

V́ctor M. Śnchez\*

# The history of a general criterium on spaceability

DOI 10.1515/math-2017-0019

Received January 12, 2017; accepted February 9, 2017.

**Abstract:** There are just a few general criteria on spaceability. This survey paper is the history of one of the first ones. Let  $I_1$  and  $I_2$  be arbitrary operator ideals and  $E$  and  $F$  be Banach spaces. The spaceability of the set of operators  $I_1(E, F) \setminus I_2(E, F)$  is studied. Before stating the criterium, the paper summarizes the main results about lineability and spaceability of differences between particular operator ideals obtained in recent years. They are the seed of the ideas contained in the general criterium.

**Keywords:** Spaceability, Operator ideal

**MSC:** 47L20

## 1 Introduction

During the last two decades many authors have been searching for large linear structures of mathematical objects enjoying certain special properties. If  $X$  is a topological vector space, a subset  $A$  of  $X$  is said to be *spaceable* if  $A \cup \{0\}$  contains an infinite-dimensional closed subspace. The subset  $A$  is called *lineable* if  $A \cup \{0\}$  contains an infinite-dimensional subspace (not necessarily closed). After these definitions were introduced by Aron, Gurariy and Seoane-Sepúlveda in [1], many authors showed that many non-linear special sets can often have these properties. See, for instance, the recent works [2–8], just to cite some. However, before the publication of [1] some authors already found large linear structures enjoying special properties even though they did not explicitly used terms like lineability or spaceability (see [9, 10]). We refer the interested reader to a very recent book on the topic ([11]) where many examples can be found and techniques are developed in several different frameworks.

Surprisingly, there exist very few general criteria on spaceability. In fact, most results on concrete properties have been proved directly and constructively. Perhaps, the first general criterium on spaceability appeared in [12] where Wilansky proved that if  $Y$  is a closed vector subspace of a Banach space  $X$ , then  $X \setminus Y$  is spaceable if and only if  $Y$  has infinite codimension. An improvement of this result, where  $X$  is a Fréchet space, is ascribed by Kitson and Timoney to Kalton (see [13, Theorem 2.2]). Kitson and Timoney used it to obtain the following theorem:

**Theorem 1.1** ([13]). *Let  $(E_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces,  $F$  a Fréchet space and  $T_n : E_n \rightarrow F$  bounded linear operators. Let  $G$  be the span of  $\bigcup_{n=1}^{\infty} T_n(E_n)$ . If  $G$  is not closed in  $F$ , then the set  $F \setminus G$  is spaceable.*

Three years before, lineability of the difference between a couple of particular operator ideals was studied for the first time. Thus, in [14], Puglisi and Seoane-Sepúlveda showed that if  $E$  and  $F$  are Banach spaces where  $E$  has the two series property, then the set  $L(E, F^*) \setminus \Pi_1(E, F^*)$  is lineable, where  $\Pi_1$  denotes the ideal of 1-summing operators.

\*Corresponding Author: V́ctor M. Śnchez: Department of Mathematical Analysis, Faculty of Mathematics, Complutense University of Madrid, 28040 Madrid, Spain, E-mail: victorms@ucm.es

The following year, and partially answering a question posed in [14], Botelho, Diniz and Pellegrino proved in [15] that if  $E$  is a superreflexive Banach space containing a complemented infinite-dimensional subspace with unconditional basis, or  $F$  is a Banach space having an infinite unconditional basic sequence, then the set  $K(E, F) \setminus \Pi_p(E, F)$  is lineable for every  $p \geq 1$ , where  $K$  denotes the ideal of all compact operators.

More generally, in [13], the spaceability of the set of operators  $K(E, F) \setminus \bigcup_{p \geq 1} \Pi_p(E, F)$  was obtained as a consequence of Theorem 1.1 when  $E$  is a superreflexive Banach space.

Finally, and recently, in [16], the authors continued this research considering general operator ideals in the sense of Pietsch  $I_1$  and  $I_2$  and Banach spaces  $E$  and  $F$ , such that the set  $I_1(E, F) \setminus I_2(E, F)$  is non-empty. They introduced the new notion of  $\sigma$ -reproducible space for obtaining the spaceability of this set when  $E$  or  $F$  belongs to this class of spaces and  $I_1(E, F)$  is complete with the usual operator norm or with another complete ideal norm. Many classical Banach spaces are  $\sigma$ -reproducible, in particular, rearrangement invariant spaces and spaces of continuous functions.

For completeness, most of the original proofs have been included.

## 2 Preliminaries

An operator ideal in the sense of Pietsch (see [17, 18]) is defined as follows:

**Definition 2.1.** Let  $\mathfrak{B}$  denote the class of all Banach spaces and let  $L$  denote the class of all bounded linear operators between Banach spaces. An **operator ideal**  $I$  is a “mapping”  $I : \mathfrak{B} \times \mathfrak{B} \rightarrow 2^L$  such that

1. For each pair of Banach spaces  $E$  and  $F$  the collection of operators  $I(E, F)$  (or  $I(E)$  if  $E = F$ ) is a subspace of the space  $L(E, F)$  (or  $L(E)$  if  $E = F$ ) of bounded linear operators from  $E$  to  $F$  containing all finite-rank operators.
2. If in a scheme of bounded linear operators  $E_0 \xrightarrow{S_1} E \xrightarrow{T} F \xrightarrow{S_2} F_0$  we have  $T \in I(E, F)$ , then  $S_2 \circ T \circ S_1 \in I(E_0, F_0)$ .

**Definition 2.2.** An **ideal norm** defined on an ideal  $I$  is a rule  $\|\cdot\|_I$  that assigns to every operator  $T \in I$  a non-negative number  $\|T\|_I$  satisfying the following conditions:

1.  $\|x^* \otimes y\|_I = \|x^*\|_{E^*} \|y\|_F$  for  $x^* \in E^*$ ,  $y \in F$  where  $(x^* \otimes y)(x) = x^*(x)y$  for  $x \in E$ .
2.  $\|S + T\|_I \leq \|S\|_I + \|T\|_I$  for  $S, T \in I(E, F)$ .
3.  $\|S_2 \circ T \circ S_1\|_I \leq \|S_2\| \|T\|_I \|S_1\|$  for  $S_2 \in L(F, F_0)$ ,  $T \in I(E, F)$  and  $S_1 \in L(E_0, E)$ .

The last condition implies that  $\|\lambda T\|_I = |\lambda| \|T\|_I$  for  $T \in I$  and  $\lambda \in \mathbb{K}$ . Thus, we have indeed a norm. Moreover,  $\|T\| \leq \|T\|_I$  where  $\|\cdot\|$  denotes the usual operator norm of  $L(E, F)$ , which is an example of ideal norm.

A classical closed operator ideal endowed with the canonical operator norm of  $L(E, F)$  is the ideal  $K$  of all compact operators.

A classical non-closed operator ideal with respect to the operator norm is the ideal  $\Pi_{q,p}$  of  $(q, p)$ -summing operators. Recall that if  $1 \leq p \leq q < \infty$ , an operator  $T \in L(E, F)$  is called  $(q, p)$ -summing (or  $p$ -summing if  $p = q$ ) if there is a constant  $C$  so that, for every choice of an integer  $n$  and vectors  $(x_i)_{i=1}^n$  in  $E$ , we have

$$\left( \sum_{i=1}^n \|T(x_i)\|^q \right)^{1/q} \leq C \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^n |x^*(x_i)|^p \right)^{1/p}.$$

The smallest possible constant  $C$  defines a complete ideal norm on this operator ideal, denoted by  $\pi_{q,p}(\cdot)$ . If  $1/p - 1/q \leq 1/r - 1/s$  and  $p \leq r$ , one has  $\Pi_{q,p} \subset \Pi_{s,r}$  (see [17, page 459]).

Let us recall now the definition of a rearrangement invariant space. Given a measure space  $(\Omega, \lambda)$ , where  $\Omega$  is the interval  $[0, 1]$  or  $[0, \infty)$  and  $\lambda$  is the Lebesgue measure, or  $\Omega = \mathbb{N}$  and  $\lambda$  is the counting measure, the distribution function  $\lambda_x$  associated to a scalar measurable function  $x$  on  $\Omega$  is defined by  $\lambda_x(s) = \lambda\{t \in \Omega : |x(t)| > s\}$ . The decreasing rearrangement function  $x^*$  of  $x$  is defined by  $x^*(t) = \inf\{s \in [0, \infty) : \lambda_x(s) \leq t\}$ . A Banach space

$(E, \|\cdot\|_E)$  of measurable functions defined on  $\Omega$  is said to be a *rearrangement invariant space* if the following conditions are satisfied:

1. If  $y \in E$  and  $|x(t)| \leq |y(t)|$   $\lambda$ -a.e. on  $\Omega$ , then  $x \in E$  and  $\|x\|_E \leq \|y\|_E$ .
2. If  $y \in E$  and  $\lambda_x = \lambda_y$ , then  $x \in E$  and  $\|x\|_E = \|y\|_E$ .

Important examples of rearrangement invariant spaces are  $L^p$ , Lorentz, Marcinkiewicz and Orlicz spaces. For properties of rearrangement invariant spaces we refer to [19–21].

### 3 Results

The first result about lineability of the difference between a couple of particular operator ideals appeared in [14]. In this paper the authors showed that if  $E$  and  $F$  are Banach spaces where  $E$  has the two series property, then  $L(E, F^*) \setminus \Pi_1(E, F^*)$  is lineable, where  $\Pi_1$  denotes the ideal of 1-summing operators.

**Definition 3.1.** A Banach space  $E$  is said to have the **two series property** provided there exist unconditionally convergent series  $\sum_{i=1}^{\infty} f_i$  in  $E^*$  and  $\sum_{i=1}^{\infty} x_i$  in  $E$  such that

$$\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right)^{1/2} = \infty.$$

For instance, every  $L^p$ -space has the two series property for  $1 < p < \infty$ .

**Lemma 3.2** ([14]). Let  $E$  be a Banach space satisfying the two series property. Let  $\sum_{i=1}^{\infty} f_i$  in  $E^*$  and  $\sum_{i=1}^{\infty} x_i$  in  $E$  such that  $\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right)^{1/2} = \infty$ . Then, there exists a sequence of countable pairwise disjoint subsets of  $\mathbb{N}$ ,  $(A_n)_{n \in \mathbb{N}}$ , such that

$$\sum_{i=1}^{\infty} \left( \sum_{j \in A_n} \frac{|f_j(x_i)|^2}{\|f_j\|} \right)^{1/2} = \infty$$

for each  $n \in \mathbb{N}$ .

**Theorem 3.3** ([14]). Let  $E$  be a Banach space satisfying the two series property. Then the set  $L(E, \ell_2) \setminus \Pi_1(E, \ell_2)$  is lineable.

*Proof.* Since  $E$  satisfies the two series property, there exist unconditionally convergent series  $\sum_{i=1}^{\infty} f_i$  in  $E^*$  and  $\sum_{i=1}^{\infty} x_i$  in  $E$  such that

$$\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right)^{1/2} = \infty.$$

Let  $(A_n)_{n \in \mathbb{N}}$  be the sequence given by Lemma 3.2. For each  $n \in \mathbb{N}$ , let us define the operator  $T_n : E \rightarrow \ell_2$  by

$$T_n(x) = \sum_{k \in A_n} \frac{f_k(x)}{\|f_k\|^{1/2}} e_k.$$

If  $\|x\|_E \leq 1$ , then

$$\|T_n(x)\| = \left( \sum_{k \in A_n} \frac{|f_k(x)|^2}{\|f_k\|} \right)^{1/2} \leq \left( \sum_{k=1}^{\infty} \frac{|f_k(x)|^2}{\|f_k\|} \right)^{1/2} \leq \left( \sup_{\|x\|=1} \sum_{k=1}^{\infty} |f_k(x)| \right)^{1/2} < \infty.$$

Thus,  $T_n$  is well-defined and  $T_n \in L(E, \ell_2)$  for every  $n \in \mathbb{N}$ .

But  $\sum_{i=1}^{\infty} \|T_n(x_i)\| = \infty$  for every  $n \in \mathbb{N}$ . Then  $T_n \notin \Pi_1(E, \ell_2)$  for each  $n \in \mathbb{N}$ .

Because of the pairwise disjointness of the sets  $A_n$ , we have that the sequence  $(T_n)_{n \in \mathbb{N}}$  is linearly independent in  $L(E, \ell_2)$ .

Finally, let us show that every nonzero bounded linear operator in the linear span of the sequence  $(T_n)_{n \in \mathbb{N}}$  does not belong to the ideal of 1-summing operators. It is enough to consider the linear combination of two elements because the general case follows similarly. Thus, if  $\lambda_1, \lambda_2 \in \mathbb{K}$  and  $n_1, n_2 \in \mathbb{N}$ , then (assuming, without loss of generality, that  $\lambda_1 \neq 0$ ) we have that

$$\sum_{i=1}^{\infty} \left\| \lambda_1 T_{n_1} \left( \frac{x_i}{\lambda_1} \right) + \lambda_2 T_{n_2} \left( \frac{x_i}{\lambda_1} \right) \right\| = \sum_{i=1}^{\infty} \left( \sum_{k \in A_{n_1}} \frac{|f_k(x_i)|^2}{\|f_k\|^2} + \sum_{k \in A_{n_2}} \frac{|f_k(\frac{\lambda_2}{\lambda_1} x_i)|^2}{\|f_k\|^2} \right)^{1/2} = \infty.$$

Since  $\sum_{i=1}^{\infty} \frac{x_i}{\lambda_1}$  is an unconditional convergent series in  $E$ , we obtain the result.  $\square$

Now, given two Banach spaces  $E$  and  $F$ , where  $E$  enjoys the two series property, a linearly independent sequence  $(\widetilde{T}_n)_{n \in \mathbb{N}}$  can be constructed from the sequence  $(T_n)_{n \in \mathbb{N}}$  made in Theorem 3.3 whose linear span is contained in  $L(E, F^*) \setminus \Pi_1(E, F^*)$ . Thus, we have

**Corollary 3.4** ([14]). *Let  $E, F$  be Banach spaces, where  $E$  has the two series property. Then the set  $L(E, F^*) \setminus \Pi_1(E, F^*)$  is lineable.*

In [22], Davis and Johnson proved that the set  $K(E, F) \setminus \Pi_p(E, F)$  is non-empty whenever  $E$  is a superreflexive Banach space and  $F$  is any Banach space. The question about the lineability of this set was posed in [14] and partially answered in [15, Theorem 2.1]:

**Theorem 3.5** ([15]). *Let  $E$  be a superreflexive Banach space. If either  $E$  contains a complemented infinite-dimensional subspace with unconditional basis or  $F$  is a Banach space having an infinite unconditional basic sequence, then  $K(E, F) \setminus \Pi_p(E, F)$  is lineable for every  $p \geq 1$ .*

*Proof.* First, let us suppose that  $E$  contains a complemented infinite-dimensional subspace  $E_0$  with unconditional basis  $(e_n)_{n \in \mathbb{N}}$ . We consider a decomposition of  $\mathbb{N}$  into infinitely many infinite pairwise disjoint subsets  $(A_k)_{k \in \mathbb{N}}$ . Since  $(e_n)_{n \in \mathbb{N}}$  is an unconditional basis, it is well known that  $(e_n)_{n \in A_k}$  is an unconditional basic sequence for every  $k \in \mathbb{N}$ . Let us denote by  $E_k$  the closed span of  $(e_n)_{n \in A_k}$ . As a subspace of a superreflexive space,  $E_k$  is superreflexive as well, and there exists  $T_k \in K(E_k, F) \setminus \Pi_p(E_k, F)$  for each  $k \in \mathbb{N}$ .

If  $C$  is the unconditional basis constant of  $(e_n)_{n \in \mathbb{N}}$ , then

$$\left\| \sum_{n=1}^{\infty} \epsilon_n a_n e_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n e_n \right\|$$

for every  $\epsilon_n = \pm 1$  and scalars  $a_n$ . We denote by  $P_k : E_0 \rightarrow E_k$  the canonical projection onto  $E_k$  for each  $k \in \mathbb{N}$ . For  $x = \sum_{n=1}^{\infty} a_n e_n \in E_0$  we have

$$2P_k(x) = \sum_{n \in A_k} 2a_n e_n = \sum_{n=1}^{\infty} \epsilon_n a_n e_n + \sum_{n=1}^{\infty} \epsilon'_n a_n e_n$$

for a suitable choice of signs  $\epsilon_n$  and  $\epsilon'_n$ . Thus,

$$\|P_k(x)\| \leq C \|x\|.$$

So each  $P_k$  is continuous,  $\|P_k\| \leq C$  and  $E_k$  is complemented in  $E_0$ .

If  $\pi : E \rightarrow E_0$  is the projection onto  $E_0$ , for each  $k \in \mathbb{N}$  we define  $\widetilde{T}_k = T_k \circ P_k \circ \pi$ . Since  $(P_k \circ \pi)(x) = x$  for every  $x \in E_k$ , then  $\widetilde{T}_k \in K(E, F) \setminus \Pi_p(E, F)$ . Given scalars  $\lambda_1, \dots, \lambda_n$  with at least one  $\lambda_k \neq 0$ , there is a weakly  $p$ -summable sequence  $(x_j)_{j \in \mathbb{N}} \subset E_k$  such that  $\sum_{j=1}^{\infty} \|T_k(x_j)\|^p = +\infty$ . The sequence  $(x_j)_{j \in \mathbb{N}}$  is

weakly  $p$ -summable in  $E$ ,  $\widetilde{T}_k(x_j) = T_k(x_j)$  and  $\widetilde{T}_i(x_j) = 0$  if  $i \neq k$  for every  $j \in \mathbb{N}$ . Thus,

$$\sum_{j=1}^{\infty} \|\lambda_1 \widetilde{T}_1(x_j) + \cdots + \lambda_n \widetilde{T}_n(x_j)\|^p = \sum_{j=1}^{\infty} \|\lambda_k T_k(x_j)\|^p = +\infty,$$

proving that the span of  $(\widetilde{T}_k)_{k \in \mathbb{N}}$  is contained in  $K(E, F) \setminus \Pi_p(E, F)$ .

Let us prove now that the set of operators  $(\widetilde{T}_k)_{k \in \mathbb{N}}$  is linearly independent. Let  $\lambda_1, \dots, \lambda_n$  be scalars such that  $\lambda_1 \widetilde{T}_1 + \cdots + \lambda_n \widetilde{T}_n = 0$ . Choosing  $x_k \in E_k$  such that  $\widetilde{T}_k(x_k) \neq 0$  for each  $k \in \{1, \dots, n\}$ , we have that

$$0 = \lambda_1 \widetilde{T}_1(x_k) + \cdots + \lambda_n \widetilde{T}_n(x_k) = \lambda_k \widetilde{T}_k(x_k).$$

It follows that  $\lambda_k = 0$  for every  $k \in \{1, \dots, n\}$ .

Now, let us suppose that  $F$  contains a subspace  $F_0$  with unconditional basis  $(e_n)_{n \in \mathbb{N}}$  having unconditional basis constant  $C$ . Considering again the subsets  $(A_k)_{k \in \mathbb{N}}$  as above, we define  $F_k$  as the closed span of  $(e_n)_{n \in A_k}$  and let  $P_k : F_0 \rightarrow F_k$  be the corresponding projections. We also obtain as above that  $\|P_k\| \leq C$ . For each  $k \in \mathbb{N}$  there exists  $T_k \in K(E, F_k) \setminus \Pi_p(E, F_k)$ .

If  $y_i \in F_i$  and  $y_j \in F_j$  with  $i \neq j$ , then

$$\|y_i\| = \|P_i(y_i + y_j)\| \leq C \|y_i + y_j\|.$$

We define now the operator  $\widetilde{T}_k$  by the composition of  $T_k$  with the inclusion from  $F_k$  to  $F$ . It is clear that  $\widetilde{T}_k \in K(E, F) \setminus \Pi_p(E, F)$  for each  $k \in \mathbb{N}$ . Since

$$\|\widetilde{T}_i(x) + \widetilde{T}_j(x)\| \geq C^{-1} \|\widetilde{T}_i(x)\|$$

for every  $x \in E$ , then  $\widetilde{T}_i + \widetilde{T}_j \in K(E, F) \setminus \Pi_p(E, F)$  for all  $i, j \in \mathbb{N}$ . It is easy to deduce that the span of  $(\widetilde{T}_k)_{k \in \mathbb{N}}$  is contained in  $K(E, F) \setminus \Pi_p(E, F)$ . The linear independence is obtained as in the first case.  $\square$

In [13], Theorem 3.5 was improved by establishing spaceability, and indeed a single infinite dimensional closed subspace valid for all  $p \geq 1$ . In order to obtain this improvement we will also need the following result:

**Proposition 3.6** ([13]). *Let  $E_n$  ( $n \in \mathbb{N}$ ) and  $F$  be Fréchet spaces and  $T_n : E_n \rightarrow F$  bounded linear operators. Let  $G$  be the span of  $\bigcup_{n=1}^{\infty} T_n(E_n)$ . If  $G$  is closed in  $F$ , then there exists  $k \in \mathbb{N}$  such that  $G$  is the span of  $\bigcup_{n=1}^k T_n(E_n)$ .*

**Theorem 3.7** ([13]). *Let  $E$  and  $F$  be infinite dimensional Banach spaces. If  $E$  is superreflexive, then*

$$K(E, F) \setminus \bigcup_{1 \leq p < \infty} \Pi_p(E, F)$$

*is spaceable.*

*Proof.* Since  $K(E, F)$  is closed in the operator norm, it follows that  $\Pi_p(E, F) \cap K(E, F)$  is closed in  $\Pi_p(E, F)$ , hence a Banach space in the norm  $\pi_p(\cdot)$  for every  $p \geq 1$ .

In the proof of [22, Theorem], it is shown that the norm induced by  $\Pi_p(E, F)$  on the finite-rank operators is not equivalent to the operator norm for  $1 \leq p < \infty$  when  $E$  is superreflexive. Thus,  $\Pi_p(E, F) \cap K(E, F)$  is not closed in  $K(E, F)$ .

Due to Proposition 3.6 the union

$$\bigcup_{1 \leq p < \infty} \Pi_p(E, F) \cap K(E, F) = \bigcup_{p \in \mathbb{N}} \Pi_p(E, F) \cap K(E, F)$$

is not closed and the result then follows from Theorem 1.1.  $\square$

Finally, in [16], the authors attained a general criterium. They introduced the notion of  $\sigma$ -reproducible space in order to obtain the spaceability of the set  $I_1(E, F) \setminus I_2(E, F)$ , where  $I_1$  and  $I_2$  are general operator ideals in the sense of Pietsch,  $E$  or  $F$  belongs to that class of spaces and  $I_1(E, F)$  is complete with the usual operator norm or with another complete ideal norm.

**Definition 3.8.** A Banach space  $E$  is said to be  $\sigma$ -reproducible if there exists a sequence  $(E_n)_{n \in \mathbb{N}}$  of complemented subspaces, where  $P_n : E \rightarrow E_n$  is a bounded projection, such that each  $E_n$  is isomorphic to  $E$ ,  $P_i \circ P_j = 0$  if  $i \neq j$ , and for all  $k \in \mathbb{N}$  the projections  $\widetilde{P}_k = \sum_{n=1}^k P_n : E \rightarrow \bigoplus_{n=1}^k E_n$  are uniformly bounded.

Notice that this notion is an isomorphic property. Also, if  $E$  and  $F$  are  $\sigma$ -reproducible Banach spaces, then  $E \oplus F$  and the dual  $E^*$  are also  $\sigma$ -reproducible (see [16, Proposition 3.2]).

Every Banach space  $E$  with a Schauder decomposition  $(E_n)_{n \in \mathbb{N}}$  (see [23, page 47]) such that each subspace  $E_n$  is isomorphic to  $E$ , is  $\sigma$ -reproducible. However, any indecomposable space is not  $\sigma$ -reproducible (recall that a Banach space  $E$  is *indecomposable* if there do not exist infinite-dimensional closed subspaces  $F$  and  $G$  of  $E$  with  $E = F \oplus G$ ).

**Proposition 3.9** ([16]). Every rearrangement invariant space  $E$  is  $\sigma$ -reproducible.

*Proof.* First, let  $E$  be a rearrangement invariant space on  $[0, 1]$ . For every  $a \in [0, 1]$  and  $r \in (0, 1 - a]$  we consider the complemented subspace  $E_{a,r} = \{x \in E : \text{supp } x \subseteq [a, a + r]\}$  and the bounded projection  $P_{a,r} : E \rightarrow E_{a,r}$  given by  $P_{a,r}(x) = x \chi_{[a, a+r]}$  for  $x \in E$ .

For a measurable function  $x$  we define the linear operators

$$T_{a,r}(x)(t) = x\left(\frac{t-a}{r}\right) \chi_{(a, a+r]}(t)$$

and

$$S_{a,r}(x)(t) = x((1-t)a + t(a+r))$$

which are bounded from  $L^\infty$  to  $L^\infty$  and from  $L^1$  to  $L^1$ . Thus, using the Calderón-Mitjagin interpolation theorem ([21, Theorem 2.a.10]),  $T_{a,r}$  and  $S_{a,r}$  are bounded from  $E$  to  $E$ .

These operators also have the following properties:

1.  $(S_{a,r} \circ T_{a,r})(x) = x$  for every  $x \in E$ .
2.  $(T_{a,r} \circ S_{a,r})(x) = x$  for every  $x \in E_{a,r}$ .
3.  $T_{a,r} : E \rightarrow E_{a,r}$  is an isomorphism.
4.  $S_{a,r} : E_{a,r} \rightarrow E$  is an isomorphism.

Let us show only that  $T_{a,r}$  is injective (it is easy to prove the rest of the properties). Indeed, if  $x \in E \setminus \{0\}$ , then there exists  $n \in \mathbb{N}$  such that  $\lambda(A_n) > 0$  where  $A_n$  is defined as  $A_n = \{t \in [0, 1] : |x(t)| > 1/n\}$ . Thus  $\lambda(A'_n) > 0$  with  $A'_n = \{(1-t)a + t(a+r) : t \in A_n\}$ . And  $|T_{a,r}(x)(s)| > 1/n$  for every  $s \in A'_n$ .

For every  $n \in \mathbb{N}$  we consider  $a_n = 1 - \frac{1}{2^{n-1}}$  and  $r_n = \frac{1}{2^n}$ . Let  $E_n = E_{a_n, r_n}$  and  $P_n = P_{a_n, r_n}$ . Since  $\|\widetilde{P}_k\| = 1$  for all  $k \in \mathbb{N}$ , we conclude that  $E$  is  $\sigma$ -reproducible.

We consider now a rearrangement invariant space  $E$  on  $[0, \infty)$ . Let  $\{A_n : n \in \mathbb{N}\}$  be a disjoint sequence of subsets of  $[0, \infty)$  where  $A_n = \bigcup_{k=1}^\infty (a_{n,k}, a_{n,k} + 1]$  for an increasing sequence  $(a_{n,k})_{k \in \mathbb{N}} \subset \mathbb{N}$ , and the complemented subspaces  $E_n = \{x \in E : \text{supp } x \subseteq A_n\}$ .

Given a measurable function  $x$ , we define the linear operators

$$T_n(x)(t) = \sum_{k=1}^\infty x(t + k - 1 - a_{n,k}) \chi_{(a_{n,k}, a_{n,k} + 1]}(t)$$

and

$$S_n(x)(t) = \sum_{k=1}^\infty x(t + a_{n,k} - k - 1) \chi_{(k-1, k]}(t).$$

Since  $(T_n(x))^* = x^*$ ,  $(S_n(x))^* \leq x^*$  and  $(S_n \circ T_n)(x) = x$ , we have that  $T_n : E \rightarrow E_n$  is an isometry and  $S_n : E_n \rightarrow E$  is an isomorphism. Now, reasoning as in the  $[0, 1]$  case we obtain the result.

Finally, we consider a symmetric sequence space. Let  $\{A_k : k \in \mathbb{N}\}$  be a disjoint partition of  $\mathbb{N}$  where the subset  $A_k$  is the range of an injective map  $\varphi_k : \mathbb{N} \rightarrow \mathbb{N}$  for every  $k \in \mathbb{N}$ .

For  $x = (x_n)_{n \in \mathbb{N}}$  we define the linear operators  $T_k(x) = (a_n)_{n \in \mathbb{N}}$  with

$$a_n = \begin{cases} x_m & \text{if } \varphi_k(m) = n \\ 0 & \text{if } n \notin A_k \end{cases}$$

and  $S_k(x) = (x_{\varphi_k(n)})_{n \in \mathbb{N}}$ .

If  $E_k = \{x \in E : \text{supp } x \subseteq A_k\}$ , we have that  $T_k : E \rightarrow E_k$  is an isometry and  $S_k : E_k \rightarrow E$  is an isomorphism. And reasoning again as in the  $[0, 1]$  case we obtain the result.  $\square$

Also, the space  $C[0, 1]$  is  $\sigma$ -reproducible (see [16, Proposition 3.4]). As a consequence, the space  $C(K)$  is  $\sigma$ -reproducible for any uncountable compact metric space  $K$  (see [24, Theorem 4.4.8]).

The following lemma will be useful in the proof of the general criterium:

**Lemma 3.10** ([16]). *Let  $I$  be an operator ideal.*

1. *If  $E_1$  and  $E_2$  are isomorphic Banach spaces, then there exists a bijection between  $I(E_1, F)$  and  $I(E_2, F)$  for every Banach space  $F$ .*
2. *If  $G$  is a closed subspace of the Banach space  $E$  and  $T \in I(E, F)$ , then the restriction  $T|_G \in I(G, F)$  for every Banach space  $F$ .*

**Theorem 3.11** ([16]). *Let  $I_1$  and  $I_2$  be operator ideals such that  $I_1(E, F) \setminus I_2(E, F)$  is non-empty for a couple of Banach spaces  $E$  and  $F$ . If  $E$  or  $F$  is  $\sigma$ -reproducible and  $I_1(E, F)$  is complete for an ideal norm, then  $I_1(E, F) \setminus I_2(E, F)$  is spaceable.*

*Proof.* Let  $T \in I_1(E, F) \setminus I_2(E, F)$ . If  $E$  is a  $\sigma$ -reproducible Banach space with isomorphisms  $\phi_n : E_n \rightarrow E$  and bounded projections  $P_n : E \rightarrow E_n$ , for every  $n \in \mathbb{N}$  we consider the operator  $T_n = T \circ \phi_n \circ P_n$  which belongs to  $I_1(E, F) \setminus I_2(E, F)$ . Indeed, using Lemma 3.10, if  $T_n \in I_2(E, F)$ , then we have that  $T_n|_{E_n} = T \circ \phi_n \in I_2(E_n, F)$ , but this is not true. The sequence  $(T_n)_{n \in \mathbb{N}}$  is formed by linearly independent operators. To show this, if  $\sum_{n=1}^k a_n T_n = 0$ , restricting to  $E_j$  we obtain  $a_j = 0$  with  $1 \leq j \leq k$ . In the same way, it can be showed that  $\sum_{n=1}^k a_n T_n$  cannot belong to  $I_2(E, F)$ . Thus,  $I_1(E, F) \setminus I_2(E, F)$  is lineable.

Furthermore,  $(T_n)_{n \in \mathbb{N}}$  is a basic sequence in  $I_1(E, F)$ . Indeed, for any integers  $k < m$  and any choice of scalars  $(\lambda_n)_{n \in \mathbb{N}}$  we have

$$\left\| \sum_{n=1}^k \lambda_n T_n \right\|_{I_1} = \left\| \sum_{n=1}^m \lambda_n T_n \circ \widetilde{P}_k \right\|_{I_1} \leq \left\| \sum_{n=1}^m \lambda_n T_n \right\|_{I_1} \|\widetilde{P}_k\|.$$

Let  $S \in \overline{[T_n : n \in \mathbb{N}]} \subset I_1(E, F)$  with  $S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0$ . Then there exists  $n_0 \in \mathbb{N}$  such that  $\lambda_{n_0} \neq 0$ . We have that  $S|_{E_{n_0}} = \lambda_{n_0} T \circ \phi_{n_0} \notin I_2(E_{n_0}, F)$ . Thus,  $S \notin I_2(E, F)$  and  $\overline{[T_n : n \in \mathbb{N}]} \subset I_1(E, F) \setminus I_2(E, F)$ .

If  $F$  is  $\sigma$ -reproducible with isomorphisms  $(\phi_n)_{n \in \mathbb{N}}$ , for each  $n \in \mathbb{N}$  we consider the operator  $T_n = \phi_n^{-1} \circ T$  which belongs to  $I_1(E, F) \setminus I_2(E, F)$ . The sequence  $(T_n)_{n \in \mathbb{N}}$  is formed by linearly independent operators. Thus, we obtain that  $I_1(E, F) \setminus I_2(E, F)$  is lineable.

And  $(T_n)_{n \in \mathbb{N}}$  is a basic sequence. Indeed, for any integers  $k < m$  and any choice of scalars  $(\lambda_n)_{n \in \mathbb{N}}$  we have

$$\left\| \sum_{n=1}^k \lambda_n T_n \right\|_{I_1} = \left\| \widetilde{P}_k \circ \sum_{n=1}^m \lambda_n T_n \right\|_{I_1} \leq \|\widetilde{P}_k\| \left\| \sum_{n=1}^m \lambda_n T_n \right\|_{I_1}.$$

Let  $S \in \overline{[T_n : n \in \mathbb{N}]} \subset I_1(E, F)$  with  $S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0$ . There exists  $n_0 \in \mathbb{N}$  such that  $\lambda_{n_0} \neq 0$ . If  $S \in I_2(E, F)$ , then  $P_{n_0} \circ S \in I_2(E, F)$ , but this is not true because  $P_{n_0} \circ S = \lambda_{n_0} T_{n_0}$ . Then  $\overline{[T_n : n \in \mathbb{N}]} \subset I_1(E, F) \setminus I_2(E, F)$ .  $\square$

The general criterium can be extended in the following way:

**Theorem 3.12** ([16]). *If  $E$  or  $F$  is a  $\sigma$ -reproducible Banach space,  $I$  is an operator ideal such that  $I(E, F)$  is complete for an ideal norm, and  $(I_n)_{n \in \mathbb{N}}$  is a sequence of operator ideals such that the set  $I(E, F) \setminus I_n(E, F)$  is non-empty for every  $n \in \mathbb{N}$ , then the set  $I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F)$  is spaceable.*



*Proof.* Let  $S_n \in I(E, F) \setminus I_n(E, F)$  for every  $n \in \mathbb{N}$ .

If  $E$  is a  $\sigma$ -reproducible Banach space with isomorphisms  $(\phi_n)_{n \in \mathbb{N}}$  and bounded projections  $(P_n)_{n \in \mathbb{N}}$ , let us consider the operators  $S_n \circ \phi_n \circ P_n \in I(E, F) \setminus I_n(E, F)$  for every  $n \in \mathbb{N}$ . Then

$$T = \sum_{n=1}^{\infty} \frac{S_n \circ \phi_n \circ P_n}{2^n \|S_n \circ \phi_n \circ P_n\|_I}$$

belongs to  $I(E, F) \setminus I_n(E, F)$  for every  $n \in \mathbb{N}$ .

Now, reasoning as in the proof of Theorem 3.11 we can construct a sequence  $(T_k)_{k \in \mathbb{N}}$  such that  $\overline{[T_k : k \in \mathbb{N}]} \subset I(E, F) \setminus I_n(E, F)$  for every  $n \in \mathbb{N}$ .

If  $F$  is a  $\sigma$ -reproducible Banach space with isomorphisms  $(\phi_n)_{n \in \mathbb{N}}$ , let us consider the operators  $\phi_n^{-1} \circ S_n \in I(E, F) \setminus I_n(E, F)$  for every  $n \in \mathbb{N}$ . Then

$$T = \sum_{n=1}^{\infty} \frac{\phi_n^{-1} \circ S_n}{2^n \|\phi_n^{-1} \circ S_n\|_I}$$

belongs to  $I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F)$ . □

**Corollary 3.13** ([16]). *Let  $E$  and  $F$  be Banach spaces, and  $\{I_p : p \in [a, b]\}$  be a family of operator ideals such that  $I_p(E, F) \subsetneq I_q(E, F)$  if  $p < q$  with continuous inclusion. If  $E$  or  $F$  is a  $\sigma$ -reproducible Banach space and  $I_b(E, F)$  is complete for an ideal norm, then the set  $I_b(E, F) \setminus \bigcup_{p < b} I_p(E, F)$  is spaceable.*

In general, Theorem 3.11 does not hold for arbitrary Banach spaces. Consider for instance the spaces with few operators given in [25, 26]. They are hereditary indecomposable Banach spaces on which every bounded linear operator is a compact perturbation of a scalar multiple of the identity.

Finally, let us remark that in the special case of considering Banach operator ideals  $I_1$  and  $I_2$  with proper continuous inclusion  $I_2 \subset I_1$  where the ideal  $I_2$  is not closed in  $I_1$ , the set of operators  $I_1(E, F) \setminus I_2(E, F)$  is always spaceable. This follows from Theorem 1.1.

Many applications of the general criterium can be found in [16] where a good number of particular operator ideals are considered: compact operators, strictly singular operators, strictly co-singular operators, finitely strictly singular operators,  $(q, p)$ -summing operators and Schatten operator classes.

## References

- [1] Aron R.M., Gurariy V.I., Seoane-Sepúlveda J.B., Lineability and spaceability of sets of functions on  $\mathbb{R}$ , Proc. Amer. Math. Soc., 2005, 133, 795-803
- [2] Aron R.M., García-Pacheco F.J., Pérez-García D., Seoane-Sepúlveda J.B., On dense-lineability of sets of functions on  $\mathbb{R}$ , Topology, 2009, 48, 149-156
- [3] Azagra D., Muñoz-Fernández G.A., Sánchez V.M., Seoane-Sepúlveda J.B., Riemann integrability and Lebesgue measurability of the composite function, J. Math. Anal. Appl., 2009, 354, 229-233
- [4] Bernal-González L., Ordoñez M., Lineability criteria, with applications, J. Funct. Anal., 2014, 266, 3997-4025
- [5] Cariello D., Seoane-Sepúlveda J.B., Basic sequences and spaceability in  $\ell_p$  spaces, J. Funct. Anal., 2014, 266, 3797-3814
- [6] Enflo P.H., Gurariy V.I., Seoane-Sepúlveda J.B., Some results and open questions on spaceability in function spaces, Trans. Amer. Math. Soc., 2014, 366, 611-625
- [7] Gámez-Merino J.L., Muñoz-Fernández G.A., Sánchez V.M., Seoane-Sepúlveda J.B., Sierpiński-Zygmund functions and other problems on lineability, Proc. Amer. Math. Soc., 2010, 138, 3863-3876
- [8] Ruiz C., Sánchez V.M., Nonlinear subsets of function spaces and spaceability, Linear Algebra Appl., 2014, 463, 56-67
- [9] Gurariy V.I., Subspaces and bases in spaces of continuous functions, Dokl. Akad. Nauk SSSR, 1966, 167, 971-973, in Russian
- [10] Levin B., Milman D., On linear sets in the space  $C$  consisting of functions of bounded variation, Zapiski Inst. Mat. Mekh. Kharkov, 1940, 16, 102-105, in Russian, English summary
- [11] Aron R.M., Bernal-González L., Pellegrino D., Seoane-Sepúlveda J.B., Lineability: the search for linearity in mathematics, CRC Press, 2016
- [12] Wilansky A., Semi-Fredholm maps in  $FK$  spaces, Math. Z., 1975, 144, 9-12
- [13] Kitson D., Timoney R.M., Operator ranges and spaceability, J. Math. Anal. Appl., 2011, 378, 680-686



- [14] Puglisi D., Seoane-Sepúlveda J.B., Bounded linear non-absolutely summing operators, *J. Math. Anal. Appl.*, 2008, 338, 292-298
- [15] Botelho G., Diniz D., Pellegrino D., Lineability of the set of bounded linear non-absolutely summing operators, *J. Math. Anal. Appl.*, 2009, 357, 171-175
- [16] Hernández F.L., Ruiz C., Sánchez V.M., Spaceability and operator ideals, *J. Math. Anal. Appl.*, 2015, 431, 1035-1044
- [17] Diestel J., Jarchow H., Pietsch A., Operator ideals, *Handbook of the geometry of Banach spaces*, volume 1, 437-496, North-Holland, 2001
- [18] Pietsch A., Operator ideals, North-Holland, 1980
- [19] Bennett C., Sharpley R., Interpolation of operators, Academic Press, 1988
- [20] Kreĭn S.G., Petunin Ju.I., Semenov E.M., Interpolation of linear operators, A.M.S., 1982
- [21] Lindenstrauss J., Tzafriri L., Classical Banach spaces II, Function spaces, Springer, 1979
- [22] Davis W.J., Johnson W.B., Compact, non-nuclear operators, *Studia Math.*, 1974, 51, 81-85
- [23] Lindenstrauss J., Tzafriri L., Classical Banach spaces I, Sequence spaces, Springer, 1977
- [24] Albiac F., Kalton N.J., Topics in Banach space theory, Graduate Texts in Mathematics, 233, Springer, 2006
- [25] Argyros S.A., Haydon R.G., A hereditarily indecomposable  $\mathcal{L}^\infty$ -space that solves the scalar-plus-compact problem, *Acta Math.*, 2011, 206, 1-54
- [26] Tarbard M., Hereditarily indecomposable, separable  $\mathfrak{L}^\infty$  Banach spaces with  $\ell^1$  dual having few but not very few operators, *J. London Math. Soc.*, 2012, 85, 737-764