On the topological classification of starlike bodies in Banach spaces

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Abstract

Starlike bodies are interesting in nonlinear analysis because they are strongly related to polynomials and smooth bump functions, and their topological and geometrical properties are therefore worth studying. In this note we consider the question as to what extent the known results on topological classification of convex bodies can be generalized for the class of starlike bodies, and we obtain two main results in this line, one which follows the traditional Bessaga–Klee scheme for the classification of convex bodies (and which in this new setting happens to be valid only for starlike bodies whose characteristic cones are convex), and another one which uses a new classification scheme in terms of the homotopy type of the boundaries of the starlike bodies (and which holds in full generality provided the Banach space is infinite-dimensional).

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A closed subset A of a Banach space X is said to be a starlike body if there exists a point x_0 in the interior of A such that every ray emanating from x_0 meets ∂A, the boundary of A, at most once. Up to a suitable translation, we can always assume (and we will do so)

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that $x_0 = 0$ is the origin of X. For a starlike body A, we define the characteristic cone of A as

$$ccA = \{x \in X \mid rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A as

$$\mu_A(x) = \inf\left\{\lambda > 0 \,\middle|\, \frac{1}{\lambda}x \in A\right\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional μ_A is a continuous function which satisfies $\mu_A(rx) = r\mu_A(x)$ for every $r \geqslant 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X \mid \mu_A(x) \leqslant 1\}$, and $\partial A = \{x \in X \mid \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(\lambda x) = \lambda\psi(x)$ for all $\lambda \geqslant 0$, then $A_\psi = \{x \in X \mid \psi(x) \leqslant 1\}$ is a starlike body. More generally, for a continuous function $\psi : X \to [0, \infty)$ such that $\psi_x(\lambda) = \psi(\lambda x)$, $\lambda > 0$, is increasing and $\sup\{\psi_x(\lambda) \colon \lambda > 0\} > \varepsilon$ for every $x \in X \setminus \psi^{-1}(0)$, the set $\psi^{-1}([0, \varepsilon])$ is a starlike body whose characteristic cone is $\psi^{-1}(0)$.

A familiar important class of starlike bodies are *convex bodies*, that is, starlike bodies that are convex. For a convex body U, ccU is always a convex set, but in general the characteristic cone of a starlike body is not convex.

We will say that A is a C^p smooth starlike body provided its Minkowski functional μ_A is C^p smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. A starlike body A is said to be Lipschitz provided its Minkowski functional μ_A is a Lipschitz function. Finally, two (smooth) starlike bodies A, B in a Banach space X are relatively homeomorphic (relatively diffeomorphic) whenever there is a self-homeomorphism (diffeomorphism) $g : X \to X$ so that $g(A) = B$. It is clear that "being relatively homeomorphic" (respectively diffeomorphic) endows the set of starlike bodies of a Banach space with an equivalence relationship.

Starlike bodies often appear in nonlinear functional analysis as natural substitutes of convex bodies or in connection with bump functions and with polynomials; more precisely, for every n-homogeneous polynomial $P : X \to \mathbb{R}$ the set $\{x \in X \mid P(x) \leqslant c\}$, $c > 0$, is either a (real-analytic) starlike body or its complement is the interior of such a body (see [4]). It is therefore reasonable to ask to what extent the geometrical properties of convex bodies are shared with the more general class of starlike bodies. In [4] the question of whether James' theorem on the characterization of reflexivity (one of the deepest classical results of functional analysis) is true for starlike bodies was answered in the negative. In [3] it was shown that the boundary of a smooth Lipschitz bounded starlike body in an infinite-dimensional Banach space is smoothly Lipschitz contractible; furthermore, the boundary is a smooth Lipschitz retract of the body. Here, we deal with the question as to what extent the known results on the topological classification of convex bodies can be generalized for the class of starlike bodies.

It was Klee [18] that first gave a topological classification of the convex bodies of a Hilbert space. This result was generalized for every Banach space with the help of Bessaga's non-complete norm technique (see the book by Bessaga and Pelczynski [8], Chapters III and V). To get a better insight in the history of the topological classification of convex bodies the reader should have a look at the papers by Stocker [22], Corson and

Klee [10], Bessaga and Klee [6,7], and Dobrowolski [13]. These results have recently been sharpened to get a full classification of the C^p smooth convex bodies of every Banach space [5]. In its most general form the result on a classification of (smooth) convex bodies reads as follows (see [5]); here, as in the whole paper, $p = 0, 1, 2, \ldots, \infty$, and "$C^0$ diffeomorphic" means just "homeomorphic".

Theorem 1. *Let U be a C^p convex body in a Banach space X.*

(a) *If ccU is a linear subspace of finite codimension (say $X = ccU \oplus Z$, with Z finite-dimensional), then U is C^p relatively diffeomorphic to $ccU + B_Z$, where B_Z is an Euclidean ball in Z.*
(b) *If ccU is not a linear subspace or ccU is a linear subspace such that the quotient space X/ccU is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space (that is, $\{x \in X \mid x^*(x) \geqslant 0\}$, for some $x^* \in X^*$).*

Our aim in this paper is to discuss to what extent this result can be generalized for (smooth) starlike bodies. The following example shows that part (b) of Theorem 1 is not true for starlike bodies whose characteristic cones are not convex sets.

Example 2. Let $A = \{(x, y) \in \mathbb{R}^2\colon\ |xy| \leqslant 1\}$. It is plain that A is a starlike body in the plane $\mathbb{R}^2$, and its characteristic cone is the pair of lines defined by the equation $xy = 0$. Then A cannot be relatively diffeomorphic (not even relatively homeomorphic) to a half-plane of $\mathbb{R}^2$. Indeed, ∂A is not connected, while the boundary of a closed half-plane (that is to say, a line) is always connected. Similar examples show that for every $n \in \mathbb{N}$ there exists a starlike body A_n in the plane $\mathbb{R}^2$ such that ∂A_n has exactly n connected components. Hence A_n is not relatively homeomorphic to A_m whenever $n \neq m$.

However, it seems natural to think that every two (smooth) starlike bodies with the same characteristic cone should be diffeomorphic. This is indeed true and it is a fact that, though elementary, will help us to unravel the tangle of starlike bodies and get a first generalization of Theorem 1. Let us state and prove this fact.

Proposition 3. *Let X be a Banach space, and let A_1, A_2 be C^p smooth starlike bodies such that $ccA_1 = ccA_2$. Then there exists a C^p diffeomorphism $g : X \to X$ such that $g(A_1) = A_2$, $g(\partial A_1) = \partial A_2$, and $g(0) = 0$. Moreover, $g(x) = \eta(x)x$, where $\eta : X \to [0, \infty)$, and hence g preserves the rays emanating from the origin.*

Proof. First of all let us see that the statement is true if we make the additional assumption that $A_1 \subseteq A_2$. So, let us suppose that A and B are starlike bodies such that the origin is an interior point of both A and B, $ccA = ccB$, and $A \subseteq B$ (so that $\mu_B(x) \leqslant \mu_A(x)$ for every x, where μ_A and μ_B are the Minkowski functionals of A and B, respectively), and see that there exists a C^p diffeomorphism $g : X \to X$ such that $g(A) = B$, $g(0) = 0$, and $g(\partial A) = \partial B$.

Let $\lambda(t)$ be a non-decreasing real function of class C^∞ defined for $t > 0$, such that $\lambda(t) = 0$ for $t \leqslant 1/2$ and $\lambda(t) = 1$ for $t \geqslant 1$. Let

$$g(x) = \left[\lambda\big(\mu_A(x)\big)\frac{\mu_A(x)}{\mu_B(x)} + 1 - \lambda\big(\mu_A(x)\big)\right]x$$

for $x \notin ccA$, and $g(x) = x$ whenever $\mu_B(x) = 0$. It is clear that g is a C^p smooth mapping. Let $y \notin ccA$ be an arbitrary vector of X and put

$$G_y(t) = \left[\lambda\big(t\mu_A(y)\big)\frac{\mu_A(y)}{\mu_B(y)} + 1 - \lambda\big(t\mu_A(y)\big)\right]t$$

for $t > 0$. Note that $G_y(t)$ is strictly increasing and satisfies $\lim_{t\to 0^+} G_y(t) = 0$, and $\lim_{t\to\infty} G_y(t) = \infty$. This implies that for every $y \in X \setminus ccA$ a number $t(y) > 0$ such that $G_y(t(y)) = 1$ is uniquely determined, which means that g is a one-to-one mapping from $X \setminus ccA$ onto $X \setminus ccA$, with $g^{-1}(y) = t(y)y$. It is also clear that g fixes all the points in ccA, so that g is a bijection from X onto X. Let us define $\Phi : (X \setminus ccA) \times (0, \infty) \to \mathbb{R}$ by

$$\Phi(y,t) = \left[\lambda\big(t\mu_A(y)\big)\frac{\mu_A(y)}{\mu_B(y)} + 1 - \lambda\big(t\mu_A(y)\big)\right]t.$$

Taking into account that $\mu_B(x) \leqslant \mu_A(x)$ and λ is non-decreasing, one can easily check that $\frac{\partial\Phi}{\partial t}(y,t) \geqslant 1 > 0$. Then, using the implicit function theorem we obtain that $y \to t(y)$ is a C^p smooth function on $X \setminus ccA$, and therefore so is g^{-1}. On the other hand, from the definition above it is clear that the map g restricts to the identity on a neighbourhood of the cone ccA, and hence both g and g^{-1} are C^p smooth on the whole of X. Thus, g is a C^p diffeomorphism from X onto X, and it is obvious that g transforms the body $A = \{x \in X \mid \mu_A(x) \leqslant 1\}$ onto $B = \{x \in X \mid \mu_B(x) \leqslant 1\}$, and its boundary $\partial A = \{x \in X \mid \mu_A(x) = 1\}$ onto $\partial B = \{x \in X \mid \mu_B(x) = 1\}$.

Now let us consider the general case. Let $A = \{x \in X \mid \mu_{A_1}(x) + \mu_{A_2}(x) \leqslant 1\}$, which is a C^p smooth starlike body satisfying $ccA = ccA_j$ and $A \subseteq A_j$, for $j = 1, 2$. From the first part of the proof we know that there exist self-diffeomorphisms of X, g_1 and g_2, such that $g_j(A) = A_j$ and $g_j(\partial A) = \partial A_j$, $j = 1, 2$. Then, if we put $g = g_2 \circ g_1^{-1}$, we get a self-diffeomorphism of X transforming A_1 onto A_2 and ∂A_1 onto ∂A_2. □

As said above, one cannot dream of extending part (b) of Theorem 1 to the class of general starlike bodies. The complexity of the characteristic cones of (unbounded) starlike bodies really makes a difference that forces us to devise a new classification scheme suitable for all starlike bodies, whatever their characteristic cones may be. If one wants to stick to the Bessaga–Klee classification scheme then the best result one can aim at is that Theorem 1 still holds for the class of starlike bodies whose characteristic cones are *convex* sets.

We will next state and prove such a result, but first we will need to establish the following proposition, which might be of independent interest (beyond the classification problem) in the theory of smoothness in Banach spaces, and which tells us that every proper closed convex cone C in a separable Banach space can regarded both as the characteristic cone of some C^∞ smooth convex body and as the set of zeros of a C^∞

smooth convex function. We say that a nonempty subset C of a Banach space X is a *cone* (respectively, a cone over a set $K \subset S$, where S is the unit sphere of X) provided $[0,\infty)C = C$ (respectively, $C = [0,\infty)K$). The cone C is proper if $C \neq X$.

Proposition 4. *For every closed convex set C in a separable Banach space X there exists a C^∞ smooth convex function $f : X \to [0,\infty)$ so that $f^{-1}(0) = C$. Moreover, when C is a cone, $U = f^{-1}([0,1])$ is a C^∞ smooth convex body in X so that $ccU = C$.*

Proof. We may obviously assume that $\emptyset \neq C \neq X$. It is well known that, as a consequence of the Hahn–Banach theorem, every such closed convex set C is the intersection of the half-spaces of X which contain C, that is,

$$C = \bigcap_{i\in I} H_i,$$

where the H_i can be assumed to be of the form $H_i = \{x \in X\colon\ x_i^*(x) \leqslant \alpha_i\}$ for some $x_i^* \in X^*$ with $\|x_i^*\| = 1$, and $\alpha_i \in \mathbb{R}$. Then we have that $X \setminus C = \bigcup_{i\in I}(X \setminus H_i)$, and since the complements $X \setminus H_i$ are open and $X \setminus C$ is a Lindelöf space, there exists a countable subcovering

$$X \setminus C = \bigcup_{n=1}^{\infty}(X \setminus H_n),$$

where the $H_n = \{x \in X\colon\ x_n^*(x) \leqslant \alpha_n\}$ form a subsequence of the family $(H_i)_{i\in I}$. Therefore, we can write C as a countable intersection of closed half-spaces,

$$C = \bigcap_{n=1}^{\infty}\bigl\{x \in X\colon\ x_n^*(x) \leqslant \alpha_n\bigr\}. \tag{1}$$

Now, let $\theta : \mathbb{R} \to [0,\infty)$ be a C^∞ smooth *convex* function so that $\theta(t) = 0$ for $t \leqslant 0$, and $\theta(t) > 0$ whenever $t > 0$; we can even demand that $\theta(t)$ be an affine function of slope 1 for $t \geqslant 1$, say $\theta(t) = t + b$ for $t \geqslant 1$, where $-1 < b < 0$. It is easy to construct such a function θ by integrating twice a suitable C^∞ smooth nonnegative function whose support is precisely the interval $[0,1]$. Define then $\theta_n : \mathbb{R} \to [0,\infty)$ by

$$\theta_n(t) = \theta(t - \alpha_n);$$

clearly θ_n is a C^∞ smooth convex function so that θ_n vanishes precisely on the interval $(-\infty, \alpha_n]$, and θ_n restricts to an affine function on $[\alpha_n + 1, \infty)$, namely $\theta_n(t) = t - \alpha_n + b$ for $t \geqslant \alpha_n + 1$.

Let us define our function $f : X \to [0,\infty)$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{\theta_n(x_n^*(x))}{(1 + |\alpha_n|)2^n}$$

for all $x \in X$. It is clear that f is a convex function. Let us see that f is well defined and C^∞ smooth. We can write f as a function series, $f(x) = \sum_{n=1}^{\infty} f_n(x)$, where

$$f_n(x) = \frac{\theta_n(x_n^*(x))}{(1 + |\alpha_n|)2^n}.$$

In order to see that f is C^∞ smooth it is enough to check that the series of derivatives $\sum_{n=1}^\infty f_n^{(j)}(x)$ converges uniformly on each ball $B(0,R)$, with $R>1$, for all $j=0,1,2,\dots$. Since the derivatives of the function θ are all bounded and θ_n is a mere translation of θ, there are constants $M_j>0$ so that $\|\theta_n^{(j)}\|_\infty=\|\theta^{(j)}\|_\infty=M_j$ for all $j=1,2,\dots$, while for $j=0$ we have

$$0\leqslant\theta_n(t)=\theta(t-\alpha_n)\leqslant\max\{\theta(1),t-\alpha_n+b\}$$

for all $t\in\mathbb{R}$. By using these bounds, and bearing in mind that $\|x_n^*\|=1$, we can estimate, for $\|x\|\leqslant R$,

$$|f_n(x)|=\left|\frac{\theta_n(x_n^*(x))}{(1+|\alpha_n|)2^n}\right|\leqslant\frac{\theta(1)+R+|\alpha_n|+|b|}{(1+|\alpha_n|)2^n}:=\delta_n^{(0)},$$

and since $\sum_{n=1}^\infty\delta_n^{(0)}<\infty$, it follows that $\sum_{n=1}^\infty f_n(x)$ converges uniformly on the ball $B(0,R)$. For $j\geqslant1$ it is easily seen that the j-linear map $f_n^{(j)}(x)\in\mathcal{L}_s^j(X;\ell_2)$ is given by

$$f_n^{(j)}(x)=\frac{\theta_n^{(j)}(x_n^*(x))}{(1+|\alpha_n|)2^n}\overbrace{x_n^*\otimes\cdots\otimes x_n^*}^{j}.$$

Then, by taking into account that $\|x_n^*\otimes\cdots\otimes x_n^*\|\leqslant1=\|x_n^*\|$, and $\|\theta_n^{(j)}\|_\infty=M_j$, we get that

$$\|f_n^{(j)}(x)\|\leqslant\frac{M_jR}{(1+|\alpha_n|)2^n}:=\delta_n^{(j)}$$

whenever $\|x\|\leqslant R$ and, since $\sum_{n=1}^\infty\delta_n^{(j)}<\infty$, this ensures that $\sum_{n=1}^\infty f_n^{(j)}$ converges uniformly on bounded sets, for all $j\in\mathbb{N}$. Therefore, f is of class C^∞.

The fact that $f^{-1}(0)=C$ follows immediately from equality (1) above and from the definitions of the functions θ_n and f.

On the other hand, every convex differentiable nonnegative function which vanishes precisely on a set C cannot have a zero derivative outside C; therefore our function f satisfies $f'(x)\neq0$ for all $x\in X\setminus C$.

Finally, when C is a cone, by bearing in mind the special construction of our function f it is not difficult to see that $U=f^{-1}([0,1])$ is a C^∞ smooth convex body in X so that $ccU=C$. Indeed, if C is a cone, we may assume that the α_i are all positive numbers. Then, for each $x\in X\setminus C$ there exists some n such that $x_n^*(x)>\alpha_n$. Now, by letting t go to ∞ we can make $x_n^*(tx)$ increase to ∞, which, by the choice of the function θ_n, means that $\theta_n(x_n^*(tx))/(1+|\alpha_n|)2^n$, the nth term of the series defining $f(tx)$, gets as large as we wish, so that the ray determined by x cannot be in the body $U=f^{-1}[0,1])$, that is, $x\in X\setminus ccU$. This shows that $ccU\subseteq C$; the other inclusion is obvious. □

Now we have arrived at the following generalization of Theorem 1.

Theorem 5. *Let A be a C^p starlike body in a separable Banach space X. Assume that ccA is a convex subset of X.*

(a) *If ccA is a linear subspace of finite codimension (say $X = ccA \oplus Z$, with Z finite-dimensional), then A is C^p relatively diffeomorphic to $ccA + B_Z$, where B_Z is an Euclidean ball in Z.*
(b) *If ccA is either not a linear subspace or else ccA is a linear subspace such that the quotient space X/ccA is infinite-dimensional, then A is C^p relatively diffeomorphic to a closed half-space.*

Moreover, in the case $p = 0$ this is true for all Banach spaces X.

Proof. To obtain (a) it is enough to apply Proposition 3 for $A_1 = A$ and $A_2 = ccA + B_Z$.

To obtain (b), write $C = ccA$, which is a closed convex cone of X. By Proposition 4 there exists a C^∞ smooth convex body U so that $ccU = C = ccA$. Then, by Proposition 3 the starlike bodies U and A are C^p relatively difeomorphic. On the other hand, by the assumption, $ccU = C$ is either not a linear subspace or else is a linear subspace such that $\dim(X/C) = \infty$. Now, part (b) of Theorem 1 tells us that U is C^p relatively diffeomorphic to a closed half-space, and hence so is A.

Finally, in the case $p = 0$, it is easy to see that, for every closed convex cone $C \subset X$, the set $U = \overline{C + B}$, where B is the unit ball of X, is a closed convex body so that $C = ccU$. Hence, the above argument applies. □

In particular, for an infinite-dimensional separable Banach space X, the boundary of every smooth bounded starlike body $A \subset X$ is C^p diffeomorphic to a hyperplane. We now apply the above result to get smooth negligibility of starlike bodies.

Corollary 6. *Let X be a separable Banach space, and let A be a C^p smooth starlike body such that its characteristic cone is a linear subspace of infinite codimension in X. Then there exists a C^p diffeomorphism from X onto $X \setminus A$.*

Proof. According to Theorem 5, there exists a C^p self-diffeomorphism of X mapping A onto a closed half-space. Therefore $X \setminus A$ is C^p diffeomorphic to an open half-space. Since an open half-space is obviously C^∞ diffeomorphic to the whole space, we may conclude that $X \setminus A$ and X are C^p diffeomorphic. □

As said above, examples like 2 show that the classification scheme used in Theorem 5 is useless when one wants to cover such cases as those of starlike bodies with nonconvex characteristic cones. Let us have a closer look at those examples. In the case of the bodies A_n whose construction is hinted in Example 2, and whose boundary has n connected components, one could wonder whether every starlike body in $\mathbb{R}^k$ whose boundary has exactly n connected components must be relatively homeomorphic to A_n.

More generally, it is natural to ask whether for every couple of starlike bodies A and B in a Banach space X with homeomorphic boundaries ∂A and ∂B it happens that A and B are relatively homeomorphic.

Surprisingly enough, the answers to these questions are all negative in the finite-dimensional setting, as we will show later on (see Examples 16, 17 and 18 below).

However, in infinite dimensions things turn out less complicated, topologically speaking. The following theorem answers the above question in the affirmative, providing a full classification of starlike bodies in terms of the homotopy type of their boundaries in infinite-dimensional Banach spaces.

Theorem 7. *Let X be an infinite-dimensional Banach space and let A, B be starlike bodies in X, with boundaries ∂A and ∂B. The following statements are equivalent:*

(1) *∂A has the same homotopy type as ∂B;*
(2) *∂A and ∂B are homeomorphic;*
(3) *A and B are relatively homeomorphic.*

Proof. Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). In order to show that (1) $\Rightarrow$ (3), we shall make use of the theory of Z-sets in infinite-dimensional topology (see [8], for instance). To begin with, notice that a starlike body is an infinite-dimensional manifold, which is a space locally homeomorphic to a fixed infinite-dimensional Hilbert space, and is contractible. In fact, topologically, it is just the Hilbert space, since every two homotopically equivalent infinite-dimensional Hilbert manifolds are topologically equivalent (see [8, p. 316]). The boundary ∂A of a starlike body A is always a Z-set in the body, since it is closed and it fulfills the standard definition. Recall that a closed subset $C \subset A$ is said to be a Z-set in A provided every continuous function $f : I^n \to A$ (where I^n is the unit cube in $\mathbb{R}^n$) can be uniformly approximated by continuous functions $g_k : I^n \to A$ so that $g_k(I^n) \subseteq A \setminus C$, that is, the images of the approximations g_k avoid the set C.

Given a starlike body A in X and a continuous function $f : I^n \to A$, the composition with a radial push provides a required approximation whose image avoids the boundary $C = \partial A$. Indeed, the sequence of functions f_k defined by

$$f_k(x) = \left(1 - \frac{1}{k}\right) f(x)$$

converges to f in the space $C(I^n, A)$ with the sup norm and, since $f(x) \in A$ (that is, $\mu_A(f(x)) \leqslant 1$) for all $x \in I^n$, and

$$\mu_A\big(f_k(x)\big) = \left(1 - \frac{1}{k}\right)\mu_A\big(f(x)\big) \leqslant \left(1 - \frac{1}{k}\right) < 1,$$

it is clear that $f_k(I^n) \subseteq A \setminus \partial A$. Hence ∂A is a Z-set in A.

A similar argument (taking $f_k(x) = (1 + 1/k) f(x)$ instead), shows that ∂A is a Z-set in $X \setminus \operatorname{int}(A)$ (which is also an infinite-dimensional manifold).

So, given two starlike bodies A and B in X, we know that ∂A is a Z-set in both A and $X \setminus \operatorname{int}(A)$, and ∂B is a Z-set in B and in $X \setminus \operatorname{int}(B)$. Now, we can make use of the so-called Z-set extension homeomorphism theorem [2], which tells us that a homeomorphism between two Z-sets can be extended to a homeomorphism between the infinite-dimensional manifolds with respect to which those sets are Z-sets.

Since ∂A and ∂B are homotopically equivalent, the above mentioned result that every two such infinite-dimensional manifolds are topologically equivalent tells us that ∂A and ∂B are, in fact, homeomorphic. Let $f : \partial A \to \partial B$ be a homeomorphism. Then, taking into account that ∂A and ∂B are Z-sets in A and B, respectively, the Z-set extension homeomorphism theorem tells us that there exists a homeomorphism $F: A \to B$ so that F restricts to f on ∂A. On the other hand, since ∂A and ∂B are also Z-sets in $X \setminus \text{int}(A)$ and $X \setminus \text{int}(B)$ respectively, using again the extension theorem, there exists a homeomorphism $G: X \setminus \text{int}(A) \to X \setminus \text{int}(B)$ so that G also restricts to f on ∂A. Therefore,

$$H(x) = \begin{cases} F(x) & \text{if } x \in A, \\ G(x) & \text{if } x \in X \setminus A \end{cases}$$

is a self-homeomorphism of X so that $H(A) = B$ (and f restricts to f on ∂A). □

In the case of the Hilbert space we can improve Theorem 7 by extending it to the C^∞ smooth category.

Theorem 8. *Let A, B be C^∞ smooth starlike bodies in the separable Hilbert space, with boundaries ∂A and ∂B. The following statements are equivalent:*

(1) *∂A has the same homotopy type as ∂B;*
(2) *∂A and ∂B are homeomorphic;*
(3) *A and B are C^∞ relatively diffeomorphic.*

Proof. By Theorem 7 we already know that (1) and (2) are equivalent and, furthermore, that either of these statements implies that the bodies A and B are relatively homeomorphic. We only need to show that in this case A and B are in fact C^∞ relatively diffeomorphic.

To this end, let us first observe that the bodies A and B and their boundaries ∂A and ∂B are paralelizable manifolds, that is, their tangent space, which is always our Hilbert space ℓ_2, has a contractible general linear group. In what follows, whenever M is a boundary or a finite union of boundaries of starlike bodies, the symbol TM stands for the tangent bundle. Since all of those manifolds are paralelizable, TM is always trivial.

Now we are in a position to apply a result of Elworthy's which reads as follows (see [15, Theorem 24]).

Suppose M and X are paralelizable C^∞ manifolds modelled on the Hilbert space, and $f_0, f_1: M \to X$ are closed C^∞ embeddings. Assume that

(1) f_0 and f_1 are homotopic, and
(2) f_0 and f_1 are tangentially homotopic.

Then there exists a C^∞ isotopy $\Phi: \mathbb{R} \times X \to \mathbb{R} \times X$ so that $\Phi(1, f_0(x)) = f_1(x)$ and $\Phi(0, x) = x$.

This result will give us what we want. Indeed, consider $M = \partial A \cup \partial(\frac{1}{2}A)$, where $\frac{1}{2}A = \{x \in X\colon \mu_A(x) \leqslant 1/2\}$, $X = \ell_2$, let f_0 be the identity on M and let $f_1: \partial A \cup \partial(\frac{1}{2}A) \to \partial B \cup \partial(\frac{1}{2}B)$ be a diffeomorphism sending ∂A onto ∂B, and $\partial(\frac{1}{2}A)$ onto $\partial(\frac{1}{2}B)$. The existence of f_1 is guaranteed by the fact that the boundaries of those starlike bodies are all homotopically equivalent, and from the classic result that two homotopic Hilbert manifolds are always C^∞ diffeomorphic [9,14,20].

Obviously, (1) is satisfied for such f_0 and f_1. So, if we only check (2), then Φ_1 will be a C^∞ self-diffeomorphism of ℓ_2 such that $\Phi_1 \circ f_0 = f_1$. Since f_0 is the identity, we have $\Phi_1(x) = f_1(x)$ for every $x \in M$, and therefore Φ_1 takes ∂A onto ∂B, and $\partial(\frac{1}{2}A)$ onto $\partial(\frac{1}{2}B)$. This in turn implies that Φ_1 takes the starlike body A onto B and hence A and B are C^∞ relatively diffeomorphic. Indeed, if one point of A is sent to a point outside B then the whole interior of A is sent outside B: suppose that, for some points $x, y \in A$, x is sent outside B and y is sent inside B; since the interior of A is path connected there is an arc joining x and y in the interior of A, and this arc must be sent by Φ_1 to another arc in X which connects the points $\Phi_1(x) \in X \setminus B$ and $\Phi_1(y) \in B$; such arc must intersect the boundary of B, but this is impossible because if it did a point in the interior of A would be sent into the boundary of B and therefore ϕ_1 would not be injective. Since there are many points inside A which are sent inside B (for instance, any of the points of $\partial(\frac{1}{2}A)$), we can be certain that Φ_1 takes A onto B.

So, in order to conclude the proof we only need to check (2). Let $f : [0,1] \times M \to X$ be the homotopy joining f_0 and f_1. The condition (2) calls to find a bundle map $\alpha : [0,1] \times TM \to f^*(TX)$ which is a homotopy between Tf_0 and Tf_1; here the Tf_i are the induced maps on the tangent bundles. In our case, these bundles are all trivial. Moreover, Tf_0 is just the identity, and Tf_1 is a closed embedding onto $B \times \ell_2$. So such α does exist. □

The starlike bodies of a Banach space X are, in some sense, in one-to-one correspondence with the closed subsets K (respectively the open subsets U) of the unit sphere S of X. Let A be a starlike body in X. Let $r : X \setminus \{0\} \to S$ be the radial retraction. Clearly, $S(A) = r(ccA \setminus \{0\})$ is a closed subset of S such that $ccA = [0,\infty)S(A)$, the cone over $S(A)$, while $r(\partial A) = S \setminus S(A)$ is an open subset of S. As it is easily seen below, a closed subset K of S gives rise to a starlike body whose characteristic cone is the cone over K.

Proposition 9. *Let K be a closed subset of S, there exists a starlike body $A = A_K$ such that $S(A) = K$. If X is separable and C^p smooth, then we may require that the body A is C^p smooth as well.*

Proof. Take any continuous function $\lambda : S \to [0,1]$ with $\lambda^{-1}(0) = K$. Define $\psi(x) = \|x\|\lambda(x/\|x\|)$ for $x \neq 0$ and $\psi(0) = 0$. We see that $\psi : X \to [0,\infty)$ is a positively homogeneous continuous function with $\psi^{-1}(0) = [0,\infty)K$. It is enough to set $A = \psi^{-1}([0,1])$. In the smooth case, if X is C^p smooth, there exists a bounded C^p smooth starlike body whose characteristic cone is $\{0\}$. Let μ stand for the Minkowski functional of this body. Using the fact that X admits C^p smooth partitions of unity, one can find a continuous function $\lambda : X \to [0,1]$ which is C^p smooth off $\lambda^{-1}(0) = [0,\infty)K$. Define $\psi(x) = \mu(x)\lambda(x/\mu(x))$ for $x \neq 0$ and $\psi(0) = 0$. Clearly, $\psi : X \to [0,\infty)$ is a positively homogeneous continuous function which is C^p smooth off $\psi^{-1}(0) = [0,\infty)K$. Set $A = \psi^{-1}([0,1])$. □

Remark 10. The smooth assertion holds true if one replaces the separability assumption by the existence of C^p smooth partitions of unity.

In the proof of Proposition 9, instead of using the functional μ, we could have used a weak Hilbertian norm ω on the separable space X, that is, a continuous norm of the form $\omega(x) = \|T(x)\|$ that is determined by an injective continuous linear operator $T : X \to \ell_2$. In such a case, ω is real-analytic off $\omega^{-1}(0)$. If K is a compact subset of S, then $K_0 = ([0,\infty)K) \cap S_\omega$, where S_ω is the unit ω-sphere, is also compact. Hence, $T(K_0)$ is compact in ℓ_2 and, by [12], there exists a continuous function $\lambda : S_\omega \to [0,1]$ that is real-analytic off $\lambda^{-1}(0) = K_0$. Letting $\psi(x) = \omega(x)\lambda(x/\omega(x))$ for $x \neq 0$ and $\psi(0) = 0$, the set $A = \psi^{-1}([0,1])$ is a real-analytic starlike body with $ccA = [0,\infty)K$. As a consequence, we have:

Remark 11. In a separable Banach space, for every starlike body A with a locally compact characteristic cone ccA, there exists a real-analytic starlike body A_0 with $ccA_0 = ccA$.

We do not know whether this last statement holds for an arbitrary starlike body A. However, if ccA is weakly closed, then we can find a weak Hilbertian norm ω so that ccA is ω-closed. We can then construct a continuous function $\lambda : S_\omega \to [0,1]$ that is C^∞ off $\lambda^{-1}(0) = ccA \cap S_\omega$. Since the characteristic cone of a weakly closed starlike body is weakly closed, we have the following:

Remark 12. For a starlike body A in a separable Banach space, which is closed in the weak topology, there exists a C^∞ starlike body A_0 with $ccA = ccA_0$.

According to Lemma 3, for a fixed closed set $K \subset S$, all (smooth) starlike bodies of the form A_K are relatively (diffeormorphic) homeomorphic. In the infinite-dimensional setting we also have:

Corollary 13. *For two closed sets $K_1, K_2 \subset S$ in an infinite-dimensional Banach space X, the starlike bodies A_{K_1} and A_{K_2} are relatively homeomorphic if and only if the complements $S \setminus K_1$ and $S \setminus K_2$ have the same homotopy type.*

Proof. This is a consequence of Theorem 7 because the boundary of A_{K_i} is homeomorphic to $S \setminus K_i$, $i = 1, 2$. □

We do not know what necessary and sufficient conditions for K_i, $i = 1, 2$, one has to impose in order their complements in S have the same homotopy type. If K is a Z-set in S (e.g., K is compact), then the complement of K is homeomorphic to S; hence, in such a case A_K is relatively homeomorphic to the unit ball. If K_1 is a one-point set and K_2 is a small closed ball intersected with S, then K_1 is a Z-set, while K_2 is not a Z-set, but the complements of K_1 and K_2 have the same homotopy type (they are contractible), and therefore A_{K_1} and A_{K_2} are relatively homeomorphic (with the unit ball). The following example shows that the contractibility of K_1 and K_2 does not suffice to obtain the same homotopy type of their complements.

Example 14. Let $K_1 \subset S$ be a one point set and $K_2 = S \cap X_0$, where X_0 is a codimension 1 vector subspace of X. Then, K_1 and K_2 are contractible, but the complement of K_2 is disconnected, while the complement of K_1 is contractible (even homeomorphic to X). We see that A_{K_1} is relatively homeomorphic to the unit ball in X, while $ccA_{K_2} = X_0$ and, consequently, A_{K_2} is relatively homeomorphic to $X_0 \times [-1, 1]$, which, in turn, (having disconnected boundary in $X_0 \times \mathbb{R}$) is not homomorphic to the unit ball in X.

Since, for a Z_σ-set Z (that is, Z is a countable union of Z-sets) in S, the spaces $S \setminus Z$ and S are homeomorphic, one can hope that if K_1 and K_2 have the same homotopy type modulo Z_σ-set, then the complements of K_i, $i = 1, 2$, have the same homotopy type. (Two closed sets P_1, P_2 are meant to have the same homotopy type modulo Z_σ-set if there are closed sets $P_i' \subset P_i$, $i = 1, 2$, such that P_i', $i = 1, 2$, have the same homotopy type and both $P_1 \setminus P_1'$ and $P_2 \setminus P_2'$ are Z_σ-sets.) This, however, is not the case because the sets K_1 and K_2 of Example 14 have the same homotopy type modulo Z_σ-set.

The finite-dimensional case

Below we provide several examples showing that Corollary 13 and Theorem 7 cannot be extended in any reasonable way for a finite-dimensional space X.

Example 15. Let $S = S^1$ and B be the unit sphere and the unit ball in $X = \mathbb{R}^2$, respectively. Consider two compacta K_1 and K_2 in S; K_1 is a copy of an infinite convergent sequence space and K_2 is a copy of the Cantor set. Then, the bodies A_{K_1} and A_{K_2} (having their boundaries homeomorphic) are not homeomorphic.

To see this it suffices to notice that each A_{K_i} is homeomorphic to $B \setminus K_i$. It is then clear that any nonisolated point of K_1 has a basis of neighborhoods (in A_{K_1}) that can be chosen to be topologically different from any neighborhood of any point of K_2. We can obviously make those starlike bodies to be real-analytic, so an improvement in smothness is not any help.

In higher dimensions, one can provide more regular examples.

Example 16. Let $S = S^2$ be the unit sphere in $X = \mathbb{R}^3$. Consider $C_1 = U_1 \cup U_2 \cup U_3$, where $U_1 = \{(x, y, z) \in S \mid |z| < 1/8\}$, $U_2 = \{(x, y, z) \in S \mid |z-1| < 1/8\}$, and $U_3 = -U_2$, and $C_2 = U_1 \cup U_2 \cup U_3'$, where $U_3' = \{(x, y, z) \in S \mid |z - 1/2| < 1/8,\ y > 0\}$. Letting $K_i = S \setminus C_i$, $i = 1, 2$, we see that the boundaries of the starlike bodies A_{K_i} (being homeomorphic to C_i) are homeomorphic. However, there is no homeomorphism of A_{K_1} onto A_{K_2}.

In $\mathbb{R}^4$, we have the following.

Example 17. Let $S = S^3$ be the unit sphere in $X = \mathbb{R}^4$. Let K be the (doubled) Fox-Artin arc in S, that is, K is a topological arc whose complement is a contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$, see [21, p. 68]. Then, for a starlike body $A = A_K$, ccA is a cone over an arc, therefore, it is contractible. Moreover, A_K is not homeomorphic to a half-space in $\mathbb{R}^4$ though both bodies have contractible boundaries.

In general, for every $n \geqslant 4$, the sphere $S = S^{n-1}$ in $X = \mathbb{R}^n$ contains an open contractible $(n-1)$-manifold U that is not homeomorphic to $\mathbb{R}^{n-1}$. In case $n = 4$, one can take U to be the so-called Whitehead manifold W in S^3. Actually, in each dimension $n \geqslant 3$, there are uncountably many topologically distinct contractible n-manifolds; the construction is due to McMillan [19] for $n = 3$, Glaser [16] for $n = 4$, and Curtis and Kwun [11] for $n \geqslant 5$. The complement $S^3 \setminus W$ is a continuum that is not contractible. For $n > 4$, one can always pick U so that $S^{n-1} \setminus U$ is a contractible $(n-1)$-manifold. To see this, let M be a contractible $(n-1)$-manifold with non-simply connected boundary; the existence of M is due to N.H.A. Newman for $n > 5$ (see [17]), and due to B. Mazur and V. Poenaru for $n = 5$. Gluing together two copies of M along their boundaries we obtain the double space N, which is a topological copy of S^{n-1} (cf. [1, p. 2, items (4) and (9)]). The complement of one copy of M in N is just the interior of the other copy, which yields a requested manifold U. Since U is not simply connected at infinity, U is not homeomorphic to $\mathbb{R}^{n-1}$; moreover, the manifold U, being the interior of a contractible manifold, is itself contractible.

Example 18. Write $K = S \setminus U$. Any starlike body A_K in $\mathbb{R}^n$, $n > 4$, has both ccA_K and ∂A_K contractible. However, A_K is not homeomorphic to a half-space.

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References

[1] F.D. Ancel, C.R. Guilbault, Compact contractible n-manifolds have arc of spines ($n > 5$), Pacific J. Math. 168 (1995) 1–10.

[2] R.D. Anderson, J.D. McCharen, On extending homeomorphisms to Fréchet manifolds, Proc. Amer. Math. Soc. 25 (1970) 283–289.

[3] D. Azagra, M. Cepedello, Smooth Lipschitz retractions of starlike bodies onto their boundaries in infinite-dimensional Banach spaces, Bull. London Math. Soc. 33 (2001) 443–453.

[4] D. Azagra, R. Deville, James' theorem fails for starlike bodies, J. Funct. Anal. 180 (2001) 328–346.

[5] D. Azagra, T. Dobrowolski, Smooth negligibility of compact sets in infinite-dimensional Banach spaces, with applications, Math. Ann. 312 (3) (1998) 445–463.

[6] C. Bessaga, V.L. Klee, Two topological properties of topological linear spaces, Israel J. Math. 2 (1964) 211–220.

[7] C. Bessaga, V.L. Klee, Every non-normable Fréchet space is homeomorphic with all of its closed convex bodies, Math. Ann. 163 (1966) 161–166.

[8] C. Bessaga, A. Pelczynski, Selected Topics in Infinite-Dimensional Topology, in: Monografie Matematyczne, Warszawa, 1975.

[9] D. Burghelea, N.H. Kuiper, Hilbert manifolds, Ann. of Math. 90 (1969) 379–417.

[10] H.H. Corson, V.L. Klee, Topological classification of convex sets, in: Proc. Symp. Pure Math., Vol. 7. Convexity, American Mathematical Society, Providence, RI, 1963, pp. 37–51.

[11] M.L. Curtis, K.W. Kwun, Infinite sums of manifolds, Topology 6 (1965) 31–42.

[12] T. Dobrowolski, Smooth and R-analytic negligibility of subsets and extension of homeomorphism in Banach spaces, Studia Math. 65 (1979) 115–139.
[13] T. Dobrowolski, Relative classification of smooth convex bodies, Bull. Acad. Polon. Sci. Sér. Sci. Math. 25 (1977) 309–312.
[14] J. Eells, D. Elworthy, Open embeddings of certain Banach manifolds, Ann. of Math. 91 (1970) 465–485.
[15] D. Elworthy, Embeddings, isotopy and stability of Banach manifolds, Compositio Math. 24 (1972) 175–226.
[16] L.C. Glaser, Uncountably many contractible open 4-manifolds, Topology 6 (1967) 37–42.
[17] C.R. Guilbault, Some compact contractible manifold containing disjoint spines, Topology 34 (1995) 99–108.
[18] V.L. Klee, Convex bodies and periodic homeomorphisms in Hilbert space, Trans. Amer. Math. Soc. 74 (1953) 10–43.
[19] D.R. McMillan, Some contractible open 3-manifolds, Trans. Amer. Math. Soc. 102 (1962) 373–382.
[20] N. Moulis, Sur les variétés hilbertiennes et les fonctions non dégénérées, Indag. Math. 30 (1968) 497–511.
[21] T.B. Rushing, Topological Embeddings, Academic Press, New York, 1973.
[22] J.J. Stoker, Unbounded convex point sets, Amer. J. Math. 62 (1940) 165–179.

Further reading

[1] D. Azagra, Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces, Studia Math. 125 (2) (1997) 179–186.
[2] C. Bessaga, Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 14 (1966) 27–31.
[3] C. Bessaga, Interplay between infinite-dimensional topology and functional analysis. Mappings defined by explicit formulas and their applications, Topology Proc. 19 (1994).
[4] H. Cartan, Calcul Différentiel, Hermann, Paris, 1967.
[5] R. Deville, G. Godefroy, V. Zizler, Smoothness and renormings in Banach spaces, in: Pitman Monograph. Surveys Pure Appl. Math., Vol. 64, Longman, Harlow, 1993.
[6] R.G. Haydon, A counterexample to several questions about scattered compact spaces, Bull. London Math. Soc. 22 (1990) 261–268.