Particle production from nonlocal gravitational effective action

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(Received 5 March 1999; published 27 October 1999)

In this paper we show how the nonlocal effective action for gravity, obtained after integrating out the matter fields, can be used to compute particle production and spectra for different space-time metrics. Applying this technique to several examples, we find that the perturbative calculation of the effective action up to second order in curvatures yields exactly the same results for the total number of particles as the Bogolyubov transformations method, in the case of massless scalar fields propagating in a Robertson-Walker space-time. Using an adiabatic approximation we also obtain the corresponding spectra and compare the results with the traditional WKB approximation. [S0556-2821(99)02920-3]

PACS number(s): 04.62.+v, 98.80.Cq

I. INTRODUCTION

In recent years the phenomenon of particle creation from classical sources has experienced a growing interest, mainly motivated by its numerous applications in cosmology, but also in other areas of physics. In cosmology, it plays a fundamental role in the mechanism of reheating after inflation [1] which is believed to be responsible for the creation of almost all the particles that populate the universe today. In the reheating models, the oscillations of a homogeneous scalar field (inflaton) around the minimum of its potential give rise to an explosive creation of a large amount of particles. On the other hand, the same methods are applied to the generation of primordial density inhomogeneities in the early universe that later on grew to create the present galactic structure [2]. In addition, the cosmological expansion can give rise to the production of a stochastic background of gravitational waves [3]. Bounds on the density of these waves are very useful to constraint the different cosmological models [4]. In all these applications, the method which is used for the calculation of the rates and spectra of the particles produced is the traditional mode-mixing Bogolyubov technique [5].

On the other hand the notion of effective action (EA) has proved to be a very useful tool for the development of the so-called phenomenological Lagrangians. Typically, effective actions are obtained in theories with heavy and light fields by functional integration of the heavy modes to find the effective low-energy theory for the light modes after some momentum expansion. Usual applications of those techniques include low-energy hadron dynamics (the so-called chiral perturbation theory), the symmetry breaking sector of the standard model, and low-energy quantum gravity (see Ref. [6] for a recent review, and references therein). Effective actions use to have a real and, in general, divergent part, that give rise to modifications of the classical equations of motion due to quantum effects. Eventually, the corresponding vacuum solutions could not exhibit some of the symmetries of the classical theory, thus giving rise to the well-known phenomenon of spontaneous radiative symmetry breaking. In addition, nonlocal finite terms also appear in the EA which contribute to the imaginary part. This imaginary part is physically important since it is connected with the possibility of having particle production [7,8]. By this we mean the production of the quanta corresponding to the fields that have been integrated out.

In this paper, we consider the production of scalar particles from classical gravitational backgrounds from the effective action point of view. We show how a perturbative calculation up to second order in the curvatures in the case of masless scalar fields, reproduces the well-known general results of particle production in Robertson-Walker space-times and can give rise to the exact amount of particles at least in the models we have considered. The paper is organized as follows: in Sec. II, we review the Euler-Heisenberg Lagrangian for QED, but paying special attention to its nonlocal part. We show how the perturbative calculation up to second order in the coupling constant yields the correct expression for the imaginary part in the massless case. In Sec. III, we introduce the nonlocal gravitational effective action for scalar fields and discuss some of the conditions for its application. Section IV is devoted to the actual calculation of the total number of particles produced due to the expansion in several Robertson-Walker models and the results are compared with those obtained by the Bogolyubov technique. In Sec. V we study how to obtain the spectrum of the particles and compare the results with the WKB approximation. Finally, Sec. VI contains the main conclusions of the work.

II. THE NONLOCAL EULER-HEISENBERG LAGRANGIAN

Let us consider the well-known Euler-Heisenberg Lagrangian for QED in flat space-time [9]. When the momentum p of photons is much smaller than the electron mass M, the one-loop effects, such as vacuum polarization, can be taken into account by adding local nonlinear terms to the classical electromagnetic Lagrangian. Consider the QED effective action given by

$$e^{iW[A]}=\int [d\psi][d\bar{\psi}]\times\exp-\frac{i}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}\times\exp\left(i\int d^4x\bar{\psi}(i\not{D}-M+i\epsilon)\psi\right)=\exp\left(-\frac{i}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}\det(i\not{D}-M+i\epsilon)\right), \quad (1)$$

where as usual $\not{D}=\gamma^{\mu}(\partial_{\mu}-ieA_{\mu})$. From Eq. (1) we can write the effective action as

$$W[A]=-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}-i\,\mathrm{Tr}\log[(i\not{D}-M+i\epsilon)]. \quad (2)$$

Expanding in a formal way the logarithm we obtain

$$W[A]=-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}+i\sum_{k=1}^{\infty}\frac{(-e)^k}{k}\mathrm{Tr}[(i\not{\partial}-M)^{-1}\not{A}]^k. \quad (3)$$

Using dimensional regularization, it is possible to find the following expression up to quadratic terms in the photon field:

$$W[A]=\int d^4x\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{e^2}{3(4\pi)^2}\Delta F^{\mu\nu}F_{\mu\nu}-\frac{2e^2}{(4\pi)^2}F^{\mu\nu}\left[-\frac{2}{3}\frac{M^2}{\Box}-\frac{1}{6}\left(1-2\frac{M^2}{\Box}\right)\times F(-\Box;M^2)\right]F_{\mu\nu}\right\}+\mathcal{O}(A^4), \quad (4)$$

where $\Delta=N_{\epsilon}-\log(M^2/\mu^2)$, $N_{\epsilon}=2/\epsilon-\gamma+\log 4\pi$ is the well known constant appearing in dimensional regularization, and we have used the expression

$$F(-\Box;M^2)F_{\mu\nu}(x)=\int d^4y\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}F(p^2;M^2)F_{\mu\nu}(y) \quad (5)$$

with

$$F(p^2;M^2)=2+\int_0^1 dt\log\left(1-\frac{p^2}{M^2}t(1-t)\right). \quad (6)$$

In the $p^2>4M^2$ case, this function can be written as

$$F(p^2;M^2)=\sqrt{1-\frac{4M^2}{p^2}}\log\frac{\sqrt{1-4M^2/p^2}+1}{\sqrt{1-4M^2/p^2}-1}. \quad (7)$$

In a similar way, the inverse operator $1/\Box$ can be defined with the usual boundary conditions on the fields as

$$\frac{1}{-\Box}F_{\mu\nu}(x)=\int d^4y\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{1}{p^2+i\epsilon}F_{\mu\nu}(y). \quad (8)$$

The expression (4) for the EA has a regular massless limit. In fact, for small p compared with M, the Mandelstam function $F(p^2;M^2)$ behaves as

$$F(p^2;M^2)=-\log\left(\frac{M^2}{-p^2-i\epsilon}\right)+\mathcal{O}(M^2). \quad (9)$$

From Eq. (4) we can see that the only contributions in the massless limit are those coming, on one hand from the Δ factor and, on the other hand, from the Mandelstam function. Both logarithmic contributions equal, up to sign, so that they cancel each other and we obtain

$$W[A]=\int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{e^2}{3(4\pi)^2}F^{\mu\nu}\Gamma(\Box)F_{\mu\nu}\right)+\mathcal{O}(A^4), \quad (10)$$

where we have used the following notation:

$$\Gamma(\Box)=N_{\epsilon}-\log\left(\frac{\Box}{\mu^2}\right) \quad (11)$$

to be understood as in the previous cases through the corresponding Fourier transform, with the $i\epsilon$ factor as shown in Eq. (9). We see that the massless limit of the EA is a nonlocal but analytical functional in the gauge curvatures $F_{\mu\nu}$.

The EA (4) allows us to derive in an exact fashion the photon two-point one loop Green functions. This, in turn, allows us to obtain for example the vacuum polarization. In the massive case, the EA can be expanded as a power series in p^2/M^2, and also in A to obtain the well-known local Euler-Heisenberg Lagrangian [9]

$$\mathcal{L}_{\mathrm{eff}}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{e^2}{3(4\pi)^2}\Delta F^{\mu\nu}F_{\mu\nu}-\frac{e^2}{15(4\pi)^2M^2}F^{\mu\nu}\Box F_{\mu\nu}+\frac{e^4}{90(4\pi)^2M^4}\times\left((F^{\mu\nu}F_{\mu\nu})^2+\frac{7}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right)+\mathcal{O}\left(\frac{p^2}{M^2}\right)^3+\mathcal{O}(A^6). \quad (12)$$

The EA (4) possesses a nonvanishing imaginary part coming from the Mandelstam function (7). This imaginary part provides the pair production rate [8]. In the massless case (10) we get

$$\mathrm{Im}\, W[A] = \mathrm{Im} \int d^4x \mathcal{L}_{\mathrm{eff}}$$

$$= -\frac{e^2}{48\pi} \int d^4x\, d^4y \frac{d^4p}{(2\pi)^4} \times e^{ip(x-y)} F_{\mu\nu}(x) F^{\mu\nu}(y)\, \theta(p^2), \tag{13}$$

where

$$\theta(x) = \begin{cases} 0, & x<0, \\ 1/2, & x=0, \\ 1, & x>0. \end{cases} \tag{14}$$

The 1/2 value arises as a consequence of the $-i\epsilon$ factor in Eq. (9). For constant electric fields and in absence of magnetic fields, the previous expression gives the probability per unit time and unit volume that at least one electron-positron pair is created by the electric field

$$p \simeq 2\, \mathrm{Im}\, \mathcal{L}_{\mathrm{eff}} = \frac{e^2}{24\pi} \vec{E}^2. \tag{15}$$

Let us compare this result with exact expression for the imaginary part obtained by Schwinger [8]:

$$p \simeq 2\, \mathrm{Im}\, \mathcal{L}_{\mathrm{eff}} = \frac{e^2 E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-m^2 n\pi/eE}. \tag{16}$$

The dependence in the electric fields appears in both a quadratic term and a nonanalytical contribution $\exp(-m^2 n\pi/eE)$. This latter term shows the importance of the nonperturbative effects in the particle production phenomenon [1]. However, in the massless limit the nonanalytical pieces disappear and the result exactly agrees with the perturbative calculation in (15). Notice that in this case, gauge invariance and the dimension of the effective Lagrangian constraint the result to be quadratic in eE and that is the reason why the second order perturbative calculation gives rise to the exact result. Accordingly, in the massless limit, the perturbative calculation can provide, in some cases, all the relevant information about the particle production processes. In the gravitational case that we will study in the next sections, we will show that the same effect takes place.

III. THE EFFECTIVE ACTION FOR GRAVITY

Let us consider a real scalar field in a curved space-time with an arbitrary nonminimal coupling to the curvature. The corresponding classical action is given by

$$S[\phi] = -\frac{1}{2} \int d^4x \sqrt{g}\, \phi(\Box + m^2 + \xi R)\phi, \tag{17}$$

where

$$\Box\phi = g^{\mu\nu} \nabla_\mu \partial_\nu \phi = \frac{1}{\sqrt{g}} \partial_\mu (g^{\mu\nu} \sqrt{g}\, \partial_\nu \phi). \tag{18}$$

The EA for the gravitational fields that arises after integrating out the real scalar matter fields is given by the following expression in Lorentzian signature:

$$\begin{aligned} \langle 0,\mathrm{out}|0,\mathrm{in}\rangle &= Z[g_{\mu\nu}] = e^{iW[g_{\mu\nu}]} \\ &= \int [d\phi] e^{iS[g_{\mu\nu},\phi]} \\ &= \int [d\phi] \\ &\quad \times \exp -\frac{i}{2} \int dx \sqrt{g}\, \phi(\Box + m^2 + \xi R - i\epsilon)\phi \\ &= (\det O)^{-1/2}, \end{aligned} \tag{19}$$

where $O_{xy}(m^2) = [-\Box_y - m^2 - \xi R(y) + i\epsilon]\delta^0(x,y)$ with $\delta^0(x,y)$ being the covariant delta $\delta^0(x,y) = g^{-1/2}(x)\delta(x,y)$. Thus we see that, following the analogy with flat space-time we could interpret $Z[g_{\mu\nu}]$ as the vacuum persistence amplitude. Thus we have

$$W[g_{\mu\nu}] = \frac{i}{2} \log \det O(m^2) = \frac{i}{2} \mathrm{Tr} \log O(m^2). \tag{20}$$

In this expression we have integrated the scalars but the gravitational field is treated classically. Accordingly, this EA must be added to the classical action for the gravitational field and it includes the quantum effects due to the matter fields. In addition, Eq. (19) is the generating functional of the Green functions containing scalar loops only and external gravitational legs.

Once one knows the EA at least in some limit, we have all the information concerning the semiclassical gravitational evolution in this limit. As we mentioned in the Introduction, the EA could have a nonvanishing imaginary part, which is related to the pair production probability. In fact, the probability P that at least one pair particle-antiparticle is created by the gravitational field is given by [8]

$$P = 1 - |\langle 0,\mathrm{out}|0,\mathrm{in}\rangle|^2_{g_{\mu\nu}} = 1 - |e^{iW[g_{\mu\nu}]}|^2 = 1 - e^{-2\, Im\, W[g_{\mu\nu}]} \tag{21}$$

for small values of $W[g_{\mu\nu}]$ we have

$$P \simeq 2\, \mathrm{Im}\, W[g_{\mu\nu}]. \tag{22}$$

Concerning the applicability of this equation, let us compare the EA method with the traditional Bogolyubov technique. The classical equations of motion for the scalar field are

$$(\Box + m^2 + \xi R)\phi = 0. \tag{23}$$

Unlike flat space-time, there is no natural set of mode solutions to this equation, rather we can expand its solutions in different ways, i.e.,

$$\phi = \sum_k (a_k u_k + a_k^\dagger u_k^*) = \sum_k (\overline{a_k}\,\overline{u_k} + \overline{a_k^\dagger}\,\overline{u_k^*}). \tag{24}$$

Each of these expansions will give rise to different Fock spaces when interpreting the coefficients $a_k, a_k^\dagger$ and $\overline{a}_k, \overline{a}_k^\dagger$ as creation and annhilation operators. A problem arises when we try to identify which of these Fock spaces corresponds to our usual notion of particle. In general, this question can only be answered when we have a high degree of symmetry (conformal invariance) or if the space-time is flat in the asymptotic in and out regions. However, in most of the interesting situations, these two conditions are not present. A solution to this problem was suggested in a series of works (see Ref. [5], and references therein) in which the notion of adiabatic vacuum is introduced. In the cosmological space-times in which we will be mainly interested, in order to define an adiabatic vacuum it is only required that asymptotically in the past and in the future the rate of expansion vanishes, i.e., $\dot{a}/a\rightarrow 0$ with $a(t)$ the universe scale factor. Expressing this statement in a covariant way, it would be equivalent to require that the curvatures and all their covariant derivatives vanish in the far past and future.

In the effective action approach, $Z[g_{\mu\nu}]$ can be interpreted as vacuum persistence amplitude in principle only when the vacuum states $|0,\text{in}\rangle$ and $|0,\text{out}\rangle$ can be defined in regions with a temporal separation [10]. When this does not occur, it is not obvious what the interpretation of the effective action is. However, we will show in the following, that the naive calculation of the effective action, in those situations in which an adiabatic vaccum can be defined although the space-time is not asymptotically Minkowskian, yield the same result for the particle production as the standard Bogolyubov technique. As a consequence, in these cases, we could try to interpret $Z[g_{\mu\nu}]$ as adiabatic vacuum persistence amplitude.

The nonlocal effective action for gravity has been evaluated in different works using several techniques. Thus in Ref. [11] it was suggested what would be the form of the two-point form factors. In Ref. [12] the effective action is derived by means of the so called covariant perturbation theory, valid in asymptotically flat manifolds, in Ref. [13] the same result is obtained by means of the partial resummation of the Schwinger-DeWitt series. The result in all these cases up to second order in curvatures can be written in the massless case as

$$W[g_{\mu\nu}]=\frac{1}{32\pi^2}\int d^4x\sqrt{g}\Bigg[\frac{1}{180}R^{\mu\nu\lambda\rho}(x)\Gamma(\Box)R_{\mu\nu\lambda\rho}(x) -\frac{1}{180}R^{\mu\nu}(x)\Gamma(\Box)R_{\mu\nu}(x) +\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R(x)\Gamma(\Box)R(x)\Bigg]+\mathcal{O}(\mathcal{R}^3), \quad (25)$$

where the form factor $\Gamma(\Box)$ is given in Eq. (11). The local finite pieces as usual depend on the different renormalization schemes and they are not relevant for our calculations, although in general their coefficients are important to fix the form of the linear terms in the trace anomaly. The nonlocal contributions are in any case unambiguous. An appropriate representation of the nonlocal form factors is provided by the use of the Riemann normal coordinates (the details of this approach will be given elsewhere [14]). Thus, taking normal coordinates (x^μ) with origin at y_0 the action of the form factors is understood through the correspondig Fourier transform

$$\mathcal{R}(y_0)\log\left(\frac{\Box}{\mu^2}\right)\mathcal{R}(y_0) =\int d^4x\frac{d^4p}{(2\pi)^4}e^{ipx}\mathcal{R}(y_0)\log\left(\frac{-p^2-i\epsilon}{\mu^2}\right)\mathcal{R}(x) \quad (26)$$

and $\mathcal{R}$ denotes generically the scalar curvature, the Ricci or Riemann tensors. For the sake of simplicity we will study massless scalar particles propagating in a cosmological background, whose metric is that of Friedmann, Robertson, and Walker (FRW):

$$ds^2=dt^2-a(t)^2\left(\frac{dr^2}{1-Kr^2}+r^2d\theta^2+r^2\sin^2(\theta)d\phi^2\right) \quad (27)$$

where K determines the spatial curvature sign [15] and $a(t)$ is the universe scale factor.

The EA imaginary part comes from the logarithms in Eq. (25). Due to the homogeneity and isotropy of space in the present case, the different curvatures appearing in that expression only depend on the time coordinate. Thus, we can perform the spatial coordinates integration in Eq. (26) and generically we will obtain

$$\begin{aligned}\text{Im}\int d^4x\frac{d^4p}{(2\pi)^4}e^{ipx}\mathcal{R}(y_0)\log\left(\frac{-p^2-i\epsilon}{\mu^2}\right)\mathcal{R}(x) &=\text{Im}\int dx^0\frac{dp_0}{(2\pi)}e^{ip_0x^0}\mathcal{R}(y_0)\log\left(\frac{-p_0^2-i\epsilon}{\mu^2}\right)\mathcal{R}(x^0)\\ &=-\pi\mathcal{R}(y_0)\mathcal{R}(y_0). \end{aligned}\quad (28)$$

Let us momentarily consider a general metric, not necessarily FRW. It is easy to see from the first term in this equation that when the metric is static, i.e., only depending on spatial coordinates, the argument in the logarithm would only contain $\vec{p}^2-i\epsilon$. Therefore the imaginary part would be zero and we would recover the well-known result of absence of particle production in general (inhomogeneous) static backgrounds.

FRW metrics. Returning to the FRW metric we obtain from Eq. (28) the general expression

$$\text{Im}\,W[g_{\mu\nu}]=\frac{1}{32\pi}\int d^4x\sqrt{g}\left[\frac{1}{180}R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} -\frac{1}{180}R^{\mu\nu}R_{\mu\nu}+\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2\right]+\mathcal{O}(\mathcal{R}^3). \quad (29)$$

This result is only valid for homogeneous and isotropic metrics. Comparing this result with the divergences, we see that both have the same form. Notice that Eq. (29) is a linear combination of $R^2_{\mu\nu\lambda\rho}$, $R^2_{\mu\nu}$, and R^2, but we can choose a different basis to write it. In particular, we can take the one made out of R^2, $C^2_{\mu\nu\lambda\rho}$, and E, where $C_{\mu\nu\lambda\rho}$ is the Weyl tensor and $E=R^2_{\mu\nu\lambda\rho}-4R^2_{\mu\nu}+R^2$ is the Gauss-Bonnet term. In this basis we have

$$a_1R^2_{\mu\nu\lambda\rho}+a_2R^2_{\mu\nu}+a_3R^2=-\left(a_1+\frac{1}{2}a_2\right)E+\left(2a_1+\frac{a_2}{2}\right)C^2+\left(\frac{1}{3}a_1+\frac{1}{3}a_2+a_3\right)R^2. \tag{30}$$

In our case, $a_1=-a_2$. On the other hand, the FRW metric is locally conformal to the Minkowski metric and hence its Weyl tensor vanishes. Therefore Eq. (29) only contains the scalar curvature and the Gauss-Bonnet terms, but the integral of the latter also vanishes in the class of asymptotically flat metrics. Moreover, the asymptotic flatness is not a neccessary condition for the Gauss-Bonnet term to vanish and, in fact, examples can be found which are not asymptotically flat, but still they have a zero Gauss-Bonnet term contribution (see below). To summarize, the imaginary part in these cases reduces to

$$\operatorname{Im} W[g_{\mu\nu}]=\frac{1}{32\pi}\int d^4x\sqrt{g}\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2+\mathcal{O}(\mathcal{R}^3). \tag{31}$$

Conformal coupling. In the conformal case ($\xi=1/6$) it is evident from the above expression that the EA imaginary part is zero and accordingly there will be no particle production. This is a well-known result and has been proved by studying the positive-energy modes of the corresponding Klein-Gordon equation [16] for the scalar field. The EA provides in this case a simple way to prove a general result.

Radiation dominated universe. Conformal invariance is not the only case in which there is no particle production in a FRW background. From the above arguments, we have seen that the only piece contributing to the EA imaginary part is the R^2 term. If this term vanishes, there would not be particle creation. For a FRW metric with $K=0$ this implies the following condition:

$$\dot{H}=-2H^2, \tag{32}$$

where $H=\dot{a}/a$ is the Hubble parameter. The solution is simply $a(t)=A(t-t_0)^{1/2}$ with A and t_0 arbitrary constants. In fact taking traces in the Einstein equations (with the stress tensor corresponding to a perfect fluid), it is obvious that $R=0$ implies $R=8\pi G(3p-\rho)=0$, with p and ρ the pressure and density of the fluid. Accordingly $\rho=3p$, which is nothing but the state equation for a fluid of highly relativistic particles. Therefore a radiation dominated universe is a stable solution of Einstein equations against pair emission. This result was obtained in Refs. [16–18] by means of the Bogolyubov technique, where in order to circumvent the problem of the initial singularity, it was assumed that when $t\to t_P$ with t_P the Planck time, the scale factor smoothly tends to a constant. This allows us to define an initial vacuum state in the problem. Again the out vacuum is chosen as an adiabatic vacuum. Notice that in this case the Gauss-Bonnet term can also be neglected.

Homogeneous anisotropic metrics. Consider now a general homogeneous but anisotropic metric of the Bianchi type I:

$$ds^2=C^2(\eta)[d\eta^2-g_{ij}(\eta)dx^idx^j], \tag{33}$$

where the three-metric g_{ij} only depends on the time coordinate. Since, as it happened with the FRW, the curvatures only depend on the time coordinate, it is possible to explicitly perform the spatial coordinate integration in Eq. (26). Therefore we obtain the same combination of curvature tensors as in Eq. (29) for the EA imaginary part. In this case the metric is not conformal to the Minkowski one and accordingly it is not possible to drop the Weyl term from Eq. (30). The Gauss-Bonnet term continues vanishing under the same assumptions about the metric. To summarize, the resulting EA imaginary part can be written for this kind of metrics as

$$\operatorname{Im} W[g_{\mu\nu}]=\frac{1}{32\pi}\int d^4x\sqrt{g}\left[\frac{1}{120}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2R^2\right]+\mathcal{O}(\mathcal{R}^3). \tag{34}$$

This result agrees with that of Zel'dovich and Starobinski [19] (see also Ref. [20]) obtained by using standard Bogolyubov techniques. In fact, assuming $g_{ij}=\delta_{ij}[1+h_i(\eta)]$, neglecting terms of order $\mathcal{O}(h^3)$ in Eq. (34) and imposing that asymptotically the anisotropies vanish, we recover their results.

From the above expressions we can extract another consequence. Particle production only takes place when curvature is nonvanishing, i.e., in the presence of a genuine gravitational field and not merely by means of a coordinate change as it happens for an accelerated observer [5], in the latter case the creation could be considered as fictitious. Therefore, for the boundary conditions in the space-time geometry that we mentioned before, the EA provides an invariant criterium (independent of the observer) to decide when particle production takes place.

IV. SPECIFIC EXAMPLES WITH MINIMAL COUPLING

In order to illustrate the previous results we will show several examples in which the EA allows us to make physical predictions. In some cases it will be possible to compare these results with those obtained by means of the traditional Bogolyubov transformations. Exact results from the Bogolyubov transformation have been obtained for a very limited number of models in the literature.

Model 1. We will now consider a *complex* scalar field and the FRW metric with $K=0$. It will be useful, in order to compare with other results, to work with the new time coordinate defined by

$$\tau=\int^{t}a^{-3}(t')dt', \tag{35}$$

which allows us to write the d'Alembertian operator acting on time dependent functions as

$$\Box f(\tau)=\frac{1}{a^{6}}\partial_{\tau}\partial_{\tau}f(\tau). \tag{36}$$

First we consider the model proposed in Ref. [18]. The scale factor is given by

$$a^{4}(\tau)\simeq a_{1}^{4}+e^{\tau/s}[(a_{2}^{4}-a_{1}^{4})(e^{\tau/s}+1)+b](e^{\tau/s}+1)^{-2}. \tag{37}$$

For $\tau\rightarrow\infty(-\infty)$, $a(\tau)$ smoothly tends to a constant $a_2(a_1)$, i.e., it is possible to unambiguously define initial and final vacuum states. On the other hand, a_1, a_2, s, and b are arbitrary parameters. For $a_2\gg a_1$ and using quantum mechanics methods, it is possible to calculate the Bogolyubov coefficients and hence the number density of produced particles [18]:

$$\langle N_k\rangle=\frac{1}{e^{4\pi s a_1^2 k}-1}, \tag{38}$$

where the number of created pairs per unit coordinate volume and unit trimomentum volume in the k mode is related to the Bogolyubov coefficients by means of $\langle N_k\rangle=|\beta_k|^2$ [5].

The relation between $\langle N_k\rangle$ and the pair production probability per unit coordinate volume $p_{\rm BOG}$ is given by this expression [21]

$$p_{\rm BOG}\simeq\int\frac{d^3k}{(2\pi)^3}[\pm\log(1\pm\langle N_k\rangle)], \tag{39}$$

where the $+$ sign is used for bosons and $-$ for fermions. In the present model $\langle N_k\rangle$ does not depend on a_2 nor b. Using Eq. (39) we find for the probability density

$$p_{\rm BOG}=\int\frac{d^3k}{(2\pi)^3}\log(1+\langle N_k\rangle)$$
$$=\int dk\,\frac{k^2}{2\pi^2}\log\left(1+\frac{1}{e^{4\pi s a_1^2 k}-1}\right). \tag{40}$$

On the other hand, the EA method provides from Eq. (31):

$$p_{\rm EA}\simeq 2\frac{1}{36}\,\frac{1}{32\pi}\int d\tau a^6(\tau)R^2(\tau) \tag{41}$$

with

$$R(\tau)=-12\frac{\dot{a}^2}{a^8}+6\frac{\ddot{a}}{a^7}. \tag{42}$$

It is possible to perform the integrals in Eqs. (40) and (41) in an explicit way, so that we can compare both results at the analytical level. They yield *exactly* the same result

$$p_{\rm EA}=p_{\rm BOG}=\frac{1}{5760\pi a_1^6 s^3}, \tag{43}$$

where we have taken the limits $b=0$ and $a_2\rightarrow\infty$.

Model 2. The second model we will study is that proposed in Ref. [22]. The scale factor is now given by

$$a^4(\tau)=A^2\tau^2+B^2, \tag{44}$$

where A and B are arbitrary constants. In this case, space-time is not asymptotically flat and therefore the Bogolyubov calculation is based on the definition of adiabatic vacua. However, the Gauss-Bonnet contribution vanishes and thus we can use again Eq. (31). The number of created pairs in the k mode is given by

$$\langle N_k\rangle=e^{-\pi B^2 k/A}. \tag{45}$$

Once again both methods yield the same results for the probability densities

$$p_{\rm BOG}=p_{\rm EA}=\frac{7A^3}{360B^6\pi}. \tag{46}$$

Since in Eq. (41) we have neglected higher order terms in curvatures, we can conclude that in these two cases they do not contribute to the EA imaginary part. As we found in the QED case, here again the second order perturbative calculation is exact. To check this fact, we should calculate the complete expression for the EA as we did in Sec. II, however, the very same arguments used in that section suggest that in the absence of a mass term, since in both cases there is just one-dimensional parameter, it is not possible to build any other term with the appropriate dimension.

V. SPECTRUM AND WKB APPROXIMATION

The traditional Bogolyubov method for particle production gives information, not only on the total number of created particles, but also on their energy distribution. However, only in very specific cases, closed analytical expressions can be written. As we have seen, the EA method provides a closed expression for the total number of particles that is obtained from the curvatures and, therefore, can be evaluated for arbitrary scale factors in a very easy way. In this respect the EA method is obviously more advantageous than the Bogolyubov method. However, it is not obvious how to derive the spectra in this formalism. Let us try to clarify this issue with a simple example and compare our result with the one obtained from the traditional method.

Consider the Klein-Gordon equation for a minimally coupled massless complex scalar field

$$\Box\phi=0. \tag{47}$$

Introducing the FRW metric (27) with time coordinate τ and $K=0$, we look for solutions by means of variable separation $\phi(\tau,\vec{x})=\chi_k(\tau)e^{i\vec{k}\vec{x}}$. Hence the temporal equation can be written as

$$\frac{d^2\chi_k}{d\tau^2}+a^4(\tau)k^2\chi_k=0, \tag{48}$$

where $k^2=\vec{k}\vec{k}$. In the simple example we are going to consider, the scale factor is made of two step functions

$$a^4(\tau)=1+v^2[\theta(\tau+T)-\theta(\tau-T)] \tag{49}$$

with v and T being arbitrary parameters. The Bogolyubov coefficients provide the following value for the number of created pairs per unit coordinate volume and unit trimomentum volume in the k mode

$$\langle N_k\rangle=|\beta_k|^2=\frac{v^4}{4(1+v^2)}\sin^2(2Tk\sqrt{1+v^2}). \tag{50}$$

Expanding the right-hand side of Eq. (39) using (50) up to $\mathcal{O}(v^4)$ we find

$$p_{\mathrm{BOG}}\simeq\frac{v^4}{8\pi^2}\int_0^\infty dk k^2\sin^2(2kT). \tag{51}$$

The integrand gives the probability density per unit trimomentum volume. On the other hand, the EA method gives the following result from Eq. (31):

$$\begin{aligned}p_{\mathrm{EA}}&\simeq 2\ \mathrm{Im}\, w\simeq 4\frac{1}{32\pi}\frac{1}{72}\int d\tau a^6(\tau)R^2\\&=4\frac{v^4}{32\pi}\frac{1}{72}\int d\tau\frac{9}{4}[\delta'(\tau-T)-\delta'(\tau+T)]^2.\end{aligned} \tag{52}$$

We have introduced a global 2 factor in the EA because now the field is complex. The spectrum can be obtained by introducing a complete set of plane waves

$$\begin{aligned}p_{\mathrm{EA}}&\simeq 4\frac{1}{32\pi}\frac{1}{72}\int d\tau d\tau'\\&\quad\times\int_{-\infty}^{\infty}\frac{dp}{2\pi}a^3(\tau)R(\tau)a^3(\tau')R(\tau')e^{-ip(\tau-\tau')}\\&=\frac{v^4}{64\pi^2}\int_0^\infty dp p^2\sin^2(pT).\end{aligned} \tag{53}$$

Comparing p_{BOG} with p_{EA} we find that both integrands agree by identifying $p=2k$. This is sensible and represents the energy conservation in the pair creation, since k is the single particle energy and p is the energy of the gravitational field oscillations producing particles.

From the complementary point of view, given the number density of created particles $\langle N_k\rangle$, it is also possible to reconstruct the scale factor evolution by inverting the previous steps

$$\begin{aligned}a^6(\tau)R^2(\tau)&=36\left|\int_{-\infty}^{\infty}\frac{dp}{2\pi}p\sqrt{\log(1+\langle N_{p/2}\rangle)}e^{-ip\tau}\right|^2+\mathcal{O}(v^6)\\&=\frac{9v^4}{4}\left|\int_{-\infty}^{\infty}\frac{dp}{2\pi}p\frac{\sin(pT)}{2}e^{-ip\tau}\right|^2+\mathcal{O}(v^6)\\&=\frac{9v^4}{4}[\delta'(\tau-T)-\delta'(\tau+T)]^2+\mathcal{O}(v^6).\end{aligned} \tag{54}$$

This result agrees with the calculation from Eq. (49).

Let us try to generalize the above results for arbitrary scale factor evolution. In the above example, it can be shown that the difference in the results using plane waves or a complete set of solutions of the Klein-Gordon equation is $\mathcal{O}(v^6)$. Therefore the former is a good approximation. Now we have to take into account the presence of the curvature. With that purpose we take variable-frequency plane waves such that in the vanishing curvature limit they tend to the usual plane waves.

Let us consider the temporal part of the Klein-Gordon equation (48). This is a harmonic-oscillator equation but with a time-dependent frequency $\omega_k(\tau)=ka^2(\tau)$. Changing to the new time coordinate $d\eta=a^2(\tau)d\tau$, the equation can be written as

$$\frac{d^2\chi_k}{d\eta^2}+2\frac{\dot{a}}{a}\frac{d\chi_k}{d\eta}+k^2\chi_k=0. \tag{55}$$

In the limit in which the expansion rate $\dot{a}/a$ is much smaller than the frequency of the oscillations k, the equation reduces to the flat space-time form. Therefore let us consider that limit and let us introduce a complete set of plane waves corresponding to the new time coordinate η:

$$\begin{aligned}p_{\mathrm{EA}}&\simeq 2\frac{1}{36}\frac{1}{32\pi}\int d\tau a^6(\tau)R^2(\tau)\\&=2\frac{1}{36}\frac{1}{32\pi}\int d\eta a^4(\eta)R(\eta)^2\\&=2\frac{1}{36}\frac{1}{32\pi}\int d\eta d\eta' a^2(\eta)a^2(\eta')R(\eta)R(\eta')\\&\quad\times\int\frac{dp}{2\pi}e^{-ip(\eta-\eta')}.\end{aligned} \tag{56}$$

Changing again to the old coordinate τ we have

$$p_{\mathrm{EA}}\simeq\frac{1}{8\pi^2}\int_0^\infty dk\left|\int d\tau\left(\frac{\ddot{a}}{a^3}-2\frac{\dot{a}^2}{a^4}\right)e^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2, \tag{57}$$

where we have used Eq. (42). According to the above discussion, the introduction of the plane waves only makes

TABLE I. Number densities corresponding to the model (37) $s=2(a.u)^{-1}$, $a_1=1$, $a_2=500$, and $b=0$. BOG denotes the Bogolyubov method and EA the effective action.

k(a.u.)	$\langle N_k\rangle_{\rm BOG}$	$\langle N_k\rangle_{\rm EA}$	$\langle N_k\rangle_{\rm WKB}$
0.3	5.31×10^{-4}	4.15×10^{-4}	4.96×10^{-4}
0.4	4.31×10^{-5}	3.33×10^{-5}	4.13×10^{-5}
0.7	2.28×10^{-8}	1.75×10^{-8}	2.28×10^{-8}
1.0	1.22×10^{-11}	0.93×10^{-11}	1.24×10^{-11}
1.2	7.97×10^{-14}	6.48×10^{-14}	8.64×10^{-14}
1.3	6.44×10^{-15}	5.91×10^{-15}	7.76×10^{-15}
1.4	4.44×10^{-16}	4.90×10^{-16}	6.19×10^{-16}

sense in the adiabatic limit and therefore this expression is valid only in the limit $\dot a/a\ll k$. As in the step function example, we have used $p=2k$. The pairs density $\langle N_k\rangle$ can be calculated in a very easy way by identifying Eq. (57) with Eq. (39). Obviously from the equality of the integrals we cannot obtain the equality of the integrands, however, the covariance and dimensionality of the integrands allows us to constrain them. In fact, from the equality of Eqs. (57) and (39) we know that the integrands differ at most in a function $f(p)$ such that $\int dp f(p)=0$. Now let us assume that such function can be written as [up to $\mathcal{O}(\mathcal{R}^2)$]

$$f(p)=\int d\eta d\eta' e^{ip(\eta-\eta')}a^2(\eta)a^2(\eta')F(\eta)\tilde F(\eta'), \tag{58}$$

where $F(\eta)$ and $\tilde F(\eta')$ are some appropriate functions. The condition $\int dp f(p)=0$ implies

$$\int d\eta a^4(\eta)F(\eta)\tilde F(\eta)=0. \tag{59}$$

Since the integrand has to be a dimension-4 operator and a scalar function then the only possibility satisfying that condition is

$$F\tilde F=\alpha E, \tag{60}$$

where E is the Gauss-Bonnet term defined before, which is a total derivative, and α is some arbitrary constant. However we know that in a radiation dominated universe, where $R=0$, the spectrum is identically zero $\langle N_k\rangle=0$ (see Refs. [16–18]). This implies that the contribution from the Gauss-Bonnet term should also vanish. This fact allows us to fix the constant $\alpha=0$. As a consequence the result in Eq. (57) gives the correct spectrum up to $\mathcal{O}(\mathcal{R}^2)$ at least in the adiabatic limit we are considering. In fact, as shown in Table I the results are in good agreement with the Bogolyubov method especially for large values of k.

Since we have used an adiabatic approximation in the last step, we can try to find which are the differences with respect to the usual WKB approximation in Ref. [23]. In this method, the solutions of Eq. (48) are taken to be

$$\chi_k(\tau)=\alpha(\tau)e^{-i\psi(\tau)}+\beta(\tau)e^{i\psi(\tau)}, \tag{61}$$

with

$$\psi(\tau)=k\int_0^\tau d\tau' a^2(\tau'), \tag{62}$$

with boundary conditons $\alpha(-\infty)=1$, $\beta(\infty)=0$. Putting this ansatz back into the equation of motion, we get

$$\dot\alpha e^{-i\psi(\tau)}-\dot\beta e^{i\psi(\tau)}=-\left(\frac{2\dot a}{a}\right)[\alpha(\tau)e^{-i\psi(\tau)}-\beta(\tau)e^{i\psi(\tau)}], \tag{63}$$

where the condition $\dot\alpha e^{-i\psi}+\dot\beta e^{i\psi}=0$ is used (see Ref. [23] for details). The solution to first adiabatic order is given by

$$\beta(\tau)=-\int_\tau^\infty d\tau'\frac{\dot a}{a}e^{-2i\psi(\tau')}. \tag{64}$$

The probability density is given by

$$p_{\rm WKB}=\frac{1}{2\pi}\int\frac{d^3k}{4\pi^2}|\beta(-\infty)|^2. \tag{65}$$

From Eqs. (64) and (65) we see that

$$\begin{aligned} p_{\rm WKB}&=\frac{1}{8\pi^3}\int d^3k|\beta(-\infty)|^2\\ &=\frac{1}{8\pi^3}\int_0^\infty dk4\pi k^2\left|\int d\tau\frac{\dot a}{a}e^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2\\ &=\frac{1}{2\pi^2}\int dk\left|\int d\tau\frac{\dot a}{a}ke^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2\\ &=\frac{1}{2\pi^2}\int dk\left|\int d\tau\frac{\dot a}{a}\frac{1}{-2ia^2}\frac{d}{d\tau}e^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2\\ &=\frac{1}{8\pi^2}\int dk\left|\int d\tau\frac{d}{d\tau}\left(\frac{\dot a}{a^3}\right)e^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2. \end{aligned} \tag{66}$$

In the last step we have used integration by parts assuming that $\dot a/a^3$ vanishes for $\tau\rightarrow\pm\infty$ [which is the same condition as for the vanishing of the Gauss-Bonnet term contribution in Eq. (29)]. This condition is satisfied in the models we have considered in the paper. Therefore we get

$$p_{\rm WKB}\simeq\frac{1}{8\pi^2}\int_0^\infty dk\left|\int d\tau\left(\frac{\ddot a}{a^3}-3\frac{\dot a^2}{a^4}\right)e^{-2ik\int_0^\tau a^2(\tau')d\tau'}\right|^2 \tag{67}$$

which is valid again only in the adiabatic limit.

In Table I some values of the number densities are shown for the different methods. The results have been obtained from Eqs. (57) and (67) for the model (37) by numeric integration. Due to the strongly oscillating integrals, the results can only be given for small momenta. Both methods give similar results to those obtained with the Bogolyubov coefficients. Notice that, despite the fact that the WKB and EA expression in Eqs. (57) and (67) are not identical, there is no contradiction in this. These results come from different approximations. The WKB method comes from a derivative expansion and is not covariant, whereas the EA is an expansion in curvatures and covariance is imposed from the beginning. From Table I, we see that the contribution from the $(\dot{a}/a^2)^2$ terms in Eqs. (57) and (67) is always smaller than the contribution from $\ddot{a}/a^3$.

VI. CONCLUSIONS

In this work we have shown how to use the nonlocal form of the gravitational EA [up to $\mathcal{O}(\mathcal{R}^2)$] for the computation of massless scalar particle production. For FRW backgrounds in which the expansion rate asymptotically vanishes, it is shown that the particle production probabilities only depend on the scalar curvature. As a consequence and as expected there is no particle creation in a radiation dominated universe. This is also the case for conformally coupled scalar fields. For anisotropic homogeneous metrics we reobtain the well-known expression of Zel'dovich and Starobinski. We compare our results with those obtained by means of the well-known Bogolyubov transformations. In the examples considered, the agreement between both methods is complete for the probability densities. This fact is quite remarkable since we have used a perturbative expression for the EA, whereas the Bogolyubov results are exact. This indicates that in some cases a perturbative calculation may contain all the relevant information about the particle production processes. In principle, the EA is defined for asymptotically flat manifolds, however, it is interesting to notice that the naive extension to those manifolds in which adiabatic vacua can be defined, properly reproduces the correct results. Finally we have also compared the different spectra derived with the EA, Bogolyubov, and WKB techniques.

In principle the EA method can be extended to more general metrics (not necessarily homogeneous) in a straightforward way. This fact could make it valuable in those areas in which the Bogolyubov technique has been traditionally used. In addition this method can also be applied to the production of higher spin particles such as Dirac and Weyl fermions, gravitons, gravitinos, etc. Finally, in a recent work [24] the relevance of the nonlocal EA for particle creation has also been stressed from a different point of view based on the energy-momentum tensor expectation values.

ACKNOWLEDGMENTS

A.L.M acknowledges support from SEUID-Royal Society. This work has been partially supported by Ministerio de Educación y Ciencia (Spain) CICYT (AEN96-1634).

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