arXiv:1306.2165v1 [math.CV] 10 Jun 2013

ON THE GENUS OF MEROMORPHIC FUNCTIONS

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Abstract. We define the class of Left Located Divisor (LLD) meromorphic functions and their vertical order $m_0(f)$ and their convergence exponent $d(f)$. When $m_0(f) \leq d(f)$ we prove that their Weierstrass genus is minimal. This explains the phenomena that many classical functions have minimal Weierstrass genus, for example Dirichlet series, the Γ-function, and trigonometric functions.

1. LLD meromorphic functions. Meromorphic functions f on $\mathbb{C}$, of the variable $s \in \mathbb{C}$, considered in this article are assumed to be of finite order $o = o(f)$. We recall that the order $o(f)$ is defined as

$$o(f) = \limsup_{R \to +\infty} \frac{\log \log ||f||_{C^0(B(0,R))}}{\log R} \ .$$

We study in this article Dirichlet series, and more generally the class of meromorphic of finite order with Left Located Divisor (LLD), which we call LLD meromorphic functions:

Definition 1. ***(LLD meromorphic functions)*** *A LLD meromorphic function is a function f of finite order and left located divisor*

$$\sigma_1 = \sup_{\rho \in f^{-1}(\{0,\infty\})} \Re \rho < +\infty \ .$$

The properties that we establish in this article are invariant by a real translation. Thus considering $g(s) = f(s + \sigma_1)$ instead of f we will assume that $\sigma_1 = 0$.

Examples of LLD meromorphic functions are Dirichlet series, that we normalize in this article such that $f(s) \to 1$ when $\Re s \to +\infty$. A Dirichlet series is of the form

$$f(s) = 1 + \sum_{n \geq 1} a_n \ e^{-\lambda_n s} \ , \tag{1}$$

with $a_n \in \mathbb{C}$ and

$$0 < \lambda_1 < \lambda_2 < \dots$$

2010 *Mathematics Subject Classification.* Primary: 30D30. Secondary: 30B50, 30D15.
Key words and phrases. Dirichlet series, Poisson-Newton formula, Hadamard factorization.
Partially supported through Spanish MICINN grant MTM2010-17389.

with (λ_n) a discrete set, that is either finite or $\lambda_n \to +\infty$, and such that we have a half plane of absolute convergence, i.e., for some $\bar{\sigma} \in \mathbb{R}$ we have

$$\sum_{n\geq 1} |a_n| \; e^{-\lambda_n \bar{\sigma}} < +\infty \, .$$

We refer to [6] for classical background on Dirichlet series.

2. **Convergence exponent.** We denote by (ρ) the set of zeros and poles of f, and the integer n_ρ is the multiplicity of ρ (positive for zeros and negative for poles, with the convention $n_\rho = 0$ if ρ is neither a zero nor a pole).

Definition 2. ***(Convergence exponent)*** *The convergence exponent of f is the minimum integer $d = d(f) \geq 0$ such that*

$$\sum_{\rho\neq 0} |n_\rho| \, |\rho|^{-d} < +\infty \; .$$

We have $d = 0$ if and only if f has a finite divisor, i.e. it is a rational function multiplied by the exponential of a polynomial, otherwise $d \geq 1$.

It is classical that the convergence exponent satisfies $d \leq [o] + 1$ (see [1]), thus it is finite for functions of finite order, but there is no upper bound of the order by the convergence exponent since we can always multiply by $\exp P$, where P is a polynomial, increasing the order without changing the divisor, hence keeping the same convergence exponent.

3. **Genus.** When f is a meromorphic function of finite order we have the Hadamard factorization of f (see [1], p.208)

$$f(s) = s^{n_0} e^{Q_f(s)} \prod_{\rho\neq 0} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \; ,$$

where

$$E_n(z) = (1-z) e^{z + \frac{1}{2}z^2 + \ldots + \frac{1}{n}z^n} \; ,$$

and Q_f is a polynomial, the Weierstrass polynomial, uniquely defined up to the addition of an integer multiple of $2\pi i$.

The *discrepancy polynomial* of the meromorphic function f is

$$P_f = -Q_f' \; .$$

We define the Hadamard part of f as

$$f_H(s) = s^{n_0} \prod_{\rho\neq 0} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \; . \tag{2}$$

Note that $f'/f = f_H'/f_H - P_f$.

The degree $g_W = \deg Q_f$ is the Weierstrass genus. The genus of f is defined as the integer

$$g = g(f) = \max(g_W(f), g_H(f)) \ .$$

where $g_H(f) = d(f) - 1$ is the Hadamard genus, which is the degree of the polynomials in the exponential of the factors $E_n(z)$. From the definition we have $d \leq g + 1$, and $g \leq o \leq g + 1$ (see [1], p.209).

We set the following useful definition:

Definition 3. ***(Hadamard and Weierstrass type)*** *A meromorphic function f is of Hadamard type when $g(f) = g_H(f) = d(f) - 1 \geq g_W(f)$. It is of Weierstrass type when $g(f) = g_W(f) > g_H(f)$.*

Many classical functions are of Hadamard type. One of the purposes of the article is to explain why this holds.

4. Vertical order. For a LLD meromorphic function we look at the growth of its logarithmic derivative on the right half plane. This growth is always polynomial (proof in Appendix 1).

Proposition 4. *The logarithmic derivative of a LLD meromorphic function has polynomial growth on a right half plane, i.e. for $\sigma_2 > \max(0, \sigma_1)$, and for $\Re s > \sigma_2$,*

$$\left| \frac{f'(s)}{f(s)} \right| \leq C_0 |s|^{\max(d, g_W - 1)} \ ,$$

more precisely we have

$$\left| \frac{f_H'(s)}{f_H(s)} \right| \leq C_0 |s|^d$$

Remark 5. The exponent d is best possible in the last estimate (see the example constructed in Appendix 2).

We define the *vertical order* as follows:

Definition 6. ***(Vertical order)*** *The vertical order of a meromorphic function f with left located divisor is the minimal integer $m_0 = m_0(f) \geq 0$ such that for $c > \sigma_1$, $c \neq 0$,*

$$|c + it|^{-m_0} \frac{f'}{f}(c + it) \in L^1(\mathbb{R}) \ .$$

Lemma 7. *This definition does not depend on the choice of c.*

This Lemma is proved in Appendix 3.

From the estimate in Proposition 4 we have that $m_0(f_H) \leq d + 2$. But we can do better:

Proposition 8. *We have* $m_0(f_H) \leq d+1$.

For a Dirichlet series normalized as in (1) we have that $f(s) \to 1$ and $f'(s) \sim -\lambda_1 a_1 e^{-\lambda_1 s}$ uniformly with $\Re s \to +\infty$, thus $m_0(f) = 2$.

In this article we say that a distribution has order n if n is the minimal integer such that it is the n-th derivative of a continuous function (there is no consensus in the classical literature on the definition of order of a distribution, for example see [8] and [10]). Proposition 4 implies that the inverse Laplace transform $\mathcal{L}^{-1}(f'/f)$ is a distribution of finite order. This is because we have an explicit formula for the inverse Laplace transform. We recall (see [10]) that

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} F(c+iu)e^{(c+iu)t}\ du\ ,$$

if the integral is convergent, and

$$\mathcal{L}^{-1}(F)(t) = \mathcal{L}_c^{-1}(F)(t) = \frac{1}{2\pi}\frac{D^n}{Dt^n}\int_{\mathbb{R}} \frac{F(c+iu)}{(c+iu)^n}e^{(c+iu)t}\ du\ ,$$

in general (the derivative is taken in distributional sense) which holds for some n when F is holomorphic with polynomial growth on $\{\Re s > \sigma_2\}$ and it is independent of $c > \sigma_2 > \sigma_1$.

A closely related integer to the vertical order is the *distributional vertical order*.

Definition 9. ***(Distributional vertical order)*** *The distributional vertical order of a LLD meromorphic function f is the minimal integer $m \geq 0$ such that the inverse Laplace transform*

$$\mathcal{L}^{-1}(f'/f)$$

is a distribution of order m.

It is clear that:

Proposition 10. *We have $m(f) \leq m_0(f)$.*

5. Main results.

Theorem 11. *For a LLD meromorphic function f we have that if $m(f) \neq g_W(f)+1$ then f is of Hadamard type, i.e. $g_W(f) \leq g_H(f) = g(f)$.*

Moreover, any Dirichlet series f is of Hadamard type, i.e. $g_W(f) \leq g_H(f) = g(f)$ unconditionally.

Corollary 12. *If a LLD meromorphic function f is of Weierstrass type then $m(f) = g_W(f)+1$.*

The next corollary gives an analytic criterium to determine if a meromorphic function is of Hadamard type.

Corollary 13. *If $m_0(f) \leq d(f)$ then f is of Hadamard type.*

The same argument used in the proof of the main theorem gives:

Theorem 14. *Let f be a non-constant Dirichlet series. Then we have*

$$d(f) \geq 2 \ ,$$

and

$$o(f) \geq 1 \ .$$

Before proving these results we need to introduce the Newton-Cramer distribution and Poisson-Newton formula.

6. Newton-Cramer Distribution. In [7] we associate to the divisor $\mathrm{div}(f) = \sum n_\rho \, \rho$ its Newton-Cramer distribution, which is given by the series

$$W(f) = \sum n_\rho \, e^{\rho t}$$

on $\mathbb{R}_+^*$. This sum is only converging in $\mathbb{R}_+^*$ in the distribution sense. The distribution $W(f)$ vanishes in $\mathbb{R}_-^*$, and has some structure at 0. The precise definition follows (we assume, in order to simplify, that $\rho = 0$ is not part of the divisor).

Definition 15. ***(Newton-Cramer distribution)*** *The Newton-Cramer distribution is*

$$W(f) = \frac{D^d}{Dt^d} \left(L_d(t) \right) ,$$

where L_d is the continuous function on $\mathbb{R}$ defined on $\mathbb{R}_+$ by

$$L_d(t) = \sum_{\rho \neq 0} \frac{n_\rho}{\rho^d} (e^{\rho t} - 1) \boldsymbol{1}_{\mathbb{R}_+} \ .$$

It is easy to see that the sum converges for $t \geq 0$.

In this article, only the order of distributions plays a role, and the space of test funcitons for which the distribution belong to the dual is not so important. The distribution $W(f)$ is Laplace transformable, that is, it can be paired with e^{-st} on $\mathbb{R}_+$, on some half-plane $\Re s > \sigma_0$. Hence, the appropriate space of distributions to use is the dual of the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}$ which decay faster than $Ce^{\alpha|t|}$, for some $C > 0$, $\alpha > 0$.

The main property of the Newton-Cramer distribution that we need follows from its definition:

Proposition 16. *The Newton-Cramer distribution is the d-th derivative of a continuous function.*

7. Poisson-Newton formula. The Newton-Cramer distribution of f is linked to the inverse Laplace transform of the logarithmic derivative f'/f by the Poisson-Newton formula (see [7]):

Theorem 17. ***(Poisson-Newton formula)*** *For a LLD meromorphic function f we have on $\mathbb{R}$*

$$W(f) = \sum_{l=0}^{g_W-1} c_l \delta_0^{(l)} + \mathcal{L}^{-1}(f'/f) \ ,$$

where $P_f(s) = c_0 + c_1 s + \ldots + c_{g_W-1} s^{g_W-1} = -Q'_f(s)$ is the discrepancy polynomial.

When f is a Dirichlet function, the Laplace transform $\mathcal{L}^{-1}(f'/f)$ is purely atomic with atoms in $\mathbb{R}_+^*$. We can compute it explicitely as follows. On the half plane $\Re s > \sigma_1$, $\log f(s)$ is well defined taking the principal branch of the logarithm. Then we can define the coefficients $(b_{\mathbf{k}})$ by

$$-\log f(s) = -\log\left(1 + \sum_{n\geq 1} a_n \ e^{-\lambda_n s}\right) = \sum_{\mathbf{k}\in\Lambda} b_{\mathbf{k}} \, e^{-\langle \boldsymbol{\lambda}, \mathbf{k}\rangle s} \ , \tag{3}$$

where $\Lambda = \{\mathbf{k} = (k_n)_{n\geq 1} \,|\, k_n \in \mathbb{N}, ||\mathbf{k}|| = \sum |k_n| < \infty, ||\mathbf{k}|| \geq 1\}$, and $\langle \boldsymbol{\lambda}, \mathbf{k}\rangle = \lambda_1 k_1 + \ldots + \lambda_l k_l$, where $k_n = 0$ for $n > l$. Note that the coefficients $(b_{\mathbf{k}})$ are polynomials on the (a_n). More precisely, we have

$$b_{\mathbf{k}} = \frac{(-1)^{||\mathbf{k}||}}{||\mathbf{k}||} \frac{||\mathbf{k}||!}{\prod_j k_j!} \prod_j a_j^{k_j} \ . \tag{4}$$

Note that if the λ_n are $\mathbb{Q}$-dependent then there are repetitions in the exponents of (3).

Since $\mathcal{L}(e^{-\lambda s}) = \delta_\lambda$, we have

$$\mathcal{L}^{-1}(f'/f) = \sum_{\mathbf{k}\in\Lambda} \langle \lambda, \mathbf{k}\rangle \, b_{\mathbf{k}} \ \delta_{\langle \boldsymbol{\lambda}, \mathbf{k}\rangle} \ .$$

Note in particular that $\operatorname{supp} \mathcal{L}^{-1}(f'/f) \subset [\epsilon, +\infty[$ for some $\epsilon > 0$.

8. Proof of the main results. The proof of Theorem 11 consists on inspecting the orders of the distributions in both sides of the Poisson-Newton equation:

$$W(f) = \sum_{l=0}^{g_W-1} c_l \delta_0^{(l)} + \mathcal{L}^{-1}(f'/f) \ .$$

We will use that for two distributions U and V, if $\mathrm{ord}(U) \neq \mathrm{ord}(V)$ then

$$\mathrm{ord}(U+V) = \max(\mathrm{ord}(U), \mathrm{ord}(V)) \ .$$

The left hand side is of order $\leq d$ since $W(f)$ is the d-th derivative of a continuous function.

Observe that the Dirac δ_0 is of order 2, and $\delta_0^{(l)}$ is of order $l+2$. In particular, the first term of the right hand side in Poisson-Newton equation is of order $g_W + 1$.

The second term of the right hand side is of order $m(f)$ by definition of $m(f)$.

To prove Theorem 11 we assume first that $m < g_W + 1$. Then the order of the right hand side in Poisson-Newton formula is $g_W + 1$. Therefore $d \geq g_W + 1$ so $g = g_H \geq g_W$ and f is of Hadamard type.

We look at the second case when $m > g_W + 1$. Then the order of the right hand side is m, thus comparing with the left hand side, we get $d \geq m > g_W + 1$, therefore $g = g_H > g_W$ and f is again of Hadamard type. This proves the first statement of the main theorem.

For a Dirichlet series f the distribution $\mathcal{L}^{-1}(f'/f)$ has support away from 0, therefore looking at the local order at 0 (which is smaller or equal than the global order) of both sides of the equation we get that $d \geq g_W + 1$ unconditionally. This gives $g = g_H \geq g_W$ and f is always of Hadamard type. This ends the proof of Theorem 11.

Now Corollary 12 is a direct application of the main theorem.

For Corollary 13 we observe that $m_0(f) \leq d(f)$ gives $m(f) \leq m_0(f) \leq d(f) \leq g(f) + 1$. If the last inequality is an equality, then f is of Hadamard type and we are done. Otherwise we have $g = g_W$ and $m(f) < g_W(f) + 1$ and using the main theorem we get also that f is of Hadamard type, and $g = g_W = g_H = d - 1$.

For the proof of Theorem 14, we inspect as before the order of the distributions in the Poisson-Newton-formula. The right hand side contains Dirac distributions at the frequencies, hence it is at least a second derivative of a continuous function. In the left hand side we have $W(f)$ that is the d-th derivative of a continuous function. This gives $d \geq 2$.

Also we know that $d \leq o + 1$, hence $o \geq 1$.

9. Proof of Poisson-Newton formula. Let us prove Theorem 17. We start from the Hadamard factorization of f (assuming that $\rho = 0$ is not part of the divisor in order to simplify).

$$f(s) = e^{Q_f(s)} \prod_{\rho} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \ ,$$

We take its logarithmic derivative:

$$\begin{aligned} f'/f &= -P_f + \sum_\rho n_\rho \frac{E'_{d-1}(s/\rho)}{E_{d-1}(s/\rho)} \\ &= -P_f + \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \end{aligned} \tag{5}$$

Since for $l \geq 0$

$$\mathcal{L}(\delta_0^{(l)}) = s^l \ ,$$

the polynomial P_f is the Laplace transform

$$P_f = \mathcal{L}\left(c_0\delta_0 + c_1\delta_0' + \ldots + c_{g-1}\delta_0^{(g-1)} \right) \ .$$

It remains to prove that

$$\mathcal{L}(W(f)) = \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \ .$$

We have

$$\frac{D^d}{Dt^d}\left((e^{\rho t} - 1)\mathbf{1}_{\mathbb{R}_+} \right) = \rho^d e^{\rho t}\mathbf{1}_{\mathbb{R}_+} + \sum_{l=0}^{d} \rho^{d-1-l}\delta_0^{(l)} \ ,$$

thus for a finite set A of zeros and poles of the divisor, we have

$$\begin{aligned} W_A(f) &= \sum_{\rho \in A} n_\rho \rho^{-d} \frac{D^d}{Dt^d}\left(e^{\rho t} - 1 \right) \mathbf{1}_{\mathbb{R}_+} \\ &= \sum_{\rho \in A} n_\rho \left(e^{\rho t}\mathbf{1}_{\mathbb{R}_+} + \sum_{l=0}^{d} \rho^{-1-l}\delta_0^{(l)} \right) \ . \end{aligned}$$

Now we have

$$\mathcal{L}(e^{\rho t}\mathbf{1}_{\mathbb{R}_+}) = \frac{1}{\rho - s} \ ,$$

so

$$\mathcal{L}\left(W_A(f) \right) = \sum_{\rho \in A} n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \ ,$$

and we are done taking the inverse Laplace transform.

10. Application to trigonometric functions. We check that the sine function is of Hadamard type. For this it is enough to consider the hyperbolic sine function which is an entire function of order 1,

$$f(s) = \sinh(s) = \frac{e^s - e^{-s}}{2i} \ .$$

The zeros are, for $k \in \mathbb{Z}$,

$$\rho_k = \pi i k \ ,$$

thus sinh is a LLD entire function and $d(f) = 2$. Also we have

$$f'(s)/f(s) = \cosh(s)/\sinh(s) = \frac{1 + e^{-2s}}{1 - e^{-2s}} \to 1$$

when $\Re s \to +\infty$. Therefore $m_0(f) = 2$.

Using Corollary 13 we get

Proposition 18. *The function $f(s) = \sinh(s)$ is of Hadamard type.*

This is something that we know from its Hadamard factorisation (due to Euler)

$$\sinh(s) = s \prod_{k \in \mathbb{Z}^*} \left(1 - \frac{s}{\pi i k}\right) e^{\frac{s}{\pi i k}} \ .$$

Corollary 19. *The function $f(s) = \sin(s)$ is of Hadamard type.*

11. Application to the Γ function. We check, without computing its Hadamard factorisation, that the classical Γ function is of Hadamard type.

The Γ-function has no zeros and has simple poles at the negative integers. Thus it is a LLD meromorphic function and $d = 2$. Stirling formula indicates that we must have $m_0(\Gamma) = 2$ and we check:

Lemma 20. *For $c > 0$ we have for some constant $C_0 > 0$ and for $|u| \geq 1$*

$$\left| \frac{\Gamma'}{\Gamma}(c + iu) \right| \leq \log |u| + C_0$$

and $m_0(\Gamma) = 2$.

The classical Stirling's asymptotics holds in a right cone, but we need the estimate in a vertical line, thus we need to refine the classical estimate. We start with Binet's second formula (see [9] p.251):

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + \varphi(s) \ ,$$

where

$$\varphi(s) = 2\int_0^{+\infty} \frac{\arctan(t/s)}{e^{2\pi t}-1}\, dt \ .$$

Taking one derivative in the above formula, we get an identity for the digamma function

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \varphi'(s) \ ,$$

and

$$\varphi'(s) = -2\int_0^{+\infty} \left(\frac{s}{s^2+t^2}\right)\left(\frac{t}{e^{2\pi t}-1}\right) dt \ .$$

Since

$$\int_0^{+\infty} \frac{t}{e^{2\pi t}-1}\, dt = \frac{B_2}{4} = \frac{1}{24} \ ,$$

and if $s = c + iu$ with $c = \Re s > 0$,

$$\left|\frac{s}{s^2+t^2}\right| \leq \frac{1}{|c|} \ ,$$

we have the estimate

$$|\varphi'(s)| \leq \frac{1}{24|c|} \ ,$$

so $|\psi(s)| \leq \log|s| + C_0$, and the lemma follows.

Now we have $m_0(\Gamma) = 2 \leq d(\Gamma) = 2$ so the application of Corollary 13 gives:

Proposition 21. *The Γ function is a meromorphic function of Hadamard type.*

12. Application to the Riemann zeta function. The Riemann zeta function is a Dirichlet series,

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-s} \ ,$$

and has a meromorphic extension of order 1 to the whole complex plane. So it is a LLD meromorphic function.

We that $d(\zeta) \leq 2$ from the order 1, and $d(\zeta) \geq 2$ for the summation of the trivial zeros that lie at the even negative integers, thus $d(\zeta) = 2$.

The logarithmic derivative is bounded on vertical lines and so $m_0(\zeta) = 2$. Again, using Corollary 13 we get:

Proposition 22. *The Riemann zeta function ζ is a meromorphic function of Hadamard type.*

13. Appendix 1: Proof of propositions 4 and 8. We start by considering the analogue of (5) centered at σ_1. This is

$$f'/f = -P_f + \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{(s-\sigma_1)^l}{(\rho-\sigma_1)^{l+1}} \right)$$

We write $f'/f = -P_f + G$, where $G(s) = \sum n_\rho \, g_\rho(s)$, where

$$g_\rho(s) = \frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{(s-\sigma_1)^l}{(\rho-\sigma_1)^{l+1}} = \frac{(s-\sigma_1)^{d-1}}{(\rho-\sigma_1)^{d-1}} \frac{1}{\rho - s}$$

In order to prove Proposition 4, we need to bound $|g_\rho(s)| \le C|s-\sigma_1|^d|\rho-\sigma_1|^{-d}$, for a uniform constant C, since $\sum n_\rho \, |\rho-\sigma_1|^{-d} < \infty$. For this we need to bound uniformly

$$g_\rho(s) = \frac{\rho - \sigma_1}{(s-\sigma_1)(\rho - s)}$$

on the half-plane $\Re s > \sigma_2$.

If $|\sigma_1 - s| \le \frac{1}{2}|\rho - \sigma_1|$ then $|\rho - s| \ge |\rho - \sigma_1| - |\sigma_1 - s| \ge \frac{1}{2}|\rho - \sigma_1|$. So $|g_\rho(s)| \le \frac{2}{|s-\sigma_1|} \le C$, as $|s - \sigma_1| \ge \sigma_2 - \sigma_1$ is bounded below.

If $|\sigma_1 - s| \ge \frac{1}{2}|\rho - \sigma_1|$ then $|g_\rho(s)| \le \frac{2}{|\rho - s|} \le C$, as $|s - \rho| \ge \sigma_2 - \sigma_1$ is bounded below.

We prove now Proposition 8. Fix $c > \sigma_1$, and let $a = c - \sigma_1 > 0$. We need to see that $G(s)|s-\sigma_1|^{-d-1}$ is integrable, and it is enough to see that

$$\int_{L_c} \frac{|\rho - \sigma_1|}{|s-\sigma_1|^2|\rho - s|} ds \tag{6}$$

is bounded uniformly on ρ, for $L_c = c + i\mathbb{R}$.

We consider two sets:

- $A = \{s \in L_c \,|\, |\rho - \sigma_1| \le \frac{3}{2}|\rho - s|\}$. This is an infinite portion of L_c. The integral is bounded by
$$\frac{3}{2} \int_{L_c} \frac{1}{|s-\sigma_1|^2} ds < \infty.$$
- $B = \{s \in L_c \,|\, |\rho - \sigma_1| \ge \frac{3}{2}|\rho - s|\}$. This is the intersection of a disc of radius $\frac{2}{3}|\rho - \sigma_1|$ with L_c. So its length is bounded by $\frac{4}{3}|\rho - \sigma_1|$. The integral there is bounded by
$$\frac{4}{3} \max \left\{ \frac{|\rho - \sigma_1|^2}{|s-\sigma_1|^2|\rho - s|} \,|\, s \in B \right\}.$$

We have that $|\rho - s| \geq a$, so $|\rho - s|^{-1/2} \leq \frac{1}{\sqrt{a}} \leq \frac{1}{2}$, for $a \geq 4$. Then $|\rho - s| + |\rho - s|^{1/2} \leq \frac{3}{2}|\rho - s|$ and

$$|\rho - s| + |\rho - s|^{1/2} \leq |\rho - \sigma_1| \leq |\rho - s| + |s - \sigma_1|.$$

So $|\rho - s|^{1/2} \leq |s - \sigma_1|$ and

$$\frac{|\rho - \sigma_1|^2}{|s - \sigma_1|^2 |\rho - s|} \leq \frac{(|\rho - s| + |s - \sigma_1|)^2}{|s - \sigma_1|^2 |\rho - s|} \leq \frac{1}{|\rho - s|} + \frac{2}{|s - \sigma_1|} + \frac{|\rho - s|}{|s - \sigma_1|^2} \leq 1 + \frac{3}{a}.$$

This proves that (6) is uniformly bounded.

14. Appendix 2: The exponent d in Proposition 4 is best possible. We construct an example that has the sharp exponent.

We construct a meromorphic function with convergence exponent $d = 1$. More precisely, let f be an entire function with zeros at $\rho = n^2 2^n i$, $n \geq 1$, and with multiplicities $n_\rho = 2^n$. Then $\sum n_\rho |\rho|^{-1} < \infty$. The logarithmic derivative of such function is given by

$$g = \frac{f'}{f} = \sum \frac{n_\rho}{s - \rho}$$

Now let us see that it is not controlled as $|f'/f| \leq C|s|^{1-\epsilon}$ with $\epsilon > 0$. For this take $s = c + k^2 2^k i$, k a fixed integer, $c > 0$. We decompose

$$g(s) = \sum_{n=1}^{k-1} \frac{2^n}{c + (k^2 2^k - n^2 2^n) i} + \frac{2^k}{c} + \sum_{n=k+1}^{\infty} \frac{2^n}{c + (k^2 2^k - n^2 2^n) i}$$

The first term is bounded by

$$\sum_{n=1}^{k-1} \frac{2^n}{k^2 2^k - (k-1)^2 2^{k-1}} \leq \frac{2^{k-1}}{2^{k-1}(k^2 + 2k - 1)} < C_0,$$

for some universal constant. The third term is bounded by

$$\sum \frac{2^n}{n^2 2^n - k^2 2^k} \leq \sum \frac{2^n}{n^2 2^{n-1}} < C_1,$$

for another universal constant. Hence $|g(s)| \geq \frac{2^k}{c} - C_0 - C_1$. For fixed c, take k large enough. Then

$$\frac{|g(s)|}{|s|^{1-\epsilon}} \geq \frac{2^k/c - C_0 - C_1}{(c^2 + k^4 2^{k+1})^{(1-\epsilon)/2}} \approx \frac{2^\epsilon}{c\, k^{2-2\epsilon}},$$

which gets as large as we wish.

15. Appendix 3: Proof of Lemma 7. Fix $c > \sigma_1$ and let m_0 be the minimal integer such that

$$\left|(c+it)^{-m_0}\frac{f'}{f}(c+it)\right| \in L^1(\mathbb{R}) \ .$$

Consider the holomorphic function

$$g(s) = s^{-m_0}\frac{f'(s)}{f(s)}$$

on the right half-plane $\Re s \geq c$. The function $F(t) = g(c+it)$ satisfies the conditions of the Representation Theorem 6.5.4 in [2] with $\alpha = 0$, $c = 0$, and we get using the last inequality of that Theorem

$$\log|g(c'+iu)| \leq (c'-c)\pi^{-1}\int_{\mathbb{R}} \frac{\log|g(c+it)|}{(t-u)^2+(c'-c)^2}\ dt \ .$$

Now taking the exponential and using Jensen's convexity inequality we get

$$|g(c'+iu)| \leq (c'-c)\pi^{-1}\int_{\mathbb{R}} \frac{|g(c+it)|}{(t-u)^2+(c'-c)^2}\ dt \ .$$

Now Fubini gives

$$\int_{\mathbb{R}} |g(c'+iu)|du \leq (c'-c)\pi^{-1}\int_{\mathbb{R}}\left(|g(c+it)|\int_{\mathbb{R}}\frac{1}{(t-u)^2+(c'-c)^2}\ du\right)dt = \int_{\mathbb{R}}|g(c+it)|dt < \infty.$$

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