

# On the smoothness of weak solutions to subcritical semilinear elliptic equations in any dimension

Rosa Pardo \*

## Abstract

Let us consider a semilinear boundary value problem  $-\Delta u = f(x, u)$ , in  $\Omega$ , with Dirichlet boundary conditions, where  $\Omega \subset \mathbb{R}^N$ ,  $N > 2$ , is a bounded smooth domain. We provide sufficient conditions guarantying that semi-stable weak positive solutions to subcritical semilinear elliptic equations are smooth in any dimension, and as a consequence, classical solutions. By a subcritical nonlinearity we mean  $f(x, s)/s^{\frac{N+2}{N-2}} \rightarrow 0$  as  $s \rightarrow \infty$ , including non-power nonlinearities, and enlarging the class of subcritical nonlinearities, which is usually reserved for power like nonlinearities.

**Keywords:** semi-stable solutions, regularity for weak solutions, subcritical nonlinearities,  $L^\infty$  *a priori* bounds

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## 1 Introduction

Let us consider the following semilinear boundary value problem

$$-\Delta u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.1)$$

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\*Rosa Pardo, Universidad Complutense de Madrid, 28.040 Madrid, Spain, [rpardo@ucm.es](mailto:rpardo@ucm.es)

where  $\Omega \subset \mathbb{R}^N$ ,  $N > 2$ , is a bounded, connected open subset, with  $C^{2,\alpha}$  boundary  $\partial\Omega$ , and the non-linearity  $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and sub-critical. Let  $2^* = \frac{2N}{N-2}$  be the critical Sobolev exponent, by a *subcritical* nonlinearity we mean

$$\lim_{s \rightarrow +\infty} \frac{\max_{x \in \bar{\Omega}} f(x, s)}{s^{2^*-1}} = 0. \quad (1.2)$$

Usually the term subcritical nonlinearity is reserved for power like nonlinearities. Our analysis shows that nonlinearities satisfying (1.2), widen the class of subcritical nonlinearities sharing with power like nonlinearities properties such as  $L^\infty$  a priori estimates, (see Theorem 2.1 and Theorem 2.3), or regularity of semi-stable weak positive solutions. (see Theorem 2.4), and Theorem 2.5). Our definition of a subcritical non-linearity includes nonlinearities such as

$$f^{(1)}(s) := \frac{(1+s)^{2^*-1}}{[\log(e+s)]^\beta}, \quad \text{or} \quad f^{(2)}(s) := \frac{(1+s)^{2^*-1}}{[\log[e + \log(1+s)]]^\beta},$$

for any  $\beta > 0$ .

We focus in contributing to the problem of regularity of *weak solution* in the class of subcritical generalized problems, for any dimension  $N > 2$ .

By a *solution* we mean a weak solution  $u \in H_0^1(\Omega)$  such that  $f(x, u) \in L_{loc}^1(\Omega)$ , and

$$\int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f(x, u) \varphi, \quad \forall \varphi \in C_c^\infty(\Omega). \quad (1.3)$$

Let  $u$  be a solution to (1.1). We will say that  $u$  is *semi-stable* if  $f_s(\cdot, u) \in L_{loc}^1(\Omega)$  and

$$\int_{\Omega} |\nabla \varphi|^2 dx \geq \int_{\Omega} f_s(x, u) \varphi^2 dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega), \quad (1.4)$$

where  $f_s := \frac{\partial f}{\partial s}$ .

Cabr e, Figalli, Ros-Oton, and Serra analyze the regularity of semi-stable<sup>1</sup> solutions with a nonlinearity  $f = f(s)$  positive, non-decreasing, convex, and such that  $f(s)/s \rightarrow \infty$  as  $t \rightarrow \infty$ , and they conclude that semi-stable weak

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<sup>1</sup>They call it stable solutions, see [3, Definition 1.1]

solutions in  $H^1(\Omega)$  are smooth up to dimension  $N \leq 9$ , for domains of class  $C^3$ , see [3, Corollary 1.6]. In their arguments, it is crucial to assume  $f$  to be convex, non-decreasing and non-negative.

Our aim is to show that, in addition to dimension, subcriticality is another barrier dividing smoothness of semi-stable solutions in  $H^1(\Omega)$ . We show sufficient conditions guarantying that any set of positive semi-stable solutions is uniformly  $L^\infty(\Omega)$  a priori bounded, and so they are classical solutions.

In order to prove our result, we first estimate the  $L^\infty$ -norm of *weak solutions* in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ ,<sup>2</sup> in terms of the  $L^{\frac{2N}{N+2}}$ -norm of  $f$ , see Theorem 2.1. As we shall see below, the  $L^\infty$  a priori bound of solutions requires nor convexity neither monotonicity of the nonlinearity. Moreover, we can prove that solutions are universally bounded in terms only of their  $L^{2^*}$ -norm, with a constant independent of the solution, and surprisingly, independent of  $f$  for non-decreasing nonlinearities in a neighborhood of infinity, see Corollary 2.2. In addition, this result holds for positive, negative and changing sign solutions.

Secondly, we will approach weak solutions by smooth ones, see for instance [1, Theorem 3] and [10, Theorem 3.2.1 and Corollary 3.2.1]. With this in mind, we work on sequences of BVP. More specifically, given a sequence of nonlinearities  $f_k$ , and the corresponding sequence of BVP, we provide sufficient conditions guarantying that any set of solutions in  $H_0^1(\Omega) \cap L^\infty(\Omega)$  is uniformly  $L^\infty(\Omega)$  a priori bounded, see Theorem 2.3.

As an application, we next state another main result, concerning the global regularity of semi-stable positive solutions in any dimension, when the non-linearity is subcritical, convex and non-decreasing. Assuming that  $u^* \geq 0$  is a semi-stable weak solution to (1.1), we build a sequence of non-negative solutions in  $H_0^1(\Omega) \cap L^\infty(\Omega)$  upper bounded by  $u^*$ . The key point here is the uniform  $L^\infty$  a-priori bounds for that sequence of solutions. Thanks to that and to the elliptic regularity, we obtain a subsequence, convergent to  $\tilde{u}$  in  $C^{1,\beta}(\overline{\Omega})$  for any  $\beta < 1$ . More regularity on  $f$  guaranties more regularity on  $\tilde{u}$ . The limit  $\tilde{u}$  is clearly a solution to (1.1). To conclude that  $u^*$  is a classical solution, we need to prove that in fact, the limit is  $u^*$ , and at this point, we use the convexity of  $f$ . As a consequence, weak solutions are classical solutions, see Theorem 2.4.

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<sup>2</sup> According to elliptic regularity, if  $f$  is continuous in both variables, then  $u$  is a *strong solution* in  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ , and by Sobolev embeddings,  $u \in C^{1,\beta}(\overline{\Omega})$  for any  $\beta < 1$ .

Finally, we elude hypothesis on convexity in Theorem 2.5, using monotonicity methods, and still giving sufficient conditions so that weak solutions in  $H^1(\Omega)$  to subcritical elliptic equations are smooth in any dimension. We emphasize that this result holds for weak solutions, not necessarily semi-stable.

Smoothness of semi-stable weak solutions is a very classical topic in elliptic equations, posed by Joseph and Lundgren in [13]. They work on particular nonlinearities  $f = f(s)$ , with  $f(s) := e^s$  or  $f(s) := (1 + s)^p$ . They consider the following BVP depending on a multiplicative parameter  $\lambda \in \mathbb{R}$ ,

$$-\Delta u = \lambda f(u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.5)$$

and look for classical radial positive solutions in the unit ball  $B_1$ . Furthermore, they study singular solutions as limit of classical solutions.

When  $N > 2$ ,  $\lambda = 2(N - 2)$ , and  $f(s) := e^s$ , they obtain the explicit weak solution

$$u_1^*(x) := \log \frac{1}{|x|^2}, \quad (1.6)$$

see [13, p. 262]. It can be seen that  $u_1^* \in H_0^1(B_1)$ , and that  $u_1^*$  is a singular weak solution to (1.5) in the unit ball.

On the other hand, the Hardy inequality states that

$$\int_{\Omega} |\nabla \varphi|^2 dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{\varphi^2}{|x|^2} dx, \quad \text{for all } \varphi \in C_0^1(\Omega) \quad (1.7)$$

when  $N \geq 3$ , and then  $u_1^*$  is a singular semi-stable solution when  $N \geq 10$ . Observe that  $f(s) := e^s$  is not a subcritical non-linearity.

When  $N > 2$ ,  $f(s) := (1 + s)^p$  with  $p > \frac{N}{N-2}$ , and  $\lambda = \frac{2}{p-1}(N - \frac{2p}{p-1})$ , they also found the explicit  $L^1$ -weak solution

$$u_2^*(x) := \left( \frac{1}{|x|} \right)^{\frac{2}{p-1}} - 1, \quad (1.8)$$

see [13, (III.a)]. We will say that a function  $u$  is a  $L^1$ -weak solution to (1.1) if

$$u \in L^1(\Omega), \quad f(\cdot, u) \delta_{\Omega} \in L^1(\Omega)$$

where  $\delta_\Omega(x) := \text{dist}(x, \partial\Omega)$  is the distance function with respect to the boundary, and

$$\int_{\Omega} \left( u \Delta \varphi + f(x, u) \varphi \right) dx = 0, \quad \text{for all } \varphi \in C^2(\overline{\Omega}), \quad \varphi|_{\partial\Omega} = 0. \quad (1.9)$$

It holds that  $u_2^* \in W_0^{1, \frac{N}{N-1}}(B_1)$  for  $p > \frac{N}{N-2}$ , and  $u_2^*$  is a singular  $L^1$ -weak solution to (1.5) on the unit ball. Moreover  $u_2^* \in H_0^1(B_1)$  only when  $p > \frac{N+2}{N-2}$ .

Since the Hardy inequality (1.7), it can be checked that  $u_2^*$  is a semi-stable solution if  $\left(\frac{N-2}{2}\right)^2 \geq \frac{2p}{p-1} \left(N - \frac{2p}{p-1}\right)$ . Note that  $f(s) := (1+s)^p$  is a subcritical non-linearity whenever  $p < \frac{N+2}{N-2}$ . In the subcritical range,  $u_2^*$  is a semi-stable solution for  $p \leq \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}$ , and  $u_2^* \in W_0^{1, \frac{N}{N-1}}(B_1)$ , so  $u_2^*$  is a singular  $L^1$ -weak solution, not in  $H^1$ .

Those examples for radially symmetric solutions to BVP's on spherical domains show that the existence of singular solution(s) in  $H_0^1(\Omega)$  is not only related with the dimension, but also with the sub-critical, *critical*, or *super-critical* nature of the non-linearity. By a *critical (supercritical)* non-linearity we mean  $f(x, s) = O(s^{\frac{N+2}{N-2}})$ ,  $(f(x, s)/s^{\frac{N+2}{N-2}} \rightarrow \infty)$  respectively, as  $s \rightarrow \infty$ .

It is natural to ask for the extent of these results on singular positive solutions, over more general nonlinearities and non-spherical domains. The regularity of semi-stable solutions to semilinear elliptic equations, is initiated in [13] with the explicit examples already mentioned, continued by Keener and Keller [14], and by Crandall and Rabinowitz in [8], and rising a huge literature on the topic, see the monograph [10] for an extensive list of results and references. Crandall and Rabinowitz consider a nonlinearity  $f \in C^3$ , positive, non-decreasing, convex, and superlinear at infinity. They state that if  $N < 10$  and the following limit exists

$$\lim_{s \rightarrow \infty} \frac{f(s) f''(s)}{(f_s(x, s))^2} := L \geq 0,$$

then  $H_0^1(\Omega)$  semi-stable solutions to (1.5) are bounded. Brezis and Vázquez study singular  $L^1$ -weak solutions, unbounded in  $L^\infty$ , for nonlinearities  $f \in C^2$ , positive, non-decreasing, convex, and superlinear at infinity, see [2]. When  $f(s) = s^{\frac{N}{N-2}}$ , Pacard in [18] prove the existence of positive  $L^1$ -weak solutions with prescribed singular set. Rébai in [21] study the existence of

positive  $L^1$ -weak solutions which are singular either at exactly  $N$  points, for  $N \geq 2$ , or on a prescribed  $(N - m)$ -dimensional compact submanifold  $\Sigma \subset \Omega$  without boundary, with  $N > m > 2$ , when  $f(s) = s^p$  for  $p > \frac{m}{m-2}$  and close to that number. In both cases, those  $L^1$ -weak solutions are not in  $H^1(\Omega)$ . Results on supercritical problems and their singular sets can be read in [12] and references therein.

This paper is organized in the following way. In Section 2 we state our main results. In Section 3 we include some preliminaries and known results. Section 4 contains the proofs of Theorem 2.1 and Theorem 2.3. Section 5 is devoted to prove Theorem 2.4 and Theorem 2.5.

## 2 Main results: Estimates of the $L^\infty$ -norm of the solutions and Regularity of semi-stable weak solutions

In this Section, we state our main results. We will do it in two parts. In the first part we estimate the  $L^\infty$ -norm of the solutions, in terms of the  $L^{\frac{2N}{N+2}}$ -norm of  $f$ , see Theorem 2.1. Also, given a sequence of nonlinearities  $f_k$ , and the corresponding sequence of BVP, we estimate the  $L^\infty$ -norm of solutions to the sequence of BVP in terms of the  $L^{\frac{2N}{N+2}}$ -norm of  $f_k$ , see Theorem 2.3. This results hold for positive, negative and changing sign solutions.

In the second part, we show that positive semi-stable weak solutions can be approximated by smooth ones, when  $f$  is convex, see Theorem 2.4. We also show that weak solutions can be approximated by smooth ones, when  $f_s > \lambda_1$ , see Theorem 2.5.

### 2.1 Part I. Estimates of the $L^\infty$ -norm of the solutions

We assume that the nonlinearity  $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous in both variables and satisfy the following assumptions

(H1)  $f$  is subcritical, that is  $\lim_{|s| \rightarrow \infty} \frac{\max_{x \in \bar{\Omega}} |f(x, s)|}{|s|^{2^*-1}} = 0$  where  $2^* = \frac{2N}{N-2}$  is the critical Sobolev exponent.

(H2) there exists a uniform constant  $c_0 > 0$  such that

$$\limsup_{s \rightarrow +\infty} \frac{\max_{\overline{\Omega} \times [-s, s]} |f|}{\max_{\overline{\Omega} \times \{-s, s\}} |f|} \leq c_0. \quad (2.1)$$

(H3) there exists two constants  $M_0 > 0$  and  $s_0 > 0$  such that

$$\max_{x \in \overline{\Omega}} |f(x, s)| > M_0 \quad \text{for } |s| > s_0. \quad (2.2)$$

Let us define

$$h(s) := \frac{|s|^{2^*-1}}{\max_{\overline{\Omega} \times \{-s, s\}} |f|}, \quad \text{for } |s| > s_0. \quad (2.3)$$

Our first main results is the following theorem. Let  $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$  be any weak solution to (1.1).<sup>4</sup> Under hypothesis (H1)-(H3), we establish an estimate for the function  $h$  applied to the  $L^\infty(\Omega)$ -norms of the solution, in terms of the  $L^{\frac{2N}{N+2}}(\Omega)$  norm of  $f$ .

From now on,  $C$  denotes several constants that may change from line to line, and are independent of  $u$ .

**Theorem 2.1.** *Assume that  $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function in both variables satisfying (H1)-(H3).*

*Then, there exists a constant  $C > 0$  such that for any  $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$  weak solution to (1.1), the following holds:*

- (i) *either  $\|u\|_\infty \leq C$ , where  $C$  is independent of the solution  $u$ ,*
- (ii) *or for any  $\theta \in (0, 1]$*

$$h(\|u\|_\infty) \leq C (\|u\|_{2^*})^{\frac{N+2}{N-2} \frac{2(1-\theta)}{N-2\theta}} \left( \|f(\cdot, u)\|_{\frac{2N}{N+2}} \right)^{\frac{2(1+\theta)}{N-2\theta}}, \quad (2.4)$$

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<sup>3</sup>Observe that in particular, if  $|f(x, \cdot)|$  is monotone for all  $x \in \Omega$ , then (H2) is obviously satisfied with  $c_0 = 1$ .

<sup>4</sup>Despite the regularity inherent to an  $L^\infty$  bound, (see footnote 2), we keep the notation as above (weak solution in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ ), in order to clarify which hypothesis are specifically involved in each statement.

where  $h$  is defined by (2.3), and  $C$  depends only on  $\Omega$ ,  $\theta$  and  $N$  and it is independent of the solution  $u$ .

In particular (for  $\theta = 1$ )

$$h(\|u\|_\infty) \leq C \left( \|f(\cdot, u)\|_{\frac{2N}{N+2}} \right)^{\frac{4}{N-2}}, \quad (2.5)$$

where  $C$  depends only on  $\Omega$ , and  $N$  and it is independent of  $u$ .

As an immediate corollary, we prove that any sequence of solutions uniformly bounded in the  $L^{2^*}(\Omega)$ -norm, is also uniformly bounded in the  $L^\infty(\Omega)$ -norm.

**Corollary 2.2.** *Let  $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function in both variables satisfying (H1)–(H3).*

*Let  $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$  be any sequence of solutions to (1.1) such that there exists a constant  $C_0 > 0$  satisfying*

$$\|u_k\|_{2^*} \leq C_0. \quad (2.6)$$

*Then, there exists a constant  $C > 0$  such that*

$$\|u_k\|_\infty \leq C. \quad (2.7)$$

*Proof.* We reason by contradiction, assuming that (2.7) does not hold. Indeed, by subcriticality, and (2.6),

$$\left( \|f(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{N+2}} \leq C \left( 1 + \int_\Omega |u_k|^{2^*} dx \right) \leq C.$$

Now part (ii) of the Theorem 2.1 implies that

$$h(\|u_k\|_\infty) \leq C. \quad (2.8)$$

From (2.3) and hypothesis (H1), for any  $\varepsilon_0 > 0$  there exists  $s_0 > 0$  such that  $h(s) \geq 1/\varepsilon_0$  for any  $s \geq s_0$ . This, joint with (2.8) ends the proof.  $\square$

### 2.1.1 Estimates of the $L^\infty$ -norm of solutions to sequences of BVP

Next, we state our second main result. It concerns sequences of subcritical BVP.

Let us now consider a sequence  $f_k : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  of continuous functions in both variables satisfying the following conditions:

(H2)<sub>k</sub> There exists a uniform constant  $c_1 > 0$  such that

$$\limsup_{s \rightarrow +\infty} \frac{\max_{\overline{\Omega} \times [-s, s]} |f_k|}{\max_{x \in \overline{\Omega}} |f_k(x, s)|} \leq c_1. \quad (2.9)$$

(H3)<sub>k</sub> there exists two constants  $M_0 > 0$  and  $s_0 > 0$  such that

$$\max_{x \in \overline{\Omega}} |f_k(x, s)| > M_0 \quad \text{for } |s| > s_0. \quad (2.10)$$

Let us also consider the corresponding sequence of elliptic equations:

$$\begin{cases} -\Delta u = f_k(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.11)$$

For each  $k \in \mathbb{N}$ , let  $u_k \in H_0^1(\Omega) \cap L^\infty(\Omega)$  be a solution to (2.11)<sub>k</sub>. Consider a sequence  $\{u_k\}$  of those solutions.

In the following Theorem, we state sufficient conditions for having a uniform  $L^\infty$  estimate for sequences of solutions  $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$  to (2.11)<sub>k</sub>.

**Theorem 2.3.** *Assume that  $f_k : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of continuous functions in both variables satisfying (H1) for  $f = f_k$ , and (H2)<sub>k</sub>-(H3)<sub>k</sub>.*

*Then, there exists a constant  $C > 0$  such that for any sequence  $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$  of solutions to (2.11)<sub>k</sub>, the following holds:*

(i) either  $\|u_k\|_\infty \leq C$ ,

(ii) or for any  $\theta \in (0, 1]$

$$h_k(\|u_k\|_\infty) \leq C (\|u_k\|_{2^*})^{\frac{N+2}{N-2} \frac{2(1-\theta)}{N-2\theta}} \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2(1+\theta)}{N-2\theta}}, \quad (2.12)$$

where  $h_k$  is defined by (2.3) for  $f = f_k$ , and  $C = C(\Omega, \theta, N)$  and it is independent of  $k$ .

In particular (for  $\theta = 1$ )

$$h_k(\|u_k\|_\infty) \leq C \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{4}{N-2}}, \quad (2.13)$$

where  $C = C(\Omega, N)$  and it is independent of  $k$ .

The arguments are inspired in the equivalence between uniform  $L^{2^*}(\Omega)$  *a priori* bound and uniform  $L^\infty(\Omega)$  *a priori* bound for solutions to subcritical elliptic equations, see [4, Theorem 1.2] for the semilinear case and  $f = f(u)$ , and [16, Theorem 1.3] for the  $p$ -laplacian and  $f = f(x, u)$ . Related results can be found in [5–7, 9, 15, 19, 20].

## 2.2 Part II. Approximation of some weak solutions by sequences of classical solutions

In this second Part, we apply the above results on  $L^\infty$  *a priori* bounds, to positive semi-stable weak solutions. We consider non-negative functions. The Maximum Principle ensures that solutions are now non-negative.

We assume that the nonlinearity  $f : \overline{\Omega} \times [0, \infty) \rightarrow [0, \infty)$  satisfy some of the following assumptions:

(H4) There exist a constant  $c_0 > 1$ , such that

$$\liminf_{s \rightarrow +\infty} \frac{s f_s(x, s)}{f(x, s)} \geq c_0 > 1, \quad \text{where } f_s(x, s) := \frac{\partial f}{\partial s}(x, s). \quad (2.14)$$

(H5) There exist a positive constant  $c_1$ , such that

$$\inf_{\overline{\Omega} \times \mathbb{R}} f_s(x, s) \geq c_1 > \lambda_1, \quad \forall (x, s) \in \overline{\Omega} \times \mathbb{R},$$

where  $\lambda_1 := \lambda_1(-\Delta; \Omega)$  is the first eigenvalue of  $-\Delta$  acting on  $H_0^1(\Omega)$

Our first main result is the following Theorem, showing that any positive weak solution  $u^* \in H_0^1(\Omega)$  of a subcritical elliptic problem, with  $f$  non-negative, non-decreasing, convex and satisfying (H4), is in fact a classical solution. Roughly speaking, this result is known for nonlinearities  $f = f(u)$  (non-negative, non-decreasing, and convex, but not necessarily subcritical), when  $N \leq 9$ , see [3, Corollary 1.6].

**Theorem 2.4.** *Let  $f : \overline{\Omega} \times [0, \infty) \rightarrow [0, \infty)$  be continuous in both variables, and continuously derivable with respect to the second variable. Assume also that  $\forall x \in \Omega$ ,  $f(x, \cdot)$  is non-decreasing, and convex, that  $f(\cdot, s) \in C^\alpha(\overline{\Omega})$  for all  $s \geq 0$ , and that  $f$  satisfies (H1)-(H4).*

*Let  $u^* \in H_0^1(\Omega)$  be a semi-stable weak solution to (1.1).*

*If*

$$f(x, 0) \geq 0, \tag{2.15}$$

*then,  $u^* \in C^{2,\alpha}(\overline{\Omega})$  is a classical solution.*

A sketch of the proof is the following: a weak solution  $u^* \in H_0^1(\Omega)$  is the limit of a curve of smooth solutions to approximate equations. A uniform  $L^{2^*}$  bound is the key. The equivalence between a uniform  $L^{2^*}(\Omega)$  *a priori* bound and a uniform  $L^\infty(\Omega)$  *a priori* bound for weak solutions to subcritical elliptic equations implies a uniform  $L^\infty(\Omega)$  *a priori* bound (see [4, Theorem 1.2] for the semilinear case and  $f = f(u)$ , and [16, Theorem 1.3] for the  $p$ -laplacian and  $f = f(x, u)$ ). By Schauder elliptic regularity, the sequence is uniformly bounded in  $C^{2,\alpha}(\overline{\Omega})$ . By compactness and monotonicity, the approximate solutions converges to  $\tilde{u} \leq u^*$  in  $C^{2,\beta}(\overline{\Omega})$  for any  $\beta < \alpha$ . If  $u^*$  is semi-stable, hypothesis (H4) and convexity play an overriding role to prove that in fact  $\tilde{u} = u^*$ .

Our second main result focuses on proving regularity, excluding convexity. We use instead hypothesis (H5) and sub and supersolution methods, proving the existence of a minimal and a maximal solution and that the sequence of maximal solutions converge to  $u^*$ . It is applicable to any positive weak solution, independently of its stability.

**Theorem 2.5.** *Let  $f : \overline{\Omega} \times [0, \infty) \rightarrow [0, \infty)$  be continuous in both variables, and continuously derivable with respect to the second variable. Assume also that  $f(\cdot, s) \in C^\alpha(\overline{\Omega})$  for all  $s \geq 0$ , and  $f$  satisfies (H1)-(H3), and (H5).*

*Let  $u^* \in H_0^1(\Omega)$  be a non-negative weak solution to (1.1).*

*If (2.15) is satisfied, then  $u^* \in C^{2,\alpha}(\overline{\Omega})$  is a classical solution.*

### 3 Preliminaries and known results

Consider the Hilbert space  $H_0^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega, \mathbb{R}^N)\}$  with its usual inner product and norm,

$$\langle u, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v, \quad \|u\| := \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}.$$

We will denote by  $\|\cdot\|_p$  the standard  $L^p$ -norm.

Let  $\lambda_1 := \lambda_1(-\Delta; \Omega)$ , and let  $\phi_1 > 0$  denote the corresponding eigenfunction, normalized in the  $L^\infty$ -norm.

By elliptic regularity,  $L^1$ -weak solutions to (1.1), bounded in  $L^\infty$  are strong or classical solutions.

**Proposition 3.1.** *Assume that  $\partial\Omega$  is  $C^{2,\alpha}$ . Assume that  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function in both variables. Let  $u$  be a  $L^1$ -weak solution to (1.1).*

*If  $u \in L^\infty(\Omega)$ , then the following holds:*

- (i)  *$u$  is a strong solution in  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  for any  $p > 1$ , and  $u \in C^{1,\beta}(\overline{\Omega})$  for any  $\beta < 1$  satisfies*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C \|f(\cdot, u)\|_{L^\infty(\Omega)}. \quad (3.1)$$

- (ii) *Moreover, if  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is such that for all  $R > 0$  there exists  $L = L(R) > 0$  satisfying*

$$|f(x, s) - f(y, t)| \leq L(|x - y|^\alpha + |s - t|^\alpha),$$

*for all  $s, t \in [-R, R]$ ,  $x, y \in \Omega$ , then  $u \in C^{2,\alpha}(\overline{\Omega})$  is a classical solution and*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \|f(\cdot, u)\|_{C^\alpha(\overline{\Omega})}. \quad (3.2)$$

*Proof.* (i) Since  $u \in L^\infty(\Omega)$ , and  $f$  is continuous in both variables,  $f(\cdot, u) \in L^\infty(\Omega)$ . By Agmon-Douglis-Nirenberg elliptic regularity,  $u \in W^{2,p}(\Omega)$ , for any  $p > 1$ , moreover  $\|u\|_{W^{2,p}(\Omega)} \leq C \|f(\cdot, u)\|_{L^\infty(\Omega)}$ . By Sobolev embeddings  $u \in C^{1,\beta}(\overline{\Omega})$ , for any  $\beta < 1$  and estimate (3.1) holds.

(ii) If  $\partial\Omega$  is  $C^{2,\alpha}$ , by Schauder elliptic regularity,  $u \in C^{2,\alpha}(\overline{\Omega})$  (see [11, Theorem 6.8]) and since [11, Theorem 6.6] with homogeneous Dirichlet boundary conditions, estimate (3.2) holds.  $\square$

**Lemma 3.2.** *Assume that  $f$  is subcritical. Then, for any  $u$  weak solution to (1.1) the following hold:*

$$f(\cdot, u) \in L^{\frac{2N}{N+2}}(\Omega), \quad (3.3)$$

and

$$\int_{\Omega} \left( \nabla u \cdot \nabla \varphi - f(x, u) \varphi \right) dx = 0, \quad \text{for any } \varphi \in H_0^1(\Omega). \quad (3.4)$$

*Proof.* Using (1.2), and Sobolev embeddings, for any  $u \in H_0^1(\Omega)$  there exists a constant  $C > 0$  such that

$$\int_{\Omega} |f(x, u)|^{\frac{2N}{N+2}} dx \leq C \left( 1 + \int_{\Omega} |u|^{2^*} dx \right) \leq C \left( 1 + \|u\|_{H_0^1(\Omega)}^{2^*} \right) < +\infty,$$

hence (3.3) holds.

In addition, by Holder inequality

$$\int_{\Omega} f(x, u) \varphi dx < +\infty \quad \text{for any } \varphi \in H_0^1(\Omega),$$

and by density, (3.4) holds.  $\square$

## 4 Proof of Theorems 2.1 and 2.3

In that Section, we prove Theorem 2.3 for sequences of BVP, and sequences of solutions  $\{u_k\}$  to (2.11). The proof of Theorem 2.1 is a particular case of Theorem 2.3, applied to one particular BVP, (1.1), and we omit it.

*Proof of Theorem 2.3.* Let  $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$  be a sequence of weak solution to (2.11)<sub>k</sub>. If  $\|u_k\|_\infty \leq C$ , then (i) holds.

Now, we argue on the contrary, assuming that  $\|u_k\|_\infty \rightarrow +\infty$  as  $k \rightarrow \infty$

Let  $x_k \in \Omega$  be such that

$$|u_k(x_k)| = \max_{\Omega} |u_k|.$$

Choose  $R_k$  such that

$$|u_k(x)| \geq \frac{1}{2} \|u_k\|_\infty \quad \text{for any } x \in \overline{B}(x_k, R_k),$$

and there exists  $y \in \partial B(x_k, R_k)$  such that

$$|u_k(y)| = \frac{1}{2} \|u_k\|_\infty. \quad (4.1)$$

*Step 1.  $W^{2,q}$  estimates for  $q \in (N/2, N)$ .*

Let us denote by

$$M_k := \max_{\bar{\Omega} \times \{-\|u_k\|_\infty, \|u_k\|_\infty\}} |f_k| \geq C \max_{\bar{\Omega} \times [-\|u_k\|_\infty, \|u_k\|_\infty]} |f_k|, \quad (4.2)$$

by hypothesis (H2)<sub>k</sub>, see (2.9).

For any  $q > \frac{2N}{N+2}$ ,

$$\begin{aligned} \int_{\Omega} |f_k(x, u_k(x))|^q dx &\leq \int_{\Omega} |f_k(x, u_k(x))|^{\frac{2N}{N+2}} |f_k(x, u_k(x))|^{q - \frac{2N}{N+2}} dx \\ &\leq C \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{N+2}} M_k^{q - \frac{2N}{N+2}}. \end{aligned} \quad (4.3)$$

Let us take  $q$  in the interval  $(N/2, N)$ . Combining elliptic regularity with Sobolev embedding, we have that

$$\|u_k\|_{W^{1,q^*}(\Omega)} \leq C \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1 - \frac{2N}{q(N+2)}}, \quad (4.4)$$

where  $1/q^* = 1/q - 1/N$ ; since  $q > N/2$ , then  $q^* > N$ .

*Step 2. A lower bound for the radius  $R_k$ .*

Using Morrey's Theorem, we have that

$$|u_k(x_1) - u_k(x_2)| \leq C |x_1 - x_2|^{2-N/q} \|\nabla u_k\|_{q^*}, \quad \forall x_1, x_2 \in \Omega, \quad (4.5)$$

where the constant  $C$  depends only on  $\Omega$ ,  $q$  and  $N$ . Hence, for all  $x \in \bar{B}(x_1, R) \subset \Omega$

$$|u_k(x) - u_k(x_1)| \leq C R^{2-\frac{N}{q}} \|\nabla u_k\|_{q^*}, \quad (4.6)$$

for any  $k$ . In particular, it follows that for any  $x \in \bar{B}(x_k, R_k)$ ,

$$|u_k(x) - u_k(x_k)| \leq C (R_k)^{2-\frac{N}{q}} \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1 - \frac{2N}{q(N+2)}}. \quad (4.7)$$

Taking  $x = y$  in the above inequality and from (4.1) we obtain

$$C (R_k)^{2-\frac{N}{q}} \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1-\frac{2N}{q(N+2)}} \geq \frac{1}{2} \|u_k\|_\infty, \quad (4.8)$$

which implies

$$(R_k)^{2-\frac{N}{q}} \geq \frac{1}{2C} \frac{\|u_k\|_\infty}{\left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1-\frac{2N}{q(N+2)}}}, \quad (4.9)$$

or equivalently

$$R_k \geq C \left( \frac{\|u_k\|_\infty}{\left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1-\frac{2N}{q(N+2)}}} \right)^{1/(2-\frac{N}{q})}. \quad (4.10)$$

*Step 3. A lower bound for the  $L^{2^*}$ -norms.*

Now, from definition of  $B(x_k, R_k)$ ,

$$\int_{B(x_k, R_k)} |u_k|^{2^*} \geq \left( \frac{1}{2} \|u_k\|_\infty \right)^{2^*} \omega (R_k)^N,$$

where  $\omega = \omega_N$  is the volume of the unit ball in  $\mathbb{R}^N$ .

Using the inequality (4.10), we deduce

$$\int_{\Omega} |u_k|^{2^*} \geq C \left( \frac{(\|u_k\|_\infty)^{1+2^*(\frac{2}{N}-\frac{1}{q})}}{\left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} M_k^{1-\frac{2N}{q(N+2)}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}}.$$

Denoting

$$a = 1 + 2^* \left( \frac{2}{N} - \frac{1}{q} \right), \quad b = 1 - \frac{2N}{q(N+2)},$$

observe that  $\frac{a}{b} = 2^* - 1$ . From (H2) $_k$ , and due to  $h_k$  is defined by (2.3) for  $f = f_k$ , we deduce

$$\|u_k\|_{2^*}^{2^*} \geq C \left( h_k(\|u_k\|_\infty) \right)^{\frac{1-\frac{2N}{q(N+2)}}{\frac{2}{N}-\frac{1}{q}}} \frac{1}{\|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}}^{\frac{\frac{2N}{q(N+2)}}{\frac{2}{N}-\frac{1}{q}}}}.$$

Since the above, we can write

$$h_k(\|u_k\|_\infty) \leq C \left[ (\|u_k\|_{2^*})^{2^* \left(\frac{2}{N} - \frac{1}{q}\right)} \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2N}{q(N+2)}} \right]^{\frac{1}{1 - \frac{2N}{q(N+2)}}}.$$

Writing  $\frac{1}{q} = \frac{1+\theta}{N}$  with  $\theta \in (0, 1)$  we obtain

$$h_k(\|u_k\|_\infty) \leq C (\|u_k\|_{2^*})^{\frac{N+2}{N-2} \frac{2(1-\theta)}{N-2\theta}} \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{2(1+\theta)}{N-2\theta}}.$$

Finally by elliptic regularity and Sobolev embedding,

$$\|u_k\|_{2^*} \leq C \|u_k\|_{H_0^1(\Omega)} \leq C \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}}, \quad (4.11)$$

and we deduce that

$$h_k(\|u_k\|_\infty) \leq C \left( \|f_k(\cdot, u_k)\|_{\frac{2N}{N+2}} \right)^{\frac{4}{N-2}},$$

ending the proof.  $\square$

## 5 Approximation of weak solutions. Sequences of BVP. Regularity theory of weak solutions.

In that Section, we use families of BVP, and families of classical solutions to approach some weak solutions, and prove Theorem 2.4 and Theorem 2.5.

The next Proposition provides an approximation result for some weak solutions to (1.1) with subcritical nonlinearities. It states that there exist a family of BVP, and a family of solutions, uniformly bounded and convergent to a classical solution to (1.1).

**Proposition 5.1.** *Let  $f : \bar{\Omega} \times [0, +\infty) \rightarrow [0, +\infty)$  be a continuous function in both variables, satisfying (H1) and (2.15). Assume that there exists  $s_0 > 0$  such that  $f(x, \cdot)$  non-decreasing for all  $s \geq s_0$ ,  $x \in \bar{\Omega}$ .*

*Let  $u^* \in H_0^1(\Omega)$  be a non-negative weak solution to (1.1).*

Then, for some  $\varepsilon_0 > 0$ , there exist a family of non-linearities  $\{f_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$  and a family of strong solutions  $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)} \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  for any  $p > 1$ , to

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon(x, u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

such that  $f_\varepsilon \leq f$ ,  $u_\varepsilon \leq u^*$ , and  $f_\varepsilon \uparrow f$  (pointwise in  $\bar{\Omega} \times \mathbb{R}$ ). Moreover,  $u_\varepsilon \rightarrow \tilde{u}$  in  $C^{1,\beta}(\bar{\Omega})$  as  $\varepsilon \rightarrow 0$ , for any  $\beta < 1$ , and  $\tilde{u}$  solves (1.1).

**Remark 5.2.** Once proved the above Proposition, there is still an open question: is  $\tilde{u} = u^*$ ? Theorem 2.4 and Theorem 2.5 shows two different ways to answer positively.

*Proof of Proposition 5.1.* If  $u^* \in L^\infty(\Omega)$  the proof is easily achieved with  $f_\varepsilon(x, s) := (1 - \varepsilon)f(x, s)$ .

Assume that  $u^* \notin L^\infty(\Omega)$ .

*Step 1. Construction of  $f_\varepsilon \leq f$ .*

Let us define

$$f_\varepsilon(x, s) := \begin{cases} (1 - \varepsilon)f(x, s), & s \leq 1/\varepsilon, \\ (1 - \varepsilon)f(x, 1/\varepsilon), & s \geq 1/\varepsilon. \end{cases} \quad (5.2)$$

Due to  $f$  is non-negative and  $f(x, \cdot)$  is non-decreasing for  $s \geq s_0$ , choosing  $\varepsilon_0 = 1/s_0$ ,  $f_\varepsilon$  is a non decreasing family for any  $\varepsilon \in (0, \varepsilon_0)$ ,  $f_\varepsilon \leq f$ , and  $f_\varepsilon \uparrow f$  (pointwise in  $\bar{\Omega} \times \mathbb{R}$ ) as  $\varepsilon \downarrow 0$ .

Thanks to Beppo-Levi Theorem, and Lemma 3.2,

$$f_\varepsilon(\cdot, u^*(\cdot)) \rightarrow f(\cdot, u^*(\cdot)) \quad \text{in } L^{\frac{2N}{N+2}}(\Omega). \quad (5.3)$$

Observe also that subcriticality (see (1.2)) implies in particular the following

$$0 \leq f_\varepsilon(x, s) \leq C_\varepsilon < +\infty, \quad \text{for all } s \geq 0, \quad (5.4)$$

for each  $\varepsilon \in (0, \varepsilon_0)$ .

*Step 2. Construction of  $u_\varepsilon \leq u^*$ , strong solutions to (5.1) $_\varepsilon$  in  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  for any  $p > 1$ .*

Consider now the family of BVP's (5.1) $_\varepsilon$ . From (2.15), 0 is a subsolution to (5.1) $_\varepsilon$  for any  $\varepsilon \in (0, \varepsilon_0)$ , and not a solution. On the other hand, since  $f$  is

non-negative and non-decreasing,  $u^*$  is a weak supersolution to  $(5.1)_\varepsilon$  for any  $\varepsilon \in (0, \varepsilon_0)$ .

Consequently, there exist  $L^1$ -weak solutions  $\underline{u}_\varepsilon \leq \bar{u}_\varepsilon$  of  $(5.1)_\varepsilon$  in  $[0, u^*]$  such that any  $u_\varepsilon$  solution to  $(5.1)_\varepsilon$  in the interval  $[0, u^*]$ , satisfies

$$0 \leq \underline{u}_\varepsilon \leq u_\varepsilon \leq \bar{u}_\varepsilon \leq u^* \quad \text{a.e.} \quad (5.5)$$

(see [17, Theorem 1.1]), since (5.4), for each  $\varepsilon > 0$ ,  $f_\varepsilon$  is bounded, hence  $u_\varepsilon$  are strong solutions in  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  for any  $p > 1$ , and  $u_\varepsilon \in C^{1,\beta}(\bar{\Omega})$ , see Proposition 3.1.

*Step 3. The family of solutions  $u_\varepsilon$  is uniformly bounded in  $C^{1,\beta}(\bar{\Omega})$ .*

Fix  $\varepsilon \in (0, \varepsilon_0)$ . Since  $f_\varepsilon \leq f$ ,  $f$  is non-negative, non-decreasing, and  $u_\varepsilon \leq u^*$

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon|^2 dx &= \int_{\Omega} f_\varepsilon(x, u_\varepsilon) u_\varepsilon \leq \int_{\Omega} f(x, u_\varepsilon) u^* \\ &\leq \int_{\Omega} f(x, u^*) u^* = \|u^*\|_{H_0^1(\Omega)}^2 \leq C, \end{aligned}$$

where  $C$  is only dependent on  $f$  and  $u^*$ , and it is independent of  $\varepsilon$ . Now, Theorem 2.3 implies that  $\{u_\varepsilon\}$  are uniformly  $L^\infty$  a priori bounded. By elliptic regularity (see Proposition 3.1), there exists a uniform constant  $C > 0$  such that  $\|u_\varepsilon\|_{C^{1,\beta}(\bar{\Omega})} \leq C$  for any  $\beta < 1$ .

*Step 4.  $u_\varepsilon \rightarrow \tilde{u}$  in  $C^{1,\beta}(\bar{\Omega})$  as  $\varepsilon \rightarrow 0$ , for any  $\beta < 1$ , and  $\tilde{u}$  solves (1.1).*

By compact embeddings and monotonicity, for any  $\beta' < \beta < 1$  the family  $\{u_\varepsilon\}$ , converges to  $\tilde{u}$  in  $C^{1,\beta'}(\bar{\Omega})$  as  $k \rightarrow \infty$ , see [11, Lemma 6.36].

Moreover,  $\tilde{u}$  solves (1.1). Indeed, since  $u_\varepsilon$  solves  $(5.1)_\varepsilon$ , using the Lipschitzian property of  $f$  on bounded intervals, and the uniform  $L^\infty$  bound for  $u_\varepsilon$ , and  $\tilde{u}$ , for any  $\varphi \in H_0^1(\Omega)$ :

$$\begin{aligned} \left| \int_{\Omega} \nabla \tilde{u} \nabla \varphi - f(x, \tilde{u}) \varphi dx \right| &= \left| \int_{\Omega} \nabla(u_\varepsilon - \tilde{u}) \nabla \varphi - [f_\varepsilon(x, u_\varepsilon) - f(x, \tilde{u})] \varphi dx \right| \\ &= \left| \int_{\Omega} \nabla(u_\varepsilon - \tilde{u}) \nabla \varphi - [f_\varepsilon(x, u_\varepsilon) - f(x, u_\varepsilon) + f(x, u_\varepsilon) - f(x, \tilde{u})] \varphi dx \right| \\ &\leq C \left( \|\nabla(u_\varepsilon - \tilde{u})\|_2 + \varepsilon \|f(x, u_\varepsilon)\|_{\frac{2N}{N+2}} + \|f(x, u_\varepsilon) - f(x, \tilde{u})\|_{\frac{2N}{N+2}} \right) \|\varphi\|_{H_0^1(\Omega)} \\ &\rightarrow 0, \quad \text{as } k \rightarrow \infty, \end{aligned}$$

ending the proof.  $\square$

## 5.1 Proof of Theorem 2.4

We extend the above result on smoothness to the case of weak solutions to subcritical semilinear elliptic equations in any dimension. The question is now if for any positive semi-stable weak solution  $u^*$ , we can construct a sequence of BVP and a sequence of classical solutions convergent to  $u^*$ .

*Proof of Theorem 2.4.* If  $\sup_{\Omega} u^* < +\infty$ , then  $u^* \in C^2(\Omega)$  and the proof is finished.

Assume that

$$\sup_{\Omega} u^* = +\infty, \quad (5.6)$$

By Proposition 5.1, there exists a family  $u_{\varepsilon} \rightarrow \tilde{u}$  in  $C^{2,\beta}(\overline{\Omega})$  as  $\varepsilon \rightarrow 0$ , for any  $\beta < \alpha$ , and  $\tilde{u}$  solves (1.1). We will now prove that  $\tilde{u} = u^*$ .

Assume by contradiction that  $\tilde{u} \leq u^*$ . We observe that, by a density argument and Fatou's Lemma, the semi-stability inequality holds for  $\varphi \in H_0^1(\Omega)$ . Testing it for  $u^*$  with  $u^* - \tilde{u} \geq 0$ , we obtain

$$\begin{aligned} \int_{\Omega} [f(x, u^*) - f(x, \tilde{u})](u^* - \tilde{u}) dx &= \int_{\Omega} |\nabla(u^* - \tilde{u})|^2 dx \\ &\geq \int_{\Omega} f_s(x, u^*)(u^* - \tilde{u})^2 dx. \end{aligned}$$

On the other hand, by convexity

$$f(x, u^*) - f(x, \tilde{u}) \leq f_s(x, u^*)(u^* - \tilde{u}),$$

this leads to

$$f(x, u^*) - f(x, \tilde{u}) = f_s(x, u^*)(u^* - \tilde{u}), \quad \text{a.e. in } \Omega. \quad (5.7)$$

Due to  $\tilde{u}$  is a classical solution, there exists  $\tilde{M} > 0$  such that

$$0 \leq \tilde{u} \leq \tilde{M}. \quad (5.8)$$

Hypothesis (H4) implies that given  $\varepsilon_0 = (c_0 - 1)/2 > 0$ , there exists  $s_0$  such that

$$\frac{sf_s(x, s)}{f(x, s)} \geq \frac{c_0 + 1}{2} > 1, \quad \text{for all } s \geq s_0, \text{ a.e. in } \Omega.$$

Therefore, there exists  $\delta > 0$  and  $s_1 > 0$  such that

$$\frac{f_s(x, s)(s - \tilde{M})}{f(x, s)} \geq \frac{(c_0 + 1)(s - \tilde{M})}{2s} \geq 1 + \delta, \quad \text{for all } s \geq s_1.$$

Taking into account (5.6), there exists  $\emptyset \neq \omega_1 \subset \Omega$  be such that  $u^*(x) \geq s_1$  a.e.  $x \in \omega_1$ .

From the above, and taking into account (5.7)-(5.8), we deduce

$$1 = \frac{f(x, \tilde{u}(x)) + f_s(x, u^*(x))(u^*(x) - \tilde{u}(x))}{f(x, u^*(x))} \geq 1 + \delta > 1, \quad \text{a.e. } x \in \omega_1$$

reaching a contradiction and concluding the proof.  $\square$

## 5.2 Proof of Theorem 2.5

The only difference with the proof of Theorem 2.4 are the arguments involved in proving that  $\tilde{u} = u^*$ .

*Proof of Theorem 2.5.* We apply Proposition 5.1 and get a sequence  $u_k \rightarrow u$  in  $C^{2,\beta}(\overline{\Omega})$  as  $k \rightarrow \infty$ , for any  $\beta < \alpha$ , and  $u$  solves (1.1). We can repeat the argument for the family of maximal solutions  $\bar{u}_\varepsilon$  getting a subsequence  $\{\bar{u}_k\}$  convergent to  $\bar{u} \leq u^*$  in  $C^{1,\beta}(\overline{\Omega})$ . Moreover  $\bar{u} \in C^{2,\beta}(\overline{\Omega})$  solves (1.1). If  $\bar{u} = u^*$ , then the proof is finished.

Assume on the contrary that  $\bar{u} \not\leq u^*$ . Fix  $\varepsilon \in (0, \varepsilon_0)$  small enough. We now prove that  $\bar{u}_\varepsilon = u^*$ , arguing on the contrary, and assuming that  $\bar{u}_\varepsilon \not\leq u^*$ . Observe firstly that hypothesis (H5) imply that there exists a positive constant  $\delta_0$  such that

$$f_s(x, s) \geq \lambda_1 + \delta_0, \quad \text{for any } x \in \overline{\Omega}, s \geq 0, \quad (5.9)$$

and consequently

$$f(x, s) \geq (\lambda_1 + \delta_0)s, \quad \text{for any } x \in \overline{\Omega}, s \geq 0. \quad (5.10)$$

Secondly, we see that  $\bar{u}_\varepsilon + \delta_1 \phi_1$  is a subsolution to  $(5.1)_\varepsilon$  for any  $\delta_1 > 0$ . From (5.9), and thanks to the mean value theorem, there exists a  $\theta = \theta(x)$

such that

$$\begin{aligned}
-\Delta(\bar{u}_\varepsilon + \delta_1\phi_1) &= (1 - \varepsilon)f(x, \bar{u}_\varepsilon) + \delta_1\lambda_1\phi_1 \leq (1 - \varepsilon)f(x, \bar{u}_\varepsilon + \delta_1\phi) \\
&\iff \delta_1\lambda_1\phi_1 \leq (1 - \varepsilon)[f(x, \bar{u}_\varepsilon + \delta_1\phi) - f(x, \bar{u}_\varepsilon)] \\
&\iff \delta_1\lambda_1\phi_1 \leq (1 - \varepsilon)f_s(x, \bar{u}_\varepsilon + \theta\delta_1\phi_1)\delta_1\phi_1 \\
&\iff \lambda_1 \leq (1 - \varepsilon)f_s(x, \bar{u}_\varepsilon + \theta\delta_1\phi_1) \checkmark.
\end{aligned}$$

Thirdly, we check that  $\bar{u}_\varepsilon + \delta_1\phi_1 \leq u^*$  for  $\delta_1$  small enough. From (5.10),  $f(x, \bar{u}_\varepsilon) \geq \lambda_1\bar{u}_\varepsilon$ , and since  $\bar{u}_\varepsilon$  is a classical solution, there exists a  $\delta_2 > 0$  such that  $\bar{u}_\varepsilon > \delta_2\phi_1$ . Hence, for any  $\varphi \in C^2(\bar{\Omega})$ ,  $\varphi|_{\partial\Omega} = 0$ ,  $\varphi > 0$ , the following holds

$$\begin{aligned}
\int_{\Omega} \nabla(u^* - \bar{u}_\varepsilon - \delta_1\phi_1) \cdot \nabla\varphi &= \int_{\Omega} [f(x, u^*) - (1 - \varepsilon)f(x, \bar{u}_\varepsilon) - \delta_1\lambda_1\phi_1] \varphi \\
&= \int_{\Omega} [f_s(x, \theta u^* + (1 - \theta)\bar{u}_\varepsilon) (u^* - \bar{u}_\varepsilon) + \varepsilon f(x, \bar{u}_\varepsilon) - \delta_1\lambda_1\phi_1] \varphi \\
&\geq \int_{\Omega} [\lambda_1 (u^* - \bar{u}_\varepsilon) + \varepsilon\lambda_1\bar{u}_\varepsilon - \delta_1\lambda_1\phi_1] \varphi \\
&\geq \int_{\Omega} [\lambda_1 (u^* - \bar{u}_\varepsilon) + (\varepsilon\delta_2 - \delta_1)\lambda_1\phi_1] \varphi \geq 0,
\end{aligned}$$

choosing  $\delta_1 \leq \varepsilon\delta_2$ . Therefore  $\bar{u}_\varepsilon + \delta_1\phi_1 \leq u^*$ . Consequently, there exist a weak solution  $\hat{u}_\varepsilon$  of  $(5.1)_\varepsilon$  in  $[\bar{u}_\varepsilon + \delta_1\phi_1, u^*]$ , contradicting that  $\bar{u}_\varepsilon$  is a maximal solution to  $(5.1)_\varepsilon$  in the interval  $[0, u^*]$ .

Consequently,  $u_\varepsilon \uparrow u^* \in C^{2,\beta}(\bar{\Omega})$ , and finally, the elliptic regularity ends the proof.  $\square$

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