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Node centrality based on its edges importance: The Position centrality ☆

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ABSTRACT

We propose a novel family of node centralities in social networks, named *family of position centralities*, which explicitly takes into account the importance of the links to assess the centrality of the nodes that support them through the Position value (Meessen, 1988). Our proposal shares with the family of Myerson centralities (Gómez et al., 2003) that it is a game-theoretic family of measures that allows to consider the functionality of the network modelled by a symmetric cooperative game. We prove that, like the Myerson centrality measures, every Position centrality measure also satisfies essential properties expected of a centrality measure. We analyse in detail the main differences between the Myerson and the position families of centrality measures. Specifically, we study the differences regarding the connection structures that share dividends and the fairness and stability properties. Along this analysis we consider the case of hub-and-spoke clusters, a prevalent model for studying transportation networks. Finally, a characterisation of the Position Attachment centrality is given, which is the Position centrality obtained when the functionality of the network is modelled by the attachment game. Some comparisons are made with the Attachment centrality introduced by Skibski et al. (2019), which is the analogue member of the family of Myerson centralities.

1. Introduction

The importance of social networks today is undeniable, extending across various fields. In sociology, they describe societal relationships, identify key members (Borgatti, 2003) and communities (Girvan and Newman, 2002), and analyse rumour spreading (Zubiaga et al., 2018). Social networks are also used, for example, in wildfire spreading (Hajian et al., 2016), genomic analysis (Horvath, 2011; Estrada, 2006), and banking contagion (Gofman, 2017). Formally, social networks are often modelled as graphs, with nodes representing agents and edges representing relationships. This structure can include additional information such as the degree of relationships or influence. A key task in social network analysis is determining the importance of nodes or agents, often based on the notion of centrality, to understand and influence network dynamics. Defining centrality measures is challenging. Sabidussi (1966) proposed criteria that a proper centrality measure should satisfy, but his criteria "eliminate most known measures of centrality" and "they do not actually attempt to explain what centrality is" (see Borgatti and Everett (2006)). The first clear idea of centrality emerged from the position of nodes in simple networks, like a star graph where the hub is the most important node due to its high degree, its closeness to the rest of the nodes, and its essential role as intermediary. Freeman (1979) distilled centrality into three measures: Degree, Closeness, and Betweenness. Borgatti and Everett (2006) extended Freeman's work, categorising centrality measures into degree-like, closeness-like, and betweenness-like types. Apart from these contributions, other attempts have been made to propose necessary properties or even axiomatic characterisations of a centrality measure, such as Garg (2009), Landherr et al. (2010), Boldi and Vigna (2014), and Bandyopadhyay et al. (2017). More recently, Bloch et al. (2023) introduced a taxonomy based on node statistics. Despite these efforts, there is no consensus on the definitive properties of centrality measures.

Centrality analysis has also been explored through cooperative game theory. Tarkowski et al. (2017) surveyed game-theoretic approaches to measuring centrality in social networks. This perspective views the network as a joint project where agents with defined interests cooperate for mutual benefits. Game theory enriches centrality measures by considering the network's functionality. Some measures combine topological information with a coalitional game that models the benefits nodes (players) gain from their interactions (e.g., Grofman and Owen (1982), Gómez et al. (2003) and del Pozo et al. (2011)). This results in families of centrality measures, each of which is tailored to specific graphs and functionalities. This approach has garnered significant attention in social network theory and applications across various fields

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(Mason and Verwoerd, 2007; Jackson, 2008; Moretti et al., 2010; Bloch et al., 2023.

One well-known game-theoretic family of centrality measures is the family of Myerson centralities (Gómez et al., 2003). For a symmetric game (N, v) and a graph $\Gamma = (N, E)$, the Myerson centrality of a node (player) is defined as its Myerson value (Myerson, 1977), which is the Shapley value in the communication game or graph restricted-game derived from v and Γ. The Myerson centralities evaluate the impact of communication structures on the benefits that players can obtain, emphasising the role of the nodes. However, by examining the importance of edges, we gain valuable insights into the flow of information, influence and relationships within the network. Edge centrality measures allow, for example, the detection of communities (Girvan and Newman, 2002; Li et al., 2019), the analysis of computer networks as well as evolving epileptic brain networks (Bröhl and Lehnertz, 2022), the design of rumour or fire-blocking algorithms (Yan et al., 2019). Moreover, its usefulness is very evident in the analysis of some networks, such as the network of domestic flights in the US (De la Cruz Cabrera et al., 2020) or road networks.

In this paper, we introduce an alternative game-theoretic family of node centrality measures: *the family of position centralities*. Let (N, v) be symmetric game which models the functionality of the network, the Position centrality measure of $\Gamma = (N, E)$ is nothing else than the well-known Position value (Meessen, 1988) for the communication situation derived from v and Γ. While the Myerson centrality focuses on a node's role in establishing communication within coalitions, Position centrality emphasises the role of edges. The importance of each edge is assessed by its communicative strength using the Shapley value of the related *link game*. This value is then divided equally between the two players at the endpoints of the edge, who end up with the sum of all the values they received from their incident edges. Although centrality measures such as PageRank, Eigenvector, Katz, HITS and Communicability indirectly consider the importance of connections to other nodes, no centrality measure explicitly focuses on the importance of edges.

For each Position centrality measure, we demonstrate that it verifies a set of desirable properties for a centrality that were formulated and justified in Gómez et al. (2003) for the family of Myerson centralities. Those properties are defined on relevant and simple structures and even can be considered as a kind of axioms reflecting the three key measures considered by Freeman (1979). Moreover, all members of both game-theoretic centrality families behave also as complex measures that account for more than one of the key dimensions considered by Borgatti and Everett (2006) as an extension of Freeman's work. Specifically, Myerson and position centralities can be rewritten to obtain more information about their behaviour as centrality measures by decomposing their values into two parts that measure two different abilities of each member in the social network: its ability to connect and its ability to intermediate (see Gómez et al. (2003, 2004)). That is, Myerson and positional centralities are simultaneously radial and medial in terms of nodal involvement in the path structure of the graph.

However, there are three key differences between both measures, which we analyse in detail. The first difference is related to the characteristics of the connection structures they consider in the decomposition of the two centralities in a connection and an intermediation measure. The other two differences concern the behaviour of the two centralities when an existing edge is removed (or a new link is added). Along this analysis, we consider the case of hub-and-spoke clusters, a prevalent model for studying transportation networks.

In addition to the study of the general properties of the family of position centralities, we consider in particular the Position Attachment centrality. For this member of the family, we study some specific properties and characterise it axiomatically, following the work of Skibski et al. (2019) on the Myerson Attachment centrality measure, which is the Myerson centrality determined by the attachment game.

In summary, in order to situate the proposed family of position centrality measures within the existing plethora of centralities, we can give the following characteristics of its members:

1. They are game-theoretic centrality measures that are based not only on the topology of the network, but also on its purpose or functionality, modelled by a symmetric cooperative game.
2. They explicitly rely on the importance of edges in order to measure the importance of the nodes that support them.
3. They verify a set of desirable and relevant properties as a measure of centrality.
4. Like the Myerson centralities, they can be decomposed into two parts that measure two different abilities of each member in the social network: its ability to connect and its ability to intermediate. However, Position centralities take into account more ways of connecting nodes than the Myerson centralities do.
5. The most relevant property that distinguishes them from the equivalent (determined by the same symmetric game) Myerson centralities is *fairness*, which is replaced by *balanced link contributions*.
6. In general, they do not satisfy the property of *stability*, which is satisfied by every member of the family of Myerson centralities. The effect on the Position centrality of two nodes, when a new edge is added between them, depends on the effect of this addition on the remaining incident edges at each node.
7. Moreover, an axiomatic characterisation is given for one of its members: the Position Attachment centrality.

The remainder of this paper is structured as follows. In Section 2, we formally define some aspects of cooperative games, graphs and the family of Myerson centralities. In Section 3, we propose the family of position centralities and some of its properties. In Section 4, we study the key differences between both families of centralities, Myerson and position. The special case of the Position Attachment centrality measure is analysed and characterised in Section 5. Finally, the last section concludes the paper. We have also included an Appendix with the proof of some results in order to facilitate the reading of the paper.

2. Preliminaries

First, we summarise the basic elements concerning both topics we deal with, graphs and coalitional games. Then, we review the concept of Myerson value (Myerson, 1977), Position value (Meessen, 1988; Borm et al., 1992; Slikker, 2005) and Myerson centrality (Owen, 1986; Gómez et al., 2003).

2.1. Graphs

Let $\Gamma = (N, E)$, be an undirected graph without loops, where N is the set of n nodes and E is the set of edges, $E \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$. Let $\mathcal{G}^N$ denote the class of all undirected graphs without loops or parallel edges with node set N. For all $i \in N$ we denote $E_i := \{e \in E \mid i \in e\}$, $E[S] := \{e \in E \mid e \subseteq S\}$, for all $S \subseteq N$, and $N[L] = \{i \in N \mid i \in \bigcup_{e \in L} e\}$, for all $L \subseteq E$. The *subgraph induced by* $S \subseteq N$ is $\Gamma_S := (S, E[S])$, whereas the *subgraph induced by* $L \subseteq E$ is $\Gamma_L := (N[L], L)$.

A *path* from node i_1 to node i_r is a sequence of different nodes $(i_1, \ldots, i_r)$, $r \geq 2$, and edges $(e_1, e_2, \ldots, e_{r-1})$, s.t. $e_h = \{i_h, i_{h+1}\} \in E$, for all $h = 1, \ldots, r-1$.

A graph is *connected* if every pair $i, j \in N$ of its nodes is connected directly or indirectly, i.e., if there is a path in the graph from node i to node j; otherwise, the graph is *not connected*. The relation of connectivity induces a partition of the node set N into *connected components*, two nodes being in the same connected component if and only if they are connected. Let us denote the collection of connected components of the graph Γ by $\mathcal{K}(\Gamma)$. Moreover, for the sake of simplicity, $\mathcal{K}(S)$ denotes the collection of connected components of the graph Γ_S, and analogously $\mathcal{K}(L)$ denotes the collection of connected components of the graph Γ_L, for all $S \subseteq N$, and all $L \subseteq E$, respectively. For every node $i \in N$, $\mathcal{K}_i(\Gamma) \in \mathcal{K}(\Gamma)$, denotes the connected component of the graph Γ to which node i belongs. Analogously, if $i \in S$, or $E_i \cap L \neq \emptyset$, $\mathcal{K}_i(S) \in \mathcal{K}(S)$, and $\mathcal{K}_i(L) \in \mathcal{K}(L)$ denote, respectively, the connected component of the corresponding graph to which node i belongs.

Throughout the paper, we call *cutvertex* a vertex whose removal increases the number of connected components, and *cutedge* an edge whose end-nodes are both cutvertices. Let $\mathcal{D}(L)$ denote the set of cutedges of L.

2.2. Coalitional games

A *coalitional game* with transferable utility (TU-game) is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\{\emptyset\}) = 0$. For each coalition, $S \subseteq N$, $v(S)$ represents the transferable utility that S can guarantee to obtain whenever its members cooperate. Let G^N be the class of all coalitional games over the finite player set N. A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$, for all $S, T \subseteq N, S \cap T = \emptyset$, and it is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. A TU-game (N, v) is *symmetric* if for all $S \subseteq N$, $v(S) = f(s)$, where s represents the cardinality of S[1]. It is *zero-normalised* if $v(\{i\}) = 0$, for all $i \in N$. In this paper we mainly consider zero-normalised, symmetric and superadditive games. We will denote by G_0^N the subclass of these games in G^N.

For each $S \in 2^N \backslash \{\emptyset\}$, the *unanimity game* $u_S \in G^N$ is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $\{u_S \,|S \in 2^N \backslash \{\emptyset\}\}$ is a basis of the vector space G^N, and each $v \in G^N$ can be expressed as follows:

$$v = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_S(v) u_S,$$

where $\{\Delta_S(v)\}_{\emptyset \neq S \subseteq N}$ is the set of the Harsanyi dividends of v, which are given by

$$\Delta_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T), \tag{1}$$

being $s = |S|$ and $t = |T|$.

A *value* φ for TU games is a function that assigns to each game $(N, v) \in G^N$ a vector $\varphi(N, v) \in \mathbb{R}^N$, where $\varphi_i(N, v) \in \mathbb{R}$ represents the value of player i, $i \in N$. Shapley (1953) defines his value as follows:

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right), \quad i \in N. \tag{2}$$

An alternative expression for the Shapley value in terms of the Harsanyi dividends is:

$$\phi_i(N, v) = \sum_{i \in S \in 2^N \setminus \{\emptyset\}} \frac{\Delta_S(v)}{s}. \tag{3}$$

2.3. Myerson and position values

The Myerson and Position values are two well known and related allocation rules for communication situations.

A *communication situation* is a triplet (N, v, E), where (N, v) is a TU-game and (N, E) is an undirected communication graph without loops or parallel arcs. CS^N denotes the class of all communication situations with player set N. CS_0^N will denote the subclass for $(N, v) \in G_0^N$.

A *solution concept* for communication situations or an *allocation rule* on CS^N is a map $\gamma : CS^N \to \mathbb{R}^N$, $\gamma_i(N, v, E)$ representing the outcome for player i in game (N, v), given the restrictions on communication imposed by the graph (N, E). We are interested in two allocation rules that are defined in a related way and share some interesting characteristics: the *Myerson value* (Myerson, 1977) and the *Position value* (Meessen, 1988). These rules are based on the Shapley value of two different types of games derived from the one and the same communication situation: node games (Myerson, 1977) and link games (cf. Borm et al., 1992).

Definition 1. Given $(N, v, E) \in CS^N$, the *node game* (N, v_E) is defined by

$$v_E(S) = \sum_{C \in \mathcal{K}(S)} v(C), \ \forall S \subseteq N.$$

Then, the *Myerson value* $\mu(N, v, E) \in \mathbb{R}^n$ is defined as:

$$\mu_i(N, v, E) = \phi_i(N, v_E), \ \forall i \in N.$$

The definition of the Position value must be restricted to the subclass CS_0^N to guarantee that the value of the empty set in the corresponding link game is 0.

Definition 2. Given $(N, v, E) \in CS_0^N$, the *link game* (E, w^v) is defined by

$$w^v(L) = \sum_{C \in \mathcal{K}(L)} v(C), \ \forall L \subseteq E.$$

Then, the *Position value* $\pi(N, v, E) \in \mathbb{R}^n$ is defined as:

$$\pi_i(N, v, E) = \frac{1}{2} \sum_{e \in E_i} \phi_e(E, w^v), \ \forall i \in N.$$

Axiomatic characterisations

The Myerson and the Position values are characterised by means of two properties. One of them, *component efficiency*, is common to the two values. The properties that differentiate them are *fairness* and *balanced link contributions*: While the Myerson value is characterised by component efficiency and fairness (Myerson, 1977), the Position value is characterised by component efficiency and balanced link contributions (Slikker, 2005).

Definition 3. Let φ be an allocation rule on CS^N. Then, φ verifies:

- *Component efficiency:* If
$$\sum_{i \in S} \varphi_i(N, v, E) = v(S), \quad \text{for all } S \in \mathcal{K}(\Gamma), \ \forall (N, v, E) \in CS^N. \tag{4}$$
- *Fairness:* If
$$\varphi_i(N, v, E) - \varphi_i(N, v, E \setminus \{e\}) = \varphi_j(N, v, E) - \varphi_j(N, v, E \setminus \{e\}), \tag{5}$$
for all $e = \{i, j\} \in E$, $(N, v, E) \in CS^N$.
- *Balanced link contributions:* If
$$\sum_{e \in E_j} \left(\varphi_i(N, v, E) - \varphi_i(N, v, E \setminus \{e\})\right) = \sum_{e \in E_i} \left(\varphi_j(N, v, E) - \varphi_j(N, v, E \setminus \{e\})\right), \tag{6}$$
for all $i, j \in N$, and all $(N, v, E) \in CS^N$.

Component efficiency assures that all the value generated by a connected component is allocated to its nodes. Fairness implies a symmetric effect of removing and edge over its both ends. In contrast, balanced link contributions, takes into account not only the removal of the edge connecting two players, but also the impact on each player of the subsequent removal of the remaining edges of the other player. Slikker (2005) makes a comparison of the balanced link contributions property with that of *balanced contributions*, introduced in Myerson (1980):

$$\phi_i(N, v) - \phi_i(N \setminus \{j\}, v|_{N \setminus \{j\}}) = \phi_j(N, v) - \phi_j(N \setminus \{i\}, v|_{N \setminus \{i\}}),$$
for all $i \neq j \in N$,

[1] Generally, as long as it is not confusing, we represent a set's cardinal by its lowercase letter.

where $v|_{N\setminus\{j\}}$ denotes the restriction of v to $N \setminus \{j\}$. Note that the removal of a node in (N, v_E) implies the simultaneous removal of all its edges. Therefore, *balanced contributions* establishes symmetry in the effect that the simultaneous removal of all edges from one player has on another.

Finally, we recall that the Position value also admits an alternative expression in terms of the Harsanyi dividends of the link game (see Slikker (2005)), which will be very useful for proving many of the properties of the proposed family of centrality measures.

$$\pi_i(N, v, E) = \sum_{e\in E_i} \frac{1}{2} \sum_{\substack{L\subseteq E\\ e\in L}} \frac{\lambda_L(E, w^v)}{l} = \sum_{L\subseteq E} \frac{1}{2}\lambda_L(E, w^v)\frac{l_i}{l}, \text{ for all } i \in N, \quad (7)$$

where $L_i = L \cap E_i$, and being $\lambda_L(E, w^v)$ the Harsanyi dividends of the link game (E, w^v).

Slikker (2005) shows that $\lambda_L(E, w^v) = \lambda_L(w^v)$, that is, it does not depend on $E \supseteq L$. This facilitates the comparison of two communication situations that differ only in the underlying graph. Moreover, in the sequel, we will work with $\lambda_L(w^v)$ without any reference to the specific graph (N, E).

2.4. The family of myerson centrality measures

Centrality is a core issue in the analysis of social networks. Its objective is to measure the relevance or importance of nodes (or edges) in a social network. Formally, a *centrality measure* for social networks with symmetric relations is an assignation which associates to each undirected graph $\Gamma = (N, E) \in \mathcal{G}^N$, a vector $\sigma(\Gamma) \in \mathbb{R}^N$, where $\sigma_i(\Gamma) \in \mathbb{R}$ represents the *centrality* of node i, $i \in N$.

As we have mentioned before, since the pioneer contribution of Grofman and Owen in 1982, several approaches to measure the centrality of nodes in a social network that rely on cooperative TU games have been proposed in the literature (see Tarkowski et al. (2017) for a survey).

The general goal of these measures is to take into account the purpose of the social network. To be specific, our approach for a game-theoretic centrality measure is based on considering an exogenous symmetric TU game (N, v) to describe the *functionality* (i.e. the purpose) of the social network (to send messages, to develop a joint project, etc.). Thus, we shall consider a social network as a communication situation with a symmetric and superadditive game. In the following, we denote the subclass of these communication situations by CSS^N.

Then, the family of *game-theoretic centralities* we shall deal with can be defined as the family of centrality measures obtained by assigning to each communication situation $(N, v, E) \in CSS^N$ the centrality of its nodes through the use of a cooperative game solution concept.

The family of *Myerson centralities* (Owen, 1986; Gómez et al., 2003) is a family indexed by the set of symmetric and superadditive games: every $(N, v) \in G^N$ symmetric and superadditive, determines a *Myerson centrality measure* that associates to every $\Gamma = (N, E) \in \mathcal{G}^N$, the vector $\mu(N, v, E) \in \mathbb{R}^N$.

For the sake of simplicity, when we refer to Myerson centrality, we are referring to any member of this family. Where confusion may arise, the nomenclature of the Myerson centrality measure explicitly states the appropriate game.

For instance, Owen (1986) and Gómez et al. (2003) consider some symmetric games modelling interesting functionalities such as sending messages between pairs of individuals (*messages game*), to develop a joint project (*overhead game*), or the ability to form groups of two or more individuals (*conferences game*).

- *Messages game*: (N, v_m), that counts the number of messages that can be sent between pairs of individuals (in both ways), $v_m(S) = s(s-1)$, for all $\emptyset \neq S \subseteq N$.
- *Overhead game on* N: (N, v_o), that accounts for the general cost that any set of players should pay to perform an action, with $v_o(S) = -1$, for all $\emptyset \neq S \subseteq N$.
- *Attachment game*: In this paper, we will consider the attachment game (N, v_a), given by $v_a(S) = 2(s-1)$, for all $\emptyset \neq S \subseteq N$, and introduced in Skibski et al. (2019), which is proportional to the zero-normalisation of the overhead game.[2]
- *Attachment-Messages game*: (N, v_{am}); defined as a sum of an attachment and a messages game: $v_{am}(S) = s^2 + s - 2$, for all $\emptyset \neq S \subseteq N$.
- *Conferences game*: (N, v_c), that counts the number of subsets in S with cardinal greater than 1, and thus $v_c(S) = 2^s - s - 1$, for all $S \subseteq N$.

Table 1
Messages and Attachment Myerson centralities.

Nodes	v_m-Myerson centrality	v_a-Myerson centrality
1, 3	29.7 (14.16%)	2.5 (8.8%)
2	28.9 (13.72%)	2.0 (7.1%)
4, 15	10.5 (5.01%)	2.1 (7.6%)
5, 9, 10, 14	11.5 (5.49%)	1.8 (6.4%)
6, 8, 11, 13	9.1 (4.34%)	1.6 (5.8%)
7, 12	9.0 (4.28%)	1.6 (5.8%)

The following example shows the impact of the functionality of the social network -modelled by a symmetric game- on the Myerson centralities of each node.

Example 1. Consider the social network shown in Fig. 1. If the functionality of the network is to send messages, we rely on the Message game, while if the goal is for the nodes to collaborate on a common project, we should use the Attachment game.

Table 1 shows the Messages Myerson centrality and the Attachment Myerson centrality for the nodes of the network in Fig. 1. Note that Node 2, which acts as an intermediary between the two subsocieties $S_1 = \{1, 4, 5, 6, 7, 8, 9\}$ and $S_2 = \{3, 10, 11, 12, 13, 14, 15\}$, is the second most central node according to the Messages Myerson centrality. However, the second most central nodes according to the Attachment Myerson centrality are the hubs of S_1 and S_2, nodes 4 and 15 respectively. Moreover, the differences between the intermediate nodes 1, 2 and 3 and the other nodes are significantly larger according to the Messages Myerson centrality than according to the Attachment Myerson centrality.

3. The family of position centrality measures

Similarly to the family of Myerson centralities, we propose an alternative family of centrality measures using the Position value (Meessen, 1988), in which the centrality assigned to a node depends on the value of the connections or links he has with other nodes.

Like the family of Myerson centralities, this new family of centrality measures retains a game-theoretic approach that takes into account the functionality of the network. In general, however, the members of this new family do not satisfy the fairness property, meaning that adding or removing an edge can impact its end nodes unequally. This can be justified, for instance, in the case of adding an edge between nodes 2 and 15 in Example 1, taking into account that they are in a very asymmetric situation. In fact, among the classical measures of centrality, only degree centrality verifies the fairness property.

After formally defining the new family of position centrality measures, we will prove that, conveniently restricting the class of games

[2] The zero-normalisation of the overhead game, $v_o^0(S) = s - 1$, is called *pure overhead game on* N in Owen (1986).

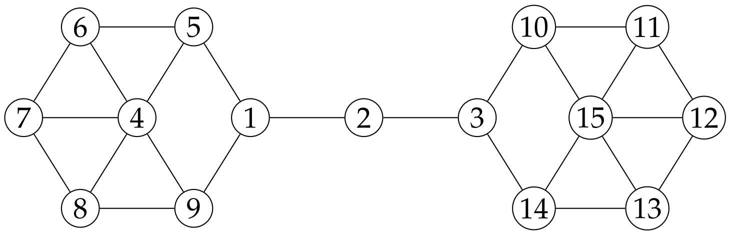


Fig. 1. Social network of Example 1.

considered, all members of the family satisfy the most desirable properties typically associated with centrality measures.

First, we will restrict to the subclass of zero-normalised, symmetric and superadditive games, in order to obtain a minimum centrality equal to zero for every isolated node.

Definition 4. The *family of position centralities* is a family indexed by the set of zero-normalised symmetric and superadditive games: every $(N,v) \in G_0^N$, determines a *Position centrality measure* that associates to each $\Gamma = (N,E) \in \mathcal{G}^N$, the vector $\pi(N,v,E) \in \mathbb{R}^N$.

For the sake of simplicity, when we refer to Position centrality, we mean any member of the family of position centralities. As before, where confusion may arise, the nomenclature of the Position centrality explicitly states the appropriate game.

3.1. Properties of the position centrality measures

Now, we prove that every member of the proposed family satisfies some of the most typical desirable properties of a centrality measure (see Gómez et al. (2003)). In particular, it attains its maximum value for the hub of a star, which is according to Freeman (1979), a defining characteristic of proper centrality measures. Some proofs of these results are based on several lemmas establishing interesting properties of the Harsanyi dividends of the link game, which are compiled in Appendix. In the sequel $f(\cdot)$ will denote the function defining the symmetric game (N,v).

First, we prove that any Position centrality is a non-negative centrality measure.

Proposition 1. *Let $(N,v,E) \in CSS_0^N$, then $\pi_i(N,v,E) \geq 0$, for all $i \in N$.*

Proof. The proof strongly depends on the superadditivity of the game. By definition

$$\pi_i(N,v,E) = \frac{1}{2}\sum_{e\in E_i} \phi_e(E,w^v),$$

where

$$\phi_e(E,w^v) = \sum_{L\subseteq E\setminus\{e\}} \frac{l!(|E|-l-1)!}{|E|!}\left(w^v(L\cup\{e\}) - w^v(L)\right),$$

with $e = \{i,j\}$.

Let $\mathcal{K}(L)$ be the set of connected components. W.l.o.g. suppose that $\mathcal{K}(L)$ has only two elements, i.e., $\mathcal{K}(L) = \{T_1, T_2\}$ and $i \in T_1$. To examine the sign of $w^v(L\cup\{e\}) - w^v(L)$, we need to consider the following three cases:

1. $j \in T_1$. Then clearly $w^v(L\cup\{e\}) - w^v(L) = 0$.
2. $j \in T_2$. Then, by superadditivity of v, we have:
$$w^v(L\cup\{e\}) - w^v(L) = v(T_1\cup T_2) - v(T_1) - v(T_2) \geq 0.$$
3. $j \notin T_1 \cup T_2$. Then, since (N,v) is superadditive and zero-normalised, it holds:
$$w^v(L\cup\{e\}) - w^v(L) = w^v(E[T_1]\cup\{e\}) - w^v(E[T_1]) = v(T_1\cup\{j\}) - v(T_1) \geq 0. \quad \square$$

Next, we show that any Position centrality measure verifies the property of *Locality*, which establishes that the centrality of a node depends only on the connected component to which it belongs and is defined as:

Definition 5. A centrality measure σ verifies *Locality* if and only if for every graph $\Gamma = (N,E) \in \mathcal{G}^N$, and every node $i \in N$ it holds

$$\sigma_i(\Gamma) = \sigma_i(\mathcal{K}_i(E), E[\mathcal{K}_i(E)]).$$

Lemma 1. *Let $(N,v,E) \in CS_0^N$. For every subset $L \subseteq E$ such that Γ_L is not connected it holds $\lambda_L(w^v) = 0$.*

Proposition 2. *Any Position centrality measure verifies Locality.*

Proof. Let $(N,v,E) \in CSS_0^N$, $\mathcal{K}(E)$ be the set of connected components in the graph (N,E) and $i \in N$.

W.l.o.g. suppose that $\mathcal{K}(E)$ has only two elements, i.e., $\mathcal{K}(E) = \{\mathcal{K}_i(E), T\}$. We are going to calculate $\pi_i(N,v,E)$ using (7):

$$\pi_i(N,v,E) = \sum_{L\subseteq E} \frac{1}{2}\lambda_L(w^v)\frac{l_i}{l}.$$

By Lemma 1, last expression can be decomposed as follows:

$$\pi_i(N,v,E) = \sum_{L\subseteq E[\mathcal{K}_i(E)]} \frac{1}{2}\lambda_L(w^v)\frac{l_i}{l} + \sum_{L\subseteq E[T]} \frac{1}{2}\lambda_L(w^v)\frac{l_i}{l}.$$

Since $L_i \subseteq E\left[\mathcal{K}_i(E)\right]$, it follows that $L_i \cap E[T] = \emptyset$. Thus,

$$\pi_i(N,v,E) = \sum_{L\subseteq E[\mathcal{K}_i(E)]} \frac{1}{2}\lambda_L(w^v)\frac{l_i}{l} = \sum_{L\subseteq E[\mathcal{K}_i(E)]} \frac{1}{2}\lambda_L(w^{v_i})\frac{l_i}{l} = \pi_i(\mathcal{K}_i(E), v_i, E[\mathcal{K}_i(E)]),$$

where v_i is the restriction of v to $\mathcal{K}_i(E)$, which completes the proof. $\square$

Proposition 3. *Any Position centrality measure verifies Component efficiency.*

Proof. Component efficiency is proven by Borm et al. (1992). $\square$

Note that Component efficiency, restricted to the class CSS_0^N, allows for comparing centralities in different graph configurations that preserve the number of connected components and their cardinalities (under the same functionality f).

The following propositions show results about the minimal and the maximal Position centrality of a node. Isolated nodes have minimal Position centrality, which equals 0, whereas maximal centrality is attained by the hub of a star.

Proposition 4. *Let $i_0 \in N$ be an isolated node in the graph (N,E), then*

$$\pi_{i_0}(N,v,E) = 0 \leq \pi_i(N,v',E'),$$

for all $i \in N$, and every $(N,v,E),(N,v',E') \in CSS_0^N$.

Proof. Let $i_0 \in N$ be an isolated node in (N, E), then it follows from (7) and the fact that $l_{i_0} = 0$ for all $L \subseteq E$, that:

$$\pi_{i_0}(N, v, E) = \sum_{L \subseteq E} \frac{1}{2} \lambda_L(w^v) \frac{l_{i_0}}{l} = 0, \text{ for all } (N, v) \in G_0^N,$$

which is the minimum Position centrality of a node, taking into account Proposition 1. □

Proposition 5. *Let $(N, v) \in G_0^N$. If $(N, E^*) \in \mathcal{G}^N$ is the star with n nodes where node 1 is the hub, then for all connected graphs $(N, E) \in \mathcal{G}^N$, and for all $i \in N$ is verified:*

$$\pi_1(N, v, E^*) \geq \pi_i(N, v, E).$$

Proof. Let $(N, E^*) \in \mathcal{G}^N$ be the star with n nodes. By symmetry of the Shapley value we have that $\phi_e(E^*, w^v) = \frac{f(n)}{n-1}$ for any e in E^*, and therefore $\pi_1(N, v, E^*) = \frac{f(n)}{2}$. Now suppose that there exists a graph (N, E) and a node $i \in N$ such that $\pi_1(N, v, E^*) < \pi_i(N, v, E)$. Then, $\pi_i(N, v, E) > \frac{f(n)}{2}$. On the other hand, by definition

$$\pi_i(N, v, E) = \frac{1}{2} \sum_{e \in E_i} \phi_e(E, w^v), \ \forall i \in N,$$

and consequently

$$\sum_{e \in E} \phi_e(E, w^v) > f(n),$$

which contradicts the efficiency property of Shapley value. □

We will now examine in Propositions 6–8 some properties related to a chain. The Position centrality of an end-node of a chain increases with the length of the chain (Proposition 6). For a given chain, the Position centrality of its nodes is maximal for the middle nodes and decreases symmetrically from the middle to the end-nodes (Proposition 7). Finally, in Proposition 8 we strengthen the result of Proposition 4 by proving that the minimum Position centrality in connected graphs is achieved by the end-nodes of a chain. Note that in this case, to ensure the above properties, the family of position centralities must be restricted to the subfamily of position centralities determined by convex games.

First, some general results about the Harsanyi dividends of the link game when the underlying graph contains no cycles are given.

Lemma 2. *Let $(N, v, E) \in CSS_0^N$. If Γ is cycle-free, then for all $\emptyset \neq L \subseteq E$ with Γ_L connected, the Harsanyi dividends of the link game can be calculated in the following way:*

$$\lambda_L(w^v) = F\left(l + 1, l - d_L\right), \tag{8}$$

where $d_L = |\mathcal{D}(L)|$, being $\mathcal{D}(L)$ the set of cutedges of L, and $F(s,r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(s-k)$, for $s, r \in \mathbb{N}$ and $s \geq r$.

Note that expression (8) only depends on l, d_L and f. Thus, all subset of edges that form a cycle-free graphs with equal number of edges and end-nodes (number of edges minus number of cutedges) have the same dividend for any fixed game. For example, the three graphs in Fig. 2 have the same Harsanyi dividend since they have 7 edges and 4 end-nodes. Then, in all cases we have:

$$\lambda_L(w^v) = F(8, 4) = f(8) - 4f(7) + 6f(6) - 4f(5) + f(4).$$

The last lemma will allow us to simplify the calculation of dividends for chains, compare the dividends of trees with l edges but with different structures, and determine the sign of the dividend under additional conditions for the game.

Lemma 3. *Let f be a real function such that $f^{(k)}(x) \geq 0$ in $[1, +\infty)$, for $k = 0, 1, \ldots, n$, then the function F verifies:*

(i) $F(s, r) \geq 0$, for all $s, r \in \mathbb{N}$ with $s > r > 0$.

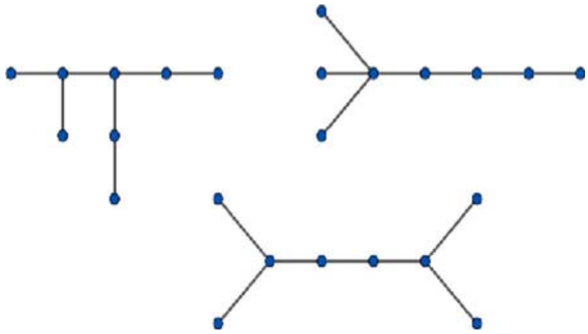


Fig. 2. Graphs with the same Harsanyi dividends.

(ii) $F(s, \cdot)$ is decreasing in the second argument for every fixed $s \in \mathbb{N}$, with $r \in \mathbb{N}, s > r$.

The next corollaries follow straightforwardly from Lemmas 2 and 3.

Corollary 1. *Let $(N, v, E) \in CSS_0^N$. If Γ is cycle-free, $f \in C^n(\mathbb{R})$ and $f^{(k)}(x) \geq 0$ in $[1, +\infty)$, for $k = 0, 1, \ldots, n$, then, for every pair of subsets $L, L' \subseteq E$, such that $l = l'$ and $d_L \leq d_{L'}$ it holds:*

$$\lambda_L(w^v) \leq \lambda_{L'}(w^v)$$

In particular, if f satisfies the required conditions of Corollary 1, the star is the structure with the lowest Harsanyi dividend, whereas the chain is the one with the highest dividend, among all the n-node trees. That is:

$$\lambda_{L^*}(w^v) \leq \lambda_L(w^v) \leq \lambda_{L^C}(w^v),$$

where L^* and L^C are two sets of edges that form a star and a chain, respectively, and being L a set of edges forming a tree with $|L^*| = |L| = |L^C|$.

With respect to the sign of the dividends, if the graph Γ is a chain with l edges, taking into account (8), the Harsanyi dividends are given by:

$$\lambda_L(w^v) = F(l+1, 2) = f(l+1) - 2f(l) + f(l-1),$$

which are always non negative if the game (N, v) is convex. However, for general cycle-free graphs, we strengthen the convexity of f in order to assure non negative dividends.

Corollary 2. *Let $(N, v, E) \in CSS_0^N$. If Γ is cycle-free, $f \in C^n(\mathbb{R})$ and $f^{(k)}(x) \geq 0$ in $[1, +\infty)$, for $k = 0, 1, \ldots, n$, then $\lambda_L(w^v) \geq 0$, for all $\emptyset \neq L \subseteq E$.*

Proposition 6. *Let $(N, v) \in G_0^N$ be convex. Let us suppose that (N_k, E_k^C) is a chain with n_k nodes numbered in the natural way. If $n_1 < n_2$, then:*

$$\pi_1\left(N_1, v, E_1^C\right) \leq \pi_1\left(N_2, v, E_2^C\right).$$

Proof. We will show that the result is true for $n_2 = n_1 + 1$. Applying (7), we have that:

$$\pi_1\left(N_k, v, E_k^C\right) = \frac{1}{2} \sum_{L \subseteq E_k^C} \frac{\lambda_L(w^v)}{l} = \frac{1}{2} \sum_{l < n_k} \frac{F(l+1, 2)}{l},$$

where the second equality follows from Lemma 1, since only the subset of edges L that form a chain have a non-zero Harsanyi dividend, and Lemma 2. Consequently,

$$\pi_1\left(N_2, v, E_2^C\right) - \pi_1\left(N_1, v, E_1^C\right) = \frac{1}{2}\left(\sum_{l < n_2} \frac{F(l+1, 2)}{l} - \sum_{l < n_1} \frac{F(l+1, 2)}{l}\right) = \frac{1}{2} \frac{\lambda_{E_2^C}(w^v)}{n_2 - 1}.$$

Finally, the result follows since $\lambda_{E_2^C}(w^v) \geq 0$ as a consequence of the convexity of v. □

Proposition 7. *If $(N, v) \in G_0^N$ is convex and (N, E^C) is a chain with n nodes numbered in the natural way, then for $1 \leq i \leq n/2$:*

$$\pi_i\left(N, v, E^C\right) \leq \pi_{i+1}\left(N, v, E^C\right).$$

Proof. Since E^C remains fixed along the proof we will write $\phi_e(E^C, w^V)$ simply as $\phi_e(w^V)$.

Let $e_i = \{i, i+1\}$, $i = 1, \ldots, n-1$. By definition:

$$\pi_i\left(N, v, E^C\right) = \frac{1}{2}\left(\phi_{e_{i-1}}(w^v) + \phi_{e_i}(w^v)\right) \text{ and}$$
$$\pi_{i+1}\left(N, v, E^C\right) = \frac{1}{2}\left(\phi_{e_i}(w^v) + \phi_{e_{i+1}}(w^v)\right).$$

Now, assuming as initial condition $\phi_{e_0}(w^v) := 0$, we prove a recursive formula for the Shapley values of the edges:

$$\phi_{e_j}(w^v) = \phi_{e_{j-1}}(w^v) + \sum_{k=j}^{n-j} \frac{F(k+1,2)}{k}, \text{ for } 1 \leq j \leq n/2. \tag{9}$$

For $j = 1$, $\phi_{e_1}(w^v)$ can be derived from expression (7), taking into account that only the subsets L such that Γ_L is a chain have a non-zero dividend. Thus,

$$\phi_{e_1}(w^v) = \sum_{e_1 \in L \subseteq E^C} \frac{\lambda_L(w^v)}{l} = \sum_{k=1}^{n-1} \frac{F(k+1,2)}{k}.$$

For $j \geq 2$, note that there is one coalition of size 1 containing e_j, two coalitions of size 2, three of size 3, and so on increasing successively up to size j; from this point, for all $s \leq n-j$ there are always exactly j coalitions of size s. Finally, for all $n-j < s \leq n-1$, the number of coalitions of size s decreases by 1 starting at $n-j+1$. Therefore:

$$\phi_{e_j}(w^v) = \sum_{k=1}^{j} k\frac{F(k+1,2)}{k} + j\sum_{k=j+1}^{n-j} \frac{F(k+1,2)}{k} + \sum_{k=n-j+1}^{n-1} (n-k)\frac{F(k+1,2)}{k}.$$

Then, it can be obtained directly that

$$\phi_{e_j}(w^v) - \phi_{e_{j-1}}(w^v) = \sum_{k=j}^{n-j} \frac{F(k+1,2)}{k} \geq 0,$$

since (N, v) is convex. Thus,

$$\pi_{i+1}\left(N, v, E^C\right) - \pi_i\left(N, v, E^C\right)$$
$$= \frac{1}{2}(\phi_{e_{i+1}}(w^v) - \phi_{e_i}(w^v) + \phi_{e_i}(w^v) - \phi_{e_{i-1}}(w^v)) \geq 0. \quad \square$$

Proposition 8. *If $(N, v) \in G_0^N$ is convex and (N, E^C) is the chain with n nodes, where node 1 is an end-node, then for all connected graphs $(N, E) \in \mathcal{G}_0^N$, and for all $i \in N$, it holds:*

$$\pi_1(N, v, E^C) \leq \pi_i(N, v, E)$$

Proof. The proof follows the same lines as the proof of Proposition 3.5 in Gómez et al. (2003). □

4. Position versus Myerson centralities

In the previous section, we showed that, when restricted to convex games, every member of the subfamily of position centralities verifies the desirable properties for a centrality measure that were formulated and justified in Gómez et al. (2003). Since the aforementioned properties were also proved for every[3] Myerson centrality, this section provides a more comprehensive examination of the key differences between the two game-theoretic centrality measures. We will compare each pair of Myerson and Position centralities, determined by the same symmetric game.

We consider three key differences. The first concerns the characteristics of the paths or subgraphs considered by the two centrality measures for communicating or mediating between two or more nodes. The analysis of this difference is based on the comparison of the decomposition of the two centralities in a communication and an intermediation measure. The second key difference considers the effect on the node's centralities of the property of fairness of the Myerson value, in contrast with the property of balanced link contributions of the Position value. For this purpose, we shall analyse the different effects of the removal of an edge on the centrality of its end nodes. The third one is also related to the effects of removing an edge. For superadditive games, the Myerson value verifies the *stability* property, which states that any two players cannot gain less by adding a new edge between them. The Position value fails to verify stability.

4.1. The myerson and position centralities decomposition: communication and betweenness

As we have mentioned before, the Myerson and Position centralities can be rewritten to obtain more information about their behaviour as a centrality measure by decomposing their values into two parts that measure two different abilities of each member in the social network: its ability to connect and its ability to intermediate (see Gómez et al. (2003, 2004)).

Let us first introduce some notation and concepts. We will follow the terminology of Dietzenbacher et al. (2017), who also give a decomposition of the Position value. Let $\emptyset \neq R \subseteq N$, a subgraph $\Gamma_L = (N[L], L)$ is called a *minimal R-connecting edge-induced subgraph* (Dietzenbacher et al., 2017) if it connects R and any $\Gamma_{L'}$ with $L' \subsetneq L$ does not connect R. Let $\mathcal{M}_\Gamma^E(R)$ denotes the collection of minimal R-connecting edge-induced subgraphs and $\mathcal{E}_\Gamma(R) = \{L_1^R, \ldots, L_{q_R}^R\}$ denotes the collection of coalitions of edges which defines them. We consider the *set of intermediaries of R through edges*, which we will denote by $Bet_\Gamma^E(R) \subseteq N \setminus R$, as the set of nodes in some minimal R-connecting edge-induced subgraph. This is:

$$Bet_\Gamma^E(R) := \{i \in N \setminus R \mid E_i \cap E[\mathcal{E}_\Gamma(R)] \neq \emptyset\}, \tag{10}$$

where $E[\mathcal{E}_\Gamma(R)] := \cup_{j=1}^{q_R} L_j^R$ is the set of edges involved in $\mathcal{E}_\Gamma(R)$. Analogously, $E[\mathcal{L}] := \bigcup_{L \in \mathcal{L}} L$, for all $\mathcal{L} \subseteq \mathcal{E}_\Gamma(R)$.

A subgraph $\Gamma_S = (S, E[S])$ is called a *minimal R-connecting vertex-induced subgraph* (Dietzenbacher et al., 2017) if it connects R and any $\Gamma_{S'}$ with $S' \subsetneq S$ does not connect R. Let $\mathcal{M}_\Gamma^N(R)$ denotes the collection of minimal R-connecting vertex-induced subgraphs and $\mathcal{S}_\Gamma(R) = \{S_1^R, \ldots, S_{m_R}^R\}$ denotes the collection of coalitions of nodes which define them. We consider the *set of intermediaries of R through nodes*, which we will denote by $Bet_\Gamma^N(R) \subseteq N \setminus R$, as the set of nodes not in R which are in some minimal R-connecting vertex-induced subgraph. This is:

$$Bet_\Gamma^N(R) := N[\mathcal{S}_\Gamma(R)] \setminus R,. \tag{11}$$

where $N[\mathcal{S}_\Gamma(R)] := \cup_{j=1}^{m_R} S_j^R$ is the set of nodes involved in $\mathcal{S}_\Gamma(R)$. For all $\mathcal{B} \subseteq \mathcal{S}_\Gamma(R)$, $N[\mathcal{B}] := \bigcup_{S \in \mathcal{B}} S$.

Note that $\mathcal{M}_\Gamma^N(R) \subseteq \mathcal{M}_\Gamma^E(R)$ and $Bet_\Gamma^N(R) \subseteq Bet_\Gamma^E(R)$.

Now, according to Gómez et al. (2003, 2004) and Dietzenbacher et al. (2017), the Position and Myerson centralities can be decomposed as follows:

$$\pi_i(N, v, E) = \sum_{\substack{R \subseteq N \\ i \in R}} \Delta_R(v) \sum_{\mathcal{L} \subseteq \mathcal{E}_\Gamma(R)} (-1)^{|\mathcal{L}|+1} \frac{|E[\mathcal{L}] \cap E_i|}{2|E[\mathcal{L}]|} +$$
$$\sum_{\substack{R \subseteq N \setminus \{i\} \\ Bet_\Gamma^E(R)}} \Delta_R(v) \sum_{\mathcal{L} \subseteq \mathcal{E}_\Gamma(R)} (-1)^{|\mathcal{L}|+1} \frac{|E[\mathcal{L}] \cap E_i|}{2|E[\mathcal{L}]|}, \tag{12}$$

[3] Convexity is also required to ensure the results on chains.

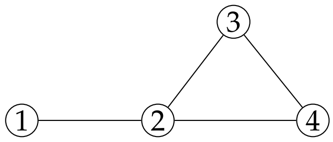


Fig. 3. Four node graph with a cycle.

and

$$\mu_i(N,v,E) = \sum_{\substack{R\subseteq N\\ i\in R}} \Delta_R(v) \sum_{\mathcal{B}\subseteq \mathcal{S}_\Gamma(R)} (-1)^{|\mathcal{B}|+1}\frac{1}{|N[\mathcal{B}]|} + \sum_{\substack{R\subseteq N\setminus\{i\}\\ i\in Bet^N_\Gamma(R)}} \Delta_R(v) \sum_{\mathcal{B}\subseteq \mathcal{S}_\Gamma(R)} (-1)^{|\mathcal{B}|+1}\frac{1}{|N[\mathcal{B}]|}. \quad (13)$$

Note that these decompositions relate each centrality with $\Delta_R(v)$, which are the Harsanyi dividends of the original game (N,v). The communication structure modifies the way in which these dividends are distributed: the Shapley value divides $\Delta_R(v)$ equally among the members of coalition R.

Both centralities measure the ability to connect (the first term) and the ability to mediate (the second term). However, they differ in the properties of the connection structures they consider. The Myerson centrality only counts subgraphs that are minimal with respect to nodes, whereas the Position centrality counts more structures as long as they are minimal with respect to edges, and therefore more vertices are considered as betweenners. Moreover, each of these structures can be properly evaluated according to the goal achieved by the members they connect, by means of the dividends of the game. Note that both measures also differ in the distribution of each dividend among the nodes involved in the connection structure. The Myerson centrality distributes them equally among all nodes regardless of structure, while the Position centrality also takes structure into account. Let us illustrate these differences with two examples. The graph of the second example is a *tree* (connected cycle-free graph), which simplifies both (13) and (12), since there is a unique R connection subgraph for each $R \subseteq N$.

Example 2. In order to illustrate the differences, let us consider the following graph with a cycle depicted in Fig. 3, and the dividend of coalition $R = \{1,4\}$, for a generic game $(N,v) \in G_0^N$.

Then, the Position centrality considers all paths connecting nodes 1 and 4, whereas the Myerson centrality only considers the minimum path between the two nodes:

$$\mathcal{E}_\Gamma(R) = \{L_1, L_2\}, \text{ where } L_1 = \{\{1,2\},\{2,4\}\} \text{ and}$$
$$L_2 = \{\{1,2\},\{2,3\},\{3,4\}\},$$
$$Bet^E_\Gamma(R) = \{2,3\},$$
$$\mathcal{S}_\Gamma(R) = \{S_1\}, \text{ where } S_1 = \{1,2,4\},$$
$$Bet^N_\Gamma(R) = \{2\}.$$

Therefore, the Myerson and Position centralities divide $\Delta_R(v)$ as seen in Table 2

Note that node 3 does not receive any share of $\Delta_R(v)$ according to the Myerson centrality, since it does not lie in any minimum path connecting nodes 1 and 4. Moreover, the Position centrality also differentiates among 1, 2 and 4.

For example, when the functionality of the social network is given by the messages game[4], the Myerson and Position centralities (determined by v_m) decompositions are shown in Table 3. Note that Position centrality gives more weight to the betweenness centrality than the Myerson centrality.

[4] $\Delta_R(v_m) \neq 0$ if, and only if, $|R| = 2$.

Table 2
$\Delta_R(v)$ distribution according to Myerson and Position centralities determined by (N,v).

$\Delta_R(v)$	1	2	3	4
Position cent.	7/24	11/24	2/24	4/24
Myerson cent.	1/3	1/3	0	1/3

Table 3
Messages Myerson and Messages Position decompositions.

Node centrality	1	2	3	4
v_m-Myerson-Closeness	7/3	3	8/3	8/3
v_m-Myerson-Betweenness		4/3		
v_m-Myerson	7/3	13/3	8/3	8/3
v_m-Position-Closeness	13/6	8/3	2	2
v_m-Position-Betweenness		13/6	1/2	1/2
v_m-Position	13/6	29/6	5/2	5/2

Table 4
$\Delta_R(v)$ Position distribution in the star and the chain.

$\Delta_R(v)$	1	2	3	4
Star	1/2	1/6	1/6	1/6
Chain	1/6	1/3	1/3	1/6

Example 3. Now, we consider cycle-free graphs where both $\mathcal{E}_\Gamma(R)$ and $\mathcal{S}_\Gamma(R)$ determine the same connection subgraph, for all $R \subseteq N$. Even in this case, the two centrality shares are different, since the Position centrality shares take into account the structure of the connection subgraph.

For instance, let be $R = \{1,2,3,4\}$ in the following two trees of Fig. 4. The Myerson centrality divides $\Delta_R(v)$ equally among the four players, whereas the Position centrality distributes $\Delta_R(v)$ among the nodes taking into account their position within the connecting subgraph. Nodes with a more central position receive a greater share (see Table 4).

4.2. Fairness versus balanced link contributions

The Myerson and Position centralities are distinguished by the properties of fairness and balanced link contributions. Fairness implies a symmetric effect of removal and edge over its two ends. In contrast, we will show that balanced link contributions, responds in a less local way to such elimination. The effect on each of the two end nodes i and j of removing an edge $\{i,j\}$ will depend on the value that the links of the other node have for it.

Let $\{i,j\} \in E$, then by condition (6), it holds:

$$\Delta\pi_i^{+ij} - \Delta\pi_j^{+ij} = \sum_{\substack{e\in E_i\\ e\neq\{i,j\}}} \Delta\pi_j^{+e} - \sum_{\substack{e\in E_j\\ e\neq\{i,j\}}} \Delta\pi_i^{+e}, \quad (14)$$

where $\Delta\pi_i^{+ij} := \pi_i(N,v,E) - \pi_i(N,v,E\setminus\{i,j\})$, $\Delta\pi_i^{+e} := \pi_i(N,v,E) - \pi_i(N,v,E\setminus e)$.

In contrast, $\Delta\mu_i^{+ij} - \Delta\mu_j^{+ij} = 0$. To illustrate the value of this difference, we consider a paradigmatic example in transportation networks, such as hub-and-spoke networks, where a critical aspect is to study the effect of connecting two hub-and-spoke systems via the hubs.[5] In this framework, measuring the effect of connecting two different hubs over the centralities of each of the original hubs as well as over the centralities of each of their satellites is a relevant question. Note, that the Position centrality also allows us to measure the importance of the added connection.

[5] See Pöschl (2006) thesis: "Furthermore, adding cities to the hub-and-spoke system would lead to exponential growth of the network. This effect can be further strengthened by connecting two hub-and-spoke systems via the hubs".

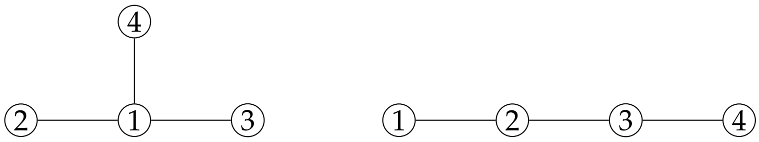


Fig. 4. 4-nodes star and chain.

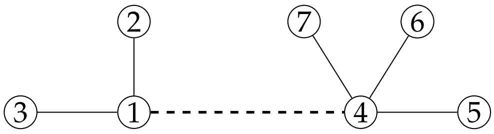


Fig. 5. Hub-and-spokes connection.

Table 5
Messages Myerson and Position centralities increments when connecting two hubs.

Nodes	Δ Myerson centrality	Δ Position centrality
Hub-k_1-star	$1+\frac{2}{3}(k_1+k_2)+\frac{k_1k_2}{2}$	$1+k_1+\frac{k_2}{2}+\frac{2k_1k_2}{3}$
Hub-k_2-star	$1+\frac{2}{3}(k_1+k_2)+\frac{k_1k_2}{2}$	$1+k_2+\frac{k_1}{2}+\frac{2k_1k_2}{3}$
Satellites-k_1-star	$\frac{2}{3}+\frac{k_2}{2}$	$\frac{1}{2}+\frac{k_2}{3}$
Satellites-k_2-star	$\frac{2}{3}+\frac{k_1}{2}$	$\frac{1}{2}+\frac{k_1}{3}$

The Fig. 5 shows an example of such a connection. In this case, we will show that Hub 4 of the larger star gains more Position centrality from connecting the two networks than Hub 1, because it brings more satellites and options to connect to Hub 1 than vice versa.

The next proposition assures that connecting two hub-and-spokes networks, asking for an additional property of the functionality of the two star networks, the increase in the Position centrality is higher for the hub of the biggest star. We also study the effect of the added connection on the satellites and prove that the increase is higher for the leaves of the smallest one. We explore this behaviour further in Example 4.

Proposition 9. *Let $(N,v)\in G_0^N$ defined by a function $f(\cdot)\in C^n(\mathbb{R})$ such that $f^{(k)}(x)\geq 0$ in $[1,+\infty)$, for all $k=1,\ldots,n$. Let (N_1,E_1^*) and (N_2,E_2^*) be two not connected stars of k_1 and k_2 leaves respectively, with $k_1\leq k_2$, and being $N=N_1\cup N_2$ and $E=E_1^*\cup E_2^*$. If an edge $b=\{c_1,c_2\}$ between the two hubs $c_1\in N_1$ and $c_2\in N_2$ is added, then it holds that:*

1. *$\Delta\pi_{c_1}\leq\Delta\pi_{c_2}$,*
2. *$\Delta\pi_i\geq\Delta\pi_j$, for every pair of leaves $i\in N_1\setminus\{c_1\}$, $j\in N_2\setminus\{c_2\}$,*

where $\Delta\pi_k=\pi_k(N,v,E\cup\{b\})-\pi_k(N,v,E)$, for all $k\in N$.

Proof. Let us first prove that the hub of the larger star gains more Position centrality than the hub of the smaller one.

From expression (7) of the Position value in terms of the Harsanyi dividends of the link game proposed in Slikker (2005), it follows that:

$$\Delta\pi_{c_1}=\frac{1}{2}\sum_{L\subseteq E}\lambda_{L\cup\{b\}}\frac{l_{c_1}+1}{l+1}, \tag{15}$$

$$\Delta\pi_{c_2}=\frac{1}{2}\sum_{L\subseteq E}\lambda_{L\cup\{b\}}\frac{l_{c_2}+1}{l+1}, \tag{16}$$

where, throughout the proof, λ_L denotes the Harsanyi dividends of the link game w^v. Note that the addends with non-zero dividend in the above two sums correspond to coalitions L of the form:

(s0) $L=\{b\}$,
(s1) $L=T_1\cup\{b\}$, with $\emptyset\neq T_1\subseteq E_1^*$,
(s2) $L=T_2\cup\{b\}$, with $\emptyset\neq T_2\subseteq E_2^*$,
(s3) $L=T_1\cup T_2\cup\{b\}$, with $\emptyset\neq T_1\subseteq E_1^*$ and $\emptyset\neq T_2\subseteq E_2^*$.

In the unique type (s0) addend, the dividend $\lambda_{\{b\}}$ appears in both increments (15) and (16) with the same coefficient 1, whereas for the remaining types, the dividends appear with different coefficients. However, taking into account Lemma 2, $\lambda_{T_1\cup\{b\}}=\lambda_{T_2\cup\{b\}}$, whenever $|T_1|=t_1=t_2=|T_2|$, all type (s1) addends have an equal one of type (s2) with T_2 s.t. $t_2\leq k_1$. Analogously, $\lambda_{T_1\cup T_2\cup\{b\}}=\lambda_{T_1'\cup T_2'\cup\{b\}}$, if $t_1+t_2=t_1'+t_2'$, and therefore, all type (s3) addends for T_2 with $t_2\leq k_1$ appear in both increments. Thus, their difference $\Delta\pi_{c_2}-\Delta\pi_{c_1}$ is given by:

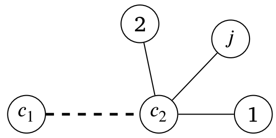
$$\frac{1}{2}\sum_{\substack{\emptyset\neq T_2\subseteq E_2^*\\ t_2>k_1}}\lambda_{T_2\cup\{b\}}\left(1-\frac{1}{t_2+1}\right)$$
$$+\frac{1}{2}\sum_{\substack{\emptyset\neq T_1\subseteq E_1^*\\ T_2\subseteq E_2^*, t_2>k_1}}\lambda_{T_1\cup T_2\cup\{b\}}\left(\frac{t_2+1}{t_1+t_2+1}-\frac{t_1+1}{t_1+t_2+1}\right),$$

which is non-negative by Corollary 2, and taking into account that $t_2>k_1\geq t_1$.

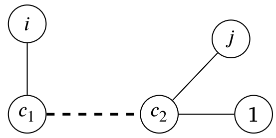
With respect to the second statement, let $i\in N_1\backslash\{c_1\}$ and $j\in N_2\backslash\{c_2\}$, be any pair of leaves. Analogously, we must consider the following sort of non-zero addends in their respective increments:

(s1) $L=T_1\cup\{\{i,c_1\},b\}$, $T_1\subseteq E_1^*\setminus\{i,c_1\}$,
(s2) $L=T_2\cup\{\{j,c_2\},b\}$, $T_2\subseteq E_2^*\setminus\{j,c_2\}$,
(s3i) $L=T_1\cup T_2\cup\{\{i,c_1\},b\}$, $T_1\subseteq E_1^*\setminus\{i,c_1\}$ and $\emptyset\neq T_2\subseteq E_2^*\setminus\{j,c_2\}$,
(s3j) $L=T_1\cup T_2\cup\{\{j,c_2\},b\}$, $\emptyset\neq T_1\subseteq E_1^*\setminus\{i,c_1\}$ and $T_2\subseteq E_2^*\setminus\{j,c_2\}$,
(s4) $L=T_1\cup T_2\cup\{\{i,c_1\},\{j,c_2\},b\}$, with $T_1\subseteq E_1^*\setminus\{i,c_1\}$ and $T_2\subseteq E_2^*\setminus\{j,c_2\}$.

In this case, comparing both increments is more subtle because it requires to compare the dividends of two different structures:



Type (s2) L, with
$l=4, d_L=0, \lambda_L=F(5,4)$



Type (s3i) L, with
$l=4, d_L=1, \lambda_L=F(5,3)$

Type (s1) addends appear only in $\Delta\pi_i$, whereas type (s2) addends appear only in $\Delta\pi_j$. Again, by Lemma 2, every type (s1) addend in $\Delta\pi_i$ has an equal one of type (s2) with T_2 s.t. $t_2\leq k_1$ in $\Delta\pi_j$. With respect to type (s3i) and (s3j) addends, both of them appear in both increments with the exception of the extreme cases: $T_1=\{\emptyset\}$ in type (s3i) addends and $T_2=\{\emptyset\}$ in type (s3j) addends, which again also coincide whenever $t_2\leq k_1-1$ in the extreme case $T_1=\{\emptyset\}$. Type (s4) addends appear in both cases. Therefore, it follows from expression (8) that:

$$\Delta\pi_i-\Delta\pi_j=\frac{1}{2}\sum_{\substack{T_2\subseteq E_2^*\setminus\{j,c_2\}\\ t_2>k_1-1}}\left(F(t_2+3,t_2+1)-F(t_2+3,t_2+2)\right)\frac{1}{t_2+2}.$$

Thus, taking into account that $F(s,\cdot)$ is a decreasing function (see Lemma 3), $\Delta\pi_i-\Delta\pi_j\geq 0$, which completes the proof. □

Example 4. In this example, we will illustrate the previous result when the functionality of the social network is given by the Messages game, which models trips between two nodes of the network, and verifies the condition of Proposition 9.

As can be seen in Table 5 the hub of the larger transportation network gains more Position centrality than the other hub. Moreover, the greater the discrepancy in size between the two networks, the greater the difference in Position centrality gain, which is equal to $\frac{k_2-k_1}{2}$. In terms of satellites, the effect is the opposite. The satellites of the smaller transportation network gain more Position centrality than the other satellites because they are now able to connect to more nodes via the other hub. Again, the greater the discrepancy in size between the two networks, the greater the difference in Position centrality gain, which is $\frac{k_2-k_1}{3}$. Although in principle this behaviour seems counterintuitive, the intuition can be found in the well known Matthew effect in social networks, where wealth or credit is distributed among individuals according to how much they already have. In proportion, however, the hub of the smaller star gains more than the hub of the larger star. The effect on satellite Myerson centrality is similar to Position centrality, but in this case satellites gain more Myerson than Position centrality.

Additionally, Position centrality allows us to measure the power of connections. The Shapley value of the edge connecting the two hubs is $2 + k_1 + k_2 + \frac{2k_1k_2}{3}$, which increases with the size of the original hub and spokes transportation networks. As all connections increase their values, it is interesting to compare the value of the added edge in relation to the value of the remaining connections. The ratios of ϕ_b over ϕ_{e_1} and over ϕ_{e_2}, where e_1 and e_2 are spokes of the k_1-star and the k_2-star, respectively, are given by:

$$\frac{\phi_b}{\phi_{e_1}} = \frac{k_2(1+\frac{2}{3}k_1)+k_1+2}{\frac{2}{3}k_2+k_1+2}, \quad \frac{\phi_b}{\phi_{e_2}} = \frac{k_2(1+\frac{2}{3}k_1)+k_1+2}{k_2+\frac{2}{3}k_1+2}.$$

Note that even in the limit case with $k_2 \to \infty$ the above ratios are greater than one: $\lim_{k_2\to\infty} \frac{\phi_b}{\phi_{e_1}} = \frac{3}{2} + k_1$, and $\lim_{k_2\to\infty} \frac{\phi_b}{\phi_{e_2}} = 1 + \frac{2}{3}k_1$, i.e. the value of the new connection remains being strictly greater than the value of any of the original spokes.

4.3. Stability

The third major difference between the Myerson and Position centralities is stability. The Myerson centrality is a stable centrality, whereas the Position centrality is not. Note that, as Boldi et al. (2023) show, some of the most commonly used centrality measures also exhibit the same behaviour: in the authors' words, "*...it is not always beneficial to get a new friend*".

Examples can be found where the Position centrality measure of one of the end nodes becomes worse. In these cases, that are not very common, the introduction of an edge may result in some of the existing edges incident on a given node becoming less valued, which could have a detrimental effect on the overall balance of the node. Obviously this will not be the case if, for instance, the newly added edge does not bypass any of the existing edges. The situation considered in Proposition 10 is one of those cases.

Proposition 10. *Let $(N,v) \in G_0^N$ be a convex game, and let (N,E) a graph with two connected components (N_1,E_1) and (N_2,E_2). If an edge $b = \{i_0,j_0\}$, with $i_0 \in N_1$ and $j_0 \in N_2$ is added, then it holds:*

$$\pi_i(N_k, v_k, E_k) = \pi_i(N, v, E) \le \pi_i(N, v, E \cup \{b\}), \tag{17}$$

for all node $i \in N_k$, $k = 1,2$, where v_k denotes the restriction of v to N_k, $k = 1,2$, and $E = E_1 \cup E_2$.

Proof. To avoid overwhelming this Section, the proof has been included in Appendix. □

Note that the previous situation of connecting two hub-and-spokes networks is a special case, and therefore connecting two such networks always increases the Position centrality of all involved nodes.

We will also prove in the next section (see Proposition 11) that when the functionality of the network is to attach nodes, the Position centrality is always stable.

5. The position attachment centrality

In this section we analyse the particular case in which the symmetric game is the *attachment game* (Skibski et al., 2019) given by $v_a(S) = f_a(s) = 2(s-1), \forall S \subset N$. Our proposal is to study some specific properties of the *Position Attachment centrality* defined, for each graph $\Gamma = (N,E)$ as the Position value of the communication situation (N, v_a, E), which we will denote as $PA(\Gamma)$.

The attachment game was used by Skibski et al. (2019) to define the graph Attachment centrality, $A(\Gamma)$, as the Myerson value of the attachment game when coalition formation is restricted by a graph Γ. The Attachment centrality of node $i \in N$ is the expected number of components created by the removal of i (multiplied by 2 for normalisation purposes) when the nodes are removed from the graph one by one in a random order. The difference between the two centrality measures is due to a difference in the measurement of marginal contributions (see Slikker (2005)). In the case of Attachment centrality, the removal of a node implies the simultaneous removal of all its edges. On the contrary, when considering the Position value, are the edges of the graph which are removed one by one in a random order. The Position Attachment centrality of $i \in N$ is measured by the sum of the expected number of components created by the removal of each of the edges incident in i.

When the graph is a tree, the Position Attachment centrality coincides with the Attachment centrality (see Borm et al. (1992)), and moreover $PA_i(\Gamma) = A_i(\Gamma) = d_i$, where d_i is the degree of node i in Γ. However, in the general case where the graph contains some cycles, the two centrality measures differ in how end-nodes of an edge are affected when such edge is removed (or added). According to Attachment centrality, the addition of an edge improves both nodes equally. On the contrary, according to the Position Attachment centrality, both nodes are also improved (or at least not worsened), but not necessarily to the same extent. Both Attachment centralities also differ on the nodes that are worsened, since the nodes considered as intermediaries by the two centralities differ (see Section 4).

Note that $(N, v_a) \in G_0^N$. Moreover, it is convex. On the contrary, we will show that the corresponding link game (N, w^a) is concave.

Lemma 4. *Let $\Gamma = (N,E)$ be a given graph, then the marginal contributions in the link game (E, w^a) defined over the communication situation (N, v_a, E) are given by:*

$$w^a(L \cup \{e\}) - w^a(L) = \begin{cases} 0, & \text{if there exists a cycle in } L \cup \{e\} \\ & \quad \text{containing edge } e, \\ 2, & \text{otherwise,} \end{cases}$$

for every $e \in E$, and every $L \subseteq E \setminus \{e\}$.

Proof. Let $e = \{i,j\} \in E$, and $L \subseteq E\setminus\{e\}$. If there exists a cycle in $L\cup\{e\}$ containing edge e then $\mathcal{K}(L) = \mathcal{K}(L\cup\{e\})$ and thus $w^a(L\cup\{e\}) = w^a(L)$.

Otherwise, there exists $S_i, S_j \in \mathcal{K}(L)$ with $i \in S_i$ and $j \in S_j$ and $S_i \ne S_j$. Thus,

$$w^a(L) = v_a(S_i) + v_a(S_j) + \sum_{\substack{S\in\mathcal{K}(L)\\ S\ne S_i,S_j}} v_a(S),$$

$$w^a(L\cup\{e\}) = v_a(S_i \cup S_j) + \sum_{\substack{S\in\mathcal{K}(L)\\ S\ne S_i,S_j}} v_a(S).$$

Thus, $w^a(L\cup\{e\}) - w^a(L) = v_a(S_i\cup S_j) - v_a(S_i) - v_a(S_j) = 2.$ □

Note that the marginal contributions in the link game are non-increasing, and thus (E, w^a) is concave for every given graph $\Gamma = (N, E)$.

Proposition 11. *Let $\Gamma = (N, E)$ be a given graph. If an edge $e_0 = (i_0, j_0)$, between two nodes $i_0, j_0 \in N$ is added. Then, it holds that:*

(i) $PA_i(N, E \cup \{e_0\}) \leq PA_i(N, E)$, for every intermediary node $i \in Bet^E_\Gamma(\{i_0, j_0\})$,
(ii) $PA_i(N, E \cup \{e_0\}) = PA_i(N, E)$, for every node $i \notin Bet^E_\Gamma(\{i_0, j_0\}) \cup \{i_0, j_0\}$,
(iii) $PA_{i_0}(N, E \cup \{e_0\}) \geq PA_{i_0}(N, E)$ and $PA_{j_0}(N, E \cup \{e_0\}) \geq PA_{j_0}(N, E)$.

Proof. Taking into account Lemma 4, for every edge $e \in E$, and every $L \subseteq E \setminus \{e\}$, it is verified:

$$w^a(L \cup \{e, e_0\}) - w^a(L \cup \{e_0\}) \leq w^a(L \cup \{e\}) - w^a(L). \quad (18)$$

Then, for every edge $e \in E$ it holds:

$$\phi_e(E \cup \{e_0\}, w^a) = \sum_{L \subseteq E \setminus \{e\}} \frac{l!(m-l)!}{(m+1)!}(w^a(L \cup \{e\}) - w^a(L))$$
$$+\frac{(l+1)!(m-l-1)!}{(m+1)!}(w^a(L \cup \{e, e_0\}) - w^a(L \cup \{e_0\}))$$
$$\leq \sum_{L \subseteq E \setminus \{e\}} \frac{l!(m-l-1)!}{m!}(w^a(L \cup \{e\}) - w^a(L)) = \phi_e(E, w^a),$$

where $m = |E|$. Therefore, $PA_i(N, E \cup \{e_0\}) \leq PA_i(N, E)$, for every node $i \in N \setminus \{i_0, j_0\}$, since for the end-nodes i_0, j_0, $\frac{1}{2}\phi_{e_0}(E \cup \{e_0\}, w^a)$ must be added.

Now, in order to prove statement (ii), let $e \notin \cup_{j=1}^{q_0} L_j^0$, being $\mathcal{E}_\Gamma(\{i_0, j_0\}) = \{L_1^0, \ldots, L_{q_0}^0\}$. Then, there exists a cycle containing edge e in $L \cup \{e, e_0\}$ if, and only if, there exists a cycle containing edge e in $L \cup \{e\}$. Thus, inequality (18) is an equality for every subset $L \subseteq E \setminus \{e\}$, and therefore $\phi_e(E, w^a) = \phi_e(E \cup \{e_0\}, w^a)$. If $i \notin Bet^E_\Gamma(\{i_0, j_0\}) \cup \{i_0, j_0\}$, then $E_i(N, E \cup \{e_0\}) = E_i(N, E) \subseteq E \setminus \cup_{j=1}^{q_0} L_j^0$ and therefore $PA_i(N, E \cup \{e_0\}) = PA_i(N, E)$.

We will restrict the proof of (iii) to the case of connected graphs. Note that for not connected graphs, two cases are possible:

1. i_0 and j_0 belong to different connected components. Then, since the attachment game is convex, by Proposition 10, (iii) holds.
2. i_0 and j_0 belong to the same connected component, then taking into account that Position centrality verifies Locality (see Proposition 2), we can restrict the proof to the connected subgraph induced by their connected component.

The proof relies on the alternative description of the Shapley value (Shapley, 1953) in terms of all possible orders of arrival of the players to a meeting point. We will prove the inequality for i_0, the same reasoning applies to j_0.

Let $\theta \in \Theta^m$ be a given order of the original edges in E. Then, it holds:

$$PA_{i_0}(N, E) = \frac{1}{2 \cdot m!} \sum_{\theta \in \Theta^m} \sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta))\right), \quad (19)$$

where $Pred^e(\theta) \subseteq E \setminus \{e\}$ denotes the set of edges that precede edge e in the order θ.

Now, let $\theta \in \Theta^m$ be any given order of the original edges, we will prove that:

$$\sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta))\right) \leq \quad (20)$$
$$\sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+))\right)$$
$$+\left(w^a(Pred^{e_0}(\theta_k^+) \cup \{e_0\}) - w^a(Pred^{e_0}(\theta_k^+))\right), \quad (21)$$

for each of the $m+1$ orders of the edges in $E \cup \{e_0\}$, $\theta_k^+ \in \Theta^{m+1}$, defined by means of inserting edge e_0 in every possible position k, $1 \leq k \leq m+1$, of the given order θ.

Making use of Lemma 4, we only need to prove that for every given order $\theta \in \Theta^m$, the number of edges with a nonzero contribution in (20) is less or equal than the number of edges with a nonzero contribution in (21), for every $1 \leq k \leq m+1$. For doing it, we will define a partition of the collection of paths, $\mathcal{M}_\Gamma(\{i_0, j_0\})$, by means of their starting edge from i_0. Formally, let $E_{i_0}(j_0) \subseteq E_{i_0}$ be the collection of edges incident in i_0 that form part of some of these paths in $\mathcal{M}_\Gamma(\{i_0, j_0\})$. Then, we define the partition $\mathcal{P}(e) := \{P \in \mathcal{M}_\Gamma(\{i_0, j_0\}) \,/\, e \in P\}$, for each $e \in E_{i_0}(j_0)$.

Let $\theta \in \Theta^m$ be a given order of the original edges, and let $e \in E_{i_0}$ with $w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta)) = 2$. Then:

1. For every position $k > \theta(e)$, since $k \notin Pred^e(\theta)$, adding e_0 cannot create a cycle. Then, it holds:
$$w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta)) = w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+)).$$
2. Analogously, if $k \leq \theta(e)$, but $e \notin E_{i_0}(j_0)$, i.e. $e \notin \cup_{j=1}^{q_0} L_j^0$, then:
$$w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta)) = w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+)).$$
3. If $k \leq \theta(e)$ and $e \in E_{i_0}(j_0)$, then one of the following three conditions – which are mutually exclusive – is verified:
   (c1) There exists a path in $P \in \mathcal{P}(e)$ such that $P \setminus \{e\} \subseteq Pred^e(\theta)$.
   (c2) There exists $P \in \mathcal{P}(e')$, $e' \in E_{i_0}(j_0) \setminus \{e\}$, with $P \subseteq Pred^e(\theta)$.
   (c3) None of the two previous conditions hold.

   If (c1) holds, then $w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+)) = 0$. However, since condition (c2) is not true, then it was also not true when edge e_0 arrived, and therefore $w^a(Pred^{e_0}(\theta_k^+) \cup \{e_0\}) - w^a(Pred^{e_0}(\theta_k^+)) = 2$. Thus, e_0 takes the place of e in (21). Note that there is a unique edge e in this situation per each order $\theta \in \Theta^m$.
   On the other hand, if (c2) holds there exists a path from i_0 to j_0 in $Pred^e(\theta)$, and therefore the arrival of edge e_0 does not affect the marginal contribution of e. Trivially, if (c3) holds, again the arrival of edge e_0 does not affect the marginal contribution of e. Thus, in both cases, its is verified:
$$w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta)) = w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+)).$$

To sum up, it is satisfied:

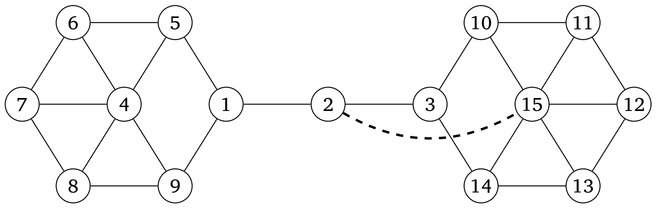
$$PA_{i_0}(N, E \cup \{e_0\}) = \frac{1}{2 \cdot (m+1)!} \sum_{\theta \in \Theta^{m+1}}$$
$$\times \Bigg[\left(w^a(Pred^{e_0}(\theta) \cup \{e_0\}) - w^a(Pred^{e_0}(\theta))\right)$$
$$+ \sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta))\right) \Bigg] =$$
$$\frac{1}{2 \cdot (m+1)!} \sum_{\theta \in \Theta^m} \sum_{k=1}^{m+1} \Bigg[\left(w^a(Pred^{e_0}(\theta_k^+) \cup \{e_0\}) - w^a(Pred^{e_0}(\theta_k^+))\right)$$
$$+ \sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta_k^+) \cup \{e\}) - w^a(Pred^e(\theta_k^+))\right) \Bigg]$$
$$\geq \frac{1}{2 \cdot (m+1)!} \sum_{\theta \in \Theta^m} (m+1) \sum_{e \in E_{i_0}} \left(w^a(Pred^e(\theta) \cup \{e\}) - w^a(Pred^e(\theta))\right)$$
$$= PA_{i_0}(N, E),$$

and statement (iii) holds. □

Example 5. To analyse the difference between both centrality measures based on attachment we consider the addition of edge $\{2, 15\}$ in the graph of Fig. 1 :

Table 6
Attachment centralities increments when adding the edge $\{2, 15\}$.

Nodes	1	2	3	4	5	6,8	7	9	10,14	11,13	12	15
$A(\Gamma)$	2.5	2.0	2.5	2.1	1.8	1.6	1.6	1.8	1.8	1.6	1.6	2.1
$PA(\Gamma)$	2.32	2.0	2.32	2.51	1.74	1.57	1.55	1.74	1.74	1.57	1.55	2.51
$\Delta A(\Gamma)$	0	0.4	−0.6	0	0	0	0	0	−0.1	0	0	0.4
$\Delta PA(\Gamma)$	0	0.26	−0.52	0	0	0	0	0	−0.12	−0.03	−0.01	0.56



In Table 6 are depicted both the Myerson and Position Attachment centralities for the original graph and their increments when an edge is added.

Note that before adding the edge $\{2, 15\}$, node 15 is more central than node 2 for both measures, since these are local measures closely related to degree centrality, as we have seen. In terms of centrality increments when the edge $\{2, 15\}$ is added, we again see the Matthew effect: node 15 gains more Position Attachment centrality than the other end-node 2 (0.56 versus 0.26), while both gain the same Attachment centrality.

With respect to the remaining nodes, only the Attachment centrality of the bypassed node 3, and that of the intermediary adjacent nodes 10 and 14, which were in the shortest paths between 2 and 15, is reduced. However, according to Position Attachment centrality, also nodes 11, 12 and 13, which were also potential intermediaries between 2 and 15 although but they did not appear in any shortest path, reduce their centrality. As we have seen in Section 4, Myerson centrality only affects nodes in minimal paths with respect to nodes, whereas Position centrality affects to nodes in minimal paths with respect to edges.

5.1. A characterisation of the position attachment centrality

The characterisation of the Position Attachment centrality in this paper is based on Position value (Slikker, 2005) and Attachment centrality (Skibski et al., 2019) characterisations. For this purpose, the following axioms for a centrality measure σ are considered.

- **Locality Axiom:** (see Definition 5)
- **Normalisation Axiom:**
  - $\sigma_i(\Gamma) \in [0, n-1]$;
  - $\sigma_i(\Gamma) = 0$ when i is isolated in Γ;
  - $\sigma_1(\Gamma^*) = n - 1$ when Γ^* is a star with n nodes and being 1 its hub.
- **Gain–Loss Axiom:** For every connected graph, $\Gamma = (N, E)$, and every pair of nodes, $i, j \in N$, adding the edge $\{i, j\} \notin E$ to E does not affect the sum of centralities.
- **Balanced link contributions Axiom:** For every $\Gamma = (N, E)$ and all $i, j \in N$

  $$\sum_{e \in E_j} \left(\sigma_i(\Gamma) - \sigma_i(\Gamma - e)\right) = \sum_{e \in E_i} \left(\sigma_j(\Gamma) - \sigma_j(\Gamma - e)\right),$$

  where $\Gamma - e := (N, E \setminus \{e\})$.

Theorem 1. *The Position Attachment centrality is the unique centrality measure that satisfies Locality, Normalisation, Gain–Loss and Balanced link contributions.*

Proof. As a Position value $PA(\Gamma)$ satisfies Locality (Proposition 2) and Balanced link contributions (Slikker, 2005). Moreover, by Proposition 3, for all connected component $C \in K(\Gamma)$

$$\sum_{i \in C} PA_i(\Gamma) = 2(c - 1).$$

Then, adding a new edge to a connected component does not affect the sum of centralities of its nodes, thus verifying Gain-Loss.

With respect to Normalisation, by Proposition 5, the node hub 1 of a star of n nodes verifies that $\pi_1(N, v_a, E^*) = f_a(n)/2$. Then, $PA_1 = n - 1$. By Proposition 4, for an isolated node k, $PA_k = 0$. Finally, by propositions 1 and 5, $0 \le PA_i(\Gamma) \le n - 1, \forall i \in N$ holds.

For proving uniqueness, assume that $\sigma(\Gamma)$ is a centrality measure that satisfies the four axioms above. First we show that for all $C \in K(\Gamma)$,

$$\sum_{i \in C} \sigma_i(\Gamma) = 2(c - 1).$$

Consider a n nodes star Γ^*, being node 1 its hub. By Normalisation $\sigma_1(\Gamma^*) = n - 1$ and $\sigma_i(\Gamma^* - \{1, i\}) = 0$ for all $i \neq 1$, and from Locality and Normalisation $\sigma_1(\Gamma^* - \{1, i\}) = n - 2$ for all $i \in N \setminus \{1\}$.

Using Balanced link contributions we will show by induction on n that $\sigma_i(\Gamma^*) = 1$ for all $i \in N \setminus \{1\}$. For $n = 3$:

$$2\sigma_i(\Gamma^*) - \sigma_i(\Gamma^* - e_i) - \sigma_i(\Gamma^* - e_j) = \sigma_1(\Gamma^*) - \sigma_1(\Gamma^* - e_i) = 1$$

and as $\sigma_i(\Gamma^* - e_i) = 0$ and $\sigma_i(\Gamma^* - e_j) = 1$, $\sigma_i(\Gamma^*) = 1$.

Assume that, $\sigma_i(\Gamma^*) = 1$ for all nodes $i \neq 1$ of a star with $n-1$ nodes. Then, by Balanced link contributions:

$$\sum_{e_j \in E_1} [\sigma_i(\Gamma^*) - \sigma_i(\Gamma^* - e_j)] = \sigma_1(\Gamma^*) - \sigma_1(\Gamma^* - e_i) = 1$$

But, by induction hypothesis, Locality and Normalisation $\sigma_i(\Gamma^* - e_i) = 0$ and $\sigma_i(\Gamma^* - e_j) = 1$ for all $j \neq i$, then $\sigma_i(\Gamma^*) = 1$.

Then $\sum_{i \in N} \sigma_i(\Gamma^*) = 2(n-1)$. Taking into account that any connected graph Γ with n nodes can be obtained adding or removing edges from a star and Gain-Loss, we obtain that $\sum_{i \in N} \sigma_i(\Gamma) = 2(n-1)$. Finally, by Locality, $\sum_{i \in S} \sigma_i(\Gamma) = 2(|S| - 1)$ for all connected component $S \subset N$ of any graph Γ.

Therefore, $\sigma(\Gamma)$ verifies Component efficiency for the game $f(S) = 2(s - 1)$ and Balanced link contributions. By Slikker (2005) characterisation, $\sigma(\Gamma) = PA(\Gamma)$ for all Γ. □

6. Some conclusions

In this work, we have proposed a new family of game-theoretic centrality measures based on the Position value (Meessen, 1988) of a given symmetric game that models the functionality of the social network. This approach allows to measure the centrality of a node by evaluating the value of its links explicitly. We have shown that this family, under certain conditions, satisfies typical properties of a centrality measure. In particular, it verifies component efficiency and it assigns a minimum value to the isolated nodes. It also ensures that, among all graphs with n nodes, maximum centrality is achieved by the hub of a star and minimum centrality in connected graphs is achieved by the end-nodes of a chain. In the case of chain, centrality increases progressively from the end-nodes towards the median nodes.

These properties are also shared by other family of game-theoretic centrality measures: he family of Myerson centralities (Grofman and

Owen, 1982; Gómez et al., 2003). Therefore, it was necessary to study in depth the key differences between them.

The first difference among the pair of Myerson and Position centralities determined by the same symmetric game we have studied is related to the respective decomposition of the two measures in closeness and betweenness centrality. In conclusion, we have shown that the Position centrality gives more weight to the betweenness centrality than the Myerson centrality. The reason for this is that the Position value considers more paths (not only minimal paths) to connect nodes.

The second difference examines the impact on the centrality of the end nodes when the edge connecting them is removed. In our approach, the fairness condition, which requires that the removal/addition of a link affects equally the centrality of its incident nodes, is not verified. For the Position centrality, that condition is replaced by the balanced link contributions condition, which could be more realistic in some cases when a clear asymmetry between the end-nodes exists. For instance, we have studied the relevant case of two hub-and-spokes networks connected by their hubs. In this particular example, we have shown that the hub of the larger transportation network gains more Position centrality than the other hub.

The third difference is that our approach does not guarantee stability, a property that estates that adding an edge always benefits its end-nodes. Although it does not happen in general, adding an edge can devalue some of the existing edges incident on a node, negatively affecting its overall balance Position centrality. This issue does not arise if the new edge does not bypass any existing edges. In the particular case of communication situations where the game is the attachment game, it is shown that the two incident nodes improve their Position Attachment centrality.

We would like to remark that, in order to demonstrate the above results, compelling properties of the Harsanyi dividends for the link game, which are interesting in themselves, have been shown.

Finally, based on the work of Skibski et al. (2019) on Attachment centrality, we have characterised the Position Attachment centrality according to four axioms: locality, normalisation, gain–loss and balanced link contributions. It would be interesting to extend this specific analysis to other prominent members of the family.

The problem of computing the Position centrality for big social networks is interesting in itself. The computations for large networks could be undertaken using Castro–Gómez–Tejada algorithm (Castro et al., 2009, 2017) for estimating the Shapley value of the link game, but more specific methods could also be developed.

Another particularly interesting and related line of research is the definition of new game-theoretic families of centrality measures. For this purpose, variants of the link game and of the allocation rule used could be considered. Afterwards, as suggested by one of the reviewers, different ways of aggregating the importance of the edges to obtain the value of the nodes could be considered.

CRediT authorship contribution statement

Susana López: Methodology, Investigation, Formal analysis, Conceptualization. **Elisenda Molina:** Methodology, Investigation, Formal analysis, Conceptualization. **Martha Saboyá:** Methodology, Investigation, Formal analysis, Conceptualization. **Juan Tejada:** Methodology, Investigation, Formal analysis, Conceptualization.

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Appendix

Lemma 1. *Let $(N, v, E) \in CS_0^N$. For every subset $L \subseteq E$ such that Γ_L is not connected it holds $\lambda_L(w^v) = 0$.*

Proof. If Γ_L is not connected, $\mathcal{K}(L)$ has at least two elements. W.l.o.g. suppose that $\mathcal{K}(L) = \{S_1, S_2\}$. Let $L_r = E[S_r]$, for $r = 1, 2$. Then, $L_1 \cap L_2 = \emptyset$, and $w^v(L) = w^v(L_1) + w^v(L_2)$. Note that each subset $T \subseteq L$ can be decomposed as $T_1 \cup T_2$ where $T_r \subseteq L_r, r = 1, 2$. Let us calculate the Harsanyi dividend of L:

$$\begin{aligned}\lambda_L(w^v) &= \sum_{T\subseteq L} (-1)^{l-t} w^v(T) = \sum_{\substack{T_1\subseteq L_1\\T_2\subseteq L_2}} (-1)^{l-t_1-t_2} w^v(T_1\cup T_2)\\ &= \sum_{\substack{T_1\subseteq L_1\\T_2\subseteq L_2}} (-1)^{l-t_1-t_2}\left(w^v(T_1)+w^v(T_2)\right)\\ &= \sum_{\substack{T_1\subseteq L_1\\T_2\subseteq L_2}} (-1)^{l-t_1-t_2} w^v(T_1) + \sum_{\substack{T_1\subseteq L_1\\T_2\subseteq L_2}} (-1)^{l-t_1-t_2} w^v(T_2).\end{aligned}$$

Note that the factor $w^v(T_1)$ appears in the summation as many times as subsets of edges of L_2, and analogously occurs with $w^v(T_2)$. Then

$$\begin{aligned}\lambda_L(w^v) &= \sum_{T_1\subseteq L_1} (-1)^{l-t_1} w^v(T_1)\left(\sum_{T_2\subseteq L_2}(-1)^{-t_2}\right)\\ &\quad+ \sum_{T_2\subseteq L_2} (-1)^{l-t_2} w^v(T_2)\left(\sum_{T_1\subseteq L_1}(-1)^{-t_1}\right)\\ &= \sum_{T_1\subseteq L_1} (-1)^{l-t_1} w^v(T_1)\left(\sum_{k=0}^{l_2}\binom{l_2}{k}(-1)^{-k}\right)\\ &\quad+ \sum_{T_2\subseteq L_2} (-1)^{l-t_2} w_E^v(T_2)\left(\sum_{k=0}^{l_1}\binom{l_1}{k}(-1)^{-k}\right)\\ &= \sum_{T_1\subseteq L_1} (-1)^{l-t_1} w^v(T_1)(1-1)^{l_2}\\ &\quad+ \sum_{T_2\subseteq L_2} (-1)^{l-t_2} w^v(T_2)(1-1)^{l_1} = 0. \quad \square\end{aligned}$$

Next lemma is needed to prove Lemma 2.

Lemma 5. *Let $(N, v, E) \in CS_0^N$. Let $L \subseteq E$, $L \neq \emptyset$, with Γ_L connected. Then the Harsanyi dividend of L in the link game w^v can be computed as follows:*

$$\lambda_L(w^v) = \sum_{\mathcal{D}(L)\subseteq T\subseteq L} (-1)^{l-t} w^v(T) \tag{A.21}$$

where $\mathcal{D}(L)$ is the set of cutedges of L.

Proof. First, we decompose the Harsanyi dividends of L as follows:

$$\begin{aligned}\lambda_L(w^v) &= \sum_{T\subseteq L} (-1)^{l-t} w^v(T)\\ &= \sum_{\mathcal{D}(L)\subseteq T\subseteq L} (-1)^{l-t} w^v(T) + \sum_{T\in\mathcal{T}(L)} (-1)^{l-t} w^v(T),\end{aligned}$$

where $\mathcal{T}(L) = \{T \subseteq L \mid \mathcal{D}(L) \nsubseteq T\}$. Then, we prove that the second summand is zero.

If $\mathcal{D}(L) = \emptyset$ or $\mathcal{T}(L) = \emptyset$, then trivially (A.21) holds. Otherwise, suppose $\mathcal{D}(L) \neq \emptyset$ and $\mathcal{T}(L) \neq \emptyset$. Due to the definition of the link game we only have to consider connected subsets $T \subseteq \mathcal{T}(L)$, since for a non-connected T its link game value would be the sum of the link game values corresponding to its components.

$w^v(T)$ will appear in the summands corresponding to all T' such that $T \subseteq T' \subseteq \mathcal{T}(L)$ and being T one of its connected components. The number of subsets T' with this property of size $t + m$ is

$$\binom{l-d(T)-t}{m},$$

where $d(T)$ is the number of adjacent links to nodes in $N[T]$ that are not in T. Thus, the coefficient of $w^v(T)$ is given by:

$$\sum_{m=0}^{l-d(T)-t} (-1)^{l-t-m} \binom{l-d(T)-t}{m} = (-1)^{l-t} (1-1)^{l-d(T)-t},$$

which equals 0, since there is at least one subset $T' \in \mathcal{T}(L)$, such that T' is not connected and $T \subset T'$ is one of its connected components. □

Lemma 2. *Let $(N,v,E) \in CSS_0^N$. If Γ is cycle-free, then for all $\emptyset \neq L \subseteq E$ with Γ_L connected, the Harsanyi dividends of the link game can be calculated in the following way:*

$$\lambda_L(w^v) = F(l+1, l-d_L), \tag{8}$$

where $d_L = |\mathcal{D}(L)|$, being $\mathcal{D}(L)$ the set of cutedges of L, and $F(s,r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(s-k)$, for $s,r \in \mathbb{N}$ and $s \geq r$.

Proof. Let $\emptyset \neq L \subseteq E$ with Γ_L connected. For all $T \subseteq L$ with $\mathcal{D}(L) \subseteq T$, since Γ is cycle-free, T is a tree with t edges. Then, taking into account the symmetry of v, $w^v(T) = f(t+1)$.

Now by (A.21), it holds

$$\lambda_L(w^v) = \sum_{\mathcal{D}(L)\subseteq T\subseteq L} (-1)^{l-t} w^v(T) = \sum_{\mathcal{D}(L)\subseteq T\subseteq L} (-1)^{l-t} f(t+1) \tag{A.22}$$

$$= \sum_{m=0}^{l-d_L} (-1)^{l-d_L-m} \binom{l-d_L}{m} f(d_L+m+1)$$

$$= F(l+1, l-d_L). \quad \square \tag{A.23}$$

Lemma 3. *Let f be a real function such that $f^{(k)}(x) \geq 0$ in $[1,+\infty)$, for $k = 0,1,\ldots,n$, then the function F verifies:*

(i) $F(s,r) \geq 0$, for all $s,r \in \mathbb{N}$ with $s > r > 0$.
(ii) $F(s,\cdot)$ is decreasing in the second argument for every fixed $s \in \mathbb{N}$, with $r \in \mathbb{N}, s > r$.

Proof. In order to prove (i), note that $F(s,r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(s-k)$, is a numerical approximation of the derivative of f of order $s-r$ in an intermediate point $x \in [s-r, s]$, for all $s,r \in \mathbb{N}$ with $s > r > 0$.

Now, let us prove (ii). We first show that $F(s,r) + F(s-1,r-1) = F(s,r-1)$, where $1 < r < s$. From definition

$$F(s,r-1) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} f(s-k)$$

$$= F(s,r) - (-1)^r f(s-r) - \sum_{k=0}^{r-1} (-1)^k \left[\binom{r}{k} - \binom{r-1}{k}\right] \times f(s-k)$$

$$= F(s,r) + (-1)^{r-1} f(s-r) + \sum_{k=1}^{r-1} (-1)^{k-1} \binom{r-1}{k-1} f(s-k)$$

$$= F(s,r) + \sum_{m=0}^{r-1} (-1)^{m-1} \binom{r-1}{m} f(s-1-m) = F(s,r) + F(s-1,r-1).$$

Then, since $F(s-1,r-1) \geq 0$ by (i), we have $F(s,r) \leq F(s,r-1)$. Thus recursively, we obtain $F(s,r) \leq F(s,r')$, whenever $r' \leq r$, and (ii) holds. □

Next corollaries follow straightforwardly from Lemmas 2 and 3.

Corollary 1. *Let $(N,v,E) \in CSS_0^N$. If Γ is cycle-free, $f \in C^n(\mathbb{R})$ and $f^{(k)}(x) \geq 0$ in $[1,+\infty)$, for $k = 0,1,\ldots,n$, then, for every pair of subsets $L, L' \subseteq E$, such that $l = l'$ and $d_L \leq d_{L'}$ it holds:*

$$\lambda_L(w^v) \leq \lambda_{L'}(w^v)$$

Corollary 2. *Let $(N,v,E) \in CSS_0^N$. If Γ is cycle-free, $f \in C^n(\mathbb{R})$ and $f^{(k)}(x) \geq 0$ in $[1,+\infty)$, for $k = 0,1,\ldots,n$, then $\lambda_L(w^v) \geq 0$, for all $\emptyset \neq L \subseteq E$.*

Proposition 10. *Let $(N,v) \in G_0^N$ be a convex game, and let (N,E) a graph with two connected components (N_1,E_1) and (N_2,E_2). If an edge $b = \{i_0,j_0\}$, with $i_0 \in N_1$ and $j_0 \in N_2$ is added, then it holds:*

$$\pi_i(N_k,v_k,E_k) = \pi_i(N,v,E) \leq \pi_i(N,v,E\cup\{b\}), \tag{17}$$

for all node $i \in N_k$, $k = 1,2$, where v_k denotes the restriction of v to N_k, $k = 1,2$, and $E = E_1 \cup E_2$.

Proof. To shorten notation, we write w instead of (E,w^v), and w^+ instead of $(E\cup\{b\},w^v)$.

Note that the first equality of (17) holds since the position value verifies *Locality* (Proposition 2). We will prove the second inequality, i.e., $\pi_i(N,v,E) \leq \pi_i(N,v,E\cup\{b\})$, for all $i \in N$, by verifying that $\phi_e(w) \leq \phi_e(w^+)$ for all $e \in E$.

Let $e \in E$ be an original edge, then it holds

$$\phi_e(w) = \sum_{L\subseteq E\setminus\{e\}} \frac{l!(m-l-1)!}{m!} \big(w(L\cup\{e\}) - w(L)\big), \tag{A.24}$$

and

$$\phi_e(w^+) = \sum_{L\subseteq E\setminus\{e\}} \Big\{ \frac{l!(m-l)!}{(m+1)!} \big(w^+(L\cup\{e\}) - w^+(L)\big) + \frac{(l+1)!(m-l-1)!}{(m+1)!} \big(w^+(L\cup\{e,b\}) - w^+(L\cup\{b\})\big) \Big\}, \tag{A.25}$$

where $m = |E|$.

Let $e \in E_1$ (the same argument applies to $e \in E_2$) and $L \subseteq E$. Clearly,

$$w^+(L\cup\{e\}) - w^+(L) = w(L\cup\{e\}) - w(L), \forall L \subseteq E. \tag{A.26}$$

Now, we will prove that

$$w^+(L\cup\{e,b\}) - w^+(L\cup\{b\}) \geq w(L\cup\{e\}) - w(L). \tag{A.27}$$

We will distinguish two cases:

(i) If $i_0 \notin \mathcal{K}_i(L\cup\{e\}) = \mathcal{K}_j(L\cup\{e\})$, with $e = \{i,j\}$ (note that then $e \notin E_{i_0}$), then $\mathcal{K}_i(L\cup\{e\}) = \mathcal{K}_i(L\cup\{e,b\})$ and thus $w^+(L\cup\{e,b\}) - w^+(L\cup\{b\}) = w(L\cup\{e\}) - w(L)$.

(ii) If $i_0 \in \mathcal{K}_i(L\cup\{e\})$, then $\mathcal{K}(L\cup\{e,b\})$ equals $\mathcal{K}(L\cup\{e\})$ replacing $\mathcal{K}_{i_0}(L\cup\{e\}) = \mathcal{K}_i(L\cup\{e\})$ by $\mathcal{K}_i(L\cup\{e\}) \cup \mathcal{R}_{j_0}(L)$, where

$$\mathcal{R}_{j_0}(L) = \begin{cases} \{j_0\}, & \text{if } L\cap E_{j_0} = \emptyset, \\ \mathcal{K}_{j_0}(L) = \mathcal{K}_{j_0}(L\cup\{e\}), & \text{otherwise.} \end{cases}$$

Note that the former component $\mathcal{K}_{j_0}(L)$ is replaced by $\mathcal{K}_{i_0}(L\cup\{e\})$ if $L\cap E_{j_0} \neq \emptyset$. Analogously, $\mathcal{K}(L\cup\{b\})$ equals $\mathcal{K}(L)$ replacing $\mathcal{K}_{i_0}(L)$ by $\mathcal{K}_{i_0}(L) \cup \mathcal{R}_{j_0}(L)$. Thus,

$$\Big(w^+(L\cup\{e,b\}) - w^+(L\cup\{b\})\Big) - \Big(w(L\cup\{e\}) - w(L)\Big) = \Big(v\big(\mathcal{K}_i(L\cup\{e\})\cup\mathcal{R}_{j_0}(L)\big) - v\big(\mathcal{K}_i(L)\cup\mathcal{R}_{j_0}(L)\big)\Big) - \Big(v\big(\mathcal{K}_i(L\cup\{e\})\big) - v\big(\mathcal{K}_i(L)\big)\Big),$$

which is non-negative since (N,v) is convex, and thus (A.27) is verified.

Therefore, taking into account (A.26) and (A.27) into expression (A.25) it follows that $\phi_e(w^+) \geq \phi_e(w)$, for all $e \in E$. Thus, for all $i \neq i_0, j_0$

$$\pi_i(N,v,E\cup\{b\}) := \frac{1}{2}\sum_{e\in E_i} \phi_e(w^+) \geq \frac{1}{2}\sum_{e\in E_i} \phi_e(w) := \pi_i(N,v,E).$$

For the end-nodes i_0 and j_0 of the added edge, the increment of the Position centrality is composed of two addends, the first one being a common contribution given by $\frac{1}{2}\phi_b(w^+) \geq 0$ (see the proof of Proposition 1), and the second one, which depends on the effect of adding the bridge b on the original edges of each end-node. We have proved that all these effects are non-negative, since the bridge b does not compete with any of the original edges:

For the end-nodes i_0 and j_0 of the added edge, the increment of the Position centrality is composed of two addends. The first is a common contribution, given by $\frac{1}{2}\phi_b(w^+) \geq 0$ (see the proof of Proposition 1). The second depends on how adding the bridge b affects the original edges of each end-node. We have demonstrated that these effects are non-negative because the bridge does not compete with any of the original edges:

$$\pi_{i_0}(N, v, E \cup \{b\}) - \pi_{i_0}(N, v, E) = \frac{1}{2}\phi_b(w^+) + \frac{1}{2}\sum_{e \in E_{i_0}} (\phi_e(w^+) - \phi_e(w)) \geq 0,$$

$$\pi_{j_0}(N, v, E \cup \{b\}) - \pi_{j_0}(N, v, E) = \frac{1}{2}\phi_b(w^+) + \frac{1}{2}\sum_{e \in E_{j_0}} (\phi_e(w^+) - \phi_e(w)) \geq 0. \quad \square$$

Data availability

No data was used for the research described in the article.

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