UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Bishop operators: invariant subspaces and spectral theory

(Operadores de Bishop: subespacios invariantes y teoría espectral)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Miguel Monsalve López

Directora

Eva Antonia Gallardo Gutiérrez

Madrid

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La distancia que separa una promesa de un acto es comparable a la existente entre una simple opinión y la demostración de un teorema. Concretamente, un mundo. Agradecer es una promesa, quizás llena de humildad y fidelidad. Hacerlo por escrito, un acto para la posteridad.

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Resumen

Operadores de Bishop: subespacios invariantes y teoría espectral

Desde hace casi un siglo, se han propuesto varias clases de operadores como posibles contraejemplos para el Problema del Subespacio Invariante: quizás, la pregunta abierta más importante en Teoría de Operadores en espacios de Banach reflexivos y, en particular, en espacios de Hilbert. Uno de los candidatos más sencillos viene dado por la familia de los operadores de Bishop definidos sobre los espacios $L^p[0,1)$ para $1 \leq p < \infty$, los cuales fueron sugeridos por Errett Bishop durante la década de los cincuenta. A pesar de su aparente sencillez, resulta que las propiedades de los operadores de Bishop T_α siguen siendo ampliamente desconocidas. En particular, hasta la fecha, es una cuestión abierta determinar si T_α dispone de subespacios invariantes no triviales en $L^p[0,1)$ para cualquier irracional $\alpha \in (0,1)$.

El objetivo principal de esta Tesis Doctoral es analizar la existencia de subespacios invariantes para todos los operadores de Bishop. En aras de una mejor comprensión, la memoria se ha dividido en dos partes bien diferenciadas. La primera parte está dedicada a introducir los preliminares necesarios (Capítulo 1); mientras que, en la segunda parte se detallarán las contribuciones más relevantes realizadas por el autor en dicho problema (Capítulos 2–5).

Al comienzo del Capítulo 2, demostramos que todos los operadores de Bishop son bicuasitriangulares, concluyendo por tanto que deben ser el límite (en la topología fuerte) de operadores nilpotentes. Posteriormente, mediante estimaciones aritméticas precisas y junto con un teorema clásico de Atzmon [**15**, Theorem 1.1], extendemos considerablemente el conjunto de los irracionales $\alpha \in (0,1)$ tales que el operador de Bishop asociado T_α posee subespacios invariantes; mejorando los resultados previos conocidos de Davie [**44**] y Flattot [**60**]. De hecho, en el Capítulo 3, establecemos el límite de las técnicas basadas en el Teorema de Atzmon en este contexto.

Posteriormente, en el Capítulo 4, con la ayuda de algunos resultados de Teoría Ergódica, probamos que una amplia gama de operadores de traslación con pesos (entre ellos, los operadores de Bishop) son *power-regular*, calculando el valor exacto de sus radios espectrales locales. Como consecuencia, deducimos que ciertas descomposiciones espectrales no pueden darse para ningún operador de Bishop. Además, caracterizamos aquellas propiedades espectrales locales satisfechas simultáneamente por todos los operadores T_α, independientemente del irracional $\alpha \in (0,1)$. En cierto sentido, esto parece indicar que los operadores de Bishop podrían carecer de un comporamiento espectral verdaderamente útil, ya que por ejemplo, nunca son descomponibles.

Finalmente, en el Capítulo 5, generalizamos el Teorema de Atzmon mediante la aplicación de modelos funcionales más débiles, los cuales permitirán construir subespacios invariantes a partir de variedades espectrales locales. Nuestra estrategia combina propiedades inherentes a las particiones de la unidad con un cálculo funcional para producir descomposiciones espectrales no nulas. En particular, en el caso concreto de los operadores de Bishop, demostramos la existencia de subespacios espectrales no triviales para cada T_α que verifique las hipótesis del Teorema de Atzmon, proporcionando descomposiciones espectrales locales no triviales para tales T_α.

Abstract

Bishop operators: invariant subspaces and spectral theory

For nearly a century, various classes of linear bounded operators have been posed as potential counterexamples to the Invariant Subspace Problem: maybe, the most important long-standing open question in Operator Theory. One of the simplest candidates consists of the family of Bishop operators T_α acting on $L^p[0,1)$ spaces, which were suggested by Errett Bishop in the fifties. Unlike their seeming simplicity, the structure and features of Bishop operators remain largely uncharted. In particular, hitherto, it is still unknown whether T_α has non-trivial invariant subspaces in $L^p[0,1)$ for each $1 \leq p < \infty$ and any irrational $\alpha \in (0,1)$.

The major purpose of the present PhD thesis is to analyse the existence of invariant subspaces for all Bishop operators. Aiming for a better comprehension of the subject, this monograph has been divided into two parts. The first part is devoted to introducing some required preliminaries (Chapter 1); while, the second part deals with our main contributions (Chapters 2–5).

At the beginning of Chapter 2, we prove that all Bishop operators are biquasitriangular and derive that they are norm-limits of nilpotent operators. Afterwards, by means of sharp arithmetical estimations along with a classical theorem of Atzmon [**15**, Theorem 1.1], the set of irrationals $\alpha \in (0,1)$ for which T_α is known to possess non-trivial invariant subspaces is considerably enlarged; extending previous results by Davie [**44**] and Flattot [**60**]. Indeed, in Chapter 3, we essentially show that when our approach fails to produce invariant subspaces for T_α, it is actually because the standard techniques no longer applies.

In Chapter 4, with the aid of some ergodic tools, it is proved that a wide class of weighted translation operators (among them, Bishop operators) are power-regular, computing the exact value of their local spectral radii. As a consequence, we deduce that certain spectral decompositions cannot hold for any Bishop operator. Furthermore, we characterize those local spectral properties enjoyed by all Bishop operators T_α, independently of the irrational α. In some sense, this seems to indicate that Bishop operators might lack of any profitable spectral behaviour since, for instance, they can never be decomposable.

Finally, in Chapter 5, we generalize Atzmon's Theorem upon considering weaker functional models which enable us to construct invariant subspaces via local spectral manifolds. Our strategy uses partitions of unity combined with a functional calculus argument in order to produce non-zero spectral decompositions. As concrete applications to Bishop operators, we prove the existence of non-trivial spectral subspaces for T_α on each $L^p[0,1)$ as long as T_α verifies the assumptions of Atzmon's Theorem, providing, in turns, a local spectral decomposition. Roughly speaking, this evinces that certain Bishop operators exhibit a weak type of decomposability.

Part I

Background: Introduction and Preliminaries

Introduction

Undoubtedly, one of the most important unsolved questions in Operator Theory is the so-called *Invariant Subspace Problem*. Although it is difficult to date accurately its formulation, it seems that this question became renowned nearly a century ago after two outstanding influential works: firstly, in the mid-thirties, the unpublished results due to von Neumann regarding the existence of invariant subspaces for compact operators on Hilbert spaces. Secondly, the complete characterization of the lattice of invariant subspaces for the shift operator acting on the Hardy space $H^2(\mathbb{D})$, given by Beurling [**25**] in 1949 using the inner-outer factorization. In spite of the significant advances made over the years, the Invariant Subspace Problem is still a conundrum for operator theorists and, according to [**36**], a full solution seems nowhere in sight, at least in the nearest future.

Nevertheless, as often happens with long-standing open questions, it is worth mentioning the wealth of mathematical tools developed around the Invariant Subspace Problem, which has incentivized major advances within many allied areas. Supporting this claim, the lack of a general method for producing invariant subspaces for an arbitrary operator usually forces operator theorists to apply a wide variety of techniques, borrowed from many distinct fields, for instance: Complex Analysis, Functional Analysis, Harmonic Analysis, or even further ones such as Homological Algebra thank to the close interplay between Spectral Theory and Sheaf Cohomology.

A complete account on the Invariant Subspace Problem and its starring role in Operator Theory is insightfully explained in the monographs [**20, 36, 105**].

For convenience, throughout the Introduction, H will stand for an arbitrary complex Hilbert space, while the symbol X shall play the same role for Banach spaces. Besides, $\mathcal{B}(X)$ will denote the space of bounded linear operators on a Banach space X.

Given a linear operator $T : \mathcal{V} \to \mathcal{V}$ acting on a vector space $\mathcal{V}$, a subset $\mathcal{M} \subseteq \mathcal{V}$ is called *invariant* if for each $v \in \mathcal{M}$, its image Tv also belongs to $\mathcal{M}$. A linear manifold in $\mathcal{V}$ which is invariant under the action of T is said to be an *invariant linear manifold* for T. In general, throughout this monograph, we shall deal with topological vector spaces (mainly with Banach spaces and Hilbert spaces); consequently, in this regard, by *invariant subspaces* of a linear operator $T : \mathcal{V} \to \mathcal{V}$, we will refer to those invariant linear manifolds of T which are closed with respect to the topology of $\mathcal{V}$. As usual, the subspaces $\{0\}$ and $\mathcal{V}$ will be known as *trivial invariant subspaces*.

Now, we are in position to state the original concern of the Invariant Subspace Problem:

Invariant Subspace Problem. *Does every linear bounded operator $T : X \to X$ acting on a complex Banach space X have a non-trivial invariant subspace?*

As explained in [**105**], one of the main motivations behind the study of invariant subspaces comes from the interest in the structure of operators: basically, all the structural information of a linear mapping is somehow encoded within its lattice of invariant subspaces. More concretely, a deep knowledge on the invariant subspaces of an operator allows to decompose it in simpler ones. This kind of constructions are highly employed in many frameworks, for instance, in Approximation Theory.

An enlightening example at this regard is the classical *Jordan Canonical Form*, which asserts that given a linear mapping T acting on the finite-dimensional Hilbert space $\mathbb{C}^d$ with *eigenvalues* $\sigma(T) = \{\lambda_1, \ldots, \lambda_\ell\}$, the operator $T : \mathbb{C}^d \to \mathbb{C}^d$ is unitarily equivalent to the upper-triangular matrix

$$T \sim \begin{pmatrix} J_{\lambda_1}^{\nu_1} & & \\ & \ddots & \\ & & J_{\lambda_\ell}^{\nu_\ell} \end{pmatrix},$$

formed by the *Jordan blocks* $J_{\lambda_j}^{\nu_j}$ (recall that for each eigenvalue λ_j may correspond several Jordan blocks), via a suitable change of basis provided by the direct sum decomposition

$$\mathbb{C}^d = \bigoplus_{j=1}^{\ell} \ker(T - \lambda_j I)^{\iota(\lambda_j)}.$$

Here, $\iota(\lambda_j)$ coincides with the size of its largest corresponding Jordan block.

In this context, although in somewhat more complex terms, the *Spectral Theorem* (see, for instance, [**39**, Thm. 2.2, IX]) may be understood under the same philosophy: given a bounded *normal operator* N on a Hilbert space H, i.e. a linear bounded operator commuting with its adjoint N^*, there exists a unique *spectral measure* E on the Borel subsets of the spectrum $\sigma(N)$, for which

$$N = \int_{\sigma(N)} \lambda \, dE(\lambda).$$

In this sense, roughly speaking, the spectral theorem can be rephrased saying that each normal operator is unitarily equivalent to the operator M_λ of multiplication by the independent variable λ subject to the direct integral decomposition

$$H = \int_{\sigma(N)}^{\oplus} H_\lambda \, dE(\lambda),$$

where, in this case, the family of Hilbert spaces $(H_\lambda)_{\lambda \in \sigma(N)}$ are constituted by *approximate eigenvectors* associated to each *approximate eigenvalue* $\lambda \in \sigma(N)$.

In the finite-dimensional setting, the *Jordan Canonical Form Theorem* grants an affirmative answer to the existence of invariant subspaces. Namely, given a linear operator $T : X \to X$ acting on a finite-dimensional complex Banach space X, the *Fundamental Theorem of Algebra* ensures the existence of eigenvalues for T as the roots of the *minimal polynomial*

$$p_T(\lambda) := \det\big(T - \lambda I\big).$$

Now, since any scalar perturbation associated to each eigenvalue $\lambda_j \in \sigma(T)$ fails to be one-to-one, all the spaces

$$\ker\big(T - \lambda_j I\big) := \{x \in X \, : \, (T - \lambda_j I)x = 0\}$$

are non-zero invariant subspaces for T.

On the contrary, the situation in the infinite-dimensional setting turns out to be much more complicated. Once again, the *spectrum* of a bounded linear operator $T : X \to X$, which is the non-empty compact set defined by the condition

$$\sigma(T) := \big\{\lambda \in \mathbb{C} \, : \, (T - \lambda I) \text{ is not invertible}\big\},$$

plays an important role in the study of invariant subspaces of T.

However, in this framework, invertibility is no longer equivalent to injectivity; thus, $\sigma(T)$ is composed by additional elements apart from eigenvalues (indeed, $\sigma(T)$ may have no eigenvalues). Consequently, it is convenient to conduct a more detailed analysis of the spectrum $\sigma(T)$.

Accordingly, in order to discuss potential candidates of operators $T : X \to X$ with no invariant subspaces, we must always look upon injective operators with dense range. Otherwise, either $\ker(T)$ or the closure of the range of T, denoted by $\overline{\operatorname{ran}(T)}$, will be non-trivial invariant subspaces for T. Concerning these notions, we may consider the sets

$$\sigma_{\mathrm{p}}(T) := \big\{\lambda \in \mathbb{C} \,:\, (T - \lambda I) \text{ is not injective}\big\},$$
$$\sigma_{\mathrm{com}}(T) := \big\{\lambda \in \mathbb{C} \,:\, (T - \lambda I)X \text{ is not dense in } X\big\},$$

usually known as *point spectrum* and *compression spectrum* respectively. Clearly, both are included in $\sigma(T)$ and are related by the identity $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{p}}(T^*)$. Even, among the simplest examples, we may find operators T exhibiting $\sigma_{\mathrm{p}}(T) = \emptyset$ and/or $\sigma_{\mathrm{com}}(T) = \emptyset$; for instance, the *unilateral shift operator* in $\ell^2(\mathbb{Z}_+)$

$$\begin{aligned} S : \ell^2(\mathbb{Z}_+) &\longrightarrow \ell^2(\mathbb{Z}_+) \\ (\lambda_0, \lambda_1, \ldots) &\longmapsto (0, \lambda_0, \lambda_1, \ldots) \end{aligned} \tag{0.1}$$

possesses $\sigma_{\mathrm{p}}(T) = \emptyset$, while $\sigma(T) = \sigma_{\mathrm{com}}(T) = \big\{\lambda \in \mathbb{C} \,:\, |\lambda| \leq 1\big\}$. In parallel, the multiplication operator $M_t f(t) := t\, f(t)$ acting on the complex Hilbert space $L^2(0,1)$ (with respect to the Lebesgue measure) has $\sigma_{\mathrm{p}}(M_t) = \sigma_{\mathrm{com}}(M_t) = \emptyset$ simultaneously.

An obvious observation links invariants subspaces to orbits of operators: recall that a non-zero vector $x \in X$ is called *cyclic* for an operator $T : X \to X$ if the *linear span* generated by its *orbit* is dense in the whole space X, i.e.

$$X = \overline{\operatorname{span}\big\{T^n x \,:\, n \geq 0\big\}}.$$

In connection to the Invariant Subspace Problem, observe that an arbitrary $T : X \to X$ has non-trivial invariant subspaces if and only if it admits non-cyclic vectors. Consequently, whenever X is a non-separable topological space, every operator $T : X \to X$ possesses non-trivial invariant subspaces.

Unfortunately, orbits of operators are neither well understood yet, giving rise to further fundamental questions in Operator Theory such as the hypercyclicity or supercyclicity phenomena.

Surprisingly, from the mid-seventies onwards, several remarkable results came into scene. In 1975, during the annual meeting of the AMS, Enflo announced a counterexample to the Invariant Subspace Problem in a separable infinite-dimensional complex Banach Space. Apparently, his result had been previously presented in the Séminaire Maurey-Schwartz at the École Polytechnique de Paris [**50**]; however, due to the high complexity of the article and a dilated reviewing process, it was finally published twelve years later [**51**].

Then, additional constructions followed the lines initiated by Enflo: firstly, Read [**106**] in 1984 and one year later, Beauzamy [**19**], who built a stronger counterexample enjoying the supercyclicity property. Afterwards, Read's construction was repeatedly strengthened and simplified (see, [**107, 108, 109**]), so as to produce several operators in the classical sequence space $\ell^1(\mathbb{Z}_+)$ (as well on the space c_0) with no invariant subspaces and satisfying further properties such as hypercyclicity. Recently, Gallardo-Gutiérrez and Read [**69**] have constructed a quasinilpotent operator $T : \ell^1(\mathbb{Z}_+) \to \ell^1(\mathbb{Z}_+)$ such that $f(T)$ has no non-trivial invariant subspaces for any non-constant analytic germ f around the origin, solving in the positive a conjecture posed by Read in 1986 (see, [**108**, Conjecture 7.1 (a)]).

Nonetheless, it is worthwhile underscoring that all existing counterexamples to the Invariant Subspaces Problem are developed, so far, in non-reflexive Banach spaces. In the light of our previous digression, the Invariant Subspaces Problem can be restated as follows:

Invariant Subspace Problem. *Does every linear bounded operator $T : X \to X$ acting on a separable infinite-dimensional reflexive complex Banach space X have a non-trivial invariant subspace?*

In particular, the Invariant Subspace Problem still remains unsolved for separable infinite-dimensional Hilbert spaces and, despite many efforts, adapting Enflo's or Read's techniques in this context fails drastically. By contrast, an approach in the positive to the Invariant Subspace Problem seems especially challenging, as we need to develop tools which apply to all Hilbert space operators simultaneously. To ease this constraint, Rota introduced in 1960 the concept of *universal operators* [**111**], which model each Hilbert space operator through their invariant subspaces. More precisely, an operator $U : H \to H$ is said to be *universal* if for every non-zero $T : H \to H$, there is a U-invariant subspace $\mathcal{M} \subseteq H$ and a non-zero $\lambda \in \mathbb{C}$ for which

$$U|\mathcal{M} \sim \lambda T.$$

Accordingly, the Invariant Subspace Problem for Hilbert spaces reduces to the apparently simpler question of showing whether all minimal invariant subspaces of a single universal operator are one-dimensional.

The original example provided by Rota [**111**] of a universal operator was the adjoint of a shift operator of infinite multiplicity, which can be regarded as a left shift operator on $L^2(0, \infty)$ by fixing a number $\beta > 0$:

$$S_\beta^* f(t) := f(t + \beta) \quad \text{for } t > 0.$$

Universal operators have attracted the attention of operator theorists for decades and additional examples have been constructed, mostly thank to a remarkable result due to Caradus [**34**] which establishes sufficient conditions for an operator to be universal. For instance, as proved by Nordgren, Rosenthal and Wintrobe [**101**], given any composition operator C_φ on the Hardy space $H^2(\mathbb{D})$, induced by a *hyperbolic automorphism* φ of the unit disk $\mathbb{D}$, i.e.

$$\begin{aligned} C_\varphi : H^2(\mathbb{D}) &\to H^2(\mathbb{D}) \\ f &\mapsto f \circ \varphi \end{aligned} \qquad \text{with} \qquad \varphi(z) = \frac{z + r}{1 + rz} \quad (0 < r < 1),$$

the operator $C_\varphi - \lambda I$ is universal for each λ in the interior of $\sigma(C_\varphi)$.

On the other hand, concrete classes of Hilbert/Banach space operators for which some powerful tools are available have received special attention over the years.

In this context, as aforementioned, one of the earliest invariant subspace theorems is the result of von Neumann for compact operators in Hilbert spaces (unpublished), extended in 1954 by Aronszajn and Smith [**12**] to the Banach space setting. Later, in 1966, Bernstein and Robinson [**22**] and Halmos [**76**] proved analogous results for polynomially compact operators.

However, in 1973, operator theorists were stunned by the generalization obtained by Lomonosov [**90**], who proved one of the most general positive results in this line, namely: *any linear bounded operator $T : X \to X$ on a Banach space X commuting with a non-zero compact operator has a non-trivial invariant subspace.* Indeed, such a T possesses a non-trivial *hyperinvariant subspace*, i.e. a closed subspace which is invariant under each operator in the commutant of T. Thereby, any bounded linear operator $T : X \to X$ which commutes with a non-scalar operator commuting with a non-zero compact operator, must admit a non-trivial invariant subspace.

But, it was not until 1980, that Hadwin, Nordgren, Radjavi and Rosenthal [**75**] proved the existence of a Hilbert space operator having non-trivial invariant subspaces to which Lomonosov's result does not apply.

Based on the work of Aronszajn and Smith [**12**], Halmos [**77**] introduced in the late 1960's the concept of *quasitriangular operators*. Recall that an operator $Q : H \to H$ acting on a separable infinite-dimensional Hilbert space H is said to be *quasitriangular* whenever there exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite-rank orthogonal projections converging strongly to the identity I and such that

$$\big\|QP_n - P_nQP_n\big\| \to 0, \quad \text{as } n \to \infty.$$

In some sense, quasitriangular operators were conceived as an attempt to transfer some of the most important features of *triangular operators* into a more general context. In this light, it is completely apparent that given a triangular operator $T : H \to H$, i.e. a linear bounded operator which admits a representation as an upper-triangular matrix with respect to a suitable orthonormal basis, there exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite-rank projections converging strongly to I and satisfying

$$TP_n - P_nTP_n = (I - P_n)TP_n = 0 \quad \text{for each } n = 1, 2, \dots.$$

Hence, roughly speaking, the definition of quasitriangularity means that Q has a sequence of "approximately invariant" finite-dimensional subspaces.

Various important classes of operators have been shown to be quasitriangular, for instance: compact operators, operators with finite spectrum, compact perturbations of normal operators or decomposable operators. On the contrary, the unilateral shift operator in $\ell^2(\mathbb{Z}_+)$ turns out to be non-quasitriangular, although its adjoint is.

Although it might seem that the Invariant Subspace Problem is more manageable for the class of quasitriangular operators, quite surprisingly, Apostol, Foiaş and Voiculescu [**10**] and, independently, Douglas and Pearcy [**46**] demonstrated precisely the opposite. In other words, the Invariant Subspace Problem for Hilbert spaces is reduced to be proved exclusively for quasitriangular operators (see Herrero's book [**79**] for more on the subject).

Another major landmark in the subject which ought to be mentioned is the advent of modern Spectral Theory. The starting point is the aforementioned *Spectral Theorem*, which characterizes all the spectral properties of normal operators on Hilbert spaces by means of a projection-valued measure supported on the spectrum. In particular, the Spectral Theorem has several far-reaching consequences: the definition of an L^∞ functional calculus, a description of the commutant of normal operators, a characterization of invariant and hyperinvariant subspaces and so on.

In this spirit, since the 1950's, a number of authors have endeavoured to develop a valid spectral theory for arbitrary Banach space operators which resembles some of the essential features of normal operators. Maybe, the earliest steps for the development of abstract Spectral Theory were initiated by Dunford [**47, 48**], who introduced the concept of spectral operators on Banach spaces in 1954. His idea relies again on the notion of spectral measure: let $\mathcal{G}$

denote the σ-algebra of Borel subsets of $\mathbb{C}$, then any mapping $E : \mathcal{G} \to \mathcal{B}(X)$ is called a *spectral measure* if:

(i) $E(\emptyset) = 0$ and $E(\mathbb{C}) = I$,

(ii) $E(F \cap G) = E(F)\, E(G)$ for each $F, G \in \mathcal{G}$,

(iii) For every countable family $(G_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel sets,

$$E\Big(\bigcup_{n\in\mathbb{N}} G_n\Big)x = \sum_{n=1}^{\infty} E(G_n)x \quad \text{for each } x \in X.$$

Recall that an operator $T \in \mathcal{B}(X)$ is said to be a *spectral operator* whenever there exists a *spectral measure* E such that

$$E(G)\, T = T\, E(G) \quad \text{and} \quad \sigma\big(T|E(G)X\big) \subseteq \overline{G} \quad \text{for each } G \in \mathcal{G}.$$

However, such severe conditions which define spectral operators narrow down considerably its scope of applicability. In fact, it may be seen that spectral operators $T \in \mathcal{B}(X)$ are precisely those admitting a sum representation $T = S + Q$, where $Q \in \mathcal{B}(X)$ is a *quasinilpotent* operator commmuting with T and $S \in \mathcal{B}(X)$ captures the spectral data of T via a spectral measure $E : \mathcal{G} \to \mathcal{B}(X)$ and the identity

$$S = \int_{\sigma(T)} \lambda\, dE(\lambda).$$

In this context, aiming for embracing additional operators still holding a profitable spectral behaviour, Foiaş [**61**] introduced the class of *generalized scalar operators* in 1960 by considering a non-analytic extension of the *Riesz functional calculus* to the Fréchet algebra $C^\infty(\mathbb{C})$ of infinitely differentiable functions. His ideas opened up a line of research, mainly focused on the application of extensions of the holomorphic functional calculus to suitable algebras $\mathcal{A}$ of functions which were large enough to hoard partitions of unity. Accordingly, an operator $T \in \mathcal{B}(X)$ is called *$\mathcal{A}$-scalar* whenever it admits an algebraic homomorphism

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(X)$$

for which $\Phi(1) = I$ and $\Phi(Z) = T$, where Z denote the identity function $z \mapsto z$. As a consequence of the fact that two non-zero functions may have pointwise zero product, this sort of constructions shall lead, in a natural way, to spectral decompositions (and, then, to invariant subspaces) for the given operator.

In what this approach refers, *Beurling algebras* have played a prominent role in the theory. Recall that the Beurling algebra $\mathcal{A}_\rho$ is defined as the Banach algebra of continuous functions on the torus $f : \mathbb{T} \to \mathbb{C}$ governed by the condition

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\, \rho_n < \infty,$$

for a suitable submultiplicative weight $\rho := (\rho_n)_{n\in\mathbb{Z}}$ such that $\rho_n^{1/n} \to 1$ as $|n| \to \infty$. A striking result due to Beurling [**24**] ensures the existence of partitions of unity in $\mathcal{A}_\rho$ whenever

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} < \infty. \tag{0.2}$$

Condition (0.2) is often referred to as *Beurling condition* and those weights $\rho := (\rho_n)_{n\in\mathbb{Z}}$ enjoying it are called *Beurling sequences*.

In 1952, just in the case of Beurling algebras, Wermer [**126**] foresaw the practicality of non-holomorphic functional calculus in order to provide invariant subspaces, anticipating part of future work done by Foiaş [**61, 62**] and Colojoară and Foiaş [**38**] a few years later:

Theorem (Wermer, [126]). *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X with $\sigma(T) \subseteq \mathbb{T}$ satisfying*

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty. \tag{0.3}$$

Then, if $\sigma(T)$ is not reduced to a singleton, the operator T has a non-trivial hyperinvariant subspace in X.

Unfortunately, as hinted by the series condition (0.3) in *Wermer's Theorem*, non-analytic functional calculus is often tightened up to rather restrictive and artificial growth constraints.

Hence, inspired by the pioneering work of Bishop [**26**], Foiaş introduced the class of decomposable operators [**62**] with the purpose of placing the spotlight exclusively on a general spectral decomposition property, discarding any specific functional calculus device. Recall that an operator $T \in \mathcal{B}(X)$ is called *decomposable*, if every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$ splits both the spectrum $\sigma(T)$ and the space X, in the sense that there exist closed T-invariant subspaces $\mathcal{M}_1, \dots, \mathcal{M}_n \subseteq X$ for which

$$\sigma(T|\mathcal{M}_j) \subseteq U_j \text{ for each } j = 1, \dots, n \quad \text{and} \quad X = \mathcal{M}_1 + \ldots + \mathcal{M}_n.$$

Somehow unexpectedly, Complex Analysis emerged as the crucial tool within this theory and decomposability turned out to be a cornerstone in Spectral Theory. In a nutshell, the reason why Complex Analysis plays such an important role in this context may be glimpsed in the subsequent construction: given an operator $T \in \mathcal{B}(X)$ on a Banach space X, for each open subset $U \subseteq \mathbb{C}$ we may define the bounded linear operator

$$\begin{aligned} T_U : \mathcal{O}(U, X) &\longrightarrow \mathcal{O}(U, X) \\ f &\longmapsto z \mapsto (T - z)f(z) \end{aligned}$$

acting on the Fréchet space $\mathcal{O}(U, X)$ of X-valued analytic functions on U. On the other hand, for each closed $F \subseteq \mathbb{C}$, we may consider

$$\begin{aligned} T^F : X &\longrightarrow \mathcal{O}(F, X)\big/ T_F\, \mathcal{O}(F, X) \\ x &\longmapsto 1 \otimes x + T_F\, \mathcal{O}(F, X). \end{aligned} \tag{0.4}$$

The great breakthrough of Bishop [**26**] was to point out that some natural spectral decompositions for $T \in \mathcal{B}(X)$ could be understood in terms of certain features regarding T_U and T^F. Moreover, he showed that a sort of duality arose between both operators T_U and T^F.

A few decades later, Albrecht and Eschmeier [**6**] confirmed all Bishop's intuitions by proving that operators $T \in \mathcal{B}(X)$ having all T^F surjective characterize, up to similarity, the quotients of decomposable operators; whilst operators $T \in \mathcal{B}(X)$ having all T_U injective with closed range characterize, up to similarity, the restrictions of decomposable operators. Furthermore, these properties, called *property* (δ) and *property* (β) respectively, turn out to be duals of each other, in the sense that an operator $T \in \mathcal{B}(X)$ has one of the properties (β) or (δ) precisely when $T^* \in \mathcal{B}(X^*)$ has the other one.

Of course, in view of many known counterexamples, we cannot expect a full success of a general spectral theory in order to address the Invariant Subspace Problem. However, most of the existing techniques for producing invariant subspaces require the operator to have a nice spectral behaviour. Hence, although in somewhat weaker terms, the tools supplied by the interplay between Complex Analysis and Spectral Theory shall be of particular relevance to understand some of the features regarding our main subject in this thesis: Bishop operators and, a bit more concretely, their invariant subspaces.

Bishop operators and weighted translation operators

According to Davie [**44**], Errett Bishop suggested one of the simplest candidates in the search for a Banach space operator with no invariant subspaces. Oddly enough, nearly half a century later, some Bishop operators still linger as possible counterexamples to the Invariant Subspace Problem.

Given an irrational number $\alpha \in (0,1)$, the *Bishop operator* T_α is defined on the complex Banach space $L^p[0,1)$ for each $1 \leq p \leq \infty$ by the assignment

$$T_\alpha f(t) := t\, f(\{t+\alpha\}), \qquad t \in [0,1),$$

where the curly brackets $\{t\} := t - \lfloor t \rfloor$ denote the *fractional part* of a real number.

Clearly, every Bishop operator is the product of two simple and well-understood operators on $L^p[0,1)$. Namely, the multiplication operator M_t by the independent variable:

$$M_t f(t) := t\, f(t), \qquad t \in [0,1),$$

which has plenty of invariant subspaces, for instance, all those of the form

$$\mathcal{M}_E := \big\{ f \in L^p[0,1) \,:\, \operatorname{supp}(f) \subseteq E \big\} \tag{0.5}$$

for each Lebesgue-measurable subset $E \subseteq [0,1)$. On the other hand, the composition operator C_{τ_α} induced by the symbol $\tau_\alpha(t) := \{t+\alpha\}$:

$$C_{\tau_\alpha} f(t) := f(\{t+\alpha\}), \qquad t \in [0,1),$$

which possesses a large set of *eigenfunctions* $e_n(t) := e^{2\pi i n t}$ for every $n \in \mathbb{Z}$, each one corresponding to the eigenvalue $e^{2\pi i n \alpha}$.

Yet, the structure of Bishop operators T_α remains largely unknown. In particular, as mentioned above, despite of partial advances over the years, the existence of invariant subspaces for all Bishop operators is still an open problem:

Problem (Bishop operators). *Does the operator T_α have non-trivial invariant subspaces in each $L^p[0,1)$ space $(1 \leq p < \infty)$ for every irrational $\alpha \in (0,1)$?*

Bishop operators are concrete examples of the so-called weighted translation operators, defined over more general measurable settings in a similar manner. If $(\Omega, \mathcal{G}, \mu)$ is a non-atomic measure space, $\phi \in L^\infty(\Omega,\mu)$ and $\tau : (\Omega, \mathcal{G}, \mu) \to (\Omega, \mathcal{G}, \mu)$ is a probability space isomorphism; then, the *weighted translation operator* $W_{\phi,\tau}$ is determined by the equation

$$W_{\phi,\tau} f := \phi \cdot (f \circ \tau)$$

on each Banach space $L^p(\Omega,\mu)$ for $1 \leq p \leq \infty$.

Again, weighted translations $W_{\phi,\tau}$ are the composition of two nicely behaved operators: firstly, the multiplication operator $M_\phi \in \mathcal{B}(L^p(\Omega,\mu))$, known as the "weight", and secondly, the composition operator $C_\tau \in \mathcal{B}(L^p(\Omega,\mu))$, referred to as the "translation". Separately, both operators M_ϕ and C_τ own a rich lattice of hyperinvariant subspaces: for instance, the weight M_ϕ has hyperinvariant subspaces of the form $\mathcal{M}_E$ similar to those indicated in (0.5) for each E belonging to the σ-algebra $\mathcal{G}$; while, the translation C_τ always turns out to be a decomposable operator with spectrum $\sigma(C_\tau) \subseteq \mathbb{T}$.

The class consisting of all weighted translation operators is a vast one, including numerous examples such as bilateral weighted shifts and, thereby, roughly speaking, model operators. Due to the large scale of this family, a general study of weighted translation operators is not an easy task and very little is known regarding certain features such as, for instance, the existence of invariant subspaces.

Problem (Weighted translation operators). *Let $(\Omega, \mathcal{G}, \mu)$ be a non-atomic measure space, $\phi \in L^\infty(\Omega,\mu)$ and $\tau : \Omega \to \Omega$ a probability space isomorphism. Does the weighted translation operator $W_{\phi,\tau}$ have non-trivial invariant subspaces in each $L^p(\Omega,\mu)$ space?*

Following this general approach, weighted translation operators were firstly studied by Parrott [**103**] in his PhD dissertation in 1965: analysing the spectrum, numerical range and reducing subspaces of such operators. Indeed, Parrott computed the spectrum of all Bishop operators showing, in particular, that it is the disk

$$\sigma(T_\alpha) = \left\{\lambda \in \mathbb{C} \,:\, |\lambda| \le e^{-1}\right\}$$

independently of the irrational number $\alpha \in (0,1)$. In 1973, Bastian [**17**] gave unitary invariants for some weighted translation operators and studied properties such as subnormality and hyponormality among them. Later on, Petersen [**104**] showed some results on the commutant of weighted translation operators in an attempt to get a deeper insight in the general context.

As far as Bishop operators concern, one of the most striking results was proved by Davie [**44**] in 1974, who, by means of a functional calculus approach mostly inspired in Wermer's Theorem, was able to show the existence of non-trivial hyperinvariant subspaces for T_α in each $L^p[0,1)$ whenever $\alpha \in (0,1)$ is a *non-Liouville number*. Recall that an irrational α is a *Liouville number* if for every $n \in \mathbb{N}$ there exists an irreducible rational number p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n};$$

so, roughly speaking, Liouville numbers are those irrationals which can be approximated very rapidly by rational numbers. Recall that all Liouville numbers are transcendental. Moreover, *Jarník-Besicovitch Theorem* (see, for instance, [**33**, Section 5.5]) asserts that Liouville numbers form a set of vanishing Hausdorff dimension and, therefore, of zero Lebesgue measure. Furthermore, Davie also proved that T_α always lacks of point spectrum (see [**36**] for related results in this context).

Afterwards, early in the nineties, extensions strengthening Davie's theorem were due to Blecher and Davie [**27**] and MacDonald [**91**] for some *Bishop-type operators*, i.e. weighted translation operators in $L^p[0,1)$ having $\tau = \tau_\alpha$ for some irrational $\alpha \in (0,1)$:

$$W_{\phi,\alpha} f(t) := \phi(t)\, f(\{t+\alpha\}), \qquad t \in [0,1).$$

However, the brick wall still consisted of Liouville irrationals and, despite of the efforts, the interesting extensions included many weights ϕ but neither Liouville number.

Once again, their approaches relied on the functional calculus machinery supplied by Beurling algebras, which had been refined by Atzmon [**15**] in 1984 in order to deal with non-invertible operators. The big enhancement in *Atzmon's Theorem* [**15**, Thm. 1.1] with regard to Wermer's Theorem is that, given an injective dense-range linear operator $T \in \mathcal{B}(X)$, the series condition (0.3) may be replaced by its local counterpart. In other words, in order to ensure existence of invariant subspaces for the operator T, it is enough to check that there exist two non-zero vectors $x \in X$ and $y \in X^*$ such that

$$\|T^n x\|_X \leq C\rho_n \quad \text{and} \quad \|T^{*n} y\|_{X^*} \leq C\rho_n \qquad (n \in \mathbb{Z}),$$

for an absolute constant $C > 0$ and some Beurling sequence $(\rho_n)_{n\in\mathbb{Z}}$ (apart from another technical condition which will be automatically fulfilled in our case).

Shortly thereafter, MacDonald [**92**] was the first in breaking the barrier of Liouville numbers. After a meticulous estimation of the norms $\|W^n_{\phi,\alpha}\|$ for each $n \in \mathbb{Z}$, the threshold of his result (see [**92**, Thms. 3.5 and 3.6]) guarantees the decomposability of invertible Bishop-type operators $W_{\phi,\alpha}$, whenever the *convergents* $(a_n/q_n)_{n\geq 0}$ of the irrational $\alpha \in (0,1)$ in its *continued fraction* representation satisfy the condition

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty$$

and $\log|\phi|$ is of *bounded variation*, evincing an unexpected connection with *Brjuno condition*. As a matter of fact, from MacDonald's work, one can deduce that the smoother the weight ϕ is, the more manageable the iterates $W^n_{\phi,\alpha}$ are.

Nonetheless, in view of many known results, such strategies relying on functional calculus with Beurling algebras began to reveal signs of exhaustion. Indeed, as claimed in MacDonald's article *"to obtain invariant subspaces for the remaining exceptional cases, it appears that a new approach will be needed"*.

Later on, proceeding again with Atzmon's Theorem, Flattot [**60**] was able to enlarge substantially the class of irrational numbers $\alpha \in (0,1)$ for which the corresponding Bishop operator T_α has non-trivial hyperinvariant subspaces in each $L^p[0,1)$, embracing some Liouville numbers:

Theorem (Flattot, [60]). *Let $\alpha \in (0,1)$ be an irrational number whose convergents $(a_n/q_n)_{n\geq 0}$ of its continued fraction satisfy the asymptotic condition*

$$\log(q_{n+1}) = O\big(q_n^{1/2-\varepsilon}\big) \quad \text{for some } \varepsilon > 0 \text{ as } n \to \infty. \tag{0.6}$$

Then, the Bishop operator T_α has non-trivial hyperinvariant subspaces in $L^p[0,1)$ for each $1 \leq p < \infty$.

For instance, the classical *Liouville constant* fulfils the hypotheses of Flattot's result:

$$\ell := \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000\ldots$$

However, the question still remained open for either Bishop operators or Bishop-type operators over a negligible set of highly transcendental numbers $\alpha \in (0,1)$.

Finally, the latest significant achievement was recently attained by Ambrozie [**7**]. Choosing extremely smooth weights ϕ given by exponentials of a trigonometric polynomial

$$\phi(t) = \exp\left(\sum_{j=-n}^{n} c_j e^{2\pi i j t}\right), \qquad t \in [0,1) \text{ and } n \in \mathbb{N};$$

he found non-trivial invariant subspaces for the Bishop-type operator $W_{\phi,\alpha}$ for each irrational number $\alpha \in (0,1)$ (including, of course, all Liouville numbers).

Our major objective in this PhD thesis is to expose the most recent and furthest advances regarding the existence of invariant subspaces for, overall, Bishop operators; although, at some points, similar applications will be discussed for Bishop-type operators. To accomplish our purposes, we shall see how different techniques, essentially borrowed from Operator Theory, Analytic Number Theory and Local Spectral Theory, may be linked altogether in order to produce, when it succeeds, invariant subspaces for such a "simple" (as well as surprising) family of linear operators.

A general outline of the thesis

This PhD thesis is organized as follows. Chapter 1 is exclusively focused on introducing some required preliminaries for a convenient treatment of the subject. More specifically, we will provide a streamlined exposition on a wide variety of topics such as Local Spectral Theory, Banach algebras in the spirit of Gelfand Theory or continued fractions in the context of Diophantine Approximation. At the end of Chapter 1, we shall give a detailed description of previous results concerning weighted translation operators with an emphasis on their spectral properties.

Along Chapter 2, using Atzmon's Theorem, we shall considerably enlarge the set of irrationals $\alpha \in (0,1)$ such that T_α has non-trivial hyperinvariant subspaces (extending the previous results due to Davie [**44**] and Flattot [**60**]). Our procedure will consist in the development of sharper arithmetical estimations which will enable us to strengthen the analysis of certain functions associated to the functional model.

In particular, using the language of continued fractions, the limit of our approach will lead us to invariant subspaces for T_α up to the following asymptotic condition:

Theorem 1. *Let $\alpha \in (0,1)$ be any irrational and $(a_n/q_n)_{n\geq 0}$ the convergents of its continued fraction. If the following condition holds:*

$$\log q_{n+1} = O\left(\frac{q_n}{(\log q_n)^3}\right) \quad \text{as } n \to \infty. \tag{0.7}$$

Then, the Bishop operator T_α has non-trivial hyperinvariant subspaces in $L^p[0,1)$ for each $1 \leq p < \infty$.

Observe that our asymptotic requirement (0.7) relaxes significantly the restriction imposed on α by Flattot (0.6), allowing the exponent 1 instead of 1/2 and quantifying the role of ε.

Indeed, a quantitative comparison between those cases covered by Davie, Flattot and us shall be discussed in terms of the Hausdorff dimension drawn by the family of functions $(|\log t|^{-s})_{s\geq 0}$ (instead of the usual one $(t^s)_{s\geq 0}$). With such a dimension, one concludes that the set of exceptions in Davie's, Flattot's and our case have dimensions ∞, 4 and 2 respectively.

Chapter 3 is entirely devoted to demonstrating that our approach to construct invariant subspaces for T_α yields essentially the best result attainable using the standard techniques. In this sense, we shall establish an effective upper limit for the growth of the denominators of the convergents of α for the application of Atzmon's Theorem:

Theorem 2. *Let $\alpha \in (0,1)$ be an irrational number not belonging to the set*

$$\mathcal{E} := \left\{ \xi \in (0,1) \,:\, \log q_{n+1} = O\left(\frac{q_n}{\log q_n} \right) \ \text{ as } n \to \infty \right\},$$

and consider $\widetilde{T}_\alpha := e\, T_\alpha$ acting on $L^p[0,1)$ for some fixed $1 \leq p < \infty$. Then, for every non-zero $f \in L^p[0,1)$, we have

$$\sum_{n \in \mathbb{Z}} \left(\frac{\log\left(1 + \|\widetilde{T}_\alpha^n f\|_p\right)}{1+n^2} \right) = +\infty.$$

Naturally, this corroborates the need of a different perspective to look for invariant subspaces for T_α based on different tools: that will be our leitmotiv in the remaining chapters.

To do so, note that, whilst all Bishop operators share the same spectrum, not all of them are known to own invariant subspaces. In a sense, this dichotomy suggests that a deeper insight in $\sigma(T_\alpha)$ might contribute to unveil certain common features of all Bishop operators and, perhaps, lead to a conclusive characterization of the invariant subspaces of T_α. Such an exploration within $\sigma(T_\alpha)$ shall be carried out via local spectral manifolds. Recall that given an operator $T \in \mathcal{B}(X)$ on a complex Banach space X and a subset $F \subseteq \mathbb{C}$, the *local spectral manifold* $X_T(F)$ is defined as

$$X_T(F) := \left\{ x \in X \,:\, \sigma_T(x) \subseteq F \right\},$$

where, as usual, $\sigma_T(x)$ denotes the *local spectrum of T at $x \in X$*. Broadly speaking, these notions, which play a central role in Local Spectral Theory, will allow us to gain a further knowledge on what constitutes each part of the spectrum $\sigma(T_\alpha)$.

In Chapter 4, we characterize those local spectral properties fulfilled simultaneously by all Bishop operators T_α, independently of the irrational number $\alpha \in (0,1)$. This characterization will be reached basically in two different steps.

Firstly, we will generalize a theorem due to Parrott [**103**, Thm. 2.10], by bounding uniformly from below the local spectral radii of ergodic weighted translation operators:

Theorem 3. *Given an ergodic system $(\Omega, \mathcal{G}, \mu, \tau)$ and $\phi \in L^\infty(\Omega, \mu)$. Then,*

$$\liminf_{n \to +\infty} \|W_{\phi,\tau}^n f\|_{L^p(\Omega,\mu)}^{1/n} \geq \exp\left(\int_\Omega \log|\phi| \, d\mu \right)$$

for each non-zero $f \in L^p(\Omega, \mu)$.

Thereafter, as a by-product of complementary results due to MacDonald [**91**, Prop. 1.3 and 1.4], Theorem 3 will enable us to prove that a wide class of Bishop-type operators are *power-regular* (in the sense of Atzmon [**16**]). For the case of Bishop operators, we will obtain the next result:

Theorem 4. *Let $\alpha \in (0,1)$ be any irrational and consider T_α acting on $L^p[0,1)$ for fixed $1 \le p < \infty$. Then, the spectral radius of the restriction of T_α to any non-zero invariant subspace $\mathcal{M}$ is*

$$r(T_\alpha|\mathcal{M}) = e^{-1}.$$

More indeed, the same holds for T_α^*.

Seen in terms of local spectral manifolds, Theorem 4 can be rephrased saying that

$$X_{T_\alpha}(F) = \{0\} \quad \text{for each } F \subseteq \operatorname{int}(\sigma(T_\alpha)).$$

Then, upon applying a duality argument in the spirit of Albrecht and Eschmeier [**6**], we will be able to discard decomposability, property (β) and property (δ) for all Bishop operators T_α (and, by the same token, for many Bishop-type operators $W_{\phi,\alpha}$):

Theorem 5. *Let $\alpha \in (0,1)$ be any irrational number. Then, the Bishop operator T_α on $L^p[0,1)$ for $1 \le p \le \infty$ is not decomposable. Moreover, T_α has neither property (β) nor property (δ).*

Secondly, a reinterpretation of certain asymptotic bounds previously obtained in Chapter 2 will help us to establish that Bishop operators T_α can neither satisfy Dunford's property. Recall that those operators $T \in \mathcal{B}(X)$ having norm-closed local spectral manifolds are said to enjoy *Dunford's property* (also known as *property* (C)):

Theorem 6. *Let $\alpha \in (0,1)$ be any irrational number. Then, the local spectral manifold*

$$X_{T_\alpha}(\partial\sigma(T_\alpha)) = \left\{ f \in L^p[0,1) \,:\, \sigma_{T_\alpha}(f) \subseteq \partial D(0, e^{-1}) \right\}$$

is norm-dense in $L^p[0,1)$ for each $1 \le p \le \infty$. In particular, T_α does not satisfy Dunford's property (C).

In view of our results, we may appreciate that the relevance of each part of the spectrum $\sigma(T_\alpha)$ differs significantly: clearly, it is apparent that the boundary $\partial\sigma(T_\alpha)$ stores much more information about T_α than the interior $\operatorname{int}(\sigma(T_\alpha))$.

In this vein, throughout Chapter 5, the local spectral analysis of $\sigma(T_\alpha)$ will be pushed much further in order to produce spectral subspaces for all those Bishop operators T_α that, up to now, are known to have non-trivial hyperinvariant subspaces. Our proof, which is inspired in Colojoară and Foiaş' restatement of Wermer's Theorem (see [**38**, Ch. V, Thm. 3.2]), uses partitions of unity to show how $\partial\sigma(T_\alpha)$ can be split into distinct pieces to obtain non-trivial local spectral manifolds:

Theorem 7. *Let T_α be a Bishop operator acting on $L^p[0,1)$ for fixed $1 \le p < \infty$, such that the irrational $\alpha \in (0,1)$ satisfies the condition:*

$$\log q_{n+1} = O\left(\frac{q_n}{(\log q_n)^3} \right) \quad \text{as } n \to \infty.$$

Then, given any open subset $U \subseteq \mathbb{C}$ such that $U \cap \partial\sigma(T_\alpha) \neq \emptyset$ and $\partial\sigma(T_\alpha) \setminus \overline{U} \neq \emptyset$, we have

$$\{0\} \neq \overline{X_{T_\alpha}(U)} \neq L^p[0,1).$$

In particular, T_α has non-trivial hyperinvariant subspaces.

Recall that, according to aforementioned results, the lack of any profitable local spectral property for T_α appeared to be an evident fact. However, in light of Theorem 7, this happens to be just a kind of illusion (at least in some cases) caused by a misguided choice of the open coverings of $\sigma(T_\alpha)$. Indeed, as stated below, we may claim, in very accurate terms, that some Bishop operators are "decomposable" (or even, $\mathcal{A}_\rho$-scalar) with respect to a dense linear submanifold:

Theorem 8. *Let T_α be a Bishop operator acting on $L^p[0,1)$ for fixed $1 \leq p \leq \infty$, such that the irrational $\alpha \in (0,1)$ enjoys the condition:*

$$\log q_{n+1} = O\left(\frac{q_n}{(\log q_n)^3}\right) \quad \text{as } n \to \infty.$$

Then, there exists a dense linear submanifold $\mathcal{D}_{\alpha,p}$ in $L^p[0,1)$ for which

$$\mathcal{D}_{\alpha,p} \subseteq X_{T_\alpha}(U_1) + \ldots + X_{T_\alpha}(U_n)$$

for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$.

Of course, our latter theorem leaves open an intriguing question: does the failure of our approach beyond the condition

$$\log q_{n+1} = O\left(\frac{q_n}{\log q_n}\right) \quad \text{as } n \to \infty$$

mean exactly that T_α loses its good spectral behaviour? Or, on the contrary, might all Bishop operators obey similar local spectral decomposition properties?

To support our question in the positive, we recall that, as settled by Albrecht [**3**], there exist decomposable operators $T \in \mathcal{B}(X)$ whose spectral behaviour cannot be described in terms of a functional calculus from a suitable algebra. Anyway, we suggest that one possible way to solve the invariant subspace problem for all Bishop operators T_α could be understanding in depth their local spectral manifolds $X_{T_\alpha}(U)$ constructed from open sets U intersecting the boundary of $\sigma(T_\alpha)$. We pose it as a conjecture:

Conjecture. *Let $\alpha \in (0,1)$ be any irrational number and $\varepsilon > 0$ sufficiently small. Then, the local spectral manifold*

$$X_{T_\alpha}\big(D(e^{-1}, \varepsilon)\big)$$

is non-trivial and non-dense in $L^p[0,1)$ for every $1 \leq p < \infty$.

Scientific articles on which this PhD thesis is based

All the original scientific content of this PhD dissertation comes from the following three articles:

- [**37**] F. Chamizo, E. A. Gallardo-Gutiérrez, M. Monsalve-López, and A. Ubis, "Invariant subspaces for Bishop operators and beyond," *Advances in Mathematics*, vol. 375, no. 2, 2020
- [**66**] E. A. Gallardo-Gutiérrez and M. Monsalve-López, "Power-regular Bishop operators and spectral decompositions," *J. Operator Theory (in press)*
- [**67**] E. A. Gallardo-Gutiérrez and M. Monsalve-López, "Spectral decompositions arising from Atzmon's hyperinvariant subspace theorem," *(submitted)*

Additionally, part of the speech and organization of the body text can be found within the survey:

- [**68**] E. A. Gallardo-Gutiérrez and M. Monsalve-López, “A closer look at Bishop operators,” *Operator Theory, Functional Analysis and Applications (in press)*, 2020

Chapter 1
Preliminaries

Due to the diversity of topics covered throughout this thesis, our preliminary chapter serves as a kind of toolkit devoted to introducing all the necessary tools and results which will be used along the monograph. At the beginning, with the additional goal of setting some standard notation, we collect well-known facts regarding Functional Analysis and Operator Theory. Subsequently, we may find an overview regarding distinct subjects such as Spectral Theory, Banach algebras or Diophantine Approximation; this will enable us to treat the problems of this PhD thesis from different many perspectives. At the end of the chapter, we will discuss various features concerning weighted translation operators.

In the sequel, $\mathbb{Z}_+$ will denote the set of non-negative integers, while $\mathbb{N}$ will represent the subset $\{1, 2, \ldots\}$ of $\mathbb{Z}$. On the other hand, $\mathbb{T} := \big\{z \in \mathbb{C} : |z| = 1\big\}$ will stand for the unit circle of the complex plane and $\mathbb{D} := \big\{z \in \mathbb{C} : |z| < 1\big\}$ the open unit disk.

Asymptotic behaviours will be expressed in terms of *Vinogradov's notation* and *big O notation*:

- For $f, g : \mathbb{Z} \to \mathbb{C}$, the symbol $f(n) \ll g(n)$ shall mean $|f(n)| \leq C|g(n)|$ for every $n \in \mathbb{Z}$ and some absolute constant $C > 0$.
- For $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{R}_+$ increasing, $f(n) = O(g(n))$ will denote that

$$\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < +\infty.$$

Whenever these asymptotic bounds may depend on extra parameters, they will be indicated as subscripts (for instance, $\ll_\varepsilon$ or O_ε).

Unless otherwise stated, X will usually denote an arbitrary infinite-dimensional complex Banach space, whilst H will stand for any infinite-dimensional complex Hilbert space. In general, we shall often work on *separable* spaces, although we will not assume this a priori.

Let $\mathcal{B}(X)$ be the Banach algebra of *bounded linear operators* on X, equipped with the sup-*norm*

$$\|T\| := \sup\big\{\|Tx\|_X : x \in X \text{ with } \|x\| = 1\big\} \qquad \text{for all } T \in \mathcal{B}(X).$$

At some point, we shall also deal with the class $\mathcal{L}(X)$ of *unbounded operators* on X, i.e. the set of linear mappings $T : \mathrm{Dom}(T) \to X$ where $\mathrm{Dom}(T)$, the *domain* of the operator T, is a linear submanifold of X. By definition, the inclusion of sets $\mathcal{B}(X) \subseteq \mathcal{L}(X)$ holds in every Banach space X.

As usual, for every operator $T \in \mathcal{L}(X)$ defined on an appropriate domain $\mathrm{Dom}(T) \subseteq X$, the sets

$$\ker(T) := \big\{x \in \mathrm{Dom}(T) \,:\, Tx = 0\big\} \quad \text{and} \quad \mathrm{ran}(T) := \big\{Tx \,:\, x \in \mathrm{Dom}(T)\big\}$$

denote the *kernel* and the *range* of T respectively. Of course, whenever $T \in \mathcal{B}(X)$, the domain will be the whole space $\mathrm{Dom}(T) = X$ and the range will be often denoted by $T(X)$. We recall that a linear operator $T \in \mathcal{L}(X)$ is said to be *densely defined* if its domain $\mathrm{Dom}(T)$ is a dense linear manifold of X. Analogously, all previous definitions are also applicable in the Hilbert space setting.

The *topological dual space* of a Banach space X, denoted by X^*, is the space of continuous *functionals* on X, i.e. the set of all bounded linear mappings $\varphi : X \to \mathbb{C}$ equipped with the usual operations of scalar multiplication and addition. Endowed with the canonical norm

$$\|\varphi\| := \sup\big\{|\varphi(x)| \,:\, x \in X \text{ with } \|x\| = 1\big\} \quad \text{for each } \varphi \in X^*,$$

the dual space X^* becomes a complex Banach space. As a consequence of the Hahn-Banach theorem, every Banach space X can be injectively embedded into its *bidual space* X^{**} via the norm-preserving assignment $x \mapsto x^{**}$ where

$$x^{**}(\varphi) := \varphi(x) \quad \text{for all } \varphi \in X^*.$$

At this regard, a Banach space X is called *reflexive* if the mapping $x \mapsto x^{**}$ is an isometric isomorphism.

A small dissimilarity appears when Hilbert spaces H are considered: in such a case, the duality is constructed by means of an inner product

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$$

which is sesquilinear (instead of exclusively linear). Furthermore, it is well-known that given any complex Hilbert space H, the mapping $x \mapsto \varphi_x(h) := \langle h, x\rangle$ establishes an isometric anti-isomorphism between H and its dual H^*; in particular, this implies that every Hilbert space is reflexive.

For every densely defined linear operator $T \in \mathcal{L}(X)$ on a complex Banach space X (or, analogously, on a complex Hilbert space H), its *adjoint operator* $T^* \in \mathcal{L}(X^*)$ has the domain

$$\mathrm{Dom}(T^*) := \big\{\varphi \in X^* \,:\, \varphi \circ T \text{ is continuous on } \mathrm{Dom}(T)\big\}$$

and is defined by the relation $(T^*\varphi)(x) := \varphi(Tx)$ for each $\varphi \in \mathrm{Dom}(T^*)$. Clearly, whenever $T \in \mathcal{B}(X)$, the adjoint operator T^* also belongs to $\mathcal{B}(X^*)$.

Again, due to the sesquilinearity of the inner product in Hilbert spaces, a slight difference will arise regarding the algebraic properties of adjointness. Specifically:

(i) For Banach space operators $S, T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, we have $(S + \lambda T)^* = S^* + \lambda T^*$;

(ii) For Hilbert space operators $S, T \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$, we have $(S + \lambda T)^* = S^* + \overline{\lambda} T^*$.

Just mention that such a distinction might be influential, for example, as we discuss spectral properties of adjoint operators. Nonetheless, since we shall often work on Banach spaces, these particularities will be generally bypassed.

Related to duality, we may define two new topologies. The *weak topology* on a Banach space X is the coarsest topology on X with respect to which all the functionals $\varphi \in X^*$ are continuous on X. In a similar manner, the *weak-star topology* on X^* is defined as the coarsest topology on X^* with respect to which, for each $x \in X$, the linear functional $x^{**} : X^* \to \mathbb{C}$ is continuous on X^*. Maybe, sometimes, it is more natural to think of weak topologies in terms of convergence: thus, a net $(x_\lambda)_{\lambda\in\Lambda}$ in X converges to $x \in X$ in the weak topology whenever $\varphi(x_\lambda) \to \varphi(x)$ for every $\varphi \in X^*$; similarly, a net $(\varphi_\lambda)_{\lambda\in\Lambda}$ in X^* converges to $\varphi \in X^*$ in the weak-star topology precisely when $\varphi_\lambda(x) \to \varphi(x)$ for every $x \in X$.

Given a subset M of a Banach space X, its *annihilator* is

$$M^\perp := \{\varphi \in X^* \,:\, \varphi(x) = 0 \text{ for every } x \in M\};$$

likewise, the *preannihilator* of a subset N in X^* is given by

$$^\perp N := \{x \in X \,:\, \varphi(x) = 0 \text{ for every } \varphi \in N\}.$$

Sometimes, in the context of Hilbert spaces, these notions are also named as the *orthogonal complement*. A standard application of the Hahn-Banach Theorem ensures that $^\perp(M^\perp)$ is always the closure of $\mathrm{span}(M)$ in the norm topology of X, whilst $(^\perp N)^\perp$ coincides with the closure of $\mathrm{span}(N)$ in the weak-star topology of X^*.

Much of basic duality theory follows upon a remarkable feature concerning annihilators. Given any closed linear subspace $\mathcal{M}$ of a complex Banach space X, the restriction from X^* onto $\mathcal{M}^*$ and the quotient mapping from X onto $X/\mathcal{M}$ induce the identifications

$$X^*/\mathcal{M}^\perp \cong \mathcal{M}^* \quad \text{and} \quad \left(X/\mathcal{M}\right)^* \cong \mathcal{M}^\perp.$$

In a similar manner, for each weak-star closed linear subspace $\mathcal{N}$ in X^*, we have

$$X^*/\mathcal{N} \cong (^\perp\mathcal{N})^* \quad \text{and} \quad \left(X/{^\perp\mathcal{N}}\right)^* \cong \mathcal{N}.$$

When the *graph* of a densely defined linear operator $T \in \mathcal{L}(X)$, which is the set given by

$$\mathfrak{G}(T) := \{(x, Tx) \in X \times X \,:\, x \in \mathrm{Dom}(T)\},$$

is closed in the product space $X \times X$, its kernels and ranges and those of the adjoint T^* are also intimately related via annihilators. More precisely:

$$\ker(T^*) = (\mathrm{ran}(T))^\perp, \quad \ker(T) = {}^\perp(\mathrm{ran}(T^*)),$$
$$\mathrm{ran}(T) \subseteq {}^\perp(\ker(T^*)), \quad \mathrm{ran}(T^*) \subseteq (\ker(T))^\perp.$$

In general, the last two inclusions are strict. In this regard, the *Closed Range Theorem* gives necessary and sufficient conditions for equality:

Theorem 1.1 (Closed Range Theorem). *Let $T \in \mathcal{L}(X)$ be densely defined operator on a complex Banach space X having closed graph. Then, the following conditions are equivalent:*

(i) $\mathrm{ran}(T)$ *is closed in* X.
(ii) $\mathrm{ran}(T^*)$ *is closed in* X^*.
(iii) $\mathrm{ran}(T) = {}^\perp(\ker(T^*))$.
(iv) $\mathrm{ran}(T^*) = (\ker(T))^\perp$.

One of the most important families of Banach spaces occurring in this thesis is the class of *L^p spaces*: given a measure space $(\Omega, \mathcal{G}, \mu)$ (here Ω will always denote a set, $\mathcal{G}$ a σ-algebra of subsets and μ a measure), let $L^p(\Omega, \mu)$, for each $1 \le p < \infty$, denote the Banach space of (equivalence classes of) $\mathcal{G}$-measurable complex-valued functions $f : \Omega \to \mathbb{C}$ with p-norm given by

$$\|f\|_{L^p(\Omega,\mu)} := \left(\int_\Omega |f(\omega)|^p \, d\mu(\omega) \right)^{1/p}.$$

On the other hand, $L^\infty(\Omega, \mu)$ stands for the space of μ-essentially bounded complex-valued functions on Ω. In this case, the norm is defined as

$$\|f\|_{L^\infty(\Omega,\mu)} := \inf \big\{ \lambda \geq 0 \,:\, |f(\omega)| \leq \lambda \text{ for } \mu\text{-a.e. } \omega \in \Omega \big\} := \mu\text{-}\operatorname*{ess\,sup}_{\omega \in \Omega} |f(\omega)|,$$

where μ-ess sup is usually known as the *essential supremum*. Related to this notion, the *essential range* of a function $f \in L^p(\Omega, \mu)$ is the closed set given by

$$\mu\text{-}\operatorname{ess\,ran}(f) := \big\{ \lambda \in \mathbb{C} \,:\, \mu(\{\omega \in \Omega : |f(\omega) - \lambda| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0 \big\}.$$

Hereafter, to simplify notation, when the measure space is clear from the context, we will simply write $\| \cdot \|_p$ to denote the p-norm. Similarly, we shall often write ess ran and ess sup, avoiding the explicit mention to μ.

In the particular case of a weighted counting measure on $\mathbb{Z}$ (as well, on $\mathbb{Z}_+$ or $\mathbb{N}$), namely $\mu(\{n\}) := \mu_n$ for all $n \in \mathbb{Z}$, we obtain the well-known ℓ^p *weighted sequences spaces*:

$$\ell^p\big(\mathbb{Z}, (\mu_n)_{n\in\mathbb{Z}}\big) := \left\{ x := (x_n)_{n\in\mathbb{Z}} \,:\, \|x\|_p := \left(\sum_{n\in\mathbb{Z}} |x_n|^p \, \mu_n \right)^{1/p} < \infty \right\}.$$

For each $1 \leq p < \infty$, the *Hölder inequality* claims that

$$\|fg\|_{L^1(\Omega,\mu)} \leq \|f\|_{L^p(\Omega,\mu)} \|g\|_{L^q(\Omega,\mu)}$$

for every pair of $\mathcal{G}$-measurable functions f, g on Ω, where q is the *conjugate exponent* of p (i.e. $1/p + 1/q = 1$). A standard consequence of the Hölder Inequality shows that the dual space of $L^p(\Omega, \mu)$ can be identified with $L^q(\Omega, \mu)$, where q is the *conjugate exponent* of p (i.e. $1/p + 1/q = 1$). Hence, each element of $L^p(\Omega, \mu)^*$ may be regarded as a functional $\varphi_g : L^p(\Omega, \mu) \to \mathbb{C}$ of the form

$$\varphi_g(f) := \int_\Omega f(\omega) \, g(\omega) \, d\mu(\omega) \qquad \text{for all } f \in L^p(\Omega, \mu),$$

with $g \in L^q(\Omega, \mu)$ arbitrary. In particular, this implies that all L^p spaces are reflexive for each $1 < p < \infty$. Moreover, $L^2(\Omega, \mu)$ is always a complex Hilbert space.

Another remarkable inequality concerning measure spaces is the well-known *Jensen Inequality*: let $(\Omega, \mathcal{G}, \mu)$ be a *probability space* (i.e. with $\mu(\Omega) = 1$) and consider any real-valued $\mathcal{G}$-measurable function $f : \Omega \to \mathbb{R}$, then

$$\psi\left(\int_\Omega f(\omega) \, d\mu(\omega) \right) \leq \int_\Omega (\psi \circ f)(\omega) \, d\mu(\omega) \tag{1.1}$$

for each convex function $\psi : \mathbb{R} \to \mathbb{R}$.

Additionally , another important family of topological spaces will be the class of Fréchet spaces. We recall that a *Fréchet space* is a locally convex vector space whose topology is induced by a complete translation-invariant metric. A significant instance is the following: given any open set $U \subseteq \mathbb{C}$ and a complex Banach space X, let $\mathcal{O}(U, X)$ denote the vector space of all X-valued analytic functions on U. In this context, we might distinguish two different types of analyticity which, at the end, turn out to be equivalent: firstly, a function $f : U \to X$ is said to be *weakly analytic* if the composition $\varphi \circ f : U \to \mathbb{C}$ is analytic on U (in the classical sense of Complex Analysis) for every $\varphi \in X^*$; secondly, a function $f : U \to X$ is called *strongly analytic* if, for each $z \in U$, the limit

$$f'(z) := \lim_{\lambda \to z} \frac{f(\lambda) - f(z)}{\lambda - z}$$

exists in the norm topology of X. Clearly, strong analyticity implies weak analyticity. The converse is also true and may be easily derived, for instance, from the Uniform Boundedness Principle (see, for example, [**115**]). Furthermore, following the Bochner integral terminology, *Cauchy's Theorem* and *Cauchy's Integral Formula* also hold in the X-valued case. Hence, given an arbitrary analytic function $f : U \to X$ and any $z \in U$, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\lambda)}{(z-\lambda)^{n+1}}\, d\lambda \qquad \text{for each } n \in \mathbb{Z}_+,$$

where Γ is any positively oriented closed rectifiable curve inside U surrounding the point $z \in U$. Not surprisingly, continuing the analogies with the scalar case, X-valued analytic functions $f : U \to X$ may be locally represented using power series. So, for each fixed $\lambda \in U$ and every $r > 0$ with $D(\lambda, r) \subseteq U$, we have

$$f(z) = \sum_{n=0}^{\infty} x_n \, (z-\lambda)^n \qquad \text{for all } z \in D(\lambda, r),$$

where the coefficients are given by the formula $x_n = f^{(n)}(\lambda)/n!$ for every $n \in \mathbb{Z}_+$ and satisfy the *Cauchy-Hadamard Formula*:

$$\limsup_{n \to +\infty} \|x_n\|_X^{1/n} \leq \frac{1}{r}. \tag{1.2}$$

In parallel, many of the classical results from Complex Analysis can be adapted into this realm. Here, we only mention without a proof the *Liouville's Theorem* on entire functions and the *Identity Theorem*:

Theorem 1.2 (Liouville's Theorem). *Let X be a complex Banach space. Then, any X-valued analytic bounded function in $\mathcal{O}(\mathbb{C}, X)$ is constant.*

Theorem 1.3 (Identity Theorem). *Consider a closed linear subspace $\mathcal{M}$ of a complex Banach space X. Suppose that $f : U \to X$ is a X-valued holomorphic function on a connected $U \subseteq \mathbb{C}$ having $f(S) \subseteq \mathcal{M}$ for a subset $S \subseteq U$ which clusters in U. Then, $f(U) \subseteq \mathcal{M}$.*

Equipped with the *topology of uniform convergence on compact subsets* of U, the space $\mathcal{O}(U, X)$ acquires structure of Fréchet space. Indeed, it is well-known that a translation-invariant metric which induces the topology of $\mathcal{O}(U, X)$ is

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} \qquad \text{for all } f, g \in \mathcal{O}(U, X),$$

where $\|f\|_{K_n} := \max\{\|f(z)\|_X : z \in K_n\}$ and $(K_n)_{n \in \mathbb{N}}$ is an arbitrary sequence of compact subsets of U satisfying

$$K_n \subseteq \operatorname{int}(K_{n+1}) \text{ for every } n \in \mathbb{N} \quad \text{and} \quad \bigcup_{n \in \mathbb{N}} K_n = U.$$

When the image space is the field of complex numbers, we will simply denote $\mathcal{O}(U) := \mathcal{O}(U, \mathbb{C})$. Indeed, by a celebrated result of Grothendieck [**74**], as a consequence of the *nuclearity* of the space $\mathcal{O}(U)$, we may topologically identify

$$\mathcal{O}(U, X) \cong \mathcal{O}(U) \otimes X,$$

where the tensor product is equipped with the completion of the *projective tensor product topology.* This shows that every $f \in \mathcal{O}(U, X)$ admits a representation of the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n (f_n \otimes x_n)(z) \qquad (z \in U),$$

where $(\lambda_n)_{n\in\mathbb{N}}$ belongs to $\ell^1(\mathbb{N})$, the sequences $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{O}(U)$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ tend to 0 in their respective topologies, and

$$(f \otimes x)(z) := f(z)\, x \quad \text{ for each } f \in \mathcal{O}(U) \text{ and } x \in X.$$

In the same vein, given any complex Banach space X and a closed subset $F \subseteq \mathbb{C}$, by $\mathcal{O}(F, X)$ we shall mean the *strict inductive limit* of Fréchet spaces $\mathcal{O}(U, X)$ as U ranges over all open neighbourhoods of F. In this sense, roughly speaking, each element of $\mathcal{O}(F, X)$ may be regarded as an analytic function on an arbitrary open neighbourhood of F. Then, as expected, the canonical *inductive topology* on $\mathcal{O}(F, X)$ is specified by the condition that a linear mapping $T : \mathcal{O}(F, X) \to Y$, where Y denotes an arbitrary locally convex topological vector space, is continuous precisely when its restrictions $T|_{\mathcal{O}(U,X)} : \mathcal{O}(U, X) \to Y$ are continuous for every open neighbourhood $U \supseteq F$.

1.1. Banach algebras and Gelfand Theory

Along this section, we recall already known features regarding Banach algebras and Gelfand Theory. Naturally, our interest in Banach algebras comes from the fact that, via functional calculus, they emerge as one of the most useful tools for constructing invariant subspaces.

The initial part of this section is devoted to collecting some of the most basic aspects about Banach algebras and Spectral Theory. Later on, we will survey describe the rudiments of Gelfand Theory in the commutative case. Our exposition may be found in standard references (even amongst not specific ones) such as, for instance, the monographs [**42**], [**82**] and/or [**89**].

1.1.1. Banach algebras and Spectral Theory

A complex *Banach algebra* $\mathcal{A}$ is an associative normed algebra over the field of complex numbers $\mathbb{C}$ such that, at the same time, it is a Banach space for the metric induced by its norm. In general, in order to ensure the continuity of the multiplication operation on $\mathcal{A}$, the norm is required to satisfy the inequality

$$\|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}\, \|b\|_{\mathcal{A}} \quad \text{for every } a, b \in \mathcal{A}.$$

If a Banach algebra $\mathcal{A}$ possesses an identity with respect to multiplication, then it is called *unital*. Since our main interest is spectral theory, we shall usually deal with unital Banach algebras; however, further on, we may encounter some Banach algebras without identity and we will require to convert them into unital ones: this process is usually known as unitization. In general, for an arbitrary complex Banach algebra, its *unitization* $\mathcal{A}_e := \mathcal{A} \oplus \mathbb{C}e$ is defined as the complex vector space $\mathcal{A} \times \mathbb{C}$ with the multiplication given by

$$(a + \mu e) \cdot (b + \nu e) := ab + \nu a + \mu b + \mu\nu e \quad \text{for all } a, b \in \mathcal{A} \text{ and } \mu, \nu \in \mathbb{C}.$$

Endowed with the canonical norm

$$\|a + \mu e\|_{\mathcal{A}_e} := \|a\|_{\mathcal{A}} + |\mu| \quad \text{ for each } a \in \mathcal{A} \text{ and } \mu \in \mathbb{C},$$

the unitization $\mathcal{A}_e$ regains the structure of complex Banach algebra; moreover, the element $e \in \mathcal{A}_e$ now plays the role of identity in $\mathcal{A}_e$.

In the sequel, consider a complex Banach algebra $\mathcal{A}$ with identity e. For an arbitrary $a \in \mathcal{A}$, the *spectrum* of a, denoted as $\sigma(a)$, consists of the complex subset given by

$$\sigma(a) := \big\{\lambda \in \mathbb{C} \,:\, a - \lambda e \text{ is not invertible in } \mathcal{A}\big\}.$$

On the other hand, the *resolvent set*, defined as $\rho(a) := \mathbb{C} \setminus \sigma(a)$, is constituted by those $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is invertible in $\mathcal{A}$. In general, the spectrum $\sigma(a)$ is always a non-empty compact subset of $\mathbb{C}$ while, consequently, $\rho(a)$ must be open and unbounded. The *resolvent function* is the $\mathcal{A}$-valued analytic function defined by

$$\begin{aligned} R_a : \rho(a) &\longrightarrow \mathcal{A} \\ \lambda &\longmapsto (a - \lambda e)^{-1}. \end{aligned}$$

Clearly, the resolvent function fulfils the functional equation $(a - \lambda e)R_a(\lambda) = e$ for each $\lambda \in \rho(a)$ and verifies the limit $\|R_a(\lambda)\|_{\mathcal{A}} \to 0$ as $|\lambda| \to \infty$.

The *spectral radius* of an element $a \in \mathcal{A}$ is given by the quantity

$$r(a) := \max\big\{|\lambda| \,:\, \lambda \in \sigma(a)\big\}.$$

Moreover, the *spectral radius formula* (also known as *Gelfand's Formula*) states that

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \quad \text{for every } a \in \mathcal{A}. \tag{1.3}$$

In a trivial way, for each $a \in \mathcal{A}$ and given any complex polynomial

$$p(z) = c_n z^n + \ldots + c_1 z + c_0 \quad c_j \in \mathbb{C} \text{ and } n \in \mathbb{N},$$

we may define the element $p(a) := c_n a^n + \ldots + c_1 a + c_0$ in $\mathcal{A}$. This algebra homomorphism from $\mathbb{C}[z]$ to $\mathcal{A}$ is sometimes called the *polynomial functional calculus*. More generally, extending the polynomial functional calculus for each $a \in \mathcal{A}$, we may consider the *holomorphic functional calculus*. Fixed $a \in \mathcal{A}$, for each $f : U \to \mathbb{C}$ analytic on some open neighbourhood U of the spectrum $\sigma(a)$, let

$$f(a) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)\,(\lambda e - a)^{-1}\, d\lambda \quad \in \mathcal{A}, \tag{1.4}$$

where, following the standard criterion from Complex Analysis, Γ is a positively oriented closed rectifiable curve in U that surrounds $\sigma(a)$. From Cauchy theorem via the Hahn-Banach theorem, it may be deduced quite straightforward that the definition of $f(a)$ is independent of the choice of the curve Γ that surrounds the spectrum $\sigma(a)$ in U. Indeed, the functional calculus mapping

$$\begin{aligned} \Phi : \mathcal{O}\big(\sigma(a)\big) &\longrightarrow \mathcal{A} \\ f &\longmapsto f(a), \end{aligned}$$

is a unital algebra homomorphism such that $\Phi(1) = e$ and $\Phi(Z) = a$, where, as usual, Z represents the identity function $z \mapsto z$ on $\mathbb{C}$. Moreover, using standard estimates inside the integral (1.4), one may check that Φ is continuous with respect to the canonical inductive topology of Fréchet spaces on $\mathcal{O}\big(\sigma(a)\big)$ and the norm topology on $\mathcal{A}$.

One of the most remarkable facts regarding the analytic functional calculus is described by the *Spectral Mapping Theorem*:

Theorem 1.4 (Spectral Mapping Theorem). *Let $\mathcal{A}$ be a complex Banach algebra with identity. Then, given any element $a \in \mathcal{A}$, the formula*

$$\sigma\big((f(a)\big) = f\big(\sigma(a)\big)$$

holds for every function f which is holomorphic on some open neighbourhood of $\sigma(a)$.

Since our main concern along this thesis deals with Operator Theory, maybe the most important context in which we shall apply all the preceding facts is the Banach algebra $\mathcal{B}(X)$ of bounded linear operators on a complex Banach space X. For instance, in this concrete case, the holomorphic functional calculus is often called *Riesz functional calculus*. Later on, in Section 1.2, we will develop in more detail the spectral theory for operators.

1.1.2. Gelfand Theory

Throughout this subsection, $\mathcal{A}$ will always denote a commutative complex Banach algebra. A non-zero multiplicative linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a *character* of $\mathcal{A}$. The set of all characters of $\mathcal{A}$, denoted by $\Delta(\mathcal{A})$, is usually known as the *Gelfand spectrum* or *Gelfand space* of $\mathcal{A}$.

Every $\varphi \in \Delta(\mathcal{A})$ is automatically continuous on $\mathcal{A}$. A straightforward proof of this well-known fact may be the following: let $\varphi \in \Delta(\mathcal{A})$ be an arbitrary character, and suppose there exists $a \in \mathcal{A}$ such that $\|a\| < 1$ but $|\varphi(a)| = 1$. In particular, without loss of generality, we may assume that $\varphi(a) = 1$. By completeness, the element $b := \sum_{n\geq 1} a^n$ belongs to $\mathcal{A}$ and, clearly, satisfies the equation $a + ab = b$. Hence, upon applying φ, we obtain $1 + \varphi(b) = \varphi(b)$ which is a contradiction.

Consequently, the Gelfand spectrum $\Delta(\mathcal{A})$ is a subset of the closed unit ball of the topological dual $\mathcal{A}^*$; hence, as a consequence of the Banach-Alaoglu Theorem, when equipped with the relative weak-star topology, $\Delta(\mathcal{A})$ turns out to be a locally compact topological space. This is the definition of the *Gelfand topology* on $\Delta(\mathcal{A})$.

Furthermore, by the definition of the weak-star topology on $\mathcal{A}^*$, we deduce that a net of characters $(\varphi_\lambda)_{\lambda\in\Lambda}$ in $\Delta(\mathcal{A})$ converges to $\varphi \in \Delta(\mathcal{A})$ in the Gelfand topology whenever $\varphi_\lambda(a) \to \varphi(a)$ for every $a \in \mathcal{A}$. Therefore, for each $\varphi \in \Delta(\mathcal{A})$, a base of open neighbourhoods in the Gelfand topology nearby φ is given by all intersections of finitely many sets of the form

$$U_{\varphi,a,\varepsilon} := \big\{\psi \in \Delta(\mathcal{A}) \,:\, |\varphi(a) - \psi(a)| < \varepsilon\big\} \quad \text{for all } a \in \mathcal{A} \text{ and } \varepsilon > 0. \tag{1.5}$$

In particular, this shows that $\Delta(\mathcal{A})$ endowed with the Gelfand topology always enjoys the *Hausdorff separation axiom*.

However, sometimes, it is more convenient to think about the Gelfand topology in other terms. Associated to each element $a \in \mathcal{A}$, the *Gelfand transform* of a is defined as the map

$$\begin{aligned} \widehat{a} : \Delta(\mathcal{A}) &\to \mathbb{C} \\ \varphi &\mapsto \varphi(a) \end{aligned}$$

Equivalently, the Gelfand topology on $\Delta(\mathcal{A})$ now may be defined to be the coarsest topology on $\Delta(\mathcal{A})$ for which all Gelfand transforms are continuous. Moreover, by the locally compactness of $\Delta(\mathcal{A})$, one may observe that actually, for each $a \in \mathcal{A}$, the Gelfand transform $\widehat{a}$ vanishes at infinity. Hence, the mapping

$$\begin{aligned} \widehat{\ } : \mathcal{A} &\to C_0\big(\Delta(\mathcal{A})\big) \\ a &\mapsto \widehat{a} \end{aligned} \tag{1.6}$$

is a well-defined Banach algebra homomorphism which, indeed, turns out to be norm-decreasing

$$\|\widehat{a}\| := \sup\big\{|\varphi(a)| \,:\, \varphi \in \Delta(\mathcal{A})\big\} \leq \|a\|.$$

This association is usually known as the *Gelfand representation* of the Banach algebra $\mathcal{A}$. In general, the Gelfand representation is neither injective nor surjective.

In the special case where $\mathcal{A}$ has an identity e, then $\Delta(\mathcal{A})$ is compact in the Gelfand topology and the Gelfand representation $\widehat{\ }: \mathcal{A} \to C_0\big(\Delta(\mathcal{A})\big)$, $a \mapsto \widehat{a}$ is a unit-preserving homomorphism. Additionally, as a direct consequence of the *Gelfand-Mazur Theorem* which assures that the only unital complex Banach algebra which is a division algebra is the field of complex numbers $\mathbb{C}$, there is a bijection between the $\Delta(\mathcal{A})$ and the set of maximal ideals in $\mathcal{A}$ given by $\varphi \mapsto \ker(\varphi)$ (for this reason, in the unital case, $\Delta(\mathcal{A})$ is also known as *maximal ideal space* of $\mathcal{A}$) and the spectrum of every element $a \in \mathcal{A}$ coincides with

$$\sigma(a) = \widehat{a}\big(\Delta(\mathcal{A})\big). \tag{1.7}$$

Thereby, the kernel of the Gelfand representation may be identified with the *Jacobson radical* of $\mathcal{A}$, i.e. with the set of *quasi-nilpotent* elements. Hence, the algebra $\mathcal{A}$ is *semi-simple* precisely when the Gelfand representation is injective.

The non-unital case is slightly more involved. Nevertheless, we may get some insight using the unitization of Banach algebras. So, given a commutative complex Banach algebra $\mathcal{A}$ without identity, it may be seen that its Gelfand spectrum decomposes in the form

$$\Delta(\mathcal{A}_e) = \Delta(\mathcal{A}) \cup \{\varphi_\infty\}$$

where $\varphi_\infty(a + \mu e) := \mu$ for every $a \in \mathcal{A}$ and $\mu \in \mathbb{C}$. Indeed, the latter decomposition of $\Delta(\mathcal{A}_e)$ may be regarded as the *one-point compactification* of the locally compact space $\Delta(\mathcal{A})$. In this case, again via the mapping $\varphi \mapsto \ker(\varphi)$, the Gelfand spectrum $\Delta(\mathcal{A})$ may be identified with the set of maximal modular ideals of $\mathcal{A}$. We recall that an ideal $J \subseteq \mathcal{A}$ is said to be *modular* if there exists an element $c \in \mathcal{A}$ with the property that $a - ac \in J$ for all $a \in \mathcal{A}$. As before, the semi-simplicity of $\mathcal{A}$ is equivalent to the injectivity of the Gelfand representation.

An illuminating example shows that the Gelfand transform actually encompasses the classical *Fourier transform*: let $(G, +)$ be an arbitrary locally compact abelian group. A celebrated theorem by Haar ensures the existence of a non-trivial positive regular Borel measure μ which is translation-invariant under the action of $+$. Indeed, this *Haar measure* is unique up to a positive constant. Now, consider the Banach space $L^1(G)$ endowed with the usual 1-norm:

$$\|f\|_1 := \int_G |f(x)|\, d\mu(x) \quad \text{for all } f \in L^1(G).$$

Equipped with the *convolution product* given by

$$(f * g)(x) := \int_G f(x - y)\, g(y)\, d\mu(y) \quad \text{for each } x \in G,$$

it may be seen that $L^1(G)$ acquires structure of semi-simple commutative Banach algebra, usually denominated as the *group algebra* of G.

Correspondingly, one may examine the Gelfand spectrum $\Delta\big(L^1(G)\big)$ of the group algebra $L^1(G)$. A remarkable fact ensures that the set of characters $\Delta\big(L^1(G)\big)$ with the Gelfand topology is homeomorphic to the group of homomorphism

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$$

endowed with the topology of uniform convergence on compact sets. This canonical identification is usually known as *Pontryagin Duality* and $\widehat{G}$ is called the *dual group* or *Pontryagin dual* of G. For instance, given a continuous homomorphism $\gamma : G \to \mathbb{T}$, it is immediate to see that the assignment

$$\varphi_\gamma(f) := \int_G f(x)\, \gamma(-x)\, d\mu(x) \quad \text{for each } f \in L^1(G).$$

yields a non-zero multiplicative linear functional on $L^1(G)$.

In the context of classical Harmonic Analysis, the Gelfand transform of a function $f \in L^1(G)$, regarded as a function acting on the Pontryagin dual $\widehat{f} : \widehat{G} \to \mathbb{C}$, turns out to be precisely the *Fourier transform*. For instance, if we choose $G = \mathbb{T}$, its dual group is $\widehat{G} = \mathbb{Z}$ and the Fourier transform is given by

$$\widehat{f}(n) := \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, \frac{d\theta}{2\pi} \quad \text{for each } n \in \mathbb{Z}; \tag{1.8}$$

on the other hand, if $G = \mathbb{R}$, its dual group is as well $\widehat{G} = \mathbb{R}$ and the Fourier transform is

$$\widehat{f}(\xi) := \int_{-\infty}^{+\infty} f(t)\, e^{-it\xi}\, dt \quad \text{for each } \xi \in \mathbb{R}. \tag{1.9}$$

Many other transforms, such as the *Laplace transform*, can be retrieved as particular examples of Gelfand transforms over group algebras. For a complete account on Harmonic Analysis on group algebras, we refer to [**114**].

1.2. Spectral Theory and invariant subspaces

As discussed in the Introduction, since the earliest days of Operator Theory, the Invariant Subspace Problem has turned, by far, into one of the most distinguished questions in the field. Along this section, our main objective will be to illustrate in detail the important role played by the different spectral constructions in order to produce invariant subspaces.

This section is organized as follows. In Subsection 1.2.1, we collect and prove some basic facts on classical Spectral Theory of operators which will be used throughout the manuscript. Then, Subsection 1.2.2 deals with Local Spectral Theory and, therein, we describe certain remarkable spectral properties which will be essential for the analysis of weighted translation operators carried out along the forthcoming chapters.

1.2.1. Classical Spectral Theory

Hereafter, let $T \in \mathcal{B}(X)$ be a bounded linear operator on a complex Banach space X. A closed linear subspace $\mathcal{M} \subseteq X$ is said to be *invariant* for T if the inclusion $T(\mathcal{M}) \subseteq \mathcal{M}$ holds. Similarly, $\mathcal{M} \subseteq X$ is called *hyperinvariant* for T, if $S(\mathcal{M}) \subseteq \mathcal{M}$ for every $S \in \mathcal{B}(X)$ in the commutant of T.

As customary, the *lattice* of all invariant subspaces of T will be denoted by $\mathrm{Lat}(T)$. Additionally, recall that $\mathrm{Lat}(T)$ is branded as *rich* whenever there exists an infinite-dimensional complex Banach space Y such that the lattice of all closed subspaces in Y (denoted as well by $\mathrm{Lat}(Y)$) is order-isomorphic to a sublattice of $\mathrm{Lat}(T)$.

Applying the classical theory of Banach algebras to $\mathcal{B}(X)$, we define the *spectrum* of T as the non-empty compact set given by

$$\sigma(T) := \big\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{B}(X)\big\}.$$

As before, the *resolvent set*, defined as $\rho(T) := \mathbb{C} \setminus \sigma(T)$, is the unbounded open set formed by those $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is invertible in $\mathcal{B}(X)$. The *resolvent function* is the $\mathcal{B}(X)$-valued holomorphic function determined by the assignment

$$\begin{aligned} R_T : \rho(T) &\longrightarrow \mathcal{B}(X) \\ \lambda &\longmapsto (T-\lambda I)^{-1}. \end{aligned}$$

Obviously, the resolvent function satisfies the functional equation $(T-\lambda I)R_T(\lambda) = I$ for each $\lambda \in \rho(T)$ and the limit $\|R_T(\lambda)\| \to 0$ as $|\lambda| \to \infty$ holds in the strong operator topology. Indeed, nearby $\infty_{\mathbb{C}}$, the resolvent function can be represented by the power series

$$(T-\lambda I)^{-1} = -\sum_{n=0}^{\infty} \lambda^{-n-1}\, T^n, \qquad \text{for all } |\lambda| > r(T),$$

where, as previously established, $r(T)$ denotes the *spectral radius* of T

$$r(T) := \max\big\{|\lambda| \,:\, \lambda \in \sigma(T)\big\}.$$

In this context, the *spectral radius formula* (again known as *Gelfand's Formula*)

$$r(T) = \lim_{n\to+\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n}$$

shall be especially remarkable on several occasions.

Regarding this framework, as aforesaid, the *holomorphic functional calculus* for operators is often referred to as *Riesz functional calculus*. So, fixed an operator $T \in \mathcal{B}(X)$, for every function $f : U \to \mathbb{C}$ holomorphic on some open neighbourhood U of $\sigma(T)$ the integral

$$f(T) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - T)^{-1}\, d\lambda,$$

where Γ is a positively oriented closed rectifiable curve in U surrounding $\sigma(T)$, defines a continuous unital algebra homomorphism

$$\begin{aligned} \Phi : \mathcal{O}\big(\sigma(T)\big) &\longrightarrow \mathcal{B}(X) \\ f &\longmapsto f(T) \end{aligned}$$

satisfying simultaneously $\Phi(1) = I$ and $\Phi(Z) = T$. Once again, the *Spectral Mapping Theorem* describes the behaviour of the spectrum respecting the Riesz functional calculus:

Theorem 1.5 (Spectral Mapping Theorem). *Let $T \in \mathcal{B}(X)$ be a linear bounded operator on a complex Banach space. Then, the identity*

$$\sigma\big(f(T)\big) = f\big(\sigma(T)\big)$$

holds for each function f which is holomorphic on some open neighbourhood of $\sigma(T)$.

As well, it will be of interest to establish some elementary facts on the spectrum of restrictions and quotients. From now on, for every T-invariant subspaces $\mathcal{M}$, the notation $T|\mathcal{M}$ will stand for the *restriction operator* of T to $\mathcal{M}$ while $T/\mathcal{M}$ will represent the *quotient operator* of T by $\mathcal{M}$:

$$\begin{aligned} T|\mathcal{M} : \mathcal{M} &\to \mathcal{M} \\ x &\mapsto Tx, \end{aligned} \qquad\qquad \begin{aligned} T/\mathcal{M} : X/\mathcal{M} &\to X/\mathcal{M} \\ x + \mathcal{M} &\mapsto Tx + \mathcal{M}. \end{aligned}$$

In this regard, the *full spectrum* $\widehat{\sigma(T)}$, i.e. the union of $\sigma(T)$ and all bounded connected components of the resolvent set $\rho(T)$, shall emerge as a crucial concept. Note that, in the language of Complex Analysis, the full spectrum $\widehat{\sigma(T)}$ is none other than the polynomially convex hull of $\sigma(T)$.

Proposition 1.6. *Let $T \in \mathcal{B}(X)$ be an operator acting on a complex Banach space X. Suppose we are given two T-invariant subspaces $\mathcal{M}_1, \mathcal{M}_2 \subseteq X$ such that $X = \mathcal{M}_1 + \mathcal{M}_2$. Then, the following inclusions hold:*

$$\sigma(T/\mathcal{M}_1) \subseteq \sigma(T) \cup \sigma(T|\mathcal{M}_1) \subseteq \widehat{\sigma(T)} \quad \text{and} \quad \sigma(T/\mathcal{M}_2) \subseteq \widehat{\sigma(T|\mathcal{M}_1)}.$$

In particular, we have the inequalities $r(T|\mathcal{M}_1) \leq r(T)$ and $r(T/\mathcal{M}_1) \leq r(T)$.

Proof. Chosen arbitrary $\lambda \in \rho(T) \cap \rho(T|\mathcal{M}_1)$, since

$$Q(T-\lambda) = (T/\mathcal{M}_1 - \lambda)Q$$

where $Q : X \to X/\mathcal{M}_1$ denotes the canonical quotient operator, we deduce that $T/\mathcal{M}_1 - \lambda$ must be surjective. In addition, for $x \in X$ with $(T/\mathcal{M}_1 - \lambda)Qx = 0$, we clearly have that $(T-\lambda)x \in \mathcal{M}_1$; but, since $\lambda \in \rho(T|\mathcal{M}_1)$, we conclude that $x \in \mathcal{M}_1$. This establishes the injectivity of $T/\mathcal{M}_1 - \lambda$, leading us to the first inclusion

$$\sigma(T/\mathcal{M}_1) \subseteq \sigma(T) \cup \sigma(T|\mathcal{M}_1). \tag{1.10}$$

Now, expanding the resolvent function as a power series

$$(T-\lambda)^{-1} = -\sum_{n=0}^{\infty} \lambda^{-n-1} T^n, \qquad \text{for all } |\lambda| > r(T),$$

it is immediate to check that $(T-\lambda)^{-1}\mathcal{M}_1 \subseteq \mathcal{M}_1$. But, since $\mathbb{C} \setminus \widehat{\sigma(T)}$ is a connected open set, the Identity Theorem for holomorphic functions (Theorem 1.3) entails that

$$(T-\lambda)^{-1}\mathcal{M}_1 \subseteq \mathcal{M}_1 \quad \text{for all } \lambda \in \mathbb{C} \setminus \widehat{\sigma(T)};$$

and, as desired, $\sigma(T|\mathcal{M}_1) \subseteq \widehat{\sigma(T)}$.

To prove the second inclusion, observe that since $X = \mathcal{M}_1 + \mathcal{M}_2$, the mapping

$$\begin{aligned} S : \mathcal{M}_1 &\longrightarrow X/\mathcal{M}_2 \\ x &\longmapsto x + \mathcal{M}_2 \end{aligned}$$

is a surjection with $\ker(S) = \mathcal{M}_1 \cap \mathcal{M}_2$. Therefore, the associated isomorphism

$$R : \mathcal{M}_1 / (\mathcal{M}_1 \cap \mathcal{M}_2) \longrightarrow X/\mathcal{M}_2$$

establishes a similarity between $(T/\mathcal{M}_2)\, R = R\,(T|\mathcal{M}_1)/(\mathcal{M}_1 \cap \mathcal{M}_2)$. So, applying inclusion (1.10) in our current case, we conclude that

$$\sigma(T/\mathcal{M}_2) = \sigma\big((T|\mathcal{M}_1) / (\mathcal{M}_1 \cap \mathcal{M}_2)\big) \subseteq \sigma(T|\mathcal{M}_1) \cup \sigma(T|(\mathcal{M}_1 \cap \mathcal{M}_2)) \subseteq \widehat{\sigma(T|\mathcal{M}_1)}.$$

□

There are several distinguished parts of the spectrum $\sigma(T)$ which can be considered in order to conduct an appropriate study of a given operator $T \in \mathcal{B}(X)$. In this regard, the *point spectrum*

$$\sigma_{\mathrm{p}}(T) := \big\{\lambda \in \mathbb{C} : \ker(T-\lambda) \neq \{0\}\big\},$$

which is the set of *eigenvalues*, scans the loss of injectivity amongst all scalar perturbations of T. In the same vein, the *approximate point spectrum*, defined by

$$\sigma_{\mathrm{ap}}(T) := \big\{\lambda \in \mathbb{C} : \exists \text{ unit vectors } (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ such that } (T-\lambda)x_n \to 0 \text{ as } n \to \infty\big\} \tag{1.11}$$

determines the lack of boundedness from below. Each point $\lambda \in \sigma_{\mathrm{ap}}(T)$ is usually known as an *approximate eigenvalue*, while sequences $(x_n)_{n\in\mathbb{N}}$ of unit vectors satisfying the inner condition of (1.11) are called *approximate eigenvectors*.

The absence of surjectivity is accounted by the *surjectivity spectrum*

$$\sigma_{\mathrm{su}}(T) := \{\lambda \in \mathbb{C} : (T-\lambda)X \neq X\},$$

whilst, more generally, the *compression spectrum*

$$\sigma_{\mathrm{com}}(T) := \{\lambda \in \mathbb{C} : \overline{(T-\lambda)X} \neq X\}$$

evaluates the denseness of the range. It is straightforward to derive the inclusions

$$\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{ap}}(T) \quad \text{and} \quad \sigma_{\mathrm{com}}(T) \subseteq \sigma_{\mathrm{su}}(T);$$

indeed, the spectrum may be split in the next manner:

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{su}}(T) \quad \text{and} \quad \sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{com}}(T).$$

In conjunction with basic duality theory and the Hahn-Banach theorem, one can relate these sets with the corresponding parts of the spectrum $\sigma(T^*)$ of the adjoint operator T^*:

(i) $\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{com}}(T^*)$ and $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{p}}(T^*)$. Furthermore, complete duality arises when X is a reflexive space.

(ii) $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T^*)$ and $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T^*)$.

(iii) $\sigma(T) = \sigma(T^*)$.

Remark 1.7. Recall that, unlike what occurs in Banach spaces, duality in Hilbert spaces has a conjugate linear component. This provokes that the spectrum and adjointness are linked in a different way, for instance:

$$\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\},$$

for each bounded linear operator $T : H \to H$ acting on a Hilbert space H.

Additionally, general topological properties for some subsets of $\sigma(T)$ can be easily obtained. Here, we state a well known feature for the approximate point spectrum which, in particular, shows us that $\sigma_{\mathrm{ap}}(T) \neq \emptyset$ unless the Banach space X is trivially zero (see, for instance, [**39**, VII Prop. 6.7]). Of course, similar conclusions can be obtained for the surjectivity spectrum thank to duality properties:

Proposition 1.8. *Let $T \in \mathcal{B}(X)$ be an arbitrary Banach space operator. Then, both spectral parts $\sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{su}}(T)$ are closed subsets of $\sigma(T)$ which contain the boundary $\partial\sigma(T)$.*

Not surprisingly, both the approximate point spectrum and the surjectivity spectrum play an important role regarding the existence of invariant subspaces:

Corollary 1.9. *Let $T \in \mathcal{B}(X)$ be a Banach space operator and suppose that $\sigma_{\mathrm{ap}}(T) \neq \sigma(T)$ (respectively $\sigma_{\mathrm{su}}(T) \neq \sigma(T)$). Then, T has a non-trivial hyperinvariant subspace.*

Proof. Fixed any $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ap}}(T)$, note that $\mathrm{ran}(T-\lambda)$ is a closed subspace properly contained in X. Now, choosing an arbitrary $y = (T-\lambda)x$ for $x \in X$ and $S \in \mathcal{B}(X)$ commuting with T, we have:

$$Sy = S(T-\lambda)x = (T-\lambda)Sx.$$

Hence, as required, $\mathrm{ran}(T-\lambda)$ is a non-trivial hyperinvariant subspace for T. □

Another fruitful tool in Operator Theory is *Fredholm Theory*. Recall that an operator $T \in \mathcal{B}(X)$ on a Banach space X is said to be a *Fredholm operator* whenever

$$\dim(\ker(T)) < \infty \quad \text{and} \quad \dim\left(X/\mathrm{ran}(T)\right) < \infty. \tag{1.12}$$

As a matter of fact, note that the latter condition implies that $\operatorname{ran}(T)$ is norm-closed.

Roughly speaking, Fredholm operators may be regarded as "small deviations" from isomorphisms of X. This intuition can be understood in more concrete terms: let $\mathcal{K}(X)$ denote the closed ideal of compact operators on X and consider the quotient space $\mathcal{B}(X)/\mathcal{K}(X)$. Topologized under the quotient norm

$$\big\|T + \mathcal{K}(X)\big\| := \inf\big\{\|S\|_{\mathcal{B}(X)} \,:\, S - T \in \mathcal{K}(X)\big\},$$

the quotient space $\mathcal{B}(X)/\mathcal{K}(X)$ acquires structure of unital Banach algebra. This Banach algebra is named as *Calkin algebra* of X and will be denoted by $\mathcal{C}(X) := \mathcal{B}(X)/\mathcal{K}(X)$.

We shall follow the standard terminology when referring to properties of operators as elements in the Calkin algebra consisting in adding the qualifier *essential*. In this sense, an operator $T \in \mathcal{B}(X)$ is called *essentially invertible* whenever there exists $S \in \mathcal{B}(X)$ for which $ST - I \in \mathcal{K}(X)$. The following result explains why Fredholm operators can be considered almost isomorphisms on X:

Theorem 1.10. *An operator $T \in \mathcal{B}(X)$ is essentially invertible if and only if it is a Fredholm operator.*

So, the notion of *essential spectrum* of a given operator $T : X \to X$, defined by

$$\sigma_{\mathrm{e}}(T) := \big\{\lambda \in \mathbb{C} \,:\, T - \lambda I \text{ is not a Fredholm operator}\big\},$$

precisely looks upon invertibility modulo compact operators. Obviously, $\sigma_{\mathrm{e}}(T)$ is contained in $\sigma(T)$.

Naturally, one obtains the class of semi-Fredholm operators when decouples the two conditions appearing in (1.12). In this fashion, $T \in \mathcal{B}(X)$ is called a *semi-Fredholm operator* if and only if $\operatorname{ran}(T)$ is norm-closed in X and either $\ker(T)$ or $X/\operatorname{ran}(T)$ is finite-dimensional. In such a case, T has a well-defined *index* given by

$$\operatorname{ind}(T) := \dim(\ker(T)) - \dim\Big(X\big/\operatorname{ran}(T)\Big).$$

Semi-Fredholm operators lead us to the following spectral subsets of $\sigma_{\mathrm{e}}(T)$: the first one is known as *left essential spectrum*

$$\sigma_{\mathrm{le}}(T) := \big\{\lambda \in \mathbb{C} \,:\, (T-\lambda)X \text{ is not closed or } \dim\ker(T-\lambda) = \infty\big\}$$

and is related to the class of *upper semi-Fredholm operators*. The second one is named as *right essential spectrum*

$$\sigma_{\mathrm{re}}(T) := \big\{\lambda \in \mathbb{C} \,:\, \dim(T/(T-\lambda)X) = \infty\big\}$$

and is linked to the set of *lower semi-Fredholm operators*. Clearly, the essential spectrum decomposes in the form

$$\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{le}}(T) \cup \sigma_{\mathrm{re}}(T),$$

and, as one may expect, the next duality relation occurs

$$\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{re}}(T) \cup \sigma_{\mathrm{le}}(T^*).$$

For the sake of completeness, we prove the following statement, which evinces the relevance of the essential spectra in the Invariant Subspace Problem:

Proposition 1.11. *Let $T \in \mathcal{B}(X)$ be a Banach space operator. If $\sigma_{\mathrm{e}}(T) \neq \sigma(T)$, then T has non-trivial hyperinvariant subspaces. Furthermore, if $\sigma_{\mathrm{le}}(T) \neq \sigma_{\mathrm{re}}(T)$, then the lattice* $\operatorname{Lat}(T)$ *is rich.*

Proof. For the first claim, let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{e}}(T)$, then $T - \lambda$ is a non-invertible Fredholm operator. This shows that either $\ker(T-\lambda)$ or $(T-\lambda)X$ is a non-trivial hyperinvariant subspace of T.

For the second claim, suppose first the existence of a point $\lambda \in \sigma_{\mathrm{le}}(T) \setminus \sigma_{\mathrm{re}}(T)$. Then, $(T - \lambda)X$ has finite codimension and, consequently, $(T - \lambda)X$ must be closed. Hence, by the definition of $\sigma_{\mathrm{le}}(T)$, we have that $\ker(T - \lambda)$ is infinite-dimensional. Now, since each linear subspace of $\ker(T - \lambda)$ belongs to $\mathrm{Lat}(T)$, we deduce that $\mathrm{Lat}(T)$ is rich.

Conversely, given any $\lambda \in \sigma_{\mathrm{re}}(T) \setminus \sigma_{\mathrm{le}}(T)$, the quotient space $X/(T - \lambda)X$ is infinite-dimensional. Clearly, as shown in our next equation, every linear subspace $\mathcal{M} \supseteq (T - \lambda)X$ is invariant for T:

$$(T-\lambda)x \in (T-\lambda)X \subseteq \mathcal{M}, \qquad \text{for each } x \in \mathcal{M}.$$

Accordingly, $\mathrm{Lat}(T)$ contains a sublattice which is order-isomorphic to $\mathrm{Lat}(X/(T - \lambda)X)$. Hence, the lattice $\mathrm{Lat}(T)$ is also rich in this case. □

Besides, a remarkable theorem proved by Douglas and Pearcy [**46**] and Apostol, Foiaş and Voiculescu [**10**] in the context of Hilbert spaces, is the key fact relating semi-Fredholm operators to quasitriangular ones:

Theorem 1.12. *An operator $T \in \mathcal{B}(H)$ which operates on a Hilbert space H is quasitriangular if and only if $\mathrm{ind}(T-\lambda) \geq 0$ for each complex number $\lambda \in \mathbb{C}$ such that $T-\lambda$ is semi-Fredholm.*

A direct corollary of this theorem asserts that the adjoint of every non-quasitriangular operator must have non-empty point spectrum and, accordingly, non-trivial hyperinvariant subspaces. In this line, we remind the reader that a Hilbert space operator $T \in \mathcal{B}(H)$ is said to be *biquasitriangular* if and only if both T and T^* are quasitriangular operators. Evidently, T immediately possesses non-trivial hyperinvariant subspaces whenever is non-biquasitriangular.

In 1978, by developing new tools essentially relying on Fredholm Theory, S. W. Brown [**31**] established the existence of invariant subspaces for *subnormal operators* on Hilbert spaces, i.e. restrictions of normal operators to closed invariant subspaces. His method became quite notorious and influential in the community and was named as *Scott Brown technique*.

Scott Brown technique has been repeatedly strengthened over the years: firstly, in 1987, S. W. Brown [**32**] adapted his own ideas to construct non-trivial invariant subspaces for every hyponormal operator with thick spectrum (in a sense to be defined below). Recall that a Hilbert space operator $T : H \to H$ is said to be *hyponormal* whenever

$$\big\langle (T^*T - TT^*)x, x \big\rangle \geq 0, \qquad \text{for all } x \in H.$$

Later on, further refinements of the Scott Brown technique in combination with certain functional models arising in Local Spectral Theory concluded with a remarkable theorem due to Eschmeier and Prunaru [**54**], leading to existence of invariant subspaces for Banach space operators with thick spectrum and one of the properties (β) or (δ).

If $U \subseteq \mathbb{C}$ is a non-empty bounded open set and $H^\infty(U)$ denotes the commutative Banach algebra consisting of all bounded analytic complex-valued functions on U, equipped with the usual pointwise operations and the supremum norm

$$\|f\|_{H^\infty} := \sup\big\{|f(z)| \,:\, z \in U\big\};$$

a subset $S \subseteq \mathbb{C}$ is *dominating* in U (in the sense of Rubel and Shields [**113**]) if the equality

$$\|p\|_{H^\infty} = \sup\left\{|p(z)| \,:\, z \in U \cap S\right\}$$

holds for each $p \in P^\infty(U)$, where the space $P^\infty(U)$ stands for the weak-star closure of the complex polynomials on U. For example, in the case of the open unit disk $\mathbb{D}$, we have $P^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$ and it follows that a subset $S \subseteq \mathbb{D}$ is dominating in $\mathbb{D}$ precisely when almost every point in the unit circle is the non-tangential limit of a sequence in S.

Definition 1.13. *A compact set $K \subseteq \mathbb{C}$ is said to be* thick *if there exists a non-empty bounded open set $U \subseteq \mathbb{C}$ in which K is dominating.*

In particular, observe that each compact subset $K \subseteq \mathbb{C}$ having non-empty interior is automatically thick upon the choice $U = \operatorname{int}(K)$.

Gathered all this information, we are in position to state Scott Brown's results without proof for future reference:

Theorem 1.14 (Brown, [31, 32]). *Let $T \in \mathcal{B}(H)$ be a hyponormal operator on a Hilbert space H. Then, T has non-trivial invariant subspaces provided that $\sigma(T)$ is thick. Similarly, each subnormal operator on H possesses a non-trivial invariant subspace.*

1.2.2. A flavour of Local Spectral Theory

As intimated in the Introduction, the theory of decomposable operators is closely related to that of vector-valued holomorphic functions. As the title says, the branch of Spectral Theory intended for the application of (vector-valued) Complex Analysis is known as *Local Spectral Theory*. Accordingly, this part of Section 1.2 is particularly devoted to explaining the basic principles of Local Spectral Theory. More precisely, our main purpose will be to enumerate, illustrate and compare various local spectral properties which arise naturally in this framework.

For a complete account on Local Spectral Theory, we refer to the manuscripts of Aiena [**2**]; Eschmeier and Putinar [**58**], which explores some profound connections with Sheaf Theory; and, especially, the detailed exposition held by Laursen and Neumann [**89**].

As before, our very first local spectral property shall be *decomposability.* However, for later convenience, this time we give a simplified version:

Definition 1.15. *An operator $T \in \mathcal{B}(X)$ on a complex Banach space X is called* decomposable *if every open cover $\{U, V\}$ of $\mathbb{C}$ effects a splitting of both $\sigma(T)$ and X, in the sense that there exist two closed T-invariant subspaces $\mathcal{M}_U, \mathcal{M}_V \subseteq X$ for which*

$$\sigma(T|\mathcal{M}_U) \subseteq U, \qquad \sigma(T|\mathcal{M}_V) \subseteq V$$

and $X = \mathcal{M}_U + \mathcal{M}_V$.

Decomposable operators were initially introduced by Foiaş [**62**] in 1963 under somewhat more complicated terms, but it was not until much later that this definition was shown to be equivalent to the one given here (see, Albrecht [**4**]). Moreover, we underscore that the sum decomposition satisfied by decomposable operators is, in general, not direct, nor are the spectra of the restrictions necessarily disjoint.

Such a connection with Complex Analysis invites to formalize an overall study of spectral properties based on local resolvent equations. Pursuing that idea, the notion of local spectrum arises in a natural way. Thus, recall from the Introduction that, given an arbitrary operator $T \in \mathcal{B}(X)$ on a Banach space X, the *local spectrum of* T *at a vector* $x \in X$, denoted from now on by $\sigma_T(x)$, is the complement of the set of points $\lambda \in \mathbb{C}$ for which there exists an open neighbourhood $U_\lambda \ni \lambda$ and an analytic function $f \in \mathcal{O}(U_\lambda, X)$ such that

$$(T-z)f(z) = x \qquad \text{for all } z \in U_\lambda.$$

Correspondingly, the *local resolvent set* $\rho_T(x)$ *of* T *at* $x \in X$ is precisely the complement of the local spectrum

$$\rho_T(x) := \mathbb{C} \setminus \sigma_T(x).$$

By definition, for all $x \in X$, the local spectrum $\sigma_T(x)$ is always a closed subset of $\sigma(T)$ while $\rho_T(x)$ is an open set including $\rho(T)$. Namely, the solutions occurring in the definition of the local resolvent set may be regarded as analytic extensions of the X-valued holomorphic function $z \mapsto (T-z)^{-1}x$ defined on $\rho(T)$. Indeed, those solutions are usually known as *local resolvent functions*.

In general, the uniqueness of the local resolvent functions is not assured and suggests the next remarkable property:

Definition 1.16. *A Banach space operator* $T \in \mathcal{B}(X)$ *has the* single-valued extension property*, abbreviated SVEP, if fixed any* $x \in X$ *and* $\lambda \in \rho_T(x)$*, there exists a unique local resolvent function on a sufficiently small open neighbourhood* $U_\lambda \ni \lambda$.

It is worthy to note that the SVEP can be restated in these words: an operator $T \in \mathcal{B}(X)$ has the SVEP if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f \in \mathcal{O}(U, X)$ of the equation $(T-z)f(z) = 0$ for all $z \in U$ is the constant zero function on U. Likewise, in terms of the Fréchet space operators

$$\begin{aligned} T_U : \mathcal{O}(U,X) &\longrightarrow \mathcal{O}(U,X) \\ f &\longmapsto z \mapsto (T-z)f(z), \end{aligned} \tag{1.13}$$

the SVEP precisely means that, for every open set $U \subseteq \mathbb{C}$, the operator T_U is injective on the space $\mathcal{O}(U, X)$.

It is completely apparent that an operator enjoys the SVEP whenever the set of its eigenvalues has empty interior. Moreover, we highlight that any operator not having the SVEP admits automatically non-trivial closed hyperinvariant subspaces.

Regarding the SVEP, the subsequent spectral identities will be of interest in a near future. We state them without a proof (see, for instance, [**2**, Corollary 2.45]):

Proposition 1.17. *Let* $T \in \mathcal{B}(X)$ *be a bounded linear operator on a Banach space* X*. Then* $\sigma(T) = \sigma_{\mathrm{su}}(T)$ *whenever* T *has SVEP, and* $\sigma(T) = \sigma_{\mathrm{ap}}(T)$ *whenever* T^* *has SVEP.*

As seen in the Introduction, the local spectrum will enable us to explore more exhaustively the significance of the different parts constituting $\sigma(T)$. To do so, we shall employ *local spectral manifolds*, defined for arbitrary sets $F \subset \mathbb{C}$ as

$$X_T(F) := \big\{x \in X \,:\, \sigma_T(x) \subseteq F\big\}.$$

Obviously, the next properties are always fulfilled by local spectral manifolds:

(i) $X_T(F) = X_T(F \cap \sigma(T))$ for every set $F \subseteq \mathbb{C}$.

(ii) If $F \subseteq G \subseteq \mathbb{C}$, then $X_T(F) \subseteq X_T(G)$.

Furthermore, it may be immediately checked that this construction preserves arbitrary intersections:

$$X_T\Big(\bigcap F_\iota \,:\, \iota \in I\Big) = \bigcap_{\iota \in I} X_T(F_\iota).$$

As expected, one of the most remarkable attributes of local spectral manifolds is their hyperinvariance:

Proposition 1.18. *Given any operator $T \in \mathcal{B}(X)$ and $F \subseteq \mathbb{C}$, the local spectral manifold $X_T(F)$ is a T-hyperinvariant linear manifold.*

Proof. Firstly, note that $0 \in X_T(F)$ trivially since $\rho_T(0) = \mathbb{C}$. Now, for arbitrary $x_1, x_2 \in X_T(F)$ and $\lambda \in \rho_T(x_1) \cap \rho_T(x_2)$, there exists an open neighbourhood $U_\lambda \subseteq \mathbb{C} \setminus F$ of λ such that

$$(T - z)(\alpha f_1(z) + \beta f_2(z)) = \alpha x_1 + \beta x_2 \quad \text{on } U_\lambda,$$

for every pair $\alpha, \beta \in \mathbb{C}$, where $(T - z)(f_j(z)) = x_j$ on U_λ for each $j = 1, 2$. This proves that $X_T(F)$ is a linear manifold in X.

To determine the hyperinvariance of $X_T(F)$, choose any operator $S \in \mathcal{B}(X)$ which commutes with T. Given any $x \in X$ and an analytic solution for $(T - z)f(z) = x$ on some open set $U \subseteq \mathbb{C}$, we have that

$$(T - z)Sf(z) = S(T - z)f(z) = Sx, \qquad \text{for all } z \in U.$$

Bearing in mind that $S \circ f$ remains holomorphic on U, we conclude that $\sigma_T(Sx) \subseteq \sigma_T(x)$. Hence, as required, $SX_T(F) \subseteq X_T(F)$. □

In particular, it should be tempting to construct hyperinvariant subspaces for a given operator T through its local spectral manifolds. Unfortunately, in general, local spectral manifolds $X_T(F)$ need not be norm-closed, even when the corresponding subset $F \subseteq \mathbb{C}$ is closed. Such a condition, which dates back to Dunford, has played a relevant role since the earliest developments in the subject:

Definition 1.19. *A bounded linear operator $T \in \mathcal{B}(X)$ on a complex Banach space X has* Dunford's property (C) *if the local spectral manifold $X_T(F)$ is norm-closed in X for every closed subset $F \subseteq \mathbb{C}$.*

In view of many concrete operators lacking "good" spectral properties, the SVEP may be considered a quite weak condition. In fact, all operators satisfying Dunford's property (C) share SVEP. Since the content of such result seems to be quite specific for the spirit of the Preliminaries, we postpone its proof to Chapter 4.

Proposition 1.20. *An operator $T \in \mathcal{B}(X)$ has SVEP precisely when $X_T(\emptyset) = \{0\}$, and this is the case if and only if $X_T(\emptyset)$ is norm-closed in X. In particular Dunford's property (C) implies SVEP.*

Another relevant property was introduced by Bishop [**26**] in his pioneering study of local spectral decompositions:

Definition 1.21. *A bounded linear operator $T \in \mathcal{B}(X)$ on a Banach space X has* Bishop's property (β) *whenever, for every open subset $U \subseteq \mathbb{C}$ and each sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U, X)$ such that*

$$(T - z)f_n(z) \to 0 \quad \text{as } n \to \infty$$

locally uniformly on U, it follows that $f_n \to 0$ as $n \to \infty$ again locally uniformly on U.

It is clear from Definition 1.21 that any restriction of an operator having property (β) must inherit such a property. Besides, as mentioned in the Introduction, an important characterization links property (β) with the action of T transferred to the Fréchet spaces $\mathcal{O}(U, X)$ described in (1.13):

Proposition 1.22. *A bounded linear operator $T \in \mathcal{B}(X)$ has property (β) if and only if, for all open subsets $U \subseteq \mathbb{C}$, the Fréchet space operator $T_U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ is injective and has closed range.*

Proof. Suppose that T has property (β). Picked any open $U \subseteq \mathbb{C}$ we need to show that the linear operator $T_U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ is injective and has closed range:

(i) First, by considering constant functions on $\mathcal{O}(U, X)$, we easily infer that T_U is an injective mapping.

(ii) Now, to check that $\operatorname{ran}(T_U)$ is closed in $\mathcal{O}(U, X)$, consider $T_U(f_n) \to g$ as $n \to \infty$. Clearly, since T_U is continuous, we need just to prove that $(f_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{O}(U, X)$.

To do so, suppose in the contrary that $(f_n)_{n\geq 1}$ is not a Cauchy sequence. This allows us to find a subsequence $(f_{n_k})_{k\geq 1}$ such that $h_k := f_{n_{k+1}} - f_{n_k}$ does not converge to 0 in the topology of $\mathcal{O}(U, X)$. But, on the other hand, we immediately have that $T_U(h_k) \to 0$ as $k \to \infty$ which, upon applying property (β), precisely says that $(h_k)_{k\geq 1}$ must converge to 0. Such a contradiction yields that $(f_n)_{n\geq 1}$ is a Cauchy sequence and, therefore, that T_U has closed range.

Conversely, suppose that T_U is injective with $\operatorname{ran}(T_U)$ closed in $\mathcal{O}(U, X)$ for every open set $U \subseteq \mathbb{C}$. Chosen an arbitrary open subset $U \subseteq \mathbb{C}$, the Open Mapping Principle ensures that T_U admits a continuous inverse, say S_U, on its range. So, if $T_U(f_n) \to 0$ as $n \to \infty$ in $\mathcal{O}(U, X)$, then, by continuity, $f_n = S_U T_U(f_n) \to 0$ again in $\mathcal{O}(U, X)$. This shows that T has property (β). □

In particular, from Proposition 1.22, we immediately conclude that property (β) implies the SVEP. Actually, next result asserts that we have much more. As before, we postpone the proof to Chapter 4:

Proposition 1.23. *Suppose that a Banach space operator $T \in \mathcal{B}(X)$ enjoys property (β). Then, T automatically has property (C).*

Nonetheless, there is an important question which has not been answered yet: which is the relation between all aforementioned local spectral properties and decomposability? Herein, we ensure that decomposability is precisely the strongest spectral property conceived up to this point (see, for example, [**2**, Theorem 6.17] or [**89**, Theorem 1.2.7]). Once again, we would prefer to wait until Chapter 4 for a formal proof.

Proposition 1.24. *Let $T \in \mathcal{B}(X)$ be a decomposable operator. Then, T has property (β).*

We collect all the implications amongst each of the aforementioned local spectral properties in the next theorem:

Theorem 1.25. *Let $T \in \mathcal{B}(X)$ be an arbitrary operator acting on a complex Banach space X. Then, the following implications always hold:*

(i) *If T is decomposable, then T verifies property (β).*
(ii) *If T has property (β), then T enjoys Dunford's property (C).*
(iii) *Finally, if T satisfies Dunford's property, then T has SVEP.*

We emphasize that, in general, due to the existence of certain counterexamples, neither of such implications can be reversed. However, a sort of reciprocal can be obtained when we demand additional assumptions (for a proof, we refer to [**2**, Theorem 6.19] and/or [**89**, Theorem 1.2.23]):

Proposition 1.26. *An operator $T \in \mathcal{B}(X)$ is decomposable if and only if it has property (C) and the sum decomposition*

$$X = X_T(\overline{U}) + X_T(\overline{V})$$

holds for every open covering $\{U, V\}$ of $\mathbb{C}$.

When we study spectral decompositions, certain problems may happen. Overall when the corresponding operator has not the SVEP and, therefore, its local spectral manifolds are not globally defined. To ease this constraint, we need a variant of the local spectral manifolds which suits better for operators without the SVEP.

Given an arbitrary Banach space operator $T \in \mathcal{B}(X)$ and a closed subset $F \subseteq \mathbb{C}$, we define the associated *glocal spectral manifold* as

$$\mathcal{X}_T(F) := \big\{x \in X \,:\, x \in T_{\mathbb{C}\setminus F}\big(\mathcal{O}(\mathbb{C} \setminus F, X)\big)\big\}.$$

In general, we have that $\mathcal{X}_T(F) \subseteq X_T(F)$ for each closed $F \subseteq \mathbb{C}$. Besides, observe that the equality precisely occurs when T has the SVEP.

This definition invites to consider the following decomposition property:

Definition 1.27. *We say that a continuous linear operator $T \in \mathcal{B}(X)$ has the* decomposition property (δ) *whenever*

$$X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$$

for each open cover $\{U, V\}$ of the complex plane $\mathbb{C}$.

Observe that, in contrast to property (β) which is transmitted via restrictions, it turns out that the quotient of an operator with property (δ) inherits again property (δ).

To avoid any misunderstanding, recall from the Introduction (see equation (0.4)) that property (δ) admitted a reformulation in terms of the surjectivity of the linear mappings

$$\begin{aligned} T^F : X &\longrightarrow \mathcal{O}(F, X)\Big/ T_F\, \mathcal{O}(F, X) \\ x &\longmapsto 1 \otimes x + T_F\, \mathcal{O}(F, X) \end{aligned}$$

defined for each closed set $F \subseteq \mathbb{C}$ (for a proof, see [**89**, Theorem 2.2.2]). This sheaf-theoretic description of property (δ), as well as the one given in Proposition 1.22 for property (β), are highly convenient to understand the duality occurring between them. All this will be commented in more detail further down.

Next theorem precisely states that decomposability may be understood as the conjunction of the two weaker properties (β) and (δ):

Theorem 1.28. *For any operator $T \in \mathcal{B}(X)$ on a Banach space X, the following conditions are equivalent:*

(i) *T is decomposable.*
(ii) *T satisfies both properties (C) and (δ).*
(iii) *T satisfies both properties (β) and (δ).*

We end this section with a complete sketch of the different local spectral properties along with their relations:

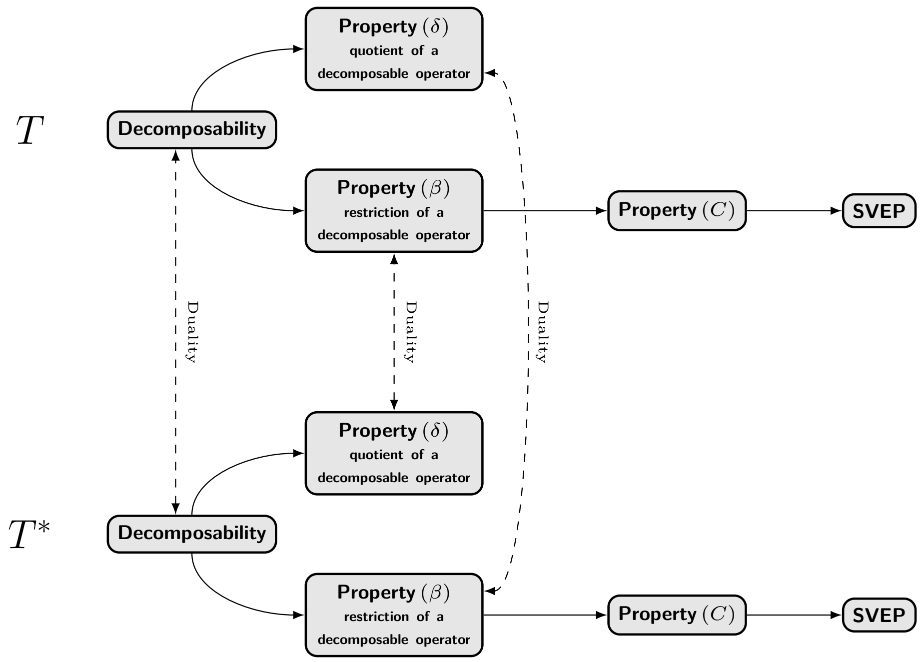

FIGURE 1. Map of local spectral properties and their duality correspondences

1.3. Beurling algebras and functional calculus

Along this section, we will describe in detail how the functional calculus from Beurling algebras can be implemented for the purpose of producing invariant subspaces. More precisely, we will focus our attention on a collection of related ideas, based on the properties of regular Beurling algebras. This sort of constructions date back to Wermer [**126**] and were lately refined by several authors such as Colojoară and Foiaş [**38**], who linked them to decomposable and generalized scalar operators; Beauzamy [**18**] and, especially, by Atzmon [**13, 15**] in the mid eighties.

As we will see later on, a classical theorem due to Atzmon (see [**15**, Theorem 1.1]) will be of particular interest for our goals, since it will turn out to be a useful machinery to produce invariant subspaces for many Bishop operators and Bishop-type operators.

1.3.1. Regularity in Beurling algebras

Given a two-sided weight sequence $\rho := (\rho_n)_{n\in\mathbb{Z}}$ lying in the real interval $[1, +\infty)$ with $\rho_0 = 1$ and such that

$$\rho_{m+n} \leq \rho_m \, \rho_n \quad \text{for every } m, n \in \mathbb{Z}, \tag{1.14}$$

$$\lim_{|n|\to\infty} \rho_n^{1/n} = 1; \tag{1.15}$$

we may consider its corresponding *Beurling algebra* $\mathcal{A}_\rho$ consisting of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ with norm defined by

$$\|f\|_\rho := \sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|\rho_n, \tag{1.16}$$

with the usual operations of pointwise addition and multiplication. In this case, the sequence of Fourier coefficients $\big(\widehat{f}(n)\big)_{n\in\mathbb{Z}}$ is given by the formula

$$\widehat{f}(n) := \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta} \, \frac{d\theta}{2\pi}.$$

Endowed with the norm $\|\cdot\|_\rho$ given by expression (1.16), $\mathcal{A}_\rho$ acquires structure of semi-simple commutative complex Banach algebra with unity and its Gelfand spectrum $\Delta(\mathcal{A}_\rho)$ can be identified with the unit circle via

$$\varphi_\theta(f) := f(e^{i\theta}) \qquad \text{for all } \theta \in [0, 2\pi) \text{ and } f \in \mathcal{A}_\rho.$$

In order to check that $(\mathcal{A}_\rho, \|\cdot\|_\rho)$ is actually a Banach algebra, one should apply the convolution formula for the Fourier series and the submultiplicativity condition (1.14):

$$\begin{aligned}
\|fg\|_\rho &= \sum_{n\in\mathbb{Z}} \big|\widehat{fg}(n)\big|\rho_n \leq \sum_{n\in\mathbb{Z}} \rho_n \sum_{m\in\mathbb{Z}} \big|\widehat{f}(m)\big| \, \big|\widehat{g}(n-m)\big| \\
&\leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \big|\widehat{f}(m)\big| \, \rho_m \, \big|\widehat{g}(n-m)\big| \, \rho_{n-m} \\
&= \Big(\sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|\rho_n\Big) \cdot \Big(\sum_{n\in\mathbb{Z}} \big|\widehat{g}(n)\big|\rho_n\Big) \\
&= \|f\|_\rho \cdot \|g\|_\rho.
\end{aligned}$$

Note that $\mathcal{A}_\rho$ is isometrically isomorphic to the weighted space $\ell^1(\mathbb{Z}, \rho)$ equipped with the convolution product given by

$$(x * y)_n := \sum_{m\in\mathbb{Z}} x_m \, y_{n-m}, \quad \text{for each } x, y \in \ell^1(\mathbb{Z}, \rho) \text{ and } n \in \mathbb{Z}.$$

Regarding Beurling algebras $\mathcal{A}_\rho$ as weighted $\ell^1(\mathbb{Z}, (\rho_n)_{n\in\mathbb{Z}})$ spaces may be useful, for instance, for identifying its dual space. Indeed, given $\varphi \in \mathcal{A}_\rho^*$ define

$$\widehat{\varphi}(n) := \langle e^{-int}, \varphi\rangle \quad (n \in \mathbb{Z}); \tag{1.17}$$

therefore, it is routine checking that the dual pairing is given by the formula

$$\langle f, \varphi \rangle = \Big\langle \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{int}, \varphi \Big\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \widehat{\varphi}(-n).$$

This shows that $\mathcal{A}_\rho^*$ is also isometrically isomorphic to the weighted space $\ell^\infty(\mathbb{Z}, (1/\rho_{-n})_{n\in\mathbb{Z}})$. Moreover, if we recall that a *hyperfunction* on $\mathbb{T}$ is an analytic function $\Phi \in \mathcal{H}(\mathbb{C}\setminus\mathbb{T})$ such that $\Phi(z) \to 0$ when $|z| \to \infty$, using the *Carleman transform*

$$\widetilde{\varphi}(z) := \begin{cases} \displaystyle\sum_{n\geq 1} \widehat{\varphi}(n) z^{n-1}, & \text{for } |z|<1; \\ \displaystyle -\sum_{n\leq 0} \widehat{\varphi}(n) z^{n-1}, & \text{for } |z|>1; \end{cases} \tag{1.18}$$

one can identify $\mathcal{A}_\rho^*$ with the set of hyperfunctions on $\mathbb{T}$ whose Fourier coefficients are in $\ell^\infty(\mathbb{Z}, (1/\rho_{-n})_{\mathbb{Z}})$.

Maybe, the best known example of a Beurling algebra is the usual *Wiener algebra*, frequently denoted by $\mathcal{A}(\mathbb{T})$, constituted by all the absolutely convergent Fourier series on the torus $\mathbb{T}$. To be specific, the Wiener algebra may be constructed as the Beurling algebra $\mathcal{A}_\rho$ associated to the constant sequence $\rho_n = 1$ for every $n \in \mathbb{Z}$.

As in the Wiener's $1/f$ theorem for $\mathcal{A}(\mathbb{T})$ [**127**] (see [**99**] for an elementary proof), one of the key facts regarding Beurling algebras $\mathcal{A}_\rho$ is that $f \in \mathcal{A}_\rho$ is invertible if and only if $f(e^{i\theta}) \neq 0$ for all $\theta \in [0, 2\pi)$. Furthermore, one of the most remarkable results in such context is the following sufficient criterion, which seems to date back to Beurling [**24**], to determine the regularity of the algebras $\mathcal{A}_\rho$. Recall that a function algebra $\mathcal{A}$ on a compact space K is said to be *regular* if for all $p \in K$ and all compact subset $M \subsetneq K$ with $p \notin M$, there exists $f \in \mathcal{A}$ such that $f(p) = 1$ and $f \equiv 0$ on M.

Theorem 1.29 (Beurling, [24]). *Let $\rho := (\rho_n)_{n\in\mathbb{Z}}$ be a real sequence satisfying both* (1.14) *and* (1.15)*. Then, the Banach algebra $\mathcal{A}_\rho$ is regular whenever*

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} < \infty. \tag{1.19}$$

For the sake of completeness, since regular Beurling algebras will be a crucial tool from now on, we will give a proof of Theorem 1.29 (see [**36**, Thm. 5.1.7] or [**38**, Thm. 2.12]):

Proof of Theorem 1.29. We must prove that given any fixed $e^{i\theta} \in \mathbb{T}$ and every $\varepsilon \in (0, \pi)$, there exists a function $f \in \mathcal{A}_\rho$ such that

$$f(e^{i\theta}) \neq 0 \quad \text{ and } \quad f(e^{it}) = 0 \;\text{ for all }\; |e^{i\theta} - e^{it}| \geq \varepsilon.$$

First, without loss of generality, we may assume that the sequence $\rho := (\rho_n)_{n\in\mathbb{Z}}$ has even symmetry; otherwise, we might proceed with the sequence $(\rho_n\, \rho_{-n})_{n\in\mathbb{Z}}$. Now, consider the even function $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ given by

$$\varphi(t) := \frac{1}{\rho_n(1+t^2)} \qquad \text{for } t \in \Big(n - \frac{1}{2}, n + \frac{1}{2}\Big) \qquad (n \in \mathbb{Z}).$$

It is quite straightforward to see that

$$\int_{-\infty}^{+\infty} \frac{|\log \varphi(t)|}{1+t^2}\, dt < \infty.$$

Then, a classical result of Paley and Wiener [**102**, Thm. XII] ensures the existence of a function $\psi \in L^2(\mathbb{R})$ supported on the half line $(-\infty, 0]$ with Fourier transform satisfying

$$\big|\widehat{\psi}(t)\big| = \varphi(t) \quad a.e. \text{ on } \mathbb{R}.$$

Moreover, note that ψ must be a non-vanishing continuous function since it is the Fourier transform of a non-zero $L^1(\mathbb{R})$ function.

Now, consider the product $g(t) := \psi(a-t)\,\psi(t-b)$ where $a, b \in \mathbb{R}$ satisfy $a < b$. It is plain to check that $g \in C(\mathbb{R})$ with $\operatorname{supp}(g) \subseteq [a, b]$. Moreover, by a suitable choice of a and b, we may ensure that $g(\theta) \neq 0$ but $g(t) = 0$ for all $|t - \theta| \geq \varepsilon$. Finally, we transfer this construction to the unit circle defining

$$f(e^{it}) := 2\pi g(t) \quad \text{ for } \; |\theta - t| < \pi.$$

It remains to show that f actually belongs to the Beurling algebra $\mathcal{A}_\rho$. To do so, we need to estimate its Fourier coefficients. First, note that, by going backwards in our construction, we have

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} f(e^{it})\, e^{-int}\, dt = \int_{-\infty}^{+\infty} g(t)\, e^{-int}\, dt = \int_{-\infty}^{+\infty} \psi(a-t)\,\psi(t-b)\, e^{-int}\, dt.$$

Whence, using the convolution formula and the translation properties of the Fourier transform, we may infer the identity

$$\widehat{f}(n) = e^{-ian}\int_{-\infty}^{+\infty} \widehat{\psi}(\xi - n)\,\widehat{\psi}(\xi)\, e^{-i(b-a)\xi}\, d\xi \qquad (n \in \mathbb{Z}).$$

Thus, taking absolute values and summing

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\,\rho_n &\leq \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} |\widehat{\psi}(\xi-n)|\,|\widehat{\psi}(\xi)|\,\rho_n\, d\xi \\
&\leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_{m-1/2}^{m+1/2} \frac{\rho_n\, d\xi}{\rho_{n-m}\rho_m\big(1+(n-\xi)^2\big)\big(1+\xi^2\big)} \\
&\leq \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \frac{d\xi}{\big(1+(n-\xi)^2\big)\big(1+\xi^2\big)} \\
&= \sum_{n\in\mathbb{Z}} \frac{2\pi}{n^2+4} < \infty.
\end{aligned}$$

Consequently, $f \in \mathcal{A}_\rho$ as required. □

Condition (1.19) is usually named as *Beurling condition* and it is closely related to the *Denjoy-Carleman Theorem* on quasi-analytic classes (see, for instance, [**116**, Chapter 19]). Likewise, similar results concerning regularity of allied function algebras were obtained by Shilov [**118**] applying similar tools. Keeping the previous result in mind, it is natural to consider the following important definition:

Definition 1.30. *A sequence of real numbers* $\rho := (\rho_n)_{n\in\mathbb{Z}}$ *such that* $\rho_0 = 1$ *and* $\rho_n \geq 1$ *for all* $n \in \mathbb{Z}$ *is called a* Beurling sequence *if*

$$\rho_{m+n} \leq \rho_m\,\rho_n \;\; (\forall\, m, n \in \mathbb{Z}) \quad \textit{and} \quad \sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} < \infty.$$

In particular, Beurling condition implies the asymptotic identities $\lim_{|n|\to\infty} \rho_n^{1/n} = 1$. This may be seen as follows: using the subadditivity of the sequence $(\log \rho_n)_{n\in\mathbb{Z}}$, as a consequence of *Fekete's Lemma* [**59**], we have that $\lim_{n\to\infty} \rho_n^{1/n} = 1$ is equivalent to

$$\lim_{n\to\infty} \frac{\log \rho_n}{n} = \inf_{n>0} \frac{\log \rho_n}{n} = 0.$$

Therefore, if we suppose that $\inf_{n>0}(\log \rho_n/n) = \delta > 0$, it is plain that

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} \geq \sum_{n>0} \frac{\log \rho_n}{n+n^2} \geq \delta \sum_{n>0} \frac{1}{n+1} = \infty,$$

and analogously when $n \to -\infty$. Thus, Beurling sequences $(\rho_n)_{n\in\mathbb{Z}}$ generate Beurling algebras $\mathcal{A}_\rho$; in other words, condition (1.15) would be redundant in Definition 1.30.

On the other hand, due to the divergence of the harmonic series, one may easily figure that Beurling sequences must obey a subexponencial growth when $|n| \to \infty$. However, there exist distinct remarkable examples of Beurling sequences. In fact, some of them will be of great interest in the results proved along Chapter 2 about invariant subspaces of Bishop operators. Below, we write down a few ones:

$$\rho_n = (1+|n|)^s \text{ for } s \geq 0, \qquad \rho_n = \exp\left(\frac{C\,|n|}{\log^\gamma(2+|n|)}\right) \text{ for } C>0 \text{ and } \gamma > 1,$$

$$\rho_n = \exp\left(|n|^\beta\right) \text{ for } 0 \leq \beta < 1, \qquad \rho_n = \exp\left(\frac{C\,|n|}{\log(2+|n|)(\log\log(5+|n|))^2}\right) \text{ for } C > 0.$$

1.3.2. The Theorems of Wermer and Atzmon

One advantage of regularity in a *function algebra* is that it enables to construct two non-zero functions whose product is identically zero; that is, regularity ensures the existence of *divisors of zero*. This idea, combined with a functional calculus argument, provides a powerful method for constructing invariant subspaces. Such a strategy, was initially studied by Wermer [**126**] for invertible operators:

Theorem 1.31 (Wermer, [126]). *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X with $\sigma(T) \subseteq \mathbb{T}$ satisfying*

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty. \tag{1.20}$$

Then, if $\sigma(T)$ is not reduced to a singleton, the operator T has a non-trivial closed hyperinvariant subspace in X.

Wermer's Theorem was the forerunner of ulterior investigations due to Colojoară and Foiaş on $\mathcal{A}$-scalar operators (see [**38**]). At this regard, we say that an algebra $\mathcal{A}$ of complex-valued functions on a complex subset $F \subseteq \mathbb{C}$ is *admissible* when $\mathcal{A}$ contains the restriction to F of all complex polynomials, and that, for every $f \in \mathcal{A}$ and $\lambda \in \mathbb{C}\setminus \operatorname{supp}(f)$, there exists another function $g \in \mathcal{A}$ such that

$$g(z) = \frac{f(z)}{z-\lambda}, \quad \text{for all } z \in F \setminus \{\lambda\}.$$

Definition 1.32. *Let $\mathcal{A}$ be an admissible algebra of complex-valued functions which owns partitions of unity. An operator $T \in \mathcal{B}(X)$ is said to be* $\mathcal{A}$-scalar *if it admits an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}(X)$ such that $\phi(1) = I$ and $\phi(Z) = T$.*

In a sense, $\mathcal{A}$-scalar operators were introduced by Colojoară and Foiaş as an attempt of approaching to decomposable operators using analytic techniques. Indeed, they actually showed that the splitting effected by the decomposability of $\mathcal{A}$-scalar operators comes implemented by the ranges of two suitably selected operators. To be specific, recall that a Banach space operator $T \in \mathcal{B}(X)$ is called *super-decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exists an operator $R \in \mathcal{B}(X)$ commuting with T for which

$$\sigma\big(T|\overline{R(X)}\big) \subseteq U \quad \text{and} \quad \sigma\big(T|\overline{(I-R)(X)}\big) \subseteq V.$$

Then, it may be seen that all $\mathcal{A}$-scalar operators are automatically super-decomposable. Evidently, super-decomposability is a more restrictive condition than decomposability.

In what Wermer's Theorem concerns, Colojoară and Foiaş (see [**38**, Chap. 5, Thm. 3.2]) realized that condition (1.20) was nothing but the specific implementation of regular Beurling algebras to the world of $\mathcal{A}$-scalar operators:

Theorem 1.33 (Colojoară-Foiaş, [38]). *Let $T \in \mathcal{B}(X)$ be an invertible operator on a complex Banach space X with $\sigma(T) \subseteq \mathbb{T}$ satisfying*

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty.$$

Then, T is a decomposable operator. In particular, if $\sigma(T)$ is not reduced to a singleton, the operator T has a non-trivial hyperinvariant subspace in X.

Since then, a number of theorems ensuring existence of hyperinvariant subspaces from boundedness conditions on the growth of the resolvent have been published occasionally. In general, such results usually required that $\sigma(T)$ was contained in a curve. This will force us to consider localized versions of Wermer's Theorem which are suitable for dealing with larger classes of operators. Such a clever trick traces back to the work of Davie [**44**] for the first time, although similar ideas had already been considered by several authors over those years, such as Sz.-Nagy and Foiaş [**97**, pg. 74] or Gellar and Herrero [**71**].

More recently, a noteworthy theorem due to Beauzamy [**18**] yields existence of non-trivial hyperinvariant subspaces for invertible operators $T \in \mathcal{B}(X)$ with $\|T\| = 1$ for which:

(i) There exists a non-zero vector $x_0 \in X$ such that the sequence $T^n x_0$ is not convergent to zero as $n \to +\infty$.
(ii) There exists a non-zero vector $y_0 \in X$ such that $\|T^{-n} y_0\|_X \le C\rho_n$ for an absolute constant $C > 0$ and some unilateral Beurling sequence $(\rho_n)_{n\ge 0}$.

Furthermore, Beauzamy proved that the hyperinvariant subspaces involved in his construction were of the form

$$\Big\{x \in X \,:\, T^n\Big(\sum_{k=-n}^{\infty} a_k T^{k+n}\Big)x = 0, \text{ as } n \to \infty\Big\},$$

where $(a_k)_{k\in\mathbb{Z}}$ is an appropriate sequence in $\ell^1(\mathbb{Z})$ (see also [**13**]). As an enlightening observation, note that for the usual bilateral shift on $\ell^2(\mathbb{Z})$, these subspaces precisely consist of sequences $(c_n)_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ for which the L^2-function $\sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ vanishes a.e. on a certain set of positive (arc length) measure in the unit circle.

Nevertheless, the strongest result at this regard was formulated by Atzmon [**15**, Thm. 1.1] in the mid-eighties. In order to state Atzmon's Theorem, we will say that a sequence $(\omega_n)_{n\in\mathbb{Z}}$ is *dominated* by another sequence $(\rho_n)_{n\in\mathbb{Z}}$ (both non-negative) if $\omega_n \leq C\,\rho_n$ for all $n \in \mathbb{Z}$ and some constant $C > 0$.

Theorem 1.34 (Atzmon, [15]). *Let $T \in \mathcal{B}(X)$ be an operator on a complex Banach space X and suppose there exist two sequences $(x_n)_{n\in\mathbb{Z}}$ in X and $(y_n)_{n\in\mathbb{Z}}$ in X^* such that $x_0 \neq 0$, $y_0 \neq 0$ and*

$$Tx_n = x_{n+1} \quad \text{and} \quad T^* y_n = y_{n+1} \qquad (\forall n \in \mathbb{Z}). \tag{1.21}$$

Suppose further that both sequences $(\|x_n\|_X)_{n\in\mathbb{Z}}$ and $(\|y_n\|_{X^*})_{n\in\mathbb{Z}}$ are dominated by a Beurling sequence, and there is at least a $\lambda \in \mathbb{T}$ at which the following functions G_x and G_y do not both possess analytic continuation into a neighbourhood of λ:

$$G_{x_0}(z) := \begin{cases} \sum_{n=1}^{\infty} x_{-n}\, z^{n-1} & \text{if } |z| < 1; \\ -\sum_{n=-\infty}^{0} x_{-n}\, z^{n-1} & \text{if } |z| > 1. \end{cases} \tag{1.22}$$

$$G_{y_0}(z) := \begin{cases} \sum_{n=1}^{\infty} y_{-n}\, z^{n-1} & \text{if } |z| < 1; \\ -\sum_{n=-\infty}^{0} y_{-n}\, z^{n-1} & \text{if } |z| > 1. \end{cases} \tag{1.23}$$

Then, either T is a multiple of the identity or it has a non-trivial closed hyperinvariant subspace in X.

Obviously, Atzmon's Theorem will always be applied to injective operators. In this context, $T^{-1}x_{n+1} = x_n$ will exactly mean that $Tx_n = x_{n+1}$, even though T is not invertible in $\mathcal{B}(X)$. Observe that, this equation makes sense for every $n \in \mathbb{Z}_+$ and each $x_0 \in X$ belonging to the *hyper-range* of the injective operator T:

$$T^{\infty}(X) := \bigcap_{n=0}^{\infty} \operatorname{ran}(T^n).$$

Observe also that the fact that the bilateral sequences $(\|x_n\|_X)_{n\in\mathbb{Z}}$ and $(\|x_n\|_X)_{n\in\mathbb{Z}}$ are dominated by Beurling sequences, ensures that the Laurent series defining G_{x_0} and G_{y_0} converge absolutely on $\mathbb{C}\setminus\mathbb{T}$. In addition, both G_{x_0} and G_{y_0} are analytic functions on $\mathbb{C}\setminus\mathbb{T}$ and at $\infty_{\mathbb{C}}$, and hence, by Liouville's Theorem on entire functions, each must possess at least one singularity on the unit circle.

1.4. Diophantine Approximation and metric properties of exceptional sets

Diophantine Approximation worries about the problem of determining the optimal rational approximants (or, a step further, algebraic approximants) of other irrational numbers. Accordingly, along the forthcoming pages, we will briefly introduce some of its most important tools and notions, focusing specially on continued fractions and Liouville numbers.

Our main references in this section shall be the broad scoped monograph by Bugeaud [**33**] and the Number Theory classic written by Hardy and Wright [**78**].

1.4.1. Approximation by rational numbers: continued fractions

Evidently, every real number ξ can be expressed in infinitely many ways as the limit of a sequence of rational numbers. Indeed, chosen any natural number $b \in \mathbb{N}$, there trivially exists another integer $a \in \mathbb{Z}$ for which $|\xi - a/b| \leq 1/2b$. However, one might hope to find infinitely many natural numbers $b \in \mathbb{N}$ such that $|\xi - a/b|$ is much smaller. With the aim of gauging the accuracy of a rational approximation, we need to compare the gap $|\xi - a/b|$ with an appropriate notion of size (or complexity) of the rational a/b. Following Dirichlet's lines, a suitable definition for this concept may be precisely its denominator:

Theorem 1.35 (Dirichlet's Theorem). *Given an irrational number $\xi \in \mathbb{R}$ there exist infinitely many irreducible rational numbers a/q such that*

$$\left| \xi - \frac{a}{q} \right| < \frac{1}{q^2}.$$

As it is well-known, a proof for *Dirichlet's Theorem* may be obtained as a standard application of the pigeon-hole principle, we refer to [**33**, Thm. 1.1] for minor details.

Remind that a complex number ξ is called *algebraic* whenever it is the root of a non-zero integer polynomial $P(z) \in \mathbb{Z}[z]$. Otherwise, $\xi \in \mathbb{C}$ is said to be a *transcendental* number. In 1844, Liouville was the first to prove the existence of transcendental numbers; moreover, he provided some explicit examples. His reasoning mainly relied on the estimation appearing in the next theorem:

Theorem 1.36 (Liouville's Theorem). *Let $\xi \in \mathbb{R}$ be a root of an irreducible integer polynomial $P(z) \in \mathbb{Z}[z]$ of degree $d \geq 2$. Then, there exists a constant $C_\xi > 0$ such that*

$$\left| \xi - \frac{a}{q} \right| \geq \frac{C_\xi}{q^d}$$

for each rational number a/q.

An immediate application of this result shows that the classical *Liouville constant*

$$\ell := \sum_{n=1}^{\infty} 10^{-n!}$$

is a transcendental number. More precisely, observing that ℓ is irrational (note that its decimal expansion is not ultimately periodic), for each integer $j \geq 2$, set $q_j = 10^{(j-1)!}$ and $a_j = q_j(10^{-1!} + \ldots + 10^{-(j-1)!})$. Then,

$$\left| \ell - \frac{a_j}{q_j} \right| = \sum_{n \geq j} 10^{-n!} \leq \frac{2}{10^{j!}} = \frac{2}{q_j^j}$$

and, consequently, as promised, ℓ is proven to be a transcendental number.

All our previous considerations lead us naturally to the following definition:

Definition 1.37. *The* index *of an irrational number* $\xi \in \mathbb{R}$ *is the quantity*

$$\iota(\xi) := \sup\left\{\omega > 0 \,:\, \textit{for every } C > 0 \textit{ there exists } a, q \in \mathbb{Z} \textit{ such that } \left|\xi - \frac{a}{q}\right| < \frac{C}{q^\omega}\right\}.$$

In particular, Liouville numbers *are defined as those irrationals* $\xi \in \mathbb{R}$ *having index* $\iota(\xi) = \infty$.

In a precise sense, observe that Liouville numbers are very well approximable by other rational numbers (maybe, in contrast to what our intuition might presume).

As a matter of fact, note that Liouville's Theorem can be restated saying that if $\iota(\xi) = \infty$, then ξ is automatically transcendental, i.e. all Liouville numbers are transcendental. Similarly, Dirichlet's Theorem may be rephrased saying that $\iota(\xi) \geq 2$ for all irrational ξ. On the other hand, just mention that a renowned theorem by Roth [**112**] asserts that $\iota(\xi) = 2$ for each irrational algebraic number ξ.

Regarding Diophantine Approximation, continued fractions arise as one of the most convenient tools to carry out a suitable analysis on irrational numbers. Recall that every irrational $\xi \in (0,1)$ can be associated to a unique infinite *continued fraction* of the form

$$\xi = \cfrac{1}{b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \dots}}}, \quad \text{where } b_0, b_1, b_2, \ldots \in \mathbb{N}. \tag{1.24}$$

This continued fraction representation corresponding to ξ can be computed via an infinite version of Euclid's Algorithm and codifies, in some sense, the distance of ξ to its closest rationals. To be precise, the following instructions allow one to calculate the continued fraction of ξ. Write $\frac{1}{\xi} = b_0 + \xi_0$ with b_0 integer and $0 < \xi_0 < 1$, namely

$$b_0 = \left\lfloor \frac{1}{\xi} \right\rfloor \quad \text{and} \quad \xi_0 = \left\{\frac{1}{\xi}\right\},$$

where, we recall that, the notations $\lfloor x \rfloor$ and $\{x\}$ stand for the *integer part* and *fractional part* of a real number x, respectively. Now, once again, proceed by splitting $1/\xi_0$ into the sum $\frac{1}{\xi_0} = b_1 + \xi_1$, where a_1 is an integer and $0 < \xi_1 < 1$, i.e.

$$b_1 = \left\lfloor \frac{1}{\xi_0} \right\rfloor \quad \text{and} \quad \xi_1 = \left\{\frac{1}{\xi_0}\right\}$$

and so on, determining by induction the infinite continued fraction representation (1.24).

Truncating the continued fraction at the j-th partial quotient, we obtain the sequence of *convergents* $(a_j/q_j)_{j\geq 0}$ of ξ:

$$\frac{a_j}{q_j} := \cfrac{1}{b_0 + \frac{1}{b_1 + \ldots + \frac{1}{b_j}}}, \qquad \text{for each } j \geq 0.$$

Broadly speaking, the convergents $(a_j/q_j)_{j\geq 0}$ of an irrational number ξ are precisely its best rational approximants. Indeed, one remarkable advantage in the usage of continuous fractions is that they allow to estimate accurately their speed of convergence. To do so, here below, we collect various useful identities. For instance, it is possible to write down explicit formulae for the convergents $(a_j/q_j)_{j\geq 0}$ exclusively in terms of the integer coefficients $(b_j)_{j\geq 0}$:

$$\frac{a_0}{q_0} = \frac{1}{b_0}, \qquad \frac{a_1}{q_1} = \frac{1}{b_0 + \frac{1}{b_1}} = \frac{b_1}{b_0 b_1 + 1}, \quad \ldots$$

More generally, it may be seen that the recurrence relations

$$a_j = b_j a_{j-1} + a_{j-2} \quad \text{and} \quad q_j = b_j q_{j-1} + q_{j-2} \tag{1.25}$$

hold for all integer $j \geq 2$. Moreover, by induction on the latter identity, we may deduce that

$$q_j a_{j-1} - a_j q_{j-1} = (-1)^{j-1}, \qquad \text{for each } j \in \mathbb{N}. \tag{1.26}$$

Next result is a slightly more precise variant of Dirichlet's Theorem. More concretely, it will provide an effective bound on how fast the sequence $(a_j/q_j)_{j\geq 0}$ converges to the irrational ξ in terms of consecutive convergents:

Proposition 1.38. *Given an irrational number $\xi \in \mathbb{R}$ with convergents $(a_j/q_j)_{j\geq 0}$, we have*

$$\frac{1}{q_j(q_j + q_{j+1})} < \left| \xi - \frac{a_j}{q_j} \right| < \frac{1}{q_j q_{j+1}}$$

for each integer $j \geq 0$.

Proof. As above, denote for every $j \geq 0$:

$$\xi = \cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}}} \qquad \text{and} \qquad \xi_j = \cfrac{1}{b_{j+1} + \cfrac{1}{b_{j+2} + \cdots}}.$$

Now, applying the identity

$$\xi = \frac{a_j/\xi_j + a_{j-1}}{q_j/\xi_j + q_{j-1}}, \qquad \text{for all } j \in \mathbb{N},$$

in conjunction with equation (1.26), we deduce the following equality:

$$\xi - \frac{a_j}{q_j} = \frac{(-1)^{j-1}}{q_j(q_j/\xi_j + q_{j-1})}, \qquad \text{for all } j \in \mathbb{N}.$$

To conclude, taking absolute values, the result follows immediately using the trivial inequality $b_{j+1} < \frac{1}{\xi_j} < b_{j+1} + 1$ and the recurrence relations (1.25). □

In this context, observe that the faster the denominators $(q_j)_{j\geq 0}$ grow to infinity, the closer the approximations $\left|\xi - a_j/q_j\right|$ turn out to be for each $j \geq 0$.

1.5. Weighted translation operators

As mentioned in the Introduction, our central subject in this dissertation is the class of weighted translation operators on L^p spaces. Specifically, we shall be particularly concerned about the existence of invariant and hyperinvariant subspaces. In order to accomplish such a task, along this section we collect several results (mainly due to Parrott [**103**], MacDonald [**91, 92**] and Chalendar, Partington and Pozzi [**35**]) regarding the spectrum and eigenvalues of some weighted translation operators. For instance, this analysis will enable us to deduce that

$$\sigma(T_\alpha) = \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| \leq e^{-1} \right\} \quad \text{and} \quad \sigma_{\mathrm{p}}(T_\alpha) = \emptyset$$

for each Bishop operator T_α acting on any $L^p[0,1)$ with $1 \leq p \leq \infty$.

In the sequel, let $(\Omega, \mathcal{G}, \mu)$ stand for an arbitrary measure space. Given an essentially bounded function $\phi \in L^\infty(\Omega, \mu)$ and a measurable transformation $\tau : \Omega \to \Omega$, the *weighted translation operator* $W_{\phi,\tau}$ is defined, on each $L^p(\Omega, \mu)$ for fixed $1 \leq p \leq \infty$, by the relation

$$\begin{aligned} W_{\phi,\tau} : L^p(\Omega, \mu) &\to L^p(\Omega, \mu) \\ f &\mapsto \phi \cdot (f \circ \tau). \end{aligned}$$

Obviously, as previously mentioned, $W_{\phi,\tau}$ is none other than the composition of two well-behaved operators. Namely, $W_{\phi,\tau} = M_\phi C_\tau$, where C_τ is a *composition operator*:

$$C_\tau f(\omega) := f(\tau(\omega)) \quad \text{for } f \in L^p(\Omega, \mu) \text{ and } \omega \in \Omega,$$

whilst M_ϕ is a *multiplication operator*

$$M_\phi f(\omega) := \phi(\omega) f(\omega) \quad \text{for } f \in L^p(\Omega, \mu) \text{ and } \omega \in \Omega.$$

Due to various technical reasons, from now on, the triple $(\Omega, \mathcal{G}, \mu)$ will always denote a non-atomic probability space arising from the Borel sets of a compact metrizable space. Naturally, the bulk of results outlined below could be adapted to finite measure spaces without difficulty (or even, with a little effort, to σ-finite spaces). However, for the sake of simplicity, we discard such a general point of view (see Parrott [**103**] for a discussion in that line).

Additionally, in the sequel, our probability space will be always equipped together with a probability space isomorphism $\tau : (\Omega, \mathcal{G}, \mu) \to (\Omega, \mathcal{G}, \mu)$, i.e. an invertible mapping $\tau : \Omega \to \Omega$ such that:

- Both τ and τ^{-1} are measurable;
- $\mu(\tau^n(E)) = \mu(E)$ for all $E \in \mathcal{G}$ and $n \in \mathbb{Z}$.

Hereinafter, this kind of settings $(\Omega, \mathcal{G}, \mu, \tau)$ will be referred to as *measure-preserving systems*.

Regarding the recurrence properties of a measure-preserving system $(\Omega, \mathcal{G}, \mu, \tau)$, we may distinguish the following cases:

(i) τ is *periodic* (with period $n \in \mathbb{N}$) if $\tau^n(E) = E$ for every $E \in \mathcal{G}$, and n is the smallest positive integer for which this holds.

(ii) τ is *aperiodic* if there is no measurable subset $E \in \mathcal{G}$, satisfying $\tau(E) = E$ and $\mu(E) > 0$, such that the restriction $\tau|_E$ is periodic.

Intuitively, a shift τ is aperiodic whenever, for every $n \in \mathbb{N}$,

$$\mu\big(\{\omega \in \Omega \,:\, \tau^n(\omega) = \omega\}\big) = 0;$$

and, clearly, a meaning in this sense can be attached to the concept of periodicity.

Our first result proves that we can find invariant subspaces for $W_{\phi,\tau}$ when it is possible to decompose the probability space $(\Omega, \mathcal{G}, \mu)$ into smaller pieces under the action of τ:

Proposition 1.39. *Let $W_{\phi,\tau}$ operate on $L^p(\Omega, \mu)$ for fixed $1 \leq p \leq \infty$. Suppose further that there is a measurable set $E \subseteq \Omega$, such that $\tau(E) = E$ and $0 < \mu(E) < 1$. Then,*

$$\mathcal{M}_E := \big\{ f \in L^p(\Omega, \mu) \,:\, \operatorname{supp}(f) \subseteq E \big\}$$

is a non-trivial closed invariant subspace for $W_{\phi,\tau}$.

Proof. Clearly, $\mathcal{M}_E$ is a norm-closed subspace of $L^p(\Omega,\mu)$. Additionally, $\mathcal{M}_E$ cannot be a trivial subspace since $0<\mu(E)<1$. Now, consider $f\in\mathcal{M}_E$, then

$$\operatorname{supp}(W_{\phi,\tau}f)\subseteq\tau^{-1}\big(\operatorname{supp}(f)\big)\subseteq\tau^{-1}(E)=E;$$

and, therefore, $W_{\phi,\tau}(\mathcal{M}_E)\subseteq\mathcal{M}_E$. □

Accordingly, *ergodicty* turns out to be a requirement for dealing with interesting examples (with regard to the existence of invariant subspaces) of weighted translation operators:

Definition 1.40. *Let $(\Omega,\mathcal{G},\mu,\tau)$ be a measure-preserving system. Then, τ is* ergodic *if any invariant set $E\subseteq\Omega$ (i.e. $\tau(E)=E$ μ-a.e.) has either full measure $\mu(E)=1$ or zero measure $\mu(E)=0$. In such a case, we will say that $(\Omega,\mathcal{G},\mu,\tau)$ is an* ergodic system.

Strictly speaking, ergodicity is a property inherent to the measure system $(\Omega,\mathcal{G},\mu,\tau)$. Nevertheless, we will sometimes commit an abuse of notation and apply the adjective "ergodic" to the transformation τ or even to a weighted translation operator $W_{\phi,\tau}$.

REMARK 1.41. Combining some well-known ergodic theorems (such as the *Individual Ergodic Theorem* or the *Mean ergodic theorem*), one can obtain various equivalent characterizations of ergodicity (see, for instance, [**122**, Theorem 2.9.7] or [**124**]):

(i) Any measurable function f such that $f\circ\tau=f$ μ-a.e., is constant μ-a.e on Ω.

(ii) For any $1<p<\infty$ and $f\in L^p(\Omega,\mu)$, the averages $\frac{1}{n}\sum_{j=0}^{n-1}(f\circ\tau^j)$ converge in L^p-norm to $\int_\Omega f\,d\mu$.

(iii) For each pair $f,g\in L^\infty(\Omega,\mu)$, the following limit holds:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_\Omega(f\circ\tau^j)\,g\,d\mu=\left(\int_\Omega f\,d\mu\right)\cdot\left(\int_\Omega g\,d\mu\right).$$

(iv) For any $f\in L^1(\Omega,\mu)$, the averages $\frac{1}{n}\sum_{j=0}^{n-1}(f\circ\tau^j)$ converge pointwise μ-a.e. to $\int_\Omega f\,d\mu$.

1.5.1. Spectrum of ergodic weighted translation operators

As one should expect, all the properties of a weighted translation operator $W_{\phi,\tau}$ depend heavily on the symbols ϕ and τ. Under our assumption that $(\Omega,\mathcal{G},\mu,\tau)$ is a measure-preserving system, one may easily check that the composition operator C_τ is a decomposable invertible isometry on each $L^p(\Omega,\mu)$ which enjoys

$$C_\tau^{-1}=C_{\tau^{-1}},\quad \sigma(C_\tau)\subseteq\mathbb{T}\quad\text{and}\quad\sigma_{\mathrm{p}}(C_\tau)\neq\emptyset.$$

Regarding its dual properties, the equation

$$\langle C_\tau f,g\rangle=\int_\Omega f(\tau(\omega))g(\omega)\,d\mu(\omega)=\int_\Omega f(\omega)g(\tau^{-1}(\omega))\,d\mu(\omega)=\langle f,C_{\tau^{-1}}g\rangle$$

exhibits that the adjoint of C_τ on $L^p(\Omega,\mu)$ for each $1\le p<\infty$ is precisely the composition operator $C_{\tau^{-1}}$ regarded on $L^q(\Omega,\mu)$ where $\frac{1}{p}+\frac{1}{q}=1$. Indeed, C_τ is always a unitary operator in $L^2(\Omega,\mu)$.

In parallel, the multiplication operator M_ϕ by an arbitrary $\phi \in L^\infty(\Omega,\mu)$ fulfils the spectral identities

$$\sigma(M_\phi) = \mu\text{-}\operatorname{ess\,ran}(\phi) \quad \text{and} \quad \sigma_{\mathrm{p}}(M_\phi) = \big\{\lambda \in \mathbb{C} \,:\, \mu\big(\phi^{-1}(\{\lambda\})\big) > 0\big\}$$

when acts on any space $L^p(\Omega,\mu)$. In particular, M_ϕ is invertible (with inverse $M_\phi^{-1} = M_{1/\phi}$) if and only if $0 \notin \mu\text{-}\operatorname{ess\,ran}(\phi)$. Moreover, both its norm and its spectral radius always coincides with the essential supremum

$$\|M_\phi\| = r(M_\phi) = \mu\text{-}\operatorname{ess\,sup}(\phi).$$

Besides, its adjoint can be also described with ease as follows:

$$\langle M_\phi f, g\rangle = \int_\Omega \phi(\omega) f(\omega) g(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\phi(\omega)g(\omega)\, d\mu(\omega) = \langle f, M_\phi g\rangle, \quad \text{whenever } p \neq 2.$$

We should remark that the situation slightly changes in the case $p = 2$, since the adjoint carries over the conjugacy; i.e., $M_\phi^* = M_{\bar\phi}$ on $L^2(\Omega,\mu)$.

A direct calculation shows the following intertwining relation between these operators:

$$M_\phi C_\tau = C_\tau M_{\phi\circ\tau^{-1}}. \tag{1.27}$$

Such identity can be used for various purposes regarding $W_{\phi,\tau}$. We enumerate some of them:

(i) Again, $W_{\phi,\tau}$ is invertible on each $L^p(\Omega,\mu)$ precisely when $0 \notin \mu\text{-}\operatorname{ess\,ran}(\phi)$. In that case, we have
$$W_{\phi,\tau}^{-1} = W_{1/(\phi\circ\tau^{-1}),\tau^{-1}}.$$

(ii) The iterates of $W_{\phi,\tau}$ are given by the formula
$$W_{\phi,\tau}^n f(\omega) = \phi(\omega)\cdot\phi(\tau(\omega))\cdots\phi(\tau^{n-1}(\omega))\cdot f(\tau^n(\omega)), \qquad \text{for all } n \geq 1.$$

(iii) Additionally, the backward iterates of $W_{\phi,\tau}$ are given by the formula
$$W_{\phi,\tau}^{-n} f(\omega) = \frac{f(\tau^{-n}(\omega))}{\phi(\tau^{-n}(\omega))\cdots\phi(\tau^{-1}(\omega))}, \qquad \text{for all } n \geq 1,$$
as long as $W_{\phi,\tau}$ is injective and $f \in \operatorname{ran}(W_{\phi,\tau}^n)$.

(iv) The adjoint of $W_{\phi,\tau} \in \mathcal{B}(L^p(\Omega,\mu))$ (with $1 \leq p < \infty$) is again a weighted translation of the form
$$W_{\phi,\tau}^* = W_{(\phi\circ\tau^{-1}),\tau^{-1}}.$$
acting on the dual space $L^q(\Omega,\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Once more, it ought to be underscored that the adjointness in $L^2(\Omega,\mu)$ should be treated separately.

Due to the rigidity imposed by ergodicity, there is a strong connexion between the eigenvalues of C_τ and those of the weighted translation operator $W_{\phi,\tau}$. This feature was initially pinpointed by Parrott [**103**, Ch. III] in his PhD thesis. For the sake of completeness, we include a simplified proof:

Theorem 1.42 (Parrott, [103]). *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator on $L^p(\Omega,\mu)$ for fixed $1 \leq p \leq \infty$. If $\lambda \in \sigma_{\mathrm{p}}(W_{\phi,\tau})$, then*

$$\sigma_{\mathrm{p}}(W_{\phi,\tau}) = \big\{c\lambda \,:\, c \in \sigma_{\mathrm{p}}(C_\tau)\big\}.$$

Proof. Depending on the measure of the zero set $Z_\phi := \{\omega \in \Omega \,:\, \phi(\omega) = 0\}$, we may encounter two differentiated cases:

(i) If $\mu(Z_\phi) > 0$, we claim that $\sigma_{\mathrm{p}}(W_{\phi,\tau}) = \{0\}$. To do so, first note that $W_{\phi,\tau}\mathbf{1}_{\tau(Z_\phi)} = 0$, where $\mathbf{1}_{\tau(Z_\phi)}$ denotes the characteristic function of the shift $\tau(Z_\phi)$. This shows that $0 \in \sigma_{\mathrm{p}}(W_{\phi,\tau})$.

Conversely, suppose for a contradiction that $\nu \in \sigma_{\mathrm{p}}(W_{\phi,\tau})$ is non-zero and let $f \in L^p(\Omega,\mu)$ be an arbitrary eigenvector, i.e.

$$W_{\phi,\tau} f = \nu f \quad \mu\text{-a.e.}$$

The latter equation implies that $f \equiv 0$ μ-a.e. on Z_ϕ. Accordingly, chosen $\omega \in \tau^{-1}(Z_\varphi)$, then $\tau(\omega) \in Z_\phi$ and the upper identity yields $f \equiv 0$ μ-a.e. on $\tau^{-1}(Z_\phi)$ as well. By induction, we see that $f \equiv 0$ on $\bigcup_{n\geq 0} \tau^{-n}(Z_\phi)$, but by the ergodicity of τ we conclude that this union fills the entire Ω.

(ii) If $\mu(Z_\phi) = 0$, consider any eigenvalue $c \in \sigma_{\mathrm{p}}(C_\tau)$ and a corresponding non-zero eigenvector $g \in L^p(\Omega,\mu)$. Since $\sigma(C_\tau) \subseteq \mathbb{T}$, note that c must be unimodular. Hence, invoking the equivalence (i) of ergodicity given above, we see that $|g|$ is constant. So, if $f \in L^p(\Omega,\mu)$ is a non-zero function such that $W_{\phi,\tau} f = \lambda f$, then

$$W_{\phi,\tau}(f\cdot g)(\omega) = \phi(\omega) f(\tau(\omega)) g(\tau(\omega)) = c\lambda f(\omega) g(\omega) \quad \mu\text{-a.e.}$$

Whence, we obtain the inclusion $\sigma_{\mathrm{p}}(W_{\phi,\tau}) \supseteq \{c\lambda : c \in \sigma_{\mathrm{p}}(C_\tau)\}$.

On the other hand, for any pair $\lambda_1, \lambda_2 \in \sigma_{\mathrm{p}}(W_{\phi,\tau})$, let f_1, f_2 be their respective eigenvectors. Clearly, $\lambda_j \neq 0$ for both $j = 1, 2$. Moreover, because of the ergodicity of τ, one can easily deduce that $f_j \neq 0$ μ-a.e. for each $j = 1, 2$. Now, since

$$\left(C_\tau \frac{f_1}{f_2}\right)(\omega) = \frac{f_1(\tau(\omega))}{f_2(\tau(\omega))} = \frac{\phi(\omega) f_1(\tau(\omega))}{\phi(\omega) f_2(\tau(\omega))} = \frac{\lambda_1}{\lambda_2}\frac{f_1(\omega)}{f_2(\omega)},$$

using again the equivalence (i) from the upper remark, we infer that f_1/f_2 is constant and that λ_1/λ_2 belongs to $\sigma_{\mathrm{p}}(C_\tau)$. This establishes the remaining inclusion.

□

As a consequence of the duality occurring between the compression spectrum and the point spectrum, any theorem concerning $\sigma_{\mathrm{p}}(W_{\phi,\tau})$ implies dual results about $\sigma_{\mathrm{com}}(W_{\phi,\tau})$:

Corollary 1.43 (Parrott, [103]). *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator on $L^p(\Omega,\mu)$ for fixed $1 \leq p < \infty$. If $\lambda \in \sigma_{\mathrm{com}}(W_{\phi,\tau})$, then*

$$\sigma_{\mathrm{com}}(W_{\phi,\tau}) = \{c\lambda : c \in \sigma_{\mathrm{p}}(C_\tau)\}.$$

Surely, the most remarkable corollary of these results is the fact that all the eigenvalues of $W_{\phi,\tau}$ (if any) have the same modulus (and, correspondingly, the same must occur for all the points belonging to $\sigma_{\mathrm{com}}(W_{\phi,\tau})$). Accordingly, since we obviously have $\mathrm{int}(\sigma_{\mathrm{p}}(W_{\phi,\tau})) = \emptyset$, this yields the SVEP for every ergodic weighted translation operator. Furthermore, recalling Proposition 1.17 one immediately concludes:

Theorem 1.44. *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator on $L^p(\Omega,\mu)$ for fixed $1 \leq p < \infty$. Then, the operator $W_{\phi,\tau}$ has the SVEP and*

$$\sigma(W_{\phi,\tau}) = \sigma_{\mathrm{ap}}(W_{\phi,\tau}) = \sigma_{\mathrm{su}}(W_{\phi,\tau}).$$

As usual, we will say that a set in the complex plane F is *circular symmetric* if $cF = F$ for every unimodular number $c \in \mathbb{C}$. Next result ensures that if the shift τ is ergodic (indeed, this hypothesis can be weakened with τ aperiodic), the approximate point spectrum $\sigma_{\mathrm{ap}}(W_{\phi,\tau})$ has circular symmetry. Correspondingly, by the identification given in Theorem 1.44, the spectrum

$\sigma(W_{\phi,\tau})$ consists of the union of closed disks and annuli (possibly degenerated) centered at the origin. To prove this, we need a previous lemma:

Lemma 1.45 (Parrott, [103]). *Let $(\Omega, \mathcal{G}, \mu, \tau)$ be an ergodic measure-preserving system and consider a measurable function $\psi : \Omega \to \mathbb{C}$ with $|\psi(\omega)| = 1$ μ-a.e. Then, for every $f \in L^1(\Omega, \mu)$, $\varepsilon > 0$ and $|c| = 1$, there exists a function g such that:*

(i) *$|g(\omega)| = 1$ for μ-a.e. $\omega \in \Omega$;*

(ii) *If $S := \big\{\omega \in \Omega \, : \, W_{\psi,\tau} g(\omega) \neq c\, g(\omega)\big\}$, then $\displaystyle\int_S |f(\omega)|\, d\mu(\omega) < \varepsilon$.*

Proof. Invoking the classical *Kakutani-Rokhlin Lemma* (see, for example, [**49**, Lemma 2.45]), we know that chosen any $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a measurable set $E \subseteq \Omega$ such that

- E, $\tau(E)$, $\dots$, $\tau^{n-1}(E)$ are pairwise disjoint sets.
- $\mu\big(E \cup \tau(E) \cup \ldots \cup \tau^{n-1}(E)\big) > 1 - \varepsilon$.

Now, we proceed to define the function g as follows:

$$g(\omega) := \frac{c^j}{\psi(\tau^{-1}(\omega)) \cdots \psi(\tau^{-j}(\omega))}, \quad \text{for each } \omega \in \tau^j(E) \;\; (\text{where } 1 \leq j \leq n-1)$$

and $g(\omega) := 1$ elsewhere. Obviously, as required in the statement (i), $|g(\omega)| = 1$ for μ-a.e. $\omega \in \Omega$. Additionally, given any $\omega \in \tau^j(E)$ for fixed $0 \leq j \leq n-2$, we have that

$$\psi(\omega) g(\tau(\omega)) = \psi(\omega) \frac{c^{j+1}}{\psi(\omega) \cdots \psi(\tau^{-j}(\omega))} = c \cdot \frac{c^j}{\psi(\tau^{-1}(\omega)) \cdots \psi(\tau^{-j}(\omega))} = c g(\omega).$$

Accordingly, the set $S = \big\{\omega \in \Omega \, : \, W_{\psi,\tau} g(\omega) \neq c\, g(\omega)\big\}$ is included in

$$S \subseteq \left(\Omega \setminus \bigcup_{j=0}^{n-2} \tau^j(E)\right) \subseteq \tau^{n-1}(E) \cup \left(\Omega \setminus \bigcup_{j=0}^{n-1} \tau^j(E)\right),$$

and, consequently, its measure can be bounded by $\mu(S) \leq 1/n + \varepsilon$. To conclude the proof, note that since $\mu(S)$ may be chosen as small as desired and f is a function in $L^1(\Omega, \mu)$, the integral $\int_S |f(\omega)|\, d\mu(\omega)$ can be made arbitrarily small. ☐

Proposition 1.46 (Parrott, [103]). *Let $W_{\phi,\tau}$ and $W_{\psi\cdot\phi,\tau}$ be weighted translation operators on $L^p(\Omega, \mu)$ for fixed $1 \leq p < \infty$, with τ ergodic and $|\psi(\omega)| = 1$ μ-a.e. Then,*

$$\sigma_{\mathrm{ap}}(W_{\phi,\tau}) = \sigma_{\mathrm{ap}}(W_{\psi\cdot\phi,\tau}).$$

In particular, $\sigma_{\mathrm{ap}}(W_{\phi,\tau})$ is always circular symmetric.

Remark 1.47. Note that the last assertion follows immediately upon choosing the constant function $\psi(\omega) := c$ with c unimodular:

$$\sigma_{\mathrm{ap}}(W_{\phi,\tau}) = \sigma_{\mathrm{ap}}(W_{c\cdot\phi,\tau}) = \sigma_{\mathrm{ap}}(c\, W_{\phi,\tau}) = c\, \sigma_{\mathrm{ap}}(W_{\phi,\tau}).$$

Proof of Proposition 1.46. Clearly, since $W_{\phi,\tau} = W_{\psi^{-1}(\psi\cdot\phi),\tau}$ and $|\psi^{-1}(\omega)| = 1$ for μ-a.e. $\omega \in \Omega$ as well, it suffices to be shown that

$$\sigma_{\mathrm{ap}}(W_{\phi,\tau}) \subseteq \sigma_{\mathrm{ap}}(W_{\psi\cdot\phi,\tau}). \tag{1.28}$$

To do so, choose any $\lambda \in \sigma_{\mathrm{ap}}(W_{\phi,\tau})$ and $\varepsilon > 0$ arbitrarily small. Since λ is an approximate eigenvalue for $W_{\phi,\tau}$, we can find a function $f \in L^p(\Omega, \mu)$ with $\|f\|_p = 1$ for which

$$\big\|(W_{\phi,\tau} - \lambda) f\big\|_p < \varepsilon/2.$$

Applying Lemma 1.45 to the function $(W_{\phi,\tau}f)^p \in L^1(\Omega,\mu)$ with $c = 1$ and the bound $\varepsilon^p/4^p$, we may infer the existence of a function g such that:

(i) $|g(\omega)| = 1$ for μ-a.e. $\omega \in \Omega$;

(ii) If $S := \big\{\omega \in \Omega : W_{\psi,\tau}g(\omega) \neq g(\omega)\big\}$, then $\displaystyle\int_S 2^p \left|W_{\phi,\tau}f(\omega)\right|^p d\mu(\omega) < \frac{\varepsilon^p}{2^p}$.

Now, we claim that the function $\omega \mapsto f(\omega)g(\omega)$ is a norm 1 approximate eigenvector (with respect to ε) for the operator $W_{\psi\cdot\phi,\tau} - \lambda$: first, a standard computation leads us to the identity

$$\big(W_{\psi\cdot\phi,\tau} - \lambda\big)(fg) = \big(W_{\psi,\tau}g - g\big) \cdot W_{\phi,\tau}f - g \cdot \big(W_{\phi,\tau}f - \lambda f\big).$$

Finally, an easy estimation shows

$$\begin{aligned}\big\|\big(W_{\psi\cdot\phi,\tau} - \lambda\big)(fg)\big\|_p &\leq \left(\int_\Omega \left|W_{\psi,\tau}g - g\right|^p \left|W_{\phi,\tau}f\right|^p d\mu\right)^{1/p} + \left(\int_\Omega \left|W_{\phi,\tau}f - \lambda f\right|^p d\mu\right)^{1/p} \\ &\leq \left(\int_S 2^p \left|W_{\phi,\tau}f\right|^p d\mu\right)^{1/p} + \big\|(W_{\phi,\tau} - \lambda)f\big\|_p < \varepsilon.\end{aligned}$$

This demonstrates that $\lambda \in \sigma_{\mathrm{ap}}(W_{\psi\cdot\phi,\tau})$. □

As shown by Parrott [**103**, Sect. II.6], one can go even further and prove that the spectrum of any ergodic weighted translation operator $\sigma(W_{\phi,\tau})$ always consists of precisely one single closed disk or annulus centered at the origin (in this spirit, the set containing only the point $\{0\}$ is counted as a degenerated disk, while any circle is conceived likewise as an annulus).

For future reference, we end this part by assembling all the preceding characterizations. Besides, we add a generalization of Theorem 1.44 including analogous identities for the essential spectra. In particular, our subsequent statement asserts that one cannot address the Invariant Subspace Problem in the context of weighted translation operators using techniques borrowed from Fredholm Theory (recall Proposition 1.11).

At some point in the forthcoming proof, we shall use the notion of semi-regular operator: recall that a Banach space operator $T \in \mathcal{B}(X)$ is said to be *semi-regular* if $\mathrm{ran}(T)$ is norm-closed in X and

$$\ker(T) \subseteq T^\infty(X).$$

Attached to this concept, we may focus on the *Apostol spectrum* (also known as *semi-regular spectrum* in the literature), which is the closed set of $\mathbb{C}$ defined by

$$\sigma_\gamma(T) := \big\{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-regular}\big\}.$$

Since for every $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, the operator $T - \lambda$ has closed range and trivial kernel, we deduce straightforwardly the inclusion $\sigma_\gamma(T) \subseteq \sigma_{\mathrm{ap}}(T)$. Moreover, any operator T having the SVEP automatically verifies $\sigma_\gamma(T) = \sigma_{\mathrm{ap}}(T)$ (see, for example, [**89**, Cor. 3.1.7]).

Theorem 1.48. *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator on $L^p(\Omega,\mu)$ for fixed $1 \leq p < \infty$. Then, $\sigma(W_{\phi,\tau})$ is a single disk or annulus centered at the origin. Moreover,*

$$\sigma(W_{\phi,\tau}) = \sigma_{\mathrm{ap}}(W_{\phi,\tau}) = \sigma_{\mathrm{su}}(W_{\phi,\tau}) = \sigma_\gamma(W_{\phi,\tau}) = \sigma_{\mathrm{e}}(W_{\phi,\tau}) = \sigma_{\mathrm{le}}(W_{\phi,\tau}) = \sigma_{\mathrm{re}}(W_{\phi,\tau}).$$

Proof. The former four equalities have been already established (see, Theorem 1.44 and the paragraph above). To prove the latter ones, it suffices to be shown that

$$\sigma_\gamma(W_{\phi,\tau}) \subseteq \sigma_{\mathrm{le}}(W_{\phi,\tau}) \cap \sigma_{\mathrm{re}}(W_{\phi,\tau}). \tag{1.29}$$

To that end, we invoke [**2**, Thm. 3.29] which ensures that all the cluster points of the Apostol spectrum $\sigma_\gamma(W_{\phi,\tau})$ belong to $\sigma_{\rm le}(W_{\phi,\tau})\cap\sigma_{\rm re}(W_{\phi,\tau})$. Due to its circular symmetry, it is obvious that each cluster point of $\sigma_\gamma(W_{\phi,\tau})$ must belong to itself. Clearly, this yields inclusion (1.29) as required. ☐

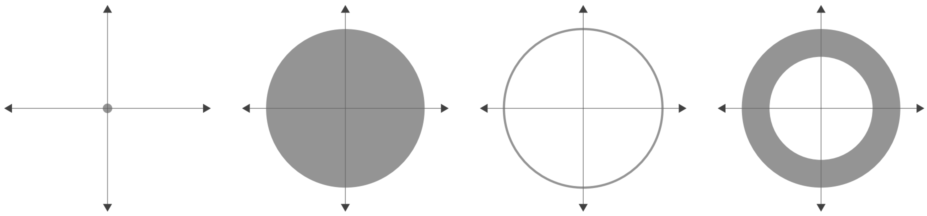

FIGURE 2. Admissible spectra for ergodic weighted translation operators

1.5.2. Unique ergodicity and irrational rotations. Bishop operators and Bishop-type operators

Some connections between topological and measure theoretic systems lead to a geometric reinterpretation of ergodicity using the theory of Banach spaces. At this point, we take a brief digression.

Recall that, given any locally compact Hausdorff space $\mathfrak{X}$, the *Riesz Representation Theorem* (see, for example, Rudin's book [**116**]) characterizes the dual space of $C_0(\mathfrak{X})$ as the Banach space of complex regular Borel measures $\mathcal{M}(\mathfrak{X})$ topologized with the *total variation norm* $\|\mu\|_{\mathcal{M}(\mathfrak{X})} := |\mu|(\mathfrak{X})$. In this one-to-one correspondence, defined by

$$\mu \mapsto \varphi_\mu(f) := \int_{\mathfrak{X}} f\,d\mu, \quad \forall f \in C_0(\mathfrak{X}),$$

it may be seen that positive linear functionals correspond to positive Borel measures.

Now, let τ be a homeomorphism on a compact metrizable space Ω. Fixed some $\omega \in \Omega$, the weak-star compactness of the unit ball in $\mathcal{M}(\Omega)$ (recall the Banach-Alaoglu Theorem) ensures that some subsequence $(\mu_{n_j})_{j\geq 0}$ of the probability measures

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{\tau^j(\omega)} \quad (n \in \mathbb{N}),$$

must converge in the weak-star topology to another Borel probability measure μ (here δ represents the *Dirac mass distribution*). Since, for each $f \in C(\Omega)$, we have the asymptotic relation

$$\int_\Omega (f\circ\tau)\,d\mu_{n_j} = \int_\Omega f\,d\mu_{n_j} + O_f\left(\frac{1}{n_j}\right) \quad \text{as } j \to \infty,$$

the Riesz Representation Theorem implies that the weak-star limit μ is a τ-invariant measure. In particular, this proves that the convex set

$$\mathcal{M}^\tau(\Omega) := \big\{\mu \in \mathcal{M}(\Omega) \,:\, \mu \text{ is a } \tau\text{-invariant Borel probability measure}\big\}$$

is always a non-empty weak-star closed subset of $\mathcal{M}(\Omega)$. Furthermore, a nice characterization identifies the ergodic measures with respect to τ as the extreme points of the convex set $\mathcal{M}^\tau(\Omega)$.

This much may be found in many standard references (see, Walters' book [**124**, Sect. 5.5] or the monograph by Einsiedler and Ward [**49**, Ch. 4]). Now, settled this context, we are in position to state the following definition:

Definition 1.49. *A homeomorphism τ on a compact metrizable space Ω is* uniquely ergodic *if there is only one τ-invariant Borel probability measure. In other words, τ is uniquely ergodic precisely when the convex set $\mathcal{M}^\tau(\Omega)$ consists of a single point.*

Of course, unique ergodicity is a stronger notion than ergodicity. As above, sometimes we shall commit the abuse of notation of adding the qualifier "uniquely ergodic" beyond its formal set-up. The next theorem reformulates unique ergodicity in terms of ergodic averages (for a proof, see [**124**, Th. 5.17]):

Theorem 1.50. *A homeomorphism τ on a compact metrizable space Ω is uniquely ergodic if and only if the averages*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\tau^j(\omega)) \longrightarrow \int_\Omega f\,d\mu$$

uniformly for all $f \in C(\Omega)$ as $n \to \infty$.

A combination of ideas due to Parrott [**103**, Sect. II.7] and MacDonald [**91**] uses Theorem 1.50 to determine exactly the spectrum of uniquely ergodic weighted translation operators. The initial argument passes through the intertwining relation (1.27) in order to prove that

$$\left\|W_{\phi,\tau}^n\right\|^{1/n} = \left\|M_{\phi\cdot(\phi\circ\tau)\cdots(\phi\circ\tau^{n-1})}\right\|^{1/n} \quad (n \in \mathbb{N}).$$

After taking logarithms and the limit as $n \to \infty$, one has

$$\log\big(r(W_{\phi,\tau})\big) = \lim_{n\to\infty}\left(\mu\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega}\ \frac{1}{n}\sum_{j=0}^{n-1}\log\left|\phi(\tau^j(\omega))\right|\right).$$

This immediately yields the inequality $r(W_{\phi,\tau}) \geq \exp\big(\int_\Omega \log|\phi|\,d\mu\big)$ as an application of the Individual Ergodic Theorem. Conversely, the upper bound $\leq$ may be obtained thank to the uniform convergence arising from Theorem 1.50. To be specific, one can deduce the next result upon reminding the circular symmetry of the spectrum:

Theorem 1.51 (MacDonald [91], Parrott [103]). *Let $W_{\phi,\tau}$ be a weighted translation operator on $L^p(\Omega,\mu)$ for some $1 \leq p \leq \infty$, with τ uniquely ergodic and ϕ continuous μ-a.e. Then:*

- *If $0 \in \mu\text{-}\operatorname{ess\,ran}(\phi)$, the spectrum of $W_{\phi,\tau}$ is*

$$\sigma(W_{\phi,\tau}) = \left\{\lambda \in \mathbb{C} \,:\, |\lambda| \leq \exp\left(\int_\Omega \log|\phi|\,d\mu\right)\right\}.$$

- *Otherwise, the spectrum of $W_{\phi,\tau}$ is*

$$\sigma(W_{\phi,\tau}) = \left\{\lambda \in \mathbb{C} \,:\, |\lambda| = \exp\left(\int_\Omega \log|\phi|\,d\mu\right)\right\}.$$

Undoubtedly, the premier examples of unique ergodicity are the equivalent systems of *irrational rotations/translations*. Let $\alpha \in (0,1)$ be an arbitrary irrational number, then:

- The *irrational rotation* $\varrho_\alpha(e^{2\pi i\theta}) := e^{2\pi i(\theta+\alpha)}$ is uniquely ergodic on the torus $\mathbb{T}$ with respect to the normalized Lebesgue measure.
- The *irrational translation* $\tau_\alpha(t) := \{t+\alpha\}$ (where recall that $\{\cdot\}$ stands for the fractional part) is uniquely ergodic on $[0,1) \cong \mathbb{R}/\mathbb{Z}$ with respect to the Lebesgue measure.

As mentioned in the Introduction, both *Bishop operators* and *Bishop-type operators* arise as the weighted translation operators corresponding to irrational translations τ_α:

Definition 1.52. *Let $\alpha \in (0,1)$ be an irrational number and $1 \leq p \leq \infty$. Then:*

- *The* Bishop operator T_α *is defined on $L^p[0,1)$ as the weighted translation operator given by*
$$T_\alpha f(t) := t\, f(\{t+\alpha\}), \qquad t \in [0,1).$$
- *For fixed $\phi \in L^\infty[0,1)$, the* Bishop-type operator $W_{\phi,\alpha}$ *acts on $L^p[0,1)$ as the weighted translation operator defined by*
$$W_{\phi,\alpha} f(t) := \phi(t)\, f(\{t+\alpha\}), \qquad t \in [0,1).$$

For instance, we point out that a direct application of Theorem 1.51 exhibits that the spectrum of each Bishop operator is always the closed disk
$$\sigma(T_\alpha) = \left\{\lambda \in \mathbb{C} \,:\, |\lambda| \leq e^{-1}\right\}.$$

Following Atzmon's/Wermer's theorems approach, several authors have established the existence of non-trivial hyperinvariant subspaces for certain Bishop operators T_α and Bishop-type operators $W_{\phi,\alpha}$. As general rule, two parameters appear to be crucial for a successful implementation of these techniques: the smoothness of the weight ϕ and the Diophantine "nature" of the irrational α.

Part II

Main contributions

Chapter 2
Invariant subspaces of Bishop operators

The present chapter is mainly inspired on the article [**37**], co-authored with Fernando Chamizo, Eva A. Gallardo-Gutiérrez and Adrián Ubis. Herein, we shall expose original contributions regarding the existence of invariant subspaces for Bishop operators. Let us just remark that, up to date, our line of research has led to the furthest achievements in this question.

The current chapter is organized as follows: at the beginning, we demonstrate that every Bishop operator T_α (and, more generally, every ergodic weighted translation operator) is biquasitriangular when acts on an L^2-space. Along Section 2.2, we illustrate how the functional calculus provided by Atzmon [**15**] can be used to produce invariant subspaces for some T_α. Our initial purpose is to present simplified proofs of those previous results due to Davie [**44**] and Flattot [**60**], indicating the key points involved in their approaches with the aim of strengthening them during our forthcoming extension. As we will see, the success of these techniques will depend strongly on the Diophantine properties of the irrational number α. Thereupon, in Section 2.3, by developing a battery of finer arithmetical devices, we are able to obtain sharper asymptotic estimations required for a broader implementation of Atzmon's Theorem. This will enable us to extend considerably the class of irrationals $\alpha \in (0,1)$ for which the corresponding Bishop operator T_α possesses non-trivial closed hyperinvariant subspaces.

Concretely, using the language of continued fractions, the main result of this chapter grants the existence of invariant subspaces for T_α up to the following condition.

Theorem. *Let $\alpha \in (0,1)$ be any irrational and $(a_j/q_j)_{j\geq 0}$ the convergents in its continued fraction. If the following condition holds:*

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right) \qquad \text{as } j \to +\infty.$$

Then, the Bishop operator T_α has non-trivial closed hyperinvariant subspaces on each $L^p[0,1)$ space for $1 \leq p < \infty$.

Apart from being an extension of Flattot's/Davie's methods, the importance of our statement should be understood in conjunction with the phenomena described in Chapter 3. Therein, we essentially prove that when our approach fails to produce invariant subspaces for Bishop operators, it is actually because Atzmon's Theorem no longer applies.

At the end of the chapter, despite of the fact that *Jarník-Besicovitch Theorem* asserts that Liouville numbers form a set of vanishing Hausdorff dimension (see Theorem 2.18 below), we will see that it is still possible to measure the size of the sets of exceptions uncovered by

Davie's, Flattot's and our approach. To this end, we will employ the theory of generalized Hausdorff dimensions with the totally ordered family of functions $(|\log t|^{-s})_{s\geq 0}$.

Before starting, we restate some already established features in the context of Bishop operators. According to Section 1.5, for each irrational number $\alpha \in (0,1)$, the corresponding Bishop operator T_α on $L^p[0,1)$ (for some $1 \leq p < \infty$) is a non-invertible weighted translation operator, determined by the uniquely ergodic mapping $\tau_\alpha(t) = \{t+\alpha\}$. Correspondingly, as claimed above, the application of Theorem 1.51 immediately ensures that

$$\sigma(T_\alpha) = \big\{\lambda \in \mathbb{C} : |\lambda| \leq e^{-1}\big\}.$$

In addition, Theorem 1.48 asserts that $\sigma(T_\alpha)$ coincides with the approximate point spectrum, the surjectivity spectrum, the Apostol spectrum as well as with each of the essential spectra. Bearing in mind that T_α lacks of point spectrum, we conclude that the inspection of the different spectral parts of T_α does not furnish any valuable information about its lattice of invariant subspaces. In particular, T_α is always injective and has dense range. On the other hand, it is evident that:

(i) The forward iterates of T_α are given by the formula
$$T_\alpha^n f(t) = t\cdot\{t+\alpha\}\cdots\{t+(n-1)\alpha\}\cdot f(\{t+n\alpha\}), \quad (n\in\mathbb{N}).$$

(ii) In a similar way, the backward iterates of T_α can be easily computed:
$$T_\alpha^{-n} f(t) = \frac{f(\{t-n\alpha\})}{\{t-n\alpha\}\cdots\{t-\alpha\}\cdot t}, \quad (n\in\mathbb{N})$$
whenever $f \in \operatorname{ran}(T_\alpha^n)$.

(iii) The adjoint of $T_\alpha \in \mathcal{B}(L^p[0,1))$ (for each $1\leq p<\infty$) is the operator
$$T_\alpha^* g(t) := \{t-\alpha\}\cdot f(\{t-\alpha\}), \qquad t\in[0,1)$$
acting on the dual space $L^q[0,1)$ where $\frac{1}{p}+\frac{1}{q}=1$. Once more, note that T_α^* is a weighted translation operator determined by a uniquely ergodic shift.

(iv) Repeating similar calculations, one has that
$$T_\alpha^{*\,n} g(t) = \{t-n\alpha\}\cdots\{t-\alpha\}\cdot g(\{t-n\alpha\}), \quad (n\in\mathbb{N}),$$
while
$$T_\alpha^{*\,-n} g(t) = \frac{g(\{t+n\alpha\})}{t\cdot\{t+\alpha\}\cdots\{t+(n-1)\alpha\}}, \quad (n\in\mathbb{N})$$
for each $g\in\operatorname{ran}(T_\alpha^{*\,n})$.

All the preceding facts will be of great importance and used repeatedly along the rest of the manuscript without reference.

2.1. Quasitriangular Bishop operators

This short section is devoted to the study of quasitriangularity in the frameworks of Bishop operators and ergodic weighted translation operators. As one may expect, the spectral identities occurring for such operators shall be determinant in this part. Subsequently, certain applications concerning approximation properties in Hilbert spaces will be discussed.

Our first theorem asserts that all Bishop operators are biquasitriangular in $L^2[0,1)$. Unfortunately, this result prevents us from using any argument involving quasitriangularity for the purpose of finding invariant subspaces for Bishop operators.

Theorem 2.1. *For every irrational $\alpha \in (0,1)$, the Bishop operator T_α on $L^2[0,1)$ is biquasitriangular.*

Proof. Let $\lambda \in \mathbb{C}$ such that $T_\alpha - \lambda$ is semi-Fredholm. Thank to equality between $\sigma(T_\alpha)$, $\sigma_{\mathrm{le}}(T_\alpha)$ and $\sigma_{\mathrm{re}}(T_\alpha)$, this is equivalent to

$$\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{le}}(T_\alpha) \cap \sigma_{\mathrm{re}}(T_\alpha) = \rho(T_\alpha).$$

In particular, both $\ker(T_\alpha - \lambda)$ and $\ker(T_\alpha^* - \overline{\lambda})$ are zero. Hence, we have

$$\operatorname{ind}(T_\alpha - \lambda) = \dim\big(\ker(T_\alpha - \lambda)\big) - \dim\big(\ker(T_\alpha - \lambda)^*\big) = 0.$$

Finally, according to Theorem 1.12, we conclude that T_α is a quasitriangular operator. An analogous argument shows the quasitriangularity of the adjoint T_α^* on $L^2[0,1)$. Consequently, the theorem is now proved. □

Although it might be slightly simplified, our preceding proof has the great advantage that can be easily extrapolated to a much more general setting. Indeed, the same argument may be reproduced verbatim for the next statement:

Corollary 2.2. *Consider an ergodic system $(\Omega, \mathcal{G}, \mu, \tau)$ and $\phi \in L^\infty(\Omega,\mu)$. Then, the weighted translation operator $W_{\phi,\tau}$ acting on $L^2(\Omega,\mu)$ is biquasitriangular.*

A few consequences may be derived in terms of approximation by linear bounded operators. For instance, an equivalent condition appearing in [**79**, Theorem 6.15] ensures that every biquasitriangular operator is the norm-limit of algebraic operators. Recall that an operator T is called *algebraic* if there exists a complex polynomial p such that $p(T)$ is the zero operator. Clearly, algebraic operators have non-trivial closed hyperinvariant subspaces.

Accordingly, one has straightforwardly that, for every irrational $\alpha \in (0,1)$, the operator T_α is the norm-limit of algebraic operators. At this regard, it is worth pointing out that indeed, each Bishop operator T_α is norm-limit of *nilpotent operators* in $L^p[0,1)$ for all $1 \le p \le \infty$. Namely, for any positive integer $n \in \mathbb{N}$, set $\phi_n(t) = t \cdot 1_{[1/n,1)}(t)$ for $t \in [0,1)$ and consider the Bishop-type operator $W_{\phi_n,\alpha}$ defined by

$$W_{\phi_n,\alpha} f(t) = \phi_n(t) f(\{t+\alpha\}), \qquad t \in [0,1),$$

for $f \in L^p[0,1)$. Obviously, one has $\|T_\alpha - W_{\phi_n,\alpha}\|_{L^p \to L^p} = 1/n$ for each $1 \le p \le \infty$. Hence, $(W_{\phi_n,\alpha})_{n\ge 1}$ are linear bounded operators in $L^p[0,1)$ converging in norm to T_α. Moreover, having in mind that τ_α is an ergodic transformation, one deduces that $W_{\phi_n,\alpha}$ is nilpotent for every $n \ge 1$.

In the same way, for an ergodic weighted translation operator $W_{\phi,\tau}$ on $L^2(\Omega,\mu)$, one immediately has that there exists a sequence $(T_n)_{n\ge1}$ of algebraic operators which converges strongly to $W_{\phi,\tau}$. Furthermore, it is possible to give a full characterization of those ergodic weighted translation operators which are approximable in norm by nilpotent operators. For that purpose, we need to invoke a theorem by Apostol, Foiaş and Voiculescu [**10**] which states that a linear bounded operator T is the norm-limit of nilpotent operators precisely when it is biquasitriangular and both its spectrum and essential spectrum are connected and contain the point 0.

Theorem 2.3. *Suppose that $W_{\phi,\tau}$ is an ergodic weighted translation operator on $L^2(\Omega,\mu)$. Then, $W_{\phi,\tau}$ is the norm-limit of nilpotent operators if and only it if is non-invertible.*

Proof. First, suppose that $W_{\phi,\tau}$ is the norm-limit of nilpotent operators. Now, using the result of Apostol, Foiaş and Voiculescu [**10**], we have that 0 belongs to $\sigma(W_{\phi,\tau})$. This precisely means that $W_{\phi,\tau}$ is non-invertible.

Conversely, now assume that $W_{\phi,\tau}$ is non-invertible. By means of the descriptions of the spectrum provided in Theorem 1.48, we know that

$$\sigma(W_{\phi,\tau}) = \sigma_{\mathrm{e}}(W_{\phi,\tau}) = \big\{\lambda \in \mathbb{C} : |\lambda| \leq r\big\} \quad \text{for some } 0 \leq r < \infty.$$

Clearly, the theorem follows immediately upon applying the characterization of Apostol, Foiaş and Voiculescu [**10**] in the opposite direction. □

2.2. Bishop operators T_α with invariant subspaces: the results of Davie and Flattot

As it was previously mentioned, many authors have tried the problem of seeking non-trivial invariant subspaces for Bishop operators and Bishop-type operators. Ignoring the technical dissimilarities amongst each of the approaches within the literature, all of them rely essentially on the same idea: the functional calculus based on regular Beurling algebras formalized by Atzmon [**15**, Thm. 1.1] in the mid eighties.

This was pursued for the first time by Davie [**44**], and refined some years later by MacDonald [**91, 92**] and Flattot [**60**]. In order to state it, we recall the notion of ρ-regularity, introduced in the work of Flattot:

Definition 2.4. *Let $\rho = (\rho_n)_{n\in\mathbb{Z}}$ be a Beurling sequence. An irrational number α is said to be ρ-*regular *if there exists $n_0 \in \mathbb{N}$ and a pair of functions $h_1, h_2 : \mathbb{N} \to \mathbb{R}$ verifying*

$$\frac{h_1(n)\,\log \rho_n}{n\,\log n} \to \infty \quad \textit{and} \quad \frac{h_2(n)\,\log n}{\log \rho_n} \to 0 \quad \textit{as } n \to \infty,$$

such that, for all $n > n_0$, there exists $p, q \in \mathbb{N}$, with $\gcd(p,q) = 1$, satisfying

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2} \quad \textit{and} \quad h_1(n) \leq q \leq h_2(n).$$

The keypoint of this definition is that, whenever $\alpha \in (0,1)$ is a ρ-regular number for certain Beurling sequence $\rho = (\rho_n)_{n\in\mathbb{Z}}$, we may obtain the required bounds in the hypothesis of Atzmon's theorem for the operator T_α. For instance, a proof of this fact may be read between lines in [**36**, Thm. 5.3.3]. Using this idea, Davie [**44**] proved his results picking the Beurling sequence $\rho_n = e^{|n|^\beta}$ for $n \in \mathbb{Z}$ with $1/2 < \beta < 1$ and the pair of functions

$$h_1(n) = n^\gamma \text{ with } 0 < \gamma < 1/2 \quad \text{and} \quad h_2(n) = \sqrt{n},$$

which characterized the non-Liouville numbers. On the other hand, Flattot [**60**, Thm. 4.6] was able to extend the techniques of Davie to a larger class of irrationals (incluiding some Liouville numbers) using the Beurling sequence $\rho_n = \exp\big(|n|/\sqrt{(\log|n|)^{2+\varepsilon}}\big)$ for every $n \in \mathbb{Z} \setminus \{0\}$ with $\varepsilon > 0$ arbitrarily small and the two functions

$$h_1(n) = (\log n)^{2+\varepsilon} \quad \text{and} \quad h_2(n) = \frac{n}{(\log n)^{2+\varepsilon}}.$$

Nevertheless, due to arithmetical issues, instead of using this terminology of ρ-regularity, we find more comfortable to work with continued fractions (recall Section 1.4). For instance, speaking in the language of continued fractions, if $(a_j/q_j)_{j\geq 0}$ denote the convergents of α, the threshold of Flattot's result (see [**60**, Remark 5.4]) grants the existence of non-trivial invariant subspaces for T_α when

$$\log q_{j+1} = O\big(q_j^{1/2-\varepsilon}\big) \quad \text{for some } \varepsilon > 0 \text{ as } j \to +\infty. \tag{2.1}$$

As an anecdote, note that condition (2.1) holds for instance for the classical Liouville number $\ell := \sum_{n=1}^{\infty} 10^{-n!}$. To see this, recall from the discussion held after Theorem 1.36, that the denominator and numerator of the convergents of the Liouville constant ℓ are respectively

$$q_j = 10^{(j-1)!} \quad \text{and} \quad a_j = q_j\big(10^{-1!} + \ldots + 10^{-(j-1)!}\big) \quad \text{for each integer } j \geq 2.$$

Now, obviously, we have the required asymptotic inequality

$$\log q_{j+1} \ll j! \ll \big(10^{(j-1)!}\big)^{1/2-\varepsilon} = O\big(q_j^{1/2-\varepsilon}\big) \quad \text{as } j \to \infty$$

for each $0 \leq \varepsilon < 1/2$. In the same fashion, Flattot studied the existence of invariant subspaces for the Bishop operators T_α defined upon Liouville numbers α of the form

$$\alpha = \sum_{n=0}^{\infty} b^{-u_n},$$

where $b \geq 2$ is some integer and $(u_n)_{n\geq 0}$ is an increasing sequence of positive integers going to infinity not too fast (see [**60**, Prop. 5.7] for explicit details).

2.2.1. A simplified proof of Davie's and Flattot's theorems

Before proving our main contributions in this line, we consider a short derivation of the results of Davie and Flattot, which highlights all the arithmetical considerations encapsulated in the Banach algebra arguments and may give some insight into the problem. In particular, it constitutes a significant simplification of the Theorem 4.6, proved by Flattot in [**60**].

Accordingly, the main goal of this subsection will be providing a careful approach to those irrationals in order to apply Atzmon's Theorem. Due to technical reasons, in the sequel, we will work with the scalar multiples of the Bishop operators T_α given by

$$\widetilde{T}_\alpha := e\, T_\alpha.$$

This choice, far from being arbitrary, is determined by the fact that $\sigma(\widetilde{T}_\alpha) = \overline{\mathbb{D}}$, which can be deduced from the Spectral Mapping Theorem (see Theorem 1.5). Of course, acting on the same L^p-space, both operators T_α and $\widetilde{T}_\alpha$ share the same lattice of invariant subspaces.

Recall that, along the construction of Atzmon's Theorem, one came across two analytic functions on $\mathbb{C} \setminus \mathbb{T}$ having singularities in the unit circle. According to the hypothesis of this result, it is necessary that such sets of singularities do not coincide in a singleton. At this regard, as observed by MacDonald [**91**, Claim pp. 307], one can ignore this requirement in the case of Bishop operators. To see that, we need the following lemma:

Lemma 2.5. *Let $\alpha \in (0,1)$ be an irrational number and T_α the corresponding Bishop operator acting on $L^p[0,1)$. Suppose further that, for some $f_0 \in L^p[0,1)$, there exists a Beurling sequence $\rho = (\rho_n)_{n\in\mathbb{Z}}$ which dominates $\big\|\widetilde{T}_\alpha^n f_0\big\|_p$ and consider the analytic functions on $\mathbb{C} \setminus \mathbb{T}$*

$$G_{f_0}(z) := \begin{cases} \sum_{n=1}^{\infty} (\widetilde{T}_\alpha^{-n} f_0)\, z^{n-1} & \text{if } |z| < 1; \\ -\sum_{n=-\infty}^{0} (\widetilde{T}_\alpha^{-n} f_0)\, z^{n-1} & \text{if } |z| > 1. \end{cases}$$

$$G_{e^{2\pi it}f_0}(z) := \begin{cases} \sum_{n=1}^{\infty} (\widetilde{T}_\alpha^{-n} e^{2\pi it} f_0)\, z^{n-1} & \text{if } |z| < 1; \\ -\sum_{n=-\infty}^{0} (\widetilde{T}_\alpha^{-n} e^{2\pi it} f_0)\, z^{n-1} & \text{if } |z| > 1. \end{cases}$$

Then, $\lambda \in \mathbb{T}$ is a singularity of G_{f_0} if and only if $e^{2\pi i\alpha}\lambda$ is a singularity of $G_{e^{2\pi it}f_0}$.

Proof. Since T_α is similar to $e^{2\pi i\alpha}T_\alpha$ via the bilateral shift $M_{e^{2\pi i}}$ on each $L^p[0,1)$ (and, indeed, unitarily equivalent in $L^2[0,1)$), we have the equation

$$M_{e^{2\pi it}} \widetilde{T}_\alpha^n = e^{-2\pi in\alpha} \widetilde{T}_\alpha^n M_{e^{2\pi it}},$$

valid for each $n \in \mathbb{Z}$. Hence, one can deduce the next functional equation relating G_{f_0} and $G_{e^{2\pi it}f_0}$:

$$G_{e^{2\pi it}f_0}(z) = e^{2\pi i(t-\alpha)}\, G_{f_0}(e^{-2\pi i\alpha} z) \qquad \text{for all } z \in \mathbb{C} \setminus \mathbb{T}.$$

Correspondingly, the set of singular points in $\mathbb{C} \setminus \mathbb{T}$ rotates as described in the statement. □

With this lemma at hand, one deduces that T_α has non-trivial hyperinvariant subspaces in $L^p[0,1)$ (for fixed $1 \le p < \infty$) as far as there exist $f_0 \in L^p[0,1)$ and $g_0 \in L^p[0,1)^*$ such that $(\|\widetilde{T}_\alpha^n f\|_p)_{n\in\mathbb{Z}}$ and $(\|\widetilde{T}_\alpha^{*\,n} g\|_q)_{n\in\mathbb{Z}}$ (here q denotes the conjugate exponent of p) are both dominated by Beurling sequences. The idea is quite simple: if, by chance, we have made such a misguided choice that both G_{f_0} and G_{g_0} possess just one singularity located at the same point in $\mathbb{T}$, it would suffice to consider $e^{2\pi it}f_0$ and invoke Atzmon's Theorem once again. We state it for later reference:

Proposition 2.6. *Let $\alpha \in (0,1)$ be any irrational number and $1 \le p < \infty$. If there exist two non-zero functions $f_0 \in L^p[0,1)$ and $g_0 \in L^p[0,1)^*$ such that*

$$\big\|\widetilde{T}_\alpha^n f_0\big\|_p \ll \rho_n \quad \text{and} \quad \big\|\widetilde{T}_\alpha^{*\,n} g_0\big\|_q \ll \rho_n \qquad (n \in \mathbb{Z}),$$

for some Beurling sequence $(\rho_n)_{n\in\mathbb{Z}}$. Then, T_α possesses non-trivial closed hyperinvariant subspaces in $L^p[0,1)$.

In what follows, given an irrational $\alpha \in (0,1)$, consider the family of real functions

$$L_n(t) := \sum_{j=0}^{n-1} \big(1 + \log(\{t + j\alpha\})\big), \quad (n \in \mathbb{N}).$$

It is plain that the functions $L_n(t)$ play a fundamental role in the asymptotic estimation of the iterates of $\widetilde{T}_\alpha$ and $\widetilde{T}_\alpha^*$, since they codify their behaviours via the equations

$$\text{(2.2)} \qquad \begin{cases} \widetilde{T}_\alpha^n f(t) = e^{L_n(t)} f(\{t+n\alpha\}), & \widetilde{T}_\alpha^{-n} f(\{t+n\alpha\}) = e^{-L_n(t)} f(t), \\[2ex] \widetilde{T}_\alpha^{*\,n} g(\{t+n\alpha\}) = e^{L_n(t)} g(t), & \widetilde{T}_\alpha^{*\,-n} g(t) = e^{-L_n(t)} g(\{t+n\alpha\}), \end{cases}$$

for $n \in \mathbb{Z}_+$ and also for $n = 0$ defining $L_0(t) := 0$.

In the light of (2.2), it is not hard to figure that, in order to control the growth of the iterates $\widetilde{T}_\alpha^n f$ and $\widetilde{T}_\alpha^{*\,n} g$ (overall the backward iterates), it might be a wise idea to construct ad hoc a function which overlooks each of the singularities arising from the summands of $L_n(t)$ with controlled gaps. To accomplish such a task, we claim that the aforementioned results by Davie and Flattot follow from Proposition 2.6 choosing the characteristic of the set

$$\mathcal{B}_\alpha := \left\{ \frac{1}{20} < t < \frac{19}{20} \,:\, \langle t - n\alpha \rangle > \frac{1}{20n^2} \ \text{ for every } n \in \mathbb{Z} \setminus \{0\} \right\},$$

where $\langle t\rangle := \min\big(\{t\}, 1-\{t\}\big)$ denotes the distance from $t\in\mathbb{R}$ to its closest integer. As a matter of fact, note that $\mathcal{B}_\alpha$ is nothing but a variant of the sets E_t appearing in those articles. We point out that replacing in the definition of $\mathcal{B}_\alpha$ the condition by $\langle q_j t\rangle > Cq_j^{-1}$, with C a suitable constant, would give a more manageable set but we prefer not to proceed in this way to keep the analogy with [**44**] and [**60**].

Our next statement ensures that the characteristic function of $\mathcal{B}_\alpha$ (which, in the sequel, will be denotes as $\mathbf{1}_{\mathcal{B}_\alpha}(t)$ for $0\le t\le 1$) can never be a trivial element of the L^p-spaces.

Proposition 2.7. *For each irrational $\alpha\in(0,1)$, the Lebesgue measure of $\mathcal{B}_\alpha$ is greater than $1/2$. In particular, the characteristic function of $\mathcal{B}_\alpha$ does not vanish identically as an element of $L^p[0,1)$ for any $1\le p\le\infty$.*

Proof. Along this proof, let m denote the Lebesgue measure. Since

$$m\big(\{t\in[0,1]\,:\,\langle t\rangle\le\delta\}\big)=2\delta\qquad\text{for each } 0\le\delta\le 1/2,$$

we conclude that the measure of the complementary of $\mathcal{B}_\alpha$ in $[0,1]$ is at most

$$\begin{aligned} m\big([0,1]\setminus\mathcal{B}_\alpha\big) &\le \frac{1}{10}+m\Bigg(\bigcup_{n\in\mathbb{Z}\setminus\{0\}}\Big\{t\in[0,1]\,:\,\langle t-n\alpha\rangle\le\frac{1}{20n^2}\Big\}\Bigg)\\ &\le \frac{1}{10}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{10n^2}=\frac{1}{10}+\frac{\pi^2}{30}. \end{aligned}$$

Consequently, as desired, we immediately have

$$m(\mathcal{B}_\alpha)>1-\left(\frac{1}{10}+\frac{\pi^2}{30}\right)>\frac{1}{2}.$$

Of course, the last assertion in the statement follows directly from this fact. □

From now on, for convenience, we are going to adopt the notations a/q and A/Q to indicate two arbitrary consecutive convergents $(a_j/q_j)_{j\ge0}$ of the irrational number α. At this regard, we remind the reader that Proposition 1.38 precisely ensures that

$$(2Q)^{-1}<\langle q\alpha\rangle<Q^{-1},\quad\text{for } q=q_j \text{ and } Q=q_{j+1}. \tag{2.3}$$

This is much precise than Dirichlet's Theorem which only assures that $\langle q\alpha\rangle<q^{-1}$ for infinitely many values of q.

At this point, we begin with our first technical result. The key point of the next lemma is that, when $n=q$, the fractional parts defining $L_n(t)$ are uniformly distributed inside the interval $[0,1]$. This fact allows one to manipulate a little the definition of $L_q(t)$.

Lemma 2.8. *Suppose that a/q and A/Q are consecutive convergents of an irrational $\alpha\in(0,1)$. Then, there exist $1/2<|\delta|<1$ and $|\delta_\ell|<1$ with the same sign such that for any $k\in\mathbb{Z}$*

$$L_q(t+kq\alpha)=\sum_{\ell=0}^{q-1}\left(1+\log\Big\{t+\frac{\ell}{q}+\frac{k\delta+\delta_\ell}{Q}\Big\}\right).$$

Proof. By (2.3), we can write $\alpha = a/q + \delta/(qQ)$ for some $1/2 < |\delta| < 1$. Introducing this equality in the fractional parts of $L_q(t+kq\alpha)$, we have

$$L_q(t+kq\alpha) = \sum_{j=0}^{q-1}\left(1+\log\left\{t+\frac{ja}{q}+\frac{k\delta}{Q}+\frac{j\delta}{qQ}\right\}\right).$$

Now, bearing in mind that $\gcd(a,q) = 1$, we conclude that the map $j \mapsto aj$ is invertible modulo q. Correspondingly, denote by $\ell \mapsto j_\ell$ its inverse with $0 \le j_\ell < q$. Finally, the result follows from taking $\delta_\ell := j_\ell\delta/q$. $\square$

Using this new expression for $L_q(t+kq\alpha)$, one is able to estimate $|L_n(t)|$ for suitable $t \in [0,1]$, in terms of the denominator q. The main idea of the proof is easily explained: we should rearrange the summands of $L_n(t)$ modulo q, with the purpose of writing $L_n(t)$ as a sum of the translations $L_q(t+kq\alpha)$ indexed in k. Then, the argument will follow from certain uniform bounds on $L_q(t+kq\alpha)$.

As a matter of fact, we remark that the following estimates for $L_n(t)$ are variations of those for $F_m(t)$ given by Davie in [**44**].

Lemma 2.9. *There exists an absolute constant $C > 0$ such that for $n \in \mathbb{Z}_+$*

$$L_n(t) \le C\Big(r + \frac{n}{q}\log(q+1)\Big) \qquad \text{for every } t \in \mathbb{R}$$

where r is the remainder when n is divided by q. Moreover, we have

$$L_n(t) \ge -C\Big(r' + \frac{n+q}{q}\log(\mu^{-1}+q)\Big) \qquad \text{if } \min_{0\le j<r'+n}\{t+j\alpha\} \ge \mu > 0$$

where $r' = 0$ if $r = 0$ and $r' = q - r$ otherwise.

Proof. Separating the last r terms in $L_n(t)$, we can rewrite it as

$$L_n(t) \le r + \sum_{j=0}^{n-r-1}\big(1+\log(\{t+j\alpha\})\big) = r + \sum_{k=0}^{\lfloor n/q\rfloor-1} L_q(t+kq\alpha).$$

Applying Lemma 2.8, since all the δ_ℓ have the same sign and $Q > q$, it turns out that on each interval $[\ell/q, (\ell+1)/q]$ there is exactly one value $\ell/q + \delta_\ell/Q$. Then, using Stirling's approximation, we have

$$L_q(t+kq\alpha) \le q + \sum_{\ell=2}^{q-1}\log\frac{\ell}{q} \le C\log(q+1),$$

which proves the first inequality.

For the second bound, we need to expand the sum until the first multiple of q not less than n. Accordingly, reasoning as above, we conclude that

$$L_n(t) = -\sum_{j=n}^{n+r'-1} + \sum_{j=0}^{n+r'-1} \ge -r' + \sum_{k=0}^{\lceil n/q\rceil-1} L_q(t+kq\alpha).$$

Now, as the values of $\ell/q + \delta_\ell/Q$ are confined into disjoint intervals of length q^{-1}, at most two of the fractional parts in L_q could nearly coincide. Consequently, since the smallest fractional part appearing in the latter sum is precisely

$$\mu := \min_{0\le j<r'+n} \{t + j\alpha\},$$

which coincides with the minimum indicated in our hypothesis, we deduce that

$$L_q(t + kq\alpha) \ge -2\log\left(\mu^{-1}\right) + q + \sum_{\ell=1}^{q-2} \log\frac{\ell}{q} \ge -C\log(\mu^{-1} + q),$$

and the result follows. $\square$

Observe that in our previous lemma, we have provided bounds for $|L_n(t)|$ exactly over the set $\mathcal{B}_\alpha$. In other words, using the equations (2.2), one can rewrite Lemma 2.9 in terms of the iterates $\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}$ and $\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}$. This is the content of our next statement.

Corollary 2.10. *Let a/q be a convergent of an irrational $\alpha \in (0,1)$. Then, for each $n \in \mathbb{Z}$*

$$\log\left(1 + \left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty + \left\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) \ll q + \frac{|n| + q}{q} \log(|n| + q + 1). \tag{2.4}$$

Proof. Of course, the result is trivial for $n = 0$. For the remaining cases, we need to invoke each of the bounds obtained in Lemma 2.9:

- For the forward iterates, we know from the first identity of (2.2) that
$$\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}(t) = e^{L_n(t)} \mathbf{1}_{\mathcal{B}_\alpha}(\{t + n\alpha\}) \quad \text{for each } n \in \mathbb{Z}_+.$$
Consequently, applying the first part of Lemma 2.9, we conclude that
$$\log\left(1 + \left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) \ll \operatorname*{ess\,sup}_{t\in[0,1]} L_n(t) \ll q + \frac{n}{q}\log(q + 1).$$
An analogous argument yields the same bound for $\log\left(\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\right)$.
- On the other hand, for the backward iterates, using again the expressions appearing in (2.2), we have
$$\widetilde{T}_\alpha^{-n} \mathbf{1}_{\mathcal{B}_\alpha}(\{t + n\alpha\}) = e^{-L_n(t)} \mathbf{1}_{\mathcal{B}_\alpha}(t) \quad \text{for fixed } n \in \mathbb{Z}_+.$$
Then, by definition
$$\log\left(\left\|\widetilde{T}_\alpha^{-n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) = \operatorname*{ess\,sup}_{t\in\mathcal{B}_\alpha} \big(-L_n(t)\big).$$
Now, taking into account that $t \in \mathcal{B}_\alpha$, we can take $\mu^{-1} = 20n^2$ and apply the second bound of Lemma 2.9 to obtain
$$\log\left(1 + \left\|\widetilde{T}_\alpha^{-n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) \ll q + \frac{n + q}{q}\log(n + q + 1).$$
Similarly, the same works for the backward iterates of the adjoint. Clearly,
$$\log\left(\left\|\widetilde{T}_\alpha^{*\,-n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) = \operatorname*{ess\,sup}_{\{t+n\alpha\}\in\mathcal{B}_\alpha} \big(-L_n(t)\big) \quad \text{for each } n \in \mathbb{Z}_+.$$
Now, $\{t + n\alpha\} \in \mathcal{B}_\alpha$ implies that $\langle t + n\alpha + \ell\alpha\rangle > 1/(20\ell)^2$ for each $-q \le \ell \le n$ such that $\ell \ne 0$. Then, $\mu^{-1} = 20(n + q)^2$ is a valid choice.

Finally, the corollary follows straightforwardly from the trivial inequality

$$\log\left(1 + \left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty + \left\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) \ll \log\left(1 + \left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) + \log\left(1 + \left\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right).$$

$\square$

Now, by means of the asymptotic inequality (2.4), if we restrict the growth of Q in terms of q, we can manage to find a Beurling sequence which dominates both $(\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty)_{n\in\mathbb{Z}}$ and $(\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty)_{n\in\mathbb{Z}}$. By doing so, at last, we can derive Flattot's result [**60**]. We state it again for the sake of readability.

Theorem 2.11. *Let $\alpha \in (0,1)$ be an irrational number whose convergents $(a_j/q_j)_{j=0}^\infty$ in its continued fraction satisfy the asymptotic condition*

$$\log q_{j+1} = O\big(q_j^{1/2-\varepsilon}\big) \quad \textit{for some } \varepsilon > 0 \textit{ as } j \to \infty. \tag{2.5}$$

Then, T_α has a non-trivial hyperinvariant subspace on each $L^p[0,1)$ for $1 \le p < \infty$.

Proof. Firstly, we claim that real sequence $\rho = (\rho_n)_{n\in\mathbb{Z}}$ given by

$$\rho_n := \exp\left(\frac{C\,|n|}{\log^\gamma(2+|n|)}\right), \quad (n \in \mathbb{Z})$$

is a Beurling sequence for each $C > 0$ and $\gamma > 1$. Obviously, $\rho_0 = 1$. On the other hand, the convergence of the series $\sum_{n\in\mathbb{Z}} \log \rho_n/(1+n^2)$ can be easily deduced from the comparison test with respect to the indefinite integral $\int_2^\infty 1/(t\log^\gamma(t))\,dt$. Finally, to see the submultiplicativity of $\rho = (\rho_n)_{n\in\mathbb{Z}}$, it suffices to check the subadditivity of the related unilateral sequence

$$\widetilde{\rho}_n := \frac{n}{\log^\gamma(2+n)}, \quad \text{for } n \in \mathbb{Z}_+. \tag{2.6}$$

This is due to the next remarkable fact: the even extension of a non-decreasing subadditive sequence on $\mathbb{Z}_+$ is subadditive on $\mathbb{Z}$. But, clearly, the subadditivity of $\widetilde{\rho}_n$ for $n \ge 0$ follows immediately from the trivial inequality

$$m \cdot \frac{\log^\gamma(2+m+n)}{\log^\gamma(2+m)} + n \cdot \frac{\log^\gamma(2+m+n)}{\log^\gamma(2+n)} \ge m+n, \quad \text{for each } m,n \ge 0.$$

Now, let $1 \le p < \infty$ be fixed and denote by $1 < p' \le \infty$ its conjugate exponent. Bearing in mind that $\|\cdot\|_\infty$ majorizes both $\|\cdot\|_p$ and $\|\cdot\|_{p'}$, by Proposition 2.6 and Corollary 2.10, it is enough to show that for $|n|$ large we can always find a convergent a/q of our irrational number α such that

$$q + \frac{|n|+q}{q}\log(|n|+q+1) = O\left(\frac{|n|}{\log^\gamma(2+|n|)}\right) \qquad \text{for some } \gamma > 1.$$

Thus, fixed some $n \in \mathbb{Z}$, make the choice $q \le |n|^{2/3} \le Q$. Clearly, by the hypothesis (2.5) on the growth of the convergents of α, we have that

$$q \gg (\log|n|)^{(1/2-\varepsilon)^{-1}} \quad \text{for some } \varepsilon > 0.$$

Observe that $(1/2-\varepsilon)^{-1} > 2$. Correspondingly, the constant $\gamma := (1/2-\varepsilon)^{-1} - 1$ is strictly bigger than 1. Then,

$$q + \frac{|n|+q}{q}\log(|n|+q+1) \ll \frac{|n|\,\log|n|}{(\log|n|)^{1+\gamma}} = O\left(\frac{|n|}{\log^\gamma(2+|n|)}\right)$$

and the theorem is proved. □

2.3. Bishop operators T_α with invariant subspaces: enlarging the set of irrationals α

Along this section, we achieve a substantial extension of the results by Davie and Flattot. To be specific, in the end, we will be able to replace the asymptotic condition in Theorem 2.11

by the much weaker one

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right) \quad \text{as } j \to \infty. \tag{2.7}$$

As mentioned above, observe that (2.7) relaxes considerably the condition provided by Flattot, allowing the power 1 instead of 1/2 and quantifying the role of ε in more precise terms. In accordance with the next chapter, we would like to underscore once more that the application of Atzmon's Theorem in the context of Bishop operators cannot be improved beyond our restriction. In other words, our forthcoming approach yields essentially the best result attainable from the standard techniques and any improvement seems to require different functional analytical tools.

Our proof follows a similar structure than the preceding one, but introduces some critical modifications which enable to sharpen the asymptotic analysis. In a few words, the main limitation of our previous method was that, when n was very much greater than q, we could not control appropriately the $\log(|n| + q + 1)$ appearing in the estimation

$$\log\left(1 + \|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty + \|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\right) \ll q + \frac{|n| + q}{q} \log(|n| + q + 1)$$

obtained in Corollary 2.10. Naturally, for some extreme irrationals $\alpha \in (0,1)$, the denominators of their convergents $(a_j/q_j)_{j\geq 0}$ are arbitrarily separated and such a situation occurs.

Unfortunately, to deal with this obstacle, we need to rework our strategy from the initial step. Recall from Lemma 2.8 that we can choose $\mu^{-1} = 20(n+q)^2$ in the lower bound

$$L_n(t) \gg -\left(q + \frac{n+q}{q}\log(\mu^{-1} + q)\right), \quad \text{for all } t \in \mathcal{B}_\alpha.$$

Accordingly, if n is very large in comparison to q, there is an asymmetry in the bounds obtained for $L_n(t)$ in this lemma, being the upper bound much stronger than the lower one. This is reasonable because inside each summand of $L_n(t)$, the fractional parts can take arbitrarily small values but are bounded from above by 1. Anyway, as we shall see, it is possible to partially recover the symmetry, getting a non-biased estimation for $|L_n(t)|$ by a more careful analysis than the one carried out in Section 2.2. The improvement is achieved when n is much larger than q, in such a way that $\log(|n| + q + 1)$ is not comparable to $\log(q+1)$ in Corollary 2.10, but is controlled by Q (see Proposition 2.14 below).

Precisely, our first lemma shows that one can obtain the desired bounds for $|L_n(t)|$ on a dilation of $\mathcal{B}_\alpha$ of controlled size, when n is a "small" multiple of q:

Lemma 2.12. *Let a/q and A/Q be two consecutive convergents of an irrational $\alpha \in (0,1)$. Suppose that $q \mid n$ and $1 \leq n \leq Q/(100q)$. Then, given any $t_0 \in \mathcal{B}_\alpha$, we have:*

$$|L_n(t)| \ll \frac{n}{q}\log(q+1), \qquad \textit{for every } t \in [0,1) \textit{ with } |t - t_0| \leq \frac{1}{100q^2}.$$

Proof. Clearly, from Lemma 2.9 we have that the unique bound to prove is the lower one. To do so, we begin by writing

$$L_n(t) = \sum_{k=0}^{n/q-1} L_q(t + kq\alpha). \tag{2.8}$$

Hence, it is enough to show that the next inequality holds

$$\min_{0\le j<q}\{t+kq\alpha+j\alpha\} > Cq^{-2} \quad \text{for some } C>0, \tag{2.9}$$

for all $k=0,\dots,n/q-1$. This is because, in such a case, the second part of Lemma 2.9 assures that each term in the sum decomposition (2.8) contributes $O\big(\log(q+1)\big)$.

To check (2.9), we invoke once again the inequality (2.3) to express $\alpha = a/q+\delta/(qQ)$ with $|\delta|<1$. Using that $t_0\in\mathcal{B}_\alpha$ and $k<n/q$, we easily deduce that

$$\{t_0+j\alpha\} \ge \langle t_0+j\alpha\rangle > \frac{1}{20q^2} \qquad \text{and} \qquad \frac{|k\delta|}{Q} < \frac{1}{100q^2}$$

for each $1\le j<q$ and $0\le k\le n/q-1$. Consequently,

$$\Big\{t_0+j\alpha+\frac{k\delta}{Q}\Big\} > \frac{1}{20q^2}-\frac{1}{100q^2}=\frac{1}{25q^2}.$$

Finally, since $|t-t_0|<1/(100q^2)$ we have

$$\{t+kq\alpha+j\alpha\} = \Big\{t_0+j\alpha+\frac{k\delta}{Q}+t-t_0\Big\} > \frac{1}{25q^2}-\frac{1}{100q^2}=\frac{3}{100q^2}$$

This establishes the inequality (2.9) with $C=3/100$ for each $k=0,\dots,n/q-1$ and $1\le j<q$. A similar argument applies for $j=0$ using that $\langle t_0\rangle > 1/20$ for $t_0\in\mathcal{B}_\alpha$. □

Now, we see that these bounds can be transfered between two $L_{n_1}(t)$ and $L_{n_2}(t)$ having congruent indices $n_1,\, n_2\in\mathbb{Z}_+$ modulo q.

Lemma 2.13. *Let a/q and A/Q be two consecutive convergents of an irrational $\alpha\in(0,1)$. Suppose that $n_1,n_2\in\mathbb{Z}_+$ are two integers such that*

$$q \mid n_2-n_1 \quad \textit{and} \quad \frac{Q}{100q}\le n_2-n_1\le Q-q.$$

Then, for every $t\in\mathcal{B}_\alpha$ we have

$$L_{n_2}(t)-L_{n_1}(t) \ll \log n_2 + \frac{n_2-n_1}{q}\log(q+1).$$

Proof. Fix an arbitrary $t\in\mathcal{B}_\alpha$. We start writing

$$L_{n_2}(t)-L_{n_1}(t)=\sum_{k=0}^{K-1} L_q(t+n_1\alpha+kq\alpha) \qquad \text{with} \quad K=\frac{n_2-n_1}{q}.$$

Let us call μ to the minimum of the fractional parts appearing in these terms, thus

$$\mu := \min_{0\le j<n_2-n_1}\{t+n_1\alpha+j\alpha\}.$$

Obviously, we have $\mu > \frac{1}{20n_2^2}$ because $t\in\mathcal{B}_\alpha$.

Now, by Lemma 2.8 and doing a translation $\ell\mapsto\ell+\ell_0$ modulo q if the minimum μ is reached for a certain $k=k_0$ and $\ell=\ell_0$, this can be expanded as

$$L_{n_2}(t)-L_{n_1}(t)=\sum_{k=0}^{K-1}\sum_{\ell=0}^{q-1}\bigg(1+\log\Big(\Big\{\mu+\frac{\ell}{q}+\frac{(k-k_0)\delta+\delta_\ell}{Q}\Big\}\Big)\bigg).$$

Note that we have employed $\delta_{\ell+\ell_0}-\delta_{\ell_0}=\delta_\ell$. In order to estimate the latter sum, we need to distinguish two different cases:

- If $\ell \neq 0$, we know that $K \leq Q/q - 1$ and recalling the properties of δ and δ_ℓ in Lemma 2.8, we deduce the upper bound
$$\left|\frac{(k-k_0)\delta+\delta_\ell}{Q}\right| < \frac{K}{Q} \leq \frac{1}{q} - \frac{1}{Q}.$$
Correspondingly, for each $\ell \neq 0$, the fractional part can be safely compared with that of ℓ/q to get
$$\sum_{\ell=0}^{q-1}\left(1+\log\left(\left\{\mu+\frac{\ell}{q}+\frac{(k-k_0)\delta+\delta_\ell}{Q}\right\}\right)\right) = O\big(\log(q+1)\big)$$
for each fixed $k = 0, \ldots, K-1$. This gives $O\big(K\log(q+1)\big)$.
- If $\ell = 0$, observe that $\delta_0 = 0$. In particular, this entails the lower bound
$$\frac{|k-k_0|}{2Q} < \left|\frac{(k-k_0)\delta}{Q}\right|.$$
Hence, the contribution of $\ell = 0$ is comparable to
$$\sum_{k=0}^{K-1}\left(1+\log\left(\left\{\mu+\frac{(k-k_0)\delta}{Q}\right\}\right)\right) \ll \log(\mu^{-1}) + K + \left|\sum_{k=1}^{K}\log\left(\frac{k}{Q}\right)\right|$$
$$\ll \log n_2 + K + \left|\sum_{k=1}^{K}\log\left(\frac{K}{Q}\right)\right| + \left|\sum_{k=1}^{K}\log\left(\frac{k}{K}\right)\right|.$$
The last sum is $O(K)$ by Stirling's approximation. Besides, noting that $Q/K \ll q^2$, the other sum contributes
$$\left|\sum_{k=1}^{K}\log\left(\frac{K}{Q}\right)\right| \ll K\log(q+1).$$

Gathering all the contributions, we obtain the expected bound. □

With these lemmas at hand, we are in position to get an improvement of Lemma 2.9. The basic idea behind the proof is to use certain multiples of q and Q as nodes for $n \in \mathbb{Z}_+$ in order to apply our preceding bounds to estimate $|L_n(t)|$:

Proposition 2.14. *Let a/q and A/Q be consecutive convergents of an irrational $\alpha \in (0,1)$. Assume that $Q \geq 4(10q)^4$, $1 \leq n \leq Q^{3/2}$ and let N be the closest multiple of Q to n. Then, for $t \in \mathcal{B}_\alpha$ we have*
$$L_n(t) \ll q + \frac{|n-N|}{q}\log(q+1) + \frac{n+Q}{Q}\log(n+1).$$

Proof. Choose any $t \in \mathcal{B}_\alpha$. First, we introduce the decomposition
$$L_n(t) = L_N(t) + \big(L_n(t) - L_{n'}(t)\big) + \big(L_{n'}(t) - L_N(t)\big), \tag{2.10}$$
with $n' = N \pm m$, where $m \in \mathbb{Z}_+$ is such that $\pm m$ is the closest multiple of q to $n - N$ (here the symbol $\pm$ indicates the sign of $n - N$). Clearly, we have $0 \leq m \leq |n-N| + q/2$.

Now, we bound each part of the decomposition (2.10) separately:

- For the first term we apply Lemma 2.9 with Q instead of q and $\mu^{-1} = 20N^2$. Thus, we obtain
$$L_N(t) \ll \frac{N}{Q}\log(N+1) \ll \frac{n}{Q}\log(n+1).$$

The trick for obtaining the last bound in the latter inequality is the following distinction of cases: if $Q \le 3n$ then $N \ll n$ and the desired bound follows straightforwardly; conversely, if $n < Q/3$ then trivially $N = 0$. A similar reasoning will be used without further indication along the proof to estimate m in terms of $|N - n|$.

- For the second term: if $n > n'$, then $L_n(t) - L_{n'}(t) = L_{n-n'}(t + n'\alpha)$; otherwise, if $n < n'$, then $L_n(t) - L_{n'}(t) = -L_{n'-n}(t + n\alpha)$. As $|n - n'| < q$, in both cases Lemma 2.9 with $\mu^{-1} = 20(n + 2q)^2$ assures

$$L_n(t) - L_{n'}(t) \ll q + \log(n + q) \ll q + \log n.$$

- To conclude, for the third term: whenever $Q/(100q) < m$, then we are under the hypotheses of Lemma 2.13 that gives

$$L_{n'}(t) - L_N(t) \ll \log(N + m) + \frac{m}{q}\log(q + 1).$$

Therefore,

$$L_{n'}(t) - L_N(t) \ll \log(n + 1) + \frac{|n - N|}{q}\log(q + 1).$$

If, on the contrary, we have $m \le Q/(100q)$, note firstly that

$$\langle N\alpha\rangle \le \frac{N}{Q}\langle Q\alpha\rangle \le \frac{N}{Q^2} \le \frac{n + Q/2}{Q^2} \le \frac{1}{100q^2}. \tag{2.11}$$

So, if $n' \ge N$ we write $L_{n'}(t) - L_n(t) = L_m(t + N\alpha)$. Now, bearing in mind that m is a multiple of q, by the previous bound (2.11) we can apply Lemma 2.12 to get

$$L_{n'}(t) - L_n(t) \ll \frac{m}{q}\log(q + 1) \ll \frac{|n - N|}{q}\log(q + 1). \tag{2.12}$$

Conversely, if $n' < N$ then

$$L_N(t) - L_{n'}(t) = \sum_{j=0}^{m-1}\big(1 + \log(\{t + N\alpha - j\alpha\})\big) + \log(\{t + n'\alpha\}) - \log(\{t + N\alpha\}).$$

The contribution of the last two terms is $O(\log(n + 1))$. Besides, the sum coincides with $L_m(t + N\alpha)$ formally changing α by $-\alpha$ in the definition of L_m. As the denominators of the convergents of α and $-\alpha$ coincide except for a unit shift in the indexes, the same argument as the one in (2.12) applies again.

Adding the contribution of the three summands of (2.10) the result is proved. □

Once we have these bounds for $L_n(t)$, the analogue of Corollary 2.10 is:

Corollary 2.15. *Let a/q be a convergent of an irrational $\alpha \in (0, 1)$. Then, for any $|n| \le Q^{3/2}$ we have*

$$\log\left(1 + \left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty + \left\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_\infty\right) \ll q + \frac{|n|}{q}\log(q + 1) + \frac{|n| + Q}{Q}\log(|n| + 2). \tag{2.13}$$

Proof. Once again, we are going to divide the proof into several cases. More precisely, given any $n \in \mathbb{Z}_+$, we will see that the desired bound holds for each of the following quantities:

- $A_n := \log\big(1 + \|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty + \|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big)$.
- $B_n := \log\big(1 + \|\widetilde{T}_\alpha^{-n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big)$.
- $C_n := \log\big(1 + \|\widetilde{T}_\alpha^{*\,-n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big)$.

First of all, the estimation for A_n can be deduced from substituting in the corresponding identities of (2.2), the former bound of Lemma 2.9. To be specific, as we did at the beginning of the proof of Corollary 2.10

$$\max\big(\log\big(\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big), \log\big(\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big)\big) \le \operatorname*{ess\,sup}_{t\in[0,1]} L_n(t).$$

Then, the aforementioned bound yields the required asymptotic inequality

$$A_n \ll q + \frac{n}{q}\log(q+1).$$

Now, if $Q < 4(10q)^4$, then $\log(|n|+q+1) \ll \log(q+1)$ and the required estimations for B_n and C_n follow directly from Corollary 2.10. More precisely,

$$\max\big(|B_n|, |C_n|\big) \ll q + \frac{n+q}{q}\log(q+1), \quad \text{whenever } Q < 4(10q)^4.$$

Conversely, if $Q \ge 4(10q)^4$, recalling the identity

$$\log\big(\|\widetilde{T}_\alpha^{-n} \mathbf{1}_{\mathcal{B}_\alpha}\|_\infty\big) = \operatorname*{ess\,sup}_{t\in\mathcal{B}_\alpha}\big(-L_n(t)\big),$$

our previous Proposition 2.14 gives the bound for B_n:

$$B_n \ll q + \frac{|n|}{q}\log(q+1) + \frac{|n|+Q}{Q}\log(|n|+2), \quad \text{whenever } Q \ge 4(10q)^4.$$

Finally, it remains to estimate C_n if $Q \ge 4(10q)^4$. With this purpose, we rewrite the last formula in (2.2) as $T_\alpha^{*-n} f\big(\{t-n\alpha\}\big) = e^{-L_n(t-n\alpha)} f(t)$ and we note

$$L_n(t-n\alpha) = -\log(\{t\}) + \log(\{t-n\alpha\}) + \sum_{j=0}^{n-1}\big(1+\log(\{t-j\alpha\})\big).$$

The sum coincides with $L_n(t)$ replacing α by $-\alpha$. As we mentioned before, the convergents of α and $-\alpha$ have the same denominators and then Proposition 2.14 applies also for this sum. On the other hand, $\log(\{t\})$ and $\log(\{t-n\alpha\})$ are $O\big(\log(|n|+1)\big)$ if $t \in \mathcal{B}_\alpha$. In sum, one also has the bound

$$C_n \ll q + \frac{|n|}{q}\log(q+1) + \frac{|n|+Q}{Q}\log(|n|+2), \quad \text{whenever } Q \ge 4(10q)^4.$$

Consequently, all the cases have been covered and the result is now proved. □

Once we have obtained this bound, the proof of our main result in this chapter parallels that of Theorem 2.11. The main difficulty will be to find a new Beurling sequence which may be adjusted for our bound (2.13):

Theorem 2.16. *Let $\alpha \in (0,1)$ be any irrational and $(a_j/q_j)_{j\ge 0}$ the convergents in its continued fraction. If the following condition holds:*

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right) \qquad \text{as } j \to +\infty. \tag{2.14}$$

Then, the Bishop operator T_α has non-trivial closed hyperinvariant subspaces on each $L^p[0,1)$ space for $1 \le p < \infty$.

Proof. As before, we claim that the real sequence $\rho := (\rho_n)_{n\in\mathbb{Z}}$ given by

$$\rho_n := \exp\left(\frac{C\,|n|}{\log(2+|n|)(\log\log(5+|n|))^2}\right), \quad (n \in \mathbb{Z})$$

is a Beurling sequence. Trivially, one has $\rho_0 = 1$. This time, the convergence of the series $\sum_{n\in\mathbb{Z}} \log \rho_n/(1+n^2)$ can be derived from the comparison test with respect to

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} \ll \sum_{n\geq 1} \frac{1}{n\log(2+n)(\log\log(5+n))^2}$$
$$= O\biggl(\int_5^\infty \frac{1}{t\log(t)(\log\log(t))^2}\,dt\biggr) = O(1).$$

To conclude, the submultiplicativity of $\rho = (\rho_n)_{n\in\mathbb{Z}}$ follows directly from the even reflection principle applied in the proof of Theorem 2.11. Here, the associated unilateral sequence is

$$\widetilde{\rho}_n := \frac{n}{\log(2+n)(\log\log(5+n))^2}, \quad \text{for } n \in \mathbb{Z}_+,$$

which is non-decreasing and subadditive on $\mathbb{Z}_+$.

Once again, let $1 \leq p < \infty$ be fixed and let $1 < p' \leq \infty$ stand for its conjugate exponent. Recall that $\|\cdot\|_\infty$ majorizes both norms $\|\cdot\|_p$ and $\|\cdot\|_{p'}$. Hence, by the same argument as in the proof of Theorem 2.11 the strategy is now clear: given $n \neq 0$, choose two consecutive convergents a/q and A/Q of the irrational α, such that $q \leq |n|^{2/3} < Q$. In this range

$$q + \frac{|n|+Q}{Q}\log(|n|+2) \ll |n|^{2/3} + |n|^{1/3}\log(|n|+2)$$

and by the condition (2.14),

$$\frac{|n|}{q}\log(q+1) \ll \frac{|n|}{\log Q(\log\log Q)^2} \ll \frac{|n|}{\log|n|(\log\log|n|)^2}.$$

Therefore, by Corollary 2.15, there exists $C > 0$ such that for every $n \in \mathbb{Z}$

$$\max\left(\bigl\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\bigr\|_\infty, \bigl\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\bigr\|_\infty\right) \leq \exp\left(\frac{C\,|n|}{\log(2+|n|)(\log\log(5+|n|))^2}\right). \tag{2.15}$$

Consequently, the result follows immediately from Proposition 2.6 since the right hand term is a Beurling sequence. ☐

Remark 2.17. Note that for Bishop-type operators of the form $W_{t^s,\alpha} f(t) = t^s f(\{t+\alpha\})$ where $s > 0$, all the bounds computed above remain true replacing $L_n(t)$ by

$$L_{s,n}(t) = \sum_{j=0}^{n-1} \bigl(s + s\log(\{t+j\alpha\})\bigr),$$

and considering again the L^p-function $\mathbf{1}_{\mathcal{B}_\alpha}$. This clearly follows from the fact $L_{s,n}(t) = sL_n(t)$. Consequently, Theorem 2.16 is also valid for every $W_{t^s,\alpha}$ with $s > 0$. In particular, we obtain a generalization of [**60**, Theorem 4.7].

2.4. Hausdorff dimensions of exceptional sets

In this section, we begin by providing some convenient tools to analyse the metric properties of sets consisting of (extremely) well-approximable irrational numbers. To accomplish such a task, we will require the notion of Hausdorff measure with generalized dimension functions. As mentioned above, our ultimate goal is to discriminate between those sets of exceptions concerning each one of the approaches developed by Davie, Flattot and us.

For a thorough treatment of metric properties of exceptional sets, we refere the reader to the books by Bugeaud [**33**] and Rogers [**110**].

One of the first metric results regarding Diophantine Approximation dates back to Khintchine [**85**], who showed that provided an *approximation function* Ψ (i.e. just a continuous mapping $\Psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$), under the extra assumption of $t \mapsto t^2\Psi(t)$ is non-increasing, the associated set

$$\mathcal{K}^*(\Psi) := \left\{ \xi \in \mathbb{R} \,:\, \left| \xi - \frac{a}{q} \right| < \Psi(q) \text{ for infinitely many rational numbers } \frac{a}{q} \right\} \tag{2.16}$$

has null Lebesgue measure if the sum $\sum_{j\geq 1} j\Psi(j)$ converges whilst has full Lebesgue measure otherwise. A specially interesting application of *Khintchine's Theorem* exhibits that the set

$$\left\{ \xi \in \mathbb{R} \,:\, \left| \xi - \frac{a}{q} \right| < \frac{1}{q^{2+\varepsilon}} \text{ for infinitely many rational numbers } \frac{a}{q} \right\}$$

is of vanishing Lebesgue measure for arbitrary fixed $\varepsilon > 0$. In particular, this entails that the set consisting of Liouville numbers has Lebesgue measure zero.

Nonetheless, in a sense to be developed below, the employment of Lebesgue measure makes us to labour somehow blindfolded because we are not able to discriminate between certain sets which are certainly much larger than others. A powerful tool for distinguishing among different sets of null Lebesgue measure are the notions of *Hausdorff measure* and *Hausdorff dimension*. In connection to our number theoretical context, it is worth mentioning that one of the earliest achievements of this metric theory was precisely the determination of the Hausdorff dimension of several sets of numbers, such as the Liouville numbers (see, for instance, Jarník [**81**] and Besicovitch [**23**]).

In the sequel, for arbitrary non-empty subset U of $\mathbb{R}^d$, its *diameter* will be given by

$$\operatorname{diam}(U) := \sup\left\{ \|x - y\|_\infty \,:\, x, y \in U \right\}.$$

In general, as we shall usually work over $\mathbb{R}$, the norm $\|\cdot\|_\infty$ is none other than the absolute value $|\cdot|$. A *δ-covering* of a set $E \subseteq \mathbb{R}^d$ is any family of open sets $(U_i)_{i\in I}$ in $\mathbb{R}^d$ such that

(i) $E \subseteq \bigcup_{i\in I} U_i$;
(ii) $0 < \operatorname{diam}(U_i) \leq \delta$ for every $i \in I$.

The standard Hausdorff measures are constructed with the usual family of functions $(t^s)_{s\geq 0}$ as follows. Given any subset $E \subseteq \mathbb{R}^d$ and fixed $s \geq 0$, set

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_{i\in I} \operatorname{diam}(U_i)^s \,:\, (U_i)_{i\in I} \text{ is a countable } \delta\text{-covering of } E \right\}.$$

Evidently, the assignment $\delta \mapsto \mathcal{H}^s_\delta(E)$ yields a non-increasing mapping for $\delta \in [0, \infty)$. Accordingly, the limit

$$\mathcal{H}^s(E) := \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E)$$

is well-defined and lies inside the interval $[0, +\infty]$. As customary, $\mathcal{H}^s(E)$ is called the *s-dimensional Hausdorff measure* of E. Clearly, we have the inequalities $0 \leq \mathcal{H}^s(E) \leq \mathcal{H}^r(E) \leq +\infty$ when $s \geq r \geq 0$. Furthermore, fixed any $E \subseteq \mathbb{R}^d$, as s ranges $[0, +\infty]$, there exists a critical value of s at which $\mathcal{H}^s(E)$ jumps from $+\infty$ to 0. This value is known as the *Hausdorff dimension* of E, i.e.

$$\dim_{\mathcal{H}^s}(E) := \inf\big\{s \geq 0 \,:\, \mathcal{H}^s(E) = 0\big\} = \sup\big\{s \geq 0 \,:\, \mathcal{H}^s(E) = +\infty\big\}.$$

As remarkable instances, recall that $\mathcal{H}^0$ coincides with the *counting measure* while $\mathcal{H}^d$ acting on $\mathbb{R}^d$ is a scalar multiple of the Lebesgue measure when applied to the Borelian subsets E of $\mathbb{R}^d$. More precisely, for each integer $d \in \mathbb{N}$, we have

$$\mathcal{H}^d(E) = \frac{2^d\,\Gamma\big(\frac{d}{2}+1\big)}{\pi^{d/2}}\,\mathrm{vol}(E),$$

where $\mathrm{vol}(E)$ here denotes the d-dimensional Lebesgue measure and $\Gamma(z)$ represents the *Gamma function*.

Theorem 2.18 (Jarník-Besicovitch, [23, 81]). *The set of Liouville numbers has Hausdorff dimension zero.*

Proof. Along this proof, let $\mathcal{L}$ stand for the set of Liouville numbers. For each $\nu > 1$, consider the set

$$\mathcal{K}^*(\nu) := \left\{\xi \in \mathbb{R} \,:\, \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2\nu}} \text{ for infinitely many rationals } \frac{p}{q}\right\}.$$

By definition, $\mathcal{L} \subseteq \mathcal{K}^*(\nu)$ holds for all $\nu > 1$.

Fixed any $\nu > 1$, set the following countable family of open sets: given an integer $q \geq 1$, for each $p \in \{0, \ldots, q\}$ define

$$U_{p,q} := \left\{\xi \in (0,1) \,:\, \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2\nu}}\right\}.$$

Clearly, each $\xi \in \mathcal{K}^*(\nu)$ is contained in infinitely many $U_{p,q}$. Now, since the series

$$\sum_{\substack{q\geq 1\\ 0\leq p\leq q}} \mathrm{diam}(U_{p,q})^s \ll \sum_{q\geq 1} \frac{2^s(q+1)}{q^{2\nu s}}$$

converges for every $s > 1/\nu$, upon applying a Borel-Cantelli Lemma type argument, we may infer that $\mathcal{H}^s(\mathcal{K}^*(\nu)) = 0$ for each $s > 1/\nu$. This gives us the upper bound

$$\dim_{\mathcal{H}^s}(\mathcal{K}^*(\nu)) \leq 1/\nu. \tag{2.17}$$

Whence, by inclusion, we have that $\dim_{\mathcal{H}^s}(\mathcal{L}) \leq 1/\nu$ for every $\nu > 1$. Obviously, this means that $\dim_{\mathcal{H}^s}(\mathcal{L}) = 0$ as desired. □

Remark 2.19. It is worth pointing out that equation (2.17) provides precisely the exact value of the Hausdorff dimension:

$$\dim_{\mathcal{H}^s}(\mathcal{K}^*(\nu)) = 1/\nu, \qquad \text{for each } \nu \geq 1.$$

Actually, the remaining lower bound requires more sophisticated methods to be proved. For instance, it may be derived by considering suitable coverings of Cantor-like sets which allow one to control their mass distribution thank to self-similarity properties (see [**33**, Ch. 5] for details).

Jarník-Besicovitch Theorem evinces that we shall deal with extremely small sets of irrational numbers which cannot be treated adequately with the standard Hausdorff measures.

To handle so negligible sets, we need to consider even more sensitive metric devices, involving further examples of dimension functions:

Definition 2.20. *Any continuous function $h:(0,\varepsilon)\to\mathbb{R}$ (for some $\varepsilon>0$) which is strictly increasing and satisfies $\lim_{t\to 0^+} h(t)=0$ is called a* dimension function.

As above, taken a dimension function h and $\delta>0$ sufficiently small, we define for every set $E\subseteq\mathbb{R}^d$

$$\mathcal{H}^h_\delta(E):=\inf\left\{\sum_{i\in I} h\big(\operatorname{diam}(U_i)\big)\ :\ (U_i)_{i\in I}\text{ is a countable }\delta\text{-covering of }E\right\}.$$

From the assumptions on the dimension function, the assignment $\delta\mapsto\mathcal{H}^h_\delta(E)$ determines a non-increasing mapping on an interval of the form $(0,\varepsilon)$. Consequently, the limit

$$\mathcal{H}^h(E):=\lim_{\delta\to 0^+}\mathcal{H}^h_\delta(E)$$

exists and is named as the $\mathcal{H}^h$*-measure* of E. It is well-known that $\mathcal{H}^h$ always defines a regular outer measure on $\mathbb{R}^d$ for which the Borelian sets are measurable.

Given two dimension functions h_1 and h_2, we say that h_1 corresponds to a *smaller generalized dimension* (and we denote it by $h_1\prec h_2$) whenever

$$\lim_{t\to 0^+}\frac{h_1(t)}{h_2(t)}=+\infty.$$

Evidently, if $h_1\prec h_2$, then h_1 increases much faster than h_2 in a neighbourhood of the origin and consequently $\mathcal{H}^{h_1}(E)\geq\mathcal{H}^{h_2}(E)$ for each $E\subseteq\mathbb{R}^d$. In general, the order induced by the relation $\prec$ is not a total order since there are dimension functions h_1 and h_2 such that

$$\liminf_{t\to 0^+}\frac{h_1(t)}{h_2(t)}=0\quad\text{and}\quad\limsup_{t\to 0^+}\frac{h_1(t)}{h_2(t)}=+\infty.$$

Nevertheless, when restricted to concrete subfamilies of dimension functions, the relation $\prec$ behaves like a total order; for instance, this is the case for the family $(|\log t|^{-s})_{s\geq 0}$ and the aforementioned $(t^s)_{s\geq 0}$.

In consideration of the next result, the notion of Hausdorff dimension still makes sense for arbitrary dimension functions:

Proposition 2.21. *Let f,g,h be dimension functions such that $f\prec g\prec h$ and $E\subseteq\mathbb{R}^d$ an arbitrary set. Then, the following assertions hold:*

(i) *If $0\leq\mathcal{H}^g(E)<+\infty$, then $\mathcal{H}^h(E)=0$.*
(ii) *If $0<\mathcal{H}^g(E)\leq+\infty$, then $\mathcal{H}^f(E)=+\infty$.*

Proof. Consider a set $E\subseteq\mathbb{R}^d$ for which $0\leq\mathcal{H}^g(E)<+\infty$ and let $\varepsilon>0$ be arbitrarily small. Now, since $g\prec h$, there exists a positive $\delta>0$ such that

$$h(t)\leq\frac{\varepsilon\, g(t)}{\mathcal{H}^g(E)+1}\qquad\text{for }0<t<\delta.$$

Picking any $0<\delta'<\delta$, there exists a countable δ'-covering $(U_i)_{i\in I}$ of E satisfying

$$\sum_{i\in I} g\big(\operatorname{diam}(U_i)\big)\leq\mathcal{H}^g(E)+1,$$

and hence

$$\sum_{i\in I} h\big(\mathrm{diam}(U_i)\big) \leq \varepsilon.$$

As desired, taking the infimum, this clearly yields $\mathcal{H}^h(E) = 0$.

For a proof of (ii), we may proceed in a similar way, replacing the functions f and h by f and g, respectively. □

So, when considered a totally ordered chain of dimension functions $(h_s)_{s\geq 0}$, we define the $\mathcal{H}^{h_s}$*-Hausdorff dimension* (or, speaking in general terms, *generalized Hausdorff dimension*) of a set $E \subseteq \mathbb{R}^d$ as

$$\dim_{\mathcal{H}^{h_s}}(E) := \inf\big\{s \geq 0 \,:\, \mathcal{H}^{h_s}(E) = 0\big\} = \sup\big\{s \geq 0 \,:\, \mathcal{H}^{h_s}(E) = +\infty\big\}.$$

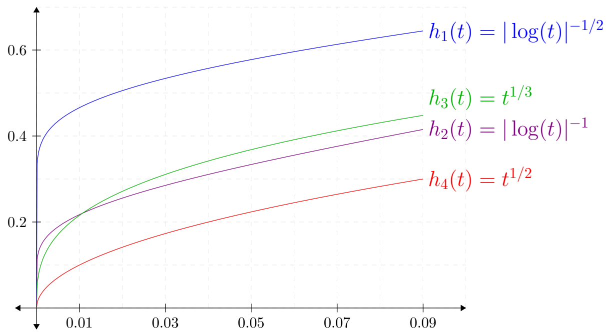

FIGURE 3. Comparison among various dimension functions near the origin

A Khintchine-type result will help us to determine the generalized Hausdorff dimensions of certain subsets of Liouville numbers constructed upon an approximation function Ψ. To state it, first remind the notation introduced in (2.16)

$$\mathcal{K}^*(\Psi) := \left\{\xi \in \mathbb{R} \,:\, \left|\xi - \frac{a}{q}\right| < \Psi(q) \text{ for infinitely many rational numbers } \frac{a}{q}\right\}.$$

Theorem 2.22 (Jarník, [81]). *Let h be a dimension function such that $t \mapsto h(t)/t$ is decreasing in a neighbourhood of the origin and $\lim_{t\to 0^+} h(t)/t = +\infty$. Assume further that the mapping $t \mapsto t^2 h(\Psi(t))$ is non-increasing. Then, if the series*

$$\sum_{q=1}^{\infty} q\, h\big(2\Psi(q)\big)$$

is convergent, we have that $\mathcal{H}^h(\mathcal{K}^*(\Psi)) = 0$. Otherwise, we have $\mathcal{H}^h(\mathcal{K}^*(\Psi)) = +\infty$.

For a proof, we refer to [**33**, Theorem 6.8]. In addition, we point out that the hypothesis in Theorem 2.22 concerning the growth of the function $t \mapsto t^2 h(\Psi(t))$ can be slightly weakened, replacing it by the condition that $t \mapsto t^2 h(\Psi(t))$ is ultimately non-increasing for sufficiently large $t \in \mathbb{R}$.

Now, once all these notions have been settled, we are in position to state the main result of this section.

Theorem 2.23. *Consider the totaly ordered family of dimension functions $h_s(t) := |\log(t)|^{-s}$ for $s \geq 0$. Respectively, let $\mathfrak{D}$, $\mathfrak{F}$ and $\mathfrak{M}$ denote the sets of exceptions in Davie's, Flattot's and our approach. Then,*

$$\dim_{\mathcal{H}^{h_s}}(\mathfrak{D}) = +\infty, \quad \dim_{\mathcal{H}^{h_s}}(\mathfrak{F}) = 4 \quad \textit{and} \quad \dim_{\mathcal{H}^{h_s}}(\mathfrak{M}) = 2.$$

Proof. Along the proof, let $\alpha \in (0,1)$ denote an irrational number with convergents $(a_j/q_j)_{j\geq 0}$. For our purposes, we shall use the improved version of Dirichlet's Theorem given at (2.3) but in the simplified form

$$\frac{1}{2q_jq_{j+1}} < \left|\alpha - \frac{a_j}{q_j}\right| < \frac{1}{q_jq_{j+1}} \quad \text{for each } j \in \mathbb{Z}_+. \tag{2.18}$$

In addition, note that our class of dimension functions $(h_s)_{s\geq 0}$ verifies each of the required hypotheses:

(i) $t \mapsto h_s(t)/t$ is decreasing on an open neighbourhood of $t = 0$ for all $s \geq 0$.
(ii) $\lim_{t\to 0^+} h_s(t)/t = +\infty$ for each $s \geq 0$.

Likewise, the families of approximation functions given by

$$\Psi_\delta(t) := \frac{1}{te^{t^\delta \log(t)}} \quad \text{and} \quad \Upsilon_\delta(t) := \frac{1}{te^{t^\delta}} \qquad \text{for every } \delta > 0$$

will appear on several occasions along the proof. By definition, the containment

$$\mathcal{K}^*(\Psi_\delta) \subseteq \mathcal{K}^*(\Upsilon_\delta)$$

holds for each $\delta > 0$. Besides, we claim that

$$\dim_{\mathcal{H}^{h_s}}\big(\mathcal{K}^*(\Psi_\delta)\big) = \dim_{\mathcal{H}^{h_s}}\big(\mathcal{K}^*(\Upsilon_\delta)\big) = 2/\delta \qquad \text{for all } \delta > 0. \tag{2.19}$$

Firstly, chosen any $s > 2/\delta$, the function $t \mapsto t^2 h_s(\Upsilon_\delta(t))$ is ultimately non-increasing and

$$\sum_{q\geq 1} q h_s\big(2\Upsilon_\delta(q)\big) \ll \sum_{q\geq 2} \frac{q}{(\log(q) + q^\delta)^s} < \sum_{q\geq 2} \frac{1}{q^{\delta s-1}} < \infty,$$

thus, by Theorem 2.22 we have $\dim_{\mathcal{H}^{h_s}}\big(\mathcal{K}^*(\Upsilon_\delta)\big) \leq 2/\delta$. Secondly, the function $t \mapsto t^2 h_{2/\delta}(\Psi_\delta(t))$ is ultimately non-increasing and

$$\sum_{q\geq 1} q h_s\big(2\Psi_\delta(q)\big) \gg \sum_{q\geq 2} \frac{1}{q\log(q)} = +\infty,$$

which yields the opposite inequality $\dim_{\mathcal{H}^{h_s}}\big(\mathcal{K}^*(\Psi_\delta)\big) \geq 2/\delta$. Accordingly, as desired, the claim (2.19) is established.

Now, with the aid of certain inclusions, we shall derive each of the cases separately:

- $\dim_{\mathcal{H}^{h_s}}(\mathfrak{D}) = +\infty$: let $\alpha \in \mathcal{K}^*(\Psi_\delta)$ for some $\delta > 0$, then

  $$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^{q^\delta+1}} \quad \text{for infinitely many } a, q \in \mathbb{N}.$$

  Obviously, this implies that its index is $\iota(\alpha) = +\infty$. Hence, it is plain that $\dim_{\mathcal{H}^{h_s}}(\mathfrak{D}) \geq 2/\delta$ and the result follows upon making $\delta > 0$ arbitrarily small.

- $\dim_{\mathcal{H}^{h_s}}(\mathfrak{F}) = 4$: by opposing on Flattot's condition (2.5), one deduces that $\alpha \in \mathfrak{F}$ if and only if

$$\limsup_{j\to\infty} \left(\frac{\log(q_{j+1})}{q_j^{1/2-\varepsilon}}\right) = +\infty \quad \text{for each } \varepsilon > 0. \tag{2.20}$$

Our goal is to check that $\mathcal{K}^*(\Psi_{1/2}) \subseteq \mathfrak{F} \subseteq \mathcal{K}^*(\Upsilon_{1/2-\varepsilon})$ for every $\varepsilon > 0$. Let us begin with the first inclusion: fixed any $\alpha \in \mathcal{K}^*(\Psi_{1/2})$, we know that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^{q^{1/2}+1}} \quad \text{for infinitely many } a, q \in \mathbb{N}.$$

Thus, by means of inequality (2.18), one can find a subsequence $(q_{j_m})_{m\geq 0}$ for which

$$\log(q_{j_m+1}) \gg q_{j_m}^{1/2} \log(q_{j_m}).$$

But, in such case, one has

$$\limsup_{j\to\infty} \left(\frac{\log(q_{j+1})}{q_j^{1/2-\varepsilon}}\right) \geq \lim_{m\to\infty} \left(\frac{\log(q_{j_m+1})}{q_{j_m}^{1/2-\varepsilon}}\right) \gg \lim_{m\to\infty} q_{j_m}^{\varepsilon} \log(q_{j_m}) = +\infty$$

for every $\varepsilon > 0$. This precisely means that $\alpha \in \mathfrak{F}$.

For the second inclusion, we pick arbitrary $\alpha \in \mathfrak{F}$ and $\varepsilon > 0$. Using condition (2.20), we derive the existence of a subsequence $(q_{j_m})_{m\geq 0}$ such that

$$q_{j_m+1} > e^{q_{j_m}^{1/2-\varepsilon}}.$$

By the upper bound within (2.18), one concludes

$$\left|\alpha - \frac{a_{j_m}}{q_{j_m}}\right| < \frac{1}{q_{j_m} q_{j_m+1}} < \frac{1}{q_{j_m} e^{q_{j_m}^{1/2-\varepsilon}}} \quad \text{for every } m \in \mathbb{Z}_+,$$

which assures that $\alpha \in \mathcal{K}^*(\Upsilon_{1/2-\varepsilon})$ as desired. Clearly, the result follows as $\varepsilon \to 0$.

- $\dim_{\mathcal{H}^{h_s}}(\mathfrak{M}) = 2$: chosen any $\varepsilon > 0$, it suffices to be proved that

$$\mathcal{K}^*(\Psi_1) \subseteq \mathfrak{M} \subseteq \mathcal{K}^*(\Upsilon_{1-\varepsilon}).$$

To that end, observe that $\alpha \in \mathfrak{M}$ precisely when

$$\limsup_{j\to\infty} \left(\frac{\log(q_{j+1})}{q_j/(\log(q_j))^3}\right) = +\infty. \tag{2.21}$$

Now, the argument is quite similar to the previous one: for the former inclusion, we select an arbitrary $\alpha \in \mathcal{K}^*(\Psi_1)$. Accordingly,

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^{q+1}} \quad \text{for infinitely many } a, q \in \mathbb{N}.$$

Invoking the lower bound within (2.18), this ensures the existence of a subsequence $(q_{j_m})_{m\geq 0}$ satisfying

$$\log(q_{j_m+1}) \gg q_{j_m} \log(q_{j_m}) \quad \text{as } m \to \infty.$$

Therefore, we have

$$\limsup_{j\to\infty} \left(\frac{\log(q_{j+1})}{q_j/(\log(q_j))^3}\right) \geq \lim_{m\to\infty} \left(\frac{\log(q_{j_m+1})}{q_{j_m}/(\log(q_{j_m}))^3}\right) \gg \lim_{m\to\infty} \log^4(q_{j_m}) = +\infty,$$

which exactly means that $\alpha \in \mathfrak{M}$.

On the other hand, given any $\alpha \in \mathfrak{M}$ and $\varepsilon > 0$, according to condition (2.21), the next inequality

$$\log(q_{j_m+1}) > \frac{q_{j_m}}{(\log(q_{j_m}))^3}$$

must hold for a subsequence $(q_{j_m})_{m\geq 0}$. Consequently, using (2.18) once more, one concludes

$$\left|\alpha - \frac{a_{j_m}}{q_{j_m}}\right| < \frac{1}{q_{j_m} q_{j_m+1}} < \frac{1}{q_{j_m} e^{q_{j_m}/(\log(q_{j_m}))^3}} < \frac{1}{q_{j_m} e^{q_{j_m}^{1-\varepsilon}}},$$

which proves that $\alpha \in \mathcal{K}^*(\Upsilon_{1-\varepsilon})$. Finally, the theorem is established upon doing $\varepsilon > 0$ arbitrarily small.

□

Chapter 3
The limits of Atzmon's Theorem

Despite of its length, this chapter stands out in this dissertation thank to its central role. Roughly speaking, along the forthcoming pages, we shall proof that those Liouville irrationals $\alpha \in (0,1)$ escaping the condition set up in Theorem 2.16 are so extreme that, essentially, Atzmon's Theorem no longer applies to the corresponding operators $\widetilde{T}_\alpha$. In other words, this result establishes an upper threshold for the application of Atzmon's Theorem to Bishop operators. Of course, from now on, we will be forced to apply different functional analytical tools or use new ideas to address the problem of finding invariant subspaces for the remaining Bishop operators. At this regard, some proposals shall be made in the remaining chapters.

This chapter is based on joint work with Fernando Chamizo, Eva A. Gallardo-Gutiérrez and Adrián Ubis [**37**].

3.1. The threshold of Atzmon's Theorem for Bishop operators

In this section we shall show that it is not possible to improve much on Theorem 2.16 by applying Atzmon's Theorem to the operators $\widetilde{T}_\alpha$. As we mentioned in the Introduction, several authors which have worked on the topic such as MacDonald [**92**], declared strong suspects about the possibility that Atzmon's Theorem had a limited scope of applicability. In contrast, we recall that Ambrozie [**7**] constructed specific weights $\phi \in L^\infty[0,1)$ with optimal smoothness properties:

$$\phi(t) = \exp\left(\sum_{j=-n}^{n} c_j e^{2\pi i j t} \right), \qquad t \in [0,1) \text{ and } n \in \mathbb{N},$$

such that all Bishop-type operators of the form $W_{\phi,\alpha}$ possess non-trivial invariant subspaces on each L^p-space for arbitrary irrational $\alpha \in (0,1)$, including all Liouville numbers. In a few words, the key point in Ambrozie's result is a theorem relating the smoothness of ϕ with the rates of convergence of the ergodic averages

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} \log |\phi(\{t + j\alpha\})| - \int_0^1 \log |\phi(t)| \, dt \right\|_\infty \quad \text{as } n \to \infty.$$

Naturally, these facts lead us to an intriguing open question, maybe closer to (or at least intersecting with) the field of Analytic Number Theory:

Problem. *Suppose that ϕ is a weight function in $L^\infty[0,1)$. Which is the set $\mathcal{E}_\phi$ of irrationals $\alpha \in (0,1)$ for which Atzmon's Theorem cannot be applied for the Bishop-type operator $W_{\phi,\alpha}$?*

Before stating the main result of the chapter, observe that if $L^0[0,1)$ denotes the space of (classes of) measurable functions defined almost everywhere on [0,1), $\widetilde{T}_\alpha$ is a bijection in $L^0[0,1)$ with inverse:

$$\widetilde{T}_\alpha^{-1} f(t) = e^{-1} \frac{f(\{t-\alpha\})}{\{t-\alpha\}}, \qquad t \in [0,1).$$

Nevertheless, in each $L^p[0,1)$ with $1 \le p < \infty$, the operator $\widetilde{T}_\alpha$ is an injective, dense range operator. Accordingly, there exists a norm-dense set of functions $g \in L^p[0,1)$ having an infinite chain of backward iterates, that is, for all $n > 0$ there is $g_n \in L^p[0,1)$, unique, such that $\widetilde{T}_\alpha^n g_n = g$ (see [**20**, Corollary 1.B.3], for instance). As an abuse of notation in the next theorem, for $f \in L^p[0,1)$ and $n > 0$, we will denote by $\|\widetilde{T}_\alpha^{-n} f\|_p$ the norm of the n-th backward iterate $\widetilde{T}_\alpha^{-n} f$ whenever it belongs to $L^p[0,1)$ or ∞, otherwise.

Now, we are in position to state the key result in this chapter:

Theorem 3.1. *Let $\alpha \in (0,1)$ be an irrational number not belonging to the set*

$$\mathcal{E} := \left\{ \xi \in (0,1) \,:\, \log q_{n+1} = O\left(\frac{q_n}{\log q_n} \right) \ \text{as } n \to \infty \right\},$$

and consider T_α acting on $L^p[0,1)$ for some fixed $1 \le p < \infty$. Then, for every non-zero $f \in L^p[0,1)$, we have

$$\sum_{n \in \mathbb{Z}} \frac{\log\big(1 + \|\widetilde{T}_\alpha^n f\|_p\big)}{1+n^2} = +\infty. \tag{3.1}$$

Note that condition (3.1) ensures that there does not exist a function in $L^p[0,1)$ whose iterates under the action of $\widetilde{T}_\alpha$ are dominated in p-norm by any Beurling sequence, as long as $\alpha \in \mathcal{E}$. In particular, we establish a threshold limit in the growth of the denominators of the convergents of α for the application of Atzmon's Theorem to Bishop operators.

In order to prove Theorem 3.1, we will show that either $\|\widetilde{T}_\alpha^n f\|_p$ or $\|\widetilde{T}_\alpha^{-n} f\|_p$ are large for many values of $n \in \mathbb{N}$. To accomplish such a task, we consider the identity

$$\|\widetilde{T}_\alpha^n f\|_p^p + \|\widetilde{T}_\alpha^{-n} f\|_p^p = \int_0^1 \big(e^{pL_n(t-n\alpha)} + e^{-pL_n(t)}\big)\, |f(t)|^p \, dt, \quad (n \in \mathbb{N}) \tag{3.2}$$

which follows directly from (2.2) and a change of variable. Roughly speaking, $\alpha \notin \mathcal{E}$ means that it is extremely well approximable by some rationals a/q, which will imply that $L_n(t-n\alpha)$ is essentially identical to $L_n(t - n\frac{a}{q}) = L_n(t)$ for any n near q and divisible by it. In this situation, it appears that the integral in (3.2) must be large unless $|L_n(t)|$ is small, which should happen rarely.

Before starting with the proof of Theorem 3.1, we need a technical lemma which provides the required lower bounds for $|L_n(t)|$:

Lemma 3.2. *Let a/q and A/Q be two consecutive convergents of an irrational $\alpha \in (0,1)$. For $\varepsilon \in (0,1/4)$ sufficiently small there exists a set $S_{q,\varepsilon} \subset [0,1)$ of measure at most 20ε such that*

$$\max\big(L_n(t-n\alpha), -L_n(t)\big) > \varepsilon\, \frac{n}{q} \log q \tag{3.3}$$

for every $t \notin S_{q,\varepsilon}$ and every $n \in [\varepsilon^{-2} q^2 \log q, \varepsilon^2 Q/q]$.

Proof. Given $\varepsilon \in (0,1/4)$, pick any $n \in [\varepsilon^{-2}q^2\log q, \varepsilon^2 Q/q]$. By (2.3) we have

$$\alpha = \frac{a}{q} + \frac{\delta}{qQ} \quad \text{with } |\delta| < 1$$

and our hypothesis assures $|j\delta/(qQ)| < \varepsilon^2/q^2$ for every $|j| \le n$. Hence, for $\langle qt\rangle > 2\varepsilon$, we have $\langle t + ja/q\rangle > 2\varepsilon/q$ and

$$\Big|\log\big(\{t+j\alpha\}\big) - \log\big(\{t+\frac{ja}{q}\}\big)\Big| \le \Big|\log\Big(\frac{2\varepsilon}{q} - \frac{\varepsilon^2}{q^2}\Big) - \log\Big(\frac{2\varepsilon}{q}\Big)\Big| \le \frac{\varepsilon}{q}, \quad \text{for } |j| \le n.$$

With this and the q-periodicity in j of $\log(\{t + ja/q\})$, we deduce

$$\Big|L_n(t) - \Big\lfloor\frac{n}{q}\Big\rfloor L(\{qt\})\Big| \le \varepsilon\,\frac{n}{q} + \Big|\sum_{j=n'}^{n-1}\Big(1+\log\{t+\frac{ja}{q}\}\Big)\Big|$$

where $L(x) = \sum_{\ell=0}^{q-1}\big(1+\log((x+\ell)/q)\big)$ and $n' = q\lfloor n/q\rfloor$. The trivial bound for the last term is $q\big(1-\log(2\varepsilon/q)\big)$ which is less than $2\varepsilon n/q$ in our range. A similar argument applies for $L_n(t-n\alpha)$. Therefore,

$$\tag{3.4} \begin{cases} -L_n(t) \ge -\Big\lfloor\dfrac{n}{q}\Big\rfloor L(\{qt\}) - 3\varepsilon\,\dfrac{n}{q}, \\ L_n(t-n\alpha) \ge \Big\lfloor\dfrac{n}{q}\Big\rfloor L(\{qt\}) - 3\varepsilon\,\dfrac{n}{q}, \end{cases} \qquad \text{for every } \langle qt\rangle > 2\varepsilon.$$

Clearly, the function $L(x)$ is continuous and increasing in the interval $(0,1)$. Moreover, the following absolute lower bound holds for the derivative

$$L'(x) = \sum_{\ell=0}^{q-1}\frac{1}{q}\,\frac{q}{x+\ell} \ge \sum_{\ell=1}^{q}\frac{1}{\ell} > \log q.$$

Correspondingly, the measure of the interval $\big\{x \in (0,1) : |L(x)| \le 8\varepsilon\log q\big\}$ is at most 16ε. Hence, for $\varepsilon > 0$ sufficiently small, the inequalities (3.4) give the expected bound except in the set

$$S_{q,\varepsilon} = \{t \in [0,1) : \langle qt\rangle \le 2\varepsilon\} \cup \{t \in [0,1) : |L(\{qt\})| \le 8\varepsilon\log q\},$$

which has measure lower or equal to 20ε. □

With this lemma at hand, we are ready to prove Theorem 3.1. The basic idea behind the proof is easily seen from the statement of Lemma 3.2: as long as Q increases much more rapidly than q, the lower bound (3.3) is true for really large sets of $n \in \mathbb{N}$, forcing the series (3.1) to be divergent.

Proof of Theorem 3.1. Without loss of generality, assume that $f \in L^p[0,1)$ has an infinite chain of backward iterates $\widetilde{T}_\alpha^{-n} f \in L^p[0,1)$ and suppose $\|f\|_p = 1$. Clearly, $\alpha \notin \mathcal{E}$ precisely means that

$$\limsup_{j\to\infty}\Big(\frac{\log q_{j+1}}{q_j/\log q_j}\Big) = +\infty.$$

Hence, there exists a subsequence $(q_{j_m})_{m\in\mathbb{N}}$ such that

$$\frac{\log Q_{j_m}}{q_{j_m}/\log q_{j_m}} > m^2, \quad \text{with } Q_{j_m} = q_{j_m+1}$$

for every $m > 2$. Now, consider the family of sets given by

$$S_{m_*} := \bigcup_{m\ge m_*} S_{q_{j_m},1/m^2} \qquad \text{for each } m_* > 2,$$

with $S_{q,\varepsilon}$ defined as in Lemma 3.2. Since the series $\sum_{m>2} 1/m^2 < \infty$, we have that

$$\lim_{m_*\to\infty} \int_{S_{m_*}} |f(t)|^p \, dt = 0$$

so there must exist an $m_* \in \mathbb{N}$ sufficiently large for which $\int_{S_{m_*}} |f(t)|^p dt < 1/2$. Using this fact and equation (3.2), we may infer the inequality

$$\begin{aligned}\|\widetilde{T}_\alpha^n f\|_p^p + \|\widetilde{T}_\alpha^{-n} f\|_p^p &\geq \int_{[0,1]\setminus S_{m^*}} \left(e^{pL_n(t-n\alpha)} + e^{-pL_n(t)}\right) |f(t)|^p \, dt \\ &\geq \frac{1}{2} \inf_{t\notin S_{m_*}} \left(e^{pL_n(t-n\alpha)} + e^{-pL_n(t)}\right) \geq \frac{1}{2} \max_{t\notin S_{m_*}} \left(e^{pL_n(t-n\alpha)}, e^{-pL_n(t)}\right).\end{aligned}$$

Now, by Lemma 3.2 with $q = q_{j_m}$, $m \geq m_*$ and $\varepsilon = 1/m^2$, we obtain the bound

$$\|\widetilde{T}_\alpha^n f\|_p^p + \|\widetilde{T}_\alpha^{-n} f\|_p^p \geq \frac{1}{2} \exp\left(\frac{pn \log q_{j_m}}{m^2 q_{j_m}}\right)$$

for any $n \in [m^2 q_{j_m}^2 \log q_{j_m}, m^{-2} Q_{j_m}/q_{j_m}]$, so that

$$\sum_{m^2 q_{j_m}^3 < |n| < m^{-2} Q_{j_m}/q_{j_m}} \frac{\log\left(1 + \|\widetilde{T}_\alpha^n f\|_p\right)}{1+n^2} \gg \log\left(\frac{Q_{j_m}}{m^4 q_{j_m}^4}\right) \frac{\log q_{j_m}}{q_{j_m}} \frac{1}{m^2} \gg 1 \tag{3.5}$$

for any m sufficiently large. Finally, as a consequence of the following chain of inequalities

$$\frac{Q_{j_m}}{m^2 q_{j_m}} < Q_{j_m} = q_{j_m+1} \leq q_{j_{m+1}} \leq (m+1)^2 q_{j_{m+1}}^3,$$

we observe that the intervals defined by the indexes of the sum in (3.5) do not overlap for different values of m. Hence, the theorem follows. □

Chapter 4

Local spectral properties of Bishop operators

In this chapter, we aim to analyse Bishop operators using techniques borrowed from Local Spectral Theory. Heuristically, the reason to proceed this way is suggested by the apparent dichotomy arising from the facts presented above: while all Bishop operators seem to share an equivalent spectral structure, Atzmon's Theorem only grants the existence of invariant subspaces for a few of them. Accordingly, maybe, a deeper insight in $\sigma(T_\alpha)$ might help us to deal with those cases which cannot be covered by means of Atzmon's Theorem. In general, this naive attempt of studying invariant subspaces via spectral subsets rarely succeeds or turns out to be very difficult to implement. Nevertheless, as we shall argue at the end of this PhD thesis, our suspicions on the importance of having a better knowledge of $\sigma(T_\alpha)$ will be strongly confirmed in Chapter 5.

The first part of the present chapter is devoted to deepening into certain specific aspects of Local Spectral Theory. Besides, we shall introduce the crucial notion of power-regularity. Later on, we will be able to characterize those local spectral properties fulfilled simultaneously by all Bishop operators T_α, independently of the irrational symbol $\alpha \in (0,1)$. Indeed, some of these characterizations will be also true for many other weighted translation operators $W_{\phi,\tau}$ whose shift τ is uniquely ergodic.

The bulk of results in this chapter may be found in a joint work with Eva A. Gallardo [**66**]. Likewise, our last results concerning Dunford's property in the context of Bishop operators come from the article [**37**].

4.1. A deeper insight on Local Spectral Theory and power-regular operators

Recall that in Subsection 1.2.2, we left open several questions about the relation among the main local spectral properties. Herein, we will establish those implications between decomposability, property (β), Dunford's property and SVEP. Further on, we will hold a thorough discussion about the role of duality theory in Local Spectral Theory and power-regular operators.

For the sake of completeness, we begin by proving some connections between properties in Local Spectral Theory (see also the monographs by Aiena [**2**] and by Laursen and Neumann [**89**]), which will be of use in our approach.

Theorem 4.1. *Let $T \in \mathcal{B}(X)$ be an arbitrary operator acting on a complex Banach space X. Then, the following implications always hold:*

(i) *If T is decomposable, then T verifies property (β).*
(ii) *If T has property (β), then T enjoys Dunford's property (C).*
(iii) *Finally, T has the SVEP if and only if $X_T(\emptyset) = 0$, and this is the case if and only if $X_T(\emptyset)$ is norm-closed in X. In particular, property (C) implies SVEP.*

Before giving a proof of this theorem, we need a previous lemma which ensures that the local spectrum is stable under the action of the local resolvent functions:

Lemma 4.2. *Let $T \in \mathcal{B}(X)$ be a linear bounded operator acting on a complex Banach space X. Fixed $x \in X$ and an open set $U \subseteq \mathbb{C}$, consider $f \in \mathcal{O}(U, X)$ verifying the equation*

$$(T - z)f(z) = x \quad \text{for all } z \in U.$$

Then, $f(\lambda) \in T_U\, \mathcal{O}(U, X)$ for each $\lambda \in U$. More precisely, $\sigma_T(x) = \sigma_T(f(\lambda))$ for all $\lambda \in U$.

Proof. Chosen any $\lambda \in U$, consider the following function in $\mathcal{O}(U, X)$:

$$g(z) := \begin{cases} \frac{f(z)-f(\lambda)}{z-\lambda}, & \text{if } z \in U \setminus \{\lambda\}, \\ f'(\lambda), & \text{if } z = \lambda. \end{cases}$$

A routine verification of the identity $(T - z)g(z) = f(\lambda)$, valid for every $z \in U$, confirms that $f(\lambda) \in T_U\, \mathcal{O}(U, X)$.

Observe that our preceding discussion also implies $U \subseteq \rho_T(f(\lambda))$. Now, for arbitrary $\mu \in \rho_T(x) \setminus U$, choose an open neighbourhood $V \subseteq \rho_T(x)$ for which $\lambda \notin V$. By definition, there exists $h \in \mathcal{O}(V, X)$ such that $(T - z)h(z) = x$ for all $z \in V$. Since the function $k \in \mathcal{O}(V, X)$ defined as

$$k(z) := \frac{h(z) - f(\lambda)}{z - \lambda}, \qquad \text{for all } z \in V,$$

verifies the equation $(T - z)k(z) = f(\lambda)$ for every $z \in V$, we conclude that $\mu \in \rho_T(f(\lambda))$. Clearly, this establishes the inclusion $\rho_T(x) \subseteq \rho_T(f(\lambda))$.

Conversely, fixed $\mu \in \rho_T(f(\lambda))$, consider a function $l \in \mathcal{O}(W, X)$ on some open neighbourhood $W \ni \mu$, such that $(T - z)l(z) = f(\lambda)$ for each $z \in W$. Then,

$$(T - z)(T - \lambda)l(z) = (T - \lambda)f(\lambda) = x, \qquad \text{for all } z \in W.$$

Of course, this proves the opposite inclusion $\rho_T(f(\lambda)) \subseteq \rho_T(x)$. □

Now, we are in position to prove all the implications appearing in Theorem 4.1:

Proof of Theorem 4.1. We prove each item separately.

(i) Suppose that $T \in \mathcal{B}(X)$ is a decomposable operator. In the sequel, let $U \subseteq \mathbb{C}$ be an arbitrary open set and consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U, X)$ verifying

$$(T - z)f_n(z) \to 0 \quad \text{as } n \to \infty$$

uniformly on compact subsets of U. Now, provided two open disks $V, W \subseteq U$ related by the inclusions $V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq U$. Applying the definition of decomposability to the open cover $\{W, \mathbb{C} \setminus \overline{V}\}$, we obtain two closed T-invariant subspaces $\mathcal{M}_1, \mathcal{M}_2 \subseteq X$ satisfying

$$X = \mathcal{M}_1 + \mathcal{M}_2, \quad \sigma(T|\mathcal{M}_1) \subseteq W \quad \text{and} \quad \sigma(T|\mathcal{M}_2) \subseteq \mathbb{C} \setminus \overline{V}.$$

As a consequence of a theorem due to Gleason [**72**], we may find two sequences $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U,\mathcal{M}_1)$ and $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U,\mathcal{M}_2)$ respectively, for which

$$f_n(z) = g_n(z) + h_n(z) \quad \text{for every } z \in U \text{ and } n \in \mathbb{N}.$$

Recall that, from Proposition 1.6, we have the spectral inclusions

$$\sigma(T/\mathcal{M}_2) \subseteq \widehat{\sigma(T|\mathcal{M}_1)} \subseteq W,$$

which, in particular, implies that $T/\mathcal{M}_2 - z$ is invertible on the quotient space $X/\mathcal{M}_2$ for each $z \in \partial W$. So, by compactness, there exists an absolute constant $C > 0$ such that

$$\max_{z\in\partial W} \left\|(T/\mathcal{M}_2 - z)^{-1}\right\| \leq C.$$

Now, consider the canonical quotient operator $Q : X \to X/\mathcal{M}_2$. Clearly,

$$Qg_n(z) = Qf_n(z) = (T/\mathcal{M}_2 - z)^{-1}Q(T - z)f_n(z) \quad \text{for every } z \in \partial W,$$

which yields the inequality

$$\left\|Qg_n(z)\right\|_{X/\mathcal{M}_2} \leq C \left\|(T - z)f_n(z)\right\|_{X/\mathcal{M}_2} \text{ for each } z \in \partial W.$$

As a consequence of the assumption on the sequence $(f_n)_{n\in\mathbb{N}}$, we conclude that $Qg_n(z) \to 0$ uniformly on ∂W. Thus, applying the *Maximum Modulus Principle*, we can extrapolate the uniform convergence of $Q \circ g_n \to 0$ to $\overline{W}$. Now, thank to the identification

$$\mathcal{O}(U, X/\mathcal{M}_2) \cong \mathcal{O}(U,X)\big/\mathcal{O}(U,\mathcal{M}_2),$$

we obtain a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U,\mathcal{M}_2)$ for which $g_n + k_n \to 0$ uniformly on compact subsets of W as $n \to \infty$, and so uniformly on $\overline{V}$.

Finally, since $f_n = g_n + h_n = (g_n + k_n) + (h_n - k_n)$, it remains to be proved that $h_n - k_n \to 0$ as $n \to \infty$ uniformly on $\overline{V}$. As above, bearing in mind that $\sigma(T|\mathcal{M}_2) \cap \overline{V} = \emptyset$, by compactness we may find a constant $K > 0$ for which

$$\max_{z\in\overline{V}} \left\|(T|\mathcal{M}_2 - z)^{-1}\right\| \leq K.$$

Consequently, for every $z \in \overline{V}$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}\left\|(h_n - k_n)(z)\right\|_X &\leq K \left\|(T - z)(h_n - k_n)(z)\right\|_X \\ &\leq K \left\|(T - z)f_n(z)\right\|_X + K \left\|(T - z)(g_n + h_n)(z)\right\|_X.\end{aligned}$$

Therefore, $h_n - k_n$ converges uniformly to 0 on $\overline{V}$. Since the closed disk $\overline{V} \subseteq U$ was elected arbitrarily, we conclude that $f_n(z) \to 0$ uniformly on compact subsets of U. Thus, T satisfies property (β), as desired.

(ii) Now, suppose that $T \in \mathcal{B}(X)$ is an operator having property (β). For any closed $F \subseteq \mathbb{C}$, denote by $Z_T(F)$ the set of all $x \in X$ such that for arbitrary compact $K \subseteq \mathbb{C} \setminus F$ and $\varepsilon > 0$ there exists an open neighbourhood $U \supseteq K$ and $f \in \mathcal{O}(U,X)$ for which

$$\left\|x - (T - z)f(z)\right\|_X < \varepsilon \quad \text{for all } z \in K.$$

Evidently, $Z_T(F)$ is a closed invariant subspace for T. So, we claim that our objective will be showing that, under our current hypothesis, $X_T(F) = Z_T(F)$.

Clearly, the inclusion $X_T(F) \subseteq Z_T(F)$ is straightforward, so we need just to prove the opposite one. To this end, consider any $x \in Z_T(F)$. Chosen arbitrary $\lambda \in \mathbb{C} \setminus F$, pick any open set $U \ni \lambda$ such that $\overline{U}$ is a compact subset of $\mathbb{C} \setminus F$.

By definition, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of X-valued analytic functions on an open neighbourhood of $\overline{U}$ verifying

$$\big\|x-(T-z)f_n(z)\big\|_X<\frac{1}{n}\quad\text{for all } z\in\overline{U}. \tag{4.1}$$

Consequently, for every pair of $m,n\in\mathbb{N}$, we have

$$\big\|(T-z)(f_m(z)-f_n(z))\big\|_X<\frac{1}{m}+\frac{1}{n}\quad\text{for all } z\in\overline{U},$$

from which we deduce that $(T-z)(f_m(z)-f_n(z))\to 0$ as $m,n\to\infty$ on compact subsets of U. Now, bearing in mind that T has property (β), we infer that $f_m(z)-f_n(z)\to 0$ as $m,n\to\infty$ again locally uniformly on U.

Of course, this is equivalent to the fact that $(f_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathcal{O}(U,X)$ which converges locally uniformly to some $f\in\mathcal{O}(U,X)$. From inequality (4.1) we easily conclude that

$$(T-z)f(z)=x\quad\text{for every } z\in U.$$

Hence $\lambda\in\rho_T(x)$, which establishes that $x\in X_T(F)$. As desired, this shows that T enjoys property (C).

(iii) First, suppose that T enjoys SVEP. Taken any $x\in X_T(\emptyset)$, we can find an entire X-valued function $f\in\mathcal{O}(\mathbb{C},X)$ such that $(T-z)f(z)=x$ for every $z\in\mathbb{C}$. Since $f(z)=(T-z)^{-1}x$ for all $z\in\rho(T)$ and $\|(T-z)^{-1}\|\to 0$ as $|z|\to\infty$, we deduce that $f\in\mathcal{O}(\mathbb{C},X)$ is also bounded. Consequently, a direct application of Liouville's Theorem (see Theorem 1.2) gives us that $f\equiv 0$ on $\mathbb{C}$. This shows that $X_T(\emptyset)=\{0\}$.

Conversely, assume that $X_T(\emptyset)=\{0\}$. Given any open set $U\subseteq\mathbb{C}$, consider $f\in\mathcal{O}(U,X)$ such that $(T-z)f(z)=0$ for all $z\in U$. Since $\sigma_T(0)=\emptyset$, we infer from Lemma 4.2 that $\sigma_T(f(\lambda))=\emptyset$ for each $\lambda\in U$. So, by our current hypothesis, we deduce that $f(\lambda)=0$ for all $\lambda\in U$. Evidently, this indicates that T has SVEP.

Now, it remains to be shown that $X_T(\emptyset)=\{0\}$ precisely when $X_T(\emptyset)$ is norm-closed in X. So, henceforth, suppose that the local spectral manifold $X_T(\emptyset)$ is norm-closed. Again, an immediate application of Lemma 4.2 shows that

$$(T-\lambda)X_T(\emptyset)=X_T(\emptyset)\qquad\text{for all }\lambda\in\mathbb{C}.$$

This exactly means that the restriction $S:=T|X_T(\emptyset)$ satisfies that $S-\lambda$ is surjective for each $\lambda\in\mathbb{C}$. Now, by the Closed Range Theorem (see Theorem 1.1), we conclude that the adjoint $S^*-\lambda$ must always have closed range and trivial kernel since

$$\ker(S^*-\lambda)=\operatorname{ran}(S-\lambda)^\perp=\{0\}.$$

Clearly, this is equivalent to $\sigma_{\mathrm{ap}}(S^*)=\emptyset$, which, as a by-product of Proposition 1.8, cannot hold unless

$$\{0\}=X_T(\emptyset)^*\cong X^*\big/X_T(\emptyset)^\perp.$$

As long as $X_T(\emptyset)$ is norm-closed, this can only occur whenever $X_T(\emptyset)=\{0\}$.

To complete the proof, observe that, by our preceding argument, now Dunford's property (C) immediately entails SVEP.

□

4.1.1. The duality correspondence in Local Spectral Theory

Just as for ordinary Spectral Theory, the significance of duality will be also major in Local Spectral Theory. Herein, our main objective will be to establish certain relations between the local spectral properties of an operator $T \in \mathcal{B}(X)$ and its adjoint $T^* \in \mathcal{B}(X^*)$.

In this spirit, our first result proves that the glocal spectral manifolds behave as expected with respect to annihilators. Originally, this statement was demonstrated by Frunză [**64**] in 1976. Here, we follow the proof provided in [**89**, Proposition 2.5.1], which is written in agreement with much modern terminology:

Proposition 4.3. *Consider a continuous linear operator $T \in \mathcal{B}(X)$ on a complex Banach space X. Then, the inclusions*

$$\mathcal{X}_T(F) \subseteq {}^{\perp}\mathcal{X}^*_{T^*}(G) \quad \textit{and} \quad \mathcal{X}^*_{T^*}(G) \subseteq \mathcal{X}_T(F)^{\perp}$$

hold for each pair of disjoint closed sets $F, G \subseteq \mathbb{C}$.

Proof. Observe that, without loss of generality, we may suppose that F and G are both subsets of $\sigma(T)$. To prove the statement, we just need to show that provided arbitrary $x \in \mathcal{X}_T(F)$ and $\varphi \in \mathcal{X}^*_{T^*}(G)$, their composition $\varphi(x)$ vanishes.

To accomplish our task, consider a global local resolvent for each element, i.e. analytic functions $f \in \mathcal{O}(\mathbb{C} \setminus F, X)$ and $\psi \in \mathcal{O}(\mathbb{C} \setminus G, X^*)$ such that

$$(T - z)f(z) = x \text{ for all } z \in \mathbb{C} \setminus F \quad \text{and} \quad (T^* - z)\psi(z) = \varphi \text{ for all } z \in \mathbb{C} \setminus G.$$

Then, taking the adjoint on the intersection of their domains, we obtain

$$\varphi(f(z)) = ((T^* - z)\psi(z))(f(z)) = \psi(z)((T - z)f(z)) = \psi(z)(x)$$

for every $z \in (\mathbb{C} \setminus F) \cap (\mathbb{C} \setminus G)$.

Consequently, the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) := \begin{cases} \varphi(f(z)) & \text{for } z \in \mathbb{C} \setminus F, \\ \psi(z)(x) & \text{for } z \in \mathbb{C} \setminus G, \end{cases}$$

is an entire holomorphic function. Now, by a standard application of Liouville's Theorem (see Theorem 1.2), we deduce that $h \equiv 0$ on the whole complex plane.

To complete the proof, we need to do a small trick: first, observe that the following limit holds in the norm-topology of X:

$$\lim_{|z| \to \infty} z\,(T - z)^{-1}x = -x.$$

So, taking the limit as $|z| \to \infty$ in the next identity

$$\varphi(z\,(T - z)^{-1}x) = \varphi(zf(z)) = zh(z) = 0 \text{ for all } z \in \rho(T),$$

by the continuity of the functional φ, we finally infer that $\varphi(x) = 0$, as required. □

Suggested by the generality of Proposition 4.3, it sounds reasonable to expect that, at least for certain well-behaved operators, one might obtain much stronger conclusions. In this fashion, decomposable operators are known to be governed by a tight dual pairing:

Theorem 4.4. *An operator $T \in \mathcal{B}(X)$ is decomposable if and only if its adjoint $T^* \in \mathcal{B}(X^*)$ is also decomposable. In such a case, we have that*

$$X_T(F) = {}^{\perp}X^*_{T^*}(\mathbb{C} \setminus F) \quad \textit{and} \quad X^*_{T^*}(F) = X_T(\mathbb{C} \setminus F)^{\perp}$$

for every closed set $F \subseteq \mathbb{C}$.

Such a duality principle built upon decomposable operators was initially explored by Frunză (see [**63, 64, 65**]) in the 1970's and successfully completed, independently, by Eschmeier [**53**] and by Wang and Liu [**125**] in 1984. Although, some of the ideas are present in the work of Bishop [**26**].

A few years later, Albrecht and Eschmeier [**6**] were able to extend this correspondence to operators having weaker spectral properties. Specifically, they proved that both properties (β) and (δ) are tied to each other by a complete duality, in the sense that an operator $T \in \mathcal{B}(X)$ enjoys either of the properties (β) or (δ) if and only if its adjoint $T^* \in \mathcal{B}(X^*)$ enjoys the other one. Furthermore, in the work of Albrecht and Eschmeier two remarkable characterizations are provided, illustrating that properties (β) and (δ) can be comprehended in terms of decomposability. Namely, those operators satisfying Bishop's property (β) are, up to similarity, the restriction of a decomposable operator; whilst, those operators having the decomposition property (δ) are, up to similarity, the quotient of a decomposable operator. We state it for later reference:

Theorem 4.5 (Albrecht-Eschmeier, [6]). *Let $T \in \mathcal{B}(X)$ be a bounded linear operator on a complex Banach space X. Then:*

(i) *T satisfies Bishop's property (β) if and only if it is similar to the restriction of a decomposable operator to one of its closed invariant subspaces.*
(ii) *T enjoys the decomposition property (δ) if and only if it is similar to the quotient of a decomposable operator with respect to one of its closed invariant subspaces.*

Moreover, T has property (β) or (δ) if and only if its adjoint T^* has the other one.

Unfortunately, even a highly heuristic approach to this theorem would force us to take a long detour (for a careful treatment, we refer to [**89**, Chapter 2]). Just mention that the strategy for the proof essentially stands upon three cornerstones. First, the characterization of the properties (β) and (δ) in terms of decomposability is achieved thank to the combination of a nice transference lemma (see [**89**, Lemmas 2.2.1 and 2.2.3]) along with the construction of certain functional models, based on vector-valued Sobolev-type and Bergman-type spaces. Finally, the dual correspondence between those functional models is obtained as an application of the *Grothendieck-Köthe Duality Principle* [**73, 87**] (see also [**80**] and/or [**88**] for a thorough discussion).

Quite remarkable, it is worth mentioning that the Albrecht-Eschmeier functional models can be blended with the Scott-Brown technique to produce invariant subspaces for operators having one of the properties (β) or (δ). In a sense, this guarantees that a non-negligible portion of the invariant subspace theory in Hilbert spaces can be successfully extended to the Banach space setting.

Theorem 4.6 (Eschmeier-Prunaru, [54]). *Let $T \in \mathcal{B}(X)$ be a continuous linear operator on a Banach space X which satisfies either property (β) or property (δ). If the spectrum $\sigma(T)$*

is thick, then T has a non-trivial invariant subspace. Furthermore, $\mathrm{Lat}(T)$ *is rich provided that $\sigma_{\mathrm{e}}(T)$ is thick.*

Clearly, this theorem subsumes the classical result due to S. W. Brown [**31, 32**] for hyponormal operators (see, Theorem 1.14), since it may be seen that every hyponormal operator on a Hilbert space automatically has property (β) (see, for instance, [**58**, Prop. 6.4.3 and Cor. 6.4.8]).

Indeed, as shown lately by Eschmeier and Putinar [**55**], this approach still admits a notable extension when the notion of localizable spectrum comes into play. Recall that the *localizable spectrum* of a Banach space operator $T \in \mathcal{B}(X)$ is the closed subset of $\sigma(T)$ given by

$$\sigma_{\mathrm{loc}}(T) := \big\{\lambda \in \mathbb{C} \,:\, X_T(\overline{U}) \neq \{0\} \text{ for each open neighbourhood } U \ni \lambda\big\}.$$

Obviously, every operator with property (δ) immediately verifies $\sigma(T) = \sigma_{\mathrm{loc}}(T)$.

Theorem 4.7 (Eschmeier-Prunaru, [55]). *Consider $T \in \mathcal{B}(X)$ acting on a complex Banach space X such that $\sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T^*) = \emptyset$. If either $\sigma_{\mathrm{loc}}(T)$ or $\sigma_{\mathrm{loc}}(T^*)$ are thick sets, then the lattice* $\mathrm{Lat}(T)$ *is rich.*

4.1.2. Spectral inclusions, local spectral radius and power-regularity

We begin this part by recalling some results in Local Spectral Theory with regard to the Riesz Functional Calculus. Subsequently, we will deliver a brief digression on various properties of the local spectral radius and power-regular operators.

Recall from Section 1.2 that the Riesz Functional Calculus also yields a spectral relationship by means of the Spectral Mapping Theorem. It is worth pointing out that similar results can be derived for many parts of the spectrum (see, for instance, the monograph by Aiena [**2**] and/or the axiomatic approach done by Kordula and Müller [**86**]). As notable cases, just mention that both the approximate point spectrum and the surjectivity spectrum admit a spectral mapping theorem (for a nice treatment, see [**2**, Theorem 2.48]). What is more, whenever our operator T enjoys the SVEP, its local spectra obey a similar law (for a proof, we refer to [**89**, Theorem 3.3.8]).

Theorem 4.8 (Local Spectral Mapping Theorem). *Let $T \in \mathcal{B}(X)$ be an operator acting on a complex Banach space X and consider an analytic function f on an open neighbourhood of $\sigma(T)$. Then,*

$$f\big(\sigma_T(x)\big) \subseteq \sigma_{f(T)}(x) \quad \text{for all } x \in X.$$

Moreover, equality holds whenever T has the SVEP.

The key step in the preceding result passes through the verification of the identity

$$\mathcal{X}_{f(T)}(F) = \mathcal{X}_T\big(f^{-1}(F)\big), \quad \text{for every closed set } F \subseteq \mathbb{C}, \tag{4.2}$$

which is valid for each admissible holomorphic function $f \in \mathcal{O}(\sigma(T))$. This establishes that glocal spectral manifolds behave canonically with respect to the Riesz Functional Calculus. Indeed, a direct use of (4.2) illustrates that all the aforementioned local spectral properties are preserved under the Riesz Functional Calculus [**89**, Theorem 3.3.6].

We remind the reader that, for any operator $T \in \mathcal{B}(X)$, the Gelfand's Formula ensures that the sequence $(\|T^n\|^{1/n})_{n\geq 0}$ is always convergent to the spectral radius $r(T)$. Notwithstanding, if we aspire to wield a local version of Gelfand's formula, we will need to carry out slight modifications, since it turns out that $(\|T^n x\|_X^{1/n})_{n\in\mathbb{Z}_+}$ may be non-convergent for certain $x \in X$. This forces us to define the *local spectral radius of T at the vector $x \in X$* as

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|_X^{1/n}.$$

Not surprisingly, the local spectral radius and the local spectrum are related by the inequality

$$r_T(x) \geq \max\{|\lambda| \,:\, \lambda \in \sigma_T(x)\}, \tag{4.3}$$

which, in fact, can be replaced by an equality as long as T satisfies the SVEP.

A general result due to Müller [**96**] asserts that given any operator $T \in \mathcal{B}(X)$, the equality $r_T(x) = r(T)$ must hold on a dense subset of X, which is indeed of the second second category (see also a well-known result by Daneš [**43**]). For the sake of completeness, below we prove that this equality actually holds for the surjectivity spectrum (see, for instance, [**2**, Theorem 2.43]):

Proposition 4.9. *For every Banach space operator $T \in \mathcal{B}(X)$ we have*

$$\sigma_{\mathrm{su}}(T) = \bigcup_{x\in X} \sigma_T(x).$$

Moreover, the set $\{x \in X \,:\, \sigma_T(x) = \sigma_{\mathrm{su}}(T)\}$ is of the second category in X.

Proof. First, observe that $T - \lambda : X \to X$ is surjective whenever $\lambda \in \rho_T(x)$ for every $x \in X$. This proves the inclusion $\sigma_{\mathrm{su}}(T) \subseteq \bigcup\{\sigma_T(x) \,:\, x \in X\}$.

Conversely, taken $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{su}}(T)$, by the *Open Mapping Principle*, we conclude that the operator $T - \lambda \in \mathcal{B}(X)$ is open; i.e. there exists an absolute constant $c > 0$ such that, for every $y \in X$, we may find a preimage $(T - \lambda)x = y$ holding the inequality $c\,\|x\|_X \leq \|y\|_X$.

Hence, for $x_0 \in X$ arbitrarily given, we can construct a sequence $(x_n)_{n\in\mathbb{Z}_+}$ in X such that

$$(T-\lambda)x_n = x_{n-1} \quad \text{and} \quad c\,\|x_n\|_X \leq \|x_{n-1}\|_X, \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, the infinite series

$$f(z) := \sum_{n=1}^{\infty} x_n (z-\lambda)^{n-1}$$

converges in the Fréchet space $\mathcal{O}(D(\lambda, c), X)$ and, according to the identity $(T - z)f(z) = x_0$ for all $z \in D(\lambda, c)$, we have that $\lambda \in \rho_T(x_0)$. Clearly, this yields the opposite inclusion.

Now, choose a countable dense subset Λ of $\sigma_{\mathrm{su}}(T)$. By definition, $\mathrm{ran}(T - \lambda) \neq X$ for each $\lambda \in \Lambda$. Moreover, an immediate application of the Open Mapping Principle prevents $\mathrm{ran}(T - \lambda)$ from being non-meager. Since meagerness is preserved by countable unions, we conclude that

$$M := \bigcup_{\lambda\in\Lambda} \mathrm{ran}(T-\lambda)$$

is again of the first category. Finally, note that any $x \in X \setminus M$ satisfies $\Lambda \subseteq \sigma_T(x)$. Consequently, by closeness, we have that $\sigma_{\mathrm{su}}(T) = \overline{\Lambda} \subseteq \sigma_T(x)$ for every $x \in X \setminus M$ and the result is proved. ∎

Corollary 4.10. *For any operator $T \in \mathcal{B}(X)$ acting on a Banach space X, the set of elements $\{x \in X \,:\, r_T(x) = r(T)\}$ is always of the second category in X.*

Proof. Choose any element $x \in X$ for which $\sigma_T(x) = \sigma_{\rm su}(T)$. Since

$$r(T) \geq r_T(x) \geq \max\big\{|\lambda| \,:\, \lambda \in \sigma_T(x)\big\},$$

and, by Proposition 1.8, the boundary $\partial\sigma(T) \subseteq \sigma_{\rm su}(T)$, the result follows immediately from the second assertion of Proposition 4.9. □

Although the local spectral radius behaves satisfactorily on a large subset of the space X, we must impose strong assumptions on T in order to ensure that the set of vectors for which $(\|T^n x\|_X^{1/n})_{n\geq 0}$ is convergent, exhausts the entire space X. For instance, this situation occurs when T is a compact operator, a selfadjoint operator or a normal operator (defined on a Hilbert space), or even for a more general class of operators with totally disconnected spectrum (see [**43**]). At this regard, Atzmon [**16**] introduced the notion of *power-regularity*:

Definition 4.11. *A Banach space operator $T \in \mathcal{B}(X)$ is said to be* power-regular *if the sequence $(\|T^n x\|_X^{1/n})_{n\geq 0}$ is convergent for all $x \in X$.*

Indeed, Atzmon proved a general criterion showing that a wide class of Banach space operators, including all those with property (β), belong to the family of power-regular operators. Consequently, all spectral operators in Dunford's sense and all operators with totally disconnected spectrum are power-regular (see [**38**], for instance). Moreover, as Atzmon points out, all operators considered in [**13**] annihilated by a non-zero analytic function are power-regular or, in particular, operators of class C_0 (see [**16**]). In addition, for every operator $T \in \mathcal{L}(X)$ belonging to one of those classes, the sequence $(\|T^n x\|_X^{1/n})_{n\geq 1}$ converges for all $x \in X$ to the spectral radius of the restriction of T to the cyclic subspace $\mathcal{M}_x := \overline{\operatorname{span}\{T^n x :\ n \geq 0\}}$.

We state Atzmon's result relating power-regularity and property (β) for later reference:

Theorem 4.12 (Atzmon, [16]). *Let $T \in \mathcal{B}(X)$ operate on a complex Banach space X. Suppose further that T has property (β). Then, T is power-regular, i.e.*

$$r_T(x) = \lim_{n\to\infty} \|T^n x\|_X^{1/n}, \quad \textit{for every } x \in X.$$

Likewise, Atzmon [**16**] introduced the concept of *radially decomposable operators*, a condition which is considerably weaker than decomposability. Namely, a Banach space operator $T \in \mathcal{B}(X)$ is radially decomposable if for every $0 < t_1 < t_2 < \infty$, there exist two invariant subspaces $\mathcal{M}_1, \mathcal{M}_2 \in \operatorname{Lat}(T)$ such that $X = \mathcal{M}_1 + \mathcal{M}_2$ and

$$\begin{aligned} i(T|\mathcal{M}_1) &= \min\big\{|\lambda| \,:\, \lambda \in \sigma(T|\mathcal{M}_1)\big\} \geq t_1, \\ r(T|\mathcal{M}_2) &= \max\big\{|\lambda| \,:\, \lambda \in \sigma(T|\mathcal{M}_2)\big\} \leq t_2. \end{aligned}$$

Atzmon conjectured that, as occurs for decomposable operators, all radially decomposable operators are likely to be power-regular. Here above, we have used the notation

$$i(T) := \min\big\{|\lambda| \,:\, \lambda \in \sigma(T)\big\}$$

to denote the *inner spectral radius* of an arbitrary operator $T \in \mathcal{B}(X)$.

4.2. A characterization of the local spectral properties for Bishop operators

The programme we follow in this section consists in identifying which local spectral properties are simultaneously fulfilled by a wide class of weighted translation operators.

More precisely, at the beginning of the section, as a nice application of ergodic theorems, we will be able to discard decomposability, property (β) and property (δ) for many ergodic weighted translations. For this purpose, the notion of power-regularity will turn out to be a crucial tool. At the end, in the specific case of Bishop operators, thank to some estimations already established in Chapter 2, we will exhibit that Bishop operators can neither satisfy Dunford's property.

4.2.1. Power-regular weighted translations and decomposability

Once more, to avoid technical problems, in the sequel we set our framework $(\Omega, \mathcal{G}, \mu, \tau)$ to be a measure-preserving system. In addition, to ensure the boundedness of the weighted translation operators $W_{\phi,\tau}$, the corresponding weights ϕ are always assumed to lie within $L^\infty(\Omega, \mu)$. In general, these assumptions will not be explicitly stated in each result.

As discussed in Section 1.5, from the pioneering work of Parrott [**103**], one can characterize several distinguished parts of the spectrum of an ergodic weighted translation operator $W_{\phi,\tau}$. At this regard, in Theorem 1.44 we stated that, as a consequence of the shape of its point spectrum $\sigma_{\mathrm{p}}(W_{\phi,\tau})$, every ergodic weighted translation $W_{\phi,\tau}$ automatically enjoys the SVEP. Not surprisingly, for the remainder of local spectral properties, we shall need to work significantly harder and provide ourselves with new estimations concerning the general behaviour of this family of operators.

In this spirit, we recall that Parrott showed in [**103**, Sect. II.7] that the spectral radius of any ergodic weighted translation $W_{\phi,\tau}$ acting on $L^p(\Omega, \mu)$ is always greater than

$$r(W_{\phi,\tau}) \geq \exp\left(\int_\Omega \log |\phi| \, d\mu \right).$$

Hereunder, we extend this result and assure that a similar bound actually holds uniformly for each non-zero function in $L^p(\Omega, \mu)$. In some sense, our next theorem is the cornerstone for all the results appearing throughout this subsection. Just mention that its proof is mainly based on a nice application of ergodic theorems:

Theorem 4.13. *Suppose that $W_{\phi,\tau}$ is an ergodic weighted translation operator acting on $L^p(\Omega, \mu)$ for some $1 \leq p < \infty$ fixed. Then, for each non-zero $f \in L^p(\Omega, \mu)$ we have*

$$\liminf_{n \to \infty} \left\| W_{\phi,\tau}^n f \right\|_p^{1/n} \geq \exp\left(\int_\Omega \log |\phi| \, d\mu \right).$$

Proof. First, observe that since $\phi \in L^\infty(\Omega, \mu)$ and $(\Omega, \mathcal{G}, \mu)$ is a probability space, then $\log |\phi| \notin L^1(\Omega, \mu)$ if and only if $\int_\Omega \log |\phi| \, d\mu = -\infty$. Hence, without loss of generality, we may assume that $\log |\phi| \in L^1(\Omega, \mu)$, since otherwise the statement holds trivially.

Fix a non-zero $f \in L^p(\Omega, \mu)$ and consider the measurable set $E := \{\omega \in \Omega : f(\omega) \neq 0\}$. Having in mind the expression for the iterates of $W_{\phi,\tau}$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\log\left(\left\|W^n_{\phi,\tau} f\right\|_p^{1/n}\right) &= \frac{1}{pn}\log\left(\int_\Omega |f(\tau^n(\omega))|^p \prod_{j=0}^{n-1} |\phi(\tau^j(\omega))|^p \, d\mu(\omega)\right) \\ &= \frac{1}{pn}\log\left(\int_\Omega |f(\omega)|^p \prod_{j=1}^{n} |\phi(\tau^{-j}(\omega))|^p \, d\mu(\omega)\right) \\ &= \frac{1}{pn}\log\left(\frac{1}{\mu(E)}\int_E |f(\omega)|^p \prod_{j=1}^{n} |\phi(\tau^{-j}(\omega))|^p \, d\mu(\omega)\right) + \frac{\log(\mu(E))}{pn}.\end{aligned}$$

Correspondingly, using *Jensen's Inequality* according to the concavity of $t \mapsto \log(t)$, we can bound from below the latter expression and obtain

$$\begin{aligned}\log\left(\left\|W^n_{\phi,\tau} f\right\|_p^{1/n}\right) \geq \frac{1}{n\mu(E)}\int_E \log|f(\omega)|\, d\mu(\omega) + \frac{1}{n}\sum_{j=1}^n \frac{1}{\mu(E)}\int_E \log|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) \\ + \frac{\log(\mu(E))}{pn}.\end{aligned} \tag{4.4}$$

Now, we claim that the following limit holds:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \frac{1}{\mu(E)}\int_E \log|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) = \int_\Omega \log|\phi(\omega)|\, d\mu(\omega). \tag{4.5}$$

In order to check the fulfilment of the limit (4.5), choose any $n \in \mathbb{N}$ and $\varepsilon > 0$. Since $\log|\phi| \in L^1(\Omega, \mu)$, by means of the *Lebesgue Dominated Convergence Theorem*, there exists $N \in \mathbb{N}$ sufficiently large such that

$$\left\| \log^{(N)}|\phi| - \log|\phi|\right\|_{L^1(\Omega,\mu)} < \varepsilon,$$

where $\log^{(N)}(t) := \max\{\log(t), -N\}$ for $t > 0$. Accordingly, one gets

$$\begin{aligned}&\left|\frac{1}{n}\sum_{j=1}^n \frac{1}{\mu(E)}\int_E \log|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) - \int_\Omega \log|\phi(\omega)|\, d\mu(\omega)\right| \\ &\quad\leq \frac{1}{n\mu(E)}\sum_{j=1}^n \left|\int_E \log|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) - \int_E \log^{(N)}|\phi(\tau^{-j}(\omega))|\, d\mu(\omega)\right| \\ &\qquad+ \left|\frac{1}{n}\sum_{j=1}^n \frac{1}{\mu(E)}\int_E \log^{(N)}|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) - \int_\Omega \log^{(N)}|\phi|\, d\mu(\omega)\right| \\ &\qquad+ \left|\int_\Omega \log^{(N)}|\phi(\omega)|\, d\mu(\omega) - \int_\Omega \log|\phi(\omega)|\, d\mu(\omega)\right|\end{aligned}$$

$$\leq \frac{\varepsilon}{\mu(E)} + \left|\frac{1}{n}\sum_{j=1}^n \frac{1}{\mu(E)}\int_E \log^{(N)}|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) - \int_\Omega \log^{(N)}|\phi(\omega)|\, d\mu(\omega)\right| + \varepsilon. \tag{4.6}$$

Letting $n \to +\infty$ and recalling the equivalence provided in Remark 1.41 (iii), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \int_E \log^{(N)}|\phi(\tau^{-j}(\omega))|\, d\mu(\omega) &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \int_\Omega \log^{(N)}|\phi(\tau^{-j}(\omega))| \cdot \mathbf{1}_E(\omega)\, d\mu(\omega) \\ &= \left(\int_\Omega \mathbf{1}_E(\omega)\, d\mu(\omega)\right) \cdot \left(\int_\Omega \log^{(N)}|\phi(\omega)|\, d\mu(\omega)\right)\end{aligned}$$

$$= \mu(E) \int_\Omega \log^{(N)} |\phi(\omega)| \, d\mu(\omega).$$

Thus, one deduces the desired limit (4.5) joining the contributions of each of the three terms in (4.6) and doing $\varepsilon > 0$ arbitrarily small.

Consequently, by taking the lim inf as $n \to +\infty$ in the inequality (4.4), it is obvious that the expected lower bound

$$\liminf_{n\to\infty} \log\left(\left\|W^n_{\phi,\tau} f\right\|_p^{1/n}\right) \geq \int_\Omega \log|\phi| \, d\mu$$

will hold whenever the next condition is verified:

$$\left| \int_E \log|f(\omega)| \, d\mu(\omega) \right| < +\infty. \tag{4.7}$$

To conclude the proof, the next simple argument completes the job: let $\widetilde{f}$ be the measurable function on Ω defined by

$$\widetilde{f}(\omega) := \frac{\|f\|_p}{2} \mathbf{1}_M(\omega)$$

where $M := \{\omega \in \Omega : |f(\omega)| \geq \|f\|_p/2\}$. It is immediate to check that the function $\widetilde{f}$ enjoys condition (4.7) and, in addition, that $|f(\omega)| \geq |\widetilde{f}(\omega)|$ for every $\omega \in \Omega$. Therefore,

$$\liminf_{n\to\infty} \log\left(\left\|W^n_{\phi,\tau} f\right\|_p^{1/n}\right) \geq \liminf_{n\to\infty} \log\left(\left\|W^n_{\phi,\tau} \widetilde{f}\right\|_p^{1/n}\right) \geq \int_\Omega \log|\phi| \, d\mu,$$

which clearly establishes Theorem 4.13. □

One the most beautiful applications of Theorem 4.13 grants the power-regularity of any ergodic weighted translations $W_{\phi,\tau}$ whose spectral radius is precisely $r(W_{\phi,\tau}) = \exp\left(\int_\Omega \log|\phi| \, d\mu\right)$:

Corollary 4.14. *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator on $L^p(\Omega,\mu)$ for some $1 \leq p < \infty$. If $r(W_{\phi,\tau}) = \exp\left(\int_\Omega \log|\phi| \, d\mu\right)$, then $W_{\phi,\tau}$ is power-regular and, in addition,*

$$\lim_{n\to\infty} \left\|W^n_{\phi,\tau} f\right\|_p^{1/n} = r(W_{\phi,\tau})$$

for all non-zero $f \in L^p(\Omega,\mu)$.

Proof. Fix any non-zero $f \in L^p(\Omega,\mu)$. As a consequence of Gelfand's Formula we have

$$\limsup_{n\to\infty} \left\|W^n_{\phi,\tau} f\right\|_p^{1/n} \leq \limsup_{n\to\infty} \left\|W^n_{\phi,\tau}\right\|^{1/n} \cdot \|f\|_p^{1/n} = r(W_{\phi,\tau}) = \exp\left(\int_\Omega \log|\phi| \, d\mu\right).$$

Now, invoking Theorem 4.13, the result follows straightforwardly. □

There are several interesting consequences which can be easily derived from our preceding results. The first one is a mere reformulation in terms of the local spectral radius:

Corollary 4.15. *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator acting on $L^p(\Omega,\mu)$ for some $1 \leq p < \infty$. Then, we have*

$$r_{W_{\phi,\tau}}(f) \geq \exp\left(\int_\Omega \log|\phi| \, d\mu\right) \quad \text{for all } f \in L^p(\Omega,\mu) \setminus \{0\}.$$

In particular, if $r(W_{\phi,\tau}) = \exp\left(\int_\Omega \log|\phi| \, d\mu\right)$, the equality $r_{W_{\phi,\tau}}(f) = r(W_{\phi,\tau})$ is true for every non-zero $f \in L^p(\Omega,\mu)$.

Observe that, by the aforementioned results due to Daneš [**43**] and Müller [**96**], given an arbitrary Banach space operator $T \in \mathcal{B}(X)$, the equality between the local spectral radii and the spectral radius $r_T(x) = r(T)$ is held for all $x \in X$ in a dense set of the second category. Accordingly, those ergodic weighted translations under the hypothesis of Corollary 4.14 are extreme examples in that sense.

Likewise, Theorem 4.13 and Corollary 4.14 still admit another striking corollary when they are rewritten in terms of the invariant subspaces:

Corollary 4.16. *Let $W_{\phi,\tau}$ be an ergodic weighted translation operator acting on $L^p(\Omega,\mu)$ for fixed $1 \le p < \infty$. Suppose that $\mathcal{M}$ is any non-zero invariant subspace for $W_{\phi,\tau}$, then*

$$r(W_{\phi,\tau}) \ge r(W_{\phi,\tau}|\mathcal{M}) \ge \exp\bigg(\int_\Omega \log|\phi|\,d\mu\bigg).$$

In particular, if $r(W_{\phi,\tau}) = \exp\big(\int_\Omega \log|\phi|\,d\mu\big)$, the equality $r(W_{\phi,\tau}|\mathcal{M}) = r(W_{\phi,\tau})$ holds for each non-zero $W_{\phi,\tau}$-invariant subspace.

Proof. The first inequality is a well-known fact previously stated in Proposition 1.6. On the other hand, chosen a non-zero element $f \in \mathcal{M}$ with $\|f\|_p = 1$, we clearly have

$$\big\|(W^n_{\phi,\tau}|\mathcal{M})\big\|^{1/n} \ge \|W^n_{\phi,\tau} f\|_p^{1/n} \quad \text{for all } n \in \mathbb{N}.$$

Taking the lim inf as n goes to $+\infty$, the second inequality follows from Theorem 4.13. □

At this stage, we recall that MacDonald [**91**] completely described the spectrum of each uniquely ergodic weighted translation operator $W_{\phi,\tau}$ on $L^p(\Omega,\mu)$ having a μ-a.e. continuous weight ϕ (see Theorem 1.51). In particular, under such assumptions, he confirmed that the spectral radius $r(W_{\phi,\tau})$ always coincides with the critical value $\exp\big(\int_\Omega \log|\phi|\,d\mu\big)$. Consequently, as a by-product of our preceding results, one can assure the following theorem.

Theorem 4.17. *Let $W_{\phi,\tau}$ be a uniquely ergodic weighted translation acting on $L^p(\Omega,\mu)$ for some $1 \le p < \infty$. Suppose further that $\phi \in L^\infty(\Omega,\mu)$ is continuous μ-a.e. Then, the operator $W_{\phi,\tau}$ is power-regular and*

$$\lim_{n\to\infty} \big\|W^n_{\phi,\tau} f\big\|_p^{1/n} = r(W_{\phi,\tau}) \quad \textit{for all } f \in L^p(\Omega,\mu)\setminus\{0\}.$$

In particular, $r(W_{\phi,\tau}|\mathcal{M}) = r(W_{\phi,\tau})$ for each non-zero invariant subspace $\mathcal{M}$.

As a remarkable instance regarding Theorem 4.17, one has that any Bishop operator T_α with $\alpha \in (0,1)$ irrational is power-regular on each $L^p[0,1)$ for $1 \le p < \infty$. In fact, it may be seen that the local spectrum $\sigma_{T_\alpha}(f)$ of any non-zero function $f \in L^p[0,1)$ always touches the boundary $\partial\sigma(T_\alpha)$. More indeed, the spectral radius of the restriction of T_α to any non-zero invariant subspace (whenever it exists) is exactly e^{-1}. In sum, a search of invariant subspaces for Bishop operators relying upon computations involving the spectral radii of the restrictions of T_α cannot be a fruitful approach.

Now, we are in position to establish the failure of decomposability, property (β) and property (δ) in the general context of ergodic weighted translation operators. The idea is quite standard and consists in taking open covers having circular symmetry with the purpose of getting a contraction from our preceding bounds on the spectral radii. We begin by the strongest one:

FIGURE 4. Admissible local spectrum touching $\partial\sigma(T_\alpha)$ (left) and non-admissible local spectrum (right) for some non-zero $f \in L^p[0,1)$

Proposition 4.18. *Let $W_{\phi,\tau}$ be a non-invertible ergodic weighted translation acting on $L^p(\Omega,\mu)$ for some $1 \leq p < \infty$. If $\log|\phi| \in L^1(\Omega,\mu)$, then the operator $W_{\phi,\tau}$ is not decomposable. Moreover, $W_{\phi,\tau}$ is not radially decomposable.*

Proof. Since $\log|\phi| \in L^1(\Omega,\mu)$, we may pick $0 < r < s < \exp\left(\int_\Omega \log|\phi|\,d\mu\right)$ and consider the open cover of $\mathbb{C}$ consisting of

$$U = D(0,s) = \big\{z \in \mathbb{C} : |z| < s\big\} \quad \text{and} \quad V = \mathbb{C} \setminus \overline{D(0,r)} = \big\{z \in \mathbb{C} : |z| > r\big\}.$$

Consequently, by Corollary 4.16, we conclude that $\mathcal{M}_U = \{0\}$ is the unique $W_{\phi,\tau}$-invariant subspace such that $\sigma(W_{\phi,\tau}|\mathcal{M}_U) \subseteq U$. In such a case, if the operator $W_{\phi,\tau}$ were (radially) decomposable, the associated sum decomposition would imply that $\mathcal{M}_V = L^p(\Omega,\mu)$. This would lead us to a contradiction since

$$\overline{D(0, r(W_{\phi,\tau}))} = \sigma\big(W_{\phi,\tau}|L^p(\Omega,\mu)\big) \not\subseteq V.$$

Consequently, as desired, $W_{\phi,\tau}$ is not a (radially) decomposable operator. □

REMARK 4.19. Clearly, the two hypotheses imposed in the statement of Proposition 4.18 ensure an enough gap in order to select an open covering of $\mathbb{C}$ consisting of circular symmetric sets. At this regard, the non-invertibility of $W_{\phi,\tau}$ could be replaced by the weaker condition on the inner spectral radius

$$i(W_{\phi,\tau}) < \exp\left(\int_\Omega \log|\phi|\,d\mu\right).$$

By contrast, the condition $\log|\phi| \in L^1(\Omega,\mu)$ is more technical and hard to control. For instance, in the particular case that τ is uniquely ergodic and ϕ continuous μ-a.e., observe that $\log|\phi| \notin L^1(\Omega,\mu)$ precisely when $W_{\phi,\tau}$ is quasinilpotent (and, trivially, decomposable).

In this light, concerning invertible ergodic weighted translations, the situation may differ completely. Indeed, since every operator $T \in \mathcal{B}(X)$ for which the set $\{|\lambda| : \lambda \in \sigma(T)\}$ has empty interior in $[0,+\infty)$ is radially decomposable (see [**16**, Proof of Theorem 4.2]), as an immediate consequence of Theorem 1.51, one deduces the following nice characterization:

Corollary 4.20. *Let $(\Omega,\mathcal{G},\mu,\tau)$ be a uniquely ergodic system and $\phi \in L^\infty(\Omega,\mu)$ a μ-a.e. continuous function. Consider $W_{\phi,\tau} \in \mathcal{B}(L^p(\Omega,\mu))$ for some $1 \leq p < \infty$ fixed and assume that $\log|\phi| \in L^1(\Omega,\mu)$. Then, $W_{\phi,\tau}$ is invertible if and only if it is radially decomposable.*

It is worthy to compare this latter result with those obtained by MacDonald in [**92**] following the decomposability criterion due to Colojoară and Foiaş (see Theorem 1.33) based on Wermer's Theorem. Firstly, observe that in the special case of Bishop-type operators $W_{\phi,\alpha}$ with $\alpha \notin \mathbb{Q}$, by Proposition 4.18 one clearly concludes:

Corollary 4.21. *Let $W_{\phi,\alpha}$ be a non-invertible Bishop-type operator on $L^p[0,1)$ for some $1 \leq p < \infty$. If $\log|\phi| \in L^1[0,1)$, then $W_{\phi,\alpha}$ is not decomposable. In particular, no Bishop operator T_α is decomposable.*

MacDonald's theorems deal with Bishop-type operators $W_{\phi,\alpha}$ when $\log|\phi|$ on $[0,1)$ is a bounded variation function, i.e. $\log|\phi| \in \mathrm{BV}[0,1)$. More precisely, speaking in the language of continued fractions, MacDonald proved the following:

Theorem 4.22 (MacDonald, [92]). *Let $\phi \in L^\infty[0,1)$ be such that $\log|\phi| \in \mathrm{BV}[0,1)$ and $\alpha \in (0,1)$ be any irrational number whose convergent denominators $(q_j)_{j\geq 0}$ satisfy*

$$\sum_{j=0}^{\infty} \frac{1}{q_j} \log\left(\frac{q_{j+1}}{q_j}\right) < \infty;$$

then, the operator $W_{\phi,\alpha}$ is decomposable.

Recalling that a function ψ has *modulus of continuity* if

$$\omega_\psi(\delta) := \sup\left\{|\psi(t) - \psi(\tilde{t})| \,:\, |t - \tilde{t}| < \delta\right\} < \infty \quad \text{for every } \delta > 0,$$

MacDonald showed the decomposability of $W_{\phi,\alpha}$ under the following assumptions:

Theorem 4.23 (MacDonald, [92]). *Let $\phi \in L^\infty[0,1)$ be such that $\log|\phi|$ has modulus of continuity $\omega_{\log|\phi|}$ and $\alpha \in (0,1)$ be any irrational number whose convergent denominators $(q_j)_{j\geq 0}$ satisfy*

$$\sum_{j=0}^{\infty} \omega_{\log|\phi|}\left(\frac{1}{q_j}\right) \cdot \log\left(\frac{q_{j+1}}{q_j}\right) < \infty;$$

then, the operator $W_{\phi,\alpha}$ is decomposable.

Note that in both theorems, MacDonald deals with invertible Bishop-type operators $W_{\phi,\alpha}$ having an a.e. continuous weight ϕ. Accordingly, some invertible Bishop operators are not only radially decomposable but also decomposable. It seems a extremely challenging open problem to characterize those invertible Bishop-type operators which are decomposable. Moreover, as suggested by Theorem 3.1, it appears that new tools (not depending on Wermer's condition) must be developed to address successfully that question.

Finally, we end this section by analysing the remaining weaker spectral properties in the context of non-invertible ergodic weighted translations, namely: the Bishop's property (β) and the decomposition property (δ). Once again, the strategy consists in considering circular symmetric coverings of the complex plane $\mathbb{C}$; however, at some point, we will need to invoke the duality correspondence due to Albrecht and Eschmeier [**6**] (see Theorem 4.5).

Theorem 4.24. *Let $W_{\phi,\tau}$ be a non-invertible ergodic weighted translation acting on $L^p(\Omega,\mu)$ for some $1 < p < \infty$. If $\log|\phi| \in L^1(\Omega,\mu)$, then the operator $W_{\phi,\tau}$ does not satisfy neither of both properties (β) and (δ).*

Remark 4.25. From now on, despite the fact that $W_{\phi,\tau}$ is defined over $L^p(\Omega,\mu)$, in order to prevent a messy notation, we prefer to keep denoting by $X_{W_{\phi,\tau}}(F)$ its local spectral manifolds and by $\mathcal{X}_{W_{\phi,\tau}}(F)$ its glocal spectral manifolds. In other words, we avoid the explicit mention to the corresponding Banach space and assume that it is clear from the context.

Proof. Fix any $1 < p < \infty$. We first prove that $W_{\phi,\tau}$ cannot enjoy property (δ) in $L^p(\Omega, \mu)$ arguing by contradiction.

Thus, suppose on the contrary that $W_{\phi,\tau}$ has property (δ). As we did in the proof of Proposition 4.18, take $0 < r < s < \exp\big(\int_\Omega \log|\phi|\, d\mu\big)$ and consider the associated open cover

$$U = D(0,s) \quad \text{and} \quad V = \mathbb{C} \setminus \overline{D(0,r)}.$$

Hence,

$$L^p(\Omega,\mu) = \mathcal{X}_{W_{\phi,\tau}}(\overline{U}) + \mathcal{X}_{W_{\phi,\tau}}(\overline{V}). \tag{4.8}$$

By Corollary 4.15, we have that $r_{W_{\phi,\tau}}(f) \geq \exp\big(\int_\Omega \log|\phi|\, d\mu\big)$ for every non-zero $f \in L^p(\Omega,\mu)$. Now, since $s < \exp\big(\int_\Omega \log|\phi|\, d\mu\big)$, using inequality (4.3), we conclude

$$\mathcal{X}_{W_{\phi,\tau}}(\overline{U}) = \{0\}.$$

But, by the sum decomposition (4.8), this would imply that

$$\sigma_{W_{\phi,\tau}}(f) \subseteq \overline{V} \cap \sigma(W_{\phi,\tau}) = \big\{\lambda \in \mathbb{C} \,:\, r \leq |\lambda| \leq r(W_{\phi,\tau})\big\}$$

for every non-zero $f \in L^p(\Omega,\mu)$. However, since $W_{\phi,\tau}$ has the SVEP, we know that $\sigma(W_{\phi,\tau}) = \sigma_{\mathrm{su}}(W_{\phi,\tau})$. Hence, by means of Proposition 4.9, this would entail that

$$\overline{D(0, r(W_{\phi,\tau}))} = \sigma_{\mathrm{su}}(W_{\phi,\tau}) = \bigcup_{f \in L^p(\Omega,\mu)} \sigma_{W_{\phi,\tau}}(f) \subseteq \big\{\lambda \in \mathbb{C} \,:\, r \leq |\lambda| \leq r(W_{\phi,\tau})\big\},$$

which is a contradiction because $r > 0$. Consequently, $W_{\phi,\tau}$ has not property (δ).

On the other hand, to prove that $W_{\phi,\tau}$ neither enjoys property (β) on $L^p(\Omega,\mu)$, observe that the function $\psi := \phi \circ \tau$ also belongs to $L^\infty(\Omega,\mu)$, $0 \in \mu\text{-}\operatorname{ess\,ran}(\psi)$ and $\log|\psi| \in L^1(\Omega,\mu)$. Moreover, the transformation τ^{-1} is also ergodic. Correspondingly, the ergodic weighted translation $W_{\psi,\tau^{-1}}$ is a bounded operator on $L^q(\Omega,\mu)$ (where $1 < q < \infty$ denotes the conjugate exponent of p) under the hypothesis of the present theorem.

According to the first part of this proof, we derive that $W_{\psi,\tau^{-1}}$ cannot have property (δ) on $L^q(\Omega,\mu)$. Since its adjoint is precisely

$$W^*_{\psi,\tau^{-1}} = W_{\phi,\tau} \ \text{ on } L^p(\Omega,\mu),$$

the desired result follows upon applying Albrecht and Eschmeier [**6**] duality between properties (β) and (δ) (see Theorem 4.5). In other words, $W_{\phi,\tau}$ does not satisfy property (β) on $L^p(\Omega,\mu)$, which completes the proof of the theorem. □

Remark 4.26. Observe that the proof of Theorem 4.24 provides also information for $p = 1$ and $p = \infty$. Indeed, under such hypotheses, if $\log|\phi| \in L^1(\Omega,\mu)$ then $W_{\phi,\tau}$ has not property (δ) in $L^1(\Omega,\mu)$ and the Bishop's property (β) in $L^\infty(\Omega,\mu)$. Consequently, $W_{\phi,\tau}$ is no longer decomposable either in $L^\infty(\Omega,\mu)$.

Recall that a result of Eschmeier and Prunaru [**54**] (see Theorem 4.6) asserts that a linear bounded operator on a Banach space with thick spectrum lacking of non-trivial closed invariant subspaces does not satisfy neither both properties (β) and (δ). Accordingly, possible candidates for operators without non-trivial closed invariant subspaces might still be among some of those non-invertible Bishop-type operators satisfying Theorem 4.24.

More indeed, all our preceding discussion also helps to delimit the localizable spectrum of many ergodic weighted translation operators. In particular, assuming the usual hypothesis appearing in Theorem 4.13, we can ensure that

$$\sigma_{\mathrm{loc}}(W_{\phi,\tau}) \subseteq \left\{ \lambda \in \mathbb{C} \,:\, |\lambda| \geq \exp\left(\int_\Omega \log|\phi| \, d\mu \right) \right\}.$$

Unfortunately, the existing methods to examine the circular symmetry of the spectral parts of $\sigma(W_{\phi,\tau})$ seem to be ineffective for the concrete case of the localizable spectrum (see Theorem 1.48). Thus, we pose that as an open question:

Problem. *Suppose that $W_{\phi,\tau}$ is an ergodic weighted translation operator acting on $L^p(\Omega,\mu)$ for $1 \leq p \leq \infty$. Must its localizable spectrum $\sigma_{\mathrm{loc}}(W_{\phi,\tau})$ be circular symmetric?*

In the specific instance of Bishop-type operators $W_{\phi,\alpha}$, the circular symmetry of the localizable spectrum $\sigma_{\mathrm{loc}}(W_{\phi,\alpha})$ is a trivial feature once we allude to the relation

$$M_{e^{2\pi i t}} W_{\phi,\alpha}^n = e^{-2\pi i n \alpha}\, W_{\phi,\alpha}^n M_{e^{2\pi i t}} \quad (n \in \mathbb{Z}).$$

In other words, the localizable spectrum $\sigma_{\mathrm{loc}}(W_{\phi,\alpha})$ is invariant under multiplication by the eigenvalues of C_{τ_α}. At this regard, we find particularly enlightening the case of Bishop operators. For later reference, in the next statement we add those already identified local spectral properties concerning Bishop operators:

Corollary 4.27. *Let $\alpha \in (0,1)$ be an irrational number and consider the Bishop operator T_α acting on $L^p[0,1)$ for some $1 < p < \infty$. Then, T_α is not decomposable and does not satisfy neither both properties (β) and (δ). Moreover, its localizable spectrum is*

$$\sigma_{\mathrm{loc}}(T_\alpha) = \emptyset \quad \text{or} \quad \sigma_{\mathrm{loc}}(T_\alpha) = \partial\sigma(T_\alpha).$$

We point out that the localizable spectrum of each Bishop operator $\sigma_{\mathrm{loc}}(T_\alpha)$ can never be a thick set for any irrational $\alpha \in (0,1)$. Accordingly, this feature prevents us from using the extension owed to Eschmeier and Prunaru [**55**] (see Theorem 4.7) as the general strategy to find invariant subspaces for all the Bishop operators.

4.2.2. On Dunford's property for Bishop operators

In general, with the exception of the trivial examples, a complete characterization of the various local spectral properties for a specific family of operators is usually an ambitious task which requires a profound knowledge on the behaviour of such class of operators.

Herein, we establish a complete identification of all the local spectral properties satisfied by any Bishop operator T_α, independently of the irrational $\alpha \in (0,1)$ and the L^p-space. To that end, we shall prove that no Bishop operator T_α can enjoy Dunford's property on any $L^p[0,1)$ with $1 \leq p \leq \infty$. In contrast to decomposability, property (β) and property (δ) which could be discarded using general ergodic principles; since Dunford's property is somewhat weaker than the aforementioned ones, we shall need the specific estimations concerning $\widetilde{T}_\alpha$ previously obtained in Chapter 2.

At this regard, an important remark is in order: given an arbitrary operator $T \in \mathcal{B}(X)$ having $\sigma_{\mathrm{p}}(T) = \emptyset$ and $\sigma_{\mathrm{p}}(T^*) = \emptyset$, it may be seen that whenever Atzmon's Theorem may be applied to the sequences

$$x_n := T^n x \quad \text{and} \quad y_n := {T^*}^n y \quad (n \in \mathbb{Z})$$

for a pair of vectors $x \in X$ and $y \in X^*$, the corresponding local spectra $\sigma_T(x)$ and $\sigma_{T^*}(y)$ must be inside the unit circle. However, the converse is far from being true. Indeed, our initial result precisely asserts that the local spectrum $\sigma_{\widetilde{T}_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ (and respectively $\sigma_{\widetilde{T}^*_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$) always lies within the unit circle $\mathbb{T}$ for every irrational $\alpha \in (0,1)$.

In some sense, this feature evinces the main gain of the asymptotic estimation (2.13) with respect to the earlier ones appearing in the works of Davie [**44**] and Flattot [**60**]. In fact, although one cannot take advantage of Atzmon's Theorem beyond the threshold established in Theorem 3.1, it results that Corollary 2.15 may be invoked independently of the chosen irrational $\alpha \in (0,1)$, even for the most extreme ones.

Theorem 4.28. *Let $\alpha \in (0,1)$ be any irrational number and T_α the associated Bishop operator acting on $L^p[0,1)$ for some $1 \le p \le \infty$. Then, the local spectrum $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ is contained in the circle of radius e^{-1}. In other words:*

$$\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha}) \subseteq \partial\sigma(T_\alpha).$$

Proof. In the sequel, let $1 \le p \le \infty$ be fixed and denote the convergents of α by $(a_j/q_j)_{j=0}^\infty$. Then, by Corollary 2.15, we know that for each $q_m \le n^{2/3} \le q_{m+1}$, we have

$$\log\left(\left\|\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\right\|_p\right) \le C \cdot \left(q_m + \frac{n}{q_m}\log(q_m+1) + \frac{n+q_{m+1}}{q_{m+1}}\log(n+2)\right),$$

where $C > 0$ is an absolute constant independent of m. Taking into account the range of n, this implies

$$\left\|\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\right\|_p \le \exp\left(C \cdot \left(n^{-1/3} + \frac{1}{q_m}\log(q_m+1) + n^{-2/3}\log(n+2)\right)\right)^n.$$

Nevertheless, for every $\varepsilon > 0$, there exists a sufficiently large m such that

$$C \cdot \left(n^{-1/3} + \frac{1}{q_m}\log(q_m+1) + n^{-2/3}\log(n+2)\right) \le \varepsilon$$

for every $q_m \le n^{2/3} \le q_{m+1}$. In particular, as a consequence of this bound, we conclude that the vector-valued function

$$R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha}) := \sum_{n\ge1}\left(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\right)\cdot z^{n-1}$$

is analytic on the open disk $D(0,e^{-\varepsilon})$. Since $R_{\widetilde{T}_\alpha}(\,\cdot\,;\mathbf{1}_{\mathcal{B}_\alpha})$ fulfils the functional equation

$$(\widetilde{T}_\alpha - zI)R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha}) = \mathbf{1}_{\mathcal{B}_\alpha} \quad \text{for each } |z| < e^{-\varepsilon},$$

this precisely means that $D(0,e^{-\varepsilon}) \subseteq \rho_{\widetilde{T}_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$. Finally, making $\varepsilon > 0$ arbitrarily small, the theorem is proved bearing in mind that $\sigma_{\widetilde{T}_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha}) = e\cdot\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$. □

Obviously, the same argument applies for the adjoint. However, to avoid any misunderstanding regarding the relation between the adjointness and the L^p-spaces, we have preferred to state each case separately.

Corollary 4.29. *Let $\alpha \in (0,1)$ be an irrational number and $T_\alpha^* f(t) = \{t-\alpha\}\cdot f(\{t-\alpha\})$ acting on $L^p[0,1)$ for some $1 \le p \le \infty$. Then, the local spectrum $\sigma_{T_\alpha^*}(\mathbf{1}_{\mathcal{B}_\alpha})$ is contained in the circle of radius e^{-1}. In other words:*

$$\sigma_{T_\alpha^*}(\mathbf{1}_{\mathcal{B}_\alpha}) \subseteq \partial\sigma(T_\alpha^*).$$

According to [**13**], since T_α enjoys the SVEP, it happens that the local spectrum $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ coincides with the singular points of its local resolvent function. This enables us to identify some of the basic features concerning $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ (and similarly of $\sigma_{T_\alpha^*}(\mathbf{1}_{\mathcal{B}_\alpha})$).

Proposition 4.30. *Let $\alpha \in (0,1)$ be any irrational number and consider T_α acting on $L^p[0,1)$ for some $1 \le p \le \infty$. Then, $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ (respectively $\sigma_{T_\alpha^*}(\mathbf{1}_{\mathcal{B}_\alpha})$) is symmetric with respect to the real axis and contains the point $\lambda = e^{-1}$.*

Proof. Fix any $1 \le p \le \infty$ and $\alpha \in (0,1)$ irrational. As above, consider the $L^p[0,1)$-valued analytic function on $\mathbb{C} \setminus \mathbb{T}$ defined by

$$R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha}) := \begin{cases} \sum\limits_{n=1}^{\infty} \big(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\big) z^{n-1}, & \text{if } |z|<1; \\ -\sum\limits_{n=-\infty}^{0} \big(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\big) z^{n-1}, & \text{if } |z|>1. \end{cases} \tag{4.9}$$

As mentioned above, the local spectrum $\sigma_{\widetilde{T}_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ coincides precisely with the singular points of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$ lying within the unit circle.

Accordingly, to establish the symmetry of $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ with respect to $\mathbb{R}$, it suffices to check the identity

$$\overline{R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha})} = R_{\widetilde{T}_\alpha}(\overline{z};\mathbf{1}_{\mathcal{B}_\alpha}) \quad \text{for each } z \in \mathbb{C}\setminus\mathbb{T}.$$

To this end, note that every iterate $\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}$ is a real-valued function in $L^p[0,1)$ for each $n \in \mathbb{Z}$. Correspondingly,

$$\overline{R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha})} = \sum_{n\ge 1} \big(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\big)\cdot \overline{z}^{n-1} = R_{\widetilde{T}_\alpha}(\overline{z};\mathbf{1}_{\mathcal{B}_\alpha}), \qquad \text{for each } |z|<1,$$

and similarly for $|z|>1$. This proves that $\sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ is symmetric with respect to the real axis.

On the other hand, to show that $e^{-1} \in \sigma_{T_\alpha}(\mathbf{1}_{\mathcal{B}_\alpha})$ we must check that $\lambda = 1$ is a singular point of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$. For this claim, note that the Taylor series of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$ nearby an arbitrary point $z_0 \in \mathbb{D}$ is

$$R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha}) = \sum_{k\ge 0} \frac{\partial^k R_{\widetilde{T}_\alpha}(z_0;\mathbf{1}_{\mathcal{B}_\alpha})}{k!}\cdot (z-z_0)^k,$$

where, as a consequence of (4.9), the derivates are given by the expressions

$$\partial^k R_{\widetilde{T}_\alpha}(z_0;\mathbf{1}_{\mathcal{B}_\alpha}) = \sum_{n\ge k+1} \frac{(n-1)!}{(n-k-1)!}\big(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\big)\, z_0^{n-k-1}, \quad \text{for each } k \in \mathbb{Z}_+. \tag{4.10}$$

Now, suppose that $e^{i\theta}$ is a singular point of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$ and choose any $0<r<1$. By hypothesis, the power series

$$R_{\widetilde{T}_\alpha}(z;\mathbf{1}_{\mathcal{B}_\alpha}) = \sum_{k\ge 0} \frac{\partial^k R_{\widetilde{T}_\alpha}(re^{i\theta};\mathbf{1}_{\mathcal{B}_\alpha})}{k!}\cdot (z-re^{i\theta})^k$$

has radius of convergence equal to $1-r$. Nevertheless, since $\widetilde{T}_\alpha^{-n}$ is a positive linear operator for each $n \in \mathbb{Z}_+$ (in the sense of *Banach lattices*) and noting that $\mathbf{1}_{\mathcal{B}_\alpha}(t) \ge 0$, we conclude that $\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}(t) \geq 0$ for every $n \in \mathbb{Z}_+$. Hence, using the series expansion of (5.4), for every $k \in \mathbb{Z}_+$ one has the following inequality

$$\begin{aligned}\left\|\partial^k R_{\widetilde{T}_\alpha}(re^{i\theta};\mathbf{1}_{\mathcal{B}_\alpha})\right\|_p &= \left\| \sum_{n\geq k+1} \frac{(n-1)!}{(n-k-1)!} \left(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\right)(re^{i\theta})^{n-k-1}\right\|_p \\ &\leq \left\| \sum_{n\geq k+1} \frac{(n-1)!}{(n-k-1)!} \left(\widetilde{T}_\alpha^{-n}\mathbf{1}_{\mathcal{B}_\alpha}\right) r^{n-k-1}\right\|_p \\ &= \left\|\partial^k R_{\widetilde{T}_\alpha}(r;\mathbf{1}_{\mathcal{B}_\alpha})\right\|_p.\end{aligned}$$

In particular, by means of the Cauchy-Hadamard Formula, this implies that the radius of convergence of the Taylor series of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$ nearby r cannot be greater than $1-r$. As desired, this entails that $\lambda = 1$ is a singular point of $R_{\widetilde{T}_\alpha}(\cdot;\mathbf{1}_{\mathcal{B}_\alpha})$. □

Remark 4.31. The second part of the preceding proof is a vector-valued analogue of a classical result in Complex Analysis, known as *Pringsheim's Theorem* (see, for instance, [**123**][Sect. 7.21]). Note that, for the latter argument to be true, the structure of Banach lattice ought to be involved at some point along the proof.

Given an arbitrary operator $T \in \mathcal{B}(X)$ and $x \in X$, it is well-known that a point $\lambda \in \mathbb{C}$ belongs to the local resolvent set $\rho_T(x)$ precisely when there exists a sequence of backward iterates $(x_n)_{n\geq 0}$ such that

$$(T-\lambda)x_n = x_{n-1} \text{ for every } n \in \mathbb{N} \quad \text{and} \quad \|x_n\|_X \leq c^n$$

for some absolute constant $c > 0$, where $x_0 := x$ (see, for instance, [**89**, Prop. 3.3.7]). Therefore, the full determination of the local spectra $\sigma_T(x)$ at the non-zero vectors $x \in X$ is often an extremely difficult problem, because it requires an exhaustive understanding on the growth of the iterates of each scalar translation $T-\lambda$ of the operator. Accordingly, we must usually settle for a partial delimitation of the local spectra $\sigma_T(x)$ as the ones achieved in Theorem 4.28 and Proposition 4.30.

Anyhow, the discovery of functions $f \in L^p[0,1)$ having non-trivial local spectra $\sigma_{T_\alpha}(f)$ for every irrational $\alpha \in (0,1)$ might be a hopeful starting point to solve the Invariant Subspace Problem in the context of Bishop operators, since the associated local spectral manifold are always hyperinvariant. Consequently, in the light of Theorem 4.28, once we are aware of

$$\{0\} \neq X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big) \neq L^p[0,1)$$

for all irrational $\alpha \in (0,1)$ and $1 \leq p \leq \infty$, the natural question which should be asked oneself is whether $X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big)$ spans densely within $L^p[0,1)$ or not. Unfortunately, hereunder we precisely establish that each $X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big)$ is always a norm-dense local spectral manifold in every L^p-space. All the same, this enables us to discard Dunford's property for Bishop operators:

Theorem 4.32. *Let $\alpha \in (0,1)$ be any irrational number. Then, the local spectral manifold*

$$X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big) = \big\{f \in L^p[0,1) : \sigma_{T_\alpha}(f) \subseteq \partial\sigma(T_\alpha)\big\}$$

is norm-dense in $L^p[0,1)$ for each $1 \leq p \leq \infty$. In particular, T_α does not enjoy Dunford's property on $L^p[0,1)$ for any $1 \leq p \leq \infty$.

Proof. In the sequel, fix any $1 \leq p \leq \infty$. Firstly, note that standard bounds lead us to the inclusion

$$\Big\{ g(t)\mathbf{1}_{\mathcal{B}_\alpha}(t) \,:\, \operatorname*{ess\,sup}_{t\in[0,1)} g(t) < \infty \Big\} \subseteq X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big).$$

Consequently, by the density of $L^\infty[0,1)$ into $L^p[0,1)$ for each $1 \leq p \leq \infty$ and taking into account that the support of $\mathbf{1}_{\mathcal{B}_\alpha}$ is precisely the set $\mathcal{B}_\alpha$, we conclude

$$\big\{ f \in L^p[0,1) \,:\, \operatorname{supp}(f) \subseteq \mathcal{B}_\alpha \big\} \subseteq \overline{X_{T_\alpha}(\partial\sigma(T_\alpha))}.$$

Now, since $T_\alpha \mathbf{1}_{\mathcal{B}_\alpha} \in X_{T_\alpha}(\partial\sigma(T_\alpha))$ as well, we deduce a similar inclusion

$$\Big\{ tg(t)\mathbf{1}_{\mathcal{B}_\alpha}(\{t+\alpha\}) \,:\, \operatorname*{ess\,sup}_{t\in[0,1)} g(t) < \infty \Big\} \subseteq X_{T_\alpha}\big(\partial\sigma(T_\alpha)\big),$$

but, noting that the multiplication operator M_t is of dense range in $L^p[0,1)$, this again entails

$$\big\{ f \in L^p[0,1) \,:\, \operatorname{supp}(f) \subseteq \tau_\alpha^{-1}(\mathcal{B}_\alpha) \big\} \subseteq \overline{X_{T_\alpha}(\partial\sigma(T_\alpha))}.$$

Repeating the same argument with $T_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}$ for each $n \in \mathbb{Z}$, we conclude that

$$\begin{aligned} \overline{X_{T_\alpha}(\partial\sigma(T_\alpha))} &\supseteq \operatorname*{span}_{j=-N,\dots,N} \big\{ f \in L^p[0,1) \,:\, \operatorname{supp}(f) \subseteq \tau_\alpha^j(\mathcal{B}_\alpha) \big\} \\ &= \Big\{ f \in L^p[0,1) \,:\, \operatorname{supp}(f) \subseteq \bigcup_{|j|\leq N} \tau_\alpha^j(\mathcal{B}_\alpha) \Big\} \end{aligned}$$

for every $N \in \mathbb{N}$. By definition, since $\mathcal{B}_\alpha$ has strictly positive measure and τ_α is ergodic, we have

$$\bigcup_{j\in\mathbb{Z}} \tau_\alpha^j(\mathcal{B}_\alpha) = [0,1).$$

Consequently, the theorem is proved. □

Of course, the same argument may be applied for the operator T_α^*. As above, we have decided to treat each case separately to avoid any misunderstanding:

Corollary 4.33. *Let $\alpha \in (0,1)$ be any irrational number and $T_\alpha^* f(t) = \{t-\alpha\} \cdot f(\{t-\alpha\})$ acting on $L^p[0,1)$ for some $1 \leq p \leq \infty$. Then, the local spectral manifold*

$$X_{T_\alpha^*}\big(\partial\sigma(T_\alpha)\big) = \big\{ f \in L^p[0,1) \,:\, \sigma_{T_\alpha^*}(f) \subseteq \partial\sigma(T_\alpha) \big\}$$

is norm-dense in $L^p[0,1)$. In particular, T_α^* does not enjoy Dunford's property on $L^p[0,1)$ for any $1 \leq p \leq \infty$.

One may clearly appreciate that the meaning behind each part of $\sigma(T_\alpha)$ differs significantly. In this light, it is plain that the boundary of $\sigma(T_\alpha)$ stores much more information about the Bishop operator T_α than the interior. Such idea will be raised into a new level throughout our next chapter: showing that those already known non-trivial invariant subspaces of T_α can be characterized as the norm-closure of certain local spectral manifolds related to $\partial\sigma(T_\alpha)$.

To conclude this chapter, using our preceding results, we are able to cover the remaining cases left open in the statement of Corollary 4.27. Naturally, a similar argument would work for the operator T_α^*.

Theorem 4.34. *Let $\alpha \in (0,1)$ be any irrational number and consider the Bishop operator T_α acting on $L^p[0,1)$ for some $1 \leq p \leq \infty$. Then:*

(i) *T_α enjoys the SVEP.*

(ii) *On the contrary,* T_α *is not decomposable. Moreover,* T_α *satisfies neither Dunford's property, property* (β) *nor property* (δ)*.*

Proof. For $1 < p < \infty$ the result has been already established. Thus:

- Set $p = 1$: as explained in Remark 4.26, we can ensure that T_α is not decomposable nor has property (δ) in $L^1[0,1)$. On the other hand, Theorem 4.32 assures that T_α enjoys none of the remaining properties (C) and (β).
- Now, for $p = \infty$: according to Corollary 4.33, the operator T_α^* on $L^1[0,1)$ cannot verify property (C). Correspondingly, by adjointness, T_α do not enjoy property (δ) on $L^\infty[0,1)$. The rest of local spectral properties have been already established by Theorem 4.32 and Remark 4.26.

□

Chapter 5

Spectral decompositions of Bishop operators

It is obvious from our preceding chapter that one cannot expect to produce invariant subspaces for any Bishop operator T_α when certain local spectral manifolds are considered. More precisely, if we select a set $F \subseteq \operatorname{int}(\sigma(T_\alpha))$, we fall too short because the associated local spectral manifold $X_{T_\alpha}(F)$ is trivially zero. On the contrary, as long as $\partial\sigma(T_\alpha) \subseteq F$, we go too far since the corresponding local spectral manifold $X_{T_\alpha}(F)$ turns out to be norm-dense.

Both occurrences restrict significantly our quest of invariant subspaces for T_α using local spectral manifolds. In fact, the only remaining alternative would be to choose subsets F intersecting the boundary $\partial\sigma(T_\alpha)$ but not covering it entirely. Surprisingly, at least for many irrationals $\alpha \in (0,1)$, this new attempt turns out to be successful: bringing about local spectral manifolds such that

$$\{0\} \neq \overline{X_{T_\alpha}(F)} \neq L^p[0,1).$$

Our strategy, which agrees with the philosophy followed along Colojoară and Foiaş' treatment of Wermer's Theorem (see [**38**, Ch. V, Thm. 3.2] and Theorem 1.33), uses partitions of unity combined with a functional calculus argument to obtain spectral decompositions over $\partial\sigma(T_\alpha)$. More indeed, in a vague sense to be specified below, certain T_α exhibit a weak type of decomposability. Broadly speaking, in a few words, the breakup of the circular symmetry in the spectrum $\sigma(T_\alpha)$ seems to result in non-trivial invariant subspaces for T_α.

Nevertheless, the news is not all good inasmuch as our approach consists in a local spectral variant of Atzmon's Theorem. Accordingly, the condition enforced by Theorem 3.1 still remains as an insurmountable constraint. Anyway, taking into consideration these new ideas, it seems completely justified to wonder about the possibility of solving the Invariant Subspace Problem in the context of Bishop operators, by means of techniques borrowed from Local Spectral Theory.

With this in mind, along the present chapter we generalize Atzmon's Theorem upon applying a weaker functional model, which enables us to construct invariant subspaces by means of local spectral decompositions. Of course, at the end of the chapter, we will discuss concrete applications regarding Bishop operators and Bishop-type operators. Additionally, for the sake of completeness, at the beginning of the chapter we will introduce several features concerning partitions of unity in the context of Gelfand Theory and their connexion with the hull-kernel topology.

This chapter is based on joint work with Eva A. Gallardo Gutiérrez [**67**].

5.1. Regularity in Banach algebras: partitions of unity and the hull-kernel topology

The aim of this section is to introduce the hull-kernel topology on the Gelfand spectrum of a Banach algebra and, more specifically, the notion of regularity. These concepts, which became fundamental in the classical theory of Banach algebras, turn out to be highly convenient for an insightful treatment of the functional calculus techniques, since they are intimately related to the theory of decomposable multipliers.

For a detailed discussion of the subject, we refer to the books [**28**], [**82**] and [**89**].

One of the pioneering works at this regard was due to Shilov [**119**], who applied the holomorphic functional calculus to prove that the characteristic function of a compact open subset of $\Delta(\mathcal{A})$ coincides with the Gelfand transform of an idempotent element in $\mathcal{A}$. We state it for later reference (for a proof see, for instance, [**82**, Sect. 3.5]):

Theorem 5.1 (Shilov's Idempotent Theorem). *Let $\mathcal{A}$ be a commutative complex Banach algebra. Suppose $K \subseteq \Delta(\mathcal{A})$ is a compact open subset in the Gelfand topology. Then, there exists an idempotent element $r \in \mathcal{A}$ such that*

$$\widehat{r} \equiv 1 \text{ on } K \quad \text{and} \quad \widehat{r} \equiv 0 \text{ on } \Delta(\mathcal{A}) \setminus K.$$

A nice application of the Shilov's Idempotent Theorem yields the following result, which is the reciprocal of a well-known fact concerning unital Banach algebras:

Corollary 5.2. *Let $\mathcal{A}$ be a semi-simple commutative complex Banach algebra. If $\Delta(\mathcal{A})$ is compact in the Gelfand topology, then $\mathcal{A}$ is unital.*

Proof. Since $\Delta(\mathcal{A})$ is Gelfand compact, a direct application of the Shilov's Idempotent Theorem reveals the existence of an element $e \in \mathcal{A}$ such that $\widehat{e} \equiv 1$ on $\Delta(\mathcal{A})$. Now, let $a \in \mathcal{A}$ be arbitrary. Clearly,

$$\widehat{(ae - a)}(\varphi) = \varphi(a)\varphi(e) - \varphi(a) = 0 \quad \text{for every } \varphi \in \Delta(\mathcal{A});$$

but, by the semi-simplicity of $\mathcal{A}$, this is equivalent to $ae - a = 0$. Since the element $a \in \mathcal{A}$ was chosen arbitrarily, e is the identity in $\mathcal{A}$. □

As one might expect from Shilov's Idempotent Theorem, there is a strong connection between topological properties of the Gelfand spectrum $\Delta(\mathcal{A})$ and the existence of certain particularly well-behaved partitions of unity in $\Delta(\mathcal{A})$. Taking into account our own interests, a thorough comprehension of partitions of unity in $\Delta(\mathcal{A})$ shall be of great usefulness, since, via a functional calculus argument, they usually lead to invariant subspaces.

In this spirit, the concept of regularity is especially important:

Definition 5.3. *A commutative complex Banach algebra $\mathcal{A}$ is called* regular *if, for each Gelfand closed set $F \subseteq \Delta(\mathcal{A})$ and every $\varphi \in \Delta(\mathcal{A}) \setminus F$, there exists an element $a \in \mathcal{A}$ whose Gelfand transform $\widehat{a}$ satisfies*

$$\widehat{a} \equiv 0 \text{ on } F \quad \text{and} \quad \widehat{a}(\varphi) \neq 0.$$

A very intuitive instance of a regular Banach algebra is exemplified by the space of continuous functions $C(K)$ over a compact subset $K \subseteq \mathbb{C}$. By contrast, as one may guess (at least in some non-pathological cases), analyticity tends to exclude regularity by means of the Identity Theorem: in fact, the *disk algebra* $A(\mathbb{D})$ fails to be a regular Banach algebra. Of course, this definition of regularity actually encompasses that one seen for function algebras, and more concretely for Beurling algebras (see Theorem 1.29).

For our purposes, in close connection with the notion of regularity, we may consider an additional topology on the Gelfand spectrum $\Delta(\mathcal{A})$. Roughly speaking, this topology is designed to measure how far is $\mathcal{A}$ from being a regular Banach algebra.

Known as the *hull-kernel topology* (abbreviated as *hk-topology* in the sequel), its definition relies on the specification of the closure of any given subset $S \subseteq \Delta(\mathcal{A})$ and involves the next two notions:

Definition 5.4. *Let $\mathcal{A}$ be a commutative complex Banach algebra. For each subset $S \subseteq \Delta(\mathcal{A})$, the* kernel *of S is the ideal in $\mathcal{A}$ given by*

$$k(S) := \big\{ a \in \mathcal{A} \,:\, \varphi(a) = 0 \text{ for every } \varphi \in S \big\}.$$

On the other hand, for every ideal $J \subseteq \mathcal{A}$, the hull *of J is defined by*

$$h(J) := \big\{ \varphi \in \Delta(\mathcal{A}) \,:\, \varphi(j) = 0 \text{ for every } j \in J \big\}.$$

Now, gathered these definitions, we can consider the following construction: for arbitrary $S \subseteq \Delta(\mathcal{A})$, the *hk-closure* of S is defined as

$$hk(S) := h(k(S)) = \big\{ \varphi \in \Delta(\mathcal{A}) \,:\, \widehat{a}(\varphi) = 0 \text{ for each } a \in \mathcal{A} \text{ such that } \widehat{a} \equiv 0 \text{ on } S \big\}.$$

Evidently, $S \subseteq hk(S)$ for every subset $S \subseteq \Delta(\mathcal{A})$. Moreover, S is said to be *hk-closed* (or, for the sake of brevity, a *hull*) whenever $S = hk(S)$.

As claimed above, the ensemble of all hk-closed subsets in $\Delta(\mathcal{A})$ constitute the closed sets of a topology which, for obvious reasons, is called *hull-kernel topology.* Naturally, the proof of this fact consists in showing that the family of hk-closed is stable under the required set-theoretic operations:

(i) $hk(\emptyset) = \emptyset$ and $hk(\Delta(\mathcal{A})) = \Delta(\mathcal{A})$;

(ii) $hk(S_1 \cup S_2) = S_1 \cup S_2$ for each pair S_1, S_2 hk-closed;

(iii) $hk\big(\bigcap_{\lambda \in \Lambda} S_\lambda\big) = \bigcap_{\lambda \in \Lambda} S_\lambda$ for arbitrary $(S_\lambda)_{\lambda \in \Lambda}$ hk-closed.

See, for instance, [**89**, Prop. 4.3.2] for details.

Proposition 5.5. *Let $\mathcal{A}$ be a commutative complex Banach algebra. Then, the hk-topology is coarser or equal than the Gelfand topology. Moreover, these topologies coincide precisely when $\mathcal{A}$ is regular.*

Proof. By definition, the complement of an arbitrary hk-closed subset $S \subseteq \Delta(\mathcal{A})$ is

$$\Delta(\mathcal{A}) \setminus hk(S) = \big\{ \varphi \in \Delta(\mathcal{A}) \,:\, \text{there exists } a \in \mathcal{A} \text{ with } \widehat{a} \equiv 0 \text{ on } S \text{ and } \widehat{a}(\varphi) \neq 0 \big\}. \tag{5.1}$$

Consequently, picked any $\varphi_0 \in \Delta(\mathcal{A}) \setminus hk(S)$, let $a_0 \in \mathcal{A}$ be an element enjoying simultaneously such conditions. Clearly, for each $0 < \varepsilon < |\varphi_0(a_0)|$, the set

$$U_{\varphi_0, a_0, \varepsilon} = \big\{ \psi \in \Delta(\mathcal{A}) \,:\, |\psi(a_0) - \varphi_0(a_0)| < \varepsilon \big\}$$

is a Gelfand open neighbourhood of φ_0 included in $\Delta(\mathcal{A}) \setminus hk(S)$ (see, for instance, equation (1.5)), which establishes the first assertion.

Now, if the Gelfand topology coincides with the hk-topology, our previous discussion implies that the Banach algebra $\mathcal{A}$ is regular. Conversely, suppose that $\mathcal{A}$ is regular and $S \subseteq \Delta(\mathcal{A})$ is Gelfand closed. Chosen any $\varphi \in \Delta(\mathcal{A}) \setminus S$, thank to regularity, there exists $a \in \mathcal{A}$ for which

$$\widehat{a} \equiv 0 \text{ on } S \quad \text{and} \quad \widehat{a}(\varphi) \neq 0.$$

According to identity (5.1), we have that $\varphi \in \Delta(\mathcal{A}) \setminus hk(S)$ or, equivalently, that $S = hk(S)$. This proves that S is also hk-closed and, therefore, that both topologies coincide. □

Our interest in the hk-topology relies on the following known result, which ensures existence of partitions of unity for hulls in $\Delta(\mathcal{A})$ under some additional assumptions:

Lemma 5.6. *Given any commutative Banach algebra $\mathcal{A}$, the following assertions hold:*

(i) *If S_1 and S_2 are disjoint hulls in $\Delta(\mathcal{A})$ and S_1 is Gelfand compact, then there exists an element $v \in \mathcal{A}$ such that $\widehat{v} \equiv 1$ on S_1 and $\widehat{v} \equiv 0$ on S_2.*

(ii) *If $S \subseteq \Delta(\mathcal{A})$ is a hull for which there exists an element $v \in \mathcal{A}$ such that*

$$|\widehat{v}| \geq \delta \text{ on } S \text{ for some } \delta > 0;$$

then, we can find $w \in \mathcal{A}$ verifying $\widehat{vw} \equiv 1$ on $\Delta(\mathcal{A})$.

Proof. Let $q : \mathcal{A} \to \mathcal{A}/k(S_1)$ be the canonical quotient mapping and consider

$$\begin{aligned} \Psi : \Delta\big(\mathcal{A}/k(S_1)\big) &\to hk(S_1) \\ \varphi &\mapsto \varphi \circ q. \end{aligned} \tag{5.2}$$

It is routine to check that Ψ is a homeomorphism with respect to both Gelfand topology and hk-topology. Moreover, since S_1 is a hull, Ψ actually establishes those topological identifications between $\Delta\big(\mathcal{A}/k(S_1)\big)$ and S_1. Consequently, the Gelfand compactness of S_1 is transferred to the space $\Delta\big(\mathcal{A}/k(S_1)\big)$. On the other hand, it is not hard to see that the equality $k(S_1) = k(hk(S_1))$ entails the semi-simplicity of $\mathcal{A}/k(S_1)$. Hence, by Corollary 5.2, $\mathcal{A}/k(S_1)$ is unital, or equivalently, that $k(S_1)$ is a modular ideal of $\mathcal{A}$.

Let $c \in \mathcal{A}$ be an element such that $a - ac \in k(S_1)$ for all $a \in \mathcal{A}$. Note that, in particular, this implies that $\widehat{c} \equiv 1$ on S_1. Now, consider the larger ideal

$$J := k(S_1) + k(S_2),$$

which, trivially, must be modular as well. Furthermore, since

$$h(J) = h\big(k(S_1) + h(S_2)\big) \subseteq hk(S_1) \cap hk(S_2) = \emptyset,$$

and each maximal modular ideal of $\mathcal{A}$ may be identified with the kernel of a character in $\Delta(\mathcal{A})$, we observe that actually $J = \mathcal{A}$. Therefore, there exists a decomposition of the form $c = u + v$ where $u \in k(S_1)$ and $v \in k(S_2)$. The element $v \in \mathcal{A}$ verifies the requirements of part (i).

To establish (ii), first, recall that $\widehat{v} \in C_0\big(\Delta(\mathcal{A})\big)$ with respect to the Gelfand topology. Hence, the inequality $|\widehat{v}| \geq \delta$ on S implies that the hull S must be Gelfand compact. As above, using the correspondence

$$\Delta\big(\mathcal{A}/k(S)\big) \cong hk(S)$$

we infer that $\mathcal{A}/k(S)$ has an identity. Moreover, since

$$\big|\varphi(v + k(S))\big| \geq \delta \quad \text{for each } \varphi \in \Delta\big(\mathcal{A}/k(S)\big),$$

we obtain that $v + k(S)$ is invertible in $\mathcal{A}/k(S)$. Hence, let $w \in \mathcal{A}$ be such that $vw + k(S)$ is the identity of $\mathcal{A}/k(S)$. Clearly, as required, this means that $\widehat{vw} \equiv 1$ on S. □

Another relevant issue when, further on, we deal with general Banach algebras (in particular, presumably non-unital), is knowing that the hk-topology behaves canonically with respect to unitization:

Proposition 5.7. *Let $\mathcal{A}$ be a commutative complex Banach algebra with unitization $\mathcal{A}_e$. Then $\mathcal{A}$ is regular if and only if $\mathcal{A}_e$ is regular. Moreover, for each $a \in \mathcal{A}$, the Gelfand transform $\widehat{a}$ is hk-continuous on $\Delta(\mathcal{A})$ precisely when $\widehat{a}$ is hk-continuous on $\Delta(\mathcal{A}_e)$.*

For the proof of Proposition 5.7, we will make a full usage of the correspondence (5.2) established between the action of the Gelfand spectrum $\Delta(\mathcal{A})$ and the ideals of $\mathcal{A}$. Specifically, given any ideal $J \subseteq \mathcal{A}$, the restriction $\Delta(\mathcal{A}) \setminus h(J) \to \Delta(J)$ and the composition with the quotient mapping $\mathcal{A} \to \mathcal{A}/J$ lead us to the following topological identifications with respect to both Gelfand topology and hk-topology:

$$\Delta(J) \cong \Delta(\mathcal{A}) \setminus h(J) \quad \text{and} \quad \Delta(\mathcal{A}/J) \cong h(J). \tag{5.3}$$

Moreover, such homeomorphisms are compatible at the level of Gelfand transforms, entailing the next decomposition

$$\Delta(\mathcal{A}) = \Delta(J) \cup \Delta(\mathcal{A}/J), \tag{5.4}$$

where the two sets on the right are disjoint.

Proof of Proposition 5.7. First, since $\mathcal{A}$ is an ideal in the unitization $\mathcal{A}_e$, as a consequence of the correspondence (5.3), it is immediate that regularity is transferred from $\mathcal{A}_e$ to $\mathcal{A}$. Conversely, suppose that $\mathcal{A}$ is regular and choose a Gelfand closed set $F \subseteq \Delta(\mathcal{A}_e)$ and a character $\varphi \in \Delta(A_e) \setminus F$. Now, two possibilities may happen:

(i) If $\varphi = \varphi_\infty$, then $F \subseteq \Delta(\mathcal{A})$ and the trivial choice e satisfies both $\widehat{e} \equiv 0$ on F and $\widehat{e}(\varphi) \neq 0$.

(ii) If $\varphi \neq \varphi_\infty$, then φ acts as well on $\Delta(\mathcal{A})$. Hence, we may apply the regularity of $\mathcal{A}$ with the Gelfand closed set $F \cap \Delta(\mathcal{A})$ to find an element $a \in \mathcal{A}$ such that

$$\widehat{a} \equiv 0 \text{ on } F \cap \Delta(\mathcal{A}) \quad \text{and} \quad \widehat{a}(\varphi) \neq 0.$$

But, since a belongs to $\mathcal{A}$, we also have $\widehat{a}(\varphi_\infty) = 0$. Therefore, $\widehat{a} \equiv 0$ on F, which gives as desired the regularity for $\mathcal{A}_e$.

As before, applying the identifications (5.3), the hk-continuity of $\widehat{a}$ is carried over from the Gelfand spectrum $\Delta(\mathcal{A}_e)$ to $\Delta(\mathcal{A})$ in a trivial way. Hence, in order to verify the full equivalence, it suffices to check that the hk-continuity of $\widehat{a} : \Delta(\mathcal{A}) \to \mathbb{C}$ is transferred when considered over $\Delta(\mathcal{A}_e)$.

So, in the sequel, let $\widehat{a}$ stand for a hk-continuous Gelfand transform acting on $\Delta(\mathcal{A})$. Given an arbitrary closed subset $G \subseteq \mathbb{C}$, consider its preimage under the action of $\widehat{a}$ in $\Delta(\mathcal{A}_e)$, i.e.

$$\widehat{a}^{-1}(G) := \{\varphi \in \Delta(\mathcal{A}_e) \, : \, \widehat{a}(\varphi) \in G\}.$$

Of course, we must confirm that $\widehat{a}^{-1}(G)$ is hk-closed in $\Delta(\mathcal{A}_e)$. Again, we need to distinguish two cases:

(i) If $0 \in G$, then, clearly, $\varphi_\infty \in \widehat{a}^{-1}(G)$. Therefore, by the hk-continuity of $\widehat{a}$ on $\Delta(\mathcal{A})$, we deduce that the set

$$\Delta(\mathcal{A}_e) \setminus \widehat{a}^{-1}(G) = \big\{\varphi \in \Delta(\mathcal{A}) \,:\, \varphi(a) \notin G\big\}$$

is hk-open when regarded in $\Delta(\mathcal{A})$. Now, since $\Delta(\mathcal{A})$ is also hk-open within $\Delta(\mathcal{A}_e)$ (this may be easily seen by means of $\Delta(\mathcal{A}) = \Delta(\mathcal{A}_e) \setminus h(\mathcal{A})$), we have that $\widehat{a}^{-1}(G)$ is actually hk-closed in $\Delta(\mathcal{A}_e)$.

(ii) If $0 \notin G$, then $\varphi_\infty \notin \widehat{a}^{-1}(G)$; thus, the preimage $\widehat{a}^{-1}(G)$ coincides with the set

$$P := \big\{\varphi \in \Delta(\mathcal{A}) \,:\, \varphi(a) \in G\big\}.$$

Clearly, by the hk-continuity of $\widehat{a} : \Delta(\mathcal{A}) \to \mathbb{C}$, we conclude that P is hk-closed in $\Delta(\mathcal{A})$. Furthermore, the uniform bound

$$|\widehat{a}| \geq \delta \;\text{ on } P,$$

holds for $\delta := \operatorname{dist}(0, G)$, which is strictly positive. Now, according to Lemma 5.6 (ii), we can find $b \in \mathcal{A}$ verifying $\widehat{ab} \equiv 1$ on P; consequently, the element $u := e - ab \in \mathcal{A}_e$ satisfies both

$$\widehat{u} \equiv 0 \;\text{ on }\; \widehat{a}^{-1}(G) \quad \text{and} \quad \widehat{u}(\varphi_\infty) = 1.$$

This proves that the character φ_∞ does not belong to the hk-closure of $\widehat{a}^{-1}(G)$ in $\Delta(\mathcal{A}_e)$. But, since P is hk-closed in $\Delta(\mathcal{A})$, we deduce that the hk-closure of $\widehat{a}^{-1}(G)$ in $\Delta(\mathcal{A}_e)$ is contained in P, which yields the hk-closeness of $\widehat{a}^{-1}(G)$ in the Gelfand spectrum $\Delta(\mathcal{A}_e)$. Of course, this establishes the hk-continuity of the mapping $\widehat{a} : \Delta(\mathcal{A}_e) \to \mathbb{C}$.

□

5.2. Algebra actions and local spectral decompositions

As mentioned above, our main objective at the present chapter is to unravel the spectral meaning hidden beneath Atzmon's Theorem (and related results from [**13**] and [**18**]), aiming for the ensuing implementation regarding the study of Bishop operators.

The preliminary step prior to understanding Atzmon's Theorem from a spectral point of view starts with the aforementioned Colojoară and Foiaş' restatement of Wermer's Theorem (see Theorem 1.33), which asserts that any invertible operator $T \in \mathcal{B}(X)$ satisfying the series condition

$$\sum_{n\in\mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty$$

is automatically decomposable. A shallow scheme of its proof is the following: firstly, note that the real sequence $\rho := (\|T^n\|)_{n\in\mathbb{Z}}$ defines a Beurling sequence. Consequently, the corresponding Beurling algebra $\mathcal{A}_\rho$ is regular. Thus, one may consider the continuous algebraic homomorphism determined by

$$\begin{aligned} \phi : \mathcal{A}_\rho &\longrightarrow \mathcal{B}(X) \\ e^{int} &\longmapsto T^n \end{aligned} \tag{5.5}$$

for each $n \in \mathbb{Z}$. Now, given an arbitrary open covering $\{U, V\}$ of the spectrum $\sigma(T)$, one can find a function $h \in \mathcal{A}_\rho$ such that

$$h \equiv 1 \;\text{ on }\; (\mathbb{C} \setminus \overline{V}) \cap \sigma(T) \quad \text{and} \quad h \equiv 0 \;\text{ on }\; (\mathbb{C} \setminus \overline{U}) \cap \sigma(T).$$

But, the manner in which h has been chosen, causes that the ranges of the operators $\phi(h)$ and $I-\phi(h)$ split the spectrum $\sigma(T)$ as follows:

$$\sigma\big(T\,\big|\,\overline{\phi(h)(X)}\big)\subseteq U \quad \text{and} \quad \sigma\big(T\,\big|\,\overline{(I-\phi(h))(X)}\big)\subseteq V.$$

In sum, as we intended to show, the operator T is proved to be decomposable.

This circle of ideas relating regular Banach algebras with decomposability was initially grasped by Colojoară and Foiaş [**38**], resulting in the definition of $\mathcal{A}$-scalar operators. Lately, a number of authors made advances in that direction such as Albrecht [**5**] or Eschmeier [**57**]. Herein, we shall work in the generality of Neumann [**98**].

At a glance, it sounds reasonable to expect that Atzmon's Theorem constitutes a sort of localized version of Colojoară and Foiaş' decomposability result. Correspondingly, one may wisely conjecture that Atzmon's Theorem should also involve a kind of decomposition on the spectrum $\sigma(T)$. Nevertheless, in this case, such spectral decomposition shall be much more subtle and less manageable. More precisely, when the invertibility of $T\in\mathcal{B}(X)$ is missing, we are compelled to consider a decomposition on $\sigma(T)$ which is exclusively built upon a proper linear submanifold of X.

For the sake of readability, we highlight the key ideas in a few lines: our intention is to modify the preceding explanation on Colojoară and Foiaş Theorem with the purpose of fitting it into the hypothesis required in Atzmon's Theorem. To that end, let $\rho:=(\rho_n)_{n\in\mathbb{Z}}$ be an arbitrary Beurling sequence. Since $T\in\mathcal{B}(X)$ needs no longer be invertible, the algebraic homomorphism (5.5) should be replaced by

$$\begin{aligned}\phi:\mathcal{A}_\rho &\longrightarrow \mathcal{L}(X)\\ e^{int} &\longmapsto T^n\end{aligned} \qquad\qquad (n\in\mathbb{Z}),$$

embracing some non-bounded linear operators. However, in order to retrieve some profitable spectral properties, we must restrict the mapping ϕ to operate on a linear submanifold of X upon which ϕ behaves nicely, for instance

$$\mathcal{D}=\big\{x\in X\,:\,\|T^n x\|_X \ll_x \rho_n \ \text{ for } \ n\in\mathbb{Z}\big\}.$$

Once again, the Beurling condition

$$\sum_{n\in\mathbb{Z}}\frac{\log\rho_n}{1+n^2}<\infty$$

enables us to employ the regularity of the Beurling algebra $\mathcal{A}_\rho$. Accordingly, for any open covering $\{U,V\}$ of the spectrum $\sigma(T)$, the regularity of $\mathcal{A}_\rho$ allows us to select a function $h\in\mathcal{A}_\rho$ having

$$h\equiv 1 \ \text{ on } \ (\mathbb{T}\setminus\overline{V}) \quad \text{and} \quad h\equiv 0 \ \text{ on } \ (\mathbb{T}\setminus\overline{U}).$$

At this point, given an arbitrary $x\in\mathcal{D}$, an adjustment on the previous estimations shows that the associated local spectra are included inside

$$\sigma_T\big(\phi(h)x\big)\subseteq U \quad \text{and} \quad \sigma_T\big((I-\phi(h))x\big)\subseteq V,$$

leading to the desired local spectral decomposition over the submanifold $\mathcal{D}$.

By all means, our preceding discussion shall be treated in full formalisation below. Actually, following the lines proposed by Neumann [**98**], Section 5.2 is devoted to developing the whole of required tools in order to devise this kind of functional models over a generic Banach algebra set-up.

In this light, aiming for adapting some of the ideas appearing in [**98**, Theorem 1] to the non-bounded case, we need to fit our framework. To that end, recall that the class of unbounded operators $\mathcal{L}(X)$ consists of those linear mappings $T : \mathrm{Dom}(T) \to X$, where $\mathrm{Dom}(T)$ (the *domain* of the operator T) is a linear submanifold of X. As is customary, for each $T \in \mathcal{L}(X)$, let $\mathrm{ran}(T) := \{Tx : x \in \mathrm{Dom}(T)\}$ denote its *range*. In contrast to $\mathcal{B}(X)$, the class $\mathcal{L}(X)$ does not constitute an algebra because the operators in $\mathcal{L}(X)$ do not share a joint domain. To this effect, in order to take advantage of a functional calculus argument, we need to restrict ourselves to a set of operators in $\mathcal{L}(X)$ acting on a joint domain:

Definition 5.8. *Let $\mathcal{A}$ be a commutative algebra and X a complex Banach space. We will say that the mapping $\phi : \mathcal{A} \to \mathcal{L}(X)$ is an* algebra action *over a linear submanifold $M \subseteq \bigcap_{a\in\mathcal{A}} \mathrm{Dom}\big(\phi(a)\big)$ if given any $x \in M$ the following holds:*

(i) *For each $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$,*
$$\phi(\lambda a + \mu b)x = \lambda\phi(a)x + \mu\phi(b)x.$$

(ii) *For each $b \in \mathcal{A}$,*
$$\phi(b)x \in M \quad \text{and} \quad \phi(ab)x = \phi(a)\phi(b)x \text{ for all } a \in \mathcal{A}.$$

Secondly, since the boundedness of the operators involved along our arguments will not be assumed, we must restrict ourselves to a subset in which the algebra action $\phi : \mathcal{A} \to \mathcal{L}(X)$ can be properly controlled. This is the goal behind our next definition:

Definition 5.9. *Let $\mathcal{A}$ be a Banach algebra and consider a map $\phi : \mathcal{A} \to \mathcal{L}(X)$. Then, a manifold $\mathcal{D}_\phi \subseteq X$ will be called a* continuity core *for ϕ if verifies the next conditions:*

(i) *$\phi : \mathcal{A} \to \mathcal{L}(X)$ is an algebra action over $\mathcal{D}_\phi$.*

(ii) *The linear operator $\mathcal{A} \to X$, $a \mapsto \phi(a)x$ is bounded for each $x \in \mathcal{D}_\phi$.*

REMARK 5.10. Note that in the previous definition, the norm of the linear mappings $\mathcal{A} \to X$, $a \mapsto \phi(a)x$ may depend on $x \in \mathcal{D}_\phi$. In other words, for each $x \in \mathcal{D}_\phi$ there exists a constant $C_x > 0$ such that
$$\|\phi(a)x\|_X \leq C_x\|a\|_{\mathcal{A}} \qquad (a \in \mathcal{A}).$$
In particular, observe that the equicontinuity of such family of mappings is not required.

Note that each continuity core $\mathcal{D}_\phi$ is a $\phi(a)$-invariant linear manifold for every $a \in \mathcal{A}$. Not surprisingly, our main interest will be those algebra actions $\phi : \mathcal{A} \to \mathcal{L}(X)$ which admit a non-zero continuity core. Indeed, in this context, an extremal case is provided by the aforementioned Wermer's Theorem [**126**], which may be regarded as the application of an algebra action with continuity core X. However, throughout the chapter, we will encounter much more general situations.

Now, we are in position to state the main lemma of this section. The underlying idea is that, the local continuity condition over $\mathcal{D}_\phi$ allows us to transfer, in a slightly weaker form, the super-decomposability of the multiplication operators on $\mathcal{A}$ (see, for instance, [**98**, Theorem 1.2]) to the operators in $\phi(\mathcal{A}) \cap \mathcal{B}(X)$. Thus, although the super-decomposability does not need to stand anymore for the bounded operators in $\phi(\mathcal{A})$, the sort of spectral decomposition arising from the forthcoming lemma will be rich enough and profitable in many cases, for instance, in order to find non-trivial non-dense local spectral subspaces for some operators lacking good local spectral properties.

Lemma 5.11. *Let $\mathcal{A}$ be a semi-simple commutative Banach algebra with identity e, X a complex Banach space and $\phi : \mathcal{A} \to \mathcal{L}(X)$ an algebra action with $\phi(e) = I$ and continuity core $\mathcal{D}_\phi$. Let $a \in \mathcal{A}$ be such that its Gelfand transform $\widehat{a}$ is hk-continuous on $\Delta(\mathcal{A})$ and $T := \phi(a)$ belongs to $\mathcal{B}(X)$. Then for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, $(n > 1)$, there exist $u_1, \ldots, u_n \in \mathcal{A}$ such that:*

(i) $\widehat{u}_1 + \ldots + \widehat{u}_n \equiv 1$ *on* $\Delta(\mathcal{A})$ *and* $\operatorname{supp} \widehat{u}_k \subseteq \widehat{a}^{-1}(U_k)$ *for every* $k = 1, \ldots, n$.

(ii) *Given any* $x \in \mathcal{D}_\phi$,

$$\sigma_T\big(\phi(u_k)\, x\big) \subseteq U_k \quad \text{and} \quad \sigma_T\big((I - \phi(u_k))\, x\big) \subseteq \bigcup_{j \neq k} U_j$$

for every $k = 1, \ldots, n$.

Proof. Proceeding by induction, it is possible to show that for every $k = 1, \ldots, n$, we can find a pair of open sets $G_k, H_k \subseteq \mathbb{C}$ verifying the chain of inclusions

$$G_k \subseteq \overline{G_k} \subseteq H_k \subseteq \overline{H_k} \subseteq U_k$$

and for which $\{G_1, \ldots, G_n\}$ is still an open cover of $\mathbb{C}$. Since $\widehat{a}$ is hk-continuous and the sets $\overline{G_k}$ and $\mathbb{C} \setminus H_k$ are closed and disjoint in $\mathbb{C}$, we deduce that $\widehat{a}^{-1}(\overline{G_k})$ and $\widehat{a}^{-1}(\mathbb{C} \setminus H_k)$ must be disjoint hulls in $\Delta(\mathcal{A})$. Furthermore, since $\mathcal{A}$ possesses an identity, both sets are also compact in the Gelfand topology of $\Delta(\mathcal{A})$. Therefore, by means of Lemma 5.6 (i), we can find an $r_k \in \mathcal{A}$ such that

$$\widehat{r}_k \equiv 1 \text{ on } \widehat{a}^{-1}(\overline{G_k}) \text{ and } \widehat{r}_k \equiv 0 \text{ on } \widehat{a}^{-1}(\mathbb{C} \setminus H_k).$$

Now, define the elements $(u_k)_{k=1}^n$ of $\mathcal{A}$ following the subsequent iteration:

$$u_1 := r_1 \text{ and } u_k := (e - r_1) \cdots (e - r_{k-1})\, r_k \text{ for } k = 2, \ldots, n.$$

Clearly, $\operatorname{supp} \widehat{u}_k \subseteq \widehat{a}^{-1}(U_k)$ for each $k = 1, \ldots, n$ and by induction it may be seen that the identity

$$\widehat{u}_1 + \ldots + \widehat{u}_k = \widehat{e} - (\widehat{e} - \widehat{r}_1) \cdots (\widehat{e} - \widehat{r}_k)$$

holds for every $k = 1, \ldots, n$. Finally, since $\{\widehat{a}^{-1}(G_1), \ldots, \widehat{a}^{-1}(G_n)\}$ must be a cover of $\Delta(\mathcal{A})$, it is deduced that

$$\widehat{u}_1 + \ldots + \widehat{u}_n \equiv 1 \text{ on } \Delta(\mathcal{A}).$$

For the second part of the proof, let $x \in \mathcal{D}_\phi$ be arbitrary and $k \in \{1, \ldots, n\}$ fixed: picked any $\lambda \in \mathbb{C} \setminus U_k$, the distance $\delta := \operatorname{dist}(\lambda, \overline{H_k})$ is strictly positive. Moreover, the bound $|\widehat{a} - \lambda \widehat{e}| \geq \delta$ holds on $\widehat{a}^{-1}(\overline{H_k})$ and therefore, by Lemma 5.6 (ii) there exists $v_\lambda \in \mathcal{A}$ for which

$$(\widehat{a} - \lambda \widehat{e})\, \widehat{v}_\lambda \equiv 1 \text{ on } \widehat{a}^{-1}(\overline{H_k}),$$

and more generally $(\widehat{a} - \lambda \widehat{e})^n\, \widehat{v}_\lambda^n \equiv 1$ on $\widehat{a}^{-1}(\overline{H_k})$ for every $n \in \mathbb{N}$. Since $\widehat{u}_k \equiv 0$ on $\widehat{a}^{-1}(\mathbb{C} \setminus H_k)$, the identity

$$(\widehat{a} - \lambda \widehat{e})^n\, \widehat{v}_\lambda^n\, \widehat{u}_k(\varphi) \equiv \widehat{u}_k(\varphi), \text{ for every } \varphi \in \Delta(\mathcal{A})$$

holds for every $n \in \mathbb{N}$. Thus, $(a - \lambda)^n\, v_\lambda^n\, u_k = u_k$ by the semi-simplicity of $\mathcal{A}$. In particular, applying the algebra action ϕ, this implies

$$(T - \lambda I)^n \phi(v_\lambda)^n \phi(u_k)\, x = \phi(u_k)\, x$$

for every $n \in \mathbb{N}$. Now, consider the X-valued analytic function given by

$$R_T\big(z; \phi(u_k)\, x\big) := \sum_{m \geq 1} \big(\phi(v_\lambda)^m \phi(u_k)\, x\big) \cdot (z - \lambda)^{m-1};$$

then, according to the boundedness of the mapping $\mathcal{A} \to X$ $u \mapsto \phi(u)x$ for each $x \in \mathcal{D}_\phi$ and the spectral radius formula, we have

$$\limsup_{m\to\infty} \left\|\phi(v_\lambda)^m \phi(u_k)\, x\right\|_X^{1/m} \le \limsup_{m\to\infty} \left\|v_\lambda^m\, u_k\right\|_{\mathcal{A}}^{1/m} \le r(v_\lambda) < +\infty.$$

This shows that the function $R_T(\,\cdot\,; \phi(u_k)\, x)$ is analytic on $D(\lambda, \varepsilon)$ for some $\varepsilon > 0$.

On the other hand, standard computations evince that $R_T(\,\cdot\,; \phi(u_k)\, x)$ is a local resolvent function

$$(T - zI)\, R_T\big(z; \phi(u_k)\, x\big) = \phi(u_k)\, x \;\text{ for every }\; z \in D(\lambda, \varepsilon),$$

and therefore, $\sigma_T\big(\phi(u_k)\, x\big) \subseteq U_k$. Finally, a similar argument shows that

$$\sigma_T\big((I - \phi(u_k))\, x\big) \subseteq \bigcup_{j\neq k} U_j,$$

which completes the proof of the lemma. □

The next step is based on a standard unitization argument which allows to extend the previous lemma to algebra actions $\phi : \mathcal{A} \to \mathcal{L}(X)$ such that either $\phi(\widetilde{e}) \neq I$, where $\widetilde{e} \in \mathcal{A}$ is the identity, or even to algebras $\mathcal{A}$ without identity.

Theorem 5.12. *Consider a semi-simple commutative Banach algebra $\mathcal{A}$, a complex Banach space X and an algebra action $\phi : \mathcal{A} \to \mathcal{L}(X)$ with continuity core $\mathcal{D}_\phi$. Let $a \in \mathcal{A}$ be such that its Gelfand transform $\widehat{a}$ is hk-continuous on $\Delta(\mathcal{A})$ and $T := \phi(a)$ belongs to $\mathcal{B}(X)$. Then for every closed subset $F \subseteq \mathbb{C}$, we have*

$$X_T(F) \supseteq \big\{\phi(u)(\mathcal{D}_\phi) \,:\, u \in \mathcal{A} \textit{ satisfies } \operatorname{supp} \widehat{u} \subseteq \widehat{a}^{-1}(F)\big\}. \tag{5.6}$$

Moreover, the inclusion

$$\mathcal{D}_\phi \subseteq X_T(U_1) + \ldots + X_T(U_n) \tag{5.7}$$

holds for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$.

As it shall be seen later on, $\mathcal{D}_\phi$ can be even a dense proper subset of X in some particular instances. Therefore, as a consequence of the inclusion (5.7), a sort of decomposability will arise in those cases for all the operators in $\phi(\mathcal{A}) \cap \mathcal{B}(X)$. On the other hand, it is worth pointing out that, unlike [**89**, Theorem 4.4.1], an equality for $X_T(F)$ cannot be obtained here, since we are restricted to work over the manifold $\mathcal{D}_\phi$. Hence, we must settle for an inclusion like (5.6).

Proof of Theorem 5.12. Regardless of $\mathcal{A}$ is a unital algebra or not, consider its unitization $\mathcal{A}_e = \mathcal{A} \oplus \mathbb{C}e$ and the extension of the action ϕ given by

$$\Phi(u + \lambda e) := \phi(u) + \lambda I, \;\text{ for every } u \in \mathcal{A} \text{ and } \lambda \in \mathbb{C},$$

whose domains are defined by $\operatorname{Dom}(\Phi(u + \lambda e)) := \operatorname{Dom}(\phi(u))$. In addition, observe that $\Phi(u) = \phi(u)$ holds for every $u \in \mathcal{A}$ and $\Phi(e) = I$.

Now, we will prove that $\mathcal{D}_\phi$ is again a continuity core for the extension Φ: first, note that the equality

$$\bigcap_{u\in\mathcal{A}} \operatorname{Dom}\big(\phi(u)\big) = \bigcap_{u\in\mathcal{A},\lambda\in\mathbb{C}} \operatorname{Dom}\big(\Phi(u + \lambda e)\big)$$

indicates, after some routine computations, that the extension $\Phi : \mathcal{A}_e \to \mathcal{L}(X)$ is still an algebra action over the manifold $\mathcal{D}_\phi$. Moreover, given any $x \in \mathcal{D}_\phi$, let C_x denote the norm of the map $\mathcal{A} \to X,\ u \mapsto \phi(u)x$; then, as a consequence of the inequality

$$\|\Phi(u+\lambda e)x\|_X \le \|\phi(u)x\|_X + |\lambda|\,\|x\|_X \le \max(C_x, \|x\|_X)\,\|u+\lambda e\|_{\mathcal{A}_e},$$

it is plain that $\mathcal{D}_\phi$ constitutes a continuity core for $\phi : \mathcal{A}_e \to \mathcal{L}(X)$. Finally, it may be seen that the semi-simplicity is carried over from $\mathcal{A}$ to its unitization $\mathcal{A}_e$.

Aiming for applying Lemma 5.6 to the algebra action $\Phi : \mathcal{A}_e \to \mathcal{L}(X)$, now consider an arbitrary open subset U such that $F \subseteq U$. Clearly, the collection $\{U, \mathbb{C} \setminus F\}$ forms an open cover of $\mathbb{C}$ and, keeping in mind that $\widehat{a}$ remains hk-continuous on $\Delta(\mathcal{A}_e)$ (see Proposition 5.7 above), Lemma 5.11 provides an element $r \in \mathcal{A}_e$ such that

$$\widehat{r} \equiv 1 \ \text{ on } \widehat{a}^{-1}(\mathbb{C}\setminus U) \ \text{ and } \ \widehat{r} \equiv 0 \ \text{ on } \widehat{a}^{-1}(F),$$

and which satisfies that, for all $x \in \mathcal{D}_\phi$

$$\sigma_T\big(\Phi(r)x\big) \subseteq \mathbb{C}\setminus F \quad \text{ and } \quad \sigma_T\big((I-\Phi(r))x\big) \subseteq U. \tag{5.8}$$

Now, take any $u \in \mathcal{A}$ with $\operatorname{supp}\widehat{u} \subseteq \widehat{a}^{-1}(F)$ on $\Delta(\mathcal{A})$; since $\operatorname{supp}\widehat{u}$ coincides trivially on both $\Delta(\mathcal{A})$ and $\Delta(\mathcal{A}_e)$, we still have

$$\operatorname{supp}\widehat{u} \subseteq \widehat{a}^{-1}(F) \ \text{ on } \Delta(\mathcal{A}_e).$$

Thus, according to their supports, it is plain that $\widehat{ur} \equiv 0$ on $\Delta(\mathcal{A}_e)$; and, as a consequence of the semi-simplicity of $\mathcal{A}_e$, it is followed that $ur = 0$. Finally, given any $x \in \mathcal{D}_\phi$, we have

$$\sigma_T\big(\phi(u)\,x\big) \subseteq \sigma_T\big(\Phi(ur)\,x\big) \cup \sigma_T\big((I-\Phi(r))\phi(u)\,x\big) = \sigma_T\big((I-\Phi(r))\phi(u)\,x\big) \subseteq U,$$

where the last inclusion is deduced from the equation (5.8) and the fact $\phi(u)x \in \mathcal{D}_\phi$. Thus, we have proved that

$$X_T(U) \supseteq \big\{\phi(u)(\mathcal{D}_\phi) \,:\, \text{for every } u \in \mathcal{A} \text{ with } \operatorname{supp}\widehat{u} \subseteq \widehat{a}^{-1}(F)\big\}$$

for arbitrary open set $U \supseteq F$. This establishes inclusion (5.6).

Finally, for the proof of inclusion (5.7), given any finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, consider the elements $u_1, \ldots, u_n \in \mathcal{A}_e$ provided by Lemma 5.11. Now, let $x \in \mathcal{D}_\phi$ be arbitrary, then

$$\Phi\Big(\sum_{k=1}^{n} u_k\Big)x = \Phi(u_1)\,x + \ldots + \Phi(u_n)\,x \in X_T(U_1) + \ldots + X_T(U_n);$$

but, since $\mathcal{A}_e$ is semi-simple and $\widehat{u}_n + \ldots + \widehat{u}_n \equiv 1$ on $\Delta(\mathcal{A}_e)$, it is plain that $u_1 + \ldots + u_n = e$. $\square$

Now, consider an arbitrary operator $T \in \mathcal{B}(X)$. As it was pointed out previously, when the techniques of our preceding results can be applied to both T and its adjoint T^* for two appropriate vectors $x \in X$ and $y \in X^*$, we manage to construct non-trivial T-hyperinvariant subspaces. In particular, this result generalizes Atzmon's Theorem [**15**, Theorem 1.1] to a wider class of functional models and, furthermore, it reveals the spectral nature of the hyperinvariant subspaces involved.

Theorem 5.13. *Let T be a bounded linear operator on a Banach space X with $\sigma_{\rm p}(T) = \sigma_{\rm p}(T^*) = \emptyset$ and $\mathcal{A}$ a semi-simple commutative Banach algebra. Suppose that there exist two algebra actions $\phi : \mathcal{A} \to \mathcal{L}(X)$ and $\psi : \mathcal{A} \to \mathcal{L}(X^*)$ with continuity cores $\mathcal{D}_\phi$ and $\mathcal{D}_\psi$, such that $T = \phi(a_1)$ and $T^* = \psi(a_2)$ for some $a_1, a_2 \in \mathcal{A}$ whose Gelfand transforms $\widehat{a}_1, \widehat{a}_2$ are hk-continuous on $\Delta(\mathcal{A})$. Then, if there exist both non-zero $x \in \mathcal{D}_\phi$ and $y \in \mathcal{D}_\psi$ such that*

$\sigma_T(x) \cup \sigma_{T^*}(y)$ is not a singleton, for every open subset $U \subseteq \mathbb{C}$ such that $U \cap \sigma_T(x) \neq \emptyset$ and $\sigma_{T^*}(y) \setminus \overline{U} \neq \emptyset$, we have

$$\{0\} \neq \overline{X_T(U)} \neq X.$$

In particular, T has a non-trivial hyperinvariant subspace.

Proof. As in the latter proof, $\mathcal{A}_e$ will stand for the unitization of the Banach algebra $\mathcal{A}$, while Φ and Ψ will represent the extensions of the actions ϕ and ψ to $\mathcal{A}_e$, respectively.

Let $\mu_1 \in \sigma_T(x) \cap U$ and $\mu_2 \in \sigma_{T^*}(y) \setminus \overline{U}$ be arbitrary and consider the following settings of open sets: $F, G \subseteq \mathbb{C}$ verifying $\mu_1 \in F \subseteq \overline{F} \subseteq G \subseteq \overline{G} \subseteq U$; and $V, W \subseteq \mathbb{C}$ verifying $\mu_2 \in V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq \mathbb{C} \setminus \overline{U}$. It is plain that both $\{G, \mathbb{C} \setminus \overline{F}\}$ and $\{W, \mathbb{C} \setminus \overline{V}\}$ form open covers of $\mathbb{C}$ and so, Lemma 5.11 provides two elements $r_1, r_2 \in \mathcal{A}_e$ such that

$$\operatorname{supp} \widehat{r}_1 \subseteq \widehat{a}_1^{-1}(G) \text{ and } \operatorname{supp}(1 - \widehat{r}_1) \subseteq \widehat{a}_1^{-1}(\mathbb{C} \setminus \overline{F}) \text{ on } \Delta(\mathcal{A}_e)$$

with $\sigma_T\big(\Phi(r_1)\, x\big) \subseteq G$ and $\sigma_T\big((I - \Phi(r_1))\, x\big) \subseteq \mathbb{C} \setminus \overline{F}$; and, analogously

$$\operatorname{supp} \widehat{r}_2 \subseteq \widehat{a}_2^{-1}(W) \text{ and } \operatorname{supp}(1 - \widehat{r}_2) \subseteq \widehat{a}_2^{-1}(\mathbb{C} \setminus \overline{V}) \text{ on } \Delta(\mathcal{A}_e)$$

with $\sigma_{T^*}\big(\Psi(r_2)\, y\big) \subseteq W$ and $\sigma_{T^*}\big((I - \Psi(r_2))\, y\big) \subseteq \mathbb{C} \setminus \overline{V}$.

Now, we claim that both $\Phi(r_1)\, x \in X$ and $\Psi(r_2)\, y \in X^*$ are non-zero: firstly, since

$$\sigma_T(x) \subseteq \sigma_T\big(\Phi(r_1)\, x\big) \cup \sigma_T\big((I - \Phi(r_1))\, x\big)$$

and $\sigma_T\big((I - \Phi(r_1))\, x\big) \subseteq \mathbb{C} \setminus \overline{F}$, we deduce that μ_1 must belong to $\sigma_T\big(\Phi(r_1)\, x\big)$. But, since $\sigma_{\mathrm{p}}(T) = \emptyset$ and this implies the SVEP for T, we have that $\Phi(r_1)\, x \neq 0$. An analogous argument applies in order to see that $\Psi(r_2)\, y \neq 0$. In particular, all this shows that both $X_T(G)$ and $X^*_{T^*}(W)$ are non-zero linear manifolds.

Finally, since $\overline{U} \cap \overline{W} = \emptyset$, by Proposition 4.3 and the SVEP for T and T^*, we have the inclusion $X_T\big(\overline{U}\big) \subseteq {}^{\perp}X^*_{T^*}\big(\overline{W}\big)$; which implies the desired inequalities

$$\{0\} \neq \overline{X_T(U)} \neq X.$$

□

Some easy remarks are in order: note that the conclusions of Theorem 5.13 holds whenever $\sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T^*) \neq \emptyset$. In addition, it plays no role if the domains of ϕ and ψ are either the same or distinct Banach algebras, or if a_1 and a_2 are either the same or distinct elements in order to draw also the same conclusions.

5.3. Local spectral decompositions of Bishop operators

We begin this section by rewriting Atzmon's Theorem in the language of Theorem 5.13 and applying it to the context of Bishop operators. For this purpose, first recall that given a suitable sequence $\rho := (\rho_n)_{n \in \mathbb{Z}}$, the associated Beurling algebra $\mathcal{A}_\rho$ consists of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ having finite

$$\|f\|_\rho := \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|\, \rho_n.$$

When $\mathcal{A}_\rho$ is endowed with the norm $\|\cdot\|_\rho$, it acquires structure of a semi-simple unital commutative complex Banach algebra with Gelfand spectrum $\Delta(\mathcal{A}_\rho) \cong \mathbb{T}$.

Now, the content of Atzmon's Theorem can be subsumed under Theorem 5.13 as follows. Consider $T \in \mathcal{B}(X)$ an injective, dense-range operator on a complex Banach space X. In particular, the injectivity of T implies that the operators $T^{-n} \in \mathcal{L}(X)$ with $\mathrm{Dom}(T^{-n}) := \mathrm{ran}(T^n)$ are densely-defined for each $n \in \mathbb{N}$. Observe that, analogously, the same holds for $T^* \in \mathcal{B}(X^*)$ since the dense range of T implies $\ker(T^*) = \{0\}$.

In this context, let $\mathcal{A}_\rho$ be a regular Beurling algebra and consider the maps taking values on the classes of linear operators $\mathcal{L}(X)$ and $\mathcal{L}(X^*)$

$$\begin{aligned} \phi : \mathcal{A}_\rho &\to \mathcal{L}(X) \\ f &\mapsto \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, T^n \end{aligned} \qquad\qquad \begin{aligned} \psi : \mathcal{A}_\rho &\to \mathcal{L}(X^*) \\ g &\mapsto \sum_{n\in\mathbb{Z}} \widehat{g}(n)\, T^{*\,n}, \end{aligned}$$

respectively. More accurately, to provide a valid definition of the operators $\phi(f) \in \mathcal{L}(X)$ and $\psi(g) \in \mathcal{L}(X^*)$, we must specify their domains. In this light, for instance, note that $\phi(f)$ is well-defined on

$$\mathrm{Dom}\big(\phi(f)\big) := \Big\{ x \in \bigcap_{n\in\mathbb{Z},\, \widehat{f}(n)\neq 0} \mathrm{Dom}(T^n) \ : \ \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\, \|T^n x\|_X < \infty \Big\},$$

due to the absolute convergence of the series. Naturally, a similar construction applies for $\psi(g)$ as well.

Finally, we need to determine one continuity core for each of the mappings $\phi : \mathcal{A}_\rho \to \mathcal{L}(X)$ and $\psi : \mathcal{A}_\rho \to \mathcal{L}(X^*)$ respectively. According to the choice made by Atzmon [**15**], consider the linear manifold

$$\mathcal{D}_\phi := \Big\{ x \in \bigcap_{n\in\mathbb{Z}} \mathrm{Dom}\big(T^n\big) \ : \ \|T^n x\|_X \ll_x \rho_n \quad (n \in \mathbb{Z}) \Big\}. \tag{5.9}$$

In order to prove that $\mathcal{D}_\phi$ is indeed a continuity core for ϕ, first observe that, as a consequence of the following inequality, we conclude that $\phi(f)x \in \mathcal{D}_\phi$ for each $f \in \mathcal{A}_\rho$ and $x \in \mathcal{D}_\phi$:

$$\big\|T^n \phi(f)x\big\|_X = \Big\| \sum_{m\in\mathbb{Z}} \widehat{f}(m) T^{m+n} x \Big\|_X \ll_x \sum_{m\in\mathbb{Z}} \big|\widehat{f}(m)\big|\, \rho_{m+n} \leq \rho_n \|f\|_{\mathcal{A}_\rho} \ll_f \rho_n \qquad (n \in \mathbb{Z}).$$

Now, the rest of algebraic requirements from Definition 5.8 follow upon standard computations considering the absolute convergence of the series. Accordingly, $\phi : \mathcal{A}_\rho \to \mathcal{L}(X)$ is an algebra action over $\mathcal{D}_\phi$. Lastly, the boundedness of the assignment $\mathcal{A}_\rho \to X$, $f \mapsto \phi(f)x$ for each $x \in \mathcal{D}_\phi$ can be checked by means of the following inequality:

$$\|\phi(f)x\|_X \leq \sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|\, \big\|T^n x\big\|_X \ll_x \sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|\, \rho_n = \|f\|_{\mathcal{A}_\rho}.$$

This proves that $\mathcal{D}_\phi$ is a continuity core for ϕ. Of course, a similar construction works for the map $\psi : \mathcal{A}_\rho \to \mathcal{L}(X^*)$.

Therefore, observe that the hypothesis on the existence of two non-zero vectors $x \in X$ and $y \in X^*$ such that

$$\|T^n x\|_X \ll \rho_n \quad \text{and} \quad \|T^{*\,n}\, y\|_{X^*} \ll \rho_n \qquad\qquad (n \in \mathbb{Z}),$$

guarantees that both continuity cores $\mathcal{D}_\phi$ and $\mathcal{D}_\psi$ given in (5.9) are non-zero. On the other hand, since $\mathcal{A}_\rho$ is a regular Beurling algebra, the hk-continuity of the elements in $\Delta(\mathcal{A}_\rho)$

is straightforward. This observation along with the technical condition regarding the local spectra $\sigma_T(x)$ and $\sigma_{T^*}(y)$ yields Atzmon's Theorem upon applying Theorem 5.13 as promised:

Theorem 5.14. *Let $T \in \mathcal{B}(X)$ be an operator on a Banach space X such that $\sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T^*) = \emptyset$. Assume that there exist non-zero $x \in X$ and $y \in X^*$ such that*

$$\|T^n x\|_X \ll \rho_n \quad \text{and} \quad \|T^{*\,n} y\|_{X^*} \ll \rho_n$$

for some Beurling sequence $\rho := (\rho_n)_{n\in\mathbb{Z}}$. Then, if $\sigma_T(x) \cup \sigma_{T^*}(y)$ is not a singleton, for every open subset $U \subseteq \mathbb{C}$ such that $U \cap \sigma_T(x) \neq \emptyset$ and $\sigma_{T^*}(y) \setminus \overline{U} \neq \emptyset$, we have

$$\{0\} \neq \overline{X_T(U)} \neq X.$$

In particular, T has a non-trivial hyperinvariant subspace.

In accordance with the conclusions presented in Chapter 4, the absence of any advantageous local spectral property seemed to be an evident feature regarding Bishop operators. Nonetheless, as evinced by Theorem 5.14, this was somehow caused by a misguided choice of the local spectral manifolds. Hereafter, we collect several applications and consequences of the results from Section 5.2 to Bishop operators:

Theorem 5.15. *Let $\alpha \in (0,1)$ be an irrational number whose convergents $(a_j/q_j)_{j=0}^{\infty}$ in its continued fraction satisfy*

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right). \tag{5.10}$$

Suppose that T_α acts on $L^p[0,1)$ for some $1 \leq p < +\infty$. Then, given any open subset $U \subseteq \mathbb{C}$ such that $U \cap \partial\sigma(T_\alpha) \neq \emptyset$ and $\partial\sigma(T_\alpha) \setminus \overline{U} \neq \emptyset$, we have

$$\{0\} \neq \overline{X_{T_\alpha}(U)} \neq L^p[0,1).$$

Proof. Fix $1 \leq p < \infty$ and denote by $1 < p' \leq \infty$ its conjugate exponent. By Theorem 2.16, we know that

$$\left\|\widetilde{T}_\alpha^n\, \mathbf{1}_{\mathcal{B}_\alpha}\right\|_p \ll \rho_n \quad \text{and} \quad \left\|\widetilde{T}_\alpha^{*\,n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_{p'} \ll \rho_n \qquad (n \in \mathbb{Z})$$

for some Beurling sequence $\rho := (\rho_n)_{n\in\mathbb{Z}}$. As a consequence of the similarity between $\widetilde{T}_\alpha$ and $e^{2\pi i\ell\alpha}\widetilde{T}_\alpha$ via the multiplication operator $M_{e^{2\pi\imath\ell t}} \in \mathcal{B}(L^p[0,1))$, we have the following identities:

$$\sigma_{\widetilde{T}_\alpha}\big(e^{2\pi\imath\ell t}\mathbf{1}_{\mathcal{B}_\alpha}\big) = e^{2\pi\imath\ell\alpha}\,\sigma_{\widetilde{T}_\alpha}\big(\mathbf{1}_{\mathcal{B}_\alpha}\big), \qquad \sigma_{\widetilde{T}_\alpha^*}\big(e^{2\pi\imath m t}\mathbf{1}_{\mathcal{B}_\alpha}\big) = e^{-2\pi\imath m\alpha}\,\sigma_{\widetilde{T}_\alpha^*}\big(\mathbf{1}_{\mathcal{B}_\alpha}\big)$$

valid for every $\ell, m \in \mathbb{Z}$. Thus, recalling that $\Delta(\mathcal{A}_\rho)$ may be identified with the unit circle, it is plain that given any open subset $U \subseteq \mathbb{C}$ such that $U \cap \mathbb{T} \neq \emptyset$ and $\mathbb{T} \setminus \overline{U} \neq \emptyset$, we can find $\ell, m \in \mathbb{Z}$ satisfying both

$$U \cap \sigma_{\widetilde{T}_\alpha}\big(e^{2\pi\imath\ell t}\mathbf{1}_{\mathcal{B}_\alpha}\big) \neq \emptyset \quad \text{and} \quad \sigma_{\widetilde{T}_\alpha^*}\big(e^{2\pi\imath m t}\mathbf{1}_{\mathcal{B}_\alpha}\big) \setminus \overline{U} \neq \emptyset.$$

Then, as desired, Theorem 5.14 yields that

$$\{0\} \neq \overline{X_{T_\alpha}(U)} \neq L^p[0,1).$$

□

Our latter result provides a deeper insight on what constitutes each of the spectral parts of $\sigma(T_\alpha)$. What is more, observe that this strongly opposes to the already seen fact:

$$X_{T_\alpha}(U) = \{0\} \text{ for every open set } U \subseteq \operatorname{int}\big(\sigma(T_\alpha)\big),$$

valid for each irrational $\alpha \in (0,1)$. Thus, along with Corollary 4.27, it is plain that Theorem 5.15 may be rephrased in terms of the localizable spectrum as:

Corollary 5.16. *Let T_α be a Bishop operator acting on $L^p[0,1)$ for some $1 \leq p < \infty$, such that the irrational $\alpha \in (0,1)$ enjoys the condition:*

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right) \quad \textit{as } j \to \infty.$$

Then, the localizable spectra $\sigma_{\text{loc}}(T_\alpha) = \partial\sigma(T_\alpha)$ and $\sigma_{\text{loc}}(T_\alpha^*) = \partial\sigma(T_\alpha)$.

On the other hand, as a direct consequence of the Local Spectral Mapping Theorem (see, Theorem 4.8), one deduces the following:

Corollary 5.17. *Let $\alpha \in (0,1)$ be an irrational satisfying (5.10) and f any holomorphic function on some open neighbourhood of $\sigma(T_\alpha)$. Then, given any $r \geq 0$ such that*

$$\min_{|\lambda|=e^{-1}} |f(\lambda)| < r < \max_{|\lambda|=e^{-1}} |f(\lambda)|,$$

the norm-closure of the linear manifold

$$\left\{g \in L^p[0,1) \,:\, \limsup_{n\to\infty} \left\|f(T_\alpha)^n g\right\|_p^{1/n} \leq r\right\}$$

is a non-trivial hyperinvariant subspace for T_α on $L^p[0,1)$ for every $1 \leq p < +\infty$.

Proof. Fix any $1 \leq p < \infty$. Since the SVEP is transfered from T_α to the operator $f(T_\alpha)$ (recall Subsection 4.1.2), the next equality is fulfilled for every $g \in L^p[0,1)$:

$$\limsup_{n\to\infty} \|f(T_\alpha)^n g\|_p^{1/n} = \max\left\{|\lambda| \,:\, \lambda \in \sigma_{f(T_\alpha)}(g)\right\}.$$

Finally, applying the Local Spectral Mapping Theorem, we have

$$X_{f(T_\alpha)}\big(\overline{D(0,r)}\big) = X_{T_\alpha}\big(f^{-1}\big(\overline{D(0,r)}\big)\big)$$

and the result follows as a consequence of Theorem 5.15, since the choice of r implies

$$f^{-1}\big(D(0,r)\big) \cap \partial\sigma(T_\alpha) \neq \emptyset \quad \text{and} \quad \partial\sigma(T_\alpha) \setminus \overline{f^{-1}\big(D(0,r)\big)} \neq \emptyset.$$

□

As last application of the results coming from Section 5.2 to Bishop operators, one has the following nice local spectral decomposition property:

Theorem 5.18. *Let T_α be a Bishop operator acting on $L^p[0,1)$ for some $1 \leq p \leq \infty$, such that the irrational $\alpha \in (0,1)$ enjoys the condition:*

$$\log q_{j+1} = O\left(\frac{q_j}{(\log q_j)^3}\right) \quad \textit{as } j \to \infty.$$

Then, there exists a dense linear manifold $\mathcal{D}_{\alpha,p}$ in $L^p[0,1)$ for which

$$\mathcal{D}_{\alpha,p} \subseteq X_{T_\alpha}(U_1) + \ldots + X_{T_\alpha}(U_n)$$

for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$.

Proof. Set any $1 \leq p \leq \infty$ and $\alpha \in (0,1)$ verifying the condition within the statement. Let $\mathcal{A}_\rho$ be a Beurling algebra whose corresponding Beurling sequence satisfies

$$\left\|\widetilde{T}_\alpha^n \mathbf{1}_{\mathcal{B}_\alpha}\right\|_p \ll \rho_n \quad \text{for every } n \in \mathbb{Z}.$$

According to the exposition held before Theorem 5.14, if we define $\mathrm{Dom}(\widetilde{T}_\alpha^{-n}) := \mathrm{ran}(\widetilde{T}_\alpha^n)$ for each $n \in \mathbb{N}$, the linear submanifold in $L^p[0,1)$ given by

$$\mathcal{D}_\phi := \left\{ h \in \bigcap_{n\in\mathbb{Z}} \mathrm{Dom}(\widetilde{T}_\alpha^n) \,:\, \left\|\widetilde{T}_\alpha^n h\right\|_p \ll_h \rho_n \quad (n \in \mathbb{Z}) \right\}$$

is a continuity core for the algebra action

$$\begin{aligned} \phi : \mathcal{A}_\rho &\longrightarrow \mathcal{L}\big(L^p[0,1)\big) \\ f &\longmapsto \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \widetilde{T}_\alpha^n. \end{aligned}$$

As argued in Theorem 4.32, due to its denseness in $L^p[0,1)$, by Theorem 5.12 it suffices to be shown that any function in

$$\mathcal{D} := \operatorname*{span}_{m\in\mathbb{Z}} \left\{ g(t)\, \widetilde{T}_\alpha^m \mathbf{1}_{\mathcal{B}_\alpha}(t) \,:\, \operatorname*{ess\,sup}_{t\in[0,1)} |g(t)| < \infty \right\}$$

belongs to the continuity core $\mathcal{D}_\phi$. Thus, chosen arbitrary $g \in L^\infty[0,1)$ and $m \in \mathbb{Z}$, one may check the equality

$$\widetilde{T}_\alpha^n\big(g \cdot \widetilde{T}_\alpha^m \mathbf{1}_{\mathcal{B}_\alpha}\big)(t) = g(\{t+n\alpha\}) \cdot \widetilde{T}_\alpha^{n+m} \mathbf{1}_{\mathcal{B}_\alpha}(t), \quad \text{for each } n \in \mathbb{Z}.$$

Correspondingly,

$$\left\|\widetilde{T}_\alpha^n\big(g \cdot \widetilde{T}_\alpha^m \mathbf{1}_{\mathcal{B}_\alpha}\big)\right\|_p \leq \|g\|_\infty \cdot \left\|\widetilde{T}_\alpha^{m+n} \mathbf{1}_{\mathcal{B}_\alpha}\right\|_p \ll_g \rho_{m+n} \ll_{g,m} \rho_n \qquad (n \in \mathbb{Z}).$$

So, provided an arbitrary open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, from Theorem 5.12 (5.7) one derives the chain of inclusions

$$\mathcal{D} \subseteq \mathcal{D}_\phi \subseteq X_{\widetilde{T}_\alpha}(U_1) + \ldots + X_{\widetilde{T}_\alpha}(U_n).$$

As desired, since $\mathcal{D}$ spans densely within $L^p[0,1)$, the theorem is finally established. □

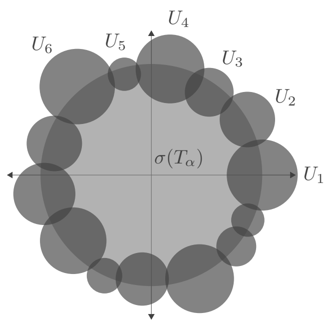

FIGURE 5. Profitable open covering $\{U_1, \ldots, U_n\}$ of $\partial\sigma(T_\alpha)$ for irrational $\alpha \in (0,1)$ satisfying (5.10)

Although the latter property cannot be actually considered a spectral decomposition because of all the operators T_α lack of property (C), our last theorem suggests that Bishop operators (as well as their adjoints) satisfy much more interesting local spectral properties than it seemed a priori in Chapter 4. This invites to research about certain weaker local spectral decompositions for T_α than the usual ones (see [**52**] for a further insight on spectral decompositions). To that end, we propose the next notion:

Definition 5.19. *Let $T \in \mathcal{B}(X)$ be an operator on a Banach space X and let $\mathcal{M} \subseteq X$ be a linear submanifold. Then, T has* property (δ) on $\mathcal{M}$ *if the inclusion*

$$\mathcal{M} \subseteq \mathcal{X}_T\big(\overline{U_1}\big) + \ldots + \mathcal{X}_T\big(\overline{U_n}\big)$$

holds for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$.

Remark 5.20. Obviously, our previous definition embraces the one of property (δ) whenever the submanifold $\mathcal{M}$ is the Banach space X. Additionally, if $\mathcal{M}$ is norm-dense (as it is the case of some Bishop operators), this notion is closely related to *quasi-decomposability* (see, [**89**, Definition 4.7.6]).

Observe that, as a consequence of our previous discussion, Bishop operators do never satisfy property (δ) (see Theorem 4.34) but they do satisfy property (δ) on a dense linear manifold as far as α is an irrational number under the hypothesis of Theorem 5.18.

To conclude, as was shown in Chapter 3, there exists a threshold limit in the growth of the denominators of the convergents of α beyond which Atzmon's Theorem cannot be applied anymore so as to produce invariant subspaces for Bishop operators. In the terminology developed in Section 5.2, such result can be translated saying that if α does not belong to the set

$$\mathcal{E} := \left\{ \alpha \in (0,1) \, : \, \log q_{j+1} = O\left(\frac{q_j}{\log q_j} \right) \right\};$$

then, the unique continuity core $\mathcal{D}_\phi$ available for an algebra action of the form

$$\begin{aligned} \phi : \mathcal{A}_\rho &\longrightarrow \mathcal{L}\big(L^p[0,1)\big) \\ f &\longmapsto \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \widetilde{T}_\alpha^n \end{aligned}$$

whose domain $\mathcal{A}_\rho$ is an arbitrary regular Beurling algebra, must be trivially $\mathcal{D}_\phi = \{0\}$.

Nevertheless, it seems also reasonable to expect that the failure of Atzmon's Theorem approach is only a defect of such particular method, since the spectral decomposition properties of the operators T_α could remain the same independently of the choice of the irrational α. In this sense, we suggest that one possible strategy to solve the invariant subspace problem for all Bishop operators might be understanding in depth their local spectral manifolds $X_{T_\alpha}(U)$ arising from the open sets $U \subseteq \mathbb{C}$ which intersect $\partial\sigma(T_\alpha)$. To support our proposal in the positive, we recall that there exist decomposable operators $T \in \mathcal{B}(X)$ whose spectral behaviour cannot be described in terms of a functional calculus from a suitable algebra (see, Albrecht [**3**]). We pose it as a conjecture:

Conjecture. *Let $\alpha \in (0,1)$ be any irrational number and $\varepsilon > 0$ sufficiently small. Then, the local spectral manifold*

$$X_{T_\alpha}\big(D(e^{-1}, \varepsilon)\big)$$

is non-trivial and non-dense in $L^p[0,1)$ for every $1 \leq p < \infty$.

Spectral decompositions of Bishop-type operators

As a final remark, we enlighten that analogous results to the ones described above can be obtained in a slightly more general framework.

Following Atzmon's/Wermer's Theorems approach, MacDonald (see [**91**, Thms. 2.5 and 2.6]) found hyperinvariant subspaces for certain Bishop-type operators $W_{\phi,\alpha}$, under somewhat restrictive assumptions on the weight $\phi \in L^\infty[0,1)$ and for non-Liouville irrationals $\alpha \in (0,1)$. To be precise, whenever ϕ and $1/\phi$ are in $L^\infty[0,1)$ (i.e. $\log|\phi| \in L^\infty[0,1)$) or more generally, when $\log|\phi|$ can be approximated by step functions $S = \sum_{j=1}^{\ell} r_j \mathbf{1}_{I_j}$ with $\sum_{j=1}^{\ell} |r_j|$ not too large.

For later reference, we summarize these results in slightly more concrete terms. Consider the class of step functions

$$\mathcal{S}_M := \left\{ S = \sum_{j=1}^{\ell} r_j \mathbf{1}_{I_j} \ : \ r_j \in \mathbb{R},\, I_j \text{ intervals and } \sum_{j=1}^{\ell} |r_j| \leq M \right\},$$

where M is a positive real number, as well as the class $\mathcal{L}$ of all real $L^\infty[0,1)$-functions f such that there exists $\gamma > 0$ and a constant $K_f > 0$ (depending only on f) such that

$$\inf \left\{ \|f - S\|_{L^\infty[0,1)} \ : \ S \in \mathcal{S}_M \right\} < K_f \, \frac{1}{M}$$

for all positive real M. Then, MacDonald proved:

Theorem 5.21 (MacDonald, [91]). *Suppose that* $\log|\phi| \in \mathcal{L}$ *and consider* $\alpha \in (0,1)$ *a non-Liouville irrational number. Then, the Bishop-type operator* $W_{\phi,\alpha}$ *has a non-trivial hyperinvariant subspace on each* $L^p[0,1)$ *for* $1 < p < \infty$.

Accordingly, using the terminology of Theorem 5.21, one can deduce the corresponding local spectral variants:

Theorem 5.22. *Suppose that* $\log|\phi| \in \mathcal{L}$ *and consider* $\alpha \in (0,1)$ *a non-Liouville irrational number. Let* $W_{\phi,\alpha}$ *be the induced Bishop-type operator acting on* $L^p[0,1)$ *for some* $1 < p < \infty$. *Then, given any open subset* $U \subseteq \mathbb{C}$ *such that* $U \cap \partial\sigma(W_{\phi,\alpha}) \neq \emptyset$ *and* $\partial\sigma(W_{\phi,\alpha}) \setminus \overline{U} \neq \emptyset$, *we have*

$$\{0\} \neq \overline{X_{W_{\phi,\alpha}}(U)} \neq L^p[0,1).$$

In particular, $W_{\phi,\alpha}$ *has a non-trivial hyperinvariant subspaces on* $L^p[0,1)$.

Theorem 5.23. *Suppose that* $\log|\phi| \in \mathcal{L}$ *and consider* $\alpha \in (0,1)$ *a non-Liouville irrational number. Let* $W_{\phi,\alpha}$ *be the induced Bishop-type operator acting on* $L^p[0,1)$ *for some* $1 < p < \infty$. *Then, there exists a dense linear manifold* $\mathcal{D}_{\phi,\alpha,p}$ *in* $L^p[0,1)$ *such that*

$$\mathcal{D}_{\phi,\alpha,p} \subseteq X_{W_{\phi,\alpha}}(U_1) + \ldots + X_{W_{\phi,\alpha}}(U_n) \tag{5.11}$$

for every finite open cover $\{U_1, \ldots, U_n\}$ *of* $\mathbb{C}$.

Of course, as a consequence of Colojoară and Foiaş' Theorem, stronger conclusions hold for Theorem 5.23 whenever the Bishop-type operator $W_{\phi,\alpha}$ is invertible; because, in that case, $W_{\phi,\alpha}$ turns out to be decomposable and the sum (5.11) comprises the whole space $L^p[0,1)$. Furthermore, recall that sharper results in terms of $\alpha \in (0,1)$ were obtained as well by MacDonald [**92**], showing decomposability for some invertible Bishop-type operators when the condition

$$\sum_{j=0}^{\infty} \frac{\log(q_{j+1})}{q_j} < \infty$$

is fulfilled by the convergents $(a_j/q_j)_{j\geq 0}$ of the irrational α. Observe that this condition embraces some Liouville numbers.

In this light, as was previously indicated in the case of Bishop operators, a better understanding of the role played by the local spectral manifolds $X_{W_{\phi,\alpha}}(U)$ constructed upon open sets $U \subseteq \mathbb{C}$ intersecting $\partial\sigma(W_{\phi,\tau})$, might be a fruitful methodology towards a solution to the Invariant Subspace Problem for larger classes of Bishop-type operators.

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