T,S},\dots,\eta_{t(S)}^{T,S}\}$. Of course $\mathcal{MCG}(T,S,\gamma)\subseteq\mathcal{MCG}(T,N,\gamma)$. Then,

$$u_T^{\gamma_w,\alpha_c}(S)=(\alpha_c)_T^S(\{w_l\})=\max_{i=1,\dots,t(S)}\min_{l\in\eta_i^{T,S}}\{w_l\}$$

and, assuming without loss of generality that $\eta_1^{T,S},\dots,\eta_{t(S)}^{T,S}$ are ordered so that $\beta_{\eta_1^{T,S}}\le\beta_{\eta_2^{T,S}}\le\cdots\le\beta_{\eta_{t(S)}^{T,S}}$,

$$\begin{aligned}
&\Bigg(\sum_{i=1}^{t(N)}\beta_{\eta_i^T}u_{D(\eta_i^T)}-\sum_{i<j}\min\{\beta_{\eta_i^T},\beta_{\eta_j^T}\}u_{D(\eta_i^T\cup\eta_j^T)}+\cdots\\
&\quad+(-1)^{t(N)-1}\min_{i=1,\dots,t(N)}\{\beta_{\eta_i^T}\}u_{D(\cup_{i=1}^{t(N)}\eta_i^T)}\Bigg)(S)\\
&=\Bigg(\sum_{i=1}^{t(S)}\beta_{\eta_i^{T,S}}u_{D(\eta_i^{T,S})}-\sum_{i<j}\min\{\beta_{\eta_i^{T,S}},\beta_{\eta_j^{T,S}}\}u_{D(\eta_i^{T,S}\cup\eta_j^{T,S})}+\cdots\\
&\quad+(-1)^{t(S)-1}\min_{i=1,\dots,t(S)}\{\beta_{\eta_i^{T,S}}\}u_{D(\cup_{i=1}^{t(S)}\eta_i^{T,S})}\Bigg)(S)\\
&=\sum_{i=1}^{t(S)}\beta_{\eta_i^{T,S}}-\sum_{i<j}\min\{\beta_{\eta_i^{T,S}},\beta_{\eta_j^{T,S}}\}+\cdots+(-1)^{t(S)-1}\min_{i=1,\dots,t(S)}\{\beta_{\eta_i^{T,S}}\}\\
&=\sum_{i=1}^{t(S)}\beta_{\eta_i^{T,S}}-\sum_{i=1}^{t(S)-1}\binom{t(S)-i}{1}\beta_{\eta_i^{T,S}}+\sum_{i=1}^{t(S)-2}\binom{t(S)-i}{2}\beta_{\eta_i^{T,S}}+\cdots\\
&\quad+(-1)^{t(S)-1}\sum_{i=1}^{1}\binom{t(S)-i}{t(S)-1}\beta_{\eta_i^{T,S}}\\
&=\beta_{\eta_1^{T,S}}\sum_{j=0}^{t(S)-1}(-1)^j\binom{t(S)-1}{j}+\beta_{\eta_2^{T,S}}\sum_{j=0}^{t(S)-2}(-1)^j\binom{t(S)-2}{j}+\cdots\\
&\quad+\beta_{\eta_{t(S)-1}^{T,S}}\sum_{j=0}^{1}(-1)^j\binom{1}{j}+\beta_{\eta_{t(S)}^{T,S}}\\
&=\beta_{\eta_{t(S)}^{T,S}}=\max_{i=1,\dots,t(S)}\beta_{\eta_i^{T,S}}=\max_{i=1,\dots,t(S)}\min_{l\in\eta_i^{T,S}}\{w_l\},\\
&\qquad\text{as }\sum_{j=0}^{t(S)-k}(-1)^j\binom{t(S)-k}{j}=0,\text{ for all }k=1,\dots,t(S)-1.
\end{aligned}$$

The proof of (ii) and (iii) is trivial. □

Corollary 3.1. *α_c is decomposable on unanimity games.*

Proof. It is sufficient to note that in this case for all $\emptyset\neq T\subseteq N$ such that $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}\neq\emptyset$, if $\{\eta_{i_1}^T,\dots,\eta_{i_s}^T\}\subseteq\mathcal{MCG}(T,N,\gamma)$, the definition of decomposability is satisfied considering $\beta_{\eta_{i_1}^T\cdots\eta_{i_s}^T}=\min\{w_l,l\in\cup_{j=1}^s\eta_{i_j}^T\}$.

The proofs of the next propositions follow the steps of the previous one and are omitted. □

Proposition 3.2. *Let $(N,u_T,\gamma_w,\alpha_d(\gamma_w))\in\mathcal{WCS}^{N,\alpha_d}$, $\emptyset\neq T\subseteq N$. If $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}$ and $\beta_{\eta_i^T}=\frac{1}{1+\sum_{l\in\eta_i^T}w_l}$ for $i=1,\dots,t(N)$, then:*

(i) $$u_T^{\gamma_w,\alpha_d}=\sum_{i=1}^{t(N)}\beta_{\eta_i^T}u_{D(\eta_i^T)}-\sum_{i<j}\min\{\beta_{\eta_i^T},\beta_{\eta_j^T}\}u_{D(\eta_i^T\cup\eta_j^T)}+\cdots+(-1)^{t(N)-1}\min_{i=1,\dots,t(N)}\{\beta_{\eta_i^T}\}u_{D(\cup_{i=1}^{t(N)}\eta_i^T)},$$
if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)\neq\emptyset$,

(ii) $u_T^{\gamma_w,\alpha_d}=0$, *if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)=\emptyset$, and*

(iii) $u_T^{\gamma_w,\alpha_d}=u_T$ *if $|T|=1$.*

Corollary 3.2. *α_d is decomposable on unanimity games.*

Proof. It is sufficient to note that in this case for all $\emptyset\neq T\subseteq N$ such that $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}\neq\emptyset$, if $\{\eta_{i_1}^T,\dots,\eta_{i_s}^T\}\subseteq\mathcal{MCG}(T,N,\gamma)$, the definition of decomposability is satisfied considering $\beta_{\eta_i^T}=\frac{1}{1+\sum_{l\in\eta_i^T}w_l}$ for $i=1,\dots,t(N)$ and, for $s\ge2$, $\beta_{\eta_{i_1}^T\cdots\eta_{i_s}^T}=\min\{\beta_{\eta_{i_1}^T},\dots,\beta_{\eta_{i_s}^T}\}$. □

Proposition 3.3. *Let $(N,u_T,\gamma_w,\alpha_f(\gamma_w))\in\mathcal{WCS}^{N,\alpha_f}$, $\emptyset\neq T\subseteq N$. If $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}$ and $\beta_{\eta_i^T}=\frac{1}{1+\max_{l\in\eta_i^T}w_l}$ for $i=1,\dots,t(N)$, then:*

(i) $$u_T^{\gamma_w,\alpha_f}=\sum_{i=1}^{t(N)}\beta_{\eta_i^T}u_{D(\eta_i^T)}-\sum_{i<j}\min\{\beta_{\eta_i^T},\beta_{\eta_j^T}\}u_{D(\eta_i^T\cup\eta_j^T)}+\cdots+(-1)^{t(N)-1}\min_{i=1,\dots,t(N)}\{\beta_{\eta_i^T}\}u_{D(\cup_{i=1}^{t(N)}\eta_i^T)},$$
if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)\neq\emptyset$,

(ii) $u_T^{\gamma_w,\alpha_f}=0$, *if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)=\emptyset$, and*

(iii) $u_T^{\gamma_w,\alpha_f}=u_T$ *if $|T|=1$.*

Corollary 3.3. *α_f is decomposable on unanimity games.*

Proof. It is sufficient to note that in this case for all $\emptyset\neq T\subseteq N$ such that $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}\neq\emptyset$, if $\{\eta_{i_1}^T,\dots,\eta_{i_s}^T\}\subseteq\mathcal{MCG}(T,N,\gamma)$, the definition of decomposability is satisfied considering $\beta_{\eta_i^T}=\frac{1}{1+\max_{l\in\eta_i^T}w_l}$ for $i=1,\dots,t(N)$ and, for $s\ge2$, $\beta_{\eta_{i_1}^T\cdots\eta_{i_s}^T}=\min\{\beta_{\eta_{i_1}^T},\dots,\beta_{\eta_{i_s}^T}\}$. □

The proof of the next proposition follows from the results in Calvo et al. (1999) and Gómez et al. (2004).

Proposition 3.4. *Let $(N,u_T,\gamma_w,\alpha_p(\gamma_w))\in\mathcal{WCS}^{N,\alpha_p}$, $\emptyset\neq T\subseteq N$. If $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}$, then:*

(i) $$u_T^{\gamma_w,\alpha_p}=\sum_{i=1}^{t(N)}\left(\prod_{l\in\eta_i^T}w_l\right)u_{D(\eta_i^T)}-\sum_{i<j}\left(\prod_{l\in\eta_i^T\cup\eta_j^T}w_l\right)u_{D(\eta_i^T\cup\eta_j^T)}+\cdots+(-1)^{t(N)-1}\left(\prod_{l\in\cup_{j=1}^{t(N)}\eta_j^T}w_l\right)u_{D(\cup_{i=1}^{t(N)}\eta_i^T)},$$
if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)\neq\emptyset$,

(ii) $u_T^{\gamma_w,\alpha_p}=0$, *if $|T|>1$ and $\mathcal{MCG}(T,N,\gamma)=\emptyset$, and*

(iii) $u_T^{\gamma_w,\alpha_p}=u_T$ *if $|T|=1$.*

Corollary 3.4. *α_p is decomposable on unanimity games.*

Proof. It is sufficient to note that in this case for all $\emptyset\neq T\subseteq N$ such that $\mathcal{MCG}(T,N,\gamma)=\{\eta_1^T,\dots,\eta_{t(N)}^T\}\neq\emptyset$, if $\{\eta_{i_1}^T,\dots,\eta_{i_s}^T\}\subseteq\mathcal{MCG}(T,N,\gamma)$, the definition of decomposability is satisfied considering $\beta_{\eta_{i_1}^T\cdots\eta_{i_s}^T}=\prod_{l\in\cup_{j=1}^s\eta_{i_j}^T}w_l$. □

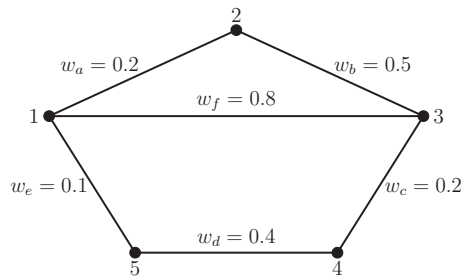

Fig. 2. A representation of the weighted network in Example 3.2.

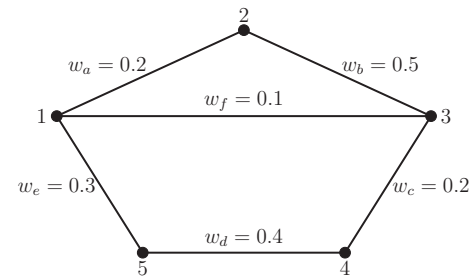

Fig. 3. A representation of the weighted network in Example 3.3.

3.3. Myerson values for weighted graph restricted games

Definition 3.5. An allocation rule ψ on $\mathcal{WCS}^{N,\alpha}$ is a map $\psi : \mathcal{WCS}^{N,\alpha} \to \mathbb{R}^n$, $\psi_i(N, v, \gamma_w, \alpha(\gamma_w))$ representing the outcome for player i in the α-weighted communication situation $(N, v, \gamma_w, \alpha(\gamma_w))$.

As our main goal in this paper is to generalize the Myerson value to the class of the α-weighted communication situations, we will use (similarly to Myerson) as point solution of the α-weighted graph restricted game the Shapley value. Other point solutions could be used but in this case the Myerson value is not generalized.

Definition 3.6. The Myerson value for α-weighted communication situations, μ^α, is the allocation rule on $\mathcal{WCS}^{N,\alpha}$ defined as $\mu^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) = \mathrm{Sh}(N, v^{\gamma_w,\alpha})$.

We will denote $\mu^{\alpha_c}, \mu^{\alpha_d}, \mu^{\alpha_f}$ and μ^{α_p} the Myerson value for α_c-weighted communication situations and respectively, for α_d, α_c and α_p-communication situations. μ^{α_p} coincides with the probabilistic Myerson value of Calvo et al. (1999).

Note that, from previous definition, the classical Myerson value is the restriction of μ^{α_c} to the set of those α_c-weighted communication situations in which the capacities are all equal to 1. Similarly, for μ^{α_d} (μ^{α_f}) and distances or costs (flows) equal to zero and, obviously, for μ^{α_p} and probabilities equal to one.

3.4. Some examples

Example 3.2. (Weights are distances or costs).

Let us consider the following weighted communication situation $(N, v, \gamma_w, \alpha_d(\gamma_w))$ in which $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,3\}}$ and γ_w is given in Fig. 2.

We will calculate $\mu^{\alpha_d}(N, v, \gamma_w, \alpha_d(\gamma_w))$. We have that $\mathcal{MCG}(N, \{1, 3\}, \gamma_w) = \{\eta_1 = \{a, b\}, \eta_2 = \{c, d, e\}, \eta_3 = \{f\}\}$ and thus[7]:

$$\begin{aligned} u_{\{13\}}^{\gamma_w,\alpha_d} &= \beta_{\eta_1} u_{D(\eta_1)} + \beta_{\eta_2} u_{D(\eta_2)} + \beta_{\eta_3} u_{D(\eta_3)} \\ &\quad - \min\{\beta_{\eta_1}, \beta_{\eta_2}\} u_{D(\eta_1\cup\eta_2)} - \min\{\beta_{\eta_1}, \beta_{\eta_3}\} u_{D(\eta_1\cup\eta_3)} \\ &\quad - \min\{\beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_2\cup\eta_3)} + \min\{\beta_{\eta_1}, \beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_1\cup\eta_2\cup\eta_3)} \\ &= \frac{1}{1+0.7} u_{\{123\}} + \frac{1}{1+0.7} u_{\{1345\}} + \frac{1}{1+0.8} u_{\{13\}} \\ &\quad - \frac{1}{1+0.7} u_{\{12345\}} - \frac{1}{1+0.8} u_{\{123\}} \\ &\quad - \frac{1}{1+0.8} u_{\{1345\}} + \frac{1}{1+0.8} u_{\{12345\}} \end{aligned}$$

Then:

$$\begin{aligned} \mu^{\alpha_d}(N, v, \gamma_w, \alpha_d(\gamma_w)) &= \mathrm{Sh}(N, u_{\{13\}}^{\gamma_w,\alpha_d}) \\ &= (0.2903, 0.0044, 0.2903, 0.0016, 0.0016). \end{aligned}$$

Example 3.3. (Weights are capacities).

Consider the following weighted communication situation $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ where $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,3\}}$ and γ_w is given in Fig. 3.

We will calculate $\mu^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w))$, i.e., the value of different players when the game is to connect players 1 and 3 taking into account the capacities of the existing communication channels. Of course, $\mathcal{MCG}(N, \{1, 3\}, \gamma_w)$ coincides with the one in the previous example and thus[8]:

$$\begin{aligned} u_{\{13\}}^{\gamma_w,\alpha_c} &= \beta_{\eta_1} u_{D(\eta_1)} + \beta_{\eta_2} u_{D(\eta_2)} + \beta_{\eta_3} u_{D(\eta_3)} \\ &\quad - \min\{\beta_{\eta_1}, \beta_{\eta_2}\} u_{D(\eta_1\cup\eta_2)} - \min\{\beta_{\eta_1}, \beta_{\eta_3}\} u_{D(\eta_1\cup\eta_3)} \\ &\quad - \min\{\beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_2\cup\eta_3)} + \min\{\beta_{\eta_1}, \beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_1\cup\eta_2\cup\eta_3)} \\ &= 0.2 u_{\{123\}} + 0.2 u_{\{1345\}} + 0.1 u_{\{13\}} - 0.2 u_{\{12345\}} - 0.1 u_{\{123\}} \\ &\quad - 0.1 u_{\{1345\}} + 0.1 u_{\{12345\}} \end{aligned}$$

and then

$$\begin{aligned} \mu^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) &= \mathrm{Sh}(N, u_{\{13\}}^{\gamma_w,\alpha_c}) \\ &= (0.0883, 0.0133, 0.0883, 0.005, 0.005). \end{aligned}$$

4. Characterizations of the weighted Myerson values

In this section we introduce two characterizations of the Myerson value for α-weighted communication situations, with α decomposable, that generalize the corresponding ones for the unweighted case.

Definition 4.1. An allocation rule ψ defined on $\mathcal{WCS}^{N,\alpha}$ satisfies component efficiency if, for all $(N, v, \gamma_w, \alpha(\gamma_w)) \in \mathcal{WCS}^{N,\alpha}$ and for all $C \in N/\gamma$,

$$\sum_{i\in C} \psi_i(N, v, \gamma_w, \alpha(\gamma_w)) = v^{\gamma_w,\alpha}(C) \text{ for all } i \in C.$$

Definition 4.2. An allocation rule ψ defined on $\mathcal{WCS}^{N,\alpha}$ satisfies fairness if, for all $(N, v, \gamma_w, \alpha(\gamma_w)) \in \mathcal{WCS}^{N,\alpha}$, and for all $l = \{i, j\} \in \gamma$,

$$\begin{aligned} &\psi_i(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_i(N, v, \gamma_w\backslash\{l\}, \alpha(\gamma_w\backslash\{l\})) \\ &\quad = \psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_j(N, v, \gamma_w\backslash\{l\}, \alpha(\gamma_w\backslash\{l\})). \end{aligned}$$

[7] We simplify the notation writing {13} instead of {1,3} and so on. Moreover, we can see here a difference with the case of the Myerson restricted game. In that case, if there are several alternative paths to connect 1 and 3 but these nodes are directly connected, all the remaining paths are ignored. In this new approach, this occurs only when the direct path is geodesic among all the possible paths. Otherwise, the nodes in shorter paths are rewarded because their contribution to the communication shortening the distance between 1 and 3. Equivalently when computing $\mu^{\alpha_c}, \mu^{\alpha_d}$ and μ^{α_p} respectively.

[8] We can see here too a difference with the case of the Myerson restricted game. In that case, if there are several alternative paths to connect 1 and 3 but these nodes are directly connected, all the remaining paths are ignored. In this new approach, this occurs only when the direct path has the greatest capacity among all the possible paths (the capacity of a path being the minimum of its links' capacities).

Definition 4.3. An allocation rule ψ defined on $\mathcal{WCS}^{N,\alpha}$ satisfies the balanced contributions property if, for all $(N, v, \gamma_w, \alpha(\gamma_w)) \in \mathcal{WCS}^{N,\alpha}$, and for all $i, j \in N$,

$$\psi_i(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_i(N, v, \gamma_w{}^{-j}, \alpha(\gamma_w{}^{-j})) = \psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_j(N, v, \gamma_w{}^{-i}, \alpha(\gamma_w{}^{-i})).$$

Theorem 4.1. *If α is decomposable then, the Myerson value for α-weighted communication situations, μ^α, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha}$ satisfying component efficiency and fairness.*

Proof. First, we will prove that μ^α satisfies component efficiency. Let $(N, v, \gamma_w, \alpha(\gamma_w)) \in \mathcal{WCS}^{N,\alpha}$. Given $C \in N/\gamma$ we have

$$\sum_{i\in C}\mu_i^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) = \sum_{i\in C}\mu_i^\alpha(C, v_{|C}, (\gamma_w)_{|C}, \alpha((\gamma_w)_{|C})) = \sum_{i\in C}\mathrm{Sh}_i(C, (v_{|C})^{(\gamma_w)_{|C},\alpha}) = (v_{|C})^{(\gamma_w)_{|C},\alpha}(C),$$

denoting $(\gamma_w)_{|C} = (\gamma_{|C})_{w_{|(\gamma_{|C})}}$ and because of the Shapley value efficiency. But, $(v_{|C})^{(\gamma_w)_{|C},\alpha}(C) = (v^{\gamma_w,\alpha})(C)$.

Second, we will prove that μ^α satisfies fairness. As is obvious from the definition, μ^α is linear in the game, so it is sufficient to prove that μ^α satisfies fairness for α-weighted communication situation in $\mathcal{WCS}^{N,\alpha}$ of the form: $(N, u_T, \gamma_w, \alpha(\gamma_w))$, with $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \ldots, \eta_{t(N)}^T\}$, as α is decomposable, the characteristic function $u_T^{\gamma_w,\alpha}$ is given by:

$$\sum_{i=1}^{t(N)}\beta_{\eta_i^T}u_{D(\eta_i^T)} - \sum_{i<j}\beta_{\eta_i^T\eta_j^T}u_{D(\eta_i^T\cup\eta_j^T)} + \cdots + (-1)^{t(N)-1}\beta_{\eta_1^T\cdots\eta_{t(N)}^T}u_{D(\cup_{i=1}^{t(N)}\eta_i^T)}$$

and thus

$$\mu^\alpha(N, u_T, \gamma_w, \alpha(\gamma_w)) = \mathrm{Sh}\left(\sum_{i=1}^{t(N)}\beta_{\eta_i^T}u_{D(\eta_i^T)} - \sum_{i<j}\beta_{\eta_i^T\eta_j^T}u_{D(\eta_i^T\cup\eta_j^T)} + \cdots + (-1)^{t(N)-1}\beta_{\eta_1^T\cdots\eta_{t(N)}^T}u_{D(\cup_{i=1}^{t(N)}\eta_i^T)}\right).$$

Suppose the weighted link $l = \{i_0, j_0\}$ is severed. Then

$$\mu^\alpha(N, u_T, \gamma_w \setminus \{l\}, \alpha(\gamma_w \setminus \{l\})) = \mathrm{Sh}\left(\sum_{j=1}^{k}\beta_{\eta_{i_j}^T}u_{D(\eta_{i_j}^T)} - \sum_{j<r}\beta_{\eta_{i_j}^T\eta_{i_r}^T}u_{D(\eta_{i_j}^T\cup\eta_{i_r}^T)} + \cdots + (-1)^{k+1}\beta_{\eta_{i_1}^T\cdots\eta_{i_k}^T}u_{D(\cup_{j=1}^{k}\eta_{i_j}^T)}\right),$$

with $\mathcal{MCG}(T, N, \gamma \setminus \{l\}) = \{\eta_{i_1}^T, \ldots, \eta_{i_k}^T\}$ being a subset of $\mathcal{MCG}(T, N, \gamma)$.

Then the difference

$$\mu^\alpha(N, u_T, \gamma_w, \alpha(\gamma_w)) - \mu^\alpha(N, u_T, \gamma_w \setminus \{l\}, \alpha(\gamma_w \setminus \{l\}))$$

is a linear combination of the Shapley value of games $u_{D(\eta)}$ with $l = \{i_0, j_0\} \in \eta$. In fact, each η is necessarily a union of minimal connection graphs of T in (N, γ) in which, at least, one of them contains l. Of course, $i_0, j_0 \in D(\eta)$ for all these graphs η and because of the Shapley value symmetry the outcome of both players i_0 and j_0 changes by the same amount.

To prove uniqueness, consider an allocation rule ψ defined on $\mathcal{WCS}^{N,\alpha}$ and satisfying efficiency and fairness. We must prove that $\psi = \mu^\alpha$. The proof uses induction on $|\gamma|$. If $|\gamma| = 0$, then each $i \in N$ forms a component in (N, γ_w), and then in (N, γ). As ψ satisfies component efficiency, for all $i \in N$, $\psi_i(N, v, \gamma_w, \alpha(\gamma_w)) = v^{\gamma_w,\alpha}(\{i\}) = \mu_i^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$ (the last equality holding because μ^α also satisfies component efficiency) and thus both allocation rules coincide.

Suppose now, by the induction hypothesis, that $\psi(N, v, \gamma_w, \alpha(\gamma_w)) = \mu^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$ for all weighted communication situations in $\mathcal{WCS}^{N,\alpha}$ with $|\gamma| \le k$ and consider $(N, v, \gamma_w, \alpha(\gamma_w)) \in \mathcal{WCS}^{N,\alpha}$ with $|\gamma| = k+1$. Let $i \in N$ and let $C(i)$ be the class in the quotient set $N/\gamma_w = N/\gamma = \{C_1, C_2, \ldots, C_k\}$ to which i belongs. If $C(i) = \{i\}$, then similar to the case $|\gamma| = 0$ above, by efficiency in components $\psi_i(N, v, \gamma_w, \alpha(\gamma_w)) = v^{\gamma_w,\alpha}(\{i\}) = \mu_i^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$ and thus both rules coincide in i. Alternatively, suppose that $|C(i)| > 1$ and let $j \in C(i)$, $j \ne i$. By the definition of connected component, there exists a sequence of players $i_1 = i, i_2, i_3, \ldots, i_r = j$ with $i_l \in C(i)$ for $l = 1, 2, \ldots, r$ and such that $\{i_l, i_{l+1}\} \in \gamma$, for each $l = 1, 2, \ldots, r-1$. As ψ satisfies fairness,

$$\psi_{i_1}(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_{i_1}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) = \psi_{i_2}(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_{i_2}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})),$$

and thus:

$$\psi_{i_1}(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_{i_2}(N, v, \gamma_w, \alpha(\gamma_w)) = \psi_{i_1}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) - \psi_{i_2}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})).$$

As $|\gamma| \le k$, using the induction hypothesis,

$$\psi_{i_1}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) = \mu_{i_1}^\alpha(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\}))$$

and

$$\psi_{i_2}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) = \mu_{i_2}^\alpha(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\}))$$

and therefore:

$$\psi_{i_1}(N, v, \gamma_w, \alpha(\gamma_w)) - \psi_{i_2}(N, v, \gamma_w, \alpha(\gamma_w)) = \psi_{i_1}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) - \psi_{i_2}(N, v, \gamma_w\setminus\{i_1, i_2\}, \alpha(\gamma_w\setminus\{i_1, i_2\})) = \mu_{i_1}^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_{i_2}^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$$

the last equality holding because μ^α satisfies the fairness property. As a consequence, $\psi_{i_1}(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_{i_1}^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) = \psi_{i_2}(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_{i_2}^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$.

Iteratively using this previous reasoning,

$$\psi_i(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_i^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) = \psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_j^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$$

for $j \in C(i)$ and thus, there exists $h_{C(i)} \in \mathbb{R}$ such that $\psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_j^\alpha(N, v, \gamma_w, \alpha(\gamma_w)) = h_{C(i)}$ for all $j \in C(i)$. Then,

$$|C(i)|h_{C(i)} = \sum_{j\in C(i)}[\psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_j^\alpha(N, v, \gamma_w, \alpha(\gamma_w))] = \sum_{j\in C(i)}\psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \sum_{j\in C(i)}\mu_j^\alpha(N, v, \gamma_w, \alpha(\gamma_w)).$$

By component efficiency of both rules ψ and μ^α, this last expression is equal to zero and thus, $h_{C(i)} = 0 = \psi_j(N, v, \gamma_w, \alpha(\gamma_w)) - \mu_j^\alpha(N, v, \gamma_w, \alpha(\gamma_w))$ for all $j \in C(i)$ and, in particular, for i, which completes the proof. □

As a direct consequence of the previous theorem we obtain the following corollaries.

Corollary 4.1. *The Myerson value for α_c-weighted communication situations, μ^{α_c}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_c}$ satisfying component efficiency and fairness.*

Corollary 4.2. *The Myerson value for α_d-weighted communication situations, μ^{α_d}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_d}$ satisfying component efficiency and fairness.*

Corollary 4.3. *The Myerson value for α_f-weighted communication situations, μ^{α_f}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_f}$ satisfying component efficiency and fairness.*

Corollary 4.4. *(Calvo et al., 1999, Theorem 1, pp. 87–89) The Myerson value for α_p-weighted communication situations, μ^{α_p}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_p}$ satisfying component efficiency and fairness.*

The values μ^{α}, if α is decomposable can be also characterized by substituting the fairness property by the balanced contributions one.

Theorem 4.2. *If α is decomposable, the Myerson value for α-weighted communication situations, μ^{α}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha}$ satisfying component efficiency and balanced contributions.*

Proof. It is already proved that μ^{α} satisfies component efficiency.

As μ^{α} is linear in the game, we only need to prove that μ^{α} satisfies balanced contributions for α-weighted communication situations in $\mathcal{WCS}^{N,\alpha}$ of the form: $(N, u_T, \gamma_w, \alpha(\gamma_w))$, with $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \ldots, \eta_{t(N)}^T\}$, then the characteristic function $u_T^{\gamma_w,\alpha}$ is given by:

$$\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i<j} \beta_{\eta_i^T \eta_j^T} u_{D(\eta_i^T \cup \eta_j^T)} + \cdots + (-1)^{t(N)-1} \beta_{\eta_1^T \cdots \eta_{t(N)}^T} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)},$$

and thus

$$\mu^{\alpha}(N, u_T, \gamma_w, \alpha(\gamma_w)) = \mathrm{Sh}\left(\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i<j} \beta_{\eta_i^T \eta_j^T} u_{D(\eta_i^T \cup \eta_j^T)} + \cdots + (-1)^{t(N)-1} \beta_{\eta_1^T \cdots \eta_{t(N)}^T} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)}\right).$$

Suppose j_0 becomes isolated, then the difference $\mu_{i_0}^{\alpha}(N, u_T, \gamma_w, \alpha(\gamma_w)) - \mu_{i_0}^{\alpha}(N, u_T, \gamma_w^{-j_0}, \alpha(\gamma_w^{-j_0}))$ is a linear combination of the Shapley value of games $u_{D(\eta)}$ with $i_0, j_0 \in D(\eta)$. If i_0 becomes isolated, $\mu_{j_0}^{\alpha}(N, u_T, \gamma_w, \alpha(\gamma_w)) - \mu_{j_0}^{\alpha}(N, u_T, \gamma_w^{-i_0}, \alpha(\gamma_w^{-i_0}))$ is the same linear combination of the Shapley value of the same games $u_{D(\eta)}$ with $i_0, j_0 \in D(\eta)$. And by the symmetry of Shapley value these quantities coincide.

The proof of uniqueness mimics the one in Theorem 4.1 and so it is omitted. □

As a direct consequence we obtain the following corollaries.

Corollary 4.5. *The Myerson value for α_c-weighted communication situations, μ^{α_c}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_c}$ satisfying component efficiency and balanced contributions.*

Corollary 4.6. *The Myerson value for α_d-weighted communication situations, μ^{α_d}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_d}$ satisfying component efficiency and balanced contributions.*

Corollary 4.7. *The Myerson value for α_f-weighted communication situations, μ^{α_f}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_f}$ satisfying component efficiency and balanced contributions.*

Corollary 4.8. *(Calvo et al., 1999, Remark 4.1, p. 89) The Myerson value for α_p-weighted communication situations, μ^{α_p}, is the unique allocation rule on $\mathcal{WCS}^{N,\alpha_p}$ satisfying component efficiency and balanced contributions.*

5. Link/weight monotonicity

In this section we deal with the problem of determining the extent to which the defined values satisfy link/weight monotonicity in the sense that if the underlying game is superadditive then, when adding a weighted link or when varying the weight of a link, other things been equal, the value of both incident nodes changes in the appropriate way.

5.1. Link/weight monotonicity of μ^{α_c}

Given an α_c-weighted communication situation $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ we will associate to the weighted graph (N, γ_w) a set of real numbers x_h and a set of unweighted graphs (N, γ_h), $h = 0, 1, \ldots, r$, where $r \leq |\gamma|$ will be the number of different values among the weights of links in (N, γ_w). So let us define $x_0 = 0$, $(N, \gamma_0) = (N, \gamma)$ and for $h = 1, 2, \ldots, r$

$$x_h = \min_{l \in \gamma_{h-1}} \{w_l\} \text{ and } \gamma_h = \{l \in \gamma / w_l > x_h\}.$$

In the following lemma we prove that, for α_c-weighted communication situations in $\mathcal{WCS}^{N,\alpha_c}$, the corresponding α_c-weighted graph restricted game can be written as a positive linear combination of unweighted graph restricted games.

Lemma 5.1. *Given $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ with (N, v) a zero-normalized game, it holds that*

$$v^{\gamma_w,\alpha_c} = \sum_{h=0}^{r-1} (x_{h+1} - x_h) v^{\gamma_h},$$

where for $h = 0, 1, \ldots, r-1$, v^{γ_h} is the Myerson game associated with the unweighted communication situation (N, v, γ_h).

Proof. Consider $S \subseteq N$ and $R \in S/\gamma$, then

$$v^{\gamma_w,\alpha_c}(R) = \sum_{T \subseteq R} \Delta_v(T)(\alpha_c)_T^R,$$

with

$$(\alpha_c)_T^R = \max_{i=1,\ldots,t(R)} \min_{l \in \eta_i^{T,R}} \{w_l\}, \quad \text{for } |T| \geq 1,$$

where $\{\eta_1^{T,R}, \eta_2^{T,R}, \ldots, \eta_{t(R)}^{T,R}\} = \mathcal{MCG}(T, R, \gamma)$ is the family of minimal connection graphs of $T \subseteq R$ in $(R, \gamma_{|R})$. Let $\{\eta_{i_1}^{T,R}, \eta_{i_2}^{T,R}, \ldots, \eta_{i_a}^{T,R}\}$ be the subfamily of $\mathcal{MCG}(T, R, \gamma)$ such that

$$(\alpha_c)_T^R = \min_{l \in \eta_{i_j}^{T,R}} \{w_l\} \text{ for } j = 1, 2, \ldots, a.$$

To determine the coefficient that in $\sum_{h=0}^{r-1} (x_{h+1} - x_h) v^{\gamma_h}(R)$ multiplies the dividend $\Delta_v(T)$, let us consider

$$k = \max_h \left\{ h / \text{ there exists } \eta_{i_j}^{T,R} \text{ with } \eta_{i_j}^{T,R} \subseteq \gamma_h \text{ and } \eta_{i_j}^{T,R} \not\subseteq \gamma_{h+1} \text{ for all } j = 1, \ldots, a \right\}.$$

Then, that coefficient equals to $w_{k+1} = \min_{l \in \gamma_k} \{w_l\}$ and so we only need to prove that $(\alpha_c)_T^R = w_{k+1}$. As there exists $j^* \in \{1, 2, \ldots, a\}$ such that $\eta_{i_{j^*}}^{T,R} \subseteq \gamma_k$, we have

$$w_{k+1} = \min_{l \in \gamma_k} \{w_l\} \leq \min_{l \in \eta_{i_j}^{T,R}} \{w_l\} = (\alpha_c)_T^R.$$

But if $w_{k+1} < (\alpha_c)_T^R$ then, for all $l \in \eta_{i_{j^*}}^{T,R}$, $w_l > w_{k+1}$ which implies $l \in \eta_{i_{j^*}}^{T,R} \subseteq \gamma_{k+1}$. And this contradiction with the definition of k proves the result. □

Given an unweighted communication situation (N, v, γ), the graph-restricted game (N, v^{γ}) inherits the superadditivity of game v (Owen, 1986). So, as a direct consequence of the previous lemma, we obtain the following result.

Corollary 5.1. *If $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ and (N, v) is a superadditive game, then $(N, v^{\gamma_w,\alpha_c})$ is also superadditive.*

Another consequence of the previous lemma is that we can calculate the Myerson value for α_c-weighted communication situations in terms of Myerson values of appropriate unweighted communication situations. The proof is straightforward from previous lemma and then it is omitted.

Proposition 5.1. *Given* $(N, v, \gamma_w, \alpha_c(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_c}$ *with* (N, v_0) *the zero-normalization of* (N, v), *it holds that* $\mu_i^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) = \sum_{h=0}^{r-1}(x_{h+1} - x_h)\mu_i(N, v_0, \gamma_h) + v(\{i\})$ *for all* $i = 1, \ldots, n$, *where for* $h = 0, 1, \ldots, r-1$, $\mu(N, v, \gamma_h)$ *is the Myerson value associated to the unweighted communication situation* (N, v, γ).

As a consequence of the following two lemmas we can establish the link/weight monotonicity of the Myerson value for α_c-weighted communication situations.

Lemma 5.2. *Given* $(N, v, \gamma_w, \alpha_c(\gamma_w)), (N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) \in \mathcal{WCS}^{N,\alpha_c}$ *with* v *being a superadditive game and such that it exists* $l^* = \{i, j\} \in \gamma$ *with* $w_{l^*} < w'_{l^*}$ *and* $w_l = w'_l$ *for all* $l \in \gamma, l \neq l^*$, *it holds that*

$$\mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \leq \mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) \text{ for } k = i, j.$$

Proof. Changing w_{l^*} by $w'_{l^*} > w_{l^*}$ some of the pairs (x_{h+1}, γ_h) for $h = 0, \ldots, r-1$ are modified. Suppose that $w_{l^*} = x_t$ with $t < r$ (the case in which the maximum weight, x_r is increased is trivial) and let us consider six different possibilities:

(i) Several links have weight equal to w_{l^*} but $w'_{l^*} < x_{t+1}$. In this case, in the sequence (x_{h+1}, γ_h) for $h = 0, 1, \ldots, r-1$ the pair (x_{t+1}, γ_t) gives rise to the pair $(w'_{l^*}, \gamma_t \cup \{l^*\})$ and the (x_{t+1}, γ_t) itself, all other couples remaining equal. And so, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha(\gamma_w)) \\
&\quad = (w'_{l^*} - x_t)\mu_k(N, v, \gamma_t \cup \{l^*\}) + (x_{t+1} - w'_{l^*})\mu_k(N, v, \gamma_t) \\
&\qquad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) \\
&\quad = (w'_{l^*} - x_t)[\mu_k(N, v, \gamma_t \cup \{l^*\}) - \mu_k(N, v, \gamma_t)] \geq 0
\end{aligned}$$

as for $k = i, j$, $\mu_k(N, v, \gamma_t \cup \{l^*\}) \geq \mu_k(N, v, \gamma_t)$ because of the link monotonicity of the Myerson value for unweighted communication situations.

(ii) Several links have weight equal to w_{l^*} but $w'_{l^*} = x_{t+1}$. Then, (x_{t+1}, γ_t) is transformed into $(x_{t+1}, \gamma_t \cup \{l^*\})$, the other pairs being unchanged and thus, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \\
&\quad = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_t \cup \{l^*\}) - \mu_k(N, v, \gamma_t)] \geq 0
\end{aligned}$$

again because of the link monotonicity of the Myerson value for unweighted communication situations.

(iii) Several links have the same weight w_{l^*} but $x_{t+1} < w'_{l^*} < x_{t+2}$. Then, the two pairs (x_{t+1}, γ_t) and (x_{t+2}, γ_{t+1}) are transformed into the three pairs $(x'_{t+1} = x_{t+1}, \gamma'_t = \gamma_t \cup \{l^*\})$, $(x'_{t+2} = w'_{l^*}, \gamma'_{t+1} = \gamma_{t+1} \cup \{l^*\})$ and $(x'_{t+3} = x_{t+2}, \gamma'_{t+2} = \gamma_{t+1})$, the rest of the pairs being transformed one to one. Then, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \\
&\quad = (x'_{t+1} - x'_t)\mu_k(N, v, \gamma'_t) + (x'_{t+2} - x'_{t+1})\mu_k(N, v, \gamma'_{t+1}) \\
&\qquad + (x'_{t+3} - x'_{t+2})\mu_k(N, v, \gamma_{t+2}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) \\
&\qquad - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) = (x_{t+1} - x_t)\mu_k(N, v, \gamma_t \cup \{l^*\}) \\
&\qquad + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) \\
&\qquad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) \geq 0
\end{aligned}$$

as, because of the link monotonicity of the Myerson value for unweighted communication situations, for $k = i, j$,

$$\mu_k(N, v, \gamma_t \cup \{l^*\}) \geq \mu_k(N, v, \gamma_t)$$
$$\text{and } \mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) \geq \mu_k(N, v, \gamma_{t+1}).$$

(iv) Only one link has weight equal to w_{l^*} and $w'_{l^*} < x_{t+1}$. Then (x_t, γ_{t-1}) change to $(x'_t = w'_{l^*}, \gamma'_{t-1} = \gamma_{t-1})$ and thus, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \\
&\quad = (x'_t - x'_{t-1})\mu_k(N, v, \gamma'_{t-1}) + (x'_{t+1} - x'_t)\mu_k(N, v, \gamma'_t) \\
&\qquad - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) \\
&\quad = (w'_{l^*} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) + (x_{t+1} - w'_{l^*})\mu_k(N, v, \gamma_t) \\
&\qquad - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) \\
&\quad = (w'_{l^*} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_t)] \geq 0
\end{aligned}$$

because of $\gamma_t = \gamma_{t-1} \setminus \{l^*\}$ and the link monotonicity of the Myerson value which implies $\mu_k(N, v, \gamma_{t-1}) \geq \mu_k(N, v, \gamma_t)$ for $k = i, j$.

(v) Only one link in γ has weight equal to w_{l^*} and $w'_{l^*} = x_{t+1}$. In this case (x_t, γ_{t-1}) and $(x_{t+1}, \gamma_t = \gamma_{t-1} \setminus \{l^*\})$ merge in $(x'_t = w'_{l^*}, \gamma'_{t-1} = \gamma_{t-1})$ and thus, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha(\gamma_w)) \\
&\quad = (x'_t - x'_{t-1})\mu_k(N, v, \gamma'_{t-1}) - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) \\
&\qquad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) \\
&\quad = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) \\
&\qquad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) \\
&\quad = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_{t-1} \setminus \{l^*\})]
\end{aligned}$$

which is nonnegative as the link monotonicity of the Myerson value implies that $\mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) \leq \mu_k(N, v, \gamma_{t-1})$ for $k = i, j$.

(vi) No other link has weight equal to w_{l^*} and $x_{t+1} < w'_{l^*} < x_{t+2}$. Then, the pairs (x_t, γ_{t-1}) and (x_{t+1}, γ_t) are transformed respectively into (x_{t+1}, γ_{t-1}) and $(w'_{l^*}, \gamma_t = \gamma_{t+1} \cup \{l^*\})$ and thus, for $k = i, j$,

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_{w'}, \alpha_c(\gamma_{w'})) - \mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \\
&\quad = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma'_t) \\
&\qquad + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) \\
&\qquad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) \\
&\quad = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - [(x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) \\
&\qquad + (x_{t+1} - x_t)\mu_k(N, v, \gamma_t = \gamma_{t-1} \setminus \{l^*\})] \\
&\qquad + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma'_t = \gamma_{t+1} \cup \{l^*\}) \\
&\qquad + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) \\
&\quad = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_t = \gamma_{t-1} \setminus \{l^*\})] \\
&\qquad + (w'_{l^*} - x_{t+1})[\mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) - \mu_k(N, v, \gamma_{t+1})].
\end{aligned}$$

which is nonnegative because of the link monotonicity of the Myerson value which implies that for $k = i, j$, $\mu_k(N, v, \gamma_t) = \mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) \leq \mu_k(N, v, \gamma_{t-1})$ and $\mu_k(N, v, \gamma_{t+1}) \leq \mu_k(N, v, \gamma'_t) = \mu_k(N, v, \gamma_{t+1} \cup \{l^*\})$. □

Lemma 5.3. *Given* $(N, v, \gamma_w, \alpha_c(\gamma_w)), (N, v, (\gamma \cup \{l^*\})_{w'}, \alpha_c((\gamma \cup \{l^*\})_{w'})) \in \mathcal{WCS}^{N,\alpha_c}$ *with* $w'_l = w_l$ *for all* $l \in \gamma$ *and* v *being a superadditive game, then, if* $l^* = \{i, j\}$, *it holds that*

$$\begin{aligned}
&\mu_k^{\alpha_c}(N, v, \gamma_w, \alpha_c(\gamma_w)) \\
&\quad \leq \mu_k^{\alpha_c}(N, v, (\gamma \cup \{l^*\})_{w'}, \alpha_c((\gamma \cup \{l^*\})_{w'})) \text{ for } k = i, j.
\end{aligned}$$

Proof. The proof is similar to the previous one (but shorter) and so it is omitted. □

Theorem 5.1. *The Myerson value for* α_c*-weighted communication situations,* μ^{α_c}, *satisfies link/weight monotonicity, i.e., if the underlying game is superadditive, adding a new weighted link to a weighted graph or increasing the capacity of an existing link the value of both incident nodes does not decrease.*

Proof. The result follows iteratively applying both previous lemmas. □

5.2. Weight monotonicity of μ^{α_f}

The proof of the link/weight monotonicity of μ^{α_f} mimics the previous one on the link/weight monotonicity of μ^{α_c}, but associating now to each graph (N, γ_w) in a communication situation $(N, v, \gamma_w, \alpha_f(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_f}$ the set of real numbers x_h and the set of unweighted graphs (N, γ_h) for $h = 0, \ldots, r$ defined by $x_0 = 0$, $(N, \gamma_0) = (N, \gamma)$ and for $h = 1, \ldots, r$

$$x_h = \min_{l \in \gamma_{h-1}} \left\{ \frac{1}{1 + w_l} \right\}$$

and $\gamma_h = \{l \in \gamma / \frac{1}{1+w_l} > x_h\}$. Then, we have the following theorem.

Theorem 5.2. *The Myerson value for α_f-weighted communication situations, μ^{α_f}, satisfies weight monotonicity, i.e., if the underlying game is superadditive, increasing the flow of an existing link, the value of both incident nodes does not increase.*

5.3. Weight monotonicity of μ^{α_d}

Unfortunately, the proof of link/weight monotonicity obtained for the cases μ^{α_c} and μ^{α_f} does not hold for μ^{α_d} and it remains as an open problem. Nevertheless, the result is easily proved if we strengthen the requirement in the game to be almost positive (that is, all its dividends are non negative).

Theorem 5.3. *The Myerson value for α_d-weighted communication situations, μ^{α_d}, satisfies weak weight monotonicity, i.e., if the underlying game is almost positive, if the distance or cost of an existing link increases, the value of both incident nodes does not increase.*

Proof. Given $(N, v, \gamma_w, \alpha_d(\gamma_w)) \in \mathcal{WCS}^{N,\alpha_d}$ the α_d-weighted graph restricted game is defined for each $S \subseteq R$ as::

$$v^{\gamma_w,\alpha_d}(S) = \sum_{R \in S/\gamma} v^{\gamma_w,\alpha_d}(R)$$

where for all $R \in S/\gamma$,

$$v^{\gamma_w,\alpha_d}(R) = \sum_{T \subseteq R} \Delta_v(T)(\alpha_d)_T^R,$$

with:

$$(\alpha_d)_T^R = \max_{i=1,\ldots,t(R)} \left\{ \frac{1}{1 + \sum_{l \in \eta_i^{T,R}} w_l} \right\},$$
$$\text{for } |T| \geq 1, \text{ and } (\alpha_d)_T^R = 1, \text{ otherwise.}$$

If the weighted communication situation $(N, v, \gamma_w, \alpha_d(\gamma_w))$ becomes $(N, v, \gamma_{w'}, \alpha_d(\gamma_{w'}))$ with $l^* = \{i, j\} \in \gamma$ such that $w'_{l^*} > w_{l^*}$ and $w'_l = w_l$ for all $l \neq l^*$ then, for $k = i, j$, and all $S \subseteq N \setminus \{k\}$, $v^{\gamma_{w'},\alpha_d}(S \cup \{k\}) \leq v^{\gamma_w,\alpha_d}(S \cup \{k\})$ and $v^{\gamma_{w'},\alpha_d}(S) = v^{\gamma_w,\alpha_d}(S)$ hold and thus:

$$\begin{aligned}\mu_k^{\alpha_d}(N, v, \gamma_{w'}, \alpha_d(\gamma_{w'})) &= \mathrm{Sh}_k(N, v^{\gamma_{w'},\alpha_d}) \leq \mathrm{Sh}_k(N, v^{\gamma_w,\alpha_d}) \\ &= \mu_k^{\alpha_d}(N, v, \gamma_w, \alpha_d(\gamma_w)). \quad \square\end{aligned}$$

6. Conclusions and final remarks

So far we have generalized the Myerson value to the framework in which direct relations among actors are not dichotomous but fuzzy or weighted. As the associated weight of a link can admit different meanings, we have adapted the definition of the graph restricted game to take into account these different interpretations. Then, the Shapley value of these weighted graph games is used to obtain point solutions for different players. Classical properties and characterizations of Myerson value are transferred to the extent possible. We have analyzed the extent to which the link monotonicity of the Myerson value for unweighted communication situations holds in this new framework. This property is satisfied for weighted communication situations in which the weights represent links capacities, intensity in the relations or flows. But when we assume that weights measure distances or costs, the hypothesis of superadditivity must be strengthened to almost positivity. In the way to obtain these results we have proved that μ^{α_c} and μ^{α_f} can be calculated in terms of a linear combination of the Myerson values of certain unweighted communication situations.

The results obtained permit us to rank nodes in a weighted graph using a game theoretical approach. For weighted graphs representing social networks with different levels of intimacy in the relations among actors, the defined value can be used to obtain a family of centrality measures for such actors.

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