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On interpolation of weakly compact bilinear operators

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Abstract
We study the interpolation properties of weakly compact bilinear operators by the real method and also by the complex method. We also study the factorization property of weakly compact bilinear operators through reflexive Banach spaces.

KEYWORDS
complex interpolation, duality for bilinear operators, real interpolation, weakly compact bilinear operators

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1 | INTRODUCTION

Boundedness, compactness or weak compactness are important examples of useful properties that an operator may have. The question how they behave under interpolation is a classical problem that has attracted the attention of many authors, and the results have found interesting applications in operator theory. The research started already in the seminal papers by Calderón [6] and Lions and Peetre [31] on the complex and the real method, respectively. Concerning compactness and real interpolation, the final result was achieved by Cwikel [14] and Cobos, Kühn and Schonbek [13]. As for weak compactness, the proof of the famous factorization result for weakly compact operators of Davis, Figiel, Johnson and Pelczyński [15] contains interpolation ideas that later Beauzamy [1] and Heinrich [23] developed in a more abstract way, establishing the factorization property for other kind of operators, as Banach–Saks operators or Asplund operators. The real interpolation of weakly compact operators has been studied by Maligranda and Quevedo [32] and Mastyło [35]. A quantitative approach based on the measure of weak noncompactness γ was undertaken by Kryczka, Prus and Szczepanik [29] and further developed by Kryczka and Prus [28], covering the case of complex interpolation of weakly compact operators. Other results in this direction can be seen in the papers by Fan [19] and Szwedek [40].

We are interesting here in bilinear operators. Boundedness of bilinear operators was considered by Calderón [6], Lions and Peetre [31] and afterwards by many authors as, for example, Karadzhov [25], König [26], Janson [24] or Mastyło [36]. The study of interpolation properties of compact bilinear operators was also initiated by Calderón [6] and it is receiving considerable attention recently. See, for example, the papers by Fernandez and Silva [20], Fernández-Cabrera and Martínez [21, 22], Mastyło and Silva [37, 38] and Cobos, Fernández-Cabrera and Martínez [9–11].

However, as far as we know, the research on interpolation properties of weakly compact bilinear operators has just started. The properties of these operators (see [5]) allow to apply the abstract results of Manzano, Rueda and Sánchez-Pérez [33, 34] to derive interpolation results for weakly compact bilinear operators in certain degenerate cases. That is, when the couple in the target reduces to a single Banach space or both couples in the source reduce to single Banach spaces. The operator T should be bounded acting among the sums of the spaces of the couples.

We investigate here the interpolation properties of weakly compact bilinear operators for arbitrary Banach couples and assuming a weaker condition on the boundedness of T. More precisely, we assume that T is defined from

$$\left(A_0 \cap A_1\right) \times \left(B_0 \cap B_1\right) \quad \text{into} \quad E_0 \cap E_1$$

with

$$\|T(a,b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1.$$

If any of the extensions $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ is weakly compact, then we derive that T admits a unique weakly compact extension to the complex interpolation spaces

$$T : \left[A_0, A_1\right]_\theta \times \left[B_0, B_1\right]_\theta \longrightarrow \left[E_0, E_1\right]_\theta$$

and also to the real interpolation spaces

$$T : \left(A_0, A_1\right)_{\theta,p} \times \left(B_0, B_1\right)_{\theta,q} \longrightarrow \left(E_0, E_1\right)_{\theta,r}.$$

Here $0 < \theta < 1$, $1 \leq p, q < \infty$, $1 < r < \infty$ and $1/p + 1/q = 1 + 1/r$.

Our techniques are based on duality results for bilinear operators (see [39]). This allows us to reduce the problem to study whether or not a certain linear operator $T^\times$ is weakly compact. Then the known results for linear operators can be used.

In the last section of the paper we follow a similar strategy to study the factorization property of weakly compact bilinear operators through reflexive Banach spaces.

The organization of the paper is as follows. In Section 2 we revise some basic notions and establish some preliminary results. Section 3 deals with interpolation of weakly compact bilinear operators by the real method. The complex interpolation results are contained in Section 4. Finally, factorization is considered in Section 5.

2 | PRELIMINARIES

Let A, B, E be Banach spaces. We put U_A for the closed unit ball of A, A^* for the dual space of A and $\mathcal{L}(A, B)$ for the space of all bounded linear operators from A into B. If $R \in \mathcal{L}(A, B)$ then $R^* \in \mathcal{L}(B^*, A^*)$ stands for the adjoint operator. We write $\mathcal{L}(A \times B, E)$ for the space of all bounded bilinear operators from $A \times B$ into E. As usual, we put

$$\|R\|_{A,B} = \sup\left\{\|Ra\|_B \ :\ \|a\|_A \leq 1\right\}, \quad R \in \mathcal{L}(A, B),$$

$$\|T\|_{A\times B,E} = \sup\left\{\|T(a,b)\|_E \ :\ \|a\|_A \leq 1\,,\ \|b\|_B \leq 1\right\}, \quad T \in \mathcal{L}(A \times B, E),$$

$$T(V, W) = \left\{T(v, w) \ :\ v \in V,\ w \in W\right\}, \quad V \subseteq A\,,\ W \subseteq B.$$

We say that $T \in \mathcal{L}(A \times B, E)$ (respectively, $R \in \mathcal{L}(A, B)$) is *weakly compact* if $T\left(U_A, U_B\right)$ $\left(\text{respectively, } R\left(U_A\right)\right)$ is relatively weakly compact in E (respectively, in B).

For $u \in A \otimes B$, we write

$$\pi(u) = \inf\left\{\sum_{k=1}^{n} \|a_k\|_A \|b_k\|_B \ :\ u = \sum_{k=1}^{n} a_k \otimes b_k\right\}.$$

The *projective tensor product* $A\widetilde{\otimes}_\pi B$ of A and B is the completion of $(A \otimes B, \pi)$ (see [16] or [17]). We still denote by π the norm of $A\widetilde{\otimes}_\pi B$.

Recall that if $u \in A\widetilde{\otimes}_\pi B$ and $\varepsilon > 0$, then there exist sequences $(a_k) \subseteq A$, $(b_k) \subseteq B$ such that $\lim_{k\to\infty} \|a_k\|_A = 0 = \lim_{k\to\infty} \|b_k\|_B$, $u = \sum_{k=1}^{\infty} a_k \otimes b_k$ and

$$\pi(u) \leq \sum_{k=1}^{\infty} \|a_k\|_A \|b_k\|_B \leq \pi(u) + \varepsilon \tag{2.1}$$

(see, for example, [17, Proposition 9, p. 227]).

We also remind that to any $T \in \mathcal{L}(A \times B, E)$ we can associate the bounded linear operator $\widetilde{T} \in \mathcal{L}\left(A\widetilde{\otimes}_\pi B, E\right)$ defined by

$$\widetilde{T}\left(\sum_{k=1}^{n} a_k \otimes b_k\right) = \sum_{k=1}^{n} T\left(a_k, b_k\right), \quad \sum_{k=1}^{n} a_k \otimes b_k \in A \otimes B.$$

The correspondence $T \longrightarrow \widetilde{T}$ is an isometric isomorphism between $\mathcal{L}(A \times B, E)$ and $\mathcal{L}\left(A\widetilde{\otimes}_\pi B, E\right)$ (see [17, Theorem 1, p. 230]). In particular, for the dual space of $A\widetilde{\otimes}_\pi B$ we have

$$\left(A\widetilde{\otimes}_\pi B\right)^* = \mathcal{L}(A \times B, \mathbb{K})$$

where $\mathbb{K}$ stands for the scalar field.

Let $T \in \mathcal{L}(A \times B, E)$. According to Ramanujan and Schock [39], the *adjoint operator* $T^\times$ of T is the linear map $T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})$ defined by

$$\left(T^\times f\right)(a, b) = f[T(a, b)], \quad f \in E^*, \quad (a, b) \in A \times B.$$

We write $T^{\widetilde{\times}}$ for the composition of $T^\times$ with the isometric isomorphism between $\mathcal{L}(A \times B, \mathbb{K})$ and $\left(A\widetilde{\otimes}_\pi B\right)^*$. That is to say, for $f \in E^*$, $T^{\widetilde{\times}} f = \widetilde{T^\times f}$.

The next result describes a useful connection between these operators.

Lemma 2.1. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then we have*

$$T^{\widetilde{\times}} = \widetilde{T}^*. \tag{2.2}$$

Proof. Note that both operators $T^{\widetilde{\times}}, \widetilde{T}^*$ act boundedly from E^* into $\left(A\widetilde{\otimes}_\pi B\right)^*$. Given any $f \in E^*$, $a \in A$ and $b \in B$, we have

$$\left(T^{\widetilde{\times}} f\right)(a \otimes b) = \left(\widetilde{T^\times f}\right)(a \otimes b) = \left(T^\times f\right)(a, b) = f[T(a, b)] = f\left[\widetilde{T}(a \otimes b)\right] = \left(\widetilde{T}^* f\right)(a \otimes b). \qquad \square$$

Let $S \subseteq A$. We write $co(S)$ for the convex hull of S, formed by all finite linear combinations of elements of S with nonnegative scalars, having the scalars sum equal to one.

Lemma 2.2. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then*

$$T\left(U_A, U_B\right) \subseteq \widetilde{T}\left(U_{A\widetilde{\otimes}_\pi B}\right) \subseteq \overline{co\left(T\left(U_A, U_B\right)\right)}.$$

Proof. The first inclusion is clear. To establish the second inclusion, take any $u \in U_{A\widetilde{\otimes}_\pi B}$ and any $\varepsilon > 0$. We should show that there is $z \in co\left(T\left(U_A, U_B\right)\right)$ such that $\left\|\widetilde{T}u - z\right\|_E < \varepsilon$. We distinguish several cases. Assume first that $u \in A \otimes B$. Then there exist $\left\{a_j\right\}_{j=1}^{n} \subseteq A$, $\left\{b_j\right\}_{j=1}^{n} \subseteq B$ such that

$$u = \sum_{j=1}^{n} a_j \otimes b_j \quad \text{and} \quad \lambda := \sum_{j=1}^{n} \left\|a_j\right\|_A \left\|b_j\right\|_B < 1 + \frac{\varepsilon}{\|T\|_{A \times B, E}}.$$

If $\lambda \geq 1$, we put

$$z = \sum_{j=1}^{n} \frac{\|a_j\|_A \|b_j\|_B}{\lambda} T\left(\frac{a_j}{\|a_j\|_A}, \frac{b_j}{\|b_j\|_B}\right).$$

Then $z \in co(T(U_A, U_B))$ and

$$\begin{aligned}\widetilde{T}u - z &= \sum_{j=1}^{n} T(a_j, b_j) - \sum_{j=1}^{n} \frac{\|a_j\|_A \|b_j\|_B}{\lambda} T\left(\frac{a_j}{\|a_j\|_A}, \frac{b_j}{\|b_j\|_B}\right) \\ &= \sum_{j=1}^{n} \left(\|a_j\|_A \|b_j\|_B - \frac{\|a_j\|_A \|b_j\|_B}{\lambda}\right) T\left(\frac{a_j}{\|a_j\|_A}, \frac{b_j}{\|b_j\|_B}\right).\end{aligned}$$

Therefore

$$\|\widetilde{T}u - z\|_E \leq \left(1 - \frac{1}{\lambda}\right) \sum_{j=1}^{n} \|a_j\|_A \|b_j\|_B \|T\|_{A\times B,E} \leq (\lambda - 1)\|T\|_{A\times B,E} < \varepsilon.$$

If $\lambda < 1$ then we pick $\eta > 1$ such that $\eta\lambda = \eta \sum_{j=1}^{n} \|a_j\|_A \|b_j\|_B = 1$. Since

$$\widetilde{T}u = \sum_{j=1}^{n} T(a_j, b_j) = \sum_{j=1}^{n} \eta \|a_j\|_A \|b_j\|_B T\left(\frac{a_j}{\eta\|a_j\|_A}, \frac{b_j}{\|b_j\|_B}\right)$$

we derive that $\widetilde{T}u$ belongs to $co(T(U_A, U_B))$.

Assume now that $u \in A\widetilde{\otimes}_\pi B$. By (2.1), there are sequences $(a_k) \subseteq A$, $(b_k) \subseteq B$ with $u = \sum_{k=1}^{\infty} a_k \otimes b_k$ and

$$\sum_{k=1}^{\infty} \|a_k\|_A \|b_k\|_B \leq \pi(u) + \frac{\varepsilon}{2\|T\|_{A\times B,E}} \leq 1 + \frac{\varepsilon}{2\|T\|_{A\times B,E}}.$$

Let $N \in \mathbb{N}$ such that $\pi\left(u - \sum_{k=1}^{N} a_k \otimes b_k\right) < \varepsilon/2\|\widetilde{T}\|_{A\widetilde{\otimes}_\pi B,E}$. Then

$$\left\|\widetilde{T}u - \widetilde{T}\left(\sum_{k=1}^{N} a_k \otimes b_k\right)\right\|_E \leq \|\widetilde{T}\|_{A\widetilde{\otimes}_\pi B,E}\, \pi\left(u - \sum_{k=1}^{N} a_k \otimes b_k\right) < \frac{\varepsilon}{2}.$$

Since $\sum_{k=1}^{N} a_k \otimes b_k \in A \otimes B$ and

$$\sum_{k=1}^{N} \|a_k\|_A \|b_k\|_B \leq \sum_{k=1}^{\infty} \|a_k\|_A \|b_k\|_B \leq 1 + \frac{\varepsilon}{2\|T\|_{A\times B,E}},$$

then the previous step shows that there is $z \in co(T(U_A, U_B))$ with $\left\|\widetilde{T}\left(\sum_{k=1}^{N} a_k \otimes b_k\right) - z\right\|_E < \varepsilon/2$. Consequently,

$$\|\widetilde{T}u - z\|_E \leq \left\|\widetilde{T}u - \widetilde{T}\left(\sum_{k=1}^{N} a_k \otimes b_k\right)\right\|_E + \left\|\widetilde{T}\left(\sum_{k=1}^{N} a_k \otimes b_k\right) - z\right\|_E < \varepsilon. \qquad \square$$

Ramanujan and Schock [39] have shown that $\|T\|_{A\times B,E} = \|T^\times\|_{E^*,\mathcal{L}(A\times B,\mathbb{K})}$ and that T is compact if and only if $T^\times$ is compact. Next we establish the connection between weak compactness of T and $T^\times$.

Lemma 2.3. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then T is weakly compact if and only if $T^{\times}$ is weakly compact.*

Proof. Assume that T is weakly compact. Then $T(U_A, U_B)$ is relatively weakly compact. Having in mind that the closed convex hull of a weakly compact subset of E is also weakly compact (see [18, Theorem V.6.4]), it follows from Lemma 2.2 that $\widetilde{T}$ is weakly compact. By (2.2), $T^{\widetilde{\times}} = \widetilde{T}^{*}$. Therefore, according to Gantmacher's theorem (see [18, Theorem VI.4.8] or [27, p. 205]), we get that $T^{\widetilde{\times}}$ is weakly compact. Since $T^{\widetilde{\times}}$ is the composition of $T^{\times}$ with an isometry, we conclude that $T^{\times}$ is weakly compact.

Conversely, if $T^{\times}$ is weakly compact, so it is $T^{\widetilde{\times}}$. Equality $T^{\widetilde{\times}} = \widetilde{T}^{*}$ yields that $\widetilde{T}^{*}$ is weakly compact. Then, using again Gantmacher's theorem, we have that $\widetilde{T}$ is weakly compact. Let $\chi(a, b) = a \otimes b$. Factorization

$$\begin{tikzcd} A \times B \arrow[rr, "T"] \arrow[dr, "\chi"'] & & E \\ & A \widetilde{\otimes}_{\pi} B \arrow[ur, "\widetilde{T}"'] & \end{tikzcd}$$

implies that T is weakly compact. □

3 | REAL INTERPOLATION

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, two Banach spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space. Let $A_0 + A_1$ be their sum and let $A_0 \cap A_1$ be their intersection. The norms of these spaces are

$$\|a\|_{A_0+A_1} = \inf\left\{ \|a_0\|_{A_0} + \|a_1\|_{A_1} \,:\, a = a_0 + a_1 \,,\ a_j \in A_j \right\}$$

and

$$\|a\|_{A_0 \cap A_1} = \max\left\{ \|a\|_{A_0} \,,\ \|a\|_{A_1} \right\}.$$

The *Peetre's K-functional* is given by

$$K(t, a) = \inf\left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \,:\, a = a_0 + a_1,\ a_j \in A_j \right\},\ a \in A_0 + A_1.$$

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^{\infty} \left(t^{-\theta} K(t, a)\right)^q \frac{dt}{t} \right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). See [2, 41].

Subsequently, if $A_0 \cap A_1$ is continuously embedded in a Banach space A, we put A° for the closure of $A_0 \cap A_1$ in A. Using the description of $(A_0, A_1)_{\theta,q}$ by means of the dual functional of the K-functional, it follows that

$$(A_0^{\circ}, A_1^{\circ})_{\theta,q} = (A_0, A_1)_{\theta,q} \tag{3.1}$$

and that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ if $1 \leq q < \infty$. Hence, $(A_0, A_1)_{\theta,q}^{\circ} = (A_0, A_1)_{\theta,q}$ if $1 \leq q < \infty$.

The Banach couple $\bar{A} = (A_0, A_1)$ is said to be *regular* if $A_j^\circ = A_j$ for $j = 0, 1$. If $\bar{A}$ is regular, then (A_0^*, A_1^*) is also a Banach couple and the following formula holds with equivalence of norms

$$(A_0, A_1)_{\theta,r}^* = (A_0^*, A_1^*)_{\theta,r'} \quad \text{provided that} \quad 1 \le r < \infty, \; 1/r + 1/r' = 1 \quad \text{and} \quad 0 < \theta < 1. \tag{3.2}$$

See [2, Theorem 3.7.1]

Let $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be other Banach couples. We write $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ if T is a bilinear operator defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ such that there are constants $M_j > 0$ with

$$\|T(a, b)\|_{E_j} \le M_j \|a\|_{A_j} \|b\|_{B_j}, \; a \in A_0 \cap A_1, \; b \in B_0 \cap B_1, \; j = 0, 1. \tag{3.3}$$

It follows from (3.3) that T may be uniquely extended to a bilinear operator $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ with $\|T\|_{A_j^\circ \times B_j^\circ, E_j^\circ} \le M_j$.

Next we describe the behaviour of weakly compact bilinear operators under the real method.

Theorem 3.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples. Assume that $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ satisfies that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ is weakly compact for $j = 0$ or 1. Let $0 < \theta < 1$, $1 \le p, q < \infty$ and $1 < r < \infty$ with $1/p + 1/q = 1 + 1/r$. Then T may be uniquely extended to a weakly compact bilinear operator from $(A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}$ to $(E_0, E_1)_{\theta,r}$.*

Proof. Applying the bilinear interpolation theorem (see [22, Theorem 4.1]), we have that T may be uniquely extended to a bounded bilinear operator

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r}.$$

By Lemma 2.3, in order to check that this extension is weakly compact, it suffices to show that

$$T^\times : (E_0, E_1)_{\theta,r}^* \longrightarrow \mathcal{L}\Big((A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}, \mathbb{K}\Big) \tag{3.4}$$

is weakly compact. With this aim, put $X_j = \mathcal{L}\Big(A_j^\circ \times B_j^\circ, \mathbb{K}\Big)$. The assumption on T yields that

$$T^\times : \Big(E_j^\circ\Big)^* \longrightarrow X_j \quad \text{boundedly for} \;\; j = 0, 1.$$

Moreover, by Lemma 2.3, one of these two restrictions of $T^\times$ is weakly compact. Whence, using the interpolation theorem for weakly compact linear operators [32, Theorem 1], we obtain that

$$T^\times : \Big(\big(E_0^\circ\big)^*, \big(E_1^\circ\big)^*\Big)_{\theta,r'} \longrightarrow (X_0, X_1)_{\theta,r'}$$

is weakly compact, where $1/r + 1/r' = 1$. By (3.2) and (3.1), the source of this operator is

$$\Big(\big(E_0^\circ\big)^*, \big(E_1^\circ\big)^*\Big)_{\theta,r'} = (E_0^\circ, E_1^\circ)_{\theta,r}^* = (E_0, E_1)_{\theta,r}^*.$$

On the other hand, it is shown in [10, p. 5] that

$$(X_0, X_1)_{\theta,r'} \hookrightarrow \mathcal{L}\Big((A_0^\circ, A_1^\circ)_{\theta,p} \times (B_0^\circ, B_1^\circ)_{\theta,q}, \mathbb{K}\Big),$$

where $\hookrightarrow$ means continuous embedding. Since

$$\mathcal{L}\Big((A_0^\circ, A_1^\circ)_{\theta,p} \times (B_0^\circ, B_1^\circ)_{\theta,q}, \mathbb{K}\Big) = \mathcal{L}\Big((A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}, \mathbb{K}\Big),$$

this yields (3.4) and completes the proof. $\square$

Remark 3.2. For compact bilinear operators among Banach couples interpolated by the real method, the counterpart of Theorem 3.1 holds for any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$ provided that we replace $(X_0, X_1)_{\theta,\infty}$ by $(X_0, X_1)^{\circ}_{\theta,\infty}$ (see [12, Theorem 4.3]). However, in the assumptions of Theorem 3.1 if the parameters p or q take the value ∞ or if $r = 1$ then T may fail to be weakly compact acting from $(A_0, A_1)^{\circ}_{\theta,p} \times (B_0, B_1)^{\circ}_{\theta,q}$ to $(E_0, E_1)^{\circ}_{\theta,r}$. We will show it next by means of examples.

Let $1 < s_0 < s_1 < \infty$, $0 < \theta < 1$, $1/s = (1-\theta)/s_0 + \theta/s_1$, $1/s'_j = 1 - 1/s_j$ and $1 \le v \le \infty$. Then $(\ell_{s_0}, \ell_{s_1})_{\theta,v} = \ell_{s,v}$ (see [41, Theorem 1.18.3/2, p. 127]). Each space ℓ_{s_j} is reflexive but $\ell^{\circ}_{s,\infty}$ is not reflexive because using [2, Remark on p. 55] we get

$$\left(\ell^{\circ}_{s,\infty}\right)^{**} = \left(\left(\ell_{s_0}, \ell_{s_1}\right)^{\circ}_{\theta,\infty}\right)^{**} = \left(\ell_{s'_0}, \ell_{s'_1}\right)^{*}_{\theta,1} = \left(\ell_{s_0}, \ell_{s_1}\right)_{\theta,\infty} = \ell_{s,\infty} \supsetneq \ell^{\circ}_{s,\infty}.$$

The space $\ell_{s,1} = (\ell_{s_0}, \ell_{s_1})_{\theta,1}$ also fails to be reflexive because it contains a subspace isomorphic to ℓ_1 (see [30]).

Assume that $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$. If $p = \infty$ then q should take the value 1 and r should be ∞. Consider the couples $\bar{A} = \bar{E} = (\ell_{s_0}, \ell_{s_1})$, $\bar{B} = (\mathbb{K}, \mathbb{K})$ and let $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ be the bilinear operator defined by $T(a, \lambda) = \lambda a$ for $a \in \ell_{s_1}$ and $\lambda \in \mathbb{K}$. Since ℓ_{s_j} is reflexive, then $T : A^{\circ}_j \times B^{\circ}_j \longrightarrow E^{\circ}_j$ is weakly compact for $j = 0, 1$. However, the interpolated operator

$$T : (A_0, A_1)^{\circ}_{\theta,\infty} \times (B_0, B_1)_{\theta,1} \longrightarrow (E_0, E_1)^{\circ}_{\theta,\infty}$$

is $T : \ell^{\circ}_{s,\infty} \times \mathbb{K} \longrightarrow \ell^{\circ}_{s,\infty}$ which is not weakly compact because $\ell^{\circ}_{s,\infty}$ is not reflexive.

If $q = \infty$ then p should be 1 and r should be ∞. So, we can proceed as before taking now $\bar{A} = (\mathbb{K}, \mathbb{K})$, $\bar{B} = \bar{E} = (\ell_{s_0}, \ell_{s_1})$ and $T(\lambda, b) = \lambda b$.

If $r = 1$ then p and q should be 1. Take again $\bar{A} = \bar{E} = (\ell_{s_0}, \ell_{s_1})$, $\bar{B} = (\mathbb{K}, \mathbb{K})$ and $T(a, \lambda) = \lambda a$. As we said before, $T : A^{\circ}_j \times B^{\circ}_j \longrightarrow E^{\circ}_j$ is weakly compact for $j = 0, 1$. But the interpolated operator is $T : \ell_{s,1} \times \mathbb{K} \longrightarrow \ell_{s,1}$ which is not weakly compact because $\ell_{s,1}$ is not reflexive.

4 | COMPLEX INTERPOLATION

In what follows we work with complex scalars and complex Banach spaces. If $\bar{A} = (A_0, A_1)$ is a Banach couple, we put $\mathcal{F}(\bar{A})$ for the space of all functions g from the closed strip $D = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ into $A_0 + A_1$ such that g is bounded and continuous on D, analytic on the interior of D, with $g(j + it) \in A_j$ for all $t \in \mathbb{R}$, $j = 0, 1$, and the functions $t \to g(j + it)$ are continuous from $\mathbb{R}$ into A_j and tend to zero as $|t| \to \infty$. We put

$$\|g\|_{\mathcal{F}(\bar{A})} = \max_{j=0,1} \left\{ \sup_{t \in \mathbb{R}} \|g(j + it)\|_{A_j} \right\}.$$

For $0 < \theta < 1$, the *complex interpolation space* $[A_0, A_1]_\theta$ is formed by all those $a \in A_0 + A_1$ such that $a = g(\theta)$ for some $g \in \mathcal{F}(\bar{A})$. The norm in $[A_0, A_1]_\theta$ is

$$\|a\|_{[A_0,A_1]_\theta} = \inf\left\{ \|g\|_{\mathcal{F}(\bar{A})} : g(\theta) = a,\, g \in \mathcal{F}(\bar{A}) \right\}.$$

See [2].

It turns out that $A_0 \cap A_1$ is dense in $[A_0, A_1]_\theta$ and that

$$[A_0, A_1]_\theta = [A^{\circ}_0, A^{\circ}_1]_\theta \quad \text{(see [2, Theorem 4.2.2]).} \tag{4.1}$$

We are going to need also the so-called *upper complex interpolation space*

$$[A_0, A_1]^\theta = \left\{ a \in A_0 + A_1 : a = h'(\theta),\, h \in \mathcal{G}(\bar{A}) \right\}$$

where $\mathcal{G}(\bar{A})$ is the space of all continuous functions h from D into $A_0 + A_1$ which are analytic on the interior of D, with $h(j+it) - h(j+is) \in A_j$ for all $t, s \in \mathbb{R}$, with $\|h(z)\|_{A_0+A_1} \le c(1+|z|)$ for all $0 \le |z| \le 1$ and

$$\|h\|_{\mathcal{G}(\bar{A})} = \max_{j=0,1} \left\{ \sup_{t,s\in\mathbb{R}} \left\| \frac{h(j+it)-h(j+is)}{t-s} \right\|_{A_j} \right\} < \infty.$$

We endow $[A_0, A_1]^\theta$ with the norm

$$\|a\|_{[A_0,A_1]^\theta} = \inf\{ \|h\|_{\mathcal{G}(\bar{A})} \ :\ h'(\theta) = a,\ h \in \mathcal{G}(\bar{A}) \}.$$

We have

$$[A_0, A_1]_\theta = [A_0, A_1]^\theta \quad \text{(equal norms) provided that at least one of } A_0 \text{ or } A_1 \text{ is reflexive (see [2, Theorem 4.3.1]).} \tag{4.2}$$

Moreover

$$[A_0, A_1]_\theta^* = \left[\left(A_0^\circ\right)^*, \left(A_1^\circ\right)^* \right]^\theta \quad \text{(see [2, Theorem 4.5.1]).} \tag{4.3}$$

Calderón [6, 12.2] has show that $[A_0, A_1]_\theta$ is reflexive if at least one of A_0 and A_1 is reflexive. Combining this fact with (4.2) we get

$$[A_0, A_1]^\theta \quad \text{is reflexive provided that } A_0 \text{ or } A_1 \text{ is reflexive.} \tag{4.4}$$

Next we establish an auxiliary result involving linear operators and the upper complex method. For the proof we follow an idea of [7, Corollary 3.4].

Lemma 4.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $R : A_0 + A_1 \longrightarrow B_0 + B_1$ be a linear operator such that the restrictions $R : A_j \longrightarrow B_j$ are bounded for $j = 0, 1$, and one of the two restrictions is weakly compact. Then*

$$R : [A_0, A_1]^\theta \longrightarrow [B_0, B_1]^\theta \quad \textit{is weakly compact.}$$

Proof. Assume that $R : A_1 \longrightarrow B_1$ is weakly compact. We start by factorizing this operator according to the diagram

$$\begin{tikzcd} A_1 \arrow[r, "R"] \arrow[d, "S"'] & B_1 \\ X \arrow[r, "j"'] & B_1. \arrow[u, "I"'] \end{tikzcd}$$

Here $X = A_1/\mathrm{Ker}(R)$ is the quotient space, $S(x) = [x]$ is the quotient mapping, $j[x] = Rx$ and I is the identity map. Let $W = R(A_1)$ endowed with the norm

$$\|Rx\|_W = \|[x]\|_X = \inf\{ \|y\|_{A_1} \ :\ Rx = Ry \}.$$

The space W is isometric to X and therefore it is a Banach space. Moreover the embedding

$$W \hookrightarrow B_1$$

is continuous. Hence, the pair (W, B_1) is a Banach couple and we have the factorization

$$\begin{tikzcd} A_1 \arrow[r, "R"] \arrow[d, "jS"'] & B_1 \\ W \arrow[r, hook] & B_1. \arrow[u, "I"'] \end{tikzcd}$$

Furthermore, since the ideal of weakly compact operators is surjective and $R : A_1 \longrightarrow B_1$ is weakly compact, then the embedding $W \hookrightarrow B_1$ is weakly compact. It follows from [1, Théorème 1, p. 40] that $Y = (W, B_1)_{1/2,2}$ is reflexive. Note also that we have the continuous embedding $Y \hookrightarrow B_0 + B_1$, therefore (B_0, Y) is a Banach couple. Now consider the diagrams

$$\begin{tikzcd} A_0 \arrow[rr, "R"] \arrow[dr, "R"'] & & B_0 \\ & B_0 \arrow[ur, "I"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[rr, "R"] \arrow[dr, "R"'] & & B_1 \\ & Y \arrow[ur, "I"'] & \end{tikzcd}$$

Since the upper complex method has the interpolation property for linear operators [2, Theorem 4.1.4], we obtain that $R : [A_0, A_1]^\theta \longrightarrow [B_0, B_1]^\theta$ can be factorized as

$$\begin{tikzcd} {[A_0, A_1]^\theta} \arrow[rr, "R"] \arrow[dr, "R"'] & & {[B_0, B_1]^\theta} \\ & {[B_0, Y]^\theta} \arrow[ur, "I"'] & \end{tikzcd}$$

and the space $[B_0, Y]^\theta$ is reflexive by (4.4). Consequently, we derive that $R : [A_0, A_1]^\theta \longrightarrow [B_0, B_1]^\theta$ is weakly compact.

The case when $R : A_0 \longrightarrow B_0$ is weakly compact can be treated analogously. □

Next we describe the behaviour under complex interpolation of weakly compact bilinear operators.

Theorem 4.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and $0 < \theta < 1$. Assume that $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ satisfies that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ is weakly compact for $j = 0$ or 1. Then T may be uniquely extended to a weakly compact bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to $[E_0, E_1]_\theta$.*

Proof. We know by (4.1) that $[A_0, A_1]_\theta = [A_0^\circ, A_1^\circ]_\theta$ and a similar equality holds for the couples $\bar{B}$ and $\bar{E}$. The complex bilinear interpolation theorem [2, Theorem 4.4.1] yields that T may be uniquely extended to a bounded bilinear operator

$$T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow [E_0, E_1]_\theta.$$

Hence, by Lemma 2.3, to establish the theorem it is enough to prove that

$$T^\times : [E_0, E_1]_\theta^* \longrightarrow \mathcal{L}\Big([A_0, A_1]_\theta \times [B_0, B_1]_\theta, \mathbb{C}\Big) \tag{4.5}$$

is weakly compact.

Since $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ is bounded for $j = 0, 1$ and weakly compact for, say, $j = 1$, using Lemma 2.3 we get that

$$T^\times : (E_j^\circ)^* \longrightarrow \mathcal{L}(A_j^\circ \times B_j^\circ, \mathbb{C}) = X_j$$

is bounded for $j = 0, 1$ and weakly compact for $j = 1$. Note that $X_j \hookrightarrow \mathcal{L}((A_0 \cap A_1) \times (B_0 \cap B_1), \mathbb{C})$, $j = 0, 1$. Applying Lemma 4.1 we obtain that

$$T^\times : \left[(E_0^\circ)^*, (E_1^\circ)^*\right]^\theta \longrightarrow [X_0, X_1]^\theta$$

is weakly compact. By (4.3), the source of this operator is $[E_0, E_1]_\theta^*$. Consequently, to derive (4.5) it suffices to prove that

$$[X_0, X_1]^\theta \hookrightarrow \mathcal{L}\Big([A_0, A_1]_\theta \times [B_0, B_1]_\theta, \mathbb{C}\Big). \tag{4.6}$$

Embedding (4.6) is an improvement of [10, (2.6) in p. 5] (see also [37, Theorem 2.1]). We are going to establish it by refining the argument in [10].

Let Φ be the bilinear operator assigning to any $R \in \mathcal{L}\big((A_0 \cap A_1) \times (B_0 \cap B_1), \mathbb{C}\big)$ and $a \in A_0 \cap A_1$ the functional of $\big(B_0 \cap B_1\big)^*$ defined by $\Phi(R,a)(b) = R(a,b)$ where $b \in B_0 \cap B_1$. We have

$$\|\Phi(R,a)\|_{(B_0\cap B_1)^*} = \sup\big\{|R(a,b)| \ : \ \|b\|_{B_0\cap B_1} \le 1\big\} \le \|R\|_{(A_0\cap A_1)\times(B_0\cap B_1),\mathbb{C}}\,\|a\|_{A_0\cap A_1}.$$

Hence,

$$\Phi \,:\, \mathcal{L}\big((A_0\cap A_1)\times(B_0\cap B_1),\mathbb{C}\big)\times\big(A_0\cap A_1\big) \longrightarrow \big(B_0\cap B_1\big)^* = \big(B_0^\circ\big)^* + \big(B_1^\circ\big)^*$$

is bounded with norm less than or equal to 1. Moreover, if $R \in X_j$, then we have

$$\|\Phi(R,a)\|_{\left(B_j^\circ\right)^*} = \sup\Big\{|R(a,b)| \ : \ \|b\|_{B_j} \le 1\,,\, b \in B_0\cap B_1\Big\} \le \|R\|_{X_j}\|a\|_{A_j}.$$

Therefore,

$$\Phi \,:\, X_j \times \big(A_0\cap A_1, \|\cdot\|_{A_j}\big) \longrightarrow \Big(\big(B_0^\circ\big)^* + \big(B_1^\circ\big)^*, \|\cdot\|_{\left(B_j^\circ\right)^*}\Big)$$

is also bounded for $j = 0,1$ with norm less than or equal to 1. According to [2, Theorem 4.4.2], the operator Φ may be extended uniquely to a bounded bilinear map from $\big[X_0,X_1\big]^\theta \times \big[A_0,A_1\big]_\theta$ to $\Big[\big(B_0^\circ\big)^*,\big(B_1^\circ\big)^*\Big]^\theta$ with norm less than or equal to 1. By (4.3), the last space is $\big[B_0,B_1\big]_\theta^*$. Therefore, for any $a \in A_0\cap A_1$, $b \in B_0 \cap B_1$ and $R \in \big[X_0,X_1\big]^\theta$ we have

$$|R(a,b)| = |\Phi(R,a)(b)| \le \|R\|_{[X_0,X_1]^\theta}\|a\|_{[A_0,A_1]_\theta}\|b\|_{[B_0,B_1]_\theta}.$$

This yields (4.6) and finishes the proof. □

5 | FACTORIZATION OF WEAKLY COMPACT BILINEAR OPERATORS

In this final section we show that Lemma 2.2 allows to extend the factorization result of Davis, Figiel, Johnson and Pelczyński [15] to bilinear operators.

Theorem 5.1. *Let A,B,E be Banach spaces and let $T \,:\, A\times B \longrightarrow E$ be a weakly compact bilinear operator. Then there is a reflexive Banach space W, a bounded bilinear operator $R \in \mathcal{L}(A\times B, W)$ and a bounded linear operator $S \in \mathcal{L}(W,E)$ such that the following diagram commutes*

$$\begin{tikzcd} A\times B \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, "S"'] & \end{tikzcd}$$

Proof. Since T is weakly compact, it follows from Lemma 2.2 and [18, Theorem V.6.4] that the linear operator $\widetilde{T} \in \mathcal{L}\big(A\widetilde{\otimes}_\pi B, E\big)$ is weakly compact. Applying the factorization theorem [15, Corollary 1] to $\widetilde{T}$ we obtain that there is a reflexive Banach space W and bounded linear operators $S_1 \in \mathcal{L}\big(A\widetilde{\otimes}_\pi B, W\big)$ and $S_2 \in \mathcal{L}(W,E)$ such that $\widetilde{T} = S_2 \circ S_1$. Consider the bounded bilinear operator $\chi \,:\, A\times B \longrightarrow A\widetilde{\otimes}_\pi B$ given by $\chi(x,u) = x\otimes u$. Then $S_1\circ\chi \in \mathcal{L}(A\times B, W)$ and the following factorization holds for T

$$\begin{tikzcd} A\times B \arrow[rr, "T"] \arrow[dr, "S_1\circ\chi"'] & & E \\ & W \arrow[ur, "S_2"'] & \end{tikzcd}$$

□

The factorization result for weakly compact linear operators has been carried over to the setting of Banach algebras by Blanco, Kaijser and Ransford [3]. They showed that every weakly compact homomorphism between Banach algebras $\Phi : A \longrightarrow B$ factors through a reflexive Banach algebra W with Banach-algebra homomorphisms as factors. See also the paper [8] of the present authors.

In order to extend Theorem 5.1 to Banach algebras, we assume that A, B, E are Banach algebras and that $T \in \mathcal{L}(A \times B, E)$ satisfies the following property:

$$T(xy, uv) = T(x,u)T(y,v) \text{ for any } x, y \in A \text{ and } u, v \in B. \tag{5.1}$$

Theorem 5.2. *Let A, B, E be Banach algebras and let $T : A \times B \longrightarrow E$ be a weakly compact bilinear operator satisfying (5.1). Then there is a reflexive Banach algebra W, a bounded bilinear operator $R \in \mathcal{L}(A \times B, W)$ satisfying (5.1) and a Banach-algebra homomorphism $\Psi \in \mathcal{L}(W, E)$ such that the following diagram commutes*

$$\begin{tikzcd} A \times B \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, "\Psi"'] & \end{tikzcd}$$

Proof. We know that $A\widetilde{\otimes}_\pi B$ is a Banach algebra (see, for example, [4, Proposition VI.42.18, p. 236]), the product in $A \otimes B$ being

$$\left(\sum_{i=1}^m x_i \otimes u_i\right)\left(\sum_{j=1}^n y_j \otimes v_j\right) = \sum_{i=1}^m \sum_{j=1}^n x_i y_j \otimes u_i v_j \,,$$

which may be extended to $A\widetilde{\otimes}_\pi B$. Since T satisfies (5.1), then $\widetilde{T} : A\widetilde{\otimes}_\pi B \longrightarrow E$ is a Banach-algebra homomorphism because

$$\begin{aligned} \widetilde{T}\left(\left(\sum_{i=1}^m x_i \otimes u_i\right)\left(\sum_{j=1}^n y_j \otimes v_j\right)\right) &= \widetilde{T}\left(\sum_{i=1}^m \sum_{j=1}^n x_i y_j \otimes u_i v_j\right) \\ &= \sum_{i=1}^m \sum_{j=1}^n T\left(x_i y_j, u_i v_j\right) \\ &= \sum_{i=1}^m \sum_{j=1}^n T\left(x_i, u_i\right) T\left(y_j, v_j\right) \\ &= \left(\sum_{i=1}^m T\left(x_i, u_i\right)\right)\left(\sum_{j=1}^n T\left(y_j, v_j\right)\right) \\ &= \widetilde{T}\left(\sum_{i=1}^m x_i \otimes u_i\right)\widetilde{T}\left(\sum_{j=1}^n y_j \otimes v_j\right), \end{aligned}$$

and, by (2.1), we may extend this equality from the elements of $A \otimes B$ to $A\widetilde{\otimes}_\pi B$. Consequently, by [3, Theorem 4.1], there is a reflexive Banach algebra W and there are Banach-algebra homomorphisms $\Phi : A\widetilde{\otimes}_\pi B \longrightarrow W$, $\Psi : W \longrightarrow E$ such that $\widetilde{T} = \Psi \circ \Phi$. Put $R = \Phi \circ \chi$. Then $R \in \mathcal{L}(A \times B, W)$ and R satisfies (5.1) because

$$R(xy, uv) = \Phi(xy \otimes uv) = \Phi\big((x \otimes u)(y \otimes v)\big) = \Phi(x \otimes u)\Phi(y \otimes v) = R(x,y)R(u,v).$$

Finally, T admits the following factorization

$$\begin{tikzcd} A \times B \arrow[rr, "T"] \arrow[dr, "R"'] & & E \\ & W \arrow[ur, "\Psi"'] & \end{tikzcd}$$

□

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