Robust inference for one-shot device testing data under exponential lifetime model with multiple stresses

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Abstract

Balakrishann et al.[1] introduced robust density-based estimators in the context of one-shot devices with exponential lifetimes under a single stress factor. However, it is usual to have several stress factors in industrial experiments involving one-shot devices. In this paper, the weighted minimum density power divergence estimators are developed as a natural extension of the classical maximum likelihood estimators for one-shot device testing data under exponential lifetime model with multiple stresses. Based on these estimators, Wald-type test statistics are also developed. Through a simulation study, it is shown that some WMDPDEs have a better performance than the MLE in relation to robustness. Two examples with multiple stresses show the usefulness of the model and, in particular, of the proposed estimators, both in Engineering and Medicine.

Keywords and phrases: Exponential distribution, Minimum density power divergence estimator, Multiple stresses, One-shot devices, Robustness, Wald-type tests.

Short title: Robust one-shot device testing under exponential lifetime with multiple stresses

1 Introduction

One-shot devices, which are products that will get destroyed immediately after use, have been widely studied in the recent years, mainly motivated by the work of Fan et al.[2] In this paper, an extensive study of the reliability of electro-explosive devices (which are indeed one-shot devices) is done, taking into account that after a succesfull detonation, the device can not be used any further, and that in case of failure, the exact time when it failed is unknown. Therefore, the data from such devices consists of both left censored (failure) and right censored (success) observations at pre-fixed inspection time points, and the observations are binary data (failure or not) instead of real failure time data. Other examples of one-shot devices are automobile air-bags, missiles (Olwell and Sorell[3]), and fire extinguishers (Newby[4]). This kind of data is also known as current status data in the survival analysis literature.

In the paper of Fan et al.,[2] a Bayesian approach was presented for the inference on the failure rate and reliability of devices. They found the normal prior to be the best one when the failure observations are rare, which is indeed the case when the devices are highly reliable. Subsequently, Balakrishnan and Ling[5] developed an EM algorithm for the determination of the MLEs of model parameters under exponential lifetime distribution for devices with a single stress factor. Balakrishnan and Ling[6] further extended their work to a model with multiple stress factors. Balakrishnan and Ling[7] developed more general inferential results for devices with Weibull lifetimes under non-constant shape parameters, while Balakrishnan and Ling[8] provided inferential work for devices with gamma lifetimes. Another interesting approach in the Bayesian framework has been given by Fan and Chang.[9]

Most of the above results are based on maximum likelihood estimators (MLE) and it is well known that they are efficient, but also non-robust. Therefore, space testing procedures based on MLEs face serious robustness problems. To avoid that problem of robustness without a strong loss of efficiency, Balakrishann et al.[1] developed the minimum density power divergence estimators (MDPDE) in the context of one-shot devices with exponential lifetimes under a single stress factor. However, it is common in industrial experiments to have several stress factors while analyzing one-shot devices, overall in accelerated life testing (ALT). An ALT plan implies the devices to be tested under high stress levels, in order to shorten their lifetimes, which are usually quite large under normal conditions because of the advances in technology. An experiment with scarce failures will result in insignificant inferential results. In this sense, stress factors such as air pressure, temperature or humidity can be easily controlled in an experimental laboratory setup. Inference and design optimization for the step-stress ALT have attracted great attention in the reliability and engineering literature. King[10] has focused the robustness of asymptotic optimal test planning for accelerated life testing to small sample setting while Han[11] has considered optimal design of ALT under progressive type I censoring. Some other pertinent references can be seen in these papers.

In this paper, we extend the results of Balakrishann et al.[1] to the more practical and realistic situation wherein there are multiple stress factors. In contrast with the initial approach, we will not necessarily consider that all the testing conditions have the same sample weight. In Section 2, we present the formulation of the problem and then introduce the MLEs. The weighted minimum density power divergence estimators (WMDPDE), a natural extension of the MLEs, is introduced in Section 3 and its asymptotic distribution and influence function are then obtained. Based on WMDPDE, in Section 4, we introduce Wald-type tests for testing hypotheses in relation to the model under consideration. Section 5 presents the results of a simulation study and in Section 6 two illustrative examples are presented. Finally, some concluding remarks are made in Section 7.

2 Model formulation and maximum likelihood estimator

Suppose the devices are stratified into I testing conditions and that in the i-th testing condition K_i units are tested with J types of stress factors being maintained at certain levels, and the working conditions of those units are then observed at pre-specified inspection times t_i, for $i = 1, \dots, I$. The data thus observed can be summarized as follows.

For $i = 1, \dots, I$ and $j = 1, \dots, J$, suppose the lifetimes of devices in the i-th testing condition are exponentially distributed with failure rate λ_i, and are subjected to various stress levels w_{ij}. Let us denote $w_{i0} = 1$, $\boldsymbol{w}_i = (w_{i0}, ..., w_{iJ})^T$ and let $\boldsymbol{a} = (a_0, a_1, \dots a_J)^T$ be the model parameter vector, that relates λ_i with stress factors $\boldsymbol{w}_i$ by

$$\lambda_i = \exp\left(\boldsymbol{w}_i^T \boldsymbol{a}\right).$$

Then, the probability density function of the lifetimes is given by

$$f(t_i; \boldsymbol{a}, \boldsymbol{w}_i) = \lambda_i \exp(-t_i \lambda_i) = \exp\left(\boldsymbol{w}_i^T \boldsymbol{a}\right) \exp\left(-t_i \exp\left(\boldsymbol{w}_i^T \boldsymbol{a}\right)\right) \tag{1}$$

and the corresponding distribution function is

$$F(t_i; \boldsymbol{a}, \boldsymbol{w}_i) = 1 - \exp\left(-t_i \exp\left(\boldsymbol{w}_i^T \boldsymbol{a}\right)\right). \tag{2}$$

Then, the reliability at time t_i and the mean lifetime under normal operating conditions $\boldsymbol{w}_i$ are given by

$$R(t_i; \boldsymbol{a}, \boldsymbol{w}_i) = 1 - F(t_i; \boldsymbol{a}, \boldsymbol{w}_i) = \exp\left(-t_i \exp\left(\boldsymbol{w}_i^T \boldsymbol{a}\right)\right) \tag{3}$$

and

$$E[T_i] = \frac{1}{\lambda_i} = \exp\left(-\boldsymbol{w}_i^T \boldsymbol{a}\right).$$

More details about this model can be seen in the work of Balakrishnan and Ling.[5,6]

The likelihood function based on the observed data, presented in Table 1, is given by

$$\mathcal{L}(\boldsymbol{a}; n_1, \dots, n_I) \propto \prod_{i=1}^{I} F^{n_i}(t_i; \boldsymbol{a}, \boldsymbol{w}_i) R^{K_i - n_i}(t_i; \boldsymbol{a}, \boldsymbol{w}_i). \tag{4}$$

We now introduce the following probability vectors

$$\widehat{\boldsymbol{p}}_i = (\widehat{p}_{i1}, \widehat{p}_{i2})^T, \quad i = 1, \dots, I, \tag{5}$$

$$\boldsymbol{\pi}_i(\boldsymbol{a}) = (\pi_{i1}(\boldsymbol{a}), \pi_{i2}(\boldsymbol{a}))^T, \quad i = 1, \dots, I, \tag{6}$$

with $\widehat{p}_{i1} = \frac{n_i}{K_i}$, $\widehat{p}_{i2} = 1 - \frac{n_i}{K_i}$, $\pi_{i1}(\boldsymbol{a}) = F(t_i; \boldsymbol{a}, \boldsymbol{w}_i)$ and $\pi_{i2}(\boldsymbol{a}) = R(t_i; \boldsymbol{a}, \boldsymbol{w}_i)$.

The Kullback-Leibler divergence measure, (see, for instance, Pardo[12]), between $\widehat{\boldsymbol{p}}_i$ and $\boldsymbol{\pi}_i(\boldsymbol{a})$ is given by

$$\begin{aligned} d_{KL}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) &= \widehat{p}_{i1} \log\left(\frac{\widehat{p}_{i1}}{\pi_{i1}(\boldsymbol{a})}\right) + \widehat{p}_{i2} \log\left(\frac{\widehat{p}_{i2}}{\pi_{i2}(\boldsymbol{a})}\right) \\ &= \frac{n_i}{K_i} \log\left(\frac{\frac{n_i}{K_i}}{F(t_i; \boldsymbol{a}, \boldsymbol{w}_i)}\right) + \left(1 - \frac{n_i}{K_i}\right) \log\left(\frac{1 - \frac{n_i}{K_i}}{R(t_i; \boldsymbol{a}, \boldsymbol{w}_i)}\right), \end{aligned} \tag{7}$$

and the weighted Kullback-Leibler divergence measure is given by

$$\sum_{i=1}^{I} \frac{K_i}{K} d_{KL}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = \frac{1}{K} \sum_{i=1}^{I} \left(n_i \log\left(\frac{\frac{n_i}{K_i}}{F(t_i; \boldsymbol{a}, \boldsymbol{w}_i)}\right) + (K_i - n_i) \log\left(\frac{1 - \frac{n_i}{K_i}}{R(t_i; \boldsymbol{a}, \boldsymbol{w}_i)}\right) \right),$$

where $K = K_1 + \cdots + K_I$, is the total number of devices under test.

Theorem 1 *The likelihood function $\mathcal{L}(\boldsymbol{a}; n_1, \ldots, n_I)$, given in (4), is related to the weighted Kullback-Leibler divergence measure through*

$$\sum_{i=1}^{I} \frac{K_i}{K} d_{KL}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = c - \frac{1}{K} \log \mathcal{L}(\boldsymbol{a}; n_1, \ldots, n_I),$$

with c being a constant not dependent on $\boldsymbol{a}$.

Based on Theorem 1, we have the following definition for the MLE of $\boldsymbol{a}$.

Definition 2 *The MLE of $\boldsymbol{a}$, $\widehat{\boldsymbol{a}}$, can be defined as*

$$\widehat{\boldsymbol{a}} = \arg\min_{\boldsymbol{a}} \sum_{i=1}^{I} \frac{K_i}{K} d_{KL}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})). \tag{8}$$

Based on (8), in what follows, we shall consider, for the one-shot device model considered in (4), the WMDPDEs.

3 The Weighted Minimum Density Power Divergence Estimators

Given the probability vectors $\widehat{\boldsymbol{p}}_i$ and $\boldsymbol{\pi}_i(\boldsymbol{a})$ defined in (5) and (6), respectively, the density power divergence (DPD) between the two probability vectors is given, as the function of a single tuning parameter $\beta \geq 0$, by

$$\begin{aligned} d_\beta(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = {} & \left(\pi_{i1}^{\beta+1}(\boldsymbol{a}) + \pi_{i2}^{\beta+1}(\boldsymbol{a})\right) - \frac{\beta+1}{\beta}\left(\widehat{p}_{i1}\pi_{i1}^{\beta}(\boldsymbol{a}) + \widehat{p}_{i2}\pi_{i2}^{\beta}(\boldsymbol{a})\right) \\ & + \frac{1}{\beta}\left(\widehat{p}_{i1}^{\beta+1} + \widehat{p}_{i2}^{\beta+1}\right), \quad \text{if } \beta > 0, \end{aligned} \tag{9}$$

and $d_{\beta=0}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = \lim_{\beta \to 0^+} d_\beta(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = d_{KL}(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a}))$, for $\beta = 0$.

We observe that in (9), the term $\frac{1}{\beta}\left(\widehat{p}_{i1}^{\beta+1} + \widehat{p}_{i2}^{\beta+1}\right)$ has no role in the minimization with respect to $\boldsymbol{a}$. Therefore, we can consider the equivalent measure

$$d_\beta^*(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})) = \left(\pi_{i1}^{\beta+1}(\boldsymbol{a}) + \pi_{i2}^{\beta+1}(\boldsymbol{a})\right) - \frac{\beta+1}{\beta}\left(\widehat{p}_{i1}\pi_{i1}^{\beta}(\boldsymbol{a}) + \widehat{p}_{i2}\pi_{i2}^{\beta}(\boldsymbol{a})\right). \tag{10}$$

Definition 3 *Based on (8) and (10), we can define the WMDPDE for $\boldsymbol{a}$ as*

$$\widehat{\boldsymbol{a}}_\beta = \arg\min_{\boldsymbol{a}} \sum_{i=1}^{I} \frac{K_i}{K} d_\beta^*(\widehat{\boldsymbol{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{a})), \quad \textit{for } \beta > 0,$$

and for $\beta = 0$, we consider the MLE, $\widehat{\boldsymbol{a}}$, defined in (8).

Theorem 4 *For $\beta \geq 0$, the estimating equations are given by*

$$\sum_{i=1}^{I} \left(K_i F(t_i; \boldsymbol{a}, \boldsymbol{w}_i) - n_i\right) f(t_i; \boldsymbol{a}, \boldsymbol{w}_i) t_i \boldsymbol{w}_i \left(F^{\beta-1}(t_i; \boldsymbol{a}, \boldsymbol{w}_i) + R^{\beta-1}(t_i; \boldsymbol{a}, \boldsymbol{w}_i)\right) = \boldsymbol{0}_{J+1},$$

where $f(t_i; \boldsymbol{a}, \boldsymbol{w}_i)$, $F(t_i; \boldsymbol{a}, \boldsymbol{w}_i)$ and $R(t_i; \boldsymbol{a}, \boldsymbol{w}_i)$ are given, respectively, by (1), (2) and (3), and $\boldsymbol{0}_{J+1}$ is the null column vector of dimension $J+1$.

In the following result, the asymptotic distribution of the WMDPDE of $\boldsymbol{a}$, $\widehat{\boldsymbol{a}}_\beta$, for one-shot device testing data, with multiple stresses is presented.

Theorem 5 *Let $\boldsymbol{a}^*$ be the true value of the parameter $\boldsymbol{a}$, the asymptotic distribution of the WMDPDE, $\widehat{\boldsymbol{a}}_\beta$, is given by*

$$\sqrt{K}\left(\widehat{\boldsymbol{a}}_\beta - \boldsymbol{a}^*\right) \underset{K\to\infty}{\xrightarrow{\mathcal{L}}} \mathcal{N}\left(\mathbf{0}_{J+1}, \boldsymbol{J}_\beta^{-1}(\boldsymbol{a}^*)\boldsymbol{K}_\beta(\boldsymbol{a}^*)\boldsymbol{J}_\beta^{-1}(\boldsymbol{a}^*)\right),$$

where

$$\boldsymbol{J}_\beta(\boldsymbol{a}) = \sum_{i}^{I}\frac{K_i}{K}\boldsymbol{w}_i\boldsymbol{w}_i^T f^2(t_i;\boldsymbol{a},\boldsymbol{w}_i)t_i^2\left(F^{\beta-1}(t_i;\boldsymbol{a},\boldsymbol{w}_i) + R^{\beta-1}(t_i;\boldsymbol{a},\boldsymbol{w}_i)\right), \tag{11}$$

$$\boldsymbol{K}_\beta(\boldsymbol{a}) = \sum_{i=1}^{I}\frac{K_i}{K}\boldsymbol{w}_i\boldsymbol{w}_i^T f^2(t_i;\boldsymbol{a},\boldsymbol{w}_i)t_i^2 F(t_i;\boldsymbol{a},\boldsymbol{w}_i)R(t_i;\boldsymbol{a},\boldsymbol{w}_i)\left(F^{\beta-1}(t_i;\boldsymbol{a},\boldsymbol{w}_i) + R^{\beta-1}(t_i;\boldsymbol{a},\boldsymbol{w}_i)\right)^2. \tag{12}$$

3.1 Influence Function of the WMDPDE

Let us denote by G_i the true distribution function of the response variable Y_i, for the $i-$th group of K_i observations, having mass function g_i, and by $F_{i,\boldsymbol{a}}$ the distribution function associated with the model, with probability mass function $f_{i,\boldsymbol{a}}$. In vector notation, we let $\boldsymbol{G} = (G_1 \otimes \mathbf{1}_{K_1}^T, \ldots, G_I \otimes \mathbf{1}_{K_I}^T)$ and $\boldsymbol{F}_{\boldsymbol{a}} = (F_{1,\boldsymbol{a}} \otimes \mathbf{1}_{K_1}^T, \ldots, F_{I,\boldsymbol{a}} \otimes \mathbf{1}_{K_I}^T)$. We first need to define the statistical functional $\boldsymbol{U}_\beta(\boldsymbol{G})$ corresponding to the WMDPDE as the minimizer of the weighted sum of DPDs between the true and model densities.

This is defined as the minimizer of

$$\sum_{i=1}^{I}\frac{K_i}{K}\left[\pi_{i1}^{\beta+1}(\boldsymbol{a}) + \pi_{i2}^{\beta+1}(\boldsymbol{a}) - \frac{\beta+1}{\beta}\left(\frac{N_{i1}}{K_i}\pi_{i1}^{\beta}(\boldsymbol{a}) + \frac{N_{i2}}{K_i}\pi_{i2}^{\beta}(\boldsymbol{a})\right)\right], \tag{13}$$

whenever it exists. When the assumption of the model holds with true parameter $\boldsymbol{a}^*$, we have $g_i = f_{i,\boldsymbol{a}^*}$ and (13) is minimized at $\boldsymbol{a} = \boldsymbol{a}^*$, implying the Fisher consistency of the WMDPDE functional $\boldsymbol{U}_\beta(\boldsymbol{G})$ in our model.

One can derive the influence function (IF) of the WMDPDE at $\boldsymbol{F}_{\boldsymbol{a}^*}$ with respect to the k-th element of the i_0-th group of observations, as

$$\begin{aligned}\mathcal{IF}(\delta^{(i_0,k)}, \boldsymbol{U}_\beta, \boldsymbol{F}_{\boldsymbol{a}^*}) &= \left.\frac{\partial \boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*,\varepsilon}^{(i_0,k)})}{\partial\varepsilon}\right|_{\varepsilon=0} \\ &= \boldsymbol{J}_\beta^{-1}(\boldsymbol{a}^*)\frac{K_{i_0}}{K}(F(t_{i_0};\boldsymbol{a}^*,\boldsymbol{\omega}_{i_0}) - \Lambda_{\delta^{(i_0,k)}}) \\ &\quad\times\left[F^{\beta-1}(t_{i_0};\boldsymbol{a}^*,\boldsymbol{\omega}_{i_0}) + R^{\beta-1}(t_{i_0};\boldsymbol{a}^*,\boldsymbol{\omega}_{i_0})\right] f(t_{i_0};\boldsymbol{a}^*,\boldsymbol{\omega}_{i_0})t_i\boldsymbol{\omega}_{i_0},\end{aligned} \tag{14}$$

where the contaminated probability vector $\boldsymbol{F}_{\boldsymbol{a}^*,\varepsilon}^{(i_0,k)} = (1-\varepsilon)\boldsymbol{F}_{\boldsymbol{a}^*} + \varepsilon\Lambda_\delta^{(i_0,k)}$ and $\Lambda_\delta^{(i_0,k)}$ is the degenerate probability at the outlier $\boldsymbol{\delta}$ in the position (i_0,k) in lexicographical order. The IF with respect to all the observations is given by

$$
\begin{aligned}
\mathcal{IF}(\boldsymbol{\delta}, \boldsymbol{U}_\beta, \boldsymbol{F}_{\boldsymbol{a}^*}) &= \left.\frac{\partial \boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*,\varepsilon})}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad (15)\\
&= \boldsymbol{J}_\beta^{-1}(\boldsymbol{a}^*) \sum_{i=1}^{I} \frac{K_i}{K}(F(t_i;\boldsymbol{a}^*,\boldsymbol{\omega}_i) - \Lambda_{\delta^{(i)}})\\
&\quad \times \left[F^{\beta-1}(t_i;\boldsymbol{a}^*,\boldsymbol{\omega}_i) + R^{\beta-1}(t_i;\boldsymbol{a}^*,\boldsymbol{\omega}_i)\right] f(t_i;\boldsymbol{a}^*,\boldsymbol{\omega}_i) t_i \boldsymbol{\omega}_i,
\end{aligned}
$$

where $\boldsymbol{F}_{\boldsymbol{a}^*,\varepsilon} = (1-\varepsilon)\boldsymbol{F}_{\boldsymbol{a}^*} + \varepsilon \sum_{i=1}^{I}\sum_{k=1}^{K_i} \Lambda_{\delta^{(i,k)}}$ and $\delta^{(i)} = \sum_{k=1}^{K_i} \delta^{(i,k)}$.

4 Wald-type test statistics

4.1 Definition and asymptotic results

Let us consider the function $\boldsymbol{m} : \mathbb{R}^{J+1} \longrightarrow \mathbb{R}^r$, where $r \leq J+1$. Then, $\boldsymbol{m}(\boldsymbol{a}) = \boldsymbol{0}_r$ represents a composite null hypothesis. We assume that the $(J+1)\times r$ matrix

$$
\boldsymbol{M}(\boldsymbol{a}) = \frac{\partial \boldsymbol{m}^T(\boldsymbol{a})}{\partial \boldsymbol{a}}
$$

exists and is continuous in "$\boldsymbol{a}$" and with rank $\boldsymbol{M}(\boldsymbol{a}) = r$. For testing

$$
H_0 : \boldsymbol{a} \in \boldsymbol{\Theta}_0 \text{ against } H_1 : \boldsymbol{a} \notin \boldsymbol{\Theta}_0, \qquad (16)
$$

where $\boldsymbol{\Theta}_0 = \{\boldsymbol{a} \in \boldsymbol{\Theta}_0 : \boldsymbol{m}(\boldsymbol{a}) = \boldsymbol{0}_r\}$, we can consider the following Wald-type test statistics

$$
W_K(\widehat{\boldsymbol{a}}_\beta) = K\boldsymbol{m}^T(\widehat{\boldsymbol{a}}_\beta)\left(\boldsymbol{M}^T(\widehat{\boldsymbol{a}}_\beta)\boldsymbol{\Sigma}(\widehat{\boldsymbol{a}}_\beta)\boldsymbol{M}(\widehat{\boldsymbol{a}}_\beta)\right)^{-1}\boldsymbol{m}(\widehat{\boldsymbol{a}}_\beta),
$$

where $\boldsymbol{\Sigma}_\beta(\widehat{\boldsymbol{a}}_\beta) = \boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{a}}_\beta)\boldsymbol{K}_\beta(\widehat{\boldsymbol{a}}_\beta)\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{a}}_\beta)$ and $\boldsymbol{J}_\beta^{-1}(\boldsymbol{a})$ and $\boldsymbol{K}_\beta(\boldsymbol{a})$ are as in (11) and (12), respectively. Wald-type test statistics based on WMDPDEs have been considered previously by Basu et al.[13] and Ghosh et al.[14]

In the following theorem, we present the asymptotic distribution of $W_K(\widehat{\boldsymbol{a}}_\beta)$.

Theorem 6 *We have*

$$
W_K(\widehat{\boldsymbol{a}}_\beta) \xrightarrow[K\to\infty]{\mathcal{L}} \chi^2_r.
$$

Based on Theorem 6, we will reject the null hypothesis in (16) if

$$
W_K(\widehat{\boldsymbol{a}}_\beta) > \chi^2_{r,\alpha}, \qquad (17)
$$

where $\chi^2_{r,\alpha}$ is the upper percentage point of order α of χ^2_r distribution.

4.2 Influence Function of the Wald-type tests

The functional associated with the Wald-type statistics, $W_K(\widehat{\boldsymbol{a}}_\beta)$, evaluated at $\boldsymbol{F}_{\boldsymbol{a}^*}$ is given by, ignoring the multiplier K

$$
\begin{aligned}
T_\beta(\boldsymbol{F}_{\boldsymbol{a}^*}) &= W_K(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*}))\\
&= \boldsymbol{m}(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*}))^T\left\{\boldsymbol{M}^T(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*})\boldsymbol{\Sigma}(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*}))\boldsymbol{M}(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*})))\right\}^{-1}\boldsymbol{m}(\boldsymbol{U}_\beta(\boldsymbol{F}_{\boldsymbol{a}^*})),
\end{aligned}
$$

with $\boldsymbol{\Sigma}\left(\boldsymbol{U}_\beta\left(\boldsymbol{F}_{\boldsymbol{a}^*}\right)\right)=\boldsymbol{J}_\beta^{-1}\left(\boldsymbol{U}_\beta\left(\boldsymbol{F}_{\boldsymbol{a}^*}\right)\right)\boldsymbol{K}_\beta\left(\boldsymbol{U}_\beta\left(\boldsymbol{F}_{\boldsymbol{a}^*}\right)\right)\boldsymbol{J}_\beta^{-1}\left(\boldsymbol{U}_\beta\left(\boldsymbol{F}_{\boldsymbol{a}^*}\right)\right).$ Therefore, the IF of the functional associated with the Wald-type test statistics for testing the composite null hypothesis in (16) with respect to the k-th element of the i-th group of observations, is given by

$$\mathcal{IF}(\delta^{(i_0,k)},T_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})=\left.\frac{\partial T_\beta(\boldsymbol{F}^{(i,k)}_{\boldsymbol{a}^*,\varepsilon})}{\partial\varepsilon}\right|_{\varepsilon=0}=0.$$

Similary, for all the indices, we have

$$\mathcal{IF}(\boldsymbol{\delta},T_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})=\left.\frac{\partial T_\beta(\boldsymbol{F}_{\boldsymbol{a}^*,\varepsilon})}{\partial\varepsilon}\right|_{\varepsilon=0}=0.$$

It, therefore, becomes necessary to consider the second-order influence function, as presented in the following result.

Theorem 7 *The second-order IF of the functional associated with the Wald-type test statistics, with respect to the k-th element of the i_0-th group of observations, is given by*

$$\begin{aligned}\mathcal{IF}_2(\boldsymbol{\delta}^{(i_0,k)},T_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})&=\left.\frac{\partial^2 T_\beta(\boldsymbol{F}^{(i_0,k)}_{\boldsymbol{a}^*,\varepsilon})}{\partial\varepsilon^2}\right|_{\varepsilon=0^+}\\&=2\mathcal{IF}^T(\boldsymbol{\delta}^{(i_0,k)},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})\boldsymbol{M}\left(\boldsymbol{a}^*\right)\left(\boldsymbol{M}^T\left(\boldsymbol{a}^*\right)\boldsymbol{\Sigma}\left(\boldsymbol{a}^*\right)\boldsymbol{M}\left(\boldsymbol{a}^*\right)\right)^{-1}\boldsymbol{M}^T\left(\boldsymbol{a}^*\right)\mathcal{IF}(\boldsymbol{\delta}^{(i_0,k)},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*}),\end{aligned}$$

with $\mathcal{IF}(\boldsymbol{\delta}^{(i_0,k)},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})$ being as given in (14). Similarly, in all the indices

$$\begin{aligned}\mathcal{IF}_2(\boldsymbol{\delta},T_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})&=\left.\frac{\partial^2 \boldsymbol{T}_\beta(\boldsymbol{F}^{(i_0,k)}_{\boldsymbol{a}^*,\varepsilon})}{\partial\varepsilon^2}\right|_{\varepsilon=0^+}\\&=2\mathcal{IF}^T(\boldsymbol{\delta},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})\boldsymbol{M}\left(\boldsymbol{a}^*\right)\left(\boldsymbol{M}^T\left(\boldsymbol{a}^*\right)\boldsymbol{\Sigma}\left(\boldsymbol{a}^*\right)\boldsymbol{M}\left(\boldsymbol{a}^*\right)\right)^{-1}\boldsymbol{M}^T\left(\boldsymbol{a}^*\right)\mathcal{IF}(\boldsymbol{\delta},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*}),\end{aligned}$$

with $\mathcal{IF}(\boldsymbol{\delta},\boldsymbol{U}_\beta,\boldsymbol{F}_{\boldsymbol{a}^*})$ being as given in (15).

5 Monte Carlo Simulation Study

In this section, Monte Carlo simulations of size 2,000 were carried out to examine the behavior of the WMDPDEs of the model parameters discussed in Section 3, as well as the Wald-type tests, based on WMDPDEs, detailed in Section 4.

Based on the simulation experiment proposed by Balakrishnan and Ling,[6] we considered the devices to have exponential lifetimes subjected to two types of stress factors at two different conditions each, the first one at levels 55 and 70 and the second one at levels 85 and 100, and tested at three different inspection times $IT=\{2,5,8\}$. Thus, we can consider a table, such as in Table 1, with $I=12$ rows corresponding to each of the 12 testing conditions. To evaluate the robustness of the WMDPDEs, we have studied the behavior of this model under the consideration of an outlying cell (for example, the last one) in this table. As with the concept of inflated models in distribution theory, this cell will be considered an outlier as, changing the value of the generating parameter vector, the observed number of successes observed in the cell will be larger than the expected one under the corresponding model.

5.1 The WMDPDEs

We carried out a simulation study to compare the behavior of some WMDPDEs with respect to the MLEs of the parameters in the one-shot device model under the exponential distribution with multiple stresses. In order to evaluate the performance of the proposed WMDPDEs, as well as the MLEs, we consider the root mean square errors (RMSEs). The model has been examined under $(a_0, a_1, a_2) = (-6.5, 0.03, 0.03)$, different samples sizes $K \in [40, 200]$, and different degrees of contamination. The estimates have been computed with values of the tuning parameter $\beta \in \{0, 0.2, 0.4, 0.6, 0.8\}$.

In the top of Figure 2, efficiency of WMDPDEs is measured under different samples sizes K with pure data (left) and contaminated data (right) where the observations in the $i = 12$ testing condition have been generated under $(a_0, a_1, \tilde{a}_2) = (-6.5, 0.03, 0.025)$. Same experiment is carried outfor $a_0 = -6$ (Figure 4) and by contaminating the last two testing conditions (top left of Figure 5). The efficiency is then measured for the last-cell-contaminated data, generated under $(a_0, \tilde{a}_1, \tilde{a}_2) = (-6.5, 0.025, 0.025)$ (top right of Figure 5). In the case of pure data, the MLE (at $\beta = 0$) presents the most efficient behavior having the least RMSE for each sample size, while WMDPDEs with larger β have slightly larger RMSEs. For the contaminated data, the behavior of the WMDPDEs is almost the opposite; the best behavior (least RMSE) is obtained for larger values of β. In both cases, as expected, the RMSEs decrease as the sample size increases.

The efficiency is also studied for different degrees of contamination of the parameters a_1 (left) and a_2 (right), as displayed in the top of Figure 3. Here, $K = 100$ and the degree of contamination is given by $4(1 - \frac{\tilde{a}_j}{a_j}) \in [0, 1]$ with $j \in \{1, 2\}$. In both cases, we can see how the MLEs and the WMDPDEs with small values of tuning parameter β present the smallest RMSEs for weak outliers, i.e., when the degree of contamination is close to 0 ($\tilde{a}_j$ is close to a_j). On the other hand, large values of tuning parameter β result in the WMDPDEs having the smallest RMSEs, for medium and strong outliers, i.e., when the degree of contamination away from 0 ($\tilde{a}_j$ is not close to a_j).

In view of the results achieved, we note that the MLE is very efficient when there are no outliers, but highly non-robust when outliers are present in the data. On the other hand, the WMDPDEs with moderate values of the tuning parameter β exhibit a little loss of efficiency when there are no outliers, but at the same time a considerable improvement in robustness is achieved when there are outliers in the data. Actually, these values of the tuning parameter β are the most appropriate ones for the estimators of the parameters in the model following the robustness theory: To improve in a considerable way the robustness of the estimators, a small amount of efficiency needs to be compromised.

5.2 The Wald-type tests based on WMDPDEs

Let us now empirically evaluate the robustness of the WMDPDE based Wald-type tests for the model. The simulation is performed with the same model as in Section 5.1, where $(a_0, a_1, a_2) = (-6.5, 0.03, 0.03)$. We first study the observed level (measured as the proportion of test statistics exceeding the corresponding chi-square critical value) of the test under the true null hypothesis $H_0 : a_2 = 0.03$ against the alternative $H_1 : a_2 \neq 0.03$. In the middle of Figure 2, these levels are plotted for different values of the samples sizes, for pure data (left) and for contaminated data ($\tilde{a}_2 = 0.025$, right). Same experiment is carried out by contaminating the last two testing conditions (middle left of Figure 5). The empirical levels are then measured for the last-cell-contaminated data, generated under $(a_0, \tilde{a}_1, \tilde{a}_2) = (-6.5, 0.025, 0.025)$ (middle right of Figure 5). In the middle of Figure 3, the degree of contamination for both a_1 and a_2 is changed with a fixed value of $K = 100$. Notice that when the pure data are considered, all the observed levels are quite close to the nominal level of 0.05. In the case of contaminated data, the level of the classical Wald test (at $\beta = 0$) as well as the proposed Wald-type tests with small β break down, while the WMDPDE based Wald-type tests for moderate and large values of β provide greater stability in their levels.

To investigate the power robustness of these tests (obtained in a similar manner), we change the true data generating parameter value to be $a_2 = 0.035$ and the resulting empirical powers are plotted in the bottom of Figures 2 and 3 and in the bottom left of Figure 5) (when the last two cells are contaminated). The empirical powers are then measured for the last-cell-contaminated data, generated under $(a_0, \tilde{a}_1, \tilde{a}_2) = (-6.5, 0.035, 0.025)$ (bottom right of Figure 5). Again, the classical Wald test (at $\beta = 0$) presents the best behavior under the pure data, while the Wald-type tests with larger $\beta > 0$ lead to better stability in the case of contaminated samples. Same tests are also evaluated with a higher/lower value of reliability ($a_0 = -6$) obtaining the same conclusions as detailed above (see Figure 4).

These results show the poor behavior in terms of robustness of the Wald-type tests based on the MLEs of the parameters of one-shot devices under the exponential model with multiple stresses. Additionally, the robustness properties of the Wald-type test statistics based on the WMDPDEs with large values of the tuning parameter β are often better as they maintain both level and power in a stable manner.

6 Illustrative Examples

In this Section, two numerical examples are presented to illustrate the model and the estimators developed in the preceding sections.

6.1 An application to Medicine: Mice Tumor Toxicological data

As mentioned earlier, current status data with covariates, which generally occur in the area of survival analysis, can be seen as one-shot device testing data with stress factors and we therefore apply here the methods developed in the preceding sections to a real data from a study in toxicology. These data, originally reported by Kodell and Nelson[15] (Table 1) and recently analyzed by Balakrishnan and Ling,[7,8,16] are taken from the National Center for Toxicological Research and consisted of 1816 mice, of which 553 had tumors, involving the strain of offspring (F1 or F2), gender (females or males), and concentration of benzidine dihydrochloride (60 ppm, 120 ppm, 200 ppm or 400 ppm) as the stress factors. The F1 strain consisted of offspring from matings of BALB/c males to C57BL/6 females, while the F2 strain consisted of offspring from non-brother-sister matings of the Fl progeny. For each testing condition, the numbers of mice tested and the numbers of mice that developed tumors were all recorded. Note that we consider mice with tumors as those that died of tumors, sacrificed with tumors, and died of competing risks with liver tumors.

Let a_1, a_2 and a_3 denote the parameters corresponding to the covariates of strain of offspring, gender, and square root of concentration of the chemical of benzidine dihydrochloride in the exponential distribution given in (2). The WMDPDEs with tuning parameter $\beta \in \{0, 0.2, 0.4, 0.6, 0.8\}$ were all computed and are presented in Table 2. Negative values for $\widehat{a}_1$ and $\widehat{a}_2$ indicate a greater resistance of F2 strain and male mice. As expected, a greater concentration of benzidine dihydrochloride is seen to decrease the expected lifetime.

6.2 An application to Engineering: Balakrishnan and Ling[6]

These data (Balakrishnan and Ling[6]), presented in Table 3, consist of 120 one-shot devices that were divided into four accelerated conditions with higher-than-normal temperature and electric current, and inspected at three different times. By subjecting the devices to adverse conditions, we shorten the lifetimes, observing more failures in a clear example of an accelerated life test design. This numerical example also served as a basis for the Monte Carlo study carried out earlier in Section 5.

The estimates of the model parameters are presented in Table 4, for different values of the tuning parameter $\boldsymbol{\beta}$. Reliability at different inspections times and normal testing conditions $\boldsymbol{w} = (25, 35)$, as well as the mean lifetimes, are also presented. As expected, the reliability of the devices decrease when the inspection time increases. Figure 1 displays the estimated reliabilities at a pre-fixed inspection time, $t = 30$, for different values of temperature and electric current, and two different tuning parameters: $\beta = 0$ (MLE) and a high-moderate value $\beta = 0.6$. Let us denote $\widehat{R}_0^{ij}$ and $\widehat{R}_{0.6}^{ij}$ for the estimated reliability at temperature level i and electric current level j based on the WMDPDEs with tuning parameter $\beta = 0$ and $\beta = 0.6$, which are represented in the top left and top right of Figure 1, respectively. As expected, they decrease when the testing conditions increase, becoming especially low for extreme testing levels. Left bottom of Figure 1 shows the differences between the two measures, that is, $\widehat{R}_{0.6}^{ij} - \widehat{R}_0^{ij}$, while right bottom of Figure 1 shows the standardized differences $(\widehat{R}_{0.6}^{ij} - \widehat{R}_0^{ij})/\widehat{R}_0^{ij}$. While in absolute value the biggest differences are given for moderate values of temperature and current electricity (where reliabilities are higher), the most remarkable difference (that is measured with independence on the scale) is obtained for extreme conditions both of current and temperature. Note that these are the only cases when the estimated reliability based on the MLEs is higher than the one based on the WMDPDEs with tuning parameter $\beta = 0.6$. Table 5 shows the estimated probabilities of the WMDPDEs with different tuning parameters $\beta \in [0, 1]$, compared with the observed probabilities. Last row of Table 5 shows the estimated mean absolute error of each WMDPDE considered here, e_i^{β}. MLE ($\beta = 0$) seems, in general, to be one of the worst choices to predict each testing condition. In particular, we can say that WMDPDEs with high or moderate value of the tuning parameter seem to have a better behavior than the MLEs when higher-than-normal testing conditions are considered, just as we observed a greater difference in terms of reliability (Figure 1).

6.3 The choice of the tuning parameter

In the previous sections, WMDPDEs with $\beta > 0$ have been shown to be more robust, both theoretically and empirically, than the classical MLE, overall when a high degree of contamination is present in the data. MLE has been shown to be much more efficient instead. Therefore, given any data set, it would be important to determine what would be the best tuning parameter to use, and how to select it. It is then necessary to provide a data-driven procedure for the determination of the optimal choice of the tuning parameter. The idea is as follows: in a grid of, say $R = 100$, possible tuning parameters, apply a measure of discrepancy, say E_β, to our data. Then, the tuning parameter which leads to the minimum discrepancy-statistic will be chosen as the "optimal" one (see Algorithm 1).

Balakrishan and Ling[8] proposed, in a goodness of fit context, the distance-based statistic of the form

$$E_\beta = max_i |n_i - K_i \pi_{i1}(\widehat{\boldsymbol{a}}_\beta)|, \quad i = 1, \dots, I,$$

as a discrepancy measure for evaluating the fit of the assumed model to the observed data. For example, in our first example data, the MLE ($\beta = 0$) will be chosen as the optimal estimator, while in the second example, $\beta = 0.48$ would be selected.

Balakrishnan et al.[1,17] applied the method originally presented in Warwick and Jones[18] and discussed subsequently in Ghosh and Basu[19] in the context of one-shot devices, by mini-

Algorithm 1 General algorithm for the data-driven selection of β

Goal: Optimal fitting of the model given any data set

1: **for** each β in a grid of $[0, 1]$ **do**

2: Compute the discrepancy measure E_β

3: **end for**

4: **return** $\beta_{opt} = \arg\min_{\beta} E_\beta$.

5: **compute** $\widehat{\boldsymbol{\theta}}_{\beta_{opt}}$ as the final estimate with optimally chosen tuning parameter.

mizing the estimated mean square error through a grid of possible estimators. This estimated mean square error is computed as the sum of the estimated squared bias and the variance. A similar procedure could be applied in the proposed model. However, this method implies the computation of complex matrices and would also depend on the choice of a pilot estimator for the bias estimate, which usually leads to the choice of an optimal tuning parameter near to it. The approach presented by Hong and Kim[20] avoids the problem of the selection of a pilot estimator, but does not take into account the model misspecification, leading sometimes to quite non-robust estimators. Most robust procedures require the choice of a tuning parameter, and so it seems that further work needs to be done in this regard.

7 Concluding Remarks

Multiple stress factors are common when dealing with one-shot devices, overall in the context of ALT plans. In this article, we have developed the WMDPDEs for one-shot device testing data under exponential lifetime with multiple stresses. Through a simulation study and two numerical examples, these estimators, as well as the Wald-type tests derived from them, are shown as an useful alternative to the classical MLEs in terms of robustness. However, most of the literature regarding one-shot device models assumes that there is only one possible cause of device failure. In lifetime data analysis, it is often the case in which the products under study can experience any one of various possible causes of failure. It will, therefore, be of interest to develop robust estimators and Wald-type tests based on data from one-shot devices under competing risks. Work in this direction is currently under progress and we hope to report these findings in a future paper.

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Table 1: Typical form of data observed.

Test condition	Inspection Time	Number of Devices	Number of Failures	Covariates		
				Stress 1	$\cdots$	Stress J
1	t_1	K_1	n_1	w_{11}	$\cdots$	w_{1J}
2	t_2	K_2	n_2	w_{21}	$\cdots$	w_{2J}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
I	t_I	K_I	n_I	w_{I1}	$\cdots$	w_{IJ}

Table 2: Point estimation of parameters in the mice tumor toxicological data.

β	$\widehat{a}_0$	$\widehat{a}_1$	$\widehat{a}_2$	$\widehat{a}_3$
0	-4.452	-0.126	-1.201	0.133
0.2	-4.821	-0.195	-1.300	0.148
0.4	-4.784	-0.184	-1.291	0.145
0.6	-4.753	-0.176	-1.282	0.143
0.8	-4.731	-0.170	-1.275	0.141

Table 3: Data on 120 one-shot devices subjected to 2 stress factors and inspected at 3 different times.

i	IT_i	K_i	n_i	Temeperature (x_{i1})	Electric current (x_{i2})
1	2	10	0	55	70
2	2	10	4	55	100
3	2	10	4	85	70
4	2	10	7	85	100
5	5	10	4	55	70
6	5	10	7	55	100
7	5	10	8	85	70
8	5	10	8	85	100
9	8	10	3	55	70
10	8	10	9	55	100
11	8	10	9	85	70
12	8	10	10	55	100

Table 4: Point estimation of parameters and reliabilities at time $t \in \{10, 30, 60\}$ and mean lifetimes for different tuning parameters at normal conditions $\boldsymbol{w} = (25, 35)$ for the data presented in Table 3.

β	$\widehat{a}_0$	$\widehat{a}_1$	$\widehat{a}_2$	$R(10, 25, 35)$	$R(30, 25, 35)$	$R(60, 25, 35)$	$\widehat{T}$
0	−6.5128	0.0301	0.0340	0.9018	0.7334	0.5379	96.74
0.1	−6.6100	0.0308	0.0346	0.9069	0.7460	0.5565	102.38
0.2	−6.7178	0.0315	0.0354	0.9123	0.7594	0.5767	109.00
0.3	−6.8327	0.0323	0.0362	0.9178	0.7730	0.5975	116.51
0.4	−6.9549	0.0332	0.0370	0.9232	0.7868	0.6190	125.09
0.5	−7.0759	0.0340	0.0379	0.9282	0.7997	0.6395	134.21
0.6	−7.1920	0.0348	0.0387	0.9327	0.8115	0.6585	143.60
0.7	−7.2915	0.0355	0.0394	0.9364	0.8211	0.6742	152.17
0.8	−7.3740	0.0361	0.0400	0.9393	0.8287	0.6867	159.65
0.9	−7.4387	0.0365	0.0404	0.9415	0.8345	0.6964	165.79
1	−7.4869	0.0369	0.0407	0.9430	0.8387	0.7034	170.52

Table 5: Estimated probabilities for different WMDPDEs for the data presented in Table 3.

		$\widehat{\pi}_i^\beta$										
i	$\frac{n_i}{K_i}$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
1	0	0.154	0.152	0.150	0.148	0.146	0.144	0.142	0.141	0.139	0.138	**0.137**
2	0.4	0.338	0.340	0.343	0.346	0.348	0.351	0.354	0.356	0.358	0.359	**0.360**
3	0.4	0.371	0.373	0.376	0.378	0.381	0.384	0.387	0.389	0.391	0.393	**0.394**
4	0.7	0.681	0.691	**0.703**	0.715	0.728	0.740	0.752	0.761	0.769	0.776	0.780
5	0.4	**0.342**	0.338	0.335	0.331	0.327	0.322	0.319	0.315	0.312	0.310	0.309
6	0.7	0.644	0.647	0.650	0.654	0.657	0.661	0.664	0.667	0.669	0.671	**0.672**
7	0.8	0.686	0.689	0.692	0.695	0.699	0.703	0.706	0.709	0.711	0.713	**0.714**
8	0.8	**0.943**	0.947	0.952	0.957	0.961	0.965	0.969	0.972	0.974	0.976	0.977
9	0.3	0.488	0.484	0.479	0.474	0.469	0.464	0.459	0.454	0.451	0.448	**0.446**
10	0.9	0.808	0.811	0.814	0.817	0.820	0.823	0.825	0.828	0.830	0.831	**0.832**
11	0.9	0.843	0.846	0.848	0.851	0.854	0.856	0.859	0.861	0.863	0.864	**0.865**
12	1	0.990	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.997	0.997	**0.997**
	e_i^β	0.082	0.080	0.078	0.077	0.077	0.076	0.076	0.076	0.075	0.075	**0.075**

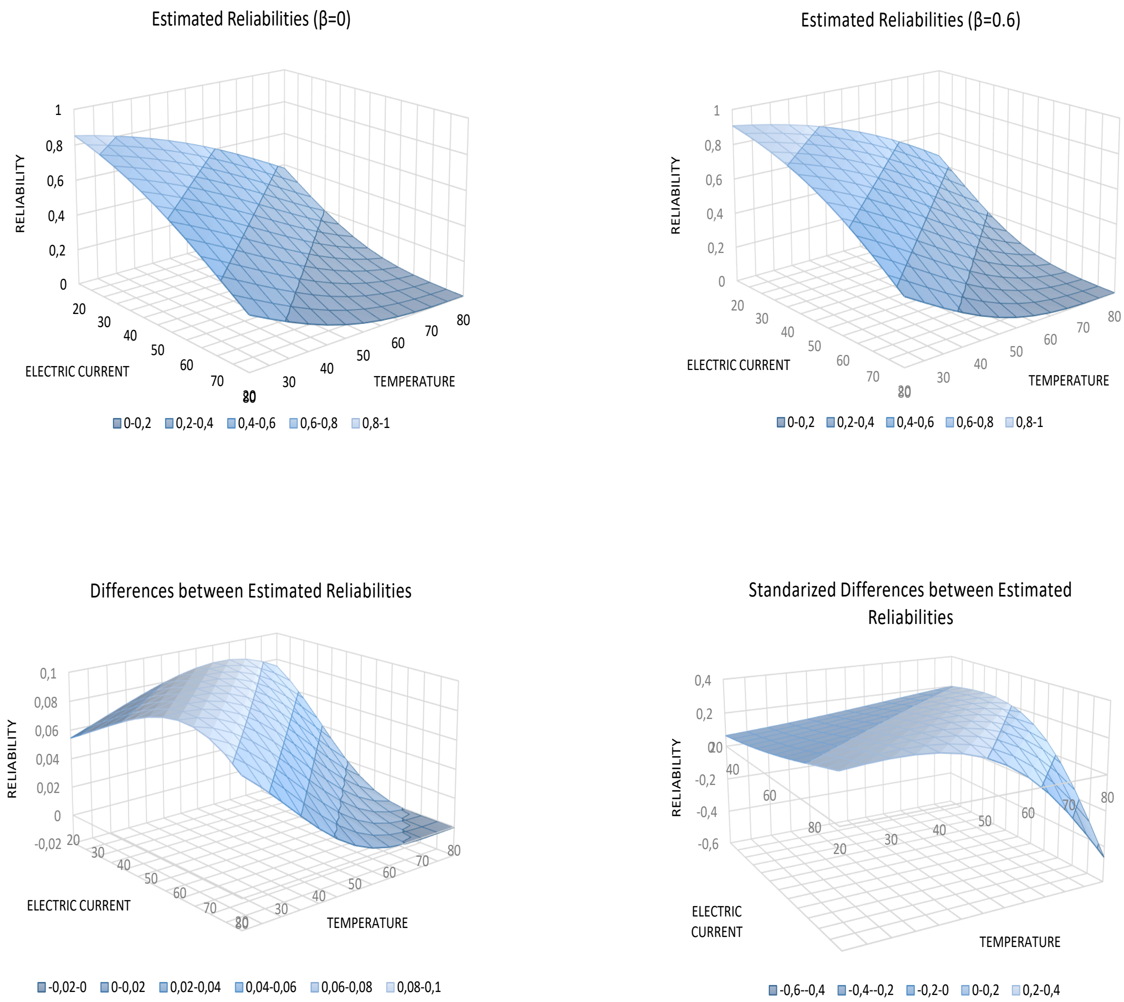

Figure 1: Estimated reliabilities based on WMDPDEs with tuning parameters $\beta = 0$ (top left) and $\beta = 0.6$ (top right) and their differences (bottom left) and standardized differences (bottom right) for the data presented in Table 3.

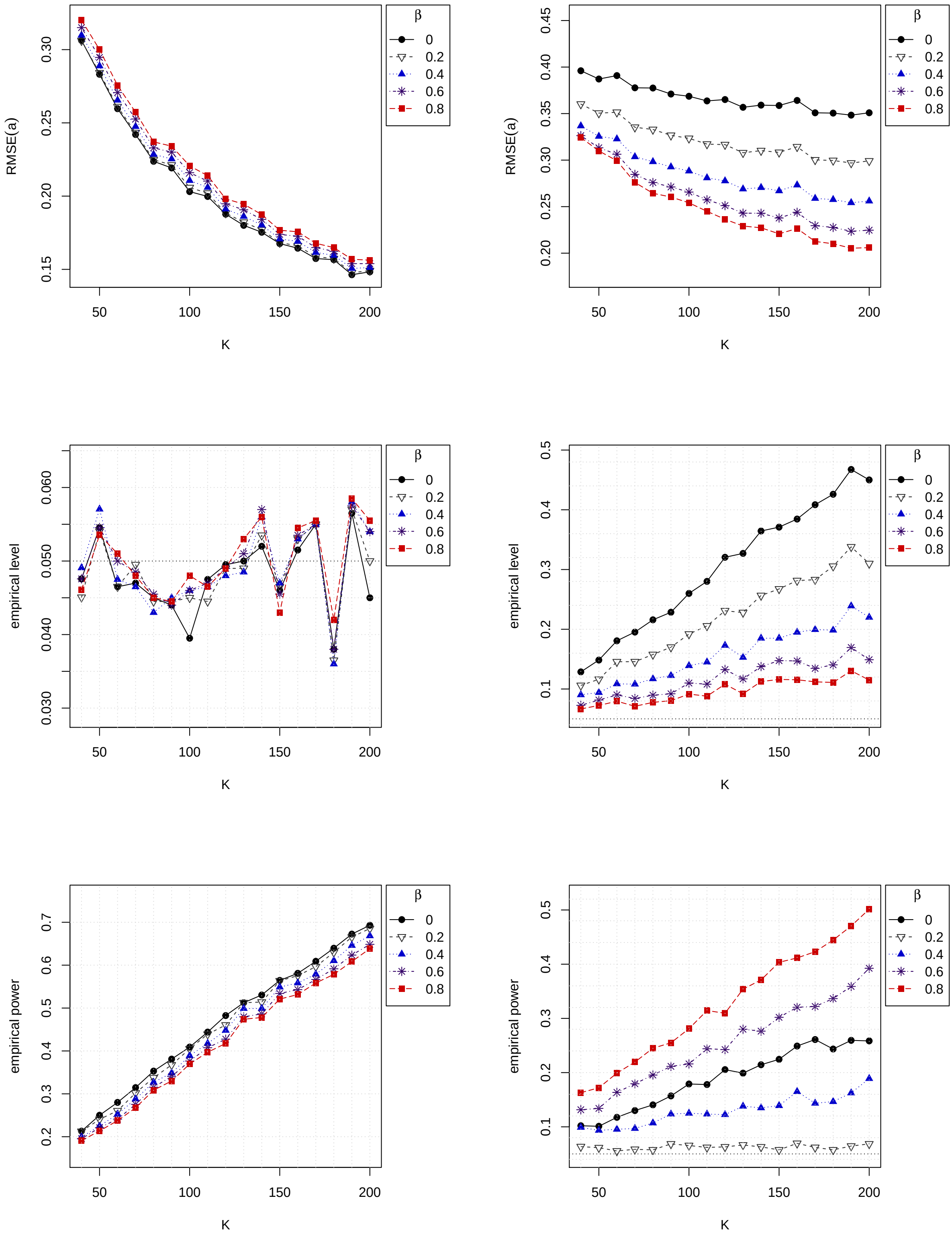

Figure 2: RMSEs (top panel) of the WMDPDEs of $\boldsymbol{a}$, the simulated levels (middle panel) and powers (bottom panel) of the Wald-type tests under the pure data (left) and under the contaminated data (right).

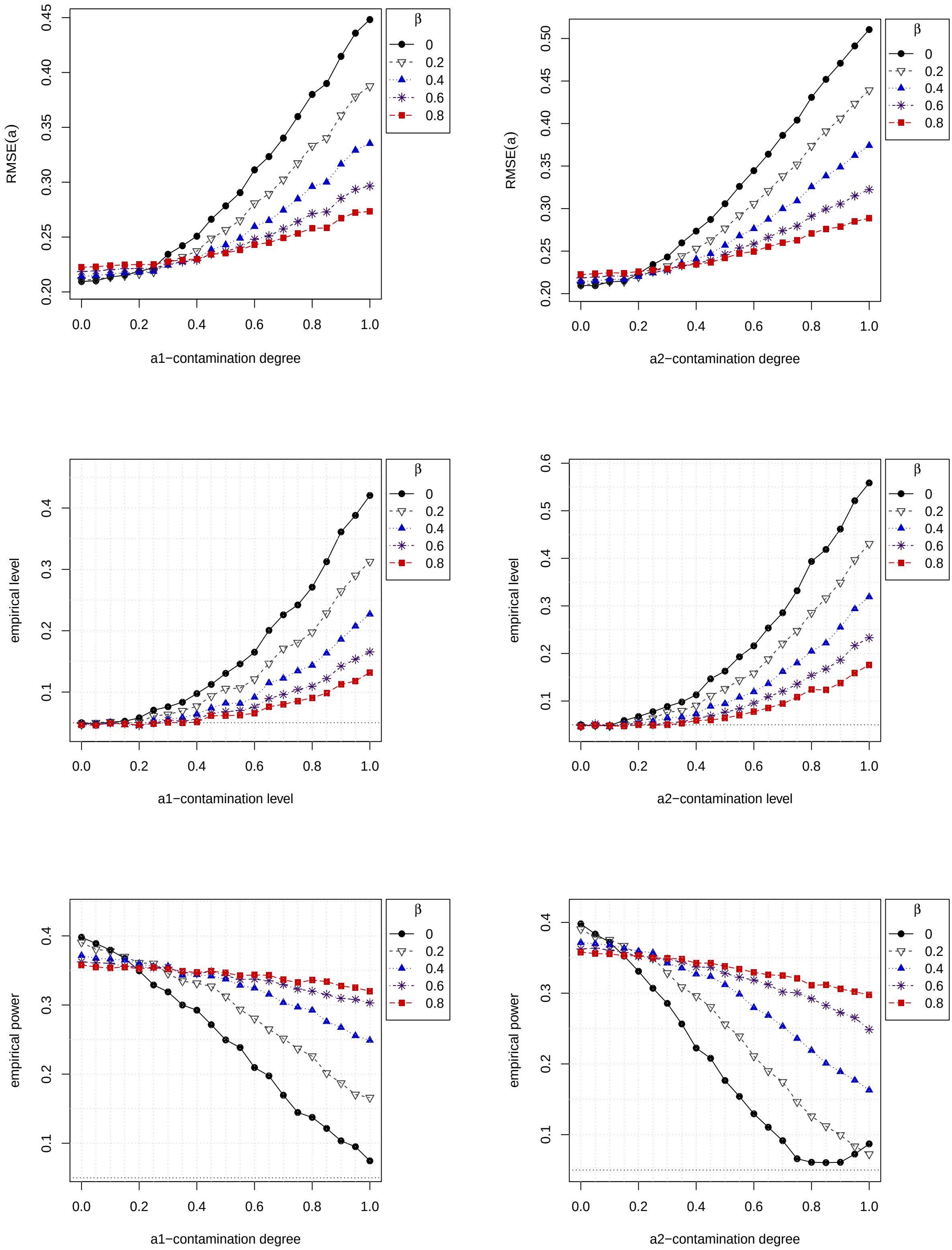

Figure 3: RMSEs (top panel) of the WMDPDEs of $\boldsymbol{a}$, the simulated levels (middle panel) and powers (bottom panel) of the Wald-type tests under the a1-contaminated data (left) and under the a2-contaminated data (right).

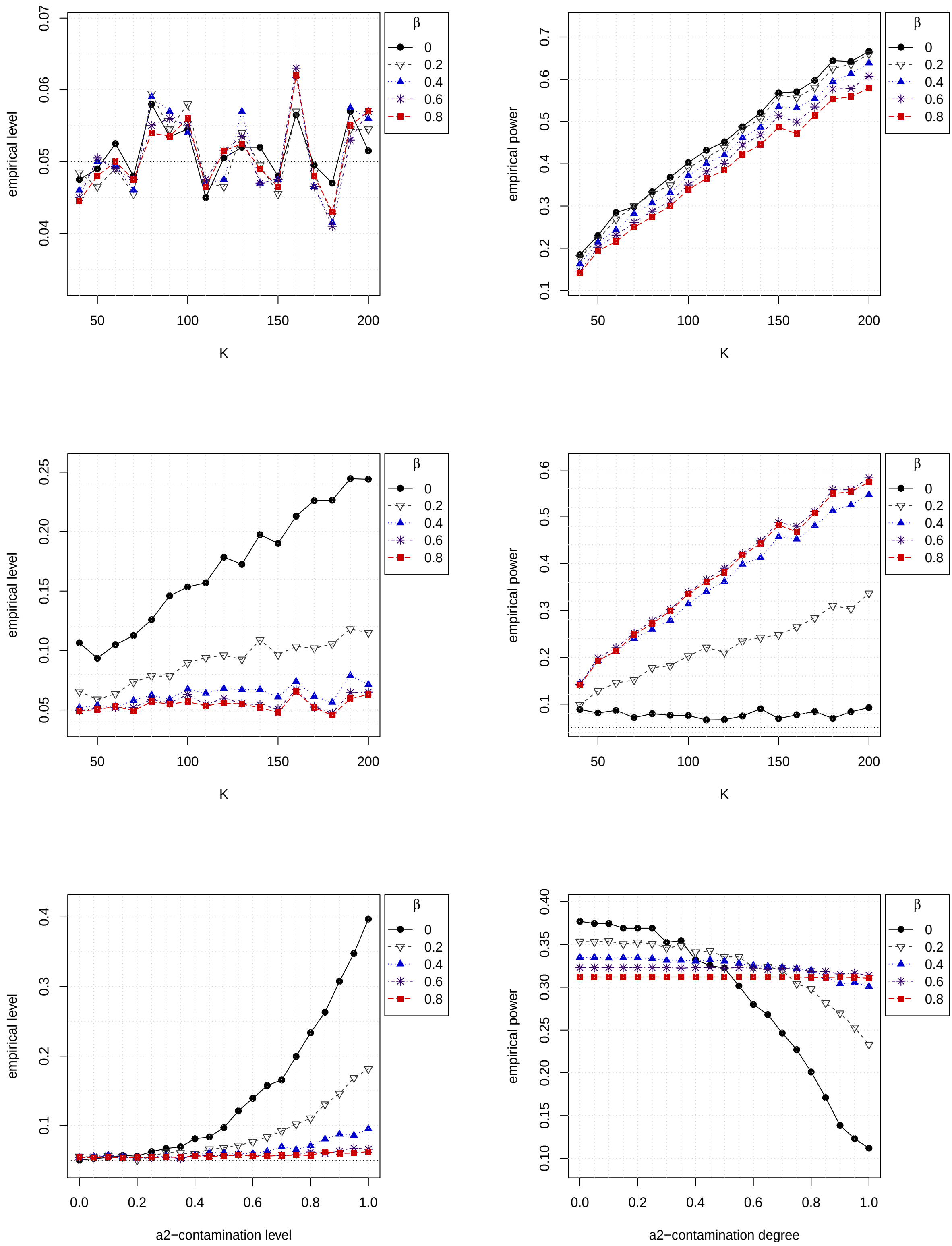

Figure 4: Empirical levels (left) and powers (right) under the pure data and under the contaminated data when parameter $a_0 = -6$

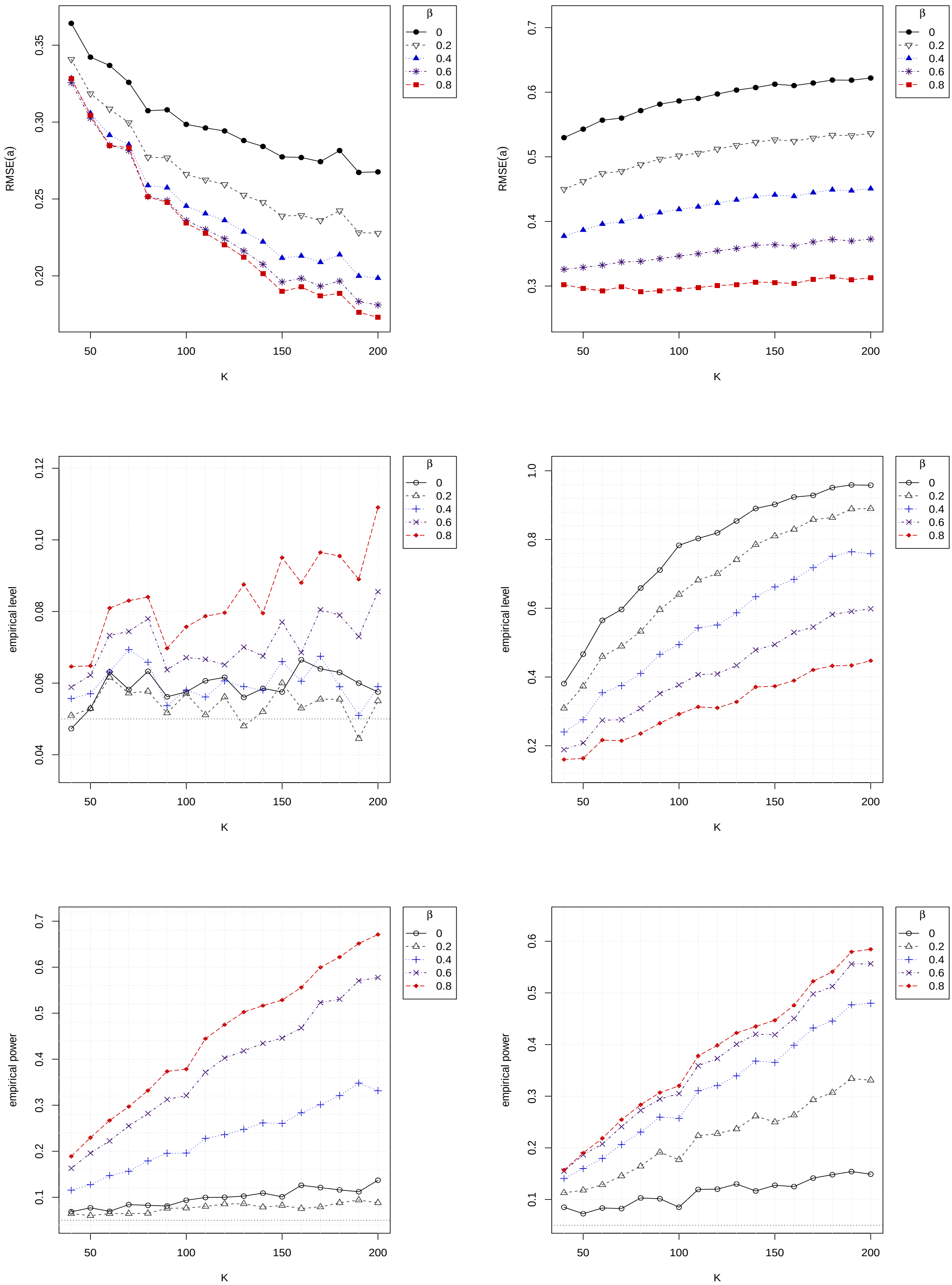

Figure 5: RMSEs (top panel), empirical levels (middle panel) and empirical powers (bottom panel) of two-cells contaminated data (left) and a1-a2-contaminated data (right), when parameter $a_0 = -6.5$.