The weak-Painlevé property as a criterion for the integrability of dynamical systems

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We investigate the validity of the weak-Painlevé property as an integrability criterion. We present an example of a time-dependent Hamiltonian system which possesses a weak-Painlevé type expansion, while presenting a chaotic behavior. However, this system presents also critical fixed singularities. The importance of the latter, as far as integrability is concerned, is discussed here.

I. INTRODUCTION

The singularity analysis has been resurrected in the past few years as an integrability criterion. Introduced a century ago, this method, usually associated to the name of Painlevé,[1] has been initially used in order to investigate integrability of nonlinear first- and second-order ordinary differential equations (ODE's). The recent use of this method concerned the integrability of nonlinear partial differential equations (PDE's).[2] However, as the original formulation of the Ablowitz–Ramani–Segur (ARS) conjecture dealt with ODE reductions of the PDE's, it was most natural to use this singularity analysis as a tool for the investigation of the integrability of dynamical systems described by ODE's. In that context, the most natural extension of the ARS conjecture would read like this: "A system of coupled nonlinear ODE's is integrable whenever it possesses the Painlevé property, i.e., the only movable singularities of the solutions in the complex t plane are poles." Several works have been devoted to the study of dynamical systems using the Painlevé property.[3,4,5,6,7] New integrable systems have thus been discovered and confirmed the particular usefulness of the Painlevé criterion. Whenever a system exhibits the Painlevé property it is integrable (although the precise meaning of integrability must be specified).

The reciprocal proposition seems less well-established. Starting from trivial examples, (e.g., Hamiltonian systems in one dimension), one can convince oneself that integrability can sometimes exist independently of a "nice" singularity structure. During the course of our investigations, we have discovered that some systems possess a particular intermediate status.[8]

They are integrable and, although they do not possess the full Painlevé property, they exhibit a simple singularity expansion in powers of $(t-t_0)^{1/r}$, with r an integer. We have called this property "weak-Painlevé." Several integrable systems have been discovered ranging from the initial 2-D Hamiltonians to N-dimensional systems[9] and even PDE's.[10]

However, recent findings, by one of us[11] make mandatory the examination of the weak-Painlevé property as integrability criterion. Namely the question we address ourselves to in this paper is whether the weak-Painlevé property is *always* sufficient for integrability.

II. PAINLEVÉ AND WEAK-PAINLEVÉ PROPERTIES ASSOCIATED TO INTEGRABILITY

In our initial work,[8,12,13] we have concentrated on autonomous systems (in fact, two-dimensional time-independent Hamiltonian systems). These systems present only movable singularities, and no fixed ones. For these systems, we believe that the weak Painlevé property suffices for integrability (although it is not always necessary).[14] When we turn now to time-dependent systems, two situations can arise: Either the fixed singularities are "nice," or they are not. The latter case is far from being an abstract one.

Consider the very simple case of the Riccati equation

$$\dot{x}=x^2+f(t). \tag{1}$$

The movable singularities of this equation are pure poles. However, if $f(t)$ has singularities at finite values t_i of t, x has *fixed* singularities at these values, these singularities depending on the behavior of f near t_i. One can easily choose f in order that these singularities be critical (i.e., not poles). It is enough for f to have double poles:

$$f(t)=\alpha/t^2,$$

α not of the form $n(n-1)$ with n integer.

However, whatever f is, this equation can be reduced to the second-order linear equation

$$\ddot{y}=f(t)\,y, \tag{2}$$

by $x=\dot{y}/y$.

This equation is considered integrable because it is linear, independent of what the singularities of f are. In general, one cannot express y explicitly (except for very special choices of f), even if f has no singularities at finite t_i's, but still this is considered as an integrable case.

In fact, linearization can even accommodate critical singularities which are movable in some sense. Consider a time-independent system where one integration is explicitly possible. This reduces the original system to a new one with one less degree of freedom and a possible explicit time dependence.

In an earlier paper,[15] we have presented such a system starting from

$$\begin{aligned}\dot{x}&=-x^2+axy+\alpha x+\beta y+\lambda,\\ \dot{y}&=-y^2+bxy+\gamma x+\delta y+\mu.\end{aligned} \tag{3}$$

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We are interested in $a = 0, \beta = 0$.

In this case the first equation for x separates:

$$\dot{x} = -x^2 + \alpha x + \lambda. \quad (4)$$

It is of Riccati type and, of course, it has the full Painlevé property.[1]

Integrating it, we obtain

$$x(t) = (r_1 + c r_2 e^{(r_2 - r_1)t})/(1 + c e^{(r_2 - r_1)t}),$$

with r_1, r_2 solutions of $r^2 - \alpha r - \lambda = 0$.

Choosing a solution for x, we can write the second equation as

$$\dot{y} = -y^2 + (bx(t) + \delta)y + (\gamma x(t) + \mu). \quad (5)$$

This equation is again a Riccati for y and its movable singularities at given x are poles. However, the Riccati for y could *a priori* have a "fixed" singularity which is worse than a pole. But this "fixed" singularity is really a movable singularity of the original system (3) because the pole of x is movable.

In Ref. 15, we have presented a detailed analysis of the conditions for the system (3) to possess the Painlevé property. However, what is clear from what we said above is that this is by no means essential for integrability: the two Riccati equations can be integrated in cascade, through the usual local linearization procedure one applies to the Riccati equations. So here we see a case where critical singularities that are fixed or even movable in the original system (although fixed in the reduced one) do not hinder integrability.

As a matter of fact, we do not know of any case of systems of nonlinear ODE's which possesses fixed critical singularities and is integrable otherwise than through a linearization.

Again let us recall that, according to the currently accepted definition of integrability, a linear ODE with variable coefficients is considered as integrable even if it presents critical fixed singularities. However, this does not necessarily mean that fixed critical singularities are not revelant for integrability. They may well be acceptable only whenever the system is linearizable.

In a recent work,[11] one of us has investigated the singularity structure of one-degree-of-freedom nonautonomous systems. One-degree-of-freedom, Hamiltonian, time-dependent systems fall in the class examined in detail by Painlevé and Gambier. There, the full-Painlevé property leads to integrability (although, sometimes at the expense of introducing new transcendents).

How about the weak Painlevé? The study of a system due to Sitnikov[16] has revealed that the weak-Painlevé property does not preclude chaotic behavior of the system. The equations of motion of the Sitnikov case are

$$\ddot{z} = -z/[z^2 + \tfrac{1}{4}(1 - \epsilon \cos\, 2t)^2]^{3/2}. \quad (6)$$

The singularity expansion of z can be written as $(\tau = t - t_0)$

$$z = z_0 + \sum_{k=4}^{\infty} \alpha_k \tau^{k/5}, \quad (7)$$

with $z_0 = \pm (i/2)(1 - \epsilon \cos 2t_0)$, t_0 free, and a_6 free (associated to the two resonances -1 and 6). A calculation of the first terms of the series yields

$$a_4 = (625/128\, z_0)^{1/5}, \quad a_5 = \pm i\epsilon \sin 2t_0, \quad a_7 = 0.$$

The above analysis concerns the movable singularities of (6).

We turn now to the fixed singularities.

It is clear that z in (6) can have a singular behavior whenever

$$\epsilon \cos 2t_0 = 1. \quad (8)$$

Expanding $1 - \epsilon \cos 2t_0$ around t_0, we obtain

$$(1 - \epsilon \cos 2t)^2 = 4B\tau^2 + \dots,$$

with $B = \epsilon^2 - 1$.

The leading behavior is, in this case,

$$z = \beta \tau^{2/3},$$

with $\beta^3 = \frac{9}{2}$.

Looking for the resonances we find them at -2 and $\frac{2}{3}$. The compatibility condition at $\frac{2}{3}$ is *not* satisfied. Thus, a logarithmic term enters the expansion at $n = \frac{2}{3}$. However, this critical singularity is fixed since t_0 is not free but given by (8).

III. CONCLUSION

So we are in the presence of an equation of motion (one dimensional, Hamiltonian, nonautonomous) which has a weak-Painlevé expansion around a movable singularity and which possesses a fixed critical singularity. Moreover, the solutions of this equation are known to exhibit chaotic behavior which makes them incompatible with integrability.

This could mean one of the two following things: Either allowing fractional powers is too weak a criterion in order to ensure integrability, or fixed singularities must also be taken into account. Our findings do not allow us to draw a clear conclusion at this stage. We can remark however that the predictive power of the weak-Painlevé property for time-independent systems (where fixed singularities do not arise) has been well established to date. On the other hand, up to second order, the full Painlevé property, i.e., movable poles only, does ensure integrability even in the presence of fixed critical singularities (e.g., Riccati), but then integrability is obtained through linearization. For higher order, however, it has not been proved yet that movable poles lead to integrability in the presence of fixed critical singularities.

It might turn out that even fixed critical singularities are not compatible with integrability for higher-order equations. One must acknowledge, at this point, Painlevé's powerful intuition. In his initial project[17] (always motivated by integrability), he was interested in equations with no critical singularities at all, although he devoted the major part of his work to equations with just no movable critical singularities.

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