ON p-PARABOLICITY OF RIEMANNIAN MANIFOLDS AND GRAPHS

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Abstract. Kanai proved that quasi-isometries between Riemannian manifolds with bounded geometry preserve many global properties, including the existence of Green's function, i.e., non-parabolicity. However, Kanai's hypotheses are too restrictive. Herein we prove the stability of p-parabolicity (with $1 < p < \infty$) by quasi-isometries between Riemannian manifolds under weaker asumptions. Also, we obtain some results on the p-parabolicity of graphs and trees; in particular, we characterize p-parabolicity for a large class of trees.

Keywords: Parabolicity; p-parabolicity; bounded local geometry; quasi-isometries.

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1. Introduction

Quasi-isometries are interesting since they are able to translate much of the large scale structure of a metric space in spite of an important local distortion. In particular, quasi-isometries need not be continuous maps. One of the key properties of quasi-isometries is that they preserve Gromov hyperbolicity of geodesic metric spaces (see, e.g., [14], [15]).

When studying large scale properties of a manifold it is a natural problem to consider which properties are preserved by quasi-isometries. In [21], [22], [23], M. Kanai studied several geometric properties (such as isoperimetric inequalities, Poincaré-Sobolev inequalities, parabolicity, growth rate of the volume of balls, and Liouville type theorems) for a large class of Riemannian manifolds whose Ricci curvature is bounded from below, and proved that these properties are preserved under quasi-isometries. Also, quasi-isometries preserve the parabolic Harnack inequality (see [6]) and several estimates on transition probabilities of random walks, such as heat kernel estimates. Moreover, Holopainen and Soardi, among other authors, (see [17], [18], [33]) proved that the existence of non-trivial solutions of a wide class of partial differential equations is also preserved under quasi-isometries.

The *injectivity radius* $\mathrm{inj}(p)$ *of* $p \in X$ is the largest radius for which the exponential map at p is a diffeomorphism. If X has non-positive sectional curvatures, then the injectivity radius can also be defined as the supremum of those $r > 0$ such that the ball $B_X(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p in X. The *injectivity radius* $\mathrm{inj}(X)$ *of* X is defined as $\mathrm{inj}(X) = \inf_{p\in X} \mathrm{inj}(p)$.

Because of the local flexibility of quasi-isometries, Kanai needed some condition to control the local geometry of the quasi-isometric Riemannian manifolds. Thus, he imposed that the injectivity radius on both manifolds were positive.

Many of these large scale properties can be studied using discrete structures approximating the Riemannian manifold. In Kanai's work this is done with a particular graph called the ε-net of the manifold. Several authors have followed Kanai's results and ideas whether to study the stability of other properties, or to prove the equivalence between a manifold and a different associated graph (see, e.g., [1] and [2] about the asymptotic behavior of p-harmonic functions on trees, [10] and [13] concerning the volume growth, its stability

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under quasi-isometries and its relation with the existence of Green's function on Riemannian surfaces, [17] and [18] about the invariance of Liouville D_p-property under quasi-isometries between Riemannian manifolds and between a Riemannian manifold and a graph, [26], [27] and [30] dealing with isoperimetric inequalities in manifolds, Riemannian surfaces and graphs, [28] and [29] characterizing Gromov hiperbolicity through decompositions, [31] connecting different conformal invariants of Riemann surfaces, [33], a book about potential theory on infinite graphs and [34] proving the equivalence of the Gromov hyperbolicity between some class of complete Riemannian surfaces and certain kind of simple graphs).

However, in the context of Riemannian manifolds, the hypothesis $\text{inj}(X) > 0$ seems too restrictive. Thus, a natural problem is to find weaker hypotheses that yield the same conclusions from Kanai's, in particular, for manifolds with $\text{inj}(X) = 0$. This was the approach of some previous works. For example, in [5] the authors extend some of the results from Kanai regarding isoperimetric inequalities and existence of Green's function to a large class of manifolds without hypotheses on the injectivity radius. They do this by using certain weighted graphs and asking that the manifolds have doubling measures and the quasi-isometry quasi-preserves the volume of the balls.

Although quasi-isometry does not preserve local topology (for example, any compact Riemannian manifold is quasi-isometric to a single point), introducing some hypothesis on the genus of the surfaces gives some stability for the injectivity radius. In particular, points with small injectivity radius are shown to be mapped onto points with small injectivity radius. See [3] and [12].

Also, the hypotheses on the injectivity radius is not necessary to prove that the linear isoperimetric inequality (positive Cheeger inequality) is preserved by quasi-isometries on Riemann surfaces with genus zero (recall that plane domains are the most important class of Riemann surfaces). Moreover, the isoperimetric inequality is preserved even when the topological hypotheses on the genus are relaxed (see [3] and [12]). Both the arguments and the results in [5], [3] and [12] are different between them and from those of this paper.

As usual, $d_X(x, y)$ will denote the distance between the points $x, y \in X$ in the Riemannian metric, and $L_X(\gamma)$ denotes the length of a curve $\gamma \subset X$ with respect to the intrinsic metric in X.

A function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if there exists a constant $c \geq 0$ such that f is c-*full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [21]).

In the first part of this paper, we study the stability of parabolicity and, more generally, p-parabolicity. This is an interesting research topic, see e.g. [11], [17], [18], [22]. Since Green's function is the fundamental solution to the Poisson equation, the problem regarding its existence is an important object, and has been extensively studied. For example, in [24] it was shown that it exists on any complete Riemannian manifold. In [4] it is proved that satisfying some necessary conditions involving the volume growth of the manifold gives the existence of a positive Green's function.

Given $1 < p < \infty$, a p-*Green's function* in a complete Riemannian manifold X is a positive fundamental solution of the p-Laplace-Beltrami operator (or, simply, the p-Laplacian) $\Delta_p u = \delta_y$, where the p-Laplacian is defined as $\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u)$ on X. Here, ∇ and div denote, respectively, the gradient and the divergence with respect to the Riemannian metric on X. When dealing with the case $p = 2$ the p will be omitted (and so, the 2-Laplacian is the "classical" Laplacian). The manifold X is p-*parabolic* if it does not have p-Green's function. A function is p-*harmonic* (respectively, p-*superharmonic*) if it is a weak solution of $\Delta_p u = 0$ (respectively, $\Delta_p u \geq 0$). Notice that p-harmonic functions are precisely the minimizers of the functional given by the p-Dirichlet integral $D_p(u) = \int |\nabla u|^p$. It is well-known that a complete manifold has p-Green's function if and only if there exists a non-constant positive p-superharmonic function (see, e.g., [32] for the case $p = 2$). In terms of Brownian motion, a complete manifold has Green's function if and only if the Brownian motion on the manifold is transient (i.e., the Brownian motion eventually escapes from any compact set with probability 1).

Green's function is also related to other topics as the heat kernel and isoperimetric inequalities: it is well-known that Green's function is the integral of the heat kernel of the manifold; Fernández proves in [8] that the existence of some kind of isoperimetric inequalities guarantees the existence of Green's function for Riemannian manifolds; the results in [9] imply that the linear isoperimetric inequality guarantees the existence of Green's function for Riemann surfaces.

One of the main results of this first part of the paper is Theorem 3.10; it states that, under some natural conditions, p-parabolicity is preserved by quasi-isometries. Note that there is no hypotheses on the injectivity radius in Theorem 3.10. This result is also new for Green's function ($p = 2$) even in the 2-dimensional case. Theorems 3.8 and 3.9 state that p-parabolicity is preserved by quasi-isometries between manifolds and graphs.

In the second part of the paper, we study the p-parabolicity of graphs and, specially, of trees. In [11], section 6, the authors give a construction of Riemannian surfaces by pasting Y-pieces following a tree pattern so that the surface is quasi-isometric to a given tree. This is used to provide examples of quasi-isometric surfaces with constant negative curvature where one is parabolic and the other is not. Thus, a deeper understanding of parabolicity on trees can be also a relevant tool for the study of necessary conditions on Riemannian manifolds so that p-parabolicity is a quasi-isometric invariant.

A graph is *uniform* if there is an upper bound for the number of neighbors a vertex can have. A tree is called *rooted* if some vertex is fixed to be the root.

Through propositions 4.4, 4.6, 4.7 and 4.9 we give sufficient conditions for uniform rooted trees to be p-parabolic and even a characterization in Proposition 4.13. Moreover, Proposition 4.5 gives also a sufficient condition for a uniform graph to be p-parabolic. Furthermore, in Theorem 4.21 we are able to give a nice characterization of p-parabolicity for a large class of trees. This result, in the particular case where $p = 2$ (i.e., the characterization of being parabolic) is proved again in Theorem 4.27 with totally different techniques using electric networks.

2. Definitions and background

Given a complete Riemannian manifold X and a domain with smooth boundary $\Omega \subset X$, define

$$\operatorname{cap}_p \Omega = \operatorname{cap}_p(\Omega, X) = \inf \Big\{ \int_X |\nabla u|^p \,:\, u \in C_c^\infty(X),\, u|_\Omega = 1 \Big\}.$$

A useful characterization of the existence of p-Green's function is:

Theorem 2.1. *Given $1 < p < \infty$, a complete Riemannian manifold is p-parabolic if and only if* $\operatorname{cap}_p \Omega = 0$ *for some (and then for every) domain with smooth boundary $\Omega \subset X$.*

The proof of Theorem 2.1 appears in [22] for $p = 2$ and in [16] for $1 < p < \infty$.

A discrete version of the previous statements is the following.

Given a function u on a graph Γ, define the *p-modulus of its discrete gradient* $|\nabla_\Gamma u|_p$ and its *discrete p-Dirichlet integral* $D_{p,\Gamma}(u)$, respectively, by

$$|\nabla_\Gamma u|_p(x) := \Big(\sum_{y \in N(x)} \big|u(y) - u(x)\big|^p \Big)^{1/p}, \qquad D_{p,\Gamma}(u) := \sum_{x \in \Gamma} |\nabla_\Gamma u|_p^p(x) = 2 \sum_{vw \in E(\Gamma)} \big|u(v) - u(w)\big|^p,$$

where the edges are considered unoriented.

For a finite subset S of Γ, the *p-capacity* of S is defined by

$$\operatorname{cap}_p S = \operatorname{cap}_p(S, \Gamma) = \inf \Big\{ D_{p,\Gamma}(u) : u \text{ function on } \Gamma \text{ with finite support, } u|_S = 1 \Big\}.$$

A graph Γ is said to be *μ-uniform* if each vertex p of V has at most μ neighbors, i.e.,

$$\sup \big\{ |N(p)| \,\big|\, p \in V(\Gamma) \big\} \le \mu.$$

If a graph Γ is μ-uniform for some constant μ we say that Γ is *uniform*

Theorem 2.2. *Given $1 < p < \infty$, a uniform graph Γ is p-parabolic if and only if $\operatorname{cap}_p S = 0$ for some (and then for every) non-empty finite subset of $S \subset \Gamma$.*

For a proof of Theorem 2.2, see [23, Proposition 6] and [17, Final remark 5.16]. Note that the definition of discrete p-Dirichlet integral in [17] is slightly different, but both are equivalent.

The following volume bounds for geodesic balls are well known:

$$\operatorname{vol} B_r(p) \geq V_0(r) \text{ for } p \in X \text{ and } r \in \Big(0, \frac{\operatorname{inj}(X)}{2}\Big], \tag{2.1}$$

and if the Ricci curvature is bounded below,

$$\operatorname{vol} B_r(p) \leq V_1(r) \text{ for } p \in X \text{ and } r > 0, \tag{2.2}$$

A subset A in a metric space (X, d) is called *ϵ-separated*, $\epsilon > 0$, if $d(a, a') \geq \epsilon$ for any distinct $a, a' \in A$. Note that if A is maximal with this property, then the union $\cup_{a\in A} B_\epsilon(a)$ covers X. A maximal ϵ-separated set A in a metric space X is called an *ϵ-approximation* of X.

Let X be a complete Riemannian manifold and denote by d the induced metric. Given any ε-approximation A of X, the graph $\Gamma_A = (A, E)$ with $E := \{xy \,|\, x, y \in A \text{ with } 0 < d(x, y) \leq 2\varepsilon\}$ is called an *ε-net*.

3. p-parabolicity on manifolds

Definition 3.1. *A metric space (X, d) is* doubling *if there is a constant M such that for every $x \in X$ and every $\varepsilon > 0$ then every ball $B(x, 2\varepsilon)$ can be covered by the union of at most M balls $B(z_i, \varepsilon)$.*

The following result is well known and easy to prove. See, for example, Lemma 2.3 in [20].

Lemma 3.2. *Given a doubling metric space (X, d), an ε-separated set S in X and some ball $B(z, k\varepsilon)$ with $x \in X$ and $k \in \mathbb{N}$, then there is a constant $\mu(k)$ such that $|S \cap B(z, k\varepsilon)| \leq \mu(k)$.*

In particular, we have the following.

Lemma 3.3. *An ε-net in a doubling metric space is uniform.*

Definition 3.4. *A Riemannian n-manifold X satisfies the* local Poincaré inequality *if for every geodesic ball $B_\varepsilon(z)$ in X and every function $u \in C_0^\infty(X)$ there is a constant $\beta = \beta(n, \varepsilon) > 0$ such that*

$$\int_{B_\varepsilon(z)} |\nabla(u)|\, dx \geq \beta \int_{B_\varepsilon(z)} |u - u^*|\, dx. \tag{3.3}$$

where $u^* = \frac{1}{\operatorname{vol} B_\varepsilon(z)} \int_{B_\varepsilon(z)} u\, dx$.

Lemma 3.5. [22, Lemma 8] [21, Lemma 2.3] *If X is a complete Riemannian manifold whose Ricci curvature is bounded below and P is an ε-net in X, then X satisfies the local Poincaré inequality and P is uniform.*

Definition 3.6. *Given $1 < p < \infty$ and $\varepsilon > 0$, we say that a complete Riemannian manifold X is (ε, p)-*nice *if there exist constants k_1, k_2 such that for every $q_0 \in X$ there is some constant $V(q_0)$ satisfying that for every $q \in \bar{B}_{3\varepsilon}(q_0)$,*

$$k_1 V(q_0)^{(p-1)/p} \leq \operatorname{vol} B_\varepsilon(q) \leq \operatorname{vol} B_{4\varepsilon}(q) \leq k_2 V(q_0). \tag{3.4}$$

Remark 3.7. *If $1 < p < \infty$, $\varepsilon > 0$ and X is a complete Riemannian manifold with positive injectivity radius whose Ricci curvature is bounded below then, by (2.1) and (2.2), X is (ε, p)-nice with $V(q_0) = 1$.*

Theorem 3.8. *Given $1 < p < \infty$ and $\varepsilon > 0$, let X be an (ε, p)-nice, doubling complete Riemannian manifold satisfying the local Poincaré inequality and P an ε-net in X. If X is p-parabolic, then P is p-parabolic.*

Proof. Suppose that P is non-p-parabolic and let us see that X is non-p-parabolic.

Fix any non-empty finite subset S of P. Then, by Theorem 2.2, $\operatorname{cap}_p S > 0$. Let Ω be a bounded domain in X with smooth boundary such that $B_{4\varepsilon}(q_0) \subset \Omega$ for every $q_0 \in S$. For an arbitrary function $u \in C_0^\infty(X)$ with $u = 1$ on Ω, define a function u^* on P by

$$u^*(q_0) = \frac{1}{\operatorname{vol} B_{4\varepsilon}(q_0)} \int_{B_{4\varepsilon}(q_0)} u\, dx.$$

It is clear that $u^*(q_0) = 1$ for every $q_0 \in S$.

Since X satisfies the local Poincaré inequality, by Hölder inequality and local Poincaré inequality, we have that for any $q \in \bar{B}_{3\varepsilon}(q_0)$

$$\big(k_2 V(q_0)\big)^{p-1} \int_{B_{4\varepsilon}(q)} |\nabla(u)|^p dx \geq \Big(\int_{B_{4\varepsilon}(q)} |\nabla(u)|\, dx \Big)^p \geq \beta^p \Big(\int_{B_{4\varepsilon}(q)} |u(x) - u^*(q)|\, dx \Big)^p.$$

Then, following Kanai's proof, we have that for any $q_0 \in P$ and any $q \in N_{q_0}$

$$\begin{aligned}
2^p \beta^{-p} \big(k_2 V(q_0)\big)^{p-1} \int_{B_{7\varepsilon}(q_0)} |\nabla(u)|^p dx &\geq \beta^{-p} \big(k_2 V(q_0)\big)^{p-1} 2^{p-1} \Big(\int_{B_{4\varepsilon}(q_0)} |\nabla(u)|^p dx + \int_{B_{4\varepsilon}(q)} |\nabla(u)|^p dx \Big) \\
&\geq 2^{p-1} \Big(\int_{B_{4\varepsilon}(q_0)} \big| u(x) - u^*(q_0) \big|\, dx \Big)^p + 2^{p-1} \Big(\int_{B_{4\varepsilon}(q)} \big| u(x) - u^*(q) \big|\, dx \Big)^p \\
&\geq \Big(\int_{B_{4\varepsilon}(q_0)} \big| u(x) - u^*(q_0) \big|\, dx + \int_{B_{4\varepsilon}(q)} \big| u(x) - u^*(q) \big|\, dx \Big)^p \\
&\geq \Big(\int_{B_{4\varepsilon}(q_0) \cap B_{4\varepsilon}(q)} \big| u^*(q) - u^*(q_0) \big|\, dx \Big)^p \\
&\geq \big| u^*(q) - u^*(q_0) \big|^p \Big(\int_{B_\varepsilon(q_0)} dx \Big)^p \\
&\geq k_1^p V(q_0)^{p-1} \big| u^*(q) - u^*(q_0) \big|^p.
\end{aligned}$$

Therefore, since P is uniform by Lemma 3.3, there is a constant C independent of u and q_0 such that for every $q_0 \in P$

$$C \int_{B_{7\varepsilon}(q_0)} |\nabla(u)|^p dx \geq |\nabla_P(u^*)|_p^p(q_0).$$

Moreover, by Lemma 3.2, there is a constant $\mu(7)$ such that

$$\mu(7) \int_X |\nabla(u)|^p dx \geq \sum_{q_0 \in P} \int_{B_{7\varepsilon}(q_0)} |\nabla(u)|^p dx.$$

Hence,

$$\mu(7\varepsilon)\, C \int_X |\nabla(u)|^p dx \geq C \sum_{q_0 \in P} \int_{B_{7\varepsilon}(q_0)} |\nabla(u)|^p dx \geq \sum_{q_0 \in P} |\nabla_P(u^*)|_p^p(q_0),$$

and we obtain that there is a constant $C_2 = \mu(7\varepsilon)C$ such that

$$C_2 \int_X |\nabla(u)|^p dx \geq D_{p,P}(u^*) = \operatorname{cap}_p S > 0,$$

proving the non-p-parabolicity of X. □

Theorem 3.9. *Given $1 < p < \infty$ and $\varepsilon > 0$, let X be an (ε, p)-nice complete Riemannian manifold with Ricci curvature bounded below and P an ε-net in X. Then, X is p-parabolic if and only if P is p-parabolic.*

Proof. If X is p-parabolic, then Lemma 3.5 and Theorem 3.8 give that P is p-parabolic. (Notice that in the proof of Theorem 3.8, since the Ricci curvature is bounded below, Lemma 3.5 gives that P is uniform and lemmas 3.2 and 3.3 are not necessary.)

Assume now that P is p-parabolic. Following Kanai's proof of [22, Theorem 1], since the hypothesis on the injectivity radius is not used in this implication, we obtain that X is p-parabolic. (Notice that in the final steps of this implication it is used that the Ricci curvature is bounded below to give an upper bound $V_1(\varepsilon)$ of the volume of $B_\varepsilon(q_0)$ for any q_0). □

Theorem 3.10. *Let X and Y be quasi-isometric complete Riemannian manifolds. Assume that X is doubling and satisfies the local Poincaré inequality, and Y has Ricci curvature bounded below. If X is p-parabolic and (ε, p)-nice for some $1 < p < \infty$ and $\varepsilon > 0$, then Y is p-parabolic.*

Proof. Since X is p-parabolic and (ε, p)-nice for some $\varepsilon > 0$, Theorem 3.8 gives that if P is an ε-net in X, then P is p-parabolic. Let Q be a net in Y. Since X and Y are quasi-isometric, P and Q are quasi-isometric to each other, and both are uniform. Thus, [22, Corollary 7], gives that Q is p-parabolic since P is p-parabolic. The argument in Kanai's proof of [22, Theorem 1], since the hypothesis on the injectivity radius is not used in this implication, gives that Y is p-parabolic. □

Corollary 3.11. *Let X and Y be quasi-isometric complete Riemannian manifolds with Ricci curvature bounded below. If X is p-parabolic and (ε, p)-nice for some $1 < p < \infty$ and $\varepsilon > 0$, then Y is p-parabolic.*

Some examples in [11] show that the conclusion of Theorem 3.10 does not hold without the hypothesis of X being (ε, p)-nice, even in the case of dimension two and constant curvature.

4. p-parabolicity on graphs and trees

Denote by P_n the path graph with length n and vertices $v_0, v_1, \dots, v_n$.

Lemma 4.1. *Consider $1 < p < \infty$ and the path graph P_n. Then, for any function u such that $u(v_0) = 1$ and $u(v_n) = 0$, we have*

$$D_{p,P_n}(u) \geq \frac{2}{n^{p-1}}\,.$$

Moreover, this lower bound is achieved when $u(v_i) = \frac{n-i}{n}$ for $0 \leq i \leq n$.

Proof. Let us consider any function u such that $u(v_0) = 1$ and $u(v_n) = 0$. Let us denote $u_i = u(v_i)$ for every i. Since $1 < p < \infty$, Hölder inequality gives

$$1 = \sum_{i=1}^{n} (u_{i-1} - u_i) \leq \Big(\sum_{i=1}^{n} |u_i - u_{i-1}|^p \Big)^{1/p} \Big(\sum_{i=1}^{n} 1^{p/(p-1)} \Big)^{(p-1)/p} = n^{(p-1)/p} \Big(\sum_{i=1}^{n} |u_i - u_{i-1}|^p \Big)^{1/p},$$

$$D_{p,P_n}(u) = 2 \sum_{i=1}^{n} |u_i - u_{i-1}|^p \geq \frac{2}{n^{p-1}}\,.$$

If $u(v_i) = \frac{n-i}{n}$ for $0 \leq i \leq n$, then we have $D_{p,P_n}(u) = \frac{2}{n^{p-1}}$. □

A *rooted tree*, (T, v), is a tree T with a fixed point $v \in T$, called the *root*.

Definition 4.2. *A rooted tree (T, v) is* geodesically complete *if every isometric embedding $f : [0, t] \to T$ with $t > 0$ and $f(0) = v$ extends to an isometric embedding $F : [0, \infty) \to T$.*

Theorem 4.3. [25] *If (T, v) is a rooted tree, then there exists a unique geodesically complete subtree $(T^\infty, v) \subseteq (T, v)$ that is maximal under inclusion.*

Proposition 4.4. *Given $1 < p < \infty$, a uniform rooted tree (T, v) and $S = v$, then $\mathrm{cap}_p(S, T^\infty) = \mathrm{cap}_p(S, T)$. Besides, (T, v) is p-parabolic if and only if (T^∞, v) is p-parabolic.*

Proof. Suppose (T, v) is a uniform rooted tree which is not geodesically complete.

For any function $u : V(T^\infty) \to \mathbb{R}$ with finite support let us define $\bar{u} : V(T) \to \mathbb{R}$ so that $\bar{u}(w) = u(w)$ for every $w \in T^\infty$. Now, notice that since T is uniform every connected component of the closure of $T \setminus T^\infty$ is a finite tree. Furthermore, since $\text{supp}(u)$ is finite, there is a finite number of components $T_1, \dots, T_k$ of the closure of $T \setminus T^\infty$ such that a unique vertex $w_j \in T_j \cap T_\infty$ satisfies that $u(w_j) > 0$. Therefore, suppose $\bar{u}$ is constant in each of this components and it is trivial to check that $\bar{u}$ has finite support and $D_{p,T^\infty}(u) = D_{p,T^\infty}(\bar{u})$. Thus, $\text{cap}_p(S, T^\infty) \geq \text{cap}_p(S, T)$.

For any function $u : V(T) \to \mathbb{R}$ with finite support let us consider its restriction $u' = u|_{V(T^\infty)}$. Therefore, $D_{p,T^\infty}(u') \leq D_{p,T}(u)$, and so, $\text{cap}_p(S, T^\infty) \leq \text{cap}_p(S, T)$.

Hence, by Theorem 2.2, T is p-parabolic if and only if T^∞ is p-parabolic. □

Proposition 4.5. *If $1 < p < \infty$ and Γ is a uniform graph with $\lim_{n\to\infty} \frac{\#E(\bar{B}(v_0,n))}{n^p} = 0$ for some $v_0 \in V(\Gamma)$, then Γ is p-parabolic.*

Proof. By Theorem 2.2, it suffices to check that $\text{cap}(v_0, \Gamma) = 0$. Let us define the family of functions $u^n : V(\Gamma) \to \mathbb{R}$ so that $u^n(v) = \frac{n - d(v_0,v)}{n}$ if $d(v_0, v) \leq n$ and $u^n(v) = 0$ otherwise.

Now, notice that for any edge $vw \in E(\Gamma)$, $|u^n(w) - u^n(v)| = \frac{1}{n}$ or $u^n(w) - u^n(v) = 0$ if the edge is contained in the closed ball $\bar{B}(v_0, n)$ and $u^n(w) - u^n(v) = 0$ otherwise. Therefore,

$$D_{p,\Gamma}(u^n) = 2 \sum_{vw \in E(\Gamma)} \big|u^n(v) - u^n(w)\big|^p \leq 2\, \frac{\#E(\bar{B}(v_0, n))}{n^p}\,.$$

Thus, if $\lim_{n\to\infty} \frac{\#E(\bar{B}(v_0,n))}{n^p} = 0$, then $\text{cap}_p(v_0, \Gamma) = 0$ and Γ is p-parabolic. □

Proposition 4.6. *If $1 < p < \infty$ and (T, v) is a uniform rooted tree such that its geodesically complete subtree (T^∞, v) satisfies that $\lim_{n\to\infty} \frac{\#E(\bar{B}(v,n))}{n^p} = 0$, then T is p-parabolic.*

Proof. Proposition 4.5 gives that T^∞ is p-parabolic, and so, Proposition 4.4 gives that T is p-parabolic. □

Proposition 4.7. *If $1 < p < \infty$ and (T, v) is a uniform rooted tree such that its geodesically complete subtree (T^∞, v) satisfies that $\lim_{n\to\infty} \frac{\#S(v,n)}{n^{p-1}} = 0$, then T is p-parabolic.*

Proof. Notice that $\#E(\bar{B}(v, n)) \leq n\, \#S(v, n)$. Thus,

$$0 \leq \liminf_{n\to\infty} \frac{\#E(\bar{B}(v, n))}{n^p} \leq \limsup_{n\to\infty} \frac{\#E(\bar{B}(v, n))}{n^p} \leq \lim_{n\to\infty} \frac{\#S(v, n)}{n^{p-1}} = 0$$

and so,

$$\lim_{n\to\infty} \frac{\#E(\bar{B}(v, n))}{n^p} = 0.$$

The result follows from Proposition 4.6. □

A *cut set* C for a rooted, geodesically complete tree (T, v) is a subset C of T such that $v \notin C$ and for every isometric embedding $f : [0, \infty) \to T$ with $f(0) = v$ there exists a unique $t_0 > 0$ such that $f(t_0) \in C$ (see [19]). Given a cut set C in a rooted tree (T, v), let us denote $[v, C]$ the set of points in T contained in some geodesic $[v, c]$ for some $c \in C$.

Lemma 4.8. *If C is a cut set for a uniform rooted tree (T, v), then $[v, C]$ is compact.*

Proof. Suppose C is a cut set such that $[v, C]$ is not compact. Then, there is a sequence $(c_n) \in C$ with $d(v, c_n) > n$ for every $n \in \mathbb{N}$. To reach a contradiction we are going to build an isometric embedding $f : [0, \infty) \to T$ with $f(0) = v$ such that $f([0, \infty)) \cap C = \emptyset$. Since T is uniform, for each $m \in \mathbb{N}$ the set $S(v, m)$ is finite. Thus, by induction, for some vertex $v_1 \in S(v, 1)$ there is an infinite subsequence $(c_n^1) \subset (c_n)$ such that $v_1 \subset [v, c_n^1] \setminus \{c_n^1\}$ for every $n \in \mathbb{N}$. Let us define an isometric embedding f with $f(0) = v$ and $f([0, 1]) = [v, v_1]$. Now, suppose f is defined in $[0, k]$ so that $f([0, k]) = [v, v_k]$ and there is an infinite subsequence $(c_n^k) \subset (c_n^{k-1})$ such that $v_k \in [v, c_n^k] \setminus \{c_n^k\}$ for every n. Again, since $S(v, k+1)$ is finite, for some vertex $v_{k+1} \in S(v, k+1)$ there is an infinite subsequence $(c_n^{k+1}) \subset (c_n^k)$ such that $v_{k+1} \in [v, c_n^{k+1}] \setminus \{c_n^{k+1}\}$ for every n. Hence, let $f([0, k+1]) = [v, v_{k+1}]$. Thus, by induction, we obtain an isometric embedding $f : [0, \infty) \to T$ so that $f(t) \notin C$ for every $t \in [0, \infty)$ leading to contradiction. □

If c is any vertex of the rooted tree (T, v), then the subtree of (T, v) determined by c is

$$T_c = \{x \in T \mid c \in [v, x]\}.$$

Let us denote by $[t]$ the upper integer part of $t \in \mathbb{R}$, i.e., the smallest integer that is greater than or equal to t.

Proposition 4.9. *Consider $1 < p < \infty$, a uniform rooted tree (T, v) and its maximal geodesically complete subtree (T^∞, v). Fix a function $g : \mathbb{N} \to (0, \infty)$ with $\lim_{n\to\infty} g(n) = \infty$. If for each n in an infinite subset $A \subseteq \mathbb{N}$ there is a cut set $C_n = \{c_1^n, \dots, c_{k_n}^n\}$ of T^∞ such that $\bar{B}_{T^\infty_{c_j^n}}(c_j^n, [g(n)k_n^{1/(p-1)}])$ is isometric to the path $P_{[g(n)k_n^{1/(p-1)}]}$ for each $1 \le j \le k_n$, then T is p-parabolic.*

Proof. First, notice that by Lemma 4.8, the set of vertices $S = V(T^\infty) \cap [v, C_n]$ is finite for every $n \in A$. For each $n \in A$, let us define a function $u^n : V(T^\infty) \to \mathbb{R}$ such that $u^n(w) = 1$ for every $w \in S$,

$$u^n(w) = \frac{\left[g(n)k_n^{1/(p-1)}\right] - d(w, c_j^n)}{\left[g(n)k_n^{1/(p-1)}\right]}$$

for every $w \in \bar{B}_{T^\infty_{c_j^n}}(c_j^n, g(n)k_n)$ and $u^n(w) = 0$ otherwise. Then, Lemma 4.1 gives that

$$D_{p,T^\infty}(u^n) = k_n \, \frac{2}{\left[g(n)k_n^{1/(p-1)}\right]^{p-1}} \le k_n \, \frac{2}{g(n)^{p-1}k_n} = \frac{2}{g(n)^{p-1}} \, ,$$

and so, $\operatorname{cap}_p(v, T^\infty) = 0$. Then, T^∞ is p-parabolic and, by Proposition 4.4, T is p-parabolic. □

Proposition 4.10. *If a uniform graph Γ contains a non-p-parabolic subgraph Γ' for some $1 < p < \infty$, then Γ is non-p-parabolic.*

Proof. Fix some vertex $v_0 \in V(\Gamma')$. Since Γ' is non-p-parabolic, Theorem 2.2 gives that $\delta = \operatorname{cap}_p(v_0, \Gamma') > 0$. Let us consider any function $u : V(\Gamma) \to \mathbb{R}$ with finite support and $u(v_0) = 1$. One can check that $D_{p,\Gamma}(u) \ge D_{p,\Gamma'}(u) \ge \delta$ and $\operatorname{cap}_p(v_0, \Gamma) \ge \delta > 0$. By Theorem 2.2, Γ is non-p-parabolic. □

Corollary 4.11. *If a uniform graph Γ is p-parabolic for some $1 < p < \infty$, then every subgraph of Γ is p-parabolic.*

Definition 4.12. *Given a rooted tree (T, v_0), a set of vertices in T, $\mathcal{C} = \{c_i\}_{i\in I}$, is called a* subcut set *in T if $T_{c_i} \cap T_{c_j} = \emptyset$ for every $i \ne j$ and $\mathcal{C}$ is not a cut set.*

Given a rooted tree (T, v_0) and a subcut set $\mathcal{C} = \{c_i\}_{i\in I}$, let us denote $T_{\mathcal{C}} = T \setminus \cup_{i\in I} T_{c_i}$.

Proposition 4.13. *Let (T, v_0) be a uniform rooted tree and $\mathcal{C}$ a subcut set in T and $1 < p < \infty$. Then T is p-parabolic if and only if $T_{\mathcal{C}}$ and T_c are p-parabolic for every $c \in \mathcal{C}$.*

Proof. By Proposition 4.10, if T is p-parabolic, then $T_{\mathcal{C}}$ and T_c are p-parabolic for every $c \in \mathcal{C}$.

Assume now that $T_{\mathcal{C}}$ and T_c are p-parabolic for every $c \in \mathcal{C}$. Then, for every n there exists some function $u^n : V(T_{\mathcal{C}}) \to \mathbb{R}$ with finite support such that $u^n(v_0) = 1$ and $D_{p,T_{\mathcal{C}}}(u^n) < \frac{1}{2n}$. Since u^n has finite support, there is a finite number of vertices $c_1, \dots, c_k \in \mathcal{C}$ adjacent to $\operatorname{supp}(u^n)$. Let w_i be the vertex in $\operatorname{supp}(u^n)$ adjacent to c_i for each $1 \le i \le k$. Then, since T_{c_i} is p-parabolic, there is a function $u_i^n : V(T_{c_i}) \to \mathbb{R}$ such that u_i^n has finite support, $u_i^n(c_i) = 1$ and $D_{p,T_{c_i}}(u_i^n) < \frac{1}{u^n(w_i)^p \cdot 2^{i+1} n}$. Let us define $\bar{u}^n : V(T) \to \mathbb{R}$ such that

$\bar{u}^n(v) = u^n(v)$ for every $v \in T_{\mathcal{C}}$, $\bar{u}^n(v) = u^n(w_i)u_i^n(v)$ for every $v \in T_{c_i}$ and every $1 \le i \le k$ and $\bar{u}^n(v) = 0$ otherwise. Notice that $D_{p,T_{c_i}}(\bar{u}^n) = u^n(w_i)^p D_{p,T_{c_i}}(u_i^n) < \frac{1}{2^{i+1}n}$. Since $\bar{u}^n(w_i) = \bar{u}^n(c_i)$ for every $1 \le i \le k$,

$$\begin{aligned} D_{p,T}(\bar{u}^n)(w_i) &= D_{p,T_{\mathcal{C}}}(u^n) + \sum_{i=1}^{k} u^n(w_i)^p D_{p,T_{c_i}(u_i^n)} \\ &< \frac{1}{2n} + \sum_{i=1}^{k} u^n(w_i)^p \frac{1}{u^n(w_i)^p \cdot 2^{i+1}n} < \frac{1}{n} \end{aligned}$$

and it is readily seen that $\bar{u}^n$ has finite support and $\bar{u}^n(v_0) = 1$. Thus, by Theorem 2.2, T is p-parabolic. □

Example 4.14. *Let (T, v_0) be a uniform rooted tree defined as follows and $1 < p < \infty$. Let $T = \{(x, y) \in \mathbb{R}^2 \mid y = 0$ and $x \ge 0$ or $x \in \mathbb{N}$ and $y \ge 0\}$ with the shortest path metric and fix $v_0 = (0, 0)$. Thus, there is a geodesic ray $f_0 : [0, \infty) \to T$ such that $f_0(0) = v_0$ and a sequence of geodesic rays $f_n : [0, \infty) \to T$ such that $f_n(0) = f_0(n)$ and $T = \cup_{k=0}^{\infty} f_k([0, \infty))$. Then, (T, v_0) does not have a cut set satisfying the conditions in Proposition 4.9 and does not contain any subtree quasi-isometric to the Cantor tree. We can check that (T, v) is p-parabolic directly using Theorem 2.2.*

Let us denote $v_i = f_0(i)$ for every $i \in \mathbb{N}$, and $w_j^i = f_i(j)$ for every $i \in \mathbb{N}$ and every $j \in \mathbb{N} \cup \{0\}$ (notice that $w_0^i = v_i$). Let us define the map $u^n : V(T) \to \mathbb{R}$ such that

- $u_i = u^n(v_i) = \frac{n-i}{n}$ *for* $1 \le i \le n$,
- $u_i = u^n(v_i) = 0$ *for every* $i > n$,
- $u_j^i = u^n(w_j^i) = \frac{n-i-j/2^i}{n}$ *for every* $1 \le j \le 2^i(n - i)$,
- $u_j^i = u^n(w_j^i) = 0$ *for every* $2^i(n - i) < j$.

Then,

$$\begin{aligned} D_{p,T}(u^n) =& 2\sum_{i=1}^{n} \Big((u_i - u_{i-1})^p + \sum_{j=1}^{2^i(n-i)} (u_j^i - u_{j-1}^i)^p \Big) \\ =& \frac{2}{n^{p-1}} + 2\sum_{i=1}^{n} \sum_{j=1}^{2^i(n-i)} \frac{1}{2^{pi}n^p} = \frac{2}{n^{p-1}} + 2\sum_{i=1}^{n} \frac{n-i}{2^{(p-1)i}n^p} \\ <& \frac{2}{n^{p-1}} + \frac{2}{n^p} \sum_{i=1}^{n} \frac{n}{2^{(p-1)i}} < \frac{2^{p-1}}{2^{p-1}-1} \frac{2}{n^{p-1}}. \end{aligned}$$

Thus, $\text{cap}_p(v_0, T) = 0$ *and, by Theorem 2.2, T is p-parabolic.*

However, using Proposition 4.13 it is immediate to check that T is p-parabolic. It suffices to see that the set $\mathcal{C} = \{f_n(1) : n \in \mathbb{N}\}$ is a subcut set and $T \setminus \cup_{n\in\mathbb{N}} T_{f_n(1)}$ is just the geodesic ray $f_0([0, \infty))$. Then, using the minimizing map on $f_0([0, n]) \approx P_n$ (resp. on $f_n([1, n + 1]) \approx P_n$) from Lemma 4.1 and Theorem 2.2, it is trivial to prove that $f_0([0, \infty))$ (resp. $f_n([1, \infty))$) is p-parabolic.

Given two sequences of positive integers $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, with $2 \le r_n \le N$ for every $n \ge 1$ and some constant N, the *Cantor tree* $(T_{L,R}, v_0)$ is a rooted tree such that the root, v_0, has degree r_1, the vertices at distance $\ell_1 + \cdots + \ell_{n-1}$ have degree $r_n + 1$, and any other vertex has degree two.

The *Cantor tree* (T_C, v_0) is a rooted tree such that the root, v_0, has degree two and any other vertex has degree three, i.e., $(T_C, v_0) = (T_{L,R}, v_0)$ with $\ell_n = 1$ and $r_n = 2$ for every $n \ge 1$. Note that (T_C, v_0) is uniform since $R = \{r_n\}_{n=1}^{\infty}$ is a bounded sequence.

Let $\mathcal{F}$ be the set of maps $u : V(T_{L,R}) \to \mathbb{R}$ such that $u(v_0) = 1$ and u has finite support. Let $\mathcal{G}$ be the set of maps in $\mathcal{F}$ such that $u(w) = u(w')$ if $d(v_0, w) = d(v_0, w')$.

Lemma 4.15. $\inf_{u\in\mathcal{F}} D_{p,T_{L,R}}(u) = \inf_{u'\in\mathcal{G}} D_{p,T_{L,R}}(u')$.

Proof. It suffices to show that given any $u \in \mathcal{F}$ we can build a function $u' \in \mathcal{G}$ such that $D_{p,T_{L,R}}(u') \le D_{p,T_{L,R}}(u)$. Given any $u \in \mathcal{F}$, since u has finite support, there is some N such that $\operatorname{supp}(u) \subset B_{T_{L,R}}(v_0, N)$. Let $v_1, \dots, v_{r_1}$ be the vertices in $S_{T_{L,R}}(v_0, \ell_1)$. Define $T^*_{v_j} = [v_0, v_j] \cup T_{v_j}$ for every $1 \le j \le r_1$. Let us assume, relabeling if necessary, that

$$\sum_{vw \in E(T^*_{v_1})} \big|u(v) - u(w)\big|^p \le \sum_{vw \in E(T^*_{v_j})} \big|u(v) - u(w)\big|^p, \qquad \text{for every } 1 < j \le r_1.$$

Let $I_j : T^*_{v_j} \to T^*_{v_1}$ be the natural isometry with $I_j(v_j) = v_1$ and let us define $u^1 : V(T_{L,R}) \to \mathbb{R}$ such that $u^1(v_0) = 1$, $u^1(v) = u(v)$ for every $v \in T^*_{v_1}$ and $u^1(v) = u(I_j(v))$ for every $v \in T^*_{v_j}$ and $1 < j \le \ell_1$. Then, it is immediate to check that $D_{p,T_{L,R}}(u^1) \le D_{p,T_{L,R}}(u)$, $u(v) = u(w)$ if $d(v_0, v) = d(v_0, w) \le \ell_1$ and its support is finite since $\operatorname{supp}(u^1) \subset B_{T_{L,R}}(v_0, N)$. Thus, $u^1 \in \mathcal{F}$. Suppose we have defined $u^k \in \mathcal{F}$ such that $u^k(v) = u^k(w)$ if $d(v_0, v) = d(v_0, w) \le \ell_1 + \dots + \ell_k$, $\operatorname{supp}(u^k) \subset B_{T_{L,R}}(v_0, N)$ and $D_{p,T_{L,R}}(u^k) \le D_{p,T_{L,R}}(u)$. Let $S_{T_{L,R}}(v_0, \ell_1 + \dots + \ell_{k+1}) = \{w_1, \dots, w_M\}$, with $M = r_1 \cdots r_{k+1}$. For each $1 \le j \le M$, we denote by w^*_j the unique vertex in $V(T_{L,R}) \setminus V(T_{w_j})$ at distance ℓ_{k+1} from w_j. Note that it is possible to have $w^*_i = w^*_j$ with $i \ne j$. Define $T^*_{w_j} = [w^*_j, w_j] \cup T_{w_j}$ for every $1 \le j \le M$.

Let us assume, relabeling if necessary, that

$$\sum_{vw \in E(T^*_{w_1})} \big|u^k(v) - u^k(w)\big|^p \le \sum_{vw \in E(T^*_{w_j})} \big|u^k(v) - u^k(w)\big|^p, \qquad \text{for every } 1 < j \le M.$$

Let $i_j : T^*_{w_j} \to T^*_{w_1}$ be the natural isometry with $i_j(w_j) = w_1$ and let us define $u^{k+1} : V(T_{L,R}) \to \mathbb{R}$ such that $u^{k+1}(v) = u^k(v)$ for every $v \in \bar{B}_{T_{L,R}}(v_0, \ell_1 + \dots + \ell_k)$, $u^{k+1}(v) = u^k(v)$ for every $v \in T^*_{w_1}$ and $u^{k+1}(v) = u^k(i_j(v))$ for every $v \in T^*_{w_j}$ and $1 < j \le M$. Then, it is immediate to check that $D_{p,T_{L,R}}(u^{k+1}) \le D_{p,T_{L,R}}(u^k) \le D_{p,T}(u)$, $u^{k+1}(v) = u^{k+1}(w)$ if $d(v_0, v) = d(v_0, w) \le \ell_1 + \dots + \ell_{k+1}$, and its support is finite since $\operatorname{supp}(u^{k+1}) \subset B_{T_{L,R}}(v_0, N)$. Thus, $u^{k+1} \in \mathcal{F}$. It is readily seen by construction that the map $u' : V(T_{L,R}) \to \mathbb{R}$ defined as $u'(v) = u^k(v)$ if $\ell_1 + \dots + \ell_{k-1} < d(v_0, v) \le \ell_1 + \dots + \ell_k$ satisfies that $u' \in \mathcal{G}$ and $D_{p,T_{L,R}}(u') \le D_{p,T_{L,R}}(u)$, finishing the proof. □

Let $\mathcal{H}_1$ be the set of maps u in $\mathcal{G}$ such that $u(v) \ge u(w)$ for every v, w such that $d(v_0, v) \le d(v_0, w)$.

Lemma 4.16.

$$\inf_{u \in \mathcal{G}} D_{p,T_{L,R}}(u) = \inf_{u' \in \mathcal{H}_1} D_{p,T_{L,R}}(u').$$

Proof. Let $u \in \mathcal{G}$ and let us denote $u(v) = u_i$ if $d(v_0, v) = i$. Let us define $u' : V(T_{L,R}) \to [0, 1]$ inductively as follows. Let $i_1 = \min_{i \ge 1}\{u_i \le u_0\}$ and define $u'_i := u_0$ for each $0 \le i < i_1$ and $u'_{i_1} := u_{i_1}$. Now, for each k, let $i_k = \min_{i > i_{k-1}}\{u_i \le u_{i_{k-1}}\}$ and define $u'_i := u_{i_{k-1}}$ for each $i_{k-1} < i < i_k$ and $u'_{i_k} := u_{i_k}$. It is straightforward that $D_{p,T_{L,R}}(u') \le D_{p,T_{L,R}}(u)$, finishing the proof. □

Let $L_k := \ell_1 + \dots + \ell_k$ for every k (and $L_0 = 0$). Let $\mathcal{H}_2$ be the set of maps u in $\mathcal{H}_1$ such that $u_{L_{k-1}+i} = u_{L_{k-1}} - \frac{u_{L_{k-1}} - u_{L_k}}{\ell_k}\, i$ for every k and $1 \le i \le \ell_k$.

Lemma 4.17.

$$\inf_{u \in \mathcal{H}_1} D_{p,T_{L,R}}(u) = \inf_{u' \in \mathcal{H}_2} D_{p,T_{L,R}}(u').$$

Proof. As we saw above, by Hölder inequality, for every k

$$\sum_{i=1}^{\ell_k} |u_{L_{k-1}+i-1} - u_{L_{k-1}+i}|^p \ge \frac{|u_{L_{k-1}} - u_{L_k}|^p}{\ell_k^{p-1}}.$$

Moreover, this minimum is attained if $u'_{L_{k-1}+i} = u_{L_{k-1}} - \frac{u_{L_{k-1}} - u_{L_k}}{\ell_k}\, i$ since $|u_{L_{k-1}+i-1} - u_{L_{k-1}+i}| = \frac{u_{L_{k-1}} - u_{L_k}}{\ell_k}$ for every $1 \le i \le \ell_k$. □

Let $\mathcal{H}_2^m$ be the set of maps u in $\mathcal{H}_2$ such that $\operatorname{supp}(u) = B(v_0, L_m)$.

Lemma 4.18.

$$\inf_{u\in\mathcal{H}_2} D_{p,T_{L,R}}(u) = \inf_{u'\in\mathcal{H}_2^m,\, m\in\mathbb{N}} D_{p,T_{L,R}}(u').$$

Proof. It is trivial to check that for every function $u \in \mathcal{H}_2$ with $\bar{B}(v_0, L_{m-1}) \subset \operatorname{supp}(u) \subset B(v_0, L_m)$ there is a function u' with $\operatorname{supp}(u') = B(v_0, L_m)$ and such that $D_{p,T_{L,R}}(u') \le D_{p,T_{L,R}}(u)$. □

Notice that for every $u \in \mathcal{H}_2^m$,

$$\begin{aligned}\frac{1}{2} D_{p,T_{L,R}}(u) &= \frac{r_1|u_{L_0} - u_{L_1}|^p}{\ell_1^{p-1}} + \frac{r_1 r_2|u_{L_1} - u_{L_2}|^p}{\ell_2^{p-1}} + \cdots + \frac{r_1\cdots r_m|u_{L_{m-1}} - u_{L_m}|^p}{\ell_m^{p-1}} \\ &= \sum_{k=1}^{m} \frac{r_1\cdots r_k|u_{L_{k-1}} - u_{L_k}|^p}{\ell_k^{p-1}}.\end{aligned} \tag{4.5}$$

Lemma 4.19. *$T_{L,R}$ is p-parabolic if and only if*

$$\inf_{u\in\mathcal{H}_2^m,\, m\in\mathbb{N}} \sum_{k=1}^{m} \frac{r_1\cdots r_k|u_{L_{k-1}} - u_{L_k}|^p}{\ell_k^{p-1}} = 0.$$

Proof. This result follows immediately from Theorem 2.2, lemmas 4.15, 4.16, 4.17, 4.18, and (4.5). □

Lemma 4.20. *Let $1 < p < \infty$, $a_1, \dots, a_n > 0$, $f : [0,\infty)^n \to \mathbb{R}$ given by*

$$f(y_1, \dots, y_n) = a_1 y_1^p + \cdots + a_n y_n^p,$$

and

$$D = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_k \ge 0 \text{ for } 1 \le k \le n \text{ and } y_1 + \cdots + y_n = 1\}.$$

Then the minimum of f on D is attained at the point y^0 with

$$y_k^0 = a_k^{-1/(p-1)} \Big(\sum_{j=1}^{n} a_j^{-1/(p-1)}\Big)^{-1}$$

for $1 \le k \le n$, and

$$\min_{y\in D} f(y) = f(y^0) = \Big(\sum_{k=1}^{n} a_k^{-1/(p-1)}\Big)^{-(p-1)}.$$

Proof. Since $1 < p < \infty$ and $a_1, \dots, a_n > 0$, the function f is strictly convex. We consider the convex set D with its induced topology of $\mathbb{R}^n$. Define $F : D \to \mathbb{R}$ as the restriction of the function f to the set D, and let $g(y_1, \dots, y_n) = y_1 + \cdots + y_n$. Note that F is a strictly convex function on D, and so, if there exists a critical point y^0 in the interior of D, the function F attains at y^0 its minimum value on D.

The method of Lagrange multipliers gives that for each critical point y of F in the interior of D, there exists $\lambda \in \mathbb{R}$ such that $\nabla f(y) = \lambda \nabla g(y)$, i.e.,

$$a_j p\, y_j^{p-1} = \lambda$$

for every $1 \le j \le n$. Thus,

$$a_j p\, y_j^{p-1} = a_1 p\, y_1^{p-1}, \qquad y_j = \Big(\frac{a_1}{a_j}\Big)^{1/(p-1)} y_1,$$

for every $1 \le j \le n$. Since $g(y) = 1$, we obtain

$$1 = \sum_{j=1}^{n} \Big(\frac{a_1}{a_j}\Big)^{1/(p-1)} y_1, \qquad a_1^{1/(p-1)} y_1 = \Big(\sum_{j=1}^{n} a_j^{-1/(p-1)}\Big)^{-1},$$

$$y_k = \Big(\frac{a_1}{a_k}\Big)^{1/(p-1)} y_1 = a_k^{-1/(p-1)} \Big(\sum_{j=1}^{n} a_j^{-1/(p-1)}\Big)^{-1}.$$

Thus, $y = y^0$ and

$$\min_{y\in D} f(y) = f(y^0) = \sum_{k=1}^n a_k\, a_k^{-p/(p-1)} \Big(\sum_{j=1}^n a_j^{-1/(p-1)} \Big)^{-p}$$
$$= \sum_{k=1}^n a_k^{-1/(p-1)} \Big(\sum_{j=1}^n a_j^{-1/(p-1)} \Big)^{-p} = \Big(\sum_{k=1}^n a_k^{-1/(p-1)} \Big)^{-(p-1)} .$$

□

Theorem 4.21. *Given $1 < p < \infty$ and sequences $L = \{\ell_n\}_{n=1}^\infty$ and $R = \{r_n\}_{n=1}^\infty$, the Cantor tree $(T_{L,R}, v_0)$ is p-parabolic if and only if*

$$\sum_{k=1}^\infty \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} = \infty .$$

Proof. By Lemma 4.19, it suffices to prove that

$$\inf_{u\in\mathcal{H}_2^m,\, m\in\mathbb{N}} \sum_{k=1}^m \frac{r_1 \cdots r_k |u_{L_{k-1}} - u_{L_k}|^p}{\ell_k^{p-1}} = 0. \tag{4.6}$$

Fix $m \in \mathbb{N}$ and $u \in \mathcal{H}_2^m$. Since $u_{L_{k-1}} - u_{L_k} \ge 0$ for $1 \le k \le m$, and

$$\sum_{k=1}^m \big(u_{L_{k-1}} - u_{L_k}\big) = u_0 - u_{L_m} = 1,$$

Lemma 4.19, with $a_k = r_1 \cdots r_k \ell_k^{-(p-1)}$, gives

$$\inf_{u\in\mathcal{H}_2^m} \sum_{k=1}^m \frac{r_1 \cdots r_k |u_{L_{k-1}} - u_{L_k}|^p}{\ell_k^{p-1}} = \Big(\sum_{k=1}^m \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} \Big)^{-(p-1)} .$$

Thus, (4.6) holds if and only if

$$\sum_{k=1}^\infty \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} = \infty .$$

□

Corollary 4.22. *Consider $1 < p < \infty$ and the sequences $L = \{\ell_n\}_{n=1}^\infty$ and $R = \{r_n\}_{n=1}^\infty$. If there is k_0 such that $\ell_{k+1}^{p-1} \ge r_{k+1}\ell_k^{p-1}$ for every $k \ge k_0$, then $(T_{L,R}, v_0)$ is p-parabolic.*

Corollary 4.23. *Let us consider two sequences $L = \{\ell_n\}_{n=1}^\infty$ and $R = \{r_n\}_{n=1}^\infty$. If $L = \{\ell_n\}_{n=1}^\infty$ is bounded, then the Cantor tree $(T_{L,R}, v_0)$ is non-p-parabolic for every $1 < p < \infty$.*

Corollary 4.24. *The Cantor tree (T_C, v_0) is non-p-parabolic for every $1 < p < \infty$.*

Corollary 4.25. *If a uniform tree T contains a subtree T' quasi-isometric to the Cantor tree (T_C, v_0), then T non-p-parabolic for each $1 < p < \infty$.*

A tree is said to be *bushy* if each vertex is a uniformly bounded distance from a vertex with degree at least three.

Proposition 4.26. *If a uniform tree T contains a uniform bushy tree without vertices of degree one, then T is non-p-parabolic for each $1 < p < \infty$.*

Proof. Suppose T contains an infinite uniform bushy tree T' without vertices of degree 1, and let us fix a root v_0 in T'. By hypothesis, there exists a constant k such that each vertex is at distance less than k from a vertex with degree at least three. Then, v_0 has two descendants, v_1, v_2 in T' with degree at least three such that $d(v_0, v_i) < k$ for $i = 1, 2$. Again, v_i has two descendants v_1^i, v_2^i with degree at least three such that $d(v_i, v_j^i) < k$ for $1 \le i, j \le 2$. Repeating the argument we obtain a tree $T'' \subset T'$ such that T'' is homeomorphic to the Cantor tree T_C. Moreover, the distance between every vertex in T'' and its two descendants is less than k. Therefore, the natural homeomorphism between T'' and T_C is a $(k, 0)$-quasi-isometry and by Corollary 4.25, T is non-p-parabolic for each $1 < p < \infty$. □

Recall that a graph is said *parabolic* if it is 2-parabolic. The following theorem is a particular case of Theorem 4.21. We include it since, in this case, a different proof is obtained by using electric networks.

Theorem 4.27. *Let us consider the sequences* $L = \{\ell_n\}_{n=1}^{\infty}$ *and* $R = \{r_n\}_{n=1}^{\infty}$. *Then the Cantor tree* $(T_{L,R}, v_0)$ *is parabolic if and only if*

$$\sum_{k=1}^{\infty} \frac{\ell_k}{r_1 \cdots r_k} = \infty.$$

Proof. It is a well-known fact that there is a strong relationship between discrete potential theory and electric networks (see, e.g., [7]). A graph G can be seen as an electric network where each edge of length ℓ has electric resistance ℓ. In this way, we can see $(T_{L,R}, v_0)$ as a rooted tree such that v_0 is the starting point of r_1 edges with length ℓ_1 ending in r_1 vertices of the first generation, for each vertex of the first generation there are r_2 edges with length ℓ_2 ending in r_2 vertices of the second generation, and for each vertex of the k-th generation there are r_{k+1} edges with length ℓ_{k+1} ending in r_{k+1} vertices of the $(k+1)$-th generation for each $k \ge 2$.

Let $R_{EFF}(T^n)$ be the effective resistance of the electric network associated to the subgraph T^n of $T_{L,R}$ induced by the set of vertices of generations up to n, between the root and the vertices of the n-th generation. By [7, Sections 3.4, 5.5 and 5.6], the random walk in $T_{L,R}$ is recurrent (i.e., $T_{L,R}$ is parabolic) if and only if $R_{EFF}(T_{L,R}) := \lim_{n\to\infty} R_{EFF}(T^n) = \infty$. Hence, it suffices to show $R_{EFF}(T_{L,R}) = \infty$. Let G^n be the graph obtained from T^n by identifying the vertices of each generation in a single vertex (G^n has $n+1$ vertices and $r_1 \cdots r_k$ edges of length ℓ_k joining the vertex of the $(k-1)$-th generation with the vertex of the k-th generation, for each $k \le n$). By symmetry, $R_{EFF}(T^n) = R_{EFF}(G^n)$. Since the $r_1 \cdots r_k$ edges of length ℓ_k joining the vertex of the $(k-1)$-th generation with the vertex of the k-th generation are parallel resistances, they can be replaced by a single edge between these vertices with resistance $\ell_k (r_1 \cdots r_k)^{-1}$. Since these new resistances are series resistances,

$$R_{EFF}(T^n) = R_{EFF}(G^n) = \sum_{k=1}^{n} \frac{\ell_k}{r_1 \cdots r_k}\,, \qquad R_{EFF}(T_{L,R}) = \sum_{k=1}^{\infty} \frac{\ell_k}{r_1 \cdots r_k}$$

(see [7, Section 3.4]). Thus, $T_{L,R}$ is parabolic if and only if

$$R_{EFF}(T_{L,R}) = \sum_{k=1}^{\infty} \frac{\ell_k}{r_1 \cdots r_k} = \infty.$$

□

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