



Rhaly operators: more on generalized Cesàro operators

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Abstract

Rhaly operators, as generalizations of the Cesàro operator, are studied from the standpoint of view of spectral theory and invariant subspaces, extending previous results by Rhaly and Leibowitz to a framework where generalized Cesàro operators arise naturally.

Keywords Cesàro operator · Rhaly operator · Invariant subspace · Spectrum · Terraced operator · Hankel operator · Contraction semigroup

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1 Introduction and preliminaries

Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence of complex numbers and

$$R_{\mathbf{a}} := \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & 0 & \cdots \\ a_3 & a_3 & a_3 & a_3 & 0 & \cdots \\ a_4 & a_4 & a_4 & a_4 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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the associated Rhaly matrix (terraced matrix). In [21], Rhaly showed that whenever the limit

$$L = \lim_{n \rightarrow \infty} (n + 1)|a_n|$$

exists, it provides a test for the boundedness of $R_{\mathbf{a}}$ as an operator acting on the classical sequence space ℓ^2 (see also [22]). More precisely, Theorem (Rhaly, 1989).

- (1) If $L < +\infty$ and D denotes the diagonal operator with diagonal entries $\{(n + 1)a_n : n = 0, 1, 2, 3, \dots\}$, then $R_{\mathbf{a}}$ is a bounded operator on ℓ^2 with

$$\|R_{\mathbf{a}}\| \leq \|D\| + \sup\{\sqrt{n(n + 1)}|a_n| : n = 0, 1, 2, 3, \dots\}.$$

Moreover, if $L = 0$, then $R_{\mathbf{a}}$ is a compact operator (with the numbers $\{a_n\}_{n \geq 0}$ in the point spectra of both $R_{\mathbf{a}}$ and its adjoint $R_{\mathbf{a}}^*$).

- (2) If $L = +\infty$, then $R_{\mathbf{a}}$ is not a bounded operator on ℓ^2 .

The classical Cesàro matrix \mathcal{C} appears as a particular instance of a Rhaly matrix where $\mathbf{a} = (1/(n + 1))_{n \geq 0}$. In particular, \mathcal{C} takes a complex sequence $\mathbf{x} = (x_0, x_1, x_2, \dots)$ to that with n th entry:

$$(\mathcal{C} \mathbf{x})_n = \frac{1}{n + 1} \sum_{k=0}^n x_k, \quad (n \geq 0),$$

and, identifying sequences with Taylor coefficients of power series, \mathcal{C} can be expressed as the integral operator acting on holomorphic functions $f(z) = \sum_{k=0}^{\infty} x_k z^k$ of the unit disc \mathbb{D} as

$$\mathcal{C}(f)(z) = \begin{cases} \frac{1}{z} \int_0^z \frac{f(\xi)}{1 - \xi} d\xi, & z \in \mathbb{D} \setminus \{0\}, \\ f(0), & z = 0; \end{cases} \tag{1.1}$$

for $z \in \mathbb{D}$. There is an extensive literature on the Cesàro operator, and more general, on integral operators, acting on a large variety of spaces of analytic functions regarding its boundedness, compactness or spectral picture (see the recent survey by Ross [23]). In the pioneering work of [4], Brown, Halmos and Shields computed the spectrum of \mathcal{C} in ℓ^2 showing that it is the closed disc:

$$\sigma(\mathcal{C}) = \{z \in \mathbb{C} : |z - 1| \leq 1\},$$

while the point spectrum of \mathcal{C} is empty and that of \mathcal{C}^* the open disc

$$\sigma_p(\mathcal{C}^*) = \{z \in \mathbb{C} : |z - 1| < 1\}.$$

Likewise, they proved that \mathcal{C} is hyponormal in ℓ^2 , namely, the conmutant $[\mathcal{C}^*, \mathcal{C}]$ is positive semi-definite. Later on, Kriete and Trutt [17] established a remarkable result proving that the Cesàro operator is, indeed, *subnormal*, namely, \mathcal{C} has a normal extension.

In the particular instances of $\mathbf{a} = (1/(n + 1)^\alpha)_{n \geq 0}$ for $\alpha > 1$, it turns out that all Rhaly matrices induce compact and nonhyponormal operators. In case that \mathbf{a} is a sequence of real numbers, Rhaly found necessary conditions for the hyponormality of $R_{\mathbf{a}}$ (see [21]) but the general question of characterizing those sequences \mathbf{a} inducing hyponormal operators $R_{\mathbf{a}}$ or moreover, subnormal operators, remains open.

Regarding the spectrum of $R_{\mathbf{a}}$, Rhaly [21] was able to extend Brown, Halmos and Shields' techniques to prove the following

Theorem (Spectrum of $R_{\mathbf{a}}$). Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence of positive numbers. Assume $a_i \neq a_j$ for all $i \neq j$ and $L = \lim_{n \rightarrow \infty} (n + 1)a_n$ exists and is finite. Then,

- (1) If $L = 0$, then $\sigma_p(R_{\mathbf{a}}^*) = \mathbf{a}$;
- (2) If $L > 0$, then

$$\mathbf{a} \cup \{z \in \mathbb{C} : |z - L| < L\} \subseteq \sigma_p(R_{\mathbf{a}}^*) \subseteq \mathbf{a} \cup \{z \in \mathbb{C} : |z - L| \leq L\} \setminus \{0\};$$

- (3) $\mathbf{a} \cap (2L, +\infty) \subseteq \sigma_p(R_{\mathbf{a}}) \subseteq \mathbf{a} \cap [2L, +\infty)$;
- (4) $\sigma(R_{\mathbf{a}}) = \mathbf{a} \cup \{z \in \mathbb{C} : |z - L| \leq L\}$.

It is worth pointing out that Leibowitz in [18] and [19] had also studied the spectrum of particular Rhaly matrices as operators acting on ℓ^p -spaces, $1 < p < \infty$. For the spectrum of the Cesàro operator in ℓ^p ($1 < p < \infty$) see, for instance, [15].

As might be expected, if the sequence \mathbf{a} corresponds to a *moment sequence* of a finite positive (or complex) Borel measure, the operator properties of $R_{\mathbf{a}}$ are more approachable. Nevertheless, in this paper we shall work mostly with the assumption that $((n + 1)a_n)_{n \geq 0}$ is a bounded sequence, not necessarily tending to a limit, which is a broader scenario. Indeed, in such a case, the linear operator $D_{\mathbf{a}}$ with diagonal entries $((n + 1)a_n)_{n \geq 0}$ is bounded, so $R_{\mathbf{a}} = D_{\mathbf{a}}\mathcal{C}$ defines also a bounded operator. In general the condition that $((n + 1)a_n)_{n \geq 0}$ is bounded is not necessary for the boundedness of $R_{\mathbf{a}}$, as shown by the example of Leibowitz [18, p. 283] with

$$a_n = \begin{cases} n^{-7/8} & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases}$$

but, it turns out to be necessary and sufficient for the boundedness of $R_{\mathbf{a}}$ if (a_n) is a moment sequence as shown by Galanopoulos, Girela and Merchán in [10]. At this regard, in both [10, 11], the authors deal with boundedness of $R_{\mathbf{a}}$ as integral operator (along the lines of (1.1)) not only in Hardy spaces, but also in weighted Bergman spaces, BMOA or the Bloch space (for boundedness of Cesàro-like operators in Hardy spaces, we refer also to the previous works by Stempak [25] and Andersen [1]).

In this setting, the main goal of this work is twofold. On one hand, it is taking further the study of the spectrum of Rhaly operators in ℓ^2 providing concrete spectral picture when \mathbf{a} is a moment sequence of a finite positive Borel measure. In particular, the results in Sect. 2 extend Rhaly’s theorem on the spectrum of $R_{\mathbf{a}}$ previously stated and exhibit examples of finite positive Borel measures μ such that the corresponding moment sequence $(\mu_n)_{n \geq 0}$ induces Rhaly operators R_{μ} in ℓ^2 with different spectral configuration. Likewise, we shall give a conceptually simpler proof that the *numerical range* of R_{μ} is always contained in the closed right half plane $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, which implies, in particular, the same containment for the spectrum $\sigma(R_{\mu})$. Consequently, it follows that R_{μ} is related to the infinitesimal generator of a contraction semigroup (see, for example, [9, Cor. II.3.17, Prop. II.3.23] and [27, Thm. IV.4.1]).

This latter fact is closely related to our second concern regarding Rhaly operators, namely, the study of their invariant subspaces along the lines of the authors’ works [13] and [14]. Relating the Cesàro operator to some semigroups, either as an infinitesimal generator or as the resolvent operator at 0, has turned out to be fruitful in order to compute norms and spectra (see Siskakis’ work [26]), determine subnormality (see Cowen’s paper [5]) or even to study local spectral properties (see the recent work [12]). In [13] the authors linked the invariant subspaces of \mathcal{C} to those of the right-shift semigroup $\{S_{\tau}\}_{\tau \geq 0}$ acting on a particular weighted $L^2(\mathbb{R}, w(y)dy)$ as an approach towards describing completely the lattice of the invariant subspaces of \mathcal{C} . In Sect. 3, we undertake this study and show a connection with the invariant subspaces of a family (not necessary a semigroup) of weighted composition operators $\{W_t\}_{t \geq 0}$ acting on the classical Hardy space H^2 .

A related family of weighted composition operators arises also naturally linked to the Hilbert matrix \mathcal{H} , as it was shown by Diamantopoulos and Siskakis in [7]. Finally, in Sect. 4, we discuss a similar approach to the invariant subspaces as well as the fact that \mathcal{H} and its generalizations are also related to the infinitesimal generator of a contraction semigroup.

2 Spectrum of Rhaly operators: a step further

In this section we study the spectrum of Rhaly operators when the induced sequence is a *moment sequence*, which extends Rhaly’s theorem recalled in the Introduction. We start by determining the point spectrum, that is, the set consisting of eigenvalues.

Theorem 2.1 *Let μ be a positive finite Borel measure in $[0, 1)$ and R_μ the Rhaly operator associated to the moment $(\mu_n)_{n \geq 0}$. Assume R_μ is a bounded operator in ℓ^2 and let*

$$s_n = \sum_{j=0}^n \mu_j = \int_0^1 \frac{1 - t^{n+1}}{1 - t} d\mu(t). \tag{2.1}$$

Then, for a given index k , $\mu_k \in \sigma_p(R_\mu)$ if and only if the sequence $(\mu_n \exp(s_n/\mu_k))_{n \geq 0}$ is in ℓ^2 . Moreover,

$$\sigma_p(R_\mu) = \{\mu_k : (\mu_n \exp(s_n/\mu_k))_{n \geq 0} \in \ell^2\}.$$

Before proceeding with the proof, note that in case μ is the Lebesgue measure m in $[0, 1)$, R_μ is the classical Cesàro operator \mathcal{C} and the sequence $(s_n)_{n \geq 1}$ grows as $\log n$. Thus $\mu_n \exp(s_n/\mu_k)$ behaves as $n^{-1}n^{k+1}$, and is not an ℓ^2 sequence for any k . That is, we recover the well-known result that the point spectrum of \mathcal{C} is empty, proved in [4].

Likewise, if μ is a positive finite Borel measure in $[0, 1)$ such that $(\mu_n) \in \ell^1$, then $(s_n)_{n \geq 1}$ is bounded and, in such a case, every μ_k is an eigenvalue of R_μ . Clearly, taking $\mu = \delta_t$, a point mass (Dirac measure) at t with $0 < t < 1$, we have the moment sequence $\mu_n = t^n$, $n = 0, 1, 2, \dots$, and $s_n = \frac{1-t^{n+1}}{1-t}$. Accordingly, the sequence $(\mu_n \exp(s_n/\mu_k))_n$ is $(t^n \exp(\frac{1-t^{n+1}}{1-t}/t^k))$, which is clearly in ℓ^2 , so every μ_k is an eigenvalue.

Besides, if we take $\mu = \delta_0 + cm$ for some $c > 0$, an easy computation yields that $\mu_0 = 1 + c$ and $\mu_n = c/(n + 1)$ for $n > 0$. So s_n grows as $c \log n$, and for $k = 0$ the sequence $(\mu_n \exp(s_n/\mu_0))_n$ behaves as $n^{-1}n^{c/(1+c)}$, which is in ℓ^2 for $0 < c < 1$.

However, for $k > 0$, $(\mu_n \exp(s_n/\mu_k))_n$ behaves as $n^{-1}n^{k+1}$, which does not belong to ℓ^2 . Consequently, if $0 < c < 1$ and $\mu = \delta_0 + cm$, the point spectrum of R_μ is simply $\{\mu_0\} = \{1 + c\}$.

In [18], Leibowitz considers Cesàro-like matrices with $\mu_n = (n + 1)^{-s}$ for $n = 0, 1, 2, \dots$, where $s \geq 1$. The case $s = 1$ gives the familiar Cesàro operator, and for $s > 1$ the corresponding operator is compact.

We remark that for $k \geq 0, n \geq 0$

$$\begin{aligned} \int_0^1 t^n (-\log t)^k dt &= \int_0^\infty e^{-nx} x^k e^{-x} dx \\ &= \int_0^\infty e^{-y} y^k (n + 1)^{-k} dy / (n + 1) \\ &= \Gamma(k + 1) / (n + 1)^{k+1} \end{aligned}$$

(using the substitutions $t = e^{-x}$ and $x = y/(n + 1)$), and so for $s > 1$ this corresponds to the measure μ on $(0, 1)$ given by $d\mu(t) = \frac{1}{\Gamma(s)}(-\log t)^{s-1} dt$.

Clearly, it is possible to provide further examples of finite positive Borel measures in $[0, 1)$ such that $\sigma_p(R_\mu)$ is either empty, finite or an infinite set.

Proof of Theorem 2.1 We start by noting that the matrix representation of the adjoint R_μ^* is upper triangular, and Apostol’s triangular decomposition of a bounded operator T states, in particular, that if $\ker(\lambda - T)^* \neq \{0\}$ then λ belongs to the diagonal of T (see [16, Corollary 3.40, (iv)], for instance). In other words, if $\ker(\bar{\lambda} - R_\mu) \neq \{0\}$, then $\bar{\lambda} \in (\mu_n)_{n \geq 0}$. Consequently,

$$\sigma_p(R_\mu) \subseteq (\mu_n)_{n \geq 0}.$$

Now, let $k \geq 0$ be fixed and μ_k the corresponding k -th moment of μ . Assume that $\mathbf{x} = (x_n)_{n \geq 0}$ is an eigenvector corresponding to the eigenvalue μ_k , namely,

$$\begin{bmatrix} \mu_0 & 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & \mu_1 & 0 & 0 & 0 & \cdots \\ \mu_2 & \mu_2 & \mu_2 & 0 & 0 & \cdots \\ \mu_3 & \mu_3 & \mu_3 & \mu_3 & 0 & \cdots \\ \mu_4 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} = \mu_k \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}.$$

The goal is to determine whether $\mathbf{x} \in \ell^2$. We have

$$\begin{aligned} \mu_0 x_0 &= \mu_k x_0, \\ \mu_1 x_0 + \mu_1 x_1 &= \mu_k x_1, \\ &\dots \dots \\ \mu_k x_0 + \mu_k x_1 + \dots + \mu_k x_k &= \mu_k x_k, \\ \mu_{k+1} x_0 + \mu_{k+1} x_1 + \dots + \mu_{k+1} x_{k+1} &= \mu_k x_{k+1} \\ &\dots \dots \end{aligned}$$

whence $x_j = 0$ for $j < k$ since the μ_j are distinct (except in the trivial case when μ is concentrated at 0 and $R_\mu f = \lambda f(0)$ for some $\lambda > 0$). If $x_k = 0$ then $\mathbf{x} = 0$, so let us suppose without loss of generality that $x_k = 1$.

Now for $n \geq k$ we have

$$\mu_n(x_0 + \dots + x_n) = \mu_k x_n, \tag{2.2}$$

$$\mu_{n+1}(x_0 + \dots + x_{n+1}) = \mu_k x_{n+1} \tag{2.3}$$

and taking $\mu_n \times (2.3) - \mu_{n+1} \times (2.2)$ we have

$$\mu_n \mu_k x_{n+1} - \mu_{n+1} \mu_k x_n = \mu_n \mu_{n+1} x_{n+1},$$

that is,

$$x_{n+1} = \frac{\mu_{n+1} \mu_k}{\mu_n (\mu_k - \mu_{n+1})} x_n.$$

For easier reading we write $\alpha = \mu_k$, and then for $m \geq k$ this gives (with $x_k = 1$)

$$x_{m+1} = \frac{\mu_{m+1}}{\alpha} \left(\prod_{n=k}^m \frac{\alpha}{\alpha - \mu_{n+1}} \right) = \frac{\mu_{m+1}}{\alpha} \left(\prod_{n=k}^m \frac{1}{1 - \mu_{n+1}/\alpha} \right). \tag{2.4}$$

To see whether $\mathbf{x} = (x_n)_{n \geq 0}$ is an ℓ^2 sequence we note that the μ_n tend to 0 as $n \rightarrow \infty$ since R_μ is a bounded operator in ℓ^2 and $\mu_n = O(1/(n + 1))$ (see [10, Thm. 1, Lem. 2], for instance). Then

$$(1 - \mu_{n+1}/\alpha)^{-1} = \exp(\mu_{n+1}/\alpha + O(\mu_{n+1}^2)). \tag{2.5}$$

From (2.4) and (2.5), noting that $(\mu_n)_{n \geq 0}$ is an ℓ^2 sequence, it follows that if $(\mu_n \exp(s_n/\alpha))_n \in \ell^2$ then $\mathbf{x} = (x_n)_{n \geq 0} \in \ell^2$ also, in which case, $\mu_k \in \sigma_p(C_\mu)$. Likewise, if $(\mu_n \exp(s_n/\alpha))_n \notin \ell^2$, then $\mathbf{x} = (x_n)$ is not in ℓ^2 either, and therefore $\mu_k \notin \sigma_p(C_\mu)$, which concludes the proof of Theorem 2.1. \square

Remark 2.2 In this proof, we only used the facts that $(\mu_n)_{n \geq 0}$ are positive and distinct and $\mu_n = O(1/(n + 1))$. Accordingly, it holds more generally for sequences \mathbf{a} under this hypotheses.

Our next task is studying the point spectrum of the adjoint of Rhaly operators. First, we state the following proposition, which follows as a direct application of Apostol’s triangular decomposition theorem and it does not require assuming that the induced sequence is a moment one (see [16, Corollary 3.40]).

Proposition 2.3 *Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence of complex numbers and $R_{\mathbf{a}}$ the associated Rhaly operator acting boundedly on ℓ^2 . Then,*

- (i) $\sigma(R_{\mathbf{a}}^*) = \sigma_l(R_{\mathbf{a}}^*) = \sigma_{lre}(R_{\mathbf{a}}^*) \cup \sigma_p(R_{\mathbf{a}}^*)$.
- (ii) Every clopen subset of $\sigma(R_{\mathbf{a}}^*)$ intersects the sequence \mathbf{a} . Moreover, every component of $\sigma(R_{\mathbf{a}}^*)$ intersects the closure of \mathbf{a} .
- (iii) Every isolated point of $\sigma(R_{\mathbf{a}}^*)$ belongs to \mathbf{a} .

Here σ_l and σ_{lre} denote the left spectrum, respectively the left–right essential spectrum (also known as the Wolf spectrum). We refer to Herrero’s book [16] for references and properties.

The next theorem provides information about the point spectrum of R_μ^* :

Theorem 2.4 *Let μ be a positive finite Borel measure in $[0, 1)$ and R_μ the Rhaly operator associated to the moment $(\mu_n)_{n \geq 0}$. Assume R_μ is a bounded operator in ℓ^2 and let*

$$s_n = \sum_{j=0}^n \mu_j = \int_0^1 \frac{1 - t^{n+1}}{1 - t} d\mu(t).$$

If $\lambda \in \sigma_p(R_\mu^)$ then $\lambda \neq 0$. Moreover, let $\gamma > 0$ such that $\exp(-(1 + \epsilon)\gamma s_n)$ is an ℓ^1 sequence for all $\epsilon > 0$. Then*

$$\{z \in \mathbb{C} : |z - 1/\gamma| < 1/\gamma\} \subseteq \sigma_p(R_\mu^*).$$

The proof follows the approach of [4] to compute the point spectrum of the adjoint of the Cesàro operator, though it is a bit more involved.

Proof Suppose $\mathbf{x} = (x_n)_{n \geq 0} \in \ell^2$ is an eigenvector of R_μ^* associated to λ , namely, $R_\mu^* \mathbf{x} = \lambda \mathbf{x}$. We thus have

$$\sum_{k=m}^\infty x_k \mu_k = \lambda x_m$$

for $m = 0, 1, 2, \dots$. By subtracting consecutive values of this expression, there follows the recurrence relation

$$\lambda(x_m - x_{m-1}) = x_m \mu_m.$$

Clearly $\lambda = 0$ gives only the trivial solution $\mathbf{x} = 0$.

For the second half of the theorem, assume $\lambda \in \{z \in \mathbb{C} : |z - 1/\gamma| < 1/\gamma\}$ and write $v = 1/\lambda$. If the sequence $(x_n)_{n \geq 0}$ satisfies

$$\begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ 0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ 0 & 0 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ 0 & 0 & 0 & \mu_3 & \mu_4 & \cdots \\ 0 & 0 & 0 & 0 & \mu_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} = \frac{1}{v} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}.$$

we obtain

$$x_n = x_0 \prod_{j=1}^n (1 - \mu_j v), \quad n = 1, 2, \dots$$

Since $|1/v - 1/\gamma| < 1/\gamma$, a little computation shows that $\gamma < 2 \operatorname{Re} v$. So, let us write $2 \operatorname{Re} v = (1 + \epsilon)\gamma > 0$ for some $\epsilon > 0$ and argue as in [4, pp. 130–131]. The key inequality here is

$$|1 - \mu_j v|^2 = 1 - 2\mu_j \operatorname{Re} v + |v|^2 \mu_j^2 \leq \exp(\mu_j^2 |v|^2) - \mu_j (1 + \epsilon)\gamma.$$

Now, since R_μ is bounded in ℓ^2 , the series $\sum_{n=0}^\infty \mu_n^2$ converges and the estimate

$$|x_n|^2 \leq |x_0|^2 \frac{\exp |v|^2 \sum_{j=1}^n \mu_j^2}{\exp(s_n (1 + \epsilon)\gamma)}$$

holds. Since $\exp(-(1 + \epsilon)\gamma s_n) \in \ell^1$ for all $\epsilon > 0$ by hypotheses, this shows that $(x_n)_{n \geq 0}$ is an ℓ^2 sequence. Accordingly, $1/v$ belongs to the point spectrum of R_μ^* , as we wished to prove. \square

With Theorem 2.4 at hand, it is possible to provide examples of Rhaly operators such that the point spectra of their adjoints differ substantially.

For instance, let $\alpha > 0$ and μ_α be the Borel measure in $[0, 1)$ with density t^α , that is, $d\mu_\alpha(t) = t^\alpha dt$. An easy computation shows that $(\mu_\alpha)_n = 1/(n + \alpha + 1)$ for $n \geq 0$. Hence, $\sum_{j=0}^n (\mu_\alpha)_j$ grows as $\log n$ and accordingly,

$$\{z \in \mathbb{C} : |z - 1| < 1\} \subseteq \sigma_p(R_{\mu_\alpha}^*).$$

On the other hand, let us consider $\mu = \delta_0 + cm$ for some $0 < c < 1$, where m is the Lebesgue measure in $[0, 1)$. Here $\mu_0 = 1 + c$ and $\mu_n = c/(n + 1)$ for $n > 0$, and as we discussed previously, $\sigma_p(R_\mu) = \{1 + c\}$. Observe that in this case s_n grows as $c \log n$, so taking $\gamma = 1/c$, it follows that

$$\{z \in \mathbb{C} : |z - c| < c\} \subseteq \sigma_p(R_\mu^*).$$

Note that $1 + c$ is not in this disc.

Remark 2.5 If R_μ is compact, clearly the spectrum does not contain a disc and it is possible to provide examples of measures such that the hypotheses of Theorem 2.4 do not hold. For example, let μ be the Lebesgue measure on $[0, r]$ for some $0 < r < 1$ and zero density in $[r, 1)$. In this case, $\mu_n = r^{n+1}/(n + 1)$ and x_n fail to tend to 0 unless $\lambda = \mu_k$ for some k . Note that no positive γ satisfies the hypotheses of Theorem 2.4.

Remark 2.6 We point out that Yildirim and coauthors [28, 29] have studied some spectral properties of particular Rhalý operators acting on ℓ^p and c_0 . Their results in the Hilbert space setting ℓ^2 regarding the essential spectrum follow as particular instances of Theorems 2.1 and 2.4 and Proposition 2.3 along with [6, Thm. 4.3.18]. This latter theorem states that if $\sigma_e(T)$ denotes the essential spectrum of a bounded operator on a Banach space, and U be the unbounded component of $\mathbb{C} \setminus \sigma_e(T)$, then $(zI - T)$ is a Fredholm operator of zero index for all $z \in U$ and $\sigma(T) \cap U$ consists of a finite or countable set of isolated eigenvalues with finite multiplicity.

We close this section by introducing a family of weighted composition operators $(W_t)_{0 \leq t < 1}$, which we use in two ways. Here, we obtain a simple proof that the numerical range of R_μ (and hence also the spectrum) is contained in closed right half-plane $\overline{\mathbb{C}_+}$. In the next section we shall obtain information on its invariant subspaces by considering the W_t .

2.1 Numerical range of R_μ

For a Hilbert space operator T on H the numerical range $W(T)$ is the image of the unit sphere of H under the quadratic form $x \rightarrow \langle Tx, x \rangle$ associated with the operator, namely:

$$W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}.$$

It is a classical fact that $W(T)$ lies in the closed disc of radius $\|T\|$ centered at the origin and contains all the eigenvalues of T . Moreover, the spectrum of an operator lies always in the closure of its numerical range and, though the numerical range is not invariant under similarities, it is invariant under unitary similarity. But, probably, the most remarkable result about the numerical range is the Toeplitz–Hausdorff Theorem, which asserts that the numerical range is always a convex set. We refer to the Bonsall and Duncan monograph [2] for more on the subject.

A more general form of the following result is given in [21, Thm. 1.2]. We provide an alternative and possibly simpler proof, introducing a decomposition method that will later be applied to Hankel operators.

Theorem 2.7 *Let μ be a positive finite Borel measure in $[0, 1)$ and R_μ the Rhalý operator associated to the moment sequence $(\mu_n)_{n \geq 0}$. Assume R_μ is a bounded operator in ℓ^2 . Then $W(R_\mu) \subset \overline{\mathbb{C}_+}$ and, consequently, $\sigma(R_\mu) \subset \overline{\mathbb{C}_+}$.*

In order to prove Theorem 2.7, we recall that the classical Hardy space H^2 consists of holomorphic functions $f(z) = \sum_{k=0}^{\infty} x_k z^k$ in the unit disc \mathbb{D} with Taylor coefficients $(x_k)_{k \geq 0} \in \ell^2$. An easy computation yields that R_μ acts on H^2 as the integral operator

$$R_\mu f(z) = \int_0^1 \frac{f(tz)}{1-tz} d\mu(t), \quad (2.6)$$

and it is bounded if and only if $\mu_n = O(1/(n+1))$ (see [10], for instance).

Clearly, (2.6) can be expressed as

$$R_\mu f(z) = \int_0^1 W_t f(z) d\mu(t), \quad (2.7)$$

where W_t (for $0 \leq t < 1$) is the weighted composition operator

$$W_t f(z) = \frac{f(tz)}{1 - tz}.$$

It is not difficult to prove that for each $0 \leq t < 1$, W_t is a bounded operator in H^2 . Moreover, for $0 < t < 1$, W_t is a Rhaly operator associated to δ_t , a point mass at t , and since $\mu_n = t^n = o(1/n)$, it is a compact operator (see [10], for instance). In such a case, the spectrum is just the closure of its set of eigenvalues; that is,

$$\sigma(W_t) = \sigma(W_t^*) = \{0\} \cup \{t^n : n \in \mathbb{N}_0\}.$$

and

$$\sigma_p(W_t) = \sigma_p(W_t^*) = \{t^n : n \in \mathbb{N}_0\}.$$

We proceed now with the proof of Theorem 2.7.

Proof of Theorem 2.7 Without loss of generality, we may argue in H^2 and consider the expression (2.7) for R_μ . Hence, an argument involving Fubini’s Theorem yields

$$\langle R_\mu f, f \rangle = \int_0^1 \langle W_t f, f \rangle d\mu(t).$$

Accordingly, it is sufficient to show that the numerical range of each weighted composition operator W_t lies in $\overline{\mathbb{C}_+}$.

Fix $t \in [0, 1)$. It is sufficient to check that $\text{Re}\langle W_t f, f \rangle \geq 0$ for all f of the form $f(z) = (1 - z)g(z)$ with $g \in H^2$; these form a dense set since $1 - z$ is an outer function (see [8] or [24], for instance, for the *inner-outer factorization* of Hardy functions). So

$$\langle W_t f, f \rangle = \langle g(tz), (1 - z)g(z) \rangle.$$

Let us write T for the self-adjoint (diagonal) operator $Tg(z) = g(t^{1/2}z)$ and S for the shift operator on H^2 ; note that $TS = t^{1/2}ST$ (each sends z^n to $t^{(n+1)/2}z^{n+1}$). Then

$$\begin{aligned} \text{Re}\langle W_t f, f \rangle &= \text{Re}\langle T^2g, g - Sg \rangle \\ &= \langle Tg, Tg \rangle - \text{Re}\langle Tg, t^{1/2}STg \rangle \\ &\geq \|Tg\|^2 - t^{1/2}\|Tg\|^2 \geq 0, \end{aligned}$$

which shows $W(W_t) \subset \overline{\mathbb{C}_+}$ for every $t \in [0, 1)$, as we wished. From here, the statement of the theorem follows. □

As a consequence, it follows from [9, Cor. II.3.17, Prop. II.3.23] and [27, Thm. IV.4.1]), the following surprising fact, which does not seem to have been previously observed, regarding Rhaly operators induced by positive finite Borel measures in $[0, 1)$:

Corollary 2.8 *If μ is a positive finite Borel measure in $[0, 1)$ and R_μ is the Rhaly operator associated to the moment sequence $(\mu_n)_{n \geq 0}$ acting boundedly in ℓ^2 , then the operator $-R_\mu$ is the infinitesimal generator of a contraction semigroup.*

We observe that it is possible to compute the expression of the contraction semigroup at least *formally*. Nevertheless, we will make use of (2.7) in the next section to study the invariant subspaces of Rhaly operators.

3 Invariant subspaces of Rhaly operators

The main aim of this section is studying the invariant subspaces of Rhaly operators. Our starting point is the following result proved in [13] for the classical Cesàro operator.

Theorem 3.1 ([13]) *Let $\{\varphi_t\}_{t \geq 0}$ be the holomorphic self-maps of \mathbb{D} given by*

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}, \quad (z \in \mathbb{D}). \quad (3.1)$$

A closed subspace M in H^2 is invariant under the Cesàro operator if and only if its orthogonal complement M^\perp is invariant under the semigroup of composition operators $\{C_{\varphi_t}\}_{t \geq 0}$.

The one-parameter family $\{\varphi_t\}_{t \geq 0}$ is, indeed, a *holomorphic flow* (or *holomorphic semiflow* by some authors), namely, a continuous family that has a semigroup property with respect to composition (see the monograph [3] for a detailed account of holomorphic flows).

Along these lines, the following result holds for any Rhaly operator when considered as acting on H^2 :

Theorem 3.2 *Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a complex sequence such that the sequence $((n+1)a_n)_{n \geq 0}$ is bounded, and $R_{\mathbf{a}}$ the associated Rhaly operator acting boundedly on the Hardy space H^2 . Let $D_{\mathbf{a}}$ be the operator with diagonal matrix $\text{diag}((n+1)a_n)_{n \geq 0}$ with respect to the standard orthonormal basis in H^2 . Then every closed subspace M invariant under $C_{\varphi_t} D_{\mathbf{a}}$ for all $t \geq 0$ is invariant under $R_{\mathbf{a}}^*$.*

Proof Note that the adjoint of the Cesàro operator acting on the orthonormal basis $\{z^n\}_{n \geq 0}$ of H^2

$$C^* : z^n \rightarrow \frac{1}{n+1} \frac{1-z^{n+1}}{1-z}$$

and the adjoint of $R_{\mathbf{a}}$ acting also on $\{z^n\}_{n \geq 0}$

$$R_{\mathbf{a}}^* : z^n \rightarrow \overline{a_n} \frac{1-z^{n+1}}{1-z},$$

are related by means of the diagonal operator $D_{\mathbf{a}}$ as follows:

$$C^* D_{\mathbf{a}}^* = C^* D_{\overline{\mathbf{a}}} = R_{\mathbf{a}}^*,$$

or, equivalently,

$$D_{\mathbf{a}} C = R_{\mathbf{a}}.$$

Now, we make use of the fact that C^* can be expressed in terms of the C_0 -semigroup $\{C_{\varphi_t}\}_{t \geq 0}$ (see [13]):

$$C^* f(z) = \int_0^\infty e^{-t} C_{\varphi_t} f(z) dt, \quad (f \in H^2),$$

to express

$$R_{\mathbf{a}}^* f(z) = \int_0^\infty e^{-t} C_{\varphi_t} D_{\overline{\mathbf{a}}} f(z) dt, \quad (f \in H^2).$$

From here it follows that the common invariant subspaces of $C_{\varphi_t} D_{\overline{\mathbf{a}}}$ are invariant subspaces of $R_{\mathbf{a}}^*$. \square

When the Rhaly operator is induced by a positive finite Borel measure μ in $[0, 1)$, equation 2.6 yields that R_μ acts on H^2 as

$$R_\mu f(z) = \int_0^1 \frac{f(tz)}{1-tz} d\mu(t), \quad (f \in H^2),$$

or in terms of the family of weighted composition operators $\{W_t\}_{0 \leq t < 1}$, where $W_t f(z) = \frac{f(tz)}{1-tz}$:

$$R_\mu f(z) = \int_0^1 W_t f(z) d\mu(t), \quad (f \in H^2).$$

Upon changing variables $t = e^{-x}$, we have that is,

$$R_\mu f(z) = \int_0^\infty \tilde{W}_x f(z) dv(x),$$

where now

$$\tilde{W}_x f(z) = \frac{f(e^{-x}z)}{1-e^{-x}z} \quad (0 < x \leq \infty), \tag{3.2}$$

and $dv(x) = d\mu(e^{-x})$.

Though the family $\{\tilde{W}_x\}_{x>0}$ does not form a semigroup, we may still conclude the following:

Proposition 3.3 *Let μ be a positive finite Borel measure μ in $[0, 1)$ and R_μ the associated Rhaly operator acting boundedly on the Hardy space H^2 . Let $\{\tilde{W}_x\}_{x>0}$ be the family of weighted composition operators in H^2 defined by (3.2). If M is a closed subspace invariant under every \tilde{W}_x , then M is invariant under R_μ .*

3.1 Common invariant subspaces for W_t

Motivated by Proposition 3.3, we characterize the common invariant subspaces for the family $\{W_t\}_{0 \leq t < 1}$.

Theorem 3.4 *The only nonzero closed subspaces invariant under every W_t , $0 \leq t < 1$, in the Hardy space H^2 are $z^k H^2$ for some $k \geq 0$.*

Proof First, we claim that for each $f \in H^2$ and $t \in (0, 1)$ the function $f(tz)$ is in the norm closed span of $W_x W_{t/x} f$ for $t < x < 1$. Note that,

$$W_x W_{t/x} f(z) = W_x f(tz/x)/(1-tz/x) = \frac{f(tz)}{(1-tz)(1-xz)}.$$

Now the closed span of the functions $1/((1-tz)(1-xz))$ for $t < x < 1$ is the whole space H^2 , since for $g \in H^2$, if g is orthogonal to these functions, then

$$\begin{aligned} \left\langle g, \frac{1}{(1-tz)(1-xz)} \right\rangle &= \frac{1}{x-t} (\langle g, -t/(1-tz) \rangle + \langle g, x/(1-xz) \rangle) \\ &= \frac{1}{x-t} (-tg(t) + xg(x)) = 0, \end{aligned}$$

which, by the isolated zeros theorem, tells us that $zg = 0$ and hence $g = 0$. Thus since $f(tz)$ is in H^∞ , we have that $f(tz)$ is in the closed span of all the $W_x W_{t/x} f$, as claimed.

Hence, it follows that any subspace invariant under all the W_t is also invariant under all the composition operators C_{ϕ_t} induced by $\phi_t(z) = tz$.

Now each operator C_{ϕ_t} is diagonal with respect to the usual orthonormal basis $(z^n)_{n \geq 0}$, and since it is compact its invariant subspaces are spanned by sets of eigenfunctions (see Chapter 1 of [20]).

Thus any common invariant subspace M for $f \mapsto f(xz)/(1 - xz)$, $t < x < 1$ is spanned by monomials (in fact it is sufficient that M is invariant under $f \mapsto f(x_n z)/(1 - x_n z)$ for a sequence (x_n) decreasing to t). But with $f(z) = z^k$ we have

$$f(xz)/(1 - xz) = x^k z^k (1 + xz + x^2 z^2 + \dots),$$

from which Theorem 3.4 follows. □

Remark 3.5 From Theorem 3.4 it follows that the only subspaces invariant under every Rhaly operator R_μ have the same form. Taking μ to be a delta point mass at t , one has that $R_{\delta_t} = W_t$. Likewise, if $e_k(z) = z^k$, note that

$$R_\mu(e_k)(z) = z^k \int_0^1 \frac{t^k}{1 - tz} d\mu(t),$$

so these spaces are also invariant under each R_μ .

4 A final remark: the Hilbert matrix

In [7], Diamantopoulos and Siskakis observe that the Hilbert matrix

$$H = \left(\frac{1}{i + j + 1} \right)_{i,j=0,1,2,\dots}$$

corresponds to an operator on H^2 defined by

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z) dt,$$

where T_t is the weighted composition operator defined by

$$T_t(f)(z) = \frac{1}{(t - 1)z + 1} f\left(\frac{t}{(t - 1)z + 1}\right). \tag{4.1}$$

Arguing as in Sect. 3, it is possible to generalize this, by taking μ a positive Borel measure on $[0, 1)$ and V_t the weighted composition operator defined by

$$V_t f(z) = \frac{f(t)}{1 - tz}. \tag{4.2}$$

Then we note that \mathcal{H}_μ defined by

$$\mathcal{H}_\mu f(z) = \int_0^1 V_t(f)(z) d\mu(t)$$

takes the function $e_n : z \mapsto z^n$ into

$$\mathcal{H}_\mu e_n(z) = \int_0^1 \frac{t^n}{1-tz} d\mu(t) = \mu_n + \mu_{n+1}z + \mu_{n+2}z^2 + \dots,$$

where, as before

$$\mu_n = \int_0^1 t^n d\mu(t),$$

and it therefore corresponds to the Hankel matrix

$$\mathcal{H}_\mu = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \cdots \\ \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The formula in (4.1) was used to calculate the norm of the original Hilbert matrix. The following theorem is similar to those stated for Rhaly operators R_μ :

Theorem 4.1 *Let μ be a positive finite Borel measure in $[0, 1)$ and \mathcal{H}_μ the associated Hankel operator. Suppose that \mathcal{H}_μ is a bounded operator in ℓ^2 . Then $W(\mathcal{H}_\mu) \subset \overline{\mathbb{C}_+}$. Consequently, $\sigma(\mathcal{H}_\mu) \subset \overline{\mathbb{C}_+}$ and the operator $-\mathcal{H}_\mu$ is the infinitesimal generator of a contraction semi-group.*

Proof Reasoning as in Theorem 2.7, it is sufficient to show that the numerical range of each rank-1 operator V_t , with $\mu_n = t^n$ for each n , lies in $\overline{\mathbb{C}_+}$. Arguing with \mathcal{H}_μ acting on the Hardy space H^2 , we observe that for every Hardy function f :

$$\langle V_t f, f \rangle = \left\langle \frac{f(t)}{1-tz}, f(z) \right\rangle = f(t) \overline{\left\langle f(z), \frac{1}{1-tz} \right\rangle} = |f(t)|^2 \geq 0,$$

using the properties of reproducing kernels in H^2 . □

Finally, regarding common invariant subspaces of the rank-1 Hankel operators $\{V_t\}_{0 \leq t < 1}$, we note that

Theorem 4.2 *Let μ be a positive finite Borel measure in $[0, 1)$ and \mathcal{H}_μ the associated Hankel operator. Assume \mathcal{H}_μ is a bounded operator in ℓ^2 . Let V_t be the weighted composition operator in H^2 given by (4.2). Then, the only closed subspaces invariant under every V_t , $0 \leq t < 1$, in the Hardy space H^2 are the trivial ones.*

Proof Let M be a common invariant subspace for all the V_t , $0 \leq t < 1$. Then if there is a function $f \in M$ with $f(t) \neq 0$, we must have $1/(1-tz) \in M$.

Now for $f \neq 0$ in M , we cannot have $f(t) = 0$ for more than a countable set $S \subset [0, 1)$. Then M contains the closed span of $1/(1-tz)$ for $t \in [0, 1) \setminus S$, which is already the whole of H^2 , since $g \in H^2$ is orthogonal to $1/(1-tz)$ if and only if $g(t) = 0$. □

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