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School choice with transferable student characteristics ☆

Carmelo Rodríguez-Álvarez [a],*, Antonio Romero-Medina [b]

[a] ICAE, Universidad Complutense de Madrid, Spain
[b] Dept. Economics, Universidad Carlos III de Madrid, Spain

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ABSTRACT

We consider school choice problems where school priorities depend on transferable student characteristics. Fair Pareto improvements can alleviate the trade-off between efficiency and stability in this framework. A group of students may improve their outcomes by exchanging their seats and transferable characteristics at the schools they are initially assigned without generating justified envy among the remaining students.

We define the student exchange with transferable characteristics (SETC) class of algorithms. Every algorithm in the SETC class starts from an initial matching of students to schools and an initial allocation of transferable characteristics. The algorithms then propose a sequence of fair Pareto improvements until the point at which any additional efficiency gain implies a violation of the school priorities that cannot be solved with a reallocation of the transferable characteristics.

1. Introduction

Studies on the school choice problem address the mechanisms that many school districts employ to assign students to public schools (see Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). The problem considers a set of students, a set of schools, and the schools' quotas, which represent the capacity of each school. Each student submits a list of preferences to a central placement authority such as a school district. Each school has a priority ranking that determines who receives a seat if a school is over-demanded. The school district decides which students attend each school using an algorithm that matches students to schools considering the students' reported preferences and the schools' priorities. A major concern regarding the design of school choice programs has been the ability to match students to schools fairly. A matching is fair if all students who obtain a seat at a given school have a higher priority than those who preferred that school over the one to which they are matched and therefore no student has justified envy. In recent years, most school districts have implemented school choice algorithms based on Gale and Shapley's deferred acceptance (DA) algorithm (see Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005; Pathak, 2016). Applying student-proposing DA algorithm to prospective students always results in a stable matching, that is, a fair,

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* Corresponding author.
E-mail addresses: carmelor@ccee.ucm.es (C. Rodríguez-Álvarez), aromero@eco.uc3m.es (A. Romero-Medina).

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individually rational, and non-wasteful matching.[1] However, the matching can be Pareto dominated by another matching that does not respect school priorities.

In this paper, we add structure to the definition of school priorities. In the canonical school choice problem, priorities are a primitive aspect of the model. However, school districts use several criteria to determine school priority orders, such as different characteristics of potential students or tie-breaker lotteries (see Abdulkadiroğlu et al., 2005). Our model relies on student characteristics as primitives. Students are endowed with characteristics specific to individual schools. Each school ranks students according to priorities defined by the individual characteristics of each student and the transferable characteristic of the student at that school. Students can exchange the characteristics of different schools and thus affect their positions in the priority rankings of those schools. In this context, a matching that Pareto improves the initial matching but that may not respect fairness may become fair after the relevant characteristics among students are exchanged. We introduce the concept of extended matching, which is a matching of students to schools and an allocation of transferable characteristics among the students. We explore the efficiency gains with respect to arbitrary initial assignments of students to schools that can be justified under schools' priority rankings after the exchanges of characteristics among the students involved in the exchange of seats (fair Pareto improvements). As far as we know, we are the first to consider this extension of the canonical school choice model formally.[2]

We propose a class of school choice algorithms: the student exchange with transferable characteristics (SETC) algorithms. Each algorithm in this class proposes a sequence of fair Pareto improvements of an initial extended matching. Suppose that the initial extended matching is individually rational and non-wasteful. Each algorithm in the class stops at an extended matching such that no further fair Pareto improvement can be obtained (Theorem 1). If the initial matching is stable, the final matching is also stable, and any possible Pareto improvement implies generating instances of justified envy, even after the exchange of transferable characteristics. We dub such an extended matching a constrained efficient extended matching. We obtain a partial version of the converse result by imposing a neutrality condition on the structure of school priorities. For every extended matching obtained as a sequence of fair Pareto improvements of a stable extended matching, an SETC algorithm selects essentially the same extended matching at some step of the algorithm (Theorem 2).

We can think of several situations where the transferability of student characteristics can improve an allocation by solving a market design problem. For example, the lottery number can be a natural transferable characteristic when schools use different tie-breaking lotteries.[3] This issue was at the heart of Amsterdam's high school assignment procedure reform in 2014. Under this reform, a system based on immediate acceptance with multiple tie-breaking and the possibility of exchanging assignments was replaced with a system based on DA with multiple tie-breaking without the possibility of exchanging assignments. In 2015, families who wished to exchange school seats unsuccessfully challenged the new allocation procedure in court. Our framework allows us to design a procedure with the lottery number as the unique exchangeable characteristic. This allows Pareto improvements that respect all but the priorities based on the tie-breaking lottery.[4]

Additionally, our model can be helpful in situations of walk-zone redistricting (see Dur et al., 2018; Casalmiglia et al., 2020). For example, for the case of Madrid, where the walk-zone priority was abolished, generating winners and losers, we could design an allocation procedure that respects the walk-zone priority but allows exchange of the walk-zone characteristic when it generates Pareto improvements (see Górtazar et al., 2023).

In general, our approach can help improve efficiency in situations where we can differentiate between allocative criteria (such as tie-breaking lotteries) and fairness constraints (such as the need for siblings to attend the same school) in the formation of priorities. An example could be the integration of transplantation programs.[5] Different programs might have different priorities or criteria on desensitization or resort to cadaveric donation. Therefore, depending on her characteristics, a patient might receive different treatments in different programs. The interaction between different programs might mean that a member will lose in favor of an outsider. A within-program assignment followed by an exchange of characteristics where the transplantation program is the only transferable characteristic allows for Pareto-improving reassignment of patients seeking treatment.

Finally, the algorithms in the SETC class allow Pareto-improving exchanges in any extended matching that is individually rational and non-wasteful. As the initial allocation does not have to be stable, an SETC algorithm can be used as a post-allocation scramble reducing instances of justified envy.

1.1. Related literature

The school choice problem was first presented by Balinski and Sönmez (1999). This paper introduces the idea of fairness in allocating school seats to students. Abdulkadiroğlu and Sönmez (2003) analyzes this problem from a mechanism design perspective. The paper shows that a student-proposing DA algorithm always selects a stable match and is strategy-proof.[6] Abdulkadiroğlu and

[1] A matching is individually rational if no student is assigned to a school that she would rather not attend. A matching is non-wasteful if every school that a student prefers to the school to which she is assigned has filled all its available seats.

[2] A similar formulation was independently proposed in Duddy (2019), which discusses the informational shortcomings of the current priority-based model and proposes a formulation based on a priority matrix.

[3] The use of multiple tie-breaking criteria can be justified since it reduces the chance that over-demanded schools will systematically reject a student with an unfavorable lottery draw (see Arnosti, 2016).

[4] For further reference, see Ashlagi et al. (2019); Ruijs and Oosterbeek (2019), and https://www.nemokennislink.nl/publicaties/schoolstrijd-in-amsterdam/(Schoolstrijd in Amsterdam) (Arnout Jaspers, Kennislink, July 1, 2015, accessed July 31, 2022).

[5] See Van der Spiegel et al. (2020) for details on the integration of national transplant programs in the European Union.

[6] A mechanism is strategy-proof if students have incentives to report their true preferences.

Sönmez (2003) also presents an adaptation of Gale's top trading cycle mechanism (TTC) mechanism by Shapley and Scarf (1974) and shows that the TTC mechanism always selects Pareto efficient matchings and is strategy-proof. Unfortunately, stable matchings are inefficient and can have severe levels of inefficiency (see Abdulkadiroğlu et al., 2009; Kesten, 2010; Dur and Morrill, 2017).

There have been attempts to alleviate the trade-off between stability and efficiency by weakening the notion of fairness. Kesten (2010) proposes the efficiency adjusted deferred acceptance algorithm (EADA) algorithm. The EADA algorithm finds a constrained efficient matching by incorporating the possibility that students may consent to renounce their priorities in schools where they cannot obtain a seat under the student-proposing DA algorithm.[7] Alcalde and Romero-Medina (2017) proposes an alternative weakening of fairness dubbed α-equitability. Ehlers and Morrill (2020) relaxes the fairness constraint and proposes a stable set of legal matchings that are not dominated in terms of fairness by any other legal matching. In the same spirit, Alva and Manjunath (2019) presents the concept of stable domination, Troyan et al. (2020) proposes the concept of essentially stable, and Tang and Zhang (2021) considers the concept of weak stability.

A different approach compares the instances of justified envy generated by different mechanisms. Hakimov and Kesten (2018) proposes a Pareto efficient and strategy-proof mechanism that eliminates justified envy due to pairwise exchanges. Abdulkadiroğlu et al. (2020) shows that TTC minimizes justified envy among all Pareto efficient and strategy-proof mechanisms in one-to-one matching. Doğan and Ehlers (2021) investigates efficient and minimally unstable Pareto improvements over the DA mechanism. Finally, Doğan and Ehlers (2022) formulates methods to compare assignments in terms of their stability in the context of priority-based allocation of objects.

With the same objective of alleviating the trade-off between stability and efficiency, other papers explore the interaction between agents' characteristics and the solution concept of the allocation problem. For example, Klaus and Klijn (2021) presents a classical school choice problem with access rights. In general, minimal access rights (for siblings, walk-zone residents, etc.) are incorporated into priorities to give students with minimal access higher priority than non-minimal access students. Klaus and Klijn (2021) weakens stability to minimal access stability, a concept that guarantees access to at most one school that guarantees the student a minimal access right. Combe (2022) studies the idea of matching with ownership in situations where ownership of an object restricts the objections of agents who are not owners. In this setting, Combe (2022) defines a notion of stability with two different ownership structures and shows that stable matchings exist in both cases.

The closest paper to ours is Dur et al. (2019), which proposes an alternative weakening of stability called partial stability. Under partial stability, specific priorities of certain students at certain schools are ignored. Then, the welfare gains can be captured by applying the improvement cycles approach proposed by Erdil and Ergin (2008) for school choice problems with weak priorities and arbitrary tie-breakers. Kitahara and Okumura (2021) modifies the stable improvement cycles algorithm introduced by Erdil and Ergin (2008) and considers a school choice problem where partial orders represent school priorities.

Like Dur et al. (2019), our paper uses improvement cycles. However, beyond this point, the two papers have considerable differences. First, the primitives in our model are not school priorities but the individual student characteristics on which those priorities are based. Second, in our case, the resulting extended matching allocates both school seats and student characteristics. Third, the possible welfare gains that we capture are derived from the exchange of characteristics. That is, the final allocation of transferable characteristics justifies the resulting extended matching of our model. Fourth, the SETC algorithms consider exchanges of characteristics; in contrast to the mechanism under the stable improvement cycle algorithm in Erdil and Ergin (2008), some students who participate in the cycles only exchange characteristics and facilitate other exchanges, and they are weakly better off. Finally, there is a technical difference. Our framework does not require additional conditions on the set of priorities that may be ignored, as in Dur et al. (2019).[8] Our results only require that school priorities are complete and neutral (monotonic) in terms of student characteristics.

The remainder of the paper is organized as follows. In Section 2, we introduce the model and notation utilized. In Section 3, we present our main results. In Section 4, we consider incentive issues and relate our framework of transferable characteristics to that of school choice with consent proposed by Kesten (2010). In Section 5, we conclude this paper. In Section 6, we prove our results.

2. Notation and definitions

We present the canonical school choice problem elements and introduce a school choice problem with school priorities that depend on transferable characteristics.

Let I be a finite set of students and S be a finite set of schools to which the students must be allocated. Each student i has a strict preference P_i over $S \cup \{\emptyset\}$, where a strict preference is a complete, transitive, and antisymmetric binary relation, and $\emptyset$ refers to the option of being unassigned. We use R_i to signify the weak preference relation associated with P_i, which is defined in the standard way. We denote by R a profile of student preferences that specifies a preference for each student, $R = (R_i)_{i \in I}$. Each school s has a quota q_s of available seats ($q_s \in \mathbb{N}$). We denote by q the vector of school quotas, $q = (q_s)_{s \in S}$.

A ***matching*** is a function $\mu : I \to S \cup \{\emptyset\}$ such that

i) for each $i \in I$, $\mu(i) \in S \cup \{\emptyset\}$.
ii) for each $s \in S$, $\#\{i \in I : \mu(i) = s\} \leq q_s$.[9]

[7] See also Tang and Yu (2014).
[8] See Assumption 1 in Dur et al. (2019).
[9] For any set A, $\#A$ stands for the cardinality of set A.

Abusing notation, for each $s \in S$, we write $\mu^{-1}(s) = \{i \in I : \mu(i) = s\}$ and represent arbitrary matchings by the list of student-school pairs:

$$\mu = [(i, \mu(i))]_{i \in I} .$$

A matching μ' ***Pareto dominates*** the matching μ if for each $i \in I$, $\mu'(i)\ R_i\ \mu(i)$ and, for some $j \in I$, $\mu'(j)\ P_j\ \mu(j)$.

The canonical school choice problem's last component is the school priorities profile. Schools rank their prospective students according to priority rankings. We consider that school priorities may depend on different student characteristics. Some of these characteristics are intrinsic to individual students, but others can be exchanged among students. The relevant priorities for schools depend on the allocation of such characteristics.

For each student i and each school s, $\omega^s(i)$ is the initial endowment of the transferable characteristic of i at s. We assume that $\omega^s(i)$ is a singleton. Let $\omega(i) = (\omega^s(i))_{s \in S}$ be the initial endowment vector of the transferable characteristics that determine the position of student i at each school s. For each school s, let $\Omega^s = \{\omega^s(i) : i \in I\}$. A permutation of the transferable characteristics for school s, $\lambda^s : I \to \Omega^s$, is a bijection from I to Ω^s and $\lambda^s(i)$ is the transferable characteristic of i at school s. Note that for each $i \in I$ and $s \in S$, there is $j \in I$ with $\lambda^s(i) = \omega^s(j)$ and, for each $j, j' \in I$ such that $j \neq j'$, $\lambda^s(j) \neq \lambda^s(j')$. We call $\lambda = (\lambda^s)_{s \in S}$ an ***allocation of transferable characteristics***. For each student i and each allocation λ, $\lambda(i) = (\lambda^s(i))_{s \in S}$. We denote by ω the initial allocation of transferable characteristics. Finally, for each allocation of transferable characteristics λ and each set of students $N \subseteq I$, $\lambda|_N$ is the restriction of λ to the students in N.

When characteristics are transferable, their allocation is relevant to define school priorities. An ***extended matching*** is a pair (μ, λ) where μ is a matching and λ is an allocation of transferable characteristics. We say that the extended matching (μ, λ) Pareto dominates the extended matching (μ', λ') if μ Pareto dominates μ'.

In a school choice problem with transferable characteristics, school priorities rank combinations of students and transferable characteristics that students present to the school choice process. Hence, school s's priority is a complete, transitive, and antisymmetric binary relation $\succ_s$ over $I \times \Omega^s$. We use the notation $\succsim_s$ to refer to the weak priority relation associated with $\succ_s$ defined in the usual way. We denote by $\succsim$ the school priorities profile $\succsim = (\succsim_s)_{s \in S}$.

Neutral priorities For each $i, j \in I$, $s \in S$, and each $l, l' \in \Omega^s$, $(i, l) \succ_s (i, l')$ if and only if $(j, l) \succ_s (j, l')$.

Under neutral priorities, for each s, the set Ω^s is naturally ordered; for each $L \subseteq \Omega^s$, we can define

$$\max\{L\} = \{l \in L : \text{ for each } i \in I, \text{ for each } l' \in L, (i, l) \succsim_s (i, l')\}.$$

Throughout this paper, we assume that, for each school s, the set Ω^s is an ordered set and the transferable characteristics affect all students neutrally. This assumption seems natural in most applications. For instance, we might assume that the transferable characteristic corresponds to tie-breaker lotteries and that each school assigns a different tie-breaker lottery number to each student ex-ante.

A ***school choice problem with transferable characteristics*** is a 6-tuple defined by the set of students, the set of schools, student preferences over schools, the number of seats available at each school, the initial allocation of transferable characteristics, and school priorities $(I, S, R, q, \omega, \succsim)$.

We now present a notion of stability for extended matchings. This notion reflects the idea that stable extended matchings should not generate justified complaints with respect to school priorities from students who want to change their assigned school.

Given an extended matching (μ, λ), student i has ***justified envy of student*** j if $\mu(j)\ P_i\ \mu(i)$ and $(i, \lambda^{\mu(j)}(i)) \succ_{\mu(j)} (j, \lambda^{\mu(j)}(j))$.

An extended matching (μ, λ) is ***stable*** if it is

i) ***fair***: no student has justified envy of any other student,
ii) ***individually rational***: for each $i \in I$, $\mu(i)\ R_i\ \varnothing$, and
iii) ***non-wasteful***: for no $i \in I$ or $s \in S$, $s\ P_i\ \mu(i)$ and $\#\{i \in I : \mu(i) = s\} = \mu^{-1}(s) < q_s$.

If an extended matching is fair, no student has a greater right (according to school priorities) to attend a particular school that she prefers to the school to which she is assigned than another student assigned to that school. If an extended matching is individually rational, no student has incentives to leave the enrollment system since she prefers to remain unassigned. Finally, if an extended matching is non-wasteful, no student prefers to be reassigned to a school that has not fulfilled its quota.

We analyze the possibility of improvements of an initial extended matching in terms of Pareto efficiency and stability. Starting from an arbitrary initial extended matching, (μ, ω), we seek a new extended matching that Pareto dominates the initial extended matching without generating new instances of justified envy thanks to the exchange of transferable characteristics.

The extended matching $(\mu', \bar{\lambda})$ is a ***fair Pareto improvement of*** (μ, λ) if

i) μ' Pareto dominates μ,
ii) for each i such that $(\mu'(i), \bar{\lambda}(i)) \neq (\mu(i), \lambda(i))$ there is no j with justified envy of i at $(\mu', \bar{\lambda})$,
iii) for each $i \in I$ and for each $s \notin \{\mu(i), \mu'(i)\}$, $\lambda^s(i) = \bar{\lambda}^s(i)$.

By items i) and ii), we require a fair Pareto improvement to generate a new matching that all students consider at least as good as the initial matching (with some strict preference). The changes in the matching and the allocation of transferable characteristics must not generate instances of justified envy. By item iii), we consider new extended matchings such that the reallocation of transferable characteristics is restricted to students and schools involved in the new assignment. Specifically, we exclude the possibility of a student abandoning her transferable characteristic at some school to which she is not assigned to accommodate the priority of another student at that school.[10]

In the following definition, we sequentially apply the notion of fair Pareto improvement and consider the possibility of obtaining extended matchings resulting from a finite sequence of fair Pareto improvements.

The extended matching (μ', λ') is a ***justifiable Pareto improvement*** of (μ, λ) if there is a sequence of extended matchings $\{(\mu_0, \lambda_0), (\mu_1, \lambda_1), \dots, (\mu_n, \lambda_n)\}$ with $(\mu_0, \lambda_0) = (\mu, \lambda)$, $(\mu_n, \lambda_n) = (\mu', \lambda')$, such that for each $t \in \{1, \dots, n\}$, (μ_t, λ_t) is a fair Pareto improvement of $(\mu_{t-1}, \lambda_{t-1})$.

When the initial transferable characteristics cannot be reallocated, the student-proposing DA algorithm selects a matching that, together with the initial allocation of transferable characteristics, is stable. For each allocation of transferable characteristics λ, we define μ_λ^{SO} as the matching obtained by the student-proposing DA algorithm for λ. We call $(\mu_\omega^{SO}, \omega)$ the ***student optimal stable extended matching (SOSEM)***. Gale and Shapley (1962) proves that the matching selected by student-proposing DA algorithm Pareto dominates all other fair, individually rational, and non-wasteful matchings under the initial allocation of transferable characteristics ω. However, it is possible to find alternative matchings that Pareto dominate the SOSEM by generating instances of justified envy.

We explore the possibility of exhausting the generation of fair Pareto improvements. Given an initial extended matching, we seek extended matchings that improve both efficiency and stability and are justified by bidirectional exchanges of seats and transferable characteristics such that further Pareto improvements necessarily generate justified envy. We focus on finding fair Pareto improvements of individually rational and non-wasteful extended matchings and specifically for stable extended matchings and the *SOSEM*, that is, stable extended matchings that are not Pareto dominated by other stable extended matchings. Therefore, any Pareto improvement will imply a violation of fairness. This notion is captured with the following definition.

An extended matching (μ, λ) is ***constrained efficient*** if it is stable and there is no fair Pareto improvement (μ', λ') of (μ, λ).

2.1. Examples

We start the analysis by providing some examples that show the possibility of finding fair Pareto improvements for the solutions to the classical school choice problem.

Example 1 provides an instance of a fair Pareto improvement above the SOSEM (and a constrained efficient extended matching) by the direct swap of transferable characteristics at different schools between two students.

Example 1. Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_{s_x} = 1$ for $x = 1, 2, 3$. Student preferences over schools are:

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3
$\varnothing$	$\varnothing$	$\varnothing$

School priorities regard non-transferable characteristics intrinsic to each student and students' tie-breaker lotteries as transferable characteristics. The tie-breaking lottery determines the school priorities of s_1 and s_2. Student i_1 has the highest priority for s_1, student i_2 has the highest priority for s_2, and student i_3 has the second-highest priority in both schools. Hence, the initial allocation of transferable characteristics and the relevant school priorities are[11]:

$\omega^s(i)$	i_1	i_2	i_3
s_1	$\mathbf{2}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\mathbf{0}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, \mathbf{2}^{s_1})$	$(i_2, \mathbf{2}^{s_2})$	$(i_3, \cdot)$
$(i_2, \mathbf{2}^{s_1})$	$(i_1, \mathbf{2}^{s_2})$	$\dots$
$(i_3, \mathbf{1}^{s_1})$	$(i_3, \mathbf{1}^{s_2})$	
$(i_2, \mathbf{0}^{s_1})$	$(i_1, \mathbf{0}^{s_2})$	

where $(i, \cdot)$ at the description of the priority of school s, $\succ_s$, denotes that student i holds this priority for any possible transferable characteristic $\lambda^s \in \Omega^s$.

Note that μ_ω^{SO} is defined by

$$\mu_\omega^{SO} = \left[(i_1, s_1), (i_2, s_2), (i_3, s_3)\right].$$

The extended matching $(\mu_\omega^{SO}, \omega)$ is stable but μ_ω^{SO} is Pareto dominated by the matching μ' such that:

[10] With this modeling option, we prevent a student from influencing the allocation of students to schools other than the one to which she was initially assigned. In Example 2, we show the implications of allowing students to distribute characteristics of schools that are not involved in direct exchanges. Furthermore, allowing such trades would add complexity to the framework.

[11] We do not present the complete school priority orders over students and transferable characteristics pairs, only the relevant comparisons.

$$\mu' = [(i_1, s_2), (i_2, s_1), (i_3, s_3)].$$

However, since $(i_3, \mathbf{1}^{s_1}) \succ_{s_1} (i_2, \mathbf{0}^{s_1})$, i_3 has justified envy of i_1, and (μ', ω) is not a fair extended matching.

When students i_1 and i_2 swap their transferable characteristics at schools s_1 and s_2, we obtain an allocation of transferable characteristics λ such that: $\lambda^{s_1}(i_1) = \omega^{s_1}(i_2)$, $\lambda^{s_1}(i_2) = \omega^{s_1}(i_1)$, $\lambda^{s_2}(i_1) = \omega^{s_2}(i_2)$, $\lambda^{s_2}(i_2) = \omega^{s_2}(i_1)$, and $\lambda^{s_3} = \omega^{s_3}$.

$\lambda^s(i)$	i_1	i_2	i_3
s_1	$\mathbf{0}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	$\mathbf{0}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{2}^{s_3}$

Note that $\mu' = \mu^{SO}_{\lambda}$. The extended matching (μ', λ) is stable and a fair Pareto improvement for the SOSEM $(\mu^{SO}_{\omega}, \omega)$. Since μ^{SO}_{λ} Pareto dominates μ^{SO}_{ω}, $(\mu^{SO}_{\omega}, \omega)$ is not constrained efficient. Since there is no matching μ'' that Pareto dominates μ', and (μ', λ) is stable, (μ', λ) is constrained efficient.

Example 2 presents the constraint that the focus on fair Pareto improvements introduces in our framework.

Example 2. Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q_{s_x} = 1$ for $x = 1, 2, 3, 4$. Student preferences over schools that are at least as good as the remaining unassigned option are:

P_{i_1}	P_{i_2}	P_{i_3}	P_4
s_2	s_1	s_1	s_4
s_1	s_2	s_2	$\emptyset$
s_3	s_3	s_3	
$\emptyset$	$\emptyset$	$\emptyset$	

The relevant initial allocation of transferable characteristics for schools ω and the relevant school priorities are:

$\omega^s(i)$	i_1	i_2	i_3	i_4
s_1	$\mathbf{0}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{3}^{s_1}$
s_2	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{3}^{s_2}$
s_3	...	...	...	...
s_4	...	...	...	...

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
$(i_1, \mathbf{0}^{s_1})$	$(i_2, \mathbf{0}^{s_2})$	$(i_3, \cdot)$	$(i_4, \cdot)$
$(i_4, \mathbf{3}^{s_1})$	$(i_4, \mathbf{3}^{s_2})$	...	...
$(i_2, \mathbf{3}^{s_1})$	$(i_1, \mathbf{3}^{s_2})$		
$(i_3, \mathbf{2}^{s_1})$	$(i_3, \mathbf{2}^{s_2})$		
$(i_2, \mathbf{1}^{s_1})$	$(i_1, \mathbf{1}^{s_2})$		

This school choice problem modifies Example 1 by adding a student with the highest ticket number from the tie-breaking lottery for schools s_1 and s_2 and school priorities such that i_1 and i_2 initially have the highest priorities at schools s_2 and s_1, respectively, but they do not have valuable lottery tickets to exchange.

The matching μ^{SO}_{ω} is defined by

$$\mu^{SO}_{\omega} = [(i_1, s_1,), (i_2, s_2), (i_3, s_3), (i_4, s_4)].$$

Note that μ^{SO}_{ω} is Pareto dominated by

$$\mu' = [(i_1, s_2), (i_2, s_1)(i_3, s_3), (i_4, s_4)].$$

Consider the allocation of transferable characteristics $\bar{\lambda}$ such that $\bar{\lambda}^s = \omega^s$ for $s \in \{s_3, s_4\}$ and

$\bar{\lambda}^s(i)$	i_1	i_2	i_3	i_4
s_1	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$
s_3	...	...	...	...
s_4	...	...	...	...

According to the allocation $\bar{\lambda}$ students i_1 and i_2 obtain i_4's transferable characteristics at schools s_1 and s_2 respectively, although student i_4 is assigned to a seat in s_4. Hence, $(\mu', \bar{\lambda})$ is not a fair Pareto improvement of $(\mu^{SO}_{\omega}, \omega)$. It turns out that $(\mu^{SO}_{\omega}, \omega)$ is constrained efficient.

Example 3 shows the possibility of different (incompatible) fair Pareto improvements over an initial extended matching, and the possibility of Pareto improvements where students willing to exchange seats need the transferable characteristic of a third student.

Example 3. Let $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, for all $s \in S \setminus \{s_2\}$, $q_s = 1$, and $q_{s_2} = 2$. Student preferences over schools that are at least as good as the remaining unassigned option are:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_1	s_1	s_1	s_5	s_5
s_4	s_2	s_2	s_4	s_2	s_2
s_1	s_3	s_4	$\varnothing$	$\varnothing$	$\varnothing$
s_3	$\varnothing$	s_3			
$\varnothing$		$\varnothing$			

The relevant initial allocation of transferable characteristics and the relevant school priorities are[12]:

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{5}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{5}^{s_2}$
s_3	...	...	...	...	...	...
s_4	$\mathbf{5}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	...	...	...	...	...	...

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
$(i_1,\mathbf{5}^{s_1})$	$(i_5,\mathbf{0}^{s_2})$	$(i_3,\cdot)$	$(i_4,\mathbf{3}^{s_4})$	$(i_5,\cdot)$
$(i_2,\mathbf{5}^{s_1})$	$(i_6,\mathbf{5}^{s_2})$		$(i_1,\mathbf{5}^{s_4})$	$(i_6,\cdot)$
$(i_4,\mathbf{5}^{s_1})$	$(i_6,\mathbf{3}^{s_2})$		$(i_3,\mathbf{4}^{s_4})$	
$(i_3,\mathbf{4}^{s_1})$	$(i_2,\mathbf{3}^{s_2})$		$(i_1,\mathbf{3}^{s_4})$	
$(i_2,\mathbf{3}^{s_1})$	$(i_1,\mathbf{5}^{s_2})$			
$(i_4,\mathbf{2}^{s_1})$	$(i_3,\mathbf{4}^{s_2})$			
	$(i_1,\mathbf{3}^{s_2})$			
	$(i_1,\mathbf{2}^{s_2})$			

The SOSEM is the matching:

$$\mu_\omega^{SO} = \left[(i_1,s_1),(i_2,s_2),(i_3,s_3),(i_4,s_4),(i_5,s_5),(i_6,s_2)\right],$$

and the extended matching (μ_ω^{SO},ω) is stable. The matching μ_ω^{SO} is Pareto dominated by two alternative matchings μ' and μ'':

$$\mu' = \left[(i_1,s_2),(i_2,s_1),(i_3,s_3),(i_4,s_4),(i_5,s_5),(i_6,s_2)\right],$$
$$\mu'' = \left[(i_1,s_4),(i_2,s_2),(i_3,s_3),(i_4,s_1),(i_5,s_5),(i_6,s_2)\right].$$

The extended matchings (μ',ω) and (μ'',ω) are not stable. Note that at (μ',ω), i_2 has justified envy of i_1 at school s_2. At (μ'',ω), i_2 has justified envy of i_4 at school s_1.

Considering the matching μ'; every fair Pareto improvement involving a swap of transferable characteristics involving only students i_1 and i_2 does not generate a stable extended matching because $\omega^{s_2}(i_2)=\mathbf{3}^{s_2}$ and $(i_3,\mathbf{4}^{s_2})\succ_{s_2}(i_1,\mathbf{3}^{s_2})$. However, if student i_6 participates in the swap of transferable characteristics, we can define the allocation of transferable characteristics λ such that for each $s\in\{s_3,s_4,s_5\}$ $\lambda^s=\omega^s$ and the allocation of transferable characteristics at schools s_1 and s_2 is:

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{3}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	$\mathbf{5}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{3}^{s_2}$
s_3	...	...	...	...	...	...
s_4	...	...	...	...	...	...
s_5	...	...	...	...	...	...

The extended matching (μ',λ) is stable and a fair Pareto improvement over (μ,ω).

On the other hand, with respect to the matching μ'', under the initial allocation of transferable characteristics, student i_1 cannot obtain a seat at school s_4 because i_4 has a higher priority at that school. In this case, student i_4 needs i_1's transferable characteristic at school s_1 to avoid generating justified envy by i_2, but i_1 does not need i_4's at s_4. Hence, we can define the allocation of transferable characteristics $\bar{\lambda}$ such that for each $s\in\{s_2,s_3,s_4,s_5\}$, $\bar{\lambda}^s=\omega^s$, and the allocation of transferable characteristics at schools s_1 and s_4 is:

$\bar{\lambda}^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{2}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{1}^{s_1}$	$\mathbf{0}^{s_1}$
s_2	...	...	...	...	...	...
s_3	...	...	...	...	...	...
s_4	$\mathbf{5}^{s_4}$	$\mathbf{2}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	...	...	...	...	...	...

The extended matching $(\mu'',\bar{\lambda})$ is stable, and $(\mu'',\bar{\lambda})$ is a fair Pareto improvement over (μ,ω). In fact, both (μ',λ) and $(\mu'',\bar{\lambda})$ are constrained efficient extended matchings.

[12] It is worth noting that the school priorities in Example 3 are consistent with the interpretation of priorities based on weak orders over students and transferable characteristics as tie-breaking lotteries. We can interpret that student i_1 always has the highest priority at school s_1. School s_1's ranking of students i_2, i_3, and i_4 depends on the respective transferable characteristics (with an irrelevant arbitrary criterion to define complete and strict priorities over all pairs of students and transferable characteristics). Similarly, student i_5 always has the highest priority and students i_6 and i_2 the second- and third-highest priority at school s_2, and the transferable characteristic determines the priority of students i_1 and i_3 at school s_2.

We conclude this section with Example 4 which complements Example 3. With Example 4 we show that the existence of students willing to exchange seats at different schools does not suffice to yield a fair Pareto improvement if they cannot involve additional students in a transferable characteristics exchange cycle.

Example 4. Consider a variation of the school choice problem defined in Example 3 with an alternative profile of student preferences. Let P' such that for each $i \neq i_5$, $P_i = P'_i$, and $s_2 \; P'_{i_5} \; s_5 \; P'_{i_5} \; \varnothing$. The matching associated with the SOSEM for the problem with the new profile of preferences is:

$$\hat{\mu} = \left[(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_2), (i_6, s_5)\right],$$

which, together with the initial allocation of transferable characteristics, forms a stable extended matching $(\hat{\mu}, \omega)$. Consider the matching $\hat{\mu}'$:

$$\hat{\mu}' = \left[(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_2), (i_6, s_5)\right].$$

Note that $\hat{\mu}'$ Pareto dominates $\hat{\mu}$, but there is no allocation of transferable characteristics λ such that $(\hat{\mu}, \lambda)$ is a fair Pareto improvement for $(\hat{\mu}, \omega)$ because $(i_3, \omega^{s_2}(i_3)) \succ_{s_2} (i_1, \max\{\omega^{s_2}(i_1), \omega^{s_2}(i_2), \omega^{s_2}(i_5)\})$.

Example 4 shows that the transferable characteristics of every student assigned to each school determine the possibility of finding fair Pareto improvements. This characteristic of our framework contrasts with the notion of partial fairness proposed by Dur et al. (2019). To Dur et al. (2019), an initial set of priority violations (instances of justified envy) is admitted. In their framework, the possibility of a swap of seats between students that generates admissible instances of justified envy does not depend on the sets of students with their transferable characteristics assigned to the same school.

3. Improvement cycles for extended matchings

This section presents a systematic method to obtain fair Pareto improvements starting from individually rational and non-wasteful extended matchings. We study the possible improvements in efficiency and fairness by allowing exchanges of transferable characteristics. If the initial extended matching is stable, the objective is to obtain constrained efficient extended matchings. Our approach follows Erdil and Ergin (2008) and Dur et al. (2019). Those papers propose a method for finding fair Pareto-improving matchings through exchange cycles. Erdil and Ergin (2008) improves the outcome of the student-proposing DA algorithm applied to priorities with indifferences with an arbitrary tie-breaker. In contrast, Dur et al. (2019) allows for similar improvements based on partially unenforceable priorities. The logic behind the fair Pareto improvement cycles in both papers relates to the idea of vacancy chains introduced by Blum et al. (1997). Suppose that a student abandons the school to which she has been matched. In this situation, her seat and transferable characteristics may be used by another student. We have a Pareto improvement only if the vacant seat is filled either by a student in the same school or by a student who prefers that seat to the school to which she has been matched. Suppose that our objective is to obtain an extended matching that improves upon the initial one in terms of both efficiency and fairness. In this case, the seat must be assigned to a student with a higher priority than the students who prefer the vacant seat to their current match. This higher priority can be the product of either using the transferable characteristic of the leaving candidate or keeping her characteristics at that school. This paper considers cycles of student exchanges of seats at schools and the associated transferable characteristics instead of vacancy chains.

The following concepts extend the graph-theoretical approach presented by Dur et al. (2019) to the framework of school choice with transferable characteristics. Unlike in the setup of Dur et al. (2019), students may be willing to move to a seat at a desirable school in our model, but an instance of justified envy may appear depending on the student who exchanges the transferable characteristics.[13] Moreover, fair Pareto improvements involving two students may require the participation of additional students who exchange transferable characteristics but do not change the school to which they are assigned.

We introduce notation for students interested in occupying another student's seat and those who may finally be assigned to a school without generating justified envy.

Given an individually rational and non-wasteful extended matching (μ, λ), for each student $j \in I$, let the set $D_{(\mu,\lambda)}(j)$ consist of the students who consider $\mu(j)$ at least as good as their current matches. Formally,

$$D_{(\mu,\lambda)}(j) = \{i \in I \setminus \{j\} : \mu(j) \; R_i \; \mu(i)\}.$$

Next, the set $\tilde{D}_{(\mu,\lambda)}(j)$ contains all the students who strictly prefer the match of student j over their current matches. That is,

$$\tilde{D}_{(\mu,\lambda)}(j) = \{i \in I : \mu(j) \; P_i \; \mu(i)\}.$$

Clearly, $\tilde{D}_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$. The students in $D_{(\mu,\lambda)}(j)$ are willing to occupy a seat at $\mu(j)$.

The set $Y_{(\mu,\lambda)}(j)$ contains every student i assigned to $\mu(j)$ such that no student in $\tilde{D}_{(\mu,\lambda)}(j)$ has justified envy of i if i stays at $\mu(j)$ with the transferable characteristic of j at $\mu(j)$.

[13] See Examples 3 and 4.

$$Y_{(\mu,\lambda)}(j) = \left\{ i \in \mu^{-1}(\mu(j)) \setminus \{j\} : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j), (i, \lambda^{\mu(j)}(j)) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k)) \right\}.$$

Analogously, the set $\tilde{Y}_{(\mu,\lambda)}(j)$ contains all the students who would enjoy an improvement by being assigned to $\mu(j)$ and who would have the highest priority among the students in $\tilde{D}_{(\mu,\lambda)}(j)$ either with their own or with j's transferable characteristic at $\mu(j)$.

$$\tilde{Y}_{(\mu,\lambda)}(j) = \left\{ \begin{array}{l} i \in \tilde{D}_{(\mu,\lambda)}(j) : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\}, \\ (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k)) \end{array} \right\}.$$

Finally, the set $X_{(\mu,\lambda)}(j)$ consists of all the students who would be willing to occupy j's seat at $\mu(j)$ without generating any additional instance of justified envy because, after a possible exchange of the transferable characteristic at $\mu(j)$, they would have higher priority than the remaining students in $\tilde{D}_{(\mu,\lambda)}(j)$ for j's seat.

$$X_{(\mu,\lambda)}(j) = Y_{(\mu,\lambda)}(j) \cup \tilde{Y}_{(\mu,\lambda)}(j).$$

Let $G = (V; E)$ be a directed application graph with the set of vertices V and the set of directed edges E, which is a set of ordered pairs of elements of V. We consider graphs with $V = I$ and directed edges consisting of ordered pairs of distinct students $ij \in I \times I$ with $i \neq j$. With slight abuse of notation, since the set of edges completely defines the graph, we write $ij \in G$ when edge ij belongs to the set of edges of G. For any directed application graph G, a set of edges $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a path if the related edges $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct, and it is a cycle if the edges $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct and $i_1 = i_{n+1}$. We generically denote an arbitrary cycle in a graph by ϕ. Student i is involved in the cycle ϕ if there is a student j such that $ij \in \phi$. For each cycle ϕ, $N(\phi)$ denotes the students involved in ϕ.

For each extended matching (μ, λ), $G(\mu, \lambda)$ is the ***directed application graph associated with*** (μ, λ) where I is the set of vertices, and the set of directed edges is defined by $ij \in G(\mu, \lambda)$ if and only if $i \in X_{(\mu,\lambda)}(j)$.

For an arbitrary (μ, λ), let ϕ be an arbitrary cycle of $G(\mu, \lambda)$. A pair formed by a cycle ϕ of $G(\mu, \lambda)$ and an allocation of transferable characteristics for the students involved in the cycle $\hat{\lambda}\,|_{N(\phi)}$, $\gamma = (\phi, \hat{\lambda}\,|_{N(\phi)})$, is an ***improvement cycle*** of $G(\mu, \lambda)$ if:

i) for some $ij \in \phi$, $\mu(i) \neq \mu(j)$, and
ii) for each $i, j \in N(\phi)$ with $ij \in \phi$,
  - for each $s \notin \{\mu(i), \mu(j)\}$, $\hat{\lambda}^s(i) = \lambda^s(i)$,
  - if $\mu(i) = \mu(j)$, then $\hat{\lambda}^{\mu(j)}(i) = \lambda^{\mu(j)}(j)$,
  - if $\mu(i) \neq \mu(j)$ then:

$$\hat{\lambda}^{\mu(j)}(i) \in \{l \in \{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\} : \text{ for each } k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\},\ (i, l) \succsim_{\mu(j)} (k, \lambda^{\mu(j)}(k))\}.$$

An improvement cycle $\gamma = (\phi, \hat{\lambda}\,|_{N(\phi)})$ is solved when for each $ij \in \phi$, student i is assigned to $\mu(j)$ with her transferable characteristics reassigned according to $\hat{\lambda}\,|_{N(\phi)}$ to yield a new extended matching. Formally, we denote the solution of a cycle by the operator $\circ$, that is, $(\nu, \bar{\lambda}) = \gamma \circ (\mu, \lambda)$ if and only if for each $i \in N(\phi)$ and $ij \in \phi$, $\nu(i) = \mu(j)$ and $\bar{\lambda}(i) = \hat{\lambda}(i)$, and for each $i' \notin N(\phi)$, $\nu(i') = \mu(i')$ and $\bar{\lambda}(i') = \lambda(i')$.

Note that for each cycle ϕ of $G(\mu, \lambda)$ that involves students initially assigned to different schools, it is possible to define at least one improvement cycle of $G(\mu, \lambda)$. Suppose that a cycle of $G(\mu, \lambda)$ generates more than one improvement cycle. In this case, solving any improvement cycle leads to an extended matching with the same matching but a different allocation of transferable characteristics.

The following algorithm is built on an extended matching and is defined by solving improvement cycles and proposing new allocations of transferable characteristics iteratively. We focus on individually rational and non-wasteful extended matching as starting points of the algorithm. Solving for individually rational and non-wasteful Pareto improvements would be trivial by assigning students to their outside option and empty seats according to the priority.

Student exchange with transferable characteristics (SETC) algorithm:

Step 0: Let (μ_0, λ_0) be an individually rational and non-wasteful extended matching.
Step $t \geq 1$: Given the extended matching $(\mu_{t-1}, \lambda_{t-1})$,

- if there is at least one improvement cycle in $G(\mu_{t-1}, \lambda_{t-1})$, solve any one of such cycles, for example, γ_t, and let $(\mu_t, \lambda_t) = \gamma_t \circ (\mu_{t-1}, \lambda_{t-1})$. Next, move to Step $t + 1$.
- if there is no improvement cycle in $G(\mu_{t-1}, \lambda_{t-1})$, then the algorithm stops and $(\mu_{t-1}, \lambda_{t-1})$ is the selected extended matching.

Note that the definition of the SETC algorithm entails a class of algorithms, as there may be several incompatible improvement cycles, and the order in which improvement cycles are solved may lead to different outcomes.

Regarding the computational efficiency of SETC algorithms, since the sets of schools and students are finite, the algorithm stops after a finite number of steps. Using the same arguments as Erdil and Ergin (2008) to compute the running time for finding stable Pareto improvements, the running time to find a fair Pareto improvement cycle is $O(\#I\#S)$, and at most, there are $\frac{1}{2}\#I(\#I - 1)$ possible Pareto improvements. Thus, the running time to solve this problem entirely is $O\left(\frac{1}{2}\#S\#I^3\right)$. Hence, the SETC algorithms are computationally efficient.

Next, we return to Example 3 to illustrate the algorithm. Example 3 shows the relevance of constructing improvement cycles for students who do not strictly benefit from exchanging their transferable characteristics.

Example 5. (Example 3 continued). Consider the school choice problem with transferable characteristics introduced in Example 3 and the corresponding SOSEM $(\mu_{\omega}^{SO}, \omega)$ and the extended matchings (μ', λ) and $(\mu'', \bar{\lambda})$ defined there to clarify the workings of the SETC algorithm. We construct the direct application graph associated with $(\mu_{\omega}^{SO}, \omega)$. Fig. 1(a) represents the possibilities of improvement for the different students. Each student points to all the students who occupy a seat at a school at least as good as the one that they have in μ_{ω}^{SO}.

Fig. 1(b) represents the associated graph $G(\mu_{\omega}^{SO}, \omega)$. We observe that there are two cycles, $\phi = \{i_1 i_6, i_6 i_2, i_2 i_1\}$ and $\phi' = \{i_1 i_4, i_4 i_1\}$, that generate two associated improvement cycles, $\gamma = (\phi, \lambda\,|_{\{1,2,6\}})$ and $\gamma' = (\phi', \bar{\lambda}\,|_{\{1,4\}})$. Student i_1 is involved in both cycles, and only one associated improvement cycle can be solved.

The extended matching (μ', λ) is the outcome of solving the improvement cycle γ, $(\mu', \lambda) = \gamma \circ (\mu_{\omega}^{SO}, \omega)$. Fig. 1(c) presents the graph $G(\mu', \lambda)$. No students point to the students assigned to schools s_3 and s_4. The students in s_2 are pointed to only by the student in s_3. Finally, the students in s_1 and s_5 do not point to any students. Hence, graph $G(\mu', \lambda)$ has no cycle. In fact, (μ', λ) is constrained efficient. Analogously, $(\mu'', \bar{\lambda}) = \gamma' \circ (\mu_{\omega}^{SO}, \omega)$. Fig. 1(d) presents the graph $G(\mu'', \bar{\lambda})$, which does not present any additional cycles.

Example 5 illustrates how the algorithms in the SETC class obtain fair Pareto improvements by seeking cycles in the directed application graph associated with an initial individually rational and non-wasteful extended matching. Our first result shows that by iteratively applying the same logic, the algorithm yields a justifiable Pareto improvement of the initial extended matching. Moreover, the SETC algorithms exhaust the possibilities of finding additional fair Pareto improvements.

Theorem 1. *Let (μ, ω) be an individually rational and non-wasteful extended matching and (μ', λ) be the outcome obtained by an SETC algorithm starting with (μ, ω). Then, the extended matching (μ', λ) is a justifiable Pareto improvement of (μ, ω) and does not admit any fair Pareto improvement.*

The proof of Theorem 1 relies on showing that if an individually rational and non-wasteful extended matching admits a fair Pareto improvement, then the directed application graph has at least one improvement cycle. The arguments in the proof are similar to Dur et al. (2019, Theorem 1), but in our framework only transferable characteristic exchanges involving specific students at each school may be mutually viable. Moreover, improvement cycles may include students who do not strictly benefit from these exchanges but are needed to facilitate reassignment through transferable characteristic trades. Our assumption of neutral priorities is not crucial to obtain the result. For non-neutral priorities, we can construct the direct application graph $G(\mu, \lambda)$ and run the SETC algorithms once we account for the fact that the transferable characteristics that imply a higher priority at each school may be different for different students.[14]

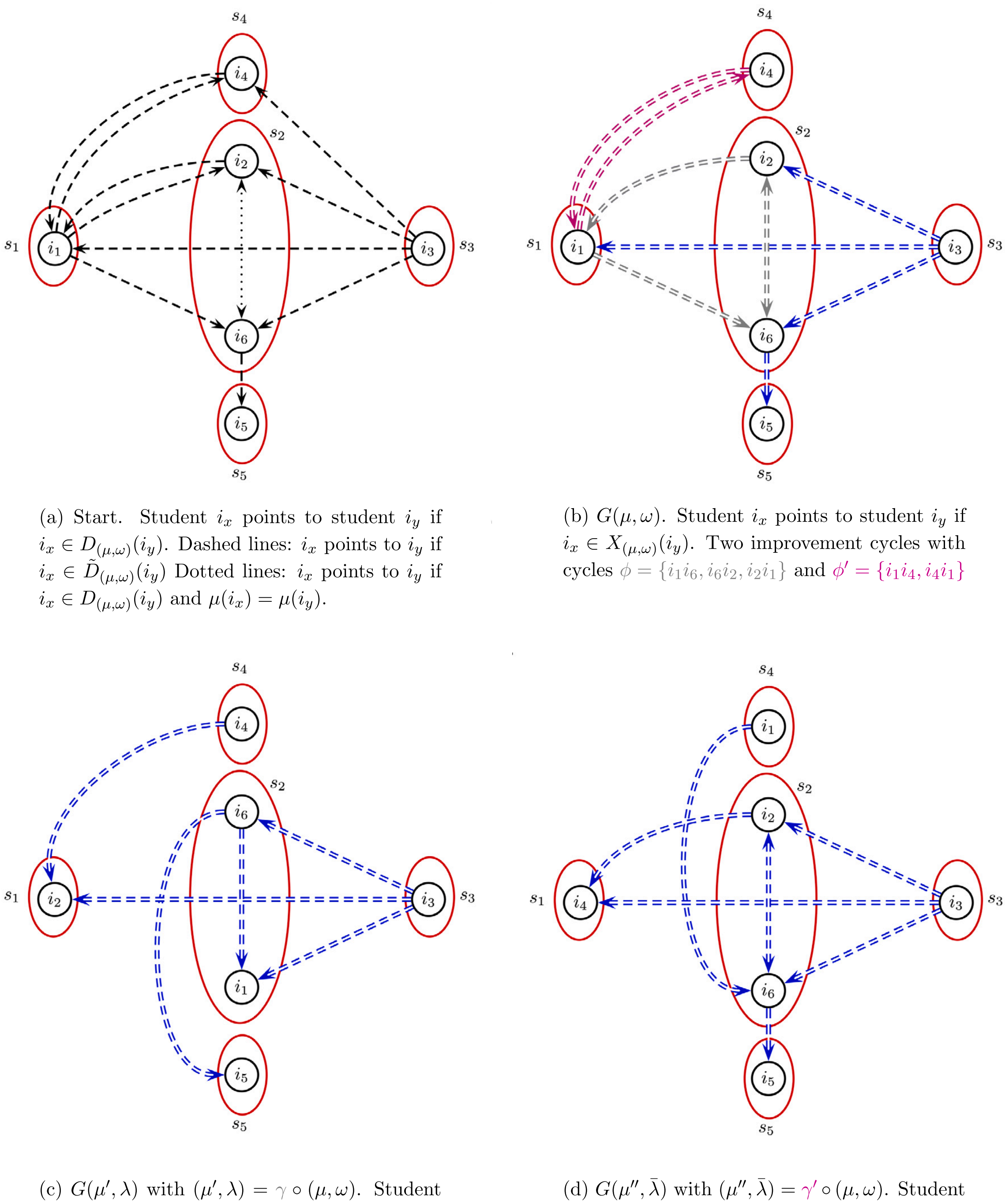
Next, we analyze the extended matchings obtained by applying an SETC algorithm starting at stable extended matchings. Since the application of the SETC algorithm yields a fair Pareto improvement at each step of the algorithm, the algorithm's outcome is a justifiable Pareto improvement of the initial extended matching. If the initial extended matching is stable, then any outcome of an SETC algorithm is also stable. Theorem 1 implies that any extended matching resulting from applying an SETC algorithm to a stable extended matching is constrained efficient. Since for every allocation of transferable characteristics λ, and every stable extended matching under λ, (μ, λ), the matching obtained by student-proposing DA algorithm Pareto dominates μ, every extended matching obtained by an SETC algorithm is the SOSEM associated with the final allocation of transferable characteristics. We formalize both implications in Corollaries 1 and 2 below.

Corollary 1. *Let (μ, ω) be a stable extended matching. If (μ', λ) is the outcome of an algorithm in the SETC class starting with (μ, ω), then (μ', λ) is constrained efficient.*

Corollary 2. *Let (μ, ω) be a stable extended matching. If (μ', λ) is the outcome of an algorithm in the SETC class starting with (μ, ω), then $(\mu', \lambda) = (\mu_{\lambda}^{SO}, \lambda)$.*

Considering Corollary 1, a natural question is whether SETC algorithms can yield all the constrained efficient extended matching that Pareto dominate an initial stable extended matching, such as the SOSEM. Note that any SETC algorithm is restricted to selecting justifiable Pareto improvements from an initial extended matching. Hence, there are school choice problems with a constrained efficient extended matching that Pareto dominates the initial extended matching (μ, ω) that cannot be obtained by applying an instance of an SETC algorithm (see Example 2). Even if we restrict our attention to constrained efficient justifiable Pareto improvements over the initial extended matching, there is an additional issue to address. At each step of any instance of an SETC algorithm, the algorithm proposes an extended matching with a specific allocation of transferable characteristics. However, an extended matching with the same matching and a different allocation of transferable characteristics could also be a fair Pareto improvement. This situation

[14] Specifically, we should define a student-specific operator max to order the transferable characteristics at each school to provide a consistent definition of the set $\tilde{Y}_{(\mu,\lambda)}(j)$.



(a) Start. Student i_x points to student i_y if $i_x \in D_{(\mu,\omega)}(i_y)$. Dashed lines: i_x points to i_y if $i_x \in \tilde{D}_{(\mu,\omega)}(i_y)$ Dotted lines: i_x points to i_y if $i_x \in D_{(\mu,\omega)}(i_y)$ and $\mu(i_x) = \mu(i_y)$.

(b) $G(\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$. Two improvement cycles with cycles $\phi = \{i_1 i_6, i_6 i_2, i_2 i_1\}$ and $\phi' = \{i_1 i_4, i_4 i_1\}$

(c) $G(\mu', \lambda)$ with $(\mu', \lambda) = \gamma \circ (\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu',\lambda)}(i_y)$.

(d) $G(\mu'', \bar{\lambda})$ with $(\mu'', \bar{\lambda}) = \gamma' \circ (\mu, \omega)$. Student i_x points to student i_y if $i_x \in X_{(\mu'',\bar{\lambda})}(i_y)$.

Fig. 1. Example 5. Construction of $G(\mu, \omega)$ and application of the SETC algorithms.

can occur when the students who initially have justified envy of other students occupying a seat at a school no longer desire that seat after being involved in cycles solved in previous steps of the SETC algorithm. In this case, the transferable characteristic of the student in that school becomes irrelevant. We define the concept of a characteristic-wise extended matching to compare different extended matchings regarding the fairness binding constraints.

Let (μ, λ) and $(\mu, \bar{\lambda})$ be an extended matching such that (μ, λ) is stable. The extended matching $(\mu, \bar{\lambda})$ is ***characteristic-wise equivalent to*** (μ, λ) if for each $i \in I$, for each $s \in S$ such that $s\ R_i\ \mu(i)$, if there is a $j \in I$ with $s\ R_j\ \mu(j)$ $(j, \lambda^s(j)) \succsim_s (i, \lambda^s(i))$ implies $(j, \bar{\lambda}^s(j)) \succsim_s (i, \bar{\lambda}^s(i))$.

Remark 1. Let (μ,λ) be a stable extended matching, if $\bar{\lambda}$ is such that for each $i \in I$, $\bar{\lambda}^{\mu(i)}(i) \geq \lambda^{\mu(i)}(i)$ and, for each $s \in S$ with $s\ P_i\ \mu(i)$, $\bar{\lambda}^s(i) = \lambda^s(i)$, then $(\mu.\bar{\lambda})$ is characteristic-wise equivalent to (μ,λ).

Example 6 illustrates the concept of characteristic-wise equivalent extended matchings.

Example 6. Let $I = \{i_1,i_2,i_3,i_4,i_5,i_6\}$, $S = \{s_1,s_2,s_3,s_4,s_5,s_6\}$, and $q_{s_x} = 1$ for $x = 1,2,3,4,5,6$. The relevant student preferences are as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_1	s_2	s_4	s_2
s_1	s_4	s_3	s_3	s_2	s_5
$\varnothing$	s_1	$\varnothing$	s_5	s_5	s_6
	s_2		s_4	$\varnothing$	$\varnothing$
	$\varnothing$		$\varnothing$		

The initial allocation of transferable characteristics ω and the relevant school priorities are:

$\omega^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{0}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{2}^{s_2}$	$\mathbf{4}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{5}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{5}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{5}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	$\mathbf{2}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{1}^{s_5}$	$\mathbf{5}^{s_5}$	$\mathbf{0}^{s_5}$
s_6	$\mathbf{5}^{s_6}$	$\mathbf{4}^{s_6}$	$\mathbf{3}^{s_6}$	$\mathbf{2}^{s_6}$	$\mathbf{1}^{s_6}$	$\mathbf{0}^{s_6}$

and

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$
$(i_1,\mathbf{0}^{s_1})$	$(i_2,\mathbf{4}^{s_2})$	$(i_3,\mathbf{5}^{s_3})$	$(i_4,\mathbf{5}^{s_4})$	$(i_5,\mathbf{5}^{s_5})$	$(i_6,\cdot)$
$(i_2,\mathbf{5}^{s_1})$	$(i_1,\mathbf{4}^{s_2})$	$(i_2,\mathbf{5}^{s_3})$	$(i_4,\mathbf{0}^{s_4})$	$(i_6,\mathbf{5}^{s_5})$	$\dots$
$(i_3,\mathbf{4}^{s_1})$	$(i_6,\mathbf{5}^{s_2})$	$(i_4,\mathbf{5}^{s_3})$	$(i_2,\mathbf{3}^{s_4})$	$(i_4,\mathbf{5}^{s_5})$	
$(i_3,\mathbf{0}^{s_1})$	$(i_5,\mathbf{5}^{s_2})$	$(i_4,\mathbf{4}^{s_3})$	$(i_5,\mathbf{5}^{s_4})$	$(i_6,\mathbf{0}^{s_5})$	
	$(i_4,\mathbf{5}^{s_2})$	$(i_2,\mathbf{3}^{s_3})$	$(i_5,\mathbf{1}^{s_4})$	$(i_4,\mathbf{1}^{s_5})$	
	$(i_5,\mathbf{1}^{s_2})$				
	$(i_4,\mathbf{3}^{s_2})$				
	$(i_1,\mathbf{2}^{s_2})$				
	$(i_5,\mathbf{0}^{s_2})$				

Note that for each $j \in \{1,\dots,6\}$, $\mu^{SO}_{\omega}(i_j) = s_j$. That is,

$$\mu^{SO}_{\omega} = \left[(i_1,s_1),(i_2,s_2),(i_3,s_3),(i_4,s_4),(i_5,s_5),(i_6,s_6)\right].$$

Consider now the extended matching (μ',λ) with

$$\mu' = \left[(i_1,s_2),(i_2,s_3),(i_3,s_1),(i_4,s_5),(i_5,s_4),(i_6,s_6)\right],$$

and λ such that:

$\lambda^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{4}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{0}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{5}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{5}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{5}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	$\mathbf{2}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{5}^{s_5}$	$\mathbf{1}^{s_5}$	$\mathbf{0}^{s_5}$
s_6	$\mathbf{5}^{s_6}$	$\mathbf{4}^{s_6}$	$\mathbf{3}^{s_6}$	$\mathbf{2}^{s_6}$	$\mathbf{1}^{s_6}$	$\mathbf{0}^{s_6}$

The extended matching (μ',λ) is a fair Pareto improvement of $(\mu^{SO}_{\omega},\omega)$, and it is a constrained efficient extended matching. However, (μ',λ) cannot be obtained by applying the SETC algorithm. Note that in the first stage of SETC algorithm there is only one improvement cycle $\{i_1i_2, i_2i_1\}$ with allocation of transferable characteristics $\bar{\lambda}$ such that $\bar{\lambda}^{s_1}(i_2) = \omega^{s_1}(i_2) = \mathbf{5}^{s_1}$ and $\bar{\lambda}^{s_2}(i_1) = \omega^{s_2}(i_2) = \mathbf{4}^{s_2}$. In the next stage of SETC algorithm, there are two cycles $\{i_2i_3, i_3i_2\}$ and $\{i_2i_4, i_4i_3, i_3i_2\}$.[15]

[15] Solving the improvement cycle with $\{i_2i_4, i_4i_3, i_3i_2\}$ does not allow further improvement cycles and stops the SETC algorithm with an extended matching (μ^*,λ^*) such that $\mu^* = \left[(i_1,s_2),(i_2,s_4),(i_3,s_1),(i_4,s_3),(i_5,s_5),(i_6,s_6)\right]$.

Solving the improvement cycle $\{i_2 i_3, i_3 i_2\}$ with allocation of transferable characteristics $\tilde{\lambda}$ such that $\tilde{\lambda}^{s_1}(i_3) = \omega^{s_1}(i_2) = \mathbf{5}^{s_1}$ and $\tilde{\lambda}^{s_3}(i_2) = \omega^{s_3}(i_3) = \mathbf{5}^{s_3}$ allows for an additional improvement cycle $\{i_4 i_5, i_5, i_4\}$ with allocation of transferable characteristics $\hat{\lambda}$ such that $\hat{\lambda}^{s_4}(i_5) = \omega^{s_4}(i_4) = \mathbf{5}^{s_4}$ and $\hat{\lambda}^{s_5}(i_4) = \omega^{s_5}(i_5) = \mathbf{5}^{s_5}$:

$\hat{\lambda}^s(i)$	i_1	i_2	i_3	i_4	i_5	i_6
s_1	$\mathbf{0}^{s_1}$	$\mathbf{4}^{s_1}$	$\mathbf{5}^{s_1}$	$\mathbf{3}^{s_1}$	$\mathbf{2}^{s_1}$	$\mathbf{1}^{s_1}$
s_2	$\mathbf{4}^{s_2}$	$\mathbf{2}^{s_2}$	$\mathbf{1}^{s_2}$	$\mathbf{3}^{s_2}$	$\mathbf{0}^{s_2}$	$\mathbf{5}^{s_2}$
s_3	$\mathbf{1}^{s_3}$	$\mathbf{5}^{s_3}$	$\mathbf{3}^{s_3}$	$\mathbf{4}^{s_3}$	$\mathbf{1}^{s_3}$	$\mathbf{0}^{s_3}$
s_4	$\mathbf{2}^{s_4}$	$\mathbf{3}^{s_4}$	$\mathbf{4}^{s_4}$	$\mathbf{1}^{s_4}$	$\mathbf{5}^{s_4}$	$\mathbf{0}^{s_4}$
s_5	$\mathbf{2}^{s_5}$	$\mathbf{4}^{s_5}$	$\mathbf{3}^{s_5}$	$\mathbf{5}^{s_5}$	$\mathbf{1}^{s_5}$	$\mathbf{0}^{s_5}$
s_6	$\mathbf{5}^{s_6}$	$\mathbf{4}^{s_6}$	$\mathbf{3}^{s_6}$	$\mathbf{2}^{s_6}$	$\mathbf{1}^{s_6}$	$\mathbf{0}^{s_6}$

Note that according to extended matching (μ', λ), $\tilde{D}_{(\mu',\lambda)}(i_3) = \emptyset$. Once all the students are assigned to a school according to μ', allocating the transferable characteristic to i_3 at s_1 is irrelevant since no student may have justified envy of i_3. Hence, we can obtain a fair Pareto improvement of the initial extended matching $(\mu_\omega^{SO}, \omega)$ that an SETC algorithm cannot yield. This fact notwithstanding, there is an SETC algorithm yielding an extended matching that is characteristic-wise equivalent to (μ', λ).

Example 6 shows that it is impossible to characterize the set of constrained efficient extended matching that are justifiable Pareto improvements of an initial extended matching as the set of outcomes of the SETC algorithm starting at that initial extended matching. However, Theorem 2 shows that for every extended matching that can be obtained by a sequence of fair Pareto improvements from an initial stable extended matching, an instance of the SETC algorithm yields either an extended matching with the same matching or an extended matching that Pareto dominates the extended matching resulting from the sequence of fair Pareto improvements.

Theorem 2. *For each stable extended matching (μ, ω) and each (μ', λ) justifiable Pareto improvement of (μ, ω), there is a stable extended matching $(\nu, \bar{\lambda})$ obtained with an algorithm of the SETC class starting at (μ, ω) such that, for each $i \in I$, $\nu(i)\ R_i\ \mu'(i)$.*

Theorem 2 follows from an intermediate result Proposition 4. If an extended matching (μ', λ) is a justifiable Pareto improvement of (μ, ω), then the outcome of an application of the SETC algorithm after t steps yields an extended matching (μ_t, λ_t) such that $\mu_t = \mu'$ and (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ). To prove this result, we check that the matching involved in any fair Pareto improvement of a stable extended matching can be obtained by an SETC algorithm after a finite number of steps (Lemma 9). However, improving students may receive an allocation of transferable characteristics such that their priority at the assigned school is higher than that prescribed by (μ', λ). Thus, it may be the case that (μ', λ) is constrained efficient but (μ_t, λ_t) is not constrained efficient and admits further fair Pareto improvement, and the outcome of the SETC algorithm may Pareto dominate (μ', λ). This is because constrained efficiency is a property of the extended matching and not of the matching.[16]

Theorem 2 applies to a stable extended matching and not to any initial non-wasteful extended matching as in Theorem 1. In this case, the restriction to fair Pareto improvements implies that no student may generate justified envy. This is an unnecessarily strong requirement. It is possible to define a weaker notion of fair Pareto improvement. For example, we can require that the set of students with justified envy of a student occupying a particular seat at a school must belong to the initial group of students with justified envy of her. With this alternative notion of fair Pareto improvements in mind, we can define an alternative class of algorithms that will be analogous to those in the SETC class and would uncover more justifiable Pareto improvements than the SETC algorithm. Finally, our assumption of neutral priorities plays a relevant role in the proof of Theorem 2. Under neutral priorities, we can construct the allocations of transferable characteristics that allow every fair Pareto improvement to be generated as a sequence of solved improvement cycles by an algorithm in the SETC class.

4. Discussion

In this section, we focus on issues related to the possibility of finding a justifiable Pareto improvement for the SOSEM. Specifically, we consider the incentives of the students to reveal their true preferences to a centralized planner and a particular class of school priorities that allows us to compare our framework with previous works.

4.1. Incentives and student transferable characteristics

We first analyze the students' incentives to reveal their true preferences when an SETC algorithm determines the allocation of school seats. For this purpose, we need to introduce further notation.

16 To obtain an example of a constrained efficient extended matching that is Pareto dominated by the outcome of the SETC algorithm, it suffices to consider a school choice problem with two independent replicas of the school choice problem in Example 6 and an additional student, where the students in the role of i_3 would like to exchange their seats at the replicas of s_1. However, the additional student may have higher priority and generate justified envy unless they exchange the relevant characteristic at s_1.

Let $\mathcal{P}$ denote the set of all student preference profiles and $\mathcal{M}$ be a set of all extended matchings. A mechanism is a mapping $\Psi : \mathcal{P} \to \mathcal{M}$.

We can construct mechanisms by the application of algorithms in the SETC class. We call the class of mechanisms defined by applying an SETC algorithm starting from the SOSEM of each preference profile the ***student optimal transferable characteristics (SOTC)*** class. Of course, for each profile of preferences, each mechanism in the SOTC class selects a stable and constrained efficient extended matching.

Strategy-proofness A mechanism Ψ satisfies ***strategy-proofness*** if for each $i \in N$, each $P, P' \in \mathcal{P}$, such that for each $j \neq i$, $P_j = P'_j$, $\Psi(P) = (\mu, \lambda)$ and $\Psi(P') = (\mu', \lambda')$, $\mu(i)\ R_i\ \mu'(i)$.

Strategy-proofness implies that no student has the capacity and the incentives to manipulate the outcome of a mechanism by misreporting her preferences regarding schools. The mechanism that selects the SOSEM for each student preference profile satisfies *strategy-proofness* but may select a Pareto dominated extended matching. According to the results in Abdulkadiroğlu et al. (2009); Kesten (2010); Alva and Manjunath (2019); Kesten and Kurino (2019), since the matching selected by any SETC algorithm that starts with the SOSEM Pareto dominates the SOSEM for the initial allocation of characteristics ω and is not Pareto dominated by any other extended matching, each mechanism in the SOTC class is manipulable for some profile of student preferences.

Proposition 1. *No mechanism in the SOTC class satisfies* strategy-proofness*.*

4.2. Fully transferable priorities

In the previous sections, we analyzed a new setting where trade-offs between stability and efficiency can be attenuated. In a school choice problem with transferable characteristics, some violations of initial priorities can be justified after exchanges of transferable characteristics. We now compare our framework with previous works that consider efficiency gains with respect to the SOSEM matching by dropping stability constraints when some students do not benefit from exercising their priority rights. In particular, the concepts of α-stability in Alcalde and Romero-Medina (2017) and of students consenting to drop their initial priorities in Kesten (2010) imply the existence of matchings that satisfy the relaxed fairness constraints and Pareto dominate the SOSEM matching.

The proposals of Alcalde and Romero-Medina (2017) and Kesten (2010) are presented in terms of the canonical school choice problem. This prevents an immediate comparison with our results because we justify the relaxation of the fairness constraints by introducing a new component of the school choice problem, student transferable characteristics. However, an extreme class of school priorities allows us to view both proposals as cases of extended matchings obtained by SETC algorithms. This is the case in the domain of *fully transferable extended priorities*, referring to situations where transferable characteristics entirely determine school priorities.

Fully transferable priorities For each $i, i', j, j' \in I$, $s \in S$, and each $l, l' \in \Omega^s$ with $l \neq l'$, if $(i, l) \succ_s (i', l')$ then $(j, l) \succ_s (j', l')$.

In the context of fully transferable priorities, analyzing the SETC algorithms is relatively straightforward. Every student who desires another student's school can obtain it with an exchange of the transferable characteristics.

Lemma 1. *Let school priorities be fully transferable, let (μ, λ) be a stable extended matching and let $G(\mu, \lambda)$ be the directed application graph associated with (μ, λ). If $\mu(j)\ P_i\ \mu(i)$, then $ij \in G(\mu, \lambda)$.*

Lemma 1 implies that under fully transferable priorities, students who exchange their characteristics but remain assigned to the same school do not need to participate in improvement cycles. Moreover, the possibility of justifying an exchange of seats that involves a violation of school priorities under the initial allocation of transferable characteristics does not depend on the students (and their transferable characteristics) initially assigned to each school. Hence, the framework under fully transferable characteristics is equivalent to the setup in Dur et al. (2019) when all potential exchange cycles are admitted under partial stability.[17] In this context, since any fair Pareto improvement of a matching can be achieved by forming disjoint cycles among students and because such cycles correspond to an improvement cycle in directed application graph $G(\mu^{SO}_\omega, \omega)$, we immediately derive the following result.

Proposition 2. *Let school priorities be fully transferable. If μ is a matching that Pareto dominates μ^{SO}_ω and μ is not Pareto dominated by any matching ν, then there is an allocation of transferable characteristics λ such that (μ, λ) is the result of applying an SETC algorithm that starts with the SOSEM.*

Alcalde and Romero-Medina (2017, Theorem 1)proves that the Pareto efficient matchings that are Pareto improvements over the initial optimal student matching coincide with a set of matchings such that under the initial priorities, no student can pose a so-called admissible objection. Hence, the set of matchings produced by an SETC algorithm coincides with the set of α-*fair* matchings produced under the assumption of fully transferable priorities.

[17] In the terms of Dur et al. (2019) this corresponds to the case where the correspondence that defines the admitted priority violations satisfies the all-or-nothing property, specifically, item i) of the all-or-nothing property for all schools.

Corollary 3. *Let school priorities be fully transferable. A matching μ is an α-fair matching if and only if there is an allocation of transferable characteristics λ such that the extended matching (μ, λ) is the result of the application of an SETC algorithm that starts with the SOSEM*

Kesten (2010) occupies a central position in analyzing Pareto efficient matching in school choice and introduces the idea of consent. Students can consent to withdraw their claims to seats they will not accept. This idea leads to a version of the student-proposing DA algorithm that yields a Pareto efficient matching with "minimal" violations of initial priorities, the EADA algorithm. Tang and Yu (2014) presents a simpler algorithm with the same outcome.[18] Under fully transferable priorities, the matching obtained by the EADA belongs to the extended matching outcome of a specific algorithm in the SETC class.

Proposition 3. *Let school priorities be fully transferable. There is an algorithm in the SETC class that, for each problem of school choice with transferable characteristics, the outcome of the algorithm starting in the SOSEM selects an extended matching (μ, λ) such that μ coincides with the EADA outcome.*

5. Conclusion

In this paper, we generalize the school choice problem by defining school priorities in terms of (possibly transferable) student characteristics. We define a class of algorithms, the SETC class. Each algorithm in this class begins with an individually rational and non-wasteful extended matching and produces an extended matching such that any extended matching that Pareto dominates it generates additional instances of justified envy. Moreover, for each constrained efficient extended matching obtained by a sequence of fair Pareto improvements from an initial stable extended matching, an algorithm in the SETC class obtains an extended matching that either is characteristic-wise equivalent to or Pareto dominates the constrained efficient extended matching.

We motivate our analysis of the allocation of objects under priorities based on individual characteristics in the school choice problem. In this framework, the tie-breaking lottery is a natural transferable characteristic when schools use multiple tie-breaking lotteries. However, the algorithms in the SETC class can be used for improving efficiency in situations where we can differentiate between allocative criteria and fairness constraints in the characteristics that define the priorities. For instance, we can avoid welfare losses by integrating different markets (walk-zone). We can study changes in the priority structure due to the redefinition of the characteristics or because of different valuations of the existing characteristics. Finally, we can use the SETC algorithms to propose an ex-post assignment scramble under mechanisms that generate instances of justified envy.

Our analysis is based on characteristics that are specific to individual schools. This is the situation in the case of the multiple tie-breaking lotteries, priorities for siblings attending the school, or legacy awarded priorities. However, other characteristics are not generally school specific, such as the walk-zone priority or all priorities associated with family circumstances, such as income or the total number of siblings. In these cases, we could adjust the algorithms in the SETC class to allow the characteristics to be valid in several schools. Still, this adjustment must be precisely defined, and it will complicate our results.

Another potential generalization of the framework would be to consider potential fair Pareto improvements when some students are initially assigned to several schools while others do not obtain any initial admission. This approach would allow for procedures such as the school choice top trade cycle mechanism proposed by Abdulkadiroğlu and Sönmez (2003). By assuming fully transferable priorities and defining an appropriate inheritance rule for admissions when students accept a school and leave the market, solving the Pareto improvement cycles corresponding to top trade cycles would result in a constrained efficient extended matching. The general case for non-specific characteristics and the analysis of fair Pareto improvements starting at assignments that are not extended matchings remain open for further research.

6. Proofs

6.1. Proof of Theorem 1

Let $(I, S, R, q, \omega, \succsim)$ be a school choice problem with transferable characteristics and let (μ, ω) be an individually rational and non-wasteful extended matching. Consider the application of an arbitrary SETC algorithm with initial extended matching $(\mu_0, \lambda_0) = (\mu, \omega)$. Let T be the last step of the SETC algorithm starting by (μ_0, λ_0). For each $t \in \{1, \dots, T\}$, let $\gamma_t = (\phi_t, \lambda_t \mid_{N(\phi_t)})$ be the improvement cycle solved at step t of the algorithm, and let $(\mu_t, \lambda_t) = \gamma_t \circ (\mu_{t-1}, \lambda_{t-1})$ be the extended matching selected at step t. Note that, the students involved in the improvement cycle are better off, some of them are strictly better off, and the students not involved in the cycle are not worse off at the extended matching obtained by solving the improvement cycle γ_t. Thus, for each $t \in \{1, \dots, T\}$, μ_t *Pareto dominates* μ_{t-1}.

Remark 2. Let $\gamma_t = (\phi_t, \lambda_t \mid_{N(\phi_t)})$ be the improvement cycle solved at step t of the application of SETC algorithm, and let $i, j \notin N(\phi_t)$.

i) $\tilde{D}_{(\mu_t, \lambda_t)}(j) \subseteq \tilde{D}_{(\mu_{t-1}, \lambda_{t-1})}(j)$.

[18] Reny (2022) characterizes the matching selected by the EADA algorithm as the unique priority-efficient matching. That is, given the initial allocation of transferable characteristics ω and the school priorities, the EADA algorithm selects the unique matching μ^* that is not Pareto dominated by any other matching, and for each matching ν such that some student with justified envy at (μ^*, ω) prefers ν to μ^*, there is another student who prefers μ^* to ν and has justified envy at (ν, ω).

ii) If $ij \in G(\mu_{t-1}, \lambda_{t-1})$ then $ij \in G(\mu_t, \lambda_t)$.

Lemma 2. *The extended matching (μ_T, λ_T) is individually rational and non-wasteful.*

Proof. Let $t \in \{0, \dots, T-1\}$ and let (μ_t, λ_t) be the extended matching obtained at step t of the algorithm. We prove the result by induction on t. The initial extended matching (μ_0, λ_0) is individually rational and non-wasteful.

First, we check that (μ_T, λ_T) is an individual rationality extended matching. Since (μ_0, λ_0) is individually rational, and each student is never worse off after each step of the algorithm, then (μ_T, λ_T) is individually rational.

We conclude by checking that (μ_T, λ_T) is non-wasteful. The initial match (μ_0, λ_0) is non-wasteful. At each step, students are assigned to better schools swapping their seats at schools, hence $\#\mu_t^{-1}(s)$ remains constant at each step of the algorithm. Therefore, if school s has an empty slot at step t, then school s has an empty slot at step 0. Since μ_0 is individually rational and non-wasteful, for each student i with $\mu_0(i) \neq s$, $\mu_0(i)\ P_i\ s$. Since for each i, $\mu_t(i)\ R_i\ \mu_0(i)$, we have that $\mu_t(i)\ R_i\ s$. Thus, (μ_t, λ_t) is non-wasteful. □

Our next result, Lemma 3 shows that the outcome of the SETC algorithm reduces the instances of justified envy.

Lemma 3. *For each $t \in \{1, \dots, T\}$, if student i does not have justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$, then student i does not have justified envy of j at (μ_t, λ_t).*

Proof. Let $i, j \in I$ such that i does not have justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$. If $\mu_t(i)\ R_i\ \mu_t(j)$, then i has not justified envy of j at (μ_t, λ_t). Hence, we assume that $\mu_t(j)\ P_i\ \mu_t(i)$. Let $\gamma_t = (\phi_t, \lambda_t\ |_{N(\phi)})$ be the improvement cycle of $G(\mu_{t-1}, \lambda_{t-1})$ solved at step t. Since (μ_t, λ_t) is the result of solving the cycle γ_t, for each student k, $\mu_t(k)\ R_k\ \mu_{t-1}(k)$, and $\mu_t(j)\ P_i\ \mu_{t-1}(i)$. Since $\mu_t(j)\ P_i\ \mu_{t-1}(i)$ and $\mu_t(j)\ P_i\ \mu_t(i)$, $\mu_t(i) \neq \mu_t(j)$ and $\mu_{t-1}(i) \neq \mu_t(j)$ imply that $\lambda_t^{\mu_t(j)}(i) = \lambda_{t-1}^{\mu_t(j)}(i)$. We consider two cases. Assume first that $j \notin N(\phi_t)$. In this case, $\mu_t(j) = \mu_{t-1}(j)$ and $\lambda_t^{\mu_t(j)}(j) = \lambda_{t-1}^{\mu_t(j)}(j)$. Since i has not justified envy of j at $(\mu_{t-1}, \lambda_{t-1})$, $(j, \lambda_{t-1}^{\mu_{t-1}(j)}(j)) \succsim_{\mu_{t-1}(j)} (i, \lambda_{t-1}^{\mu_{t-1}(j)}(i))$. Consider the second case, $j \in N(\phi)$. Let k be the student such that $jk \in \phi_t$. Note that $\mu_t(j) = \mu_{t-1}(k)$, $j \in X_{(\mu_{t-1}, \lambda_{t-1})}(k)$, and $i \in \tilde{D}_{(\mu_{t-1}, \lambda_{t-1})}(k)$. Since $j \in X_{(\mu_{t-1}, \lambda_{t-1})}(k)$ and $\lambda_t^{\mu(j)}(i) = \lambda_{t-1}^{\mu(j)}(i)$, we have $(j, \lambda_t^{\mu_t(j)}(j)) \succsim_{\mu_t(j)} (i, \lambda_{t-1}^{\mu_t(j)}(i)) = (i, \lambda_t^{\mu(j)}(i))$. Since the two cases are exhaustive and imply that $(j, \lambda_t^{\mu_t(j)}(j)) \succsim_{\mu_t(j)} (i, \lambda_t^{\mu_t(j)}(i))$, we conclude that i has not justified envy of j at (μ_t, λ_t). □

From Lemma 3 and since students may only improve at each step of the algorithm, we immediately obtain Corollaries 4 and 5.

Corollary 4. *For each $i \in I$:*

i) *If for some $t \in \{1, \dots, T\}$, $i \in N(\phi_t)$; then no student has justified envy of i at (μ_T, λ_T).*
ii) *If i is not involved in any improvement cycle solved to obtain (μ_T, λ_T) and there is a student j with justified envy of i at (μ_T, λ_T), then j has justified envy of i at (μ_0, λ_0).*

Corollary 5. *For each $t \in \{1, \dots, T\}$, (μ_t, λ_t) is a fair Pareto improvement of $(\mu_{t-1}, \lambda_{t-1})$.*

Lemma 4. *For each individually rational and non-wasteful extended matching (μ, λ) and $j \in I$, $X_{(\mu,\lambda)}(j) \cap \tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ if and only if $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$.*

Proof. If $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$, then $D_{(\mu,\lambda)}(j) = \{i \in I : \mu(i) = \mu(j)\}$. Since $X_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$, the result is immediate. If $\tilde{D}_{(\mu,\lambda)}(j) \neq \varnothing$, then by completeness and transitivity of school priorities, there is $i \in \tilde{D}_{(\mu,\lambda)}(j)$ such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$. Therefore, $i \in X_{(\mu,\lambda)}(j)$. □

From Lemma 4 and the definition of the set $\tilde{D}_{(\mu,\lambda)}(j)$ we obtain Remark 3.

Remark 3. For each individually rational and non-wasteful extended matching (μ, λ) and $j \in I$, if $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$, then for each $j' \in I$ with $\mu(j) = \mu(j')$, $\tilde{D}_{(\mu,\lambda)}(j') = \varnothing$.

Lemma 5. *Let (μ, λ) and (ν, λ') be individually rational and non-wasteful extended matchings such that (ν, λ') Pareto dominates (μ, λ). For each $s \in S$, $\#\mu^{-1}(s) = \#\nu^{-1}(s)$.*

Proof. Let $N = \{i \in I :\ \nu(i)\ P_i\ \mu(i)\}$. Since (ν, λ') Pareto dominates (μ, λ) and student preferences are strict, for each $j \in I \setminus N$, $\mu(j) = \nu(j)$. Consider an arbitrary school $s \in S$ and assume to the contrary that $\#(N \cap \nu^{-1}(s)) > \#(N \cap \mu^{-1}(s))$. This implies that $\#\mu^{-1}(s) < q_s$. For each $i \in (N \cap \nu^{-1}(s))$, $\nu(i) = s\ P_i\ \mu(i)$, which contradicts that (μ, λ) is non-wasteful. Hence, $\#(N \cap \nu^{-1}(s)) \leq \#(N \cap \mu^{-1}(s))$. Finally, there is s such that the strict inequality holds. Summing up the inequalities across schools, the number of students in N assigned to some school in matching μ is larger than the number of students in N assigned to some school in matching ν. Thus, there is a student $i \in N$ such that $\mu(i) \in S$, and $\nu(i) = \varnothing$. Since μ is an individually rational matching, we have that $\mu(i)\ P_i\ \nu(i)$, which contradicts the definition of N. □

Lemma 6. *Let (μ, λ) be an individually rational and non-wasteful extended matching, if (ν, λ') Pareto dominates (μ, λ), then (ν, λ') is individually rational and non-wasteful.*

Proof. Since (μ, λ) is individually rational and for each $i \in I$, $\nu(i)\ R_i\ \mu(i)$, (ν, λ') is an individually rational extended matching. Let $i \in I$ such that $\nu(i)\ P_i\ \mu(i)$. Since (μ, λ) is non-wasteful, there is $j \in I$ such that $\nu(j) \neq \mu(j) = \nu(i)$. Since (ν, λ') Pareto dominates (μ, λ) and $\nu(j) \neq \mu(j)$, we have $\nu(j)\ P_j\ \mu(j)$. By Lemma 5, there is $k \in I$ such that $\nu(k) \neq \mu(k) = \nu(j)$. As S is finite, for each i with $\nu(i)\ P_i\ \mu(i)$ there is a finite sequence of students $i_1, i_2, i_3, \ldots, i_n$ such that $\mu(i_i) = \nu(i_{i+1})$ and $i_1 = i_n$. Since (μ, λ) is non-wasteful, for each $i \in I$ for each $s \in S$ such that $s\ P_i\ \mu(i)$, we have $\#\mu^{-1}(s) = \#\nu^{-1}(s) = q_s$. Finally, as $s\ P_i\ \nu(i)$ implies $s\ P_i\ \mu(i)$, and for each s such that $s\ P_i\ \mu(i)$, $\#\mu^{-1}(s) = q_s$, we have that for each s such that $s\ P_i\ \nu(i)$, $\#\nu^{-1}(s) = q_s$, which suffices to prove that (ν, λ') is non-wasteful. □

Lemma 7 provides the final step in the proof of Theorem 1.

Lemma 7. *The extended matching (μ_T, λ_T) does not admit any additional fair Pareto improvement.*

Proof. Let $(\mu, \lambda) = (\mu_T, \lambda_T)$ and assume to the contrary, that (ν, λ') is a fair Pareto improvement of (μ, λ). By Lemma 6, the extended matching (ν, λ') is individually rational and non-wasteful. By the definition of the SETC algorithm, there is no improvement cycle in graph $G(\mu, \lambda)$. There are two cases:

Case 1. For each $i \in I$, $\tilde{D}_{(\mu,\lambda)}(i) = \emptyset$. Then, by Lemma 4 and Remark 3 for each $i \in I$, $X_{(\mu,\lambda)}(i) \subseteq \{i' \in I : \mu(i) = \mu(i')\}$. This implies that each student is assigned to her best school at μ, there is no improvement cycle, and ν does not Pareto dominate μ.

Case 2. There are paths in $G(\mu, \lambda)$ involving students who would like to change her assigned school, but there is no improvement cycle. This implies that some students are only pointed to by students assigned to the same school.

Assume we are in Case 2. Since there is no improvement cycle, there is a set of students not pointed to by any other student in $G(\mu, \lambda)$. Let $I_1 = \{i \in I : \tilde{D}_{(\mu,\lambda)}(i) = \emptyset\}$. Let $i_1 \in I_1$ and $s_1 = \mu(i_1)$. By Remark 3, for each j with $\mu(j) = s_1$, $\tilde{D}_{(\mu,\lambda)}(j) = \emptyset$ and $j \in I_1$. Since ν Pareto dominates μ, there is no $j' \in I$, such that $\mu(j') \neq s_1$ and $\nu(j') = s_1$. Thus $\nu^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 5, $\#\nu^{-1}(s_1) = \#\mu^{-1}(s_1)$ and we get $\nu^{-1}(s_1) = \mu^{-1}(s_1)$. Since i_1 was arbitrary, this holds for each s such that $\mu^{-1}(s) \cap I_1 \neq \emptyset$.

Next, since there is no improvement cycle in $G(\mu, \lambda)$, there is at least one student in $I \setminus I_1$ such that only students in I_1 point to her. Otherwise, there would be an improvement cycle or no path (Case 1). Let $I_2 = \{i \in I : \tilde{D}_{(\mu,\lambda)}(i) \subseteq I_1\} \setminus I_1$ Let $i_2 \in I_2$ and $s_2 = \mu(i_2)$. We first show that there is no j with $\mu(j) \neq s_2$ and $\nu(j) = s_2$. Assume to the contrary and since ν Pareto dominates μ, $s_2\ P_j\ \mu(j)$ and thus, $j \in \tilde{D}_{(\mu,\lambda)}(i_2)$. Nevertheless, by definition, i_2 is only pointed to by students in I_1. By the arguments in the previous paragraph, for each $j \in I_1$, $\mu(j) = \nu(j)$. Hence, $\nu^{-1}(s_2) \subseteq \mu^{-1}(s_2)$. By Lemma 5, $\#\mu^{-1}(s_2) = \#\nu^{-1}(s_2)$, and therefore $\mu^{-1}(s_2) = \nu^{-1}(s_2)$.

We can apply the same argument iteratively to conclude that all students in any path in $G(\mu, \lambda)$ have the same match under μ and ν. The students who are not in a path in $G(\mu, \lambda)$, are contained in I_1 and have the same match in both μ and ν. We conclude that $\mu = \nu$ and ν does not Pareto dominate μ. □

To conclude the proof of Theorem 1, by Corollary 5, $\{(\mu_0, \lambda_0), (\mu_1, \lambda_1), \ldots, (\mu_T, \lambda_T)\}$ is a sequence of extended matching such that (μ_t, λ_t) is a fair Pareto improvement of $(\mu_{t-1}, \lambda_{t-1})$. Therefore, (μ_T, λ_T) is a justifiable Pareto improvement of (μ, ω). By Lemma 7, (μ_T, λ_T) does not admit further fair Pareto improvement.

6.2. Proof of Theorem 2

Theorem 2 states that for every justifiable Pareto improvement of an initial stable extended matching, there is a specific SETC algorithm that obtains an extended matching with the same matching after a finite number of steps. The proof follows from the following intermediate result.

Proposition 4. *Let (μ, λ) be a stable extended matching and (μ', λ') a justifiable Pareto improvement of (μ, λ). There exist a natural number $t \in \mathbb{N}$ and an extended matching (μ_t, λ_t) such that (μ_t, λ_t) is the extended matching selected at step t of an application of an SETC algorithm, $\mu_t = \mu'$, and (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ').*

The key step in the proof of Proposition 4 is checking that for each fair Pareto improvement from an arbitrary stable extended matching (μ, λ), the application of an algorithm in the SETC class yields an extended matching with the same matching μ. Lemma 8 presents the structure of fair Pareto improvements and is a crucial first step for constructing improvement cycles of $G(\mu, \lambda)$.

Lemma 8. *Let (μ, λ) be a stable extended matching and $(\nu, \bar{\lambda})$ a fair Pareto improvement of (μ, λ). There exists a finite set of disjoint cycles of students $\Phi = \{\phi_1, \ldots, \phi_m\}$ such that for each $i \notin \cup_{\phi\in\Phi} N(\phi)$, $\nu(i) = \mu(i)$, and for each $j \in \cup_{\phi\in\Phi} N(\phi)$, there are j' and $m' \leq m$ with $jj' \in \phi_{m'}$ and $\nu(j) = \mu(j')$.*

Proof. Let $N \subseteq I$ be the set of students who either strictly prefer their match under ν to the match under μ or $\lambda(i) \neq \bar{\lambda}(i)$. Let us partition the set N in three disjointed sets N_1, N_2, N_3 defined by:

$$N_1 = \{i \in N: \ \mu(i) = \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},$$
$$N_2 = \{i \in N: \ \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},$$
$$N_3 = \{i \in N: \ \mu(i) \neq \nu(i) \ \& \ \bar{\lambda}^{\nu(i)}(i) = \lambda^{\nu(i)}(i)\}.$$

Let $n = \#N$. Index the students in N in such that for each $x, x', x'' \in \{1, \dots, n\}$, if $i_x \in N_1$, $i_{x'} \in N_2$, and $i_{x''} \in N_3$ then $x < x' < x''$. Moreover, for each $x, y \in \{1, \dots, n\}$ such that $i_x, i_y \in N_3$ and $\nu(i_x) = \nu(i_y)$, if $x < y$ then $(i_x, \bar{\lambda}^{\nu(i_x)}(i_x)) \succ_{\nu(i_x)} (i_y, \bar{\lambda}^{\nu(i_x)}(i_y))$.

Let $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda}')]$ be a directed graph with vertices $i \in I$ and such that its edges are constructed sequentially in the following way. For each $x \in \{1, \dots, n\}$:

i) If $i_x \in N_1$, i_x points to student j if and only if $i \neq j$ and $\bar{\lambda}^{\nu(i_x)}(i_x) = \lambda^{\nu(i_x)}(j)$.
ii) If $i_x \in N_2$, i_x points to student j if and only if $i \neq j$ and $\bar{\lambda}^{\nu(i_x)}(i_x) = \lambda^{\nu(i_x)}(j)$.
iii) If $i_x \in N_3$, i_x points to student $j \in N$ such that $\mu(j) = \nu(i_x)$, and j has not been pointed to by any i_y with $y < x$.[19]

Students that do not belong to N do not point to any other student. Note that for each $i \in N$, i always points to a student in N.

In the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$, each student is pointed to by a unique student and points to a unique student in N. Since N is finite, there is at least a cycle in the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$. Moreover, each student in N is in a cycle and no two cycles intersect. By construction, the matching ν is obtained by assigning each student to the school to which the student she points to is initially assigned. □

Lemma 8 implies that any fair Pareto improvement $(\nu, \bar{\lambda})$ of a stable extended matching (μ, λ) can be defined by a set of cycles. Without loss of generality, we can assume that none of those cycles exclusively involves students assigned to the same school according to the matching μ.[20] However, Lemma 8 does not imply that those cycles form improvement cycles of $G(\mu, \lambda)$. We show that for every fair Pareto improvement of (μ, λ), there is an improvement cycle of $G(\mu, \lambda)$ only involving agents that are in one of the cycles defined in Lemma 8. Applying this observation, Lemma 9 shows that using the SETC algorithms starting at the initial stable extended matching (μ, λ), we can obtain an extended matching that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$.

Lemma 9. *Let (μ, λ) be a stable extended matching. If $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), then there exist a finite sequence of improvement cycles $\{\gamma_1, \dots, \gamma_{t^*}\}$ and an allocation of transferable characteristics $\tilde{\lambda}$ such that:*

- *γ_1 is an improvement cycle of $G(\mu, \lambda)$.*
- *For each $t \in \{2, \dots, t^*\}$, γ_t is an improvement cycle of $G(\gamma_{t-1} \circ \dots \circ \gamma_1 \circ (\mu, \lambda))$.*
- *$(\nu, \tilde{\lambda}) = \gamma_{t^*} \circ \dots \circ \gamma_1 \circ (\mu, \lambda)$.*
- *$(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$.*

Proof. Let $(\nu, \bar{\lambda})$ be a fair Pareto improvement of (μ, λ) and $\Phi^* = \{\phi_1^*, \dots, \phi_m^*\}$ be the set of cycles of students defined in Lemma 8. Since $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ), by Lemma 8, we can construct a set of pairs consisting of disjoint cycles and allocations of transferable characteristics restricted to the students involved in the cycle, $\Pi^* = \left\{(\phi_1^*, \bar{\lambda}\,|_{N(\phi_1^*)}), \dots, (\phi_M^*, \bar{\lambda}\,|_{N(\phi_M^*)})\right\}$. The result is trivial when all the pairs in Π^* are improvement cycles of the graph $G(\mu, \lambda)$. Hence, we focus on the case where some pairs in the set Π^* are not improvement cycles of $G(\mu, \lambda)$. Let $N^* = \cup_{\phi \in \Phi^*} N(\phi)$ be the set of students involved in cycles in Φ^*. We proceed through a series of claims that can be applied iteratively:

- In Claim 1, we prove that for each $j \in N^*$ there is $k \in N^*$ such that $k \in X_{(\mu,\lambda)}(j)$. Therefore, each student in N^* is pointed to by a member of N^* at a link of $G(\mu, \lambda)$.
- In Claim 2, we construct an auxiliary graph $\bar{G}(\mu, \lambda) \subseteq G(\mu, \lambda)$ containing a cycle that defines an improvement cycle γ_1 of $G(\mu, \lambda)$.
- In Claim 3, we show that there is no student preferring the matching obtained solving γ_1 to ν.
- Finally, in Claim 4, we define the allocation of transferable characteristics $\tilde{\lambda}$ such that $(\nu, \tilde{\lambda})$ is a fair Pareto improvement over $\gamma_1 \circ (\mu, \lambda)$.

Claim 1. *For each $\phi \in \Phi^*$ and each $ij \in \phi$, there exists $k \in I$ such that $kj \in G(\mu, \lambda)$ and $k'k \in \phi'$ for some $k' \in I$ and $\phi' \in \Phi^*$.*

Consider an arbitrary $\phi \in \Phi^*$ and $ij \in \phi$. There are two cases:

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[19] Note that since $(\nu, \bar{\lambda})$ is a fair Pareto improvement of (μ, λ) such a student j exists for each $i_x \in N_3$.

[20] It may be the case that some of the cycles defined in the proof of Lemma 8 involve students assigned to the same school according to the initial extended matching (μ, λ). Such a cycle would never be solved at any stage of an algorithm in the SETC class. However, we could construct a extended matching (ν, λ') that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$ by setting $\lambda'(i) = \lambda(i)$ for each student involved in the non-improving cycle.

Case 1. If $i \in X_{(\mu,\lambda)}(j)$, then $ij \in G(\mu,\lambda)$ by construction. Moreover, i is involved in cycle ϕ, which implies there exists $k' \in I$ with $k'i \in \phi' \in \Phi^*$.

Case 2. If $i \notin X_{(\mu,\lambda)}(j)$, there exists a student i' such that $i' \in \tilde{D}_{(\mu,\lambda)}(j)$ and

$$(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i)).$$

Let $k \in \tilde{D}_{(\mu,\lambda)}(j)$ be such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$,

$$(k, \max\{\lambda^{\mu(j)}(k), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i', \max\{\lambda^{\mu(j)}(i'), \lambda^{\mu(j)}(j)\}).$$

Note that this student k exists because school priorities are complete and transitive. Note also that $k \in X_{(\mu,\lambda)}(j)$, and therefore $kj \in G(\mu,\lambda)$. Finally, we check that k is in a cycle in Φ^*. That is, there is $\phi' \in \Phi$ such that $k'k \in \phi'$ for some $k' \in I$. Assume to the contrary that $\mu(k) = \nu(k)$, $\lambda^{\mu(k)}(k) = \bar{\lambda}^{\mu(k)}(k)$, and $\mu(j)\ P_k\ \mu(k) = \nu(k)$. Since $ij \in \phi$, by Lemma 8, $\nu(i) = \mu(j)$ and $\bar{\lambda}^{\mu(j)}(i) \in \{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$. Since $k \in X_{(\mu,\lambda)}(j)$, $i \notin X_{(\mu,\lambda)}(j)$, then $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, which is a contradiction, since $(\nu,\bar{\lambda})$ is a fair Pareto improvement of (μ,λ), and $(\nu,\bar{\lambda})$ is stable. Thus, $\nu(k)\ P_k\ \mu(k)$, which implies that k is in a cycle in Φ^*.

Claim 2. *There is an improvement cycle $\gamma_1 = (\phi_1, \lambda_1)$ in $G(\mu,\lambda)$ such that $N(\phi_1) \subseteq N^*$.*

Construct the directed graph $\bar{G}(\mu,\lambda)$ with set of vertices N^* in the following way. For each $j \in N^*$ there is only one link pointing to j in $\bar{G}(\mu,\lambda)$. Note that by Claim 1, $N^* \cap X_{(\mu,\lambda)}(j) \neq \{\emptyset\}$. Let $i \in N^*$ be the unique student such that $ij \in \phi$ for some $\phi \in \Phi^*$.

i) If $ij \in G(\mu,\lambda)$, then $ij \in \bar{G}(\mu,\lambda)$.
ii) If $ij \notin G(\mu,\lambda)$, let $i^* \in N^* \cap \tilde{D}_{(\mu,\lambda)}(j)$ be the student such that for each $i' \in N^* \cap \tilde{D}_{(\mu,\lambda)}(j)$, $(i^*, \lambda^{\mu(j)}(i^*)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$; and $i^*j \in \bar{G}(\mu,\lambda)$.

That is, each student $j \in N^*$ is pointed to by the student i such that $ij \in \phi$ for some $\phi \in \Phi^*$ whenever $ij \in G(\mu,\lambda)$. Otherwise, j is pointed to by the student in $N^* \cap \tilde{D}_{(\mu,\lambda)}(j)$ with the highest priority at $\mu(j)$.

By Claim 1, for each $kj \in \bar{G}(\mu,\lambda)$, $k \in X_{(\mu,\lambda)}(j)$, and $kj \in G(\mu,\lambda)$. Since N^* is finite and each student in N^* is pointed to by only one other student in N^*, $\bar{G}(\mu,\lambda)$ has at least one cycle. If the cycle of $\bar{G}(\mu,\lambda)$ belongs to Φ^*, then by definition, it is an improvement cycle of $G(\mu,\lambda)$. If the cycle of $\bar{G}(\mu,\lambda)$ does not belong to Φ^*, then there is a pair kj in the cycle such for no $\phi' \in \Phi^*$, $kj \in \phi'$. Since $k \in \tilde{D}_{(\mu,\lambda)}(j)$, $\mu(k) \neq \mu(j)$ and the cycle of $\bar{G}(\mu,\lambda)$ is an improvement cycle of $G(\mu,\lambda)$. Let $\gamma_1 = (\phi_1, \lambda_1 |_{N(\phi^*)})$ be such that ϕ_1 is a cycle of $\bar{G}(\mu,\lambda)$ and $\lambda_1 |_{N(\phi_1)}$ is such that for each $i \in N(\phi_1)$,

- if $ij \in \phi_1$ and $\mu_1(i) = \mu(j) \neq \mu(i)$ then $\lambda_1^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(j), \lambda^{\mu(j)}(i)\}$,
- if $ij \in \phi_1$ and $\mu_1(i) = \mu(j) = \mu(i)$, then $\lambda_1^{\mu(j)}(i) = \lambda^{\mu(j)}(j)$.

By the definition of $\bar{G}(\mu,\lambda)$ and since school priorities are neutral, γ_1 is an improvement cycle of (μ,λ).

Let $(\mu_1,\lambda_1) = \gamma_1 \circ (\mu,\lambda)$. Since (μ_1,λ_1) is the outcome of solving an improvement cycle of $G(\mu,\lambda)$, by Corollary 5, (μ_1,λ_1) Pareto dominates (μ,λ). Hence, we focus on proving that no student prefers the school she is assigned to at μ_1 to $\nu(i)$.

Claim 3. *For each $i \in I$, $\nu(i)\ R_i\ \mu_1(i)\ R_i\ \mu(i)$.*

Let $\gamma_1 = (\phi_1, \lambda_1 |_{N(\phi^*)})$. If $i \notin N(\phi_1)$, $\mu_1(i) = \mu(i)$, since $(\nu,\bar{\lambda})$ Pareto dominates (μ,λ), then $\nu(i)\ R_i\ \mu_1(i)\ R_i\ \mu(i)$. Hence, assume $i \in N(\phi_1)$ and let $j \in N(\phi_1)$ be such that $ij \in \phi_1$. Note that $\mu_1(i) = \mu(j)$. We consider two cases

Case 1. If $ij \in \phi$ for some $\phi \in \Phi$, then $\nu(i) = \mu_1(i) = \mu(j)$.

Case 2. If $ij \notin \phi$ for each $\phi \in \Phi$, we claim that $\nu(i)\ R_i\ \mu(j)$. Suppose that $\mu_1(i) = \mu(j)\ P_i\ \nu(i)$. That is, $i \in \tilde{D}_{(\nu,\bar{\lambda})}(j)$. Since (μ,λ) is stable, and $(\nu,\bar{\lambda})$ is a fair Pareto improvement of (μ,λ), we have that $(\nu,\bar{\lambda})$ is stable. Consider the student $k \in I$ such that $kj \in \phi'$ for some $\phi' \in \Phi$, so $\nu(k) = \mu(j)$. By the definition of γ_1 and $\bar{G}(\mu,\lambda)$, since $ij \in \phi_1$, $ij \in \bar{G}(\mu,\lambda)$, and $kj \in \phi'$, then $k \notin X_{(\mu,\lambda)}(j)$, and also $kj \notin \bar{G}(\mu,\lambda)$. Hence, by the definition of $\bar{G}(\mu,\lambda)$, $(i, \lambda^{\mu(j)}(i)) \succ_{\mu(j)} (k, \max\{\lambda^{\mu(j)}(j), \lambda^{\mu(j)}(k)\})$, which is a contradiction because $\bar{\lambda}^{\mu(j)}(i) \in \{\lambda^{\mu(j)}(j), \lambda^{\mu(j)}(k)\}$, and $(\nu,\bar{\lambda})$ is stable.

Thus, each student j involved in γ_1 weakly prefers $\nu(j)$ to $\mu_1(j)$ to $\mu(j)$. Each remaining student is assigned to the same school to which she is assigned under μ which implies that the matching (μ_1,λ_1) Pareto dominates (μ,λ) and either $\mu_1 = \nu$, or $(\nu,\bar{\lambda})$ Pareto dominates (μ_1,λ_1).

Claim 4. *There is an extended matching $(\nu,\tilde{\lambda})$ that is a fair Pareto improvement of (μ_1,λ_1) and $(\nu,\tilde{\lambda})$ is a characteristic-wise equivalent to $(\nu,\bar{\lambda})$.*

The result is immediate if $\gamma_1 \in \Pi^*$. Hence assume that $\gamma_1 \notin \Pi^*$. Construct an allocation of transferable characteristics $\tilde{\lambda}$ in such a way that

- For each $i \notin N^*$, $\tilde{\lambda}(i) = \bar{\lambda}(i)$.
- Let $i \in N^*$ and $ij \in \phi$ for some $\phi \in \Phi^*$.
  - If there is $k \in N^*$ with $kj \in \phi_1$, then $\tilde{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda_1^{\mu(j)}(k)\}$ and $\tilde{\lambda}^{\mu(j)}(k) = \{\lambda^{\mu(j)}(i), \lambda_1^{\mu(j)}(k)\} \setminus \max\{\lambda^{\mu(j)}(i), \lambda_1^{\mu(j)}(k)\}$.
  - Otherwise, $\tilde{\lambda}^s(i) = \bar{\lambda}^s(i) = \lambda_1^s(i)$.

Note that for each $i \in I$ and $s \in S$, if $s\ P_i\ \nu(i)$, $\lambda^s(i) = \bar{\lambda}^s(i) = \tilde{\lambda}^s(i)$. Additionally, since priorities are neutral, for each $i \in I$ $(i, \tilde{\lambda}^{\nu(i)}(i)) \succsim_{\nu(i)} (i, \bar{\lambda}^{\nu(i)}(i))$. Hence, for each student i the differences between $\bar{\lambda}(i)$ and $\tilde{\lambda}(i)$ are restricted to the transferable characteristics of schools that are worse than i's final match, and to obtain a transferable characteristic that raises i's priority at her final match. Thus, $(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$.

By Claim 3, $(\nu, \tilde{\lambda})$ Pareto dominates (μ_1, λ_1). Since $(\nu, \bar{\lambda})$ is stable and $(\nu, \tilde{\lambda})$ is characteristic-wise equivalent to $(\nu, \bar{\lambda})$, $(\nu, \tilde{\lambda})$ is stable. By definition of $\tilde{\lambda}$ for each $i \in I$ and $s \notin \{\mu_1(i), \nu(i)\}$, $\lambda_1^s(i) = \tilde{\lambda}^s(i)$. Thus, $(\nu, \tilde{\lambda})$ is a fair Pareto improvement of (μ_1, λ_1).

We now conclude the proof of Lemma 9. The result is immediate if all the elements of Π^* appear in $G(\mu, \lambda)$. In that case, the elements of Π^* are improvement cycles of $G(\mu, \lambda)$ that involve disjoint sets of students. Solving the improvement cycle in an arbitrary order yields $(\nu, \bar{\lambda})$. Hence, assume to the contrary that no pair in Π^* is an improvement cycle of $G(\mu, \lambda)$. This assumption is without loss of generality because of the following observation. If a pair $\left(\phi, \bar{\lambda}\mid_{N(\phi)}\right) \in \Pi^*$ is an improvement cycle of $G(\mu, \lambda)$, then this improvement cycle is solved first. Since all the pairs in Π^* involve disjoint sets of students and whenever two students are forming a link in $G(\mu, \lambda)$, and those students are not involved in the cycle $\phi \in \Phi^*$, then the link also appears in $G((\phi, \bar{\lambda}\mid_{N(\phi)}) \circ (\mu, \lambda))$. Following this logic, whenever a subset of cycles Φ^* appear in $G(\mu, \lambda)$, these cycles are solved first, until no improvement cycle of $G(\mu, \lambda)$ remains. In that case, by Claim 2, we can find an improvement cycle in $G(\mu, \lambda)$ involving only students in N^*. By Claim 3, the extended matching obtained solving any such improvement cycle, (μ_1, λ_1), is not Pareto dominated by $(\nu, \bar{\lambda})$. By Claim 4, (μ_1, λ_1) admits a fair Pareto improvement $(\nu, \tilde{\lambda})$ that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$. We can repeat the argument solving improvement cycles until we obtain an extended matching that is characteristic-wise equivalent to $(\nu, \bar{\lambda})$. □

Lemma 10. *Let (μ, λ) be a stable extended matching, and let ϕ be a cycle of $G(\mu, \lambda)$ and let $\left(\phi, \lambda_1\mid_{N(\phi)}\right)$, $\left(\phi, \hat{\lambda}_1\mid_{N(\phi)}\right)$ be two improvement cycles of $G(\mu, \lambda)$ such that for each $i \in I$, $(i, \lambda_1^{\mu_1(i)}(i)) \succsim_{\mu_1(i)} (i, \hat{\lambda}_1^{\mu_1(i)}(i))$. Let $(\mu_1, \lambda_1) = \left(\phi, \lambda_1\mid_{N(\phi)}\right) \circ (\mu, \lambda)$ and $(\mu_1, \hat{\lambda}_1) = \left(\phi, \hat{\lambda}_1\mid_{N(\phi)}\right) \circ (\mu, \lambda)$. If there is a cycle ϕ' in $G(\mu_1, \hat{\lambda}_1)$; then ϕ' is a cycle of $G(\mu_1, \lambda_1)$.*

Proof. Note that for each $i \in I$ with $\hat{\lambda}_1^{\mu_1(i)}(i) \in \{\lambda^{\mu_1(i)}(i), \lambda^{\mu_1(i)}(j)\}$ for some j with $\mu_1(i) = \mu(j)$. For each i and each j' such that $\mu_1(j')\ P_i\ \mu_1(i)$, $\hat{\lambda}_1^{\mu_1(j')}(i) = \lambda_1^{\mu_1(j')}(i)$. If a student $j'' \in X_{(\mu_1, \hat{\lambda})}(i)$, then also $j'' \in X_{(\mu_1, \lambda_1)}(i)$. Since priorities are neutral and $k \in X_{(\mu_1, \hat{\lambda})}(i) \setminus \{j'\}$, $(j', \lambda_1^{\mu_1(i)}(i)) \succsim_{\mu_1(i)} (j', \hat{\lambda}_1^{\mu_1(i)}(i)) \succsim_{\mu_1(i)} (k, \lambda_1^{\mu_1(i)}(k))$. Hence, if there is an improvement cycle involving a cycle ϕ in $G(\mu_1, \hat{\lambda}_1)$, then $G(\mu_1, \lambda_1)$ admits an improvement cycle involving the cycle ϕ. □

Proof of Proposition 4. Let $(\nu, \bar{\lambda})$ be a justifiable Pareto improvement of (μ, ω). There is a sequence of extended matchings, $\{(\mu'_0, \lambda'_0), (\mu'_1, \lambda'_1), \ldots, (\mu'_{t*}, \lambda'_{t*})\}$ such that improvements that $(\mu'_0, \lambda'_0) = (\mu, \lambda)$, $(\mu'_{t*}, \lambda'_{t*}) = (\mu', \lambda')$ and for each $t \in \{1, \ldots, t^*\}$, (μ'_t, λ'_t) is a fair Pareto Improvement of $(\mu'_{t-1}, \lambda'_{t-1})$. By Lemma 9, the application of an SETC algorithm starting at (μ, λ) yields after a finite number of steps an extended matching (μ'_1, λ^*_1) that is characteristic-wise equivalent to (μ'_1, λ'_1). By the argument of Claim 4, in Lemma 9, we can construct an extended matching characteristic-wise equivalent to (μ'_2, λ'_2) such that is a fair Pareto improvement of (μ'_1, λ^*_1). By Lemma 10, if there is an improvement cycle in $G(\mu'_1, \lambda_1)$, then there is an improvement cycle in $G(\mu'_1, \lambda^*_1)$ involving the same cycle of students. Repeating the argument, we obtain the result. □

Proof of Theorem 2. Let (μ_t, λ_t) be the extended matching obtained after a series of t steps of the application of an SETC algorithm such that (μ_t, λ_t) is characteristic-wise equivalent to (μ', λ'). Either the algorithm stops at step t and the (μ_t, λ_t) is constrained efficient, or (μ_t, λ_t) admits a fair Pareto improvement and the outcome of the SETC algorithm Pareto dominates (μ', λ'). □

6.3. Proof of the remaining results

Proof of Proposition 1. Let A be an algorithm in the SETC class. Define the SOTC mechanism Ψ that for each profile of students' preferences selects the matching obtained through the application of A at that preference profile. By Corollary 1, the extended matching selected by Ψ is stable and constrained efficient for each preference profile. For each $P \in \mathcal{P}$, if $\Psi(P) = (\mu, \lambda)$ then either $\Psi(P)$ is the SOSEM or $\Psi(P)$ Pareto dominates the SOSEM. Hence, by Abdulkadiroğlu et al. (2009, Theorem 1), Ψ violates *strategy-proofness*. □

Proof of Lemma 1. Let $s = \mu(j)$. Since $i \in \tilde{D}_{(\mu,\lambda)}(j)$, we have that $s\ P_i\ \mu(i)$. Since (μ, λ) is stable, for each $j' \neq i$ such that $s\ P_{j'}\ \mu(j')$, $(j, \lambda^s(j)) \succ_s (j', \lambda^s(j'))$. Therefore, since priorities are fully transferable, we have $(i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (j', \lambda^s(j'))$ and $i \in X_{(\mu,\lambda)}(j)$. □

Proof of Proposition 3. We start the proof by describing an algorithm that yields the EADA matching in problems without transferable characteristics and the SETC algorithm that translates it to the extended matching framework

(Simplified) Efficiency Adjusted Deferred Acceptance Algorithm (EADA). Tang and Yu (2014):

Given a matching μ, a school s is ***underdemanded at*** μ if no student prefers s to the school to which they are assigned by μ. The simplified EADA algorithm executes the student-proposing DA algorithm iteratively after sequentially altering the preferences of students assigned to underdemanded schools. Starting with the SOSEM, as a first step, the student-proposing DA algorithm is executed a second time with the students previously assigned to underdemanded schools listing those schools as their top choices. Therefore, in this second stage, students at underdemanded schools retain their seats, and their potential priorities at schools where they cannot obtain seats become ineffective. This process is repeated until there are no underdemanded schools.

Next, we propose the EADA-SETC algorithm, a specific SETC algorithm that under transferable priorities selects the matching obtained by the (simplified) EADA algorithm. The successive selection of cycles utilized by this EADA-SETC algorithm requires: identifying the students assigned to underdemanded schools, dropping the potential cycles involving those students, and of the remaining cycles, solving those that would satisfy the school priorities under the initial allocation of transferable characteristics for the students who are not assigned to underdemanded schools first. This process is equivalent to running the student-proposing DA algorithm when students assigned to underdemanded schools report that those underdemanded schools are their preferred alternative. Running this process as many times as necessary yields a constrained efficient extended matching with the matching selected by the EADA algorithm.

EADA-SETC Algorithm:

Step *0*. Let $(\mu_0, \lambda_0) = (\mu^{SO}_\omega, \omega)$, $I_0 = I$, and let $U_0 = \left\{s \in S : \text{ for each } j \in I,\ \mu_0(j)\ R_j\ s\right\}$, be the set of underdemanded schools at μ_0.

Step *t*≥*1*. Given $(\mu_{t-1}, \lambda_{t-1})$:

Stage *t.0*. If $\cup_{\tau=0}^{t-1} U_\tau = S$, the algorithm stops and $(\mu_{t-1}, \lambda_{t-1})$ is the outcome. If $\cup_{\tau=0}^{t-1} U_\tau \neq S$, let $(\mu_t^0, \lambda_t^0) = (\mu_{t-1}, \lambda_{t-1})$, $I_t = I \setminus \left\{i \in I :\ \mu_{t-1}(i) \in \left(\cup_{\tau=0}^{t-1} U_\tau\right) \cup \varnothing\right\}$, and move to stage $t.1$.

Stage *t.t'* (*t'*≥*1*). For each extended matching (μ, λ), let the graph $G_t(\mu, \lambda)$ be such that for each $i, j \in I$, $ij \in G_t(\mu, \lambda)$ if and only if $i, j \in I_t$, $\mu(j)\ P_i\ \mu(i)$, and for each $i' \in \tilde{D}(\mu, \lambda)(j) \cap I_t$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$.

- If there is one or more cycles at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, solve one of the cycles at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, for example, γ; let $(\mu_t^{t'}, \lambda_t^{t'}) = \gamma \circ (\mu_t^{t'-1}, \lambda_t^{t'-1})$ and move to Stage $t.(t'+1)$.
- If there is no cycle at $G_t(\mu_t^{t'-1}, \lambda_t^{t'-1})$, let $(\mu_t, \lambda_t) = (\mu_t^{t'-1}, \lambda_t^{t'-1})$, and let

$$U_t = \left\{s \in S \setminus \left(\cup_{\tau=0}^{t-1} U_\tau\right) :\ \text{for each } i \in I_{t-1},\ \mu_t(j)\ R_j\ s\right\},$$

and move to step $t+1$.

Note that for each step $t \geq 1$, $\cup_{\tau=0}^{t-1} U_\tau$ is the set of underdemanded schools at μ_{t-1}, and I_t are the set of students who are not assigned to underdemanded schools at μ_{t-1}.

We now check that for each school choice problem with transferable characteristics and fully transferable priorities the extended matching obtained by the EADA-SETC algorithm selects the EADA matching.

Note first that by Lemma 1, for each $i, j \in I$, and for each natural number $t \in \mathbb{N}$, $ij \in G_t(\mu, \lambda)$ implies $ij \in G(\mu, \lambda)$. Thus, since (μ_0, λ_0) is stable, by Corollary 1, $(\mu_t^{t'}, \lambda_t^{t'})$ is also stable. Since for each t, t', and $j \in I_t$, there is at most another student i such that $ij \in G_t(\mu_t^{t'}, \lambda_t^{t'})$. This fact implies that for each t, t', all the cycles in $G_t(\mu_t^{t'}, \lambda_t^{t'})$ are disjoint, that is, iff ϕ and ϕ' are cycles in $G_t(\mu_t^{t'}, \lambda_t^{t'})$, then $\phi \cap \phi' = \varnothing$.

Moreover, if at some t, $\cup_{\tau=0}^{t-1} U_\tau = S$, by an argument similar to those in the proof of Lemma 7 and since priorities are fully transferable, then $(\mu_{t-1}, \lambda_{t-1})$ does not admit any improvement cycle, and no extended matching (μ', λ') Pareto dominates $(\mu_{t-1}, \lambda_{t-1})$. Hence, the algorithm selects a constrained efficient extended matching.

If $U_0 = S$, then every student is assigned to her best preferred school, and the algorithm stops immediately, $(\mu^{SO}_\omega, \omega)$ is constrained efficient, and μ^{SO}_ω coincides with the outcome of the EADA algorithm.

If $U_0 \neq S$, note that (μ_0, λ_0) is stable but not necessarily constrained efficient. We prove the result by comparing the graph $G_1(\mu_0.\lambda_0)$ defined at step 1 of the EADA-SETC algorithm with the directed application graph associated with (μ_0, λ_0) obtained for an alternative school choice problem for particular student preferences and school priorities. Consider the school choice problem with transferable characteristics $(I, S, R^*, q, \omega, \succsim^*)$, such that for each $i \in I_1$, $R_i^* = R_i$ and for each $j \notin I_1$ R_j^* is such that for each $s \in S \setminus \{\mu_0(j)\}$, $\mu_0(j)\ P_j^*\ s$, and for each $i, j \in I$ and each allocation of transferable characteristics λ, $(i, \lambda^s(i)) \succsim_s^* (j, \lambda^s(j))$ if and only if $(i, \omega^s(i)) \succsim_s^* (j, \omega^s(j))$. That is, students assigned to underdemanded schools under μ_0 consider that school as the best possible alternative, and school priorities are defined on the initial allocation of transferable characteristics. For each extended matching (μ, λ), let us denote by $G^*(\mu, \lambda)$ the directed application graph associated with (μ, λ) for the problem $(I, S, R^*, q, \omega, \succsim^*)$ Note that $G^*(\mu_0, \lambda_0)$ coincides with $G_1(\mu_0, \lambda_0)$. By Theorem 1 and Corollary 1, starting with a stable extended matching, the EADA-SETC algorithm yields a constrained efficient extended matching. Note that under the new student preferences and school priorities, since the transferable characteristics are irrelevant, the student-proposing DA algorithm yields the unique constrained efficient matching (see Gale and Shapley, 1962). This fact also implies that the order in which the cycles are solved at any stage $1.t$ is irrelevant and a unique extended matching (μ_1, λ_1) is obtained, and μ_1 coincides with the matching of the SOSEM for the school choice problem with student preference profile R^*.

We can iteratively repeat the argument as many times as necessary for each $t \geq 1$, and $G^*(\mu_t, \lambda_t)$ coincides with $G_t(\mu_t, \lambda_t)$, until for some $t \geq 0$, $U_t = \cup_{\tau=0}^{t-1} U_\tau = S$, which completes the proof. □

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Data availability

No data was used for the research described in the article.

References

Abdulkadiroğlu, A., Che, Y.K., Pathak, P., Roth, A.E., Tercieux, O., 2020. Efficiency, justified envy, and incentives in priority-based matching. Am. Econ. Rev. Insights 2–4, 425–442.
Abdulkadiroğlu, A., Pathak, P., Roth, A.E., 2009. Strategy-proofness versus efficiency in matching with indifferences: redesigning the NYC high school match. Am. Econ. Rev. 99 (5), 1954–1978.
Abdulkadiroğlu, A., Pathak, P., Roth, A.E., Sönmez, T., 2005. The Boston public school match. Am. Econ. Rev. 95 (2), 368–371.
Abdulkadiroğlu, A., Sönmez, T., 2003. School choice: a mechanism design approach. Am. Econ. Rev. 93 (3), 729–747.
Alcalde, J., Romero-Medina, A., 2017. Fair student placement. Theory Decis. 83, 293–307.
Alva, S., Manjunath, V., 2019. Strategy-proof Pareto improvements. J. Econ. Theory 181, 121–142.
Arnosti, N., 2016. Centralized Clearinghouse Design: a Quantity-Quality Trade-off. Working Paper. Columbia Business School.
Ashlagi, I., Nikzad, A., Romm, A., 2019. Assigning more students to their top choices: a comparison of tie-breaking rules. Games Econ. Behav. 115, 167–187.
Balinski, M., Sönmez, T., 1999. A tale of two mechanisms: student placement. J. Econ. Theory 84, 73–94.
Blum, Y., Roth, A.E., Rothblum, U., 1997. Vacancy chains and equilibration in senior-level labor markets. J. Econ. Theory 76, 362–411.
Casalmiglia, C., Fu, C., Güell, M., 2020. Structural estimation of a model of school choices: the Boston mechanism vs. its alternatives. J. Polit. Econ. 128 (2), 642–680.
Combe, J., 2022. Matching with ownership. J. Math. Econ. 98, 102563.
Doğan, B., Ehlers, L., 2021. Minimally unstable Pareto improvements over deferred acceptance. Theor. Econ. 16, 1249–1279.
Doğan, B., Ehlers, L., 2022. Robust minimal instability of the top trading cycles mechanism. Am. Econ. J. Microecon. 14 (4), 556–582.
Duddy, C., 2019. The structure of priority in the school choice problem. Econ. Philos. 35, 361–381.
Dur, U.M., Gitmez, A., Yılmaz, Ö., 2019. School choice under partial fairness. Theor. Econ. 14 (4), 1309–1346.
Dur, U.M., Kominers, S.D., Pathak, P.A., Sönmez, T., 2018. Reserve design: unintended consequences and the demise of Boston's walk zones. J. Polit. Econ. 126 (6), 2457–2479.
Dur, U.M., Morrill, T., 2017. The Impossibility of Restricting Tradeable Priorities in School Assignment. Working Paper. North Carolina State University.
Ehlers, L., Morrill, T., 2020. (Il)legal assignments in school choice. Rev. Econ. Stud. 87, 1837–1875.
Erdil, A., Ergin, H., 2008. What's the matter with tie-breaking? Improving efficiency in school choice. Am. Econ. Rev. 98 (3), 669–689.
Gale, D., Shapley, L., 1962. College admissions and the stability of marriage. Am. Math. Mon. 69, 9–15.
Górtazar, L., Mayor, D., Montalbán, J., 2023. Residence-based priorities and school choice. Econ. Educ. Rev. 95, 102384.
Hakimov, R., Kesten, O., 2018. The equitable top trading cycles mechanism for school choice. Int. Econ. Rev. 59 (4), 2219–2258.
Kesten, O., 2010. School choice with consent. Q. J. Econ. 125, 1297–1348.
Kesten, O., Kurino, M., 2019. Strategy-proof improvements upon deferred acceptance: a maximal domain for possibility. Games Econ. Behav. 117, 120–143.
Kitahara, M., Okumura, Y., 2021. Improving efficiency in school choice under partial priorities. Int. J. Game Theory 50, 971–987.
Klaus, B., Klijn, F., 2021. Minimal-Access Rights in School Choice and the Deferred Acceptance Mechanism. Barcelona GSE Working Paper Series n. 1264.
Pathak, P., 2016. What really matters in designing school choice mechanisms. In: Advances in Economics and Econometrics, 11th World Congress of the Econometric Society.
Reny, P., 2022. Efficient matching in the school choice problem. Am. Econ. Rev. 112 (6), 2025–2043.
Ruijs, N., Oosterbeek, H., 2019. School choice in Amsterdam: which schools are chosen when school choice is free? Educ. Finance Policy 14 (1), 1–30.
Shapley, L., Scarf, H., 1974. On cores and indivisibility. J. Math. Econ. 1, 23–37.
Tang, Q., Yu, J., 2014. A new perspective on Kesten's school choice with consent idea. J. Econ. Theory 154, 543–561.
Tang, Q., Zhang, Y., 2021. Weak stability and Pareto efficiency in school choice. Econ. Theory 71, 533–552.
Troyan, P., Delacrétaz, D., Kloosterman, A., 2020. Essentially stable matching. Games Econ. Behav. 120, 370–390.
Van der Spiegel, S., Schröder-Bäck, P., Brand, H., 2020. Organ transplantation and the European Union, 2009–2015 developments. Transpl. Int. 33, 603–611.