

GLOBAL INVERSION OF NONSMOOTH MAPPINGS USING PSEUDO-JACOBIAN MATRICES

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ABSTRACT. We study the global inversion of a continuous nonsmooth mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which may be non-locally Lipschitz. To this end, we use the notion of pseudo-Jacobian map associated to f , introduced by Jeyakumar and Luc, and we consider a related index of regularity for f . We obtain a characterization of global inversion in terms of its index of regularity. Furthermore, we prove that the Hadamard integral condition has a natural counterpart in this setting, providing a sufficient condition for global invertibility.

1. INTRODUCTION

Global inversion of mappings is a relevant issue in nonlinear analysis. In the case that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 -smooth mapping with everywhere nonzero Jacobian, the problem of global invertibility of f was first considered by Hadamard [4], who obtained a sufficient condition in terms of the growth of the norm of the inverse of the derivative $df(x)$, by means of his celebrated *integral condition*. Namely, he proved that f is a global diffeomorphism provided

$$\int_0^\infty \inf_{\|x\|=t} \|df(x)^{-1}\|^{-1} dt = \infty.$$

This result has found a wide number of extensions and variants in different contexts. In this way, an extension of Hadamard integral condition to the case of local diffeomorphisms between Banach spaces was given by Plastock in [12]. The global invertibility of local diffeomorphisms between Banach-Finsler manifolds was studied by Rabier in [15]. In [10], John obtained a variant of Hadamard integral condition for a local homeomorphism f between Banach spaces, using the lower scalar Dini derivative of f . More recently, the problem of global inversion of a local homeomorphism between metric spaces has been considered in [3] and [2], where some versions of Hadamard integral condition are obtained in terms of an analogue of lower scalar Dini derivative.

In the nonsmooth setting, Pourciau studied in [13] and [14] the global inversion of locally Lipschitz mappings $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ using the Clarke generalized differential, and he also obtained a variant of Hadamard integral condition in this context. The global invertibility of locally Lipschitz mappings between (finite-dimensional) Finsler

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manifolds, using an analogue of the Clarke generalized differential, has been studied in [6].

Our main purpose in this paper is to use techniques of nonsmooth analysis in order to study the global inversion of a continuous mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which may be non-locally Lipschitz, and for which the Clarke generalized differential is not necessarily defined. To this end, we will use the concept of *pseudo-Jacobian* (also called *approximate Jacobian*) of the mapping f , introduced by Jeyakumar and Luc in [7] and studied later in [8, 11, 9]. In particular, we will see that the Hadamard integral condition has a natural counterpart in this setting, and provides a sufficient condition for global invertibility.

The paper is organized as follows. In Section 2 we include some basic definitions and preliminary results which will be useful throughout the paper. In Section 3, for a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we introduce the regularity index of f related to a pseudo-Jacobian mapping Jf , and we see its connection with the lower scalar Dini derivative of f . In Section 4 we prove our main results, Theorem 4.2 gives a characterization of global inversion of a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in terms of the regularity index of f . In Corollary 4.7, we obtain an integral estimation of the domain of invertibility of f around a point. Finally, Corollary 4.8 gives a version of Hadamard integral condition using the regularity index, which provides a sufficient condition for global inversion.

2. PRELIMINARIES

Let us begin by recalling the definition of pseudo-Jacobian associated to a continuous mapping, as well as some basic properties. This notion was introduced in [7], where it was called approximate Jacobian. We refer to the book [9] for an extensive information about this concept and its applications. The notation we use is standard. The n -dimensional Euclidean space is denoted by \mathbb{R}^n , and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of all linear mappings from \mathbb{R}^n to \mathbb{R}^m , which can be regarded as $m \times n$ -matrices. The space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is endowed with its usual matrix norm. The open unit ball of \mathbb{R}^n and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ are denoted, respectively, by \mathbf{B}_n and $\mathbf{B}_{m \times n}$.

Definition 2.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous mapping. We say that a nonempty closed set of $m \times n$ matrices $Jf(x) \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a pseudo-Jacobian of f at x if for every $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ one has*

$$(vf)^+(x; u) \leq \sup_{M \in Jf(x)} \langle v, Mu \rangle,$$

where vf is the real function $(vf)(x) = \sum_{i=1}^m v_i f_i(x)$ for every $x \in \mathbb{R}^n$, (v_i being components of v and f_i being components of f), and $(vf)^+(x; u)$ is the upper Dini directional derivative of the function vf at x in the direction u , that is

$$(vf)^+(x; u) := \limsup_{t \rightarrow 0^+} \frac{(vf)(x + tu) - (vf)(x)}{t}.$$

Each element of $Jf(x)$ is called a pseudo-Jacobian matrix of f at x . If for every $x \in \mathbb{R}^n$ we have that $Jf(x)$ is a pseudo-Jacobian of f at x , we say that the set-valued map $Jf : \mathbb{R}^n \rightrightarrows \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ given by $Jf : x \mapsto Jf(x)$ is a pseudo-Jacobian map for f .

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and Gâteaux-differentiable at x with derivative $df(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then, of course, $Jf(x) := \{df(x)\}$ is a pseudo-Jacobian of f at x . Moreover, as can be seen in [9, Section 1.3], many generalized derivatives frequently used in nonsmooth analysis are examples of pseudo-Jacobians. In particular, this is the case of Clarke generalized Jacobian of a locally Lipschitz mapping. Nevertheless, there are examples of locally Lipschitz functions whose Clarke generalized Jacobian strictly contains a pseudo-Jacobian. Consider, for example, the following illustrative case taken from [9]. A pseudo-Jacobian of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (|x|, |y|)$ at $(0, 0)$ is the set

$$Jf(0, 0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Whereas the Clarke generalized Jacobian is given by

$$\partial^C f(0, 0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in [-1, 1] \right\},$$

which is also a pseudo-Jacobian of f at $(0, 0)$ and contains $Jf(0, 0)$, in fact, $\partial^C f(0, 0)$ is the convex hull of $Jf(0, 0)$. However, this is not always the case (see [9, Example 1.3.3]).

We will need the following Mean Value Theorem concerning pseudo-Jacobians, whose proof can be seen in [9, Theorem 2.2.2].

Theorem 2.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous mapping and $u, v \in \mathbb{R}^n$. Suppose that $Jf(x)$ is a pseudo-Jacobian of f at x for each x in the segment $[u, v]$. Then*

$$f(u) - f(v) \in \overline{\text{co}}(Jf([u, v])(u - v)).$$

Throughout the paper, we will be interested in pseudo-Jacobians with nice stability properties, in the sense that they are upper semicontinuous.

Definition 2.3. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. We say that F is upper semicontinuous (usc) at x if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$F(x + \delta \mathbf{B}_n) \subseteq F(x) + \varepsilon \mathbf{B}_m.$$

It is well-known that the Clarke generalized Jacobian of a locally Lipschitz mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is usc (see [1, Proposition 2.6.2]). Recall that a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is said to be *locally bounded* at x if there exist a neighborhood U of x and a constant $\alpha > 0$ such that $\|A\| \leq \alpha$ for each $A \in F(U)$. Clearly, if F is usc at x and $F(x)$ is bounded, then F is locally bounded at x . On the other hand, it is proved in [9, Proposition 2.2.8] that a continuous mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has a locally bounded pseudo-Jacobian map at x if, and only if, f is locally Lipschitz at x .

Finally, let us recall the definition of the lower and upper scalar Dini derivatives in the setting of continuous mappings between Banach spaces. These quantities were used by John [10] to study local homeomorphisms between Banach spaces, in order to obtain a version of Hadamard integral condition. Later on, this kind of scalar derivatives have been considered in [3] and [2] in a metric space setting.

Definition 2.4. Let E and F be Banach spaces, $U \subset E$ an open set and $f : U \rightarrow F$ a continuous mapping. The lower and upper scalar derivatives of f at a point $x \in U$ are defined respectively as

$$D_x^- f = \liminf_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}, \quad D_x^+ f = \limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}$$

where $y \in U$ and $y \neq x$.

We will also need the following Mean Value inequality given in [3, Proposition 3.9]. Firstly, recall that if $p : [a, b] \rightarrow E$ is a continuous path in a Banach space E , the *length* of p is defined by

$$\ell(p) = \sup \sum_{i=0}^{n-1} \|p(t_{i+1}) - p(t_i)\|,$$

where the supremum is taking over all partitions $a = t_0 \leq t_1 \leq \dots \leq t_n = b$. The path p is said to be *rectifiable* if $\ell(p) < \infty$.

Proposition 2.5. Let E and F be Banach spaces, $U \subset E$ an open set and $f : U \rightarrow F$ a continuous mapping. Suppose that $q : [a, b] \rightarrow U$ is a continuous path such that $p = f \circ q : [a, b] \rightarrow F$ is rectifiable and $q(a) \neq q(b)$. Then there exists $\tau \in [a, b]$ such that

$$\ell(p) \geq D_{q(\tau)}^- f \cdot \|q(b) - q(a)\|.$$

3. REGULARITY INDEX AND LOWER SCALAR DERIVATIVE

In this section we study the equi-invertibility of pseudo-Jacobian matrices, and how it is related to the lower scalar derivative. Let us recall that, according to Pourciau [14], the co-norm of a matrix $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is defined as

$$\|A\| = \inf_{\|u\|=1} \|Au\|.$$

It is clear that the matrix A is invertible if, and only if, $\|A\| > 0$. A subset of matrices $\mathcal{A} \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is said to be *equi-invertible* provided

$$\inf\{\|A\| : A \in \mathcal{A}\} > 0.$$

The concept of regularity for pseudo-Jacobian maps is defined as follows.

Definition 3.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping, and let $Jf : \mathbb{R}^n \rightrightarrows \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a pseudo-Jacobian map of f . The associated regularity index $\alpha_{Jf}(x)$ of f at $x \in \mathbb{R}^n$ is defined by

$$\alpha_{Jf}(x) := \inf\{\|A\| : A \in \text{co}(Jf(x))\}.$$

We say that f is *Jf-regular* at x if $\alpha_{Jf}(x) > 0$. When f is Jf-regular at x for every $x \in \mathbb{R}^n$, we say that f is *Jf-regular*.

The connection of the regularity index with the original Hadamard integral condition can be seen as follows. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -smooth mapping with everywhere nonzero Jacobian and consider the natural pseudo-Jacobian of Jf of f given by $Jf(x) := \{df(x)\}$. Then it is easy to see that

$$\alpha_{Jf}(x) = \|df(x)^{-1}\|^{-1}.$$

In order to analyze the behavior of the regularity index, we shall use the following notation:

Notation 3.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let $Jf : \mathbb{R}^n \rightrightarrows \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a pseudo-Jacobian map of f . For each $\beta > 0$, we denote:

$$\alpha_{Jf}(x, \beta) := \inf\{\|A\| : A \in \text{co}(Jf(x + \beta\mathbf{B}_n))\}.$$

We will need the following simple, but useful, lemmas.

Lemma 3.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and Jf a pseudo-Jacobian map of f . If Jf is usc at a point $x \in \mathbb{R}^n$, then*

$$\lim_{\beta \rightarrow 0^+} \alpha_{Jf}(x, \beta) = \sup_{\beta > 0} \alpha_{Jf}(x, \beta) = \alpha_{Jf}(x).$$

Proof. First of all, notice that $\alpha_{Jf}(x) \geq \alpha_{Jf}(x, \beta_1) \geq \alpha_{Jf}(x, \beta_2)$ whenever $0 < \beta_1 < \beta_2$, so the first equality is clear. Now, since Jf is usc at x , given $\varepsilon > 0$ there is some $\beta > 0$ such that

$$Jf(x + \beta\mathbf{B}_n) \subset Jf(x) + \varepsilon\mathbf{B}_{n \times n}.$$

Then,

$$\text{co}(Jf(x + \beta\mathbf{B}_n)) \subset \text{co}(Jf(x) + \varepsilon\mathbf{B}_{n \times n}) \subset \text{co}(Jf(x)) + \varepsilon\mathbf{B}_{n \times n}.$$

Therefore, for each $A \in \text{co}(Jf(x + \beta\mathbf{B}_n))$ there exist $\tilde{A} \in \text{co}(Jf(x))$ and $\tilde{B} \in \mathbf{B}_{n \times n}$ such that $A = \tilde{A} + \varepsilon\tilde{B}$. As a consequence, for every $u \in \mathbb{R}^n$ with $\|u\| = 1$, we have that

$$\|Au\| \geq \|\tilde{A}u\| - \varepsilon\|u\| \geq \alpha_{Jf}(x) - \varepsilon,$$

and hence $\|A\| \geq \alpha_{Jf}(x) - \varepsilon$. Then, it follows that

$$\alpha_{Jf}(x) \geq \alpha_{Jf}(x, \beta) \geq \alpha_{Jf}(x) - \varepsilon,$$

for all $\varepsilon > 0$, and the second equality holds. \square

Lemma 3.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . If there are $x \in \mathbb{R}^n$ and $\beta > 0$ such that $\alpha_{Jf}(x, \beta) > 0$, then*

$$\|f(x + h) - f(x)\| \geq \alpha_{Jf}(x, \beta) \cdot \|h\| \quad \text{for all } \|h\| < \beta.$$

Proof. Choose $0 < \|h\| < \beta$. By the Mean Value Theorem (Theorem 2.2) we have that, for any fixed $0 < \varepsilon \leq \alpha_{Jf}(x, \beta)$,

$$\begin{aligned} f(x + h) - f(x) &\in \overline{\text{co}}(Jf([x, x + h])(h)) \subseteq \overline{\text{co}}(Jf(x + \beta\mathbf{B}_n)(h)) \\ &\subseteq \text{co}(Jf(x + \beta\mathbf{B}_n)(h)) + \varepsilon\|h\|\mathbf{B}_n = \text{co}(Jf(x + \beta\mathbf{B}_n))(h) + \varepsilon\|h\|\mathbf{B}_n. \end{aligned}$$

Thus, there exist $A \in \text{co}(Jf(x + \beta\mathbf{B}_n))$ and $v \in \mathbf{B}_n$ such that $f(x + h) - f(x) = Ah + \varepsilon\|h\|v$. Then

$$\|f(x + h) - f(x)\| \geq \|Ah\| - \varepsilon\|h\| \geq (\alpha_{Jf}(x, \beta) - \varepsilon)\|h\|.$$

As a consequence, we obtain that $\|f(x + h) - f(x)\| \geq \alpha_{Jf}(x, \beta)\|h\|$. \square

In the next result, which is a direct consequence of Lemmas 3.3 and 3.4 above, we compare the lower scalar derivative of a continuous mapping f to the regularity index of a pseudo-Jacobian of f . This relationship will be one of the keys to obtain our global inversion results.

Theorem 3.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . If Jf is usc at $x \in \mathbb{R}^n$ and f is Jf -regular at x , then*

$$D_x^- f \geq \alpha_{Jf}(x).$$

As we have mentioned before, in the case of a continuous but non-locally Lipschitz mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we have that the pseudo-Jacobian map can be, in general, unbounded. In order to deal with this problem a useful tool is the so-called the recession cone, which is defined as follows. Consider a subset $\mathcal{A} \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$; the *recession cone* (or *asymptotic cone*) of \mathcal{A} is defined by

$$\mathcal{A}_\infty := \left\{ \lim_{j \rightarrow \infty} t_j A_j : A_j \in \mathcal{A}, t_j \downarrow 0 \right\}.$$

We refer to [9] for further information about recession cones. Suppose now that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and that $Jf(x)$ is a pseudo-Jacobian of f at x . The set $Jf(x)_\infty$ denotes the recession cone of $Jf(x)$. The elements of $Jf(x)_\infty$ are called *recession matrices* of $Jf(x)$. Recall that from [9, Proposition 3.1.6] we have that f is Jf -regular whenever the set $\overline{\text{co}}(Jf(x)) \cup \text{co}(Jf(x)_\infty \setminus \{0\})$ is invertible. Thus, we can obtain the following direct consequence.

Corollary 3.6. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . If Jf is usc at $x \in \mathbb{R}^n$ and each element of the set $\overline{\text{co}}(Jf(x)) \cup \text{co}(Jf(x)_\infty \setminus \{0\})$ is invertible, then*

$$D_x^- f \geq \alpha_{Jf}(x) > 0.$$

4. GLOBAL INVERSION

First of all, we consider the problem of local inversion. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let $x_0 \in \mathbb{R}^n$ be given. We say that f admits locally an inverse at x_0 if there exist neighborhoods U of x_0 and V of $f(x_0)$, and a continuous mapping $g : V \rightarrow \mathbb{R}^n$ such that $g(f(x)) = x$ and $f(g(y)) = y$ for every $x \in U$ and $y \in V$. Taking into account Lemmas 3.3 and 3.4, the following result follows from [11, Corollary 5.2]:

Theorem 4.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . If Jf is usc at $x_0 \in \mathbb{R}^n$ and f is Jf -regular at x_0 , then f admits locally an inverse at x_0 , which is Lipschitz at $f(x_0)$.*

Now, let us give a characterization of global inversion of a continuous mapping, in terms of the regularity index of a pseudo-Jacobian map.

Theorem 4.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . Suppose that Jf is usc and f is Jf -regular on \mathbb{R}^n . The following conditions are equivalent:*

- (a) $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a global homeomorphism.
- (b) For each compact subset $K \subset \mathbb{R}^n$ there exists $\alpha_K > 0$ such that $\alpha_{Jf}(x) > \alpha_K$ for every $x \in f^{-1}(K)$.

Proof. (a) \Rightarrow (b) Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism, and let $K \subset \mathbb{R}^n$ be a compact set. Then $f^{-1}(K)$ is also compact. Now consider

$$\alpha_K = \inf \{ \alpha_{Jf}(x) : x \in f^{-1}(K) \}.$$

We claim that $\alpha_K > 0$. Indeed, otherwise we can find a sequence (x_j) in $f^{-1}(K)$ such that $\alpha_{Jf}(x_j)$ converges to zero. By compactness, we can assume that (x_j) is convergent to some point x , and we know that $\alpha_{Jf}(x) > 0$. By Lemma 3.3, there is some $\beta > 0$ such that $\alpha_{Jf}(x, \beta) > \frac{1}{2}\alpha_{Jf}(x) > 0$. But x_j belongs to the ball $x + \beta\mathbf{B}_n$ for j large enough, which is a contradiction.

(b) \Rightarrow (a) By Theorem 4.1 we know that f is a local homeomorphism. Thus, according to [12, Theorem 1.2], it is sufficient to prove that f satisfies the following limiting condition (L):

(L) For every line segment $p : [0, 1] \rightarrow \mathbb{R}^n$, given by $p(t) = (1-t)y_0 + ty_1$ for some $y_0, y_1 \in \mathbb{R}^n$, for every $0 < b \leq 1$ and for every continuous path $q : [0, b) \rightarrow \mathbb{R}^n$ satisfying that $f(q(t)) = p(t)$ for every $t \in [0, b)$, there exists a sequence (t_j) in $[0, b)$ convergent to b and such that the sequence $\{q(t_j)\}$ is convergent in \mathbb{R}^n .

In order to verify condition (L), consider a line segment $p : [0, 1] \rightarrow \mathbb{R}^n$ given by $p(t) = (1-t)y_0 + ty_1$ for some $y_0, y_1 \in \mathbb{R}^n$, consider some $0 < b \leq 1$, and let $q : [0, b) \rightarrow \mathbb{R}^n$ be a continuous path satisfying that $f(q(t)) = p(t)$ for every $t \in [0, b)$. The set $K := p([0, 1])$ is compact in \mathbb{R}^n and $q([0, b)) \subset f^{-1}(K)$, so, by hypothesis, there exists $\alpha_K > 0$ such that $\alpha_{Jf}(x) > \alpha_K > 0$ for every $x \in q([0, b))$. Using Theorem 3.5 we deduce that

$$D_x^- f \geq \alpha_{Jf}(x) > \alpha_K > 0,$$

for every $x \in q([0, b))$. Now by Proposition 2.5 we have that, for every $s, t \in [0, b)$ with $s < t$:

$$\ell(p|_{[s, t]}) \geq \inf\{D_x^- f : x \in q([s, t])\} \cdot \|q(s) - q(t)\|.$$

Since $\ell(p|_{[s, t]}) = |s - t| \cdot \|y_0 - y_1\|$, we obtain that

$$|s - t| \cdot \|y_0 - y_1\| \geq \alpha_K \|q(s) - q(t)\|.$$

Now consider any sequence (t_j) in $[0, b)$ convergent to b . The above inequality gives that

$$\|q(t_i) - q(t_j)\| \leq \frac{1}{\alpha_K} |t_i - t_j| \cdot \|y_0 - y_1\|,$$

for every i, j . This shows that the sequence $\{q(t_j)\}$ is Cauchy, and therefore convergent in \mathbb{R}^n . \square

As a consequence of the previous result, we obtain at once the following result:

Corollary 4.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a pseudo-Jacobian map of f . Suppose that Jf is usc and there exists some $\alpha > 0$ such that $\alpha_{Jf}(x) \geq \alpha$ for every $x \in \mathbb{R}^n$. Then $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a global homeomorphism.*

It is worth noticing that by [9, Proposition 3.1.6] the Jf -regularity of a mapping f is weaker than the condition that each element of the set $\overline{\text{co}}(Jf(x)) \cup \text{co}(Jf(x)_\infty \setminus \{0\})$ is invertible. In the following example, we prove that it is, in fact, a strictly weaker condition. Thus, Theorem 4.1 is more general than the one given in [8].

Example 4.4. Consider the mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y) = (x - y, x + 3y^{1/3}).$$

It is not difficult to check that f has the following pseudo-Jacobian:

$$Jf(x, y) = \left\{ \begin{pmatrix} 1 & -1 \\ 1 & y^{-2/3} \end{pmatrix} \right\} \quad \text{for } y \neq 0, \quad \text{and} \quad Jf(x, 0) = \left\{ \begin{pmatrix} 1 & -1 \\ 1 & \beta \end{pmatrix} : \beta \geq 0 \right\}.$$

It is clear that Jf is usc. On the other hand, for every $(x, y) \in \mathbb{R}^2$, every matrix of $Jf(x, y)$ is invertible. Moreover, for each $\beta \geq 0$ we have that

$$A_\beta^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & \beta \end{pmatrix}^{-1} = \frac{1}{\beta+1} \begin{pmatrix} \beta & 1 \\ -1 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} \|A_\beta^{-1}\| &= \sup_{\|(u,v)\| \leq 1} \left\| \frac{1}{\beta+1} \begin{pmatrix} \beta & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\| \leq \sqrt{2} \sup_{\|(u,v)\|_\infty \leq 1} \left\| \frac{1}{\beta+1} \begin{pmatrix} \beta & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\infty \\ &\leq \sqrt{2} \max \left\{ 1, \frac{2}{\beta+1} \right\} \leq 2\sqrt{2}. \end{aligned}$$

Thus, $\|A_\beta\| = \|A_\beta^{-1}\|^{-1} \geq \frac{1}{2\sqrt{2}}$ for all $\beta \geq 0$. Therefore, $\alpha_{Jf}(x, y) \geq \frac{1}{2\sqrt{2}}$ for every $(x, y) \in \mathbb{R}^2$, so f is Jf -regular. Thus, as a consequence of Corollary 4.3 we obtain that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a global homeomorphism. Nevertheless, the recession cone of the set $co(Jf(0, 0))$ is given by

$$(Jf(0, 0))_\infty = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} : \beta \geq 0 \right\},$$

whose matrices are not invertible.

In the last part of the paper we are going to obtain variants of the Hadamard integral condition in terms of the lower scalar Dini derivative and of the regularity index. First, we need to recall the following concept from [10]. Suppose that E and F are Banach spaces, $D \subset E$ is an open set and $f : D \rightarrow F$ is a local homeomorphism. Then for each $x \in D$ there is a neighborhood S_x of $f(x)$ in F such that f has a local inverse f_x^{-1} in S_x . Moreover, as it was proved in [10], S_x can be chosen as to be the so-called *maximal star* with vertex $f(x)$, which is defined as the set of all points $z \in F$ for which the line segment $[f(x), z]$ can be lifted to a path γ in D starting at x , and such that f maps homeomorphically the image $Im(\gamma)$ onto the segment $[f(x), z]$. The following properties are also obtained in [10]:

- (i) (Star-shaped) S_x is an open neighborhood of $f(x)$, which is star-shaped with vertex $f(x)$, that is, for each $z \in S_x$ the whole segment $[f(x), z]$ is also contained in S_x .
- (ii) (Maximality) S_x is maximal in the sense that for every sequence $(z_n) \subset S_x$ that lies on the same ray from $f(x)$ and converges to a point $z \notin S_x$, the sequence $\{f_x^{-1}(z_n)\}$ does not converge in D .
- (iii) (Monodromy) For each path q contained in D connecting x with some point y , and such that $f(Im(q))$ is contained in S_x , we have that $f_x^{-1}(f(y)) = y$.

Our next result is a nonsmooth version of [5, Theorem 2.1], and provides a slight improvement of the condition in [10, Theorem IIA].

Theorem 4.5. *Let E and F be Banach spaces, let $\mathbf{B}(x_0, \rho)$ be an open ball of E and let $f : \mathbf{B}(x_0, \rho) \rightarrow F$ be a local homeomorphism. Suppose that*

$$(1) \quad \inf_{\|x-x_0\| \leq r} (D_x^- f) > 0 \quad \text{for } 0 \leq r < \rho,$$

and there exists a Riemann-integrable function $\eta : [0, \rho) \rightarrow (0, \infty)$ such that

$$(2) \quad 0 < \eta(t) \leq \inf_{\|x-x_0\|=t} D_x^- f \quad \text{for } 0 \leq t < \rho.$$

Then the maximal star S_{x_0} contains the open ball $\mathbf{B}(f(x_0), \sigma)$, where

$$(3) \quad \sigma = \int_0^\rho \eta(t) dt.$$

Proof. By considering the mapping $g(x) = f(x+x_0) - f(x_0)$ we may assume, without loss of generality, that $x_0 = 0$ and $f(x_0) = 0$. Suppose that the maximal star S_0 does not contain the ball $\mathbf{B}(0, \sigma)$. Then there exists a vector $w \in \mathbf{B}(0, \sigma)$ such that $w \notin S_0$. Consider $u = \frac{w}{\|w\|}$, and define

$$R := \sup\{\lambda \geq 0 : \lambda u \in S_0\}.$$

Then

$$R = \|Ru\| \leq \|w\| < \sigma.$$

Now, let

$$r := \sup\{\|f^{-1}(\lambda u)\| : 0 \leq \lambda < R\}.$$

By construction, it is clear that $r \leq \rho$. We are going to see that, in fact, $r = \rho$. Indeed, if $r < \rho$ we have that $m = \inf_{\|x\| \leq r} (D_x^- f) > 0$ by condition (1). Now, fixed $\lambda_1, \lambda_2 \in [0, R)$ with $\lambda_1 < \lambda_2$ consider the path $p(t) = tu$ defined for $t \in [\lambda_1, \lambda_2]$ and set $q = f^{-1} \circ p$. Applying Proposition 2.5, we obtain that, for some $\tau \in [\lambda_1, \lambda_2]$,

$$|\lambda_2 - \lambda_1| \geq (D_{f^{-1}(\tau u)}^- f) \|f^{-1}(\lambda_1 u) - f^{-1}(\lambda_2 u)\| \geq m \|f^{-1}(\lambda_1 u) - f^{-1}(\lambda_2 u)\|.$$

This implies that

$$\lim_{\lambda \rightarrow R} f^{-1}(\lambda u)$$

exists in the closed ball $\overline{\mathbf{B}}(x_0, r)$, and this contradicts the maximality of S_0 . Hence, $r = \rho$.

Now, since the map $\lambda \mapsto \|f^{-1}(\lambda u)\|$ is continuous, it assumes all values between 0 and ρ . Then, for any sequence of values $0 = t_0 < t_1 < \dots < t_n < \rho$ there exist $\lambda_1, \dots, \lambda_n \in [0, R)$ such that $t_i = \|f^{-1}(\lambda_i u)\|$ for $i = 1, \dots, n$. For each fixed $i \in \{1, \dots, n-1\}$, suppose that $\lambda_i < \lambda_{i+1}$, and applying [5, Lemma 2.1] we can find $\lambda'_i, \lambda'_{i+1} \in [\lambda_i, \lambda_{i+1}]$ with $\lambda'_i < \lambda'_{i+1}$ such that

$$t_i = \|f^{-1}(\lambda_i u)\| = \|f^{-1}(\lambda'_i u)\| \leq \|f^{-1}(\lambda u)\| \leq \|f^{-1}(\lambda'_{i+1} u)\| = \|f^{-1}(\lambda_{i+1} u)\| = t_{i+1}$$

for every $\lambda \in [\lambda'_i, \lambda'_{i+1}]$. Using again Proposition 2.5, there exists $\tau_i \in [\lambda'_i, \lambda'_{i+1}]$ such that

$$(\lambda_{i+1} - \lambda_i) \geq (\lambda'_{i+1} - \lambda'_i) \geq (D_{f^{-1}(\tau_i u)}^- f) \|f^{-1}(\lambda'_{i+1} u) - f^{-1}(\lambda'_i u)\|.$$

Hence,

$$\begin{aligned} t_{i+1} - t_i &= \|f^{-1}(\lambda'_{i+1}u)\| - \|f^{-1}(\lambda'_i u)\| \\ &\leq \|f^{-1}(\lambda'_{i+1}u) - f^{-1}(\lambda'_i u)\| \\ &\leq (D_{f^{-1}(\tau_i u)}^- f)^{-1}(\lambda_{i+1} - \lambda_i). \end{aligned}$$

Since $\|f^{-1}(\tau_i u)\| \in [t_i, t_{i+1}]$ for each $i \in \{1, \dots, n-1\}$, we deduce from the condition (2) that

$$\sum_{i=0}^{n-1} \eta(\|f^{-1}(\tau_i u)\|)(t_{i+1} - t_i) \leq \sum_{i=0}^{n-1} (D_{f^{-1}(\tau_i u)}^- f)(t_{i+1} - t_i) \leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) = \lambda_n < R.$$

As a consequence, we have that

$$\sigma = \int_0^\rho \eta(t) dt \leq R,$$

which is a contradiction \square

As a consequence of Theorem 4.5 above, we obtain a sufficient condition for global inversion by means of an integral condition in terms of the lower scalar Dini derivative.

Corollary 4.6. *Let E and F be Banach spaces and $f : E \rightarrow F$ a local homeomorphism such that*

$$(4) \quad \inf_{\|x\| \leq r} (D_x^- f) > 0 \quad \text{for } 0 \leq r < \infty.$$

Suppose that there exists a Riemann-integrable function $\eta : [0, \infty) \rightarrow (0, \infty)$ such that

$$(5) \quad \int_0^\infty \eta(t) dt = \infty \quad \text{and} \quad 0 < \eta(t) \leq \inf_{\|x\|=t} D_x^- f \quad \text{for } 0 \leq t < \infty.$$

Then f is a global homeomorphism from E onto F . Moreover, for each $x \in E$,

$$(6) \quad \|f(x) - f(0)\| \geq \int_0^{\|x\|} \eta(t) dt.$$

Proof. By (5) and Theorem 4.5 we have that every open ball centered at $f(0)$ in F is contained in the maximal star S_0 . Therefore S_0 is the whole space F , and f maps E onto F . On the other hand, from the monodromy property (iii) of the maximal star we obtain that f is one to one. Thus, f is a global homeomorphism between E and F .

Now, we will prove (6). Let $x \in E$ nonzero, consider $\rho = \|x\| > 0$ and $\sigma = \int_0^\rho \eta(t) dt$, and suppose that $\|f(x) - f(0)\| < \sigma$. Applying Theorem 4.5 to the restriction of f to the open ball $\mathbf{B}(0, \rho)$, we obtain that $f(x)$ belongs to the corresponding maximal star S_0 , and then, by the monodromy property (iii), we have that $x = f^{-1}(f(x))$ belongs to the open ball $\mathbf{B}(0, \rho)$, which is a contradiction. This completes the proof. \square

In the next result we give an estimate for the domain of invertibility of a continuous mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is a local homeomorphism around a point x , in terms of the regularity index of f associated to a pseudo-Jacobian map.

Corollary 4.7. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a usc pseudo-Jacobian map of f . Let $x_0 \in E$ and $\rho > 0$, and suppose that there exists a Riemann-integrable function $\eta : [0, \rho) \rightarrow (0, \infty)$ such that*

$$0 < \eta(t) \leq \inf_{\|x-x_0\|=t} \alpha_{Jf}(x) \quad \text{for } 0 \leq t < \rho.$$

Then $f(x_0 + \rho \mathbf{B}_n) \supset f(x_0) + \sigma \mathbf{B}_n$ and f admits a local inverse defined in the open ball $f(x_0) + \sigma \mathbf{B}_n$, where

$$\sigma = \int_0^\rho \eta(t) dt.$$

Proof. By Theorem 4.1 and Theorem 3.5, for applying Theorem 4.5 it only remains to show that condition (1) holds. Fix $0 < r < \rho$. For each x in the closed ball $x_0 + r \overline{\mathbf{B}}_n$, by Lemma 3.3 there exists $\beta_x > 0$ such that $\alpha_{Jf}(x, \beta_x) > 0$. The desired result follows by the compactness of $x_0 + r \overline{\mathbf{B}}_n$ and using again Theorem 3.5. \square

Finally, we obtain a version of the Hadamard integral condition for a continuous mapping, in terms of the regularity index associated to a pseudo-Jacobian map.

Corollary 4.8. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and let Jf be a usc pseudo-Jacobian map of f . Suppose that there exists a Riemann-integrable function $\eta : [0, \infty) \rightarrow (0, \infty)$ such that*

$$\int_0^\infty \eta(t) dt = \infty \quad \text{and} \quad 0 < \eta(t) \leq \inf_{\|x\|=t} \alpha_{Jf}(x) \quad \text{for } 0 \leq t < \infty.$$

Then f is a global homeomorphism. Moreover, for each $x \in E$,

$$\|f(x) - f(0)\| \geq \int_0^{\|x\|} \eta(t) dt.$$

Proof. The result can be derived from Corollary 4.6 following the lines of Corollary 4.7 above. \square

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