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SOME RESULTS ON THE $M/G/1$ QUEUE WITH N-POLICY

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This paper deals with the $M/G/1$ queue with N-policy. We show some applications of the stochastic decomposition property for the queue size. A new stochastic decomposition property for the waiting time is observed. Explicit expressions for the moments of the stationary waiting time are also obtained.

Keywords. $M/G/1$ queue, N-policy, stochastic decomposition, waiting time.

1. Introduction

Queueing models with vacations have found wide interest in theory and applications. Doshi (1986, 1990) and Teghem (1986) provide surveys of the results in this area. In this paper, we consider the $M/G/1$ queue with N-policy, which was first studied by Yadin and Naor (1963). It is assumed that customers arrive according to a Poisson process with parameter λ and their service times are nonnegative i.i.d. random variables with a distribution function $B(t)$ ($t \geq 0$). Let $\beta(s)$ be the Laplace-Stieltjes transform of $B(t)$. If the kth moment of the service time is denoted by β_k, then the traffic intensity is given by $\rho = \lambda\beta_1$. We assume that $\rho < 1$. The behaviour of the system is controlled by the N-policy. In this policy, the server is turned off when the system becomes empty and turned on when the number of units reaches N. Thus, it is clear that the particular case $N = 1$ reduces to the standard $M/G/1$ queue. Many papers on vacation models consider variants of the N-policy. Minh (1988) and Takagi (1990) analyze a combination of N-policy, setup times and closedown times. Kella (1989) investigates an $M/G/1$ queue in which the server takes vacations and is turned on depending on the number of customers present in the system at the end of the vacations periods. Lee *et al.* (1994) study a batch arrival queue with N-policy.

The N-policy is probably the most popular vacation policy. However, in a more general context, we can define a vacation model as a queueing system in which the idle time of the server may be utilized for other secondary jobs, for instance to serve customers of another system. To that end, the

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server can switched on and off according to a wide variety of rules (see Takagi, 1991). A remarkable result for queueing systems with vacations is the stochastic decomposition property. That is, under adequate conditions on the vacation, a random variable representing the number of customers (or other performance measure such as the waiting time) in the system with vacations has the same distribution as the sum of two independent random variables, where one of these is the random variable representing the same measure for the queueing system without vacations. For many single-server queueing systems, this property has been fully characterized. In particular, Fuhrmann and Cooper (1985) consider a class of $M/G/1$ queues with generalized vacations. For such class, they showed that the stationary queue length is the convolution of two random variables. The first one is the stationary number of customers present in the standard $M/G/1$ queue. The second contribution to the convolution is a random variable distributed as the number of customers in the system at a random point given that the server is on vacation. This result was generalized by Shanthikumar (1988).

Our first objective is to show some applications of the stochastic decomposition property for the queue length. In particular, we have obtained:

i) a measure of the proximity between the $M/G/1$ queue with N-policy and the standard $M/G/1$ queue,

ii) explicit expressions for the steady-state probabilities and the factorial moments of the number of customers present in the system in the $M/H_2/1$ queue with N-policy.

It is well-known that the standard decomposition for the waiting time does not hold in an $M/G/1$ queue with N-policy. However, we show that an alternative stochastic decomposition can be easily derived. From a practical point of view, it is important to know explicit expressions for the mean and variance of the most usual queueing performance characteristics. Thus, we also give explicit formulae for the second moment of the waiting time in the $M/G/1$ queue with N-policy and for higher moments in the case of exponential service times.

The results developed in this paper are based on simple mathematical tools and provide new closed form expressions for the performance of the $M/G/1$ queue with N-policy. Thus, we hope that our results will be useful in application.

2. Applications of the Stochastic Decomposition Property

We denote by P_{ij} the steady-state probability associated to the state (i, j), where $i \in \{0, 1\}$ and $j \in \mathcal{N}$. In other words, there are j customers present in the system and i available servers. We note that $P_{0j} = 0$ for

$j \geq N$ and $P_{1j} = 0$ for $j = 0$. From Yadin and Naor (1963), we have

$$P_{0j} = \frac{1-\rho}{N}, \quad \text{for} \quad 0 \leq j \leq N-1. \tag{1}$$

Let $P_i(z)$ be the generating functions of $\{P_{ij}\}_{j=0}^{\infty}$. We also consider the marginal distribution $P_j = P_{0j} + P_{1j}$ of the number of customers present in the system and denote by $P(z)$ its corresponding generating function.

From Fuhrmann and Cooper (1985), we have the following decomposition result:

$$P(z) = Q(z)\frac{P_0(z)}{1-\rho}, \tag{2a}$$

where $Q(z) = \sum_{j=0}^{\infty} Q_j z^j$ denotes the classical Pollaczek-Khintchine formula for the standard $M/G/1$ queue. In particular, we observe that $Q_0 = 1-\rho$ and $Q_1 = (1-\rho)(1-\beta(\lambda))/\beta(\lambda)$.

Then, from the definitions and (2a) we also have

$$P_1(z) = (Q(z) - 1 + \rho)\frac{P_0(z)}{1-\rho}. \tag{2b}$$

At first we will derive some bounds for the proximity between the steady-state distributions for the standard $M/G/1$ queue and the $M/G/1$ queue with N-policy. In fact, the bounds are valid for any vacation model where the expressions (2a) and (2b) hold. The significance of these bounds is to provide upper and lower estimates for the distance between the limiting distributions of the number of customers in the queueing model under consideration and in a standard $M/G/1$ queue with the same parameters.

We next show that (2a) and (2b) lead respectively to the following bounds:

$$2(1-\rho)\left(1 - \frac{P_{00}}{1-\rho}\right) \leq \sum_{j=0}^{\infty} |P_j - Q_j| \leq 2\left(1 - \frac{P_{00}}{1-\rho}\right), \tag{3a}$$

$$2(1-\rho)\left(1 - \frac{P_{00}}{1-\rho}\right)\frac{1-\beta(\lambda)}{\beta(\lambda)} \leq \sum_{j=1}^{\infty} |P_{1j} - Q_j| \leq 2\rho\left(1 - \frac{P_{00}}{1-\rho}\right). \tag{3b}$$

The proof of (3b) is as follows. The stochastic decomposition (2b) implies that P_{1j} can be expressed in terms of Q_j and P_{0j} in the form:

$$P_{1j} = \sum_{k=1}^{j} Q_k \frac{P_{0,j-k}}{1-\rho}, \quad \text{for} \quad j \geq 1. \tag{4}$$

Thus

$$P_{1j} - Q_j = Q_j\left(\frac{P_{00}}{1-\rho} - 1\right) + (1-\delta_{1j})\sum_{k=1}^{j-1} Q_k \frac{P_{0,j-k}}{1-\rho}, \tag{5}$$

where δ denotes Kronecker's function.

From (4) it follows that

$$\begin{aligned} |P_{1j} - Q_j| &\leq Q_j\left(1 - \frac{P_{00}}{1-\rho}\right) + (1-\delta_{1j})\sum_{k=1}^{j-1} Q_k \frac{P_{0,j-k}}{1-\rho} \\ &= P_{1j} + Q_j\left(1 - \frac{2P_{00}}{1-\rho}\right). \end{aligned} \tag{6}$$

Finally, we can show that

$$\sum_{j=1}^{\infty} |P_{1j} - Q_j| \leq 2\rho\left(1 - \frac{P_{00}}{1-\rho}\right). \tag{7}$$

Now we obtain a lower bound by using the inequality $|a-b| \geq a - b$, so that

$$\begin{aligned} \sum_{j=1}^{\infty} |P_{1j} - Q_j| &\geq |P_{11} - Q_1| + \sum_{j=2}^{\infty}(P_{1j} - Q_j) \\ &= Q_1\left(1 - \frac{P_{00}}{1-\rho}\right) + \sum_{j=2}^{\infty}(P_{1j} - Q_j) = 2Q_1\left(1 - \frac{P_{00}}{1-\rho}\right) \\ &= 2(1-\rho)\left(1 - \frac{P_{00}}{1-\rho}\right)\frac{1-\beta(\lambda)}{\beta(\lambda)}. \end{aligned} \tag{8}$$

The bounds (3a) can be proven by similar arguments. For the N-policy, $P_{00} = (1-\rho)/N$ and the upper bound in (3a) reduces to $2(1-N^{-1})$. Hence, for any real number $\varepsilon > 0$, we can obtain the largest integer N_ε such that $2(1-N^{-1}) \leq \varepsilon$. If $\varepsilon < 1$ or $\varepsilon \geq 2$ the solution is trivially $N_\varepsilon = 1$ or any arbitrary positive integer, respectively. The case $\varepsilon \in [1,2)$ leads to proper solutions which allow us to interpret the bounds (3a) and (3b) as results about the rate of convergence of the distribution P_j to Q_j as $N \to 1$.

We have observed that similar bounds are also true for retrial queues with two priority levels (see Falin *et al.*, 1993).

As a second application of the stochastic decomposition property we now study the factorial moments of the number of customers present in the system. Let $M^n[\{P_j\}]$ be the nth factorial moment of $\{P_j\}_{j=0}^{\infty}$. A relationship among $M^n[\{P_j\}]$ and the corresponding moments in the standard $M/G/1$ queue, $M^n[\{Q_j\}]$, follows trivially by differentiating equations (2a):

$$M^n[\{P_j\}] = \frac{1}{1-\rho}\sum_{k=0}^{n}\binom{n}{k}M^k[\{Q_j\}]M^{n-k}[\{P_{0j}\}], \quad \text{for } n \in \mathcal{N}. \tag{9}$$

Observe that explicit expressions for $M^k[\{Q_j\}]$ are only available for a few particular service time distributions but $M^k[\{Q_j\}]$ can be readily obtained from the Takács recursive formula. Nevertheless, our main purpose along this paper is to find closed form solutions. Thus, we will assume, along the rest of this section, that the service times are hyperexponentially distributed with density

$$f(t) = p_1\mu_1 exp\{-\mu_1 t\} + p_2\mu_2 exp\{-\mu_2 t\}, \quad \text{for } t \geq 0, \tag{10}$$

where $0 < p_1, p_2 < 1$, $p_1 + p_2 = 1$.

Note that the nth factorial moment of the standard $M/M/1$ queue is given by $n!\rho^n/(1-\rho)^n$, and the nth factorial moment associated to the sequence $\{P_{0j}\}_{j=0}^{\infty}$ is given by $n!(1-\rho)I_{\{0,\ldots,N-1\}}(n)\sum_{j=n}^{N-1}\binom{j}{n}/N$. Taking into account the structural form of the steady-state probabilities in the standard $M/H_2/1$ (see Morse (1958, pp. 82-84) and Kleinrock (1975, pp. 195-196)) we obtain for $n \in \mathcal{N}$

$$M^n[\{P_j\}] = \frac{n!(1-\rho)}{N}\sum_{i=1}^{2}\frac{H_iA_i^n}{(1-A_i)^{n+1}}\sum_{k=0}^{\min(n,N-1)}\left(\frac{1-A_i}{A_i}\right)^k\sum_{j=k}^{N-1}\binom{j}{k}, \tag{11}$$

where

$$H_1 = \frac{A_1 - A}{A_1 - A_2}, \quad H_2 = 1 - H_1, \quad A = \lambda(p_1\mu_1 + p_2\mu_2)/c, \tag{12a}$$

$$A_1 = -(b-(b^2-4ac)^{1/2})/2c, \quad A_2 = -(b+(b^2-4ac)^{1/2})/2c, \tag{12b}$$

$$a = \lambda^2, \quad b = -(\lambda^2 + \lambda(\mu_1+\mu_2)), \quad c = \lambda(p_1\mu_1 + p_2\mu_2) + \mu_1\mu_2. \tag{12c}$$

For the sake of completeness we also give the steady-state probabilities which are as follows:

$$P_j = \begin{cases} \dfrac{1-\rho}{N}\displaystyle\sum_{i=1}^{2}\frac{H_i(A_i^{j+1}-1)}{A_i-1}, & \text{for} \quad 0 \leq j \leq N-1, \\ \dfrac{1-\rho}{N}\displaystyle\sum_{i=1}^{2}\frac{H_iA_i^{j+1}(A_i^{-N}-1)}{1-A_i}, & \text{for} \quad j \geq N. \end{cases} \tag{13}$$

After some algebraic manipulations it is easy to prove that $0 < A_2 \leq A < A_1 < 1$. Thus, $H_1 \in (0, 1]$ and the sequence $\{P_j\}_{j=0}^{\infty}$ has a unique mode in the point $j^* = N - 1$.

If $B(t) = 1 - exp\{-\mu t\}$, for $t \geq 0$, then $\mu_1 = \mu_2 = \mu$, $A_1 = \rho$, $A_2 = A = \rho/(1+\rho)$, $H_1 = 1$ and $H_2 = 0$. Then, (11) and (13) reduce to the following expressions:

$$M^n[\{P_j\}] = \frac{n!}{N}\left(\frac{\rho}{1-\rho}\right)^n \sum_{k=0}^{\min(n,N-1)} \left(\frac{1-\rho}{\rho}\right)^k \sum_{j=k}^{N-1} \binom{j}{k}, \tag{14a}$$

$$P_j = \begin{cases} (1-\rho^{j+1})/N, & \text{for} \quad 0 \leq j \leq N-1, \\ \rho^{j+1}(\rho^{-N}-1)/N, & \text{for} \quad j \geq N. \end{cases} \tag{14b}$$

3. Waiting Time

We assume in this section that the waiting time of a customer is defined as the time between the arrival epoch and the beginning of the service time. Let $W(t)$ be the distribution function of the waiting time of a random customer in an $M/G/1$ queue with N-policy and $FCFS$ queueing discipline, and let $\tilde{W}(s)$ denote the Laplace-Stieltjes transform of $W(t)$. Let $Y(t)$ and $\tilde{Y}(s)$ be the analogous quantities for the standard $M/G/1$ queue.

For vacation systems where the waiting time of a customer does not depend on the future of the arrival process that occurs after the customer's arrival epoch, Fuhrmann and Cooper (1985) show that $\tilde{W}(s)$ can be decomposed as

$$\tilde{W}(s) = \tilde{Y}(s)P_0(1 - s\lambda^{-1})/(1-\rho). \tag{15}$$

Nevertheless, in the $M/G/1$ queue with N-policy and $N > 1$ the waiting times of those customers that arrive during the vacation period depend on the future of the arrival process. Thus, expression (15) does not hold.

The time-dependent analysis of the waiting time in an $M/G/1$ queue with N-policy and setup times was studied by Takagi (1992). In particular, the steady-state transform $\tilde{W}(s)$ for an $M/G/1$ queue with N-policy is given by

$$\tilde{W}(s) = \frac{1-\rho}{N}\left(\frac{\left(\frac{\lambda}{\lambda+s}\right)^N - \beta^N(s)}{\frac{\lambda}{\lambda+s} - \beta(s)} + \frac{\lambda(1-\beta^N(s))}{s - \lambda + \lambda\beta(s)}\right). \tag{16}$$

On the other hand, it is well-known that $\tilde{Y}(s)$ is given by

$$\tilde{Y}(s) = 1 - \rho + \frac{\lambda(1-\rho)(1-\beta(s))}{s - \lambda + \lambda\beta(s)}. \tag{17}$$

Note that (17) is obtained by adding two terms which represent the contributions of the discrete and the continuous parts of $Y(t)$. These contributions are respectively given by $\tilde{Y}_0(s) = 1 - \rho$ and $\tilde{Y}_1(s) = \lambda(1-\rho)(1-\beta(s))/(s-\lambda+\lambda\beta(s))$. In a similar way, we can divide $\tilde{W}(s)$ in two contributions associated to the state of the server (on vacation or active) at the arrival epoch of the marked customer. It is clear that the first contribution is given by

$$\tilde{W}_0(s) = \sum_{j=0}^{N-1} P_{0j}\left(\frac{\lambda}{\lambda+s}\right)^{N-j-1} \beta^j(s) = \left(\frac{\lambda}{\lambda+s}\right)^{N-1} P_0(\lambda^{-1}(\lambda+s)\beta(s)). \tag{18}$$

Observe that $P_0(z) = (1-\rho)(1-z^N)/(N(1-z))$. Then, a straightforward manipulation over (16) leads to

$$\tilde{W}(s) = \left(\frac{\lambda}{\lambda+s}\right)^{N-1} P_0(\lambda^{-1}(\lambda+s)\beta(s)) + \frac{\lambda(1-\beta(s))P_0(\beta(s))}{s-\lambda+\lambda\beta(s)}. \tag{19}$$

Hence, from (17), (18) and (19) we deduce that

$$\tilde{W}_0(s) = \tilde{Y}_0(s)\left(\frac{\lambda}{\lambda+s}\right)^{N-1} P_0(\lambda^{-1}(\lambda+s)\beta(s))/(1-\rho), \tag{20a}$$

$$\tilde{W}_1(s) = \tilde{Y}_1(s)P_0(\beta(s))/(1-\rho), \tag{20b}$$

where the subindexes 0 and 1 denote the contribution to the waiting time associated to the periods during which the server is idle or busy, respectively. The expressions (20a) and (20b) constitute two new decomposition type results.

The question arises here is whether the physical meaning of the multipliers in a decomposition result can be explained. It should be pointed out that the second multiplier in formulas (2a) and (2b), $P_0(z)(1-\rho)^{-1}$, is the probability generating function of the number of customers in the system given that the server is on vacation. On the other hand, if we turn our attention to the waiting time, then we observe that the multipler $P_0(1-s\lambda^{-1})$ can be interpreted meaningfully in some particular cases. For example, it is well-known that for the case of an $M/G/1$ queue with T-policy $P_0(1-s\lambda^{-1})$

is related to the Laplace-Stieltjes transform for the residual life of a vacation. Nevertheless, a universal interpretation of $P_0(1 - s\lambda^{-1})$, valid for any vacation model, is unknown. The same comments also hold for our new stochastic decomposition property. The expressions (20a) and (20b) reduce the solution of the distribution for the $M/G/1$ queue with N-policy to that of obtaining an additional term which is due to the effect of vacation, but it seems difficult to provide a meaningful interpretation of the additional terms.

With the help of expressions (20a) and (20b), we have determined the second moment of $W(t)$, which is as follows:

$$E[W^2] = \frac{\lambda\beta_3}{3(1-\rho)} + \frac{\lambda^2\beta_2^2}{2(1-\rho)^2} + \frac{(N-1)\beta_2}{2(1-\rho)} + \frac{(N-1)(1-\rho)}{\lambda^2} + \frac{(N-1)(N-2)}{3\lambda^2}. \qquad (21)$$

The derivation of the above expression follows after very lengthy calculations but only first principles are involved. Thus, we omit the proof. We have to point out here that formula (16) only provides a theoretical solution for the waiting time distribution, but in practice it is essential to know the moments of $W(t)$. In fact, from the first two moments of $W(t)$ it is possible to develop a computationally tractable estimation for the density of $W(t)$ by using information theoretic techniques (see Falin *et al.*, 1994, Artalejo and Gomez-Corral, 1995, and the references therein). Note that the computation of higher-moments of $W(t)$ involves the differentiation of a composite function and the algebraic efforts can be simplified by using Faa di Bruno's formula (see Riordan, 1958).

To conclude we observe that the lack of memory of the exponential law allows us to derive all moments of $W(t)$. Thus, in what follows we assume that $B(t) = 1 - exp\{-\mu t\}$, for $t \geq 0$. To compute the kth moment of $W(t)$ we condition on the state of the system upon arrival. Thus, we find that

$$E[W^k] = \sum_{j=0}^{N-1} P_{0j} E[(\xi_j + \eta_j)^k] + \sum_{j=1}^{\infty} P_{1j} E[\eta_j^k], \quad \text{for} \quad k = 1, 2, ..., \qquad (22)$$

where ξ_j and η_j are Gamma distributed with densities

$$f_{\xi_j}(t) = \lambda^{N-j-1} t^{N-j-2} exp\{-\lambda t\}/(N-j-2)!, \quad \text{for} \quad t \geq 0, \qquad (23a)$$

$$f_{\eta_j}(t) = \mu^j t^{j-1} exp\{-\mu t\}/(j-1)!, \quad \text{for} \quad t \geq 0. \qquad (23b)$$

We have assumed that ξ_{N-1} and η_0 are defined as zero. The kth moment of a Gamma distribution with mean p/a and variance p/a^2 is given by $(p+k-1)!/(a^k(p-1)!)$. This implies that

$$\sum_{j=0}^{N-1} P_{0j} E[(\xi_j+\eta_j)^k] = \frac{(1-\rho)k!}{N\lambda^k} \sum_{i=0}^{k} \rho^i \sum_{n=0}^{N-1} \binom{n+i-1}{i} \binom{N-n+k-i-2}{k-i}, \tag{24}$$

where $\binom{m-1}{m}$, for $m \in \mathcal{N}$, is defined as δ_{0m}.

From (1) and (14b) we obtain explicit expressions for $\{P_{1j}\}_{j=1}^{\infty}$. Therefore, we get

$$\sum_{j=1}^{\infty} P_{1j} E[\eta_j^k] = -N^{-1} \sum_{j=1}^{\infty} \rho^{j+1} E[\eta_j^k]$$
$$+ N^{-1} \rho I_{\{N\geq 2\}} \sum_{j=1}^{N-1} E[\eta_j^k] + N^{-1} \rho^{-N} \sum_{j=N}^{\infty} \rho^{j+1} E[\eta_j^k]. \tag{25}$$

Substituting the moments of the Gamma distribution into (25), we obtain

$$\sum_{j=1}^{\infty} P_{1j} E[\eta_j^k] = \frac{\rho k!}{N\mu^k} I_{\{N\geq 2\}} \sum_{j=0}^{N-2} \binom{j+k}{k}$$
$$+ \frac{(\rho^{-N}-1)\rho^2 k!}{N\mu^k (1-\rho)^{k+1}} - \frac{\rho^{-N+2}}{N\mu^k} \frac{\partial^k}{\partial \rho^k} \sum_{j=0}^{k+N-2} \rho^j. \tag{26}$$

Finally, we also note that

$$\sum_{j=0}^{N-2} \binom{j+k}{k} = \binom{N+k-1}{k+1}, \tag{27a}$$

and

$$\frac{\partial^k}{\partial \rho^k} \sum_{j=0}^{k+N-2} \rho^j = \frac{k!}{\rho} \sum_{j=1}^{N-1} \binom{k+j-1}{k} \rho^j. \tag{27b}$$

Hence, substituting (27a) and (27b) into (26), we get

$$E[W^k] = \frac{(1-\rho)k!}{N\lambda^k} \sum_{i=0}^{k} \rho^i \sum_{n=0}^{N-1} \binom{n+i-1}{i} \binom{N-n+k-i-2}{k-i}$$

$$+\frac{(\rho^{-N}-1)\rho^2 k!}{N\mu^k(1-\rho)^{k+1}}+\frac{\rho k!}{N\mu^k}\binom{N+k-1}{k+1}$$

$$-\frac{\rho^{-N+1}k!}{N\mu^k}\sum_{j=1}^{N-1}\binom{k+j-1}{k}\rho^j. \tag{28}$$

It should be pointed out that Li and Zhu (1994) developed a recursive method for computing the moments of the stationary waiting time in single-server queues with N-policy. Nevertheless, their method is only valid for customers who arrive to turn on the server or during the busy period.

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