

## FUZZY RATIONALITY AS A BASIS FOR GROUP DECISION MAKING

J. Montero (\*)  
Faculty of Mathematics  
Complutense University  
Madrid 28040 (Spain)

**Abstract:** This paper deals with the group decision making problem, assuming that each individual defines his/her opinion through fuzzy binary preference relations, in parallel to the classical approach due to Prof. Arrow. In particular, it is postulated that the main reason for the discouraging impossibility theorems is neither in the domain of admissible preferences or in the concept of solution (*Social Welfare Functions* versus *Social Decision Functions*), but in the underlying idea of rationality under all crisp approaches: *non complete irrational* aggregations will be possible under a fuzzy approach, in such a way that classical Arrow's theorem should be understood just as an impossibility of getting *complete rational* aggregations.

**Keywords:** aggregation rules, fuzzy opinions, fuzzy rationality, group decision making.

### 1. INTRODUCTION

As shown by Arrow [1] in his classical impossibility theorem, aggregation of individual preferences into a single social preference represents a difficult task if we want to keep a minimum of ethical rules together with a minimum of rational rules.

In his historical work, it was considered an arbitrary finite group  $D$  of -at least two- human beings, expressing their preferences about a finite set  $X$  of -at least three- feasible alternatives. The problem was stated on how to reach to a social preference representing all the group as a single individual, and it was assumed that both individual and consensus social opinions were expressed by linear orders, that is, complete, reflexive and transitive binary relations defined on the set  $X$  of alternatives. In this context it was proved the impossibility of aggregation rules assigning a social preference ordering to each possible profile of individual preference orderings -*Social Welfare Functions*- verifying some ethical conditions which seemed to be desirable at a first sight:

---

(\*) Present address: School of Business Administration, 350 Barrows Hall, Berkeley University, California 94720.

(1) *Unrestricted domain*: each individual is free to choose his/her personal preference ordering;

(2) *Non-negative Response*: if some changes are made by individuals improving their opinions about some alternative, it does not cause a final change against such alternative;

(3) *Independence of Irrelevant Alternatives*: there is no influence between disjoint subsets of alternatives;

(4) *Citizen Sovereign*: if people agrees about a particular ordering, such ordering should represent the social consensus; and

(5) *Non-dictatorship*: there is no individual imposing systematically his/her opinion.

Many efforts have been applied to this problem of amalgamating individual opinions into a consensus, since it represents a key point for the development of any social group. Research after Arrow's theorems appears as a long discussion in order to know where does the difficulty lie in fact. From a normative point of view, the objective was to search for minimal sets of conditions allowing some kind of social consensus (in some way, it could also help to understand how human societies are very often able to solve their problems, despite so lots of discouraging impossibility theorems that followed Arrow's work). Two approaches have been mainly tried:

(a) By assuming some restriction on the domain, arguing that the most important difficulty in practice is due to the variety of individual opinions (for example, by considering some condition based on Black's *single-peakedness* [2]). As a consequence, it could be said that strong contradictions in a group can be solved only by *suppression* of minorities; a more admissible consequence was given in a fuzzy context (see [3]) by considering the idea of *single-peakedness* in order to show how dangerous for a consensus are those *crisp* individuals with *too clear* opinions, not inclined anyway to accept other conclusion than their own.

(b) By relaxing the concept of consensus, arguing that the real objective in practice is just some useful decision-oriented information (for example, by considering Sen's *Social Decision Functions*, which should allow a coherent subset of alternatives that should be analyzed in a second step).

Some positive results have been obtained under both approaches, but even Sen himself recognizes that Arrow's negative philosophy still remains. But though it can be understood that there is no general methodology for

aggregating crisp individual preferences, in practice consensus are usually reached, perhaps through a dynamic decision-oriented process, or just making some kind of social pressure against *discordant* individuals to margin them in such a way that they are in fact excluded from the decision making. Our thesis is that the main difficulty under Arrow's focus is not due to how any *ethical* condition is formally written or how the social opinion should be expressed, but how the idea of *rationality* is understood: the underlying Aristotelian concept of *rationality* based upon any *crisp* transitivity or *crisp* acyclity, which provokes to think that if something is not absolutely rational, then it is absolutely irrational. In this sense, as pointed out in Montero [5], Lukasiewicz's censure to sciences based on using Aristotelian logic is also valid here.

## 2.- FUZZY ACYCLITY AS A MEASURE OF RATIONALITY

Let us assume that each preference is given by a fuzzy preference relation, that is, a mapping

$$\mu: X \times X \rightarrow [0,1]$$

where each value  $\mu(x,y)$  represents the intensity of preference of alternative  $x$  over alternative  $y$  ( $xPy$ ). If there is no problem of incomparability, we can assume that such a fuzzy preference relation is complete:

$$\mu(x,y) + \mu(y,x) \geq 1 \quad \forall x,y \in X$$

in such a way that the values

$$\mu_I(x,y) = \mu(x,y) + \mu(y,x) - 1$$

$$\mu_B(x,y) = \mu(x,y) - \mu_I(x,y)$$

$$\mu_W(x,y) = \mu(y,x) - \mu_I(x,y)$$

can be understood, respectively, as the degree of indifference between each pair of alternatives ( $xIy$ ), the degree of strict preference of alternative  $x$  over alternative  $y$  ( $xBy$ ), and the degree of strict preference of alternative  $y$  over alternative  $x$  ( $xWy$ ). It should be pointed out that under our approach, the condition  $\mu(x,x)=1 \quad \forall x \in X$  is really not required (but depending on such values  $\mu(x,x) \quad \forall x \in X$ , the degree of rationality of the relation  $\mu$  will be higher or lower, as shown later).

Our idea in this paper is to keep the complete philosophy given in Arrow [1] but allowing strength in preference intensities. A natural measure for rationality can be proposed, on the basis of the concept of acyclity due to Sen [4], allowing a quite different meaning of classical Impossibility Theorems.

Let us consider a given fuzzy preference relation  $\mu$  and let us look for a natural weighting of all acyclic chains of different alternatives: for example, if we take only the alternative  $x$ , there is only one acyclic path:

$xIx$

which can be weighted by  $\mu_1(x,x)$ ; and if we take two alternatives  $\{x,y\}$ , we can find three different acyclic paths:

$xByWx, yBxWy, xIyIx$

which can be weighted respectively by

$$\mu_B^2(x,y), \mu_B^2(y,x), \mu_1^2(x,y)$$

In general, a path  $x_1-x_2-\dots-x_k-x_1$  of  $k$  distinct alternatives will be non-acyclic if and only if  $x_1Px_2P\dots Px_kPx_1$  with some strict preference or  $x_1Px_kP\dots Px_2Px_1$  with some strict preference; in our context, a natural degree of acyclity of a given path can be defined by an addition -through all contained acyclic chains- of products of preference intensities (see [5]).

Based on those path weights, rationality -that is, acyclity- can be defined as a fuzzy property

$$A: \mathcal{R}(X) \rightarrow [0,1]$$

being  $\mathcal{R}(X)$  is the family of all complete fuzzy preference relations on  $X$ ; the value  $A(\mu)$  will be the minimum sum of weights of acyclic paths over all possible chains of alternatives:

$$A(\mu) = \min_{\mu} A_{\mu}(G)$$

where  $G=(x_1-x_2-\dots-x_k)$  represents an arbitrary path of different alternatives and, taking  $x_{k+1}=x_1$  by convenience,

$$A_{\mu}(G) = 1 - (\frac{k}{i=1} \mu(x_i, x_{i+1}) + \frac{k}{i=1} \mu(x_{i+1}, x_i) - 2 \cdot \frac{k}{i=1} \mu_1(x_i, x_{i+1}))$$

For example, if  $X=\{x,y,z\}$  with

$$\begin{aligned} \mu(x,x) &= \mu(y,y) = \mu(z,z) = 1 \\ \mu(x,y) &= 0.3, \quad \mu(y,x) = 0.8 \\ \mu(y,z) &= 0.9, \quad \mu(z,y) = 0.4 \\ \mu(z,x) &= 0.5, \quad \mu(x,z) = 0.7 \end{aligned}$$

then the degree of acyclity -rationality- of such a complete fuzzy preference relation  $\mu$  will be

$$A(\mu) = \min \{ 1, 1, 1; 0.54, 0.46, 0.38; 0.653 \} = 0.38$$

where we can find -in this order- the sum of acyclic weights for the three chains  $(x)$ ,  $(y)$  and  $(z)$  with only one alternative (in this case we find what we know as crisp reflexivity), the sum of acyclic weights for the three chains  $(x,y)$ ,  $(y,z)$  and  $(z,x)$  with two alternatives, and the acyclity for the unique chain  $(x,y,z)$  with three alternatives. In particular, the path  $G=(z,x)$  drops the lowest acyclity  $A_{\mu}(G)$ , that is, the highest irrationality.

Obviously, the number of numerical operations required for the above expression increases exponentially as the number of alternatives grows up. Moreover, in real applications, we can find that some portion of irrationality can be justified just by considering the size

of the problem (the higher number of comparisons needed, the lowest acyclity); this *structural* acyclity should be measured by some *ad hoc* sensitivity analysis. For example, very long chains fall very easily to the level 0.5 of acyclity (see [5]), suggesting a modification of the above initial definition for  $A(\mu)$ , perhaps by weighting all the values  $A_\mu(G)$ , depending on the size of the path  $G$ .

### 3.- THE AMALGAMATING PROBLEM UNDER FUZZY RATIONALITY

Following the welfare-oriented approach of Arrow, and assuming preference intensities between all pairs of alternatives, in such a way that individuals and group opinions are represented by complete fuzzy preference relations, Social Welfare Functions in this context can be defined -as a first proposal- as mappings

$$S: \mathcal{R}^n(X) \rightarrow \mathcal{R}(X)$$

( $\text{card}(D)=n$ ), verifying analogous ethical conditions to those of Arrow:

(1) *Unrestricted Domain*: such mapping is in fact defined over all possible profiles of complete fuzzy preference relations  $\mathcal{R}^n(X)$ .

(2) *Non-Negative Response*:

$$S(p^1, p^2, \dots, p^n)(x, y) \geq S(q^1, q^2, \dots, q^n)(x, y) \quad \forall x, y \in X$$

whenever  $p^i(x, y) \geq q^i(x, y) \quad \forall i \in D, \forall x, y \in X$ .

(3) *Independence of Irrelevant Alternatives*:

$$p^i(x, y) = q^i(x, y) \quad \forall i \in D, \forall x, y \in Y \subset X \quad \Rightarrow$$

$$\Rightarrow S(p^1, p^2, \dots, p^n)(x, y) \geq S(q^1, q^2, \dots, q^n)(x, y) \quad \forall x, y \in Y \subset X$$

for any non-empty subset of alternatives  $Y$ .

(4) *Citizen Sovereign*: for any given  $p \in \mathcal{R}(X)$  there exists a profile  $(p^1, p^2, \dots, p^n) \in \mathcal{R}^n(X)$  such that

$$S(p^1, p^2, \dots, p^n)(x, y) = p(x, y) \quad \forall x, y \in X$$

(5) *Non-Dictatorship*: there is no individual  $i \in D$  such that

$$S(p^1, p^2, \dots, p^n)(x, y) = p^i(x, y)$$

for all  $x, y \in X$  and any  $(p^1, p^2, \dots, p^{i-1}, p^{i+1}, \dots, p^n) \in \mathcal{R}^{n-1}(X)$ .

The condition of *Unrestricted Domain* requires a more carefully discussion in order to maintain Arrow's philosophy: when linear orders were associated to each individual and the group itself, it was assumed -from our point of view- that individuals and group were absolutely rational; and this should be the strict consequence of classical Impossibility Theorems: there is no ethical possibility of assuring absolute rationality. From our

point of view, such maximum degree of rationality can not be claimed in practical situations, neither for individuals; hence, perhaps some minimums degrees of rationality should be previously defined for the group preference relation and all individual preference relations. A simple approach could be just assume that individuals are not absolutely irrational, and then ask if it is possible to find rules assuring non absolutely irrationality for the group.

According to these last comments, let us denote

$$\mathcal{F}(X) = \{\mu \in \mathcal{R}(X) / A(\mu) > 0\}$$

the set of Non-Absolutely Irrational (NAI) complete fuzzy preference relations. A NAI Social Welfare Function will be then given by a mapping

$$S: \mathcal{F}^n(X) \rightarrow \mathcal{F}(X)$$

satisfying the above conditions (1) to (5), where  $\mathcal{R}^n(X)$  and  $\mathcal{R}(X)$  have been replaced, respectively, by  $\mathcal{F}^n(X)$  and  $\mathcal{F}(X)$ . Then we can escape from the crisp negative result, since it will be possible to find NAI Social Welfare Functions verifying such ethical rules. Though weak at first sight -it merely implies the absence of absolute irrationality, no matter how close the opinions are to absolute rationality- this positive result represents a justification for a posterior research looking for the best or a satisfying NAI aggregation rule, perhaps by adding to the model an additional criterion, as some consensus measure (see [6]).

#### 4.- TWO EXAMPLES AND A FINAL COMMENT

Two well known NAI Social Welfare Functions which appear very often in the literature, as democratic aggregation rules assuring that  $A(\sigma) > 0$  always holds for the social fuzzy preference  $\sigma$  -assuming that  $A(p^i) > 0 \forall i \in D$  also hold- are the following:

\* *Arithmetical Weighted Mean* (see [7] for an alternative axiomatic justification):

$$\sigma(x, y) = \sum_{i=1}^n p^i(x, y) / n \quad \forall x, y \in X$$

\* *Maximum* (it is a conservative rule that can be in some cases poorly decisive, in a clear parallelism with classical Social Decision Functions):

$$\sigma(x, y) = \max_{i=1, n} p^i(x, y) \quad \forall x, y \in X$$

An axiomatic study of general social fuzzy preferences based on means can be found in Ovchinnikov [8], also presenting them as a way of avoiding Arrow's paradox, but assuming that individual preferences are of crisp nature. In any case, both results are showing how useful can be Fuzzy Set Theory in order to get a better knowledge of this particular complex decision making problem.

## ACKNOWLEDGEMENT

This research has been supported by Dirección General de Investigación Científica y Técnica (National Grants number PB88-0137 and BE91-225).

## REFERENCES

- [1] K.J. Arrow (1951, 1964), *Social Choice and Individual Values*. Wiley, New York.
- [2] D. Black (1958), *The Theory of Committees and Elections*, Cambridge University Press, Cambridge.
- [3] J. Montero (1990), *Single-peakedness in weighted aggregation of fuzzy opinions in a fuzzy group*; in: J. Kacprzyk & M. Fedrizzi (Eds.), *Multiperson Decision Making using Fuzzy Sets and Possibility Theory*, Dordrecht, pp. 163-171.
- [4] A.K. Sen (1970), *Collective Choice and Social Welfare*, Holden-Day, San Francisco.
- [5] J. Montero (1987), *Arrow's theorem under fuzzy rationality*. Behavioral Science 267-273.
- [6] M. Fedrizzi (1990), *On a consensus measure in a group MCDM problem*; in: J. Kacprzyk & M. Fedrizzi (Eds.), *Multiperson Decision Making using Fuzzy Sets and Possibility Theory*, Kluwer, Dordrecht, pp. 231-241.
- [7] J. Montero (1988), *An Axiomatic Approach to Fuzzy Multicriteria Analysis*; in: M.M. Gupta and T. Yamakawa (Eds.), *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, North-Holland, Amsterdam, pp. 259-269.
- [8] S. Ovchinnikov (1990), *Means and social welfare function in fuzzy binary relation spaces*; in: J. Kacprzyk & M. Fedrizzi (Eds.), *Multiperson Decision Making using Fuzzy Sets and Possibility Theory*, Kluwer, Dordrecht, pp 143-154.