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Topología y dinámica de conjuntos non-saddle e índice de Conley en variedades

Topology and dynamics of non-saddle sets and Conley index in manifolds

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Para ti, que caminas siempre a mi lado.

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CONTENTS

RESUMEN

Introducción

Esta tesis está dedicada al estudio, utilizando técnicas topológicas, de la estructura cualitativa de un flujo cerca de un compacto invariante. Este fue uno de los primeros temas clásicos desarrollados por H. Poincaré, I. Bendixson, A. Andronov y S. Lefschetz al inicio de la teoría cualitativa de ecuaciones diferenciales, con contribuciones de autores como D.M. Grobman, P. Hartman, J.K. Hale y A. Stokes y muchos otros. Cabe destacar, por ejemplo, la descripción presentada por T. Ura e I. Kimura en [97] o las teorías de índice de Wazewski y Conley que encapsulan, en un sentido topológico, propiedades del flujo cerca de un compacto invariante (aislado).

La importancia del estudio de la estructura de un flujo cerca de un compacto invariante queda patente en las palabras de J. Auslander, N.P. Bhatia y P. Seibert [5] recogidas en el célebre artículo sobre el concepto de un atractor [60] de J. Milnor:

"En el estudio de las propiedades topológicas de las ecuaciones diferenciales ordinarias, la teoría de estabilidad de compactos invariantes (que podrían ser considerados generalizaciones de puntos críticos y ciclos límite) juega un papel central".

Para terminar, nos gustaría subrayar la riqueza de los aspectos topológicos de los sistemas dinámicos y las ecuaciones diferenciales que, a menudo, presentan propiedades inesperadas y extrañas. Como fue observado por J.A. Kennedy y J.A. Yorke en [51] "La topología extraña es natural en sistemas dinámicos".

Objetivos

Objetivo 1

Recuperar el índice de Conley de un continuo invariante aislado K utilizando solamente información de la topología de K y de la dinámica en su variedad inestable y extraer conclusiones dinámicas y topológicas de interés de la relación obtenida. Este objetivo sigue la línea de los trabajos de Robbin y Salamon [71], Sanjurjo [88] y Sánchez-Gabites [83]. Sin embargo, en esta tesis se busca obtener relaciones más sencillas utilizando la variedad inestable de K dotada de la topología heredada del espacio de fases y no con la topología intrínseca, puesto que ambas topologías son distintas en general y la topología intrínseca es más difícil de manejar.

Objetivo 2

Establecer un marco general que englobe la teoría clásica de atractores estables y repulsores negativamente estables [14] y otras teorías más recientes como la teoría de atractores inestables sin explosiones externas (ver [3,63,82]). En particular, el objetivo es desarrollar la teoría de los conjuntos non-saddle, inicialmente estudiados por N.P. Bhatia y T. Ura [96], y ver que esta teoría es efectivamente un marco natural para extender la teoría de estabilidad y atracción clásica y la teoría de atractores inestables sin explosiones externas.

Resultados

La tesis está estructurada del siguiente modo:

Capítulo 1. El primer capítulo está dedicado al estudio de la estructura de un flujo $\varphi : M \times \mathbb{R} \to M$, definido en una superficie, cerca de un continuo invariante aislado K. Todos los resultados presentados en este capítulo están contenidos en [7, 10, 11].

En la Sección 1.1 se presenta una descripción de los bloques aislantes en superficies. En particular, se prueba el Lema 1.1.3, que muestra que cerca de K el flujo es equivalente a un flujo C^1 y, como consecuencia, K admite una base de entornos compuesta de variedades bloque aislante. Otro resultado interesante es el Teorema 1.1.4, que establece que K tiene el shape de un poliedro finito.

En la Sección 1.2 se estudia la dinámica de la variedad inestable $W^u(K)$ de un compacto invariante aislado. El resultado principal de esta sección es el Teorema 1.2.3, donde se establece que dada una sección inicial de la variedad inestable truncada, el flujo en su parte inicial es paralelizable. Otro resultado importante es el Teorema 1.2.5, que garantiza que todas las secciones iniciales son homeomorfas de manera natural.

La Sección 1.3 se centra en el cálculo del índice de Conley $h(K)$ de un continuo invariante aislado K de un flujo definido en una superficie. Para este propósito probamos el Teorema 1.3.3, que garantiza que K admite una base de entornos formada por bloques aislantes regulares. Este resultado juega un papel crucial en la demostración del Teorema 1.3.7, donde se establece una clasificación completa de todos los posibles índices de Conley de K.

La Sección 1.4 versa sobre el índice cohomológico de un continuo invariante aislado K de un flujo en una superficie. El resultado principal de esta sección es el Teorema 1.4.2, en el que se muestra que el anillo de cohomología $CH^*(K;\mathbb{Z}_2)$ determina $h(K)$.

En la Sección 1.5 se introducen los conjuntos non-saddle, que son el tema principal del Capítulo 2, y se da una condición necesaria y suficiente para que un continuo invariante aislado sea non-saddle en términos de su variedad inestable (Proposición 1.5.3). El resultado principal de esta sección es el Teorema 1.5.9, en el que se establece una clasificación topológica de los continuos invariantes aislados sin puntos fijos.

La Sección 1.6 trata sobre el estudio de atractores de flujos en el plano. Destacamos el Teorema 1.6.2, que establece una caracterización del atractor global de un flujo disipativo en el plano.

Finalmente, la Sección 1.7 trata sobre el estudio de familias parametrizadas de flujos en superficies. En particular, estudiamos continuaciones y bifurcaciones de continuos invariantes aislados. Destacamos el Teorema 1.7.3, que establece condiciones para la conservación de algunas propiedades topológicas y dinámicas por continuación.

Capítulo 2. El segundo capítulo trata sobre el estudio de conjuntos non-saddle de flujos definidos en espacios métricos localmente compactos. Los resultados de este capítulo están contenidos en [6, 9, 12].

En la Sección 2.1 estudiamos propiedades elementales de los conjuntos non-saddle. Destacamos el Teorema 2.1.7, en el que se establece que ciertos non-saddle aislados en el toro tienen el shape de la circuferencia.

En la Sección 2.2 presentamos dos situaciones dinámicas en las que los conjuntos non-saddle aparecen de modo natural.

En la Sección 2.3 profundizamos en la estructura de un flujo que tiene un conjunto non-saddle aislado. En particular, se da una descripción completa de la estructura topológica y dinámica de la región de influencia de un conjunto non-saddle aislado. También demostramos el Teorema 2.3.15, que establece que, si M es una variedad cerrada, conexa y orientada con $H^1(M;\mathbb{Z}) \neq 0$, entonces admite un flujo que tiene un continuo non-saddle aislado con puntos disonantes. Por otro lado, vemos en el Teorema 2.3.16 que la situación es radicalmente opuesta si $H^1(M;G) = 0$. Terminamos esta sección demostrando el Teorema 2.3.20, que proporciona una visión global de la estructura cualitativa de un flujo en un ANR compacto y conexo que posee un non-saddle aislado conexo.

En la Sección 2.4 se introduce y estudia la noción de complejidad dinámica de la región de influencia de un non-saddle aislado conexo. Destacamos el Teorema 2.4.3 en el que se establecen cotas superiores para esta complejidad.

La Sección 2.5 se centra en el estudio de conexiones entre los conjuntos non-saddle aislados y las descomposiciones de Morse. Los resultados principales de esta sección son el Teorema 2.5.3 y el Teorema 2.5.4. Ambos resultados presentan la ecuación de Morse de la descomposición de Morse natural inducida por un non-saddle aislado conexo de un flujo en una variedad cerrada, conexa y G-orientable M con $H^1(M;G) = 0$.

En la Sección 2.6 se estudian propiedades dinámicas de los non-saddle aislados de flujos definidos en superficies. En particular, se establecen condiciones necesarias y suficientes para que un continuo invariante aislado de un flujo en el plano sea non-saddle (Teorema 2.6.3) y para que un continuo non-saddle aislado de un flujo en una superficie no tenga puntos disonantes (Teorema 2.6.5).

En la Sección 2.7 estudiamos condiciones para que la propiedad de ser non-saddle sea robusta. También estudiamos excisiones atractor-repulsor de conjuntos non-saddle. Destacamos el Teorema 2.7.5 y el Teorema 2.7.7, que relacionan la robustez dinámica con la robustez topológica y la robustez topológica fuerte respectivamente.

Conclusiones

A pesar de que es muy complicado calcular el índice de Conley de un continuo invariante aislado de un flujo en un espacio métrico localmente compacto sin utilizar pares índice, hemos visto que esto se puede hacer de manera satisfactoria en el caso de flujos en superficies utilizando información sobre la topología del conjunto invariante y sobre la dinámica de su variedad inestable.

Por otro lado, podemos concluir que la teoría de conjuntos non-saddle se presenta como el marco apropiado para generalizar la teoría de estabilidad y atracción. Hemos visto que esta teoría no solo contiene a la teoría clásica de estabilidad y atracción y la más reciente teoría de atractores aislados sin explosiones externas, sino que comparte muchas de sus buenas propiedades topológicas y dinámicas. De hecho, localmente, un conjunto non-saddle aislado es indistinguible de un atractor aislado sin explosiones externas. Sin embargo, hemos visto que en presencia de puntos disonantes los fenómenos de carácter global que pueden aparecer en el caso de los non-saddle aislados pueden ser mucho más complicados.

SUMMARY

Introduction

This dissertation is devoted to the study, using topological techniques, of the qualitative structure of a flow near a compact invariant set. This was one of the first classical subjects dealt with by H. Poincaré, I. Bendixson, A. Andronov and S. Lefschetz at the beginning of the qualitative theory of differential equations, with further contributions by authors such as D.M. Grobman, P. Hartman, J.K. Hale and A. Stokes and many others. We highlight, for instance, the description provided by T. Ura and I. Kimura in [97] or the index theories by Wazewski and Conley which encapsulate, in a topological way, local properties of the flow near (isolated) invariant sets.

The importance of studying the structure of a flow near a compact invariant set is evident in the words of J. Auslander, N.P. Bhatia and P. Seibert [5] included in the celebrated paper on the concept of an attractor [60] by J. Milnor:

"In the study of topological properties of ordinary differential equations, the stability theory of compact invariant sets (which may be regarded as generalizations of critical points and limit cycles) plays a central role".

Finally, we would like to stress the richness of the topological aspects of dynamical systems and differential equations, often presenting unexpected and strange properties. As was remarked by Kennedy and Yorke in [51] "bizarre topology is natural in dynamical systems".

Objectives

Objective 1

To recover the Conley index of an isolated invariant continuum K using only information about the topology of K and the dynamics in its unstable manifold and to extract dynamical and topological conclusions from the obtained relation. This objective follows the line of the works by Robbin and Salamon [71], Sanjurjo [88] and Sánchez-Gabites [83]. However, in this dissertation we look for simpler relations by using the unstable manifold of K endowed with the topology inherited by the phase space instead of the intrinsic topology, since both topologies do not agree in general and the intrinsic topology is harder to deal with.

Objective 2

To establish a general framework which encompasses the classical theory of stable attractors and negatively stable repellers and more recent theories like the theory of unstable attractors without external explosions (see [3, 63, 82]). In particular, the objective is to develop the theory of non-saddle sets, initially studied by N.P. Bhatia and T. Ura [96], and to see that this theory is, in fact, a natural framework to extend the classical theory of stability and attraction and the theory of unstable attractors without external explosions.

Results

The dissertation is structured as follows:

Chapter 1. The first chapter is devoted to the study of the structure of a flow $\varphi : M \times \mathbb{R} \to M$, defined on a surface, near an isolated invariant continuum K. All the results presented in this chapter are contained in [7, 10, 11].

In Section 1.1 a description of isolating blocks in surfaces is presented. In particular, we prove Lemma 1.1.3, which shows that near K the flow is equivalent to a C^1 flow and, as a consequence, K admits a neighborhood basis composed of isolating block manifolds. Another interesting result is Theorem 1.1.4, which states that K has the shape of a finite polyhedron.

In Section 1.2 we study the dynamics of the unstable manifold $W^u(K)$ of an isolated invariant compactum. The main result of this section is Theorem 1.2.3, where it is established that given an initial section of the truncated unstable manifold, the flow in its initial part is parallelizable. Another important result is Theorem 1.2.5, which ensures that all the initial sections are homeomorphic in a natural way.

Section 1.3 focuses on the computation of the Conley index $h(K)$, of an isolated invariant continuum K. For this purpose, we prove Theorem 1.3.3, which ensures that K admits a basis of neighborhoods comprised of regular isolating blocks. This result plays a key role in the proof of Theorem 1.3.7, where a complete classification of the possible Conley indices of an isolated invariant continuum of a flow on a surface is stated.

Section 1.4 deals with the cohomology index of an isolated invariant continuum K of a flow on a surface. The main result of this section is Theorem 1.4.2, in which we show that the cohomology ring $CH^*(K;\mathbb{Z}_2)$ determines $h(K)$.

In Section 1.5 we introduce non-saddle sets, which are the main topic of Chapter 2, and give a necessary and sufficient condition for an isolated invariant continuum to be non-saddle in terms of its unstable manifold (Proposition 1.5.3). The main result of this section is Theorem 1.5.9, which establishes a topological classification of isolated invariant continua without fixed points.

Section 1.6 deals with the study of attractors of flows on the plane. We highlight Theorem 1.6.2, which establishes a characterization of the global attractor of a dissipative flow on the plane.

Finally, in Section 1.7, we study parametrized families of flows on surfaces. In particular, we study continuations and bifurcations of isolated invariant continua. We highlight Theorem 1.7.3, which states conditions for the preservation of some dynamical and topological properties by continuation.

Chapter 2. The second chapter is devoted to the study of non-saddle sets of flows defined on locally compact metric spaces. The results of this chapter are contained [6, 9, 12].

In Section 2.1 we study elementary properties of non-saddle sets. We highlight Theorem 2.1.7, which states that certain isolated non-saddle sets in the torus must have the shape of a circle.

In Section 2.2 we present two different situations in which non-saddle sets arise in a natural way.

In Section 2.3 we deepen into the structure of a flow having an isolated non-saddle set. In particular, we give a complete description of the dynamical and topological structure of the region of influence of an isolated non-saddle set. We also prove Theorem 2.3.15, which states that, if M is a closed, connected and oriented manifold with $H^1(M;\mathbb{Z}) \neq 0$, then it admits a flow having an isolated non-saddle continuum with dissonant points. On the other hand, we see in Theorem 2.3.16 that, the situation is radically opposite if $H^1(M;G) = 0$. We end this section by proving Theorem 2.3.20, which provides a global vision of the qualitative structure of a flow on a compact and connected ANR having a connected isolated non-saddle set.

In Section 2.4 we introduce and study the notion of dynamical complexity of the region of influence of an isolated non-saddle continuum. We highlight Theorem 2.4.3 in which upper bounds for this complexity are established.

Section 2.5 is devoted to the study of connections between Morse decompositions and isolated non-saddle sets. The main results of this section are Theorem 2.5.3 and Theorem 2.5.4. Both results present the Morse equation of the natural Morse decomposition induced by an isolated non-saddle continuum of a flow on a closed, connected, G-orientable manifold M with $H^1(M;G) = 0$.

In Section 2.6 we deal with dynamical properties of isolated non-saddle sets of flows defined on surfaces. In particular, we state necessary and sufficient conditions for an isolated invariant continuum of a flow on the plane to be non-saddle (Theorem 2.6.3) and for an isolated non-saddle set of a flow on a surface not to have dissonant points (Theorem 2.6.5).

In Section 2.7 we study conditions for the property of being non-saddle to be a robust property. In addition, we also study attractor-repeller splittings of non-saddle sets. We highlight Theorem 2.7.5 and Theorem 2.7.7, which relate the dynamical robustness with the topological and strong topological robustness respectively.

Conclusions

In spite of the fact that, in general, it is very difficult to compute the Conley index without using index pairs, we have seen that this can be done in a succesful way in the case of flows on surfaces using information from the topology of the invariant set and the dynamics of its unstable manifold.

On the other hand, we can conclude that the theory of non-saddle sets presents itself as the appropriate framework to generalize the theory of stability and attraction. We have seen that this theory not only contains the classical theory of stability and attraction and the more recent theory of isolated attractors without external explosions, but it shares a lot of their nice topological and dynamical properties. In fact, locally, an isolated non-saddle set is indistinguishable from an isolated attractor without external explosions. Nevertheless, we have seen that in the presence of dissonant points the kind of global phenomena which may appear in the case of isolated non-saddle sets may be much more complicated.

PRELIMINARIES

Preliminaries about topology

Manifolds. An *n-dimensional manifold* M is a second countable, Hausdorff topological space satisfying that for each point $x \in M$ there exists a neighborhood U of x in M and a homeomorphism $\psi : U \to \mathbb{R}^n$. On the other hand, a second countable Hausdorff space is said to be an *n-manifold with boundary* if it satisfies that for each point $x \in M$ there exists a neighborhood U of x in M homeomorphic either to $\mathbb{R}^n$ or to the upper half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

An n-manifold (resp. n-manifold with boundary) is said to be *smooth* if it can be covered by neighborhoods $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ in such a way that for each $\alpha, \beta \in I$ the *transition map*

$$\psi_\beta \circ \psi_\alpha^{-1}|_{\psi_\alpha(U_\alpha \cap U_\beta)} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta),$$

is C^∞.

Notice that we use the term *differentiable* with the same meaning as smooth.

We are specially interested in 2-manifolds. Through this dissertation connected 2-manifolds will be called surfaces. We reccommend to the reader the book [54] and the paper [70] as references about the topology of surfaces.

Regarding the orientability of manifolds, to avoid a rather technical exposition we prefer to refer the reader to [42, pg. 233]. We shall make use of the concept of G-orientability, where G is going to be either $\mathbb{Z}$ or $\mathbb{Z}_2$. In particular, the fact that every manifold is $\mathbb{Z}_2$-orientable will be useful for us. Notice that if we do not specify the group of coefficients when talking about orientability, it will be understood that we are meaning $\mathbb{Z}$-orientability.

ANR's. A metric space X is said to be an Absolute neighborhood retract or, shortly, an *ANR* if it satisfies that whenever there exists an embedding $f : X \to Y$ of X into a metric space Y such that $f(X)$ is closed in Y, there exists a neighborhood U of $f(X)$ such that $f(X)$ is a retract of U. Some examples of ANR's are manifolds, CW-complexes and polyhedra. Besides, an open subset of an ANR is an ANR and a retract of ANR is also an ANR. For more information about ANR's we recommend [17] and [45].

Algebraic Topology. We use some topological notions through this dissertation. A good reference for this material is the book of Spanier [93]. We will use the notation H_* and H^* for the singular homology and cohomology respectively and we will denote by $\check{H}_*$ and $\check{H}^*$ the Čech homology and cohomology functors. In both cases we consider homology and cohomology taking coefficients in G, where G is either $\mathbb{Z}$ or $\mathbb{Z}_2$. We do not make explicit mention to the coefficients through the dissertation unless we need to. We recall that Čech and singular cohomology theories agree on ANR's and, from this fact, combined with a simple argument involving the long exact sequences of Čech and singular cohomology of a pair, the natural homomorphism between both cohomologies and the five lemma, it can be proved that it also holds for pairs of ANR's.

Let (X, A) be a pair of topological spaces, we recall that the *i-dimensional Betti number* $\beta_i(X, A)$ is the rank of the Čech cohomology group $\check{H}^i(X, A)$. Notice that the Universal Coefficient Theorem ensures that $\beta_i(X, A)$ agrees with the rank of $\check{H}_i(X, A)$. We would like to point out that, since we only use singular homology and cohomology when dealing with ANR's, i.e. where both Čech and singular theories agree, we will also denote by $\beta_i(X, A)$ to the rank of $H^i(X, A)$ (resp. $H_i(X, A)$) since it should not lead to any confussion.

If a pair of spaces (X, A) satisfies that its cohomology $\check{H}^k(X, A)$ is finitely generated for each k and is non-zero only for a finite number of values of k, as it happens if (X, A) is a pair of compact ANR's, its *Poincaré polinomial* is defined as

$$P_t(X, A) = \sum_{k \geq 0} \beta_k(X, A) t^k.$$

In this case, the *Euler characteristic* of the pair (X, A) is defined as

$$\chi(X, A) = P_{-1}(X, A).$$

Although each Betti number depends on the choice of G the Euler characteristic is independent of this choice. A useful property of the Euler characteristic which can be found in [93] is that

$$\chi(X) = \chi(X, A) + \chi(A).$$

Some applications of homological techniques to dynamics can be seen in the papers [8, 77, 84].

Shape theory. There is a form of homotopy which has proved to be the most convenient for the study of the global topological properties of the invariant spaces involved in dynamics, namely the *shape theory* introduced and studied by Karol Borsuk. Although we are not going to make a deep use of shape theory, we reproduce here, for the sake of completeness, the introduction made by Kapitanski and Rodnianski in [49].

Let X be a closed subset of an ANR M and Y a closed subset of an ANR N. Denote by $\mathbb{U}(X; M)$ (resp. $\mathbb{U}(Y; N)$) the set of all open neighborhoods of X in M (resp. Y in N).

Let $\mathbf{f} = \{f : U \to V\}$ be a collection of continuous maps from the neighborhoods $U \in \mathbb{U}(X; M)$ to $V \in \mathbb{U}(Y; N)$. We say that $\mathbf{f}$ is a *mutation* from X to Y if it satisfies:

1. For every $V \in \mathbb{U}(Y; N)$ there exists at least a map $f : U \to V$ in $\mathbf{f}$.

2. If $f : U \to V$ is in $\mathbf{f}$ then the restriction $f|_{U_1} : U_1 \to V_1$ is also in $\mathbf{f}$ for every neighborhood $U_1 \subset U$ and every neighborhood $V_1 \supset V$.

3. If two maps $f, f' : U \to V$ are in $\mathbf{f}$, there exists a neighborhood $U_1 \subset U$ such that the restrictions $f|_{U_1}$ and $f'|_{U_1}$ are homotopic.

An example of mutation is the *identity mutation* $\mathrm{id}_{\mathbb{U}(X;M)}$ consisting of the identity maps $\mathrm{id} : U \to U$.

Composition of mutations $\mathbf{f} = \{f : U \to V\}$, $\mathbf{g} = \{g : V \to W\}$ from X to Y and from Y to Z, respectively, is defined in the straightforward way. Two mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{f'} = \{f' : U' \to V'\}$ (both from X to Y) are said to be *homotopic* if for every pair of maps $f : U \to V$ and $f' : U' \to V$ belonging to $\mathbf{f}$ and $\mathbf{f'}$ respectively, there exists a neighborhood $U_0 \in \mathbb{U}(X;M)$, $U_0 \subset U \cap U'$ such that $f|_{U_0}$ is homotopic to $f'|_{U_0}$. It is easy to see that homotopy of mutations is an equivalence relation.

Let X and Y be metric spaces. We say that X is *shape dominated* by Y if they can be embedded as closed sets in ANR's M and N in such a way that there exist mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{g} = \{g : V \to U\}$ such that the composition $\mathbf{gf}$ is homotopic to the identity mutation $\mathrm{id}_{\mathbb{U}(X;M)}$.

Two metric spaces X and Y have the same *shape* if they can be embedded as closed sets in ANR's M and N in such a way that there exist mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{g} = \{g : V \to U\}$ such that the compositions $\mathbf{gf}$ and $\mathbf{fg}$ are homotopic to the identity mutations $\mathrm{id}_{\mathbb{U}(X;M)}$ and $\mathrm{id}_{\mathbb{U}(Y;N)}$ respectively. In this case, the mutation $\mathbf{f}$ (resp. $\mathbf{g}$) is said to be a *shape equivalence*.

We stress the following basic features whose proofs can be found in [19]:

1. The notion of shape of sets depends neither on the ANR's they are embedded in nor on the particular embeddings.

2. Spaces belonging to the same homotopy type have the same shape.

3. ANR's have the same shape if and only if they have the same homtopy type.

4. Čech homology and cohomology are shape invariants.

The following characterizations of the shape of plane continua [19] and of continua in surfaces [79, 81] will be useful in the sequel.

Theorem 1 (K. Borsuk [19])**.** *Two continua K and L contained in $\mathbb{R}^2$ have the same shape if and only if they disconnect $\mathbb{R}^2$ in the same number of connected components. More generally, the shape of K dominates the shape of L if and only if the number of connected components of $\mathbb{R}^2 \setminus L$ is less than or equal to the number of components of $\mathbb{R}^2 \setminus K$. In particular, a continuum has the shape of a point if and only if it does not disconnect $\mathbb{R}^2$. A continuum has the shape of a circle if and only if it disconnects $\mathbb{R}^2$ into two connected components. Every continuum has the shape of a wedge of circles, finite or infinite.*

Theorem 2. *Let K be a compactum contained in a compact 2-manifold (resp. in the interior of a 2-manifold with boundary) M. If the inclusion $i : K \hookrightarrow M$ induces isomorphisms $i^* : \check{H}^k(M;\mathbb{Z}_2) \to \check{H}^k(K;\mathbb{Z}_2)$ for $0 \leq k \leq 2$, then it is a shape equivalence.*

Corollary 1. *Let K be a continuum contained in a 2-manifold (resp. in the interior of a 2-manifold with boundary) M. If $\check{H}^2(K;\mathbb{Z}_2) = 0$ and $\check{H}^1(K;\mathbb{Z}_2)$ is finitely generated, then K has the shape of a wedge of $\beta_1(K;\mathbb{Z}_2)$ circumferences.*

Notice that if M is a compact and connected 2-manifold with boundary and K is a subcontinuum contained in its interior, it would be enough i^* : $\check{H}^1(M;\mathbb{Z}_2) \to \check{H}^1(K;\mathbb{Z}_2)$ to be an isomorphism to meet the assumptions of Theorem 2 and, hence, to ensure that the inclusion is a shape equivalence. On the other hand, if we only consider proper subcontinua of surfaces, Corollary 1 ensures that $\beta_1(K;\mathbb{Z}_2)$, when finite, determines the shape of K. These facts can be easily seen using Alexander duality.

We are also going to use [81, Teorema B.7], which states that if K is a continuum contained in a 2-manifold (resp. in the interior of a 2-manifold with boundary) M and N_1 and N_2 are compact and connected submanifolds with boundary of M which are neighborhoods of K in M such that the inclusions i_k : $K \hookrightarrow N_k$, $k = 1, 2$, are shape equivalences, then N_1 and N_2 are homeomorphic.

For a complete treatment of shape theory we refer the reader to [19, 24, 26, 52, 53, 85]. The use of shape in dynamics is illustrated by the papers [31–34, 37, 40, 49, 62, 71, 72, 82, 86, 87, 90, 98].

Preliminaries about dynamical systems

The main reference for the elementary concepts of dynamical systems will be [16] but we also recommend [68, 69, 73].

Flows. A *continuous dynamical system* or *flow*, defined on a locally compact metric space M, is a continuous map $\varphi : M \times \mathbb{R} \to M$ satisfying

1) $\varphi(x, 0) = x$ for each $x \in M$

2) $\varphi(x, t + s) = \varphi(\varphi(x, t), s)$ for each $x \in M$ and $t, s \in \mathbb{R}$.

Notice that, from the very definition of flow, it easily follows that for each $t \in \mathbb{R}$, the map $\varphi_t : M \to M$, $x \mapsto \varphi_t(x) = \varphi(x,t)$ is a homeomorphism isotopic to the identity.

Through this dissertation we usually use the notation xt to denote $\varphi(x,t)$. In a similar way, we denote by AJ to $\varphi(A,J)$ when $A \subset M$ and $J \subset \mathbb{R}$.

Trajectories and limit sets. We shall use the notation $\gamma(x)$ for the *trajectory* of the point x, i.e.

$$\gamma(x) = \{xt \mid t \in \mathbb{R}\}.$$

Similarly for the *positive semi-trajectory* and the *negative semi-trajectory*

$$\gamma^+(x) = \{xt \mid t \in \mathbb{R}^+\}, \quad \gamma^-(x) = \{xt \mid t \in \mathbb{R}^-\}.$$

By the *omega*-limit and the *negative omega*-limit of a set $X \subset M$ we understand respectively the sets

$$\omega(X) = \bigcap_{t>0} \overline{X \cdot [t,\infty)}, \quad \omega^*(X) = \bigcap_{t>0} \overline{X \cdot (-\infty,-t]}.$$

On the other hand, the *positive prolongational limit set* of a point x is the set

$$J^+(x) = \bigcap_{U \in \mathcal{E}(x), t>0} \overline{U[t,\infty)},$$

where $\mathcal{E}(x)$ denotes the system of neighborhoods of the point x. The *negative prolongational limit set* of a point x, $J^-(x)$ is defined in a dual fashion. Besides, we need to introduce the concept of *two-sided prolongational limit set*

Definition 1. Given $x \in M$, the *two-sided prolongational limit set* of x, is defined to be

$$J^*(x) := \{(y,z) \in M \times M \mid \text{there exist } x_n \to x,\ t_n \to \infty,\ s_n \to -\infty \\ \text{such that, } x_n t_n \to y \text{ and } x_n s_n \to z\}.$$

It is easy to see that J^* is closed in $M \times M$ and that $(yt, zs) \in J^*(x)$ for every $(y,z) \in J^*(x)$ and every $t, s \in \mathbb{R}$.

Sections and parallelizable flows. Given a flow $\varphi : M \times \mathbb{R} \to M$ by a *section* S, we mean a set which intersects each trajectory exactly in a point.

The flow φ is said to be *parallelizable* if it admits a section S such that the map $\sigma : M \to \mathbb{R}$ defined by the property $x\sigma(x) \in S$ is continuous. Notice that, if one section satisfies that condition, all of them do.

If a flow is parallelizable and S is a section, the map $h : S \times \mathbb{R} \to M$ defined by $(x, t) \mapsto xt$ is a homeomorphism. A direct consequence of these considerations is that a section S of a parallelizable flow is a strong deformation retract of M and the deformation retraction is provided by the flow.

Invariant manifolds, stability, attractors and repellers. The *stable* and *unstable manifolds* of an invariant compactum K are defined respectively as the sets

$$W^s(K) = \{x \in M \mid \emptyset \neq \omega(x) \subset K\}, \quad W^u(K) = \{x \in M \mid \emptyset \neq \omega^*(x) \subset K\}.$$

We shall also make use of the concept of region of influence of an invariant compactum.

Definition 2. If K is an invariant compactum, the *region of influence* of K is the set

$$\mathcal{I}(K) = W^s(K) \cup W^u(K).$$

An invariant compactum K is said to be *stable* if every neighborhood U of K contains a neighborhood V of K such that $V[0, \infty) \subset U$. Similarly, K is *negatively stable* if every neighborhood U of K contains a neighborhood V of K such that $V(-\infty, 0] \subset U$.

The compact invariant set K is said to be *attracting* provided that there exists a neighborhood U of K such that $\emptyset \neq \omega(x) \subset K$ for every $x \in U$ and *repelling* if there exists a neighborhood U of K such that $\emptyset \neq \omega^*(x) \subset K$ for every $x \in U$.

If K is an attracting (resp. repelling) set, its stable (resp. unstable) manifold is usually called *region (or basin) of attraction (resp. repulsion)* and denoted by $\mathcal{A}(K)$ (resp. $\mathcal{R}(K)$). It is well known that $\mathcal{A}(K)$ (resp. $\mathcal{R}(K)$) is an invariant open set. An attracting (resp. repelling) set K is *globally attracting* (resp. *globally repelling*) provided that $\mathcal{A}(K)$ (resp. $\mathcal{R}(K)$) is the whole phase space.

For the reader interested in a detailed treatment of attracting sets we recommend [63] and [82].

An *attractor* (or *asymptotically stable* compactum) is an attracting stable set and a *repeller* is a repelling negatively stable set. If K is an attractor (resp. repeller) and $\mathcal{A}(K)$ (resp. $\mathcal{R}(K)$) is the whole phase space, then K is said to be a *global attractor* (or *globally asymptotically stable* compactum) (resp. *global repeller*). We stress the fact that stability (positive or negative) is required in the definition of attractor or repeller.

If K is an attractor (resp. repeller), the restriction flow $\varphi|_{\mathcal{A}(K)\setminus K}$ (resp. $\varphi|_{\mathcal{R}(K)\setminus K}$) is parallelizable and its sections are compact.

The following result will be useful in the sequel

Theorem 3 (Morón, Sánchez-Gabites and Sanjurjo [63]). *Every connected isolated globally attracting set K in $\mathbb{R}^2$ is a global attractor.*

Isolated invariant sets and isolating blocks. A compact invariant set K is said to be an *isolated invariant set* if it possesses a so-called *isolating neighborhood*, that is, a compact neighborhood N such that K is the maximal invariant set in N, or setting

$$N^+ = \{x \in N \mid x[0, +\infty) \subset N\}, \quad N^- = \{x \in N \mid x(-\infty, 0] \subset N\};$$

such that $K = N^+ \cap N^-$. Notice that N^+ and N^- are compact and, respectively, positively and negatively invariant. For instance, attractors and repellers are isolated invariant sets.

To avoid trivial cases, when we consider an isolated invariant set, it will be implicit that it is a non-empty proper subsets of the phase space unless otherwise specified.

The dynamical structure near isolating invariant sets shall play an important role in this dissertation and it is described by the following result.

Theorem 4 (Ura-Kimura-Egawa [28,97]). *Let M be a locally compact separable metric space and φ a flow on M. Suppose K is an isolated invariant compactum. Then, one and only one of the following alternatives holds:*

1. *K is an attractor;*

2. *K is a repeller;*

3. *There exist points $x \in M \setminus K$ and $y \in M \setminus K$ such that $\emptyset \neq \omega(x) \subset K$ and $\emptyset \neq \omega^*(y) \subset K$.*

We shall make use of a special type of isolating neighborhoods, the so-called *isolating blocks*, which have good topological properties. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and exit sets, satisfying

1. $\partial N = N^i \cup N^o$,

2. for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M \setminus N$
and for every $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M \setminus N$,

3. for every $x \in \partial N \setminus N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \mathring{N}$
and for every $x \in \partial N \setminus N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \mathring{N}$.

These blocks form a neighborhood basis of K in M. We shall also use the notation $n^+ = N^+ \cap \partial N$ and $n^- = N^- \cap \partial N$.

Associated to an isolating block N there are defined two continuous functions

$$t^o : N \setminus N^+ \to [0, +\infty), \quad t^i : N \setminus N^- \to (-\infty, 0]$$

given by

$$t^o(x) := \sup\{t \geq 0 \mid x[0, t] \subset N\}, \quad t^i(x) := \inf\{t \leq 0 \mid x[t, 0] \subset N\}.$$

These functions are known as the *exit time* and the *entrance time* respectively.

If the phase space is a smooth manifold and the flow is of class C^r with $r \geq 1$, the isolating blocks can be chosen to be manifolds with boundary which contain N^i and N^o as submanifolds of their boundaries and such that $\partial N^i = \partial N^o = N^i \cap N^o$. This kind of isolating blocks will be called *isolating block manifolds*. For flows defined on surfaces, the exit set N^o of an isolating block manifold is the disjoint union of a finite number of intervals $J_1, \ldots, J_m$ and circumferences $C_1, \ldots, C_n$ and the same is true for the entrance set N^i.

Smoothing of 2-dimensional flows. We shall also make use of a classical result of C. Gutiérrez about smoothing of 2-dimensional flows.

Theorem 5 (Gutiérrez [38]). *Let $\varphi : M \times \mathbb{R} \to M$ be a continuous flow on a compact C^∞ 2-manifold M. Then there exists a C^1 flow ψ on M which is topologically equivalent to φ. Furthermore, the following conditions are equivalent:*

1. *any minimal set of φ is trivial;*

2. *φ is topologically equivalent to a C^2 flow;*

3. *φ is topologically equivalent to a C^∞ flow.*

By a trivial minimal set we understand a fixed point, a closed trajectory or the whole manifold if M is the 2-dimensional torus and φ is topologically equivalent to an irrational flow.

We readily deduce from Gutiérrez' Theorem applied to the Alexandrov compactification of the plane that continuous flows $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are topologically equivalent to C^∞ flows.

The Conley index theory. The *Conley index* of an isolated invariant set is an important tool in dynamical systems and plays a central role in this dissertation. It is intended to be a generalization of the classical Morse index of non-degenerate fixed points of gradient like vector fields.

Definition 3. Let K be an isolated invariant set and N an isolating block of K. The *Conley index* $h(K)$ of the isolated invariant set K is defined as the homotopy type of the pointed space $(N/N^o, [N^o])$.

A crucial fact concerning the previous definition is, of course, that this homotopy type does not depend on the particular choice of N.

An important property of the Conley index which will be useful in the sequel is that, if an isolated invariant set K possesses $K_1, \ldots, K_n$ connected components, then

$$h(K) = \bigvee_{i=1}^{n} h(K_i).$$

Notice that, if K is isolated so are all of his components.

We will also make use of the *cohomology index* $CH^*(K)$ defined as the cohomology $H^*(N/N^o, [N^o])$. The *homology index* is defined in an analogous way using homology. We denote by $C\check{H}^*(K)$ to the cohomology index using Čech cohomology, i.e. to $\check{H}^*(N/N^o, [N^o])$. Besides, we also denote by $CH^*_-(K)$ to the *negative cohomology index*, i.e. to $H^*(N/N^i, [N^i])$ and the *negative homology index* $CH_*^-(K)$ is defined in an analagous fashion using homology. As before, we denote by $C\check{H}_-^*(K)$ to the negative cohomology index using Čech cohomology. Notice that it can be seen that $CH^*(K) \cong H^*(N, N^o)$ and that the analogous isomorphims hold for the other homology and cohomology indices introduced.

We refer the reader to [21–23, 61, 78] for information about the Conley index theory. We also recommend the survey [46], where some connections with the classical Morse theory and the Brouwer degree are stated, and [91, 100] to see recent applications of the Conley index techniques to some problems in ecology.

Morse decompositions and equations.

We recall that if K is a compact invariant set, the finite collection $\{M_1, \ldots, M_n\}$ of pairwise disjoint invariant subcompacta of K is a *Morse decomposition* if it satisfies that

$$\text{for each } x \in \left(K \setminus \bigcup_{i=1}^{n} M_i\right), \quad \omega(x) \subset M_j \text{ and } \omega^*(x) \subset M_k \text{ with } j < k.$$

Each set M_i is said to be a *Morse set*.

If we have a flow φ on a compact metric space M and $\{M_1, \ldots, M_n\}$ is a Morse decomposition of M, then the *dual set* of M_k is defined as

$$M_k^* = \{x \in M \mid \omega(x) \not\subset M_k \text{ and } \omega^*(x) \not\subset M_k\}.$$

Therefore, $x \in M_k^*$ if and only if $\omega(x) \subset M_i$ and $\omega^*(x) \subset M_j$ with $i, j \neq k$.

Given a Morse decomposition $\{M_1, \ldots, M_k\}$ of an isolated invariant set K, there exists a polynomial $Q(t)$ whose coefficients are non-negative integers such that

$$\sum_{i=1}^{n} P_t(h(M_i)) = P_t(h(K)) + (1+t)Q(t).$$

This formula, which relates the Conley indices of the Morse sets with the Conley index of the isolated invariant set is known as the *Morse equation* of the Morse decomposition and it generalizes the classical Morse inequalities.

Continuations of isolated invariant sets.

Let M be a locally compact metric space. We say that the family of flows $\varphi_\lambda : M \times \mathbb{R} \to M$, with λ in the unit interval I, is a *parametrized family of flows* if the map $\varphi : M \times \mathbb{R} \times I \to M$ given by $\varphi(x,t,\lambda) = \varphi_\lambda(x,t)$ is continuous. If in addition M is a differentiable manifold and the map φ is differentiable we say that the family is a *differentiable parametrized family of flows.*

Let M be a locally compact metric space, and let $\varphi_\lambda : M \times \mathbb{R} \to M$ be a parametrized family of flows with $\lambda \in [0,1]$. The family $(K_\lambda)_{\lambda\in J}$, where $J \subset [0,1]$ is a closed (non-degenerate) subinterval and, for each $\lambda \in J$, K_λ is an isolated invariant set for φ_λ is said to be a *continuation* if for each $\lambda_0 \in J$ and each N_{λ_0} isolating neighborhood for K_{λ_0}, there exists $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$. We say that the family $(K_\lambda)_{\lambda\in J}$ is a continuation of K_{λ_0} for each $\lambda_0 \in J$.

Notice that [78, Lemma 6.1] ensures that if K_{λ_0} is an isolated invariant set for φ_{λ_0}, there always exists a continuation $(K_\lambda)_{\lambda\in J_{\lambda_0}}$ of K_{λ_0} for some closed (non-degenerate) subinterval $\lambda_0 \in J_{\lambda_0} \subset [0,1]$.

There is a simpler definition of continuation based on [78, Lemma 6.2]. There, it is proved that if $\varphi_\lambda : M \times \mathbb{R} \to M$ is a parametrized family of flows and if N_1 and N_2 are isolating neighborhoods of the same isolated invariant set for φ_{λ_0}, then there exists $\delta > 0$ such that N_1 and N_2 are isolating neighborhoods for φ_λ, for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$, with the property that, for every λ, the isolated invariant subsets in N_1 and N_2 which have N_1 and N_2 as isolating neighborhoods agree.

Therefore, the family $(K_\lambda)_{\lambda\in J}$, with K_λ an isolated invariant set for φ_λ, is a continuation if for every $\lambda_0 \in J$ there are an isolating neighborhood N_{λ_0} for K_{λ_0} and a $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ, for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$.

We will make use of the fact that if $(K_\lambda)_{\lambda\in J}$ is a continuation then, for each $\lambda_1, \lambda_2 \in J$, the Conley indices $h(K_{\lambda_1})$ and $h(K_{\lambda_2})$ agree (see [78, Corollary 6.8]). A consequence of this fact is that if K_{λ_0} is a non-empty attractor and $(K_\lambda)_{\lambda\in J}$ is a continuation of it, then K_λ is non-empty for each $\lambda \in J$.

We are interested in continuations $(K_\lambda)_{\lambda\in J}$, with $0 \in J$, where K_0 is a global attractor. Since K_0 is an attractor, using [87, Theorem 4] it follows that there exists $0 < \lambda_0 \in J$ such that, for $\lambda < \lambda_0$, K_λ is an attractor which has the shape of K_0. As a consequence, if the phase space is a Euclidean space, then for small values of λ, K_λ has the shape of a point and, in particular, it is connected.

Notice that, since this should not lead to any confusion, sometimes we will only say that K_λ is a continuation of K_{λ_0} without specifying the subinterval $J \subset [0,1]$ to which the parameters belong.

Planar vector fields, Brouwer degree and fixed point index.

We use some basic results about planar vector fields. For instance, we use the Poincaré-Bendixson Theorem and some properties of tranversal sections. Two good references covering this material are the book of Hirsch, Smale and Devaney [44] and the monograph of Palis and de Melo [68].

We also make use of some elementary properties of Brouwer degree and fixed point index. We suggest the references [1,47,65] for a complete treatment of the subject, the monography [48] for a clear exposition about fixed point index and the papers [43,75,76,101] for some connections with dynamical systems and the Conley index.

CHAPTER 1

UNSTABLE MANIFOLD, CONLEY INDEX AND APPLICATIONS

The aim of this chapter is to study the structure of a flow $\varphi : M\times\mathbb{R} \to M$, defined on a surface M, near an isolated invariant continuum K. We recall that, to avoid trivial cases, when we consider an isolated invariant set K, we always assume that it is a proper subset of M, i.e., $\emptyset \neq K \subsetneq M$ unless otherwise specified. An important question regarding the dynamics near an isolated invariant set K is to understand the dynamics on its unstable manifold $W^u(K)$. It turns out that K is a repelling set for $\varphi|_{W^u(K)}$ but not necessarily a repeller. In particular, the flow $\varphi|_{W^u(K)\setminus K}$ is not in general parallelizable. Some attemps have been made to give $W^u(K)$ a reasonable structure; however these attemps pass through defining the so-called *intrinsic topology* in $W^u(K)$ introduced by Robbin and Salamon in [71] (see also [4, 83, 88]). This topology does not agree in general with the topology inherited from the phase space M, so the problem remains of studying $W^u(K)$ with its natural topology to find some regularity in its structure. In particular, we see that in spite of the fact that $\varphi|_{W^u(K)\setminus K}$ is not parallelizable, there exist certain sections S for which the flow is parallelizable when restricted to an *initial part* of $W^u(K)\setminus K$, that is the part of the flow coming before S.

The problem of understanding the dynamics in $W^u(K)$ is also related with the problem of computing the Conley index of an isolated invariant set without using index pairs [83]. The Conley index is a topological tool which encapsulates some significant information about the dynamics near an isolated invariant set. However, one of the main drawbacks of the Conley index theory is that it is defined in terms of external objects called isolating blocks (or more generally, in terms of index pairs) and, in general, given two different isolating blocks of the same isolated invariant set they could be very different from the topological point of view. This makes very difficult to establish precise relations between the topology of K and the dynamics near it with its Conley index. We overcome these difficulties in the 2-dimensional case for isolated invariant continua and see that, in this case, it is possible to reconstruct the Conley index having only some information about the dynamical nature (if it is an attractor, a repeller or neither of them), the topology of K (its first Betti number or its shape) and the knowledge of an initial section $W^u(K) \setminus K$. This enables us to elaborate a complete classification of all the possibilities for the Conley index of an isolating invariant continuum in a surface. We use these facts to extract some nice consequences about the dynamics of isolated invariant continua on surfaces and about the preservation of some topological and dynamical properties under continuation.

Since through this chapter we deal with surfaces, to avoid questions about orientability we consider both Čech and singular homology and cohomology theories with $\mathbb{Z}_2$ coefficients.

All the results of this chapter are contained in [7, 10, 11].

1.1 Isolating blocks in surfaces

In this section we study the structure of a flow defined on a surface near an isolated invariant continuum K. In particular we see that K admits a neighborhood in which the flow is topologically equivalent to a C^1 flow. From this fact we deduce that K has the shape of a finite polyhedron.

The following result states some useful properties of isolating blocks which we exploit in the sequel.

Lemma 1.1.1. *Suppose that K is an isolated invariant continuum of a flow on a manifold and that N is a connected isolating block manifold of K. Then*

a) Each component of N^o must contain some component of n^-, *b) n^- has a finite number of components, and*

c) if x_0 is a point in $N^o \setminus n^-$ and U a compact neighborhood of x_0 in $N^o \setminus n^-$ then, the set

$$W = \bigcup_{x \in U} x[t^i(x), 0]$$

is homeomorphic to the product $U \times [0,1]$ via a homeomorphism which carries each trajectory segment $x[t^i(x), 0]$ to the fiber $\{x\} \times [0,1]$.

Proof. Since the inclusion $K \hookrightarrow N^-$ is a shape equivalence [49, Theorem 3.6], a straightforward application of the five lemma gives that $\check{H}^k(N,K) \cong \check{H}^k(N,N^-)$. In addition, the inclusion $N^- \cup N^o \hookrightarrow N$ is also a shape equivalence (see [88, Theorem 1]) and, reasoning as before, it follows that $\check{H}^k(N,N^-) \cong \check{H}^k(N^- \cup N^o, N^-)$. On the other hand, by the strong excision property of Čech cohomology

$$\begin{aligned} \check{H}^k(N^- \cup N^o, N^-) &\cong \check{H}^k\left(\frac{N^- \cup N^o}{N^-}, [N^-]\right) \\ &\cong \check{H}^k\left(\frac{N^o}{n^-}, [n^-]\right) \\ &\cong \check{H}^k(N^o, n^-). \end{aligned}$$

Since N and K are connected, $0 = \check{H}^0(N,K) = \check{H}^0(N^o, n^-)$ and, hence, from the long exact sequence of cohomology of the pair (N^o, n^-) we get that the homomorphism

$$\check{H}^0(N^o) \to \check{H}^0(n^-)$$

induced by the inclusion $n^- \hookrightarrow N^o$ is a monomorphism. This proves a).

Consider the long exact sequence of reduced Čech cohomology of the pair (N,K)

$$0 \to \check{H}^1(N,K) \to \check{H}^1(N) \to \check{H}^1(K) \to \check{H}^2(N,K) \to \cdots$$

Since N is a manifold, then $\check{H}^1(N)$ agrees with $H^1(N)$ and, hence, it is finitely generated. Thus, from the exact sequence we get that $\check{H}^1(N,K)$ is also finitely generated. As a consequence, $\check{H}^1(N^o, n^-)$ is finitely generated being isomorphic to $\check{H}^1(N,K)$. Moreover, since $\check{H}^0(N^o, n^-) = 0$, the long exact sequence of the pair (N^o, n^-) splits into the short exact sequence

$$0 \to \check{H}^0(N^o) \to \check{H}^0(n^-) \to \operatorname{im}\delta \to 0$$

where $\delta : \check{H}^0(n^-) \to \check{H}^1(N^o, n^-)$ is the coboundary homomorphism. In addition, the groups $\check{H}^0(N^o)$ and $\operatorname{im}\delta$ are finitely generated since N^o has a finite number of components being a compact manifold and $\operatorname{im}\delta$ being a subgroup of the finitely generated group $\check{H}^1(N^o, n^-)$. Therefore, $\check{H}^0(n^-)$ is finitely generated. This proves b).

Let $x_0 \in N^o \setminus n^-$ and U be a compact neighborhood of x_0 in $N^o \setminus n^-$. Consider for each $x \in U$ the linear homeomorphism $\sigma_x : [0,1] \to [t^i(x), 0]$ given by $\sigma_x(s) = t^i(x)(1-s)$. We define $h : U \times [0,1] \to W$ as $h(x,s) = x\sigma_x(s)$ which is clearly a bijection. We see that h is continuous. Let (x_n) and (s_n) be sequences in U and $[0,1]$ convergent to $\bar{x} \in U$ and $\bar{s} \in [0,1]$ respectively. Then, $\sigma_{x_n}(s_n) = t^i(x_n)(1-s_n)$, which by the continuity of t^i converges to $\sigma_{\bar{x}}(\bar{s})$ and, hence, $h(x_n, s_n)$ converges to $h(\bar{x}, \bar{s})$ by the continuity of the flow. Therefore, h is continuous. Let us see that h^{-1} is also continuous. Consider a sequence (y_n) of points in W convergent to a certain $\bar{y} \in W$. Each y_n is of the form $x_n\sigma_{x_n}(s_n)$ and $\bar{y} = \bar{x}\sigma_{\bar{x}}(\bar{s})$ respectively, where, x_n, $\bar{x} \in U$ and s_n, $\bar{s} \in [0,1]$. We see that x_n converges to $\bar{x}$ and s_n converges to $\bar{s}$. Since U and $[0,1]$ are compact, we can choose subsequences $x_{n_k} \to x'$ and $s_{n_k} \to s'$. Besides, the continuity of h guarantees $h(x_{n_k}, s_{n_k}) \to h(x', s')$. But, on the other hand, $h(x_{n_k}, s_{n_k}) = x_{n_k}\sigma_{x_{n_k}}(s_{n_k}) \to \bar{y}$. As a consequence we get that $\bar{y} = h(x', s')$, leading to $\bar{x}\sigma_{\bar{x}}(\bar{s}) = x'\sigma_{x'}(s')$. Then, it follows that $\bar{x} = x'$ and $\bar{s} = s'$. Indeed, suppose, arguing by contradiction, that $\bar{x} \neq x'$, then, assuming that the absolute value of $\sigma_{\bar{x}}(\bar{s})$ is greater than or equal to $\sigma_{x'}(s')$ we would have that $\bar{x}(\sigma_{\bar{x}}(\bar{s}) - \sigma_{x'}(s')) = x'$ and, since $(\sigma_{\bar{x}}(\bar{s}) - \sigma_{x'}(s')) \in (t^i(\bar{x}), 0]$, it follows that either $\bar{x} = x'$ or x' is point of internal tangency in contradiction with the definition of isolating block. It also follows that $\bar{s} = s'$ since, if not, the trajectory of $\bar{x}$ would be periodic and, thus, $\bar{x}$ would be a point of internal tangency. We have proved that every convergent subsequence of (x_n) converges to $\bar{x}$ and every convergent subsequence of (s_n) converges to $\bar{s}$. As a consequence, since U and S are compact, $x_n \to \bar{x}$ and $s_n \to \bar{s}$. This proves c). □

Remark 1.1.2. A dual statement of Lemma 1.1.1 involving n^+ and t^i can be obtained from Lemma 1.1.1 applied to the reverse flow $\varphi^* = \varphi(\cdot, -t)$.

From now on we focus on flows defined on surfaces. The next result is a local version of classical Gutiérrez' Theorem. This result can be extracted from the proof of [80, Corollary 4]. We include here a more complete proof by adapting some ideas from [63, Lemma 13] to the context of surfaces and keeping the line of proof of [80, Corollary 4].

Lemma 1.1.3. *Let $\varphi : M \times \mathbb{R} \to M$ be a flow defined on a surface and K be an isolated invariant continuum. Then, φ is topologically equivalent to a C^1 flow near K. Moreover, K admits a basis of neighborhoods comprised of connected isolating block manifolds.*

Proof. We start the proof by showing that K admits a neighborhood basis comprised of compact and connected 2-manifolds with boundary. Indeed, since M is a surface, we may assume without loss of generality that M is C^∞ (see [41, Theorem A]). We may assume, without loss of generality, that M is embedded in some Euclidean space. Consider the metric d in M induced by the Euclidean metric and the continuous map $x \mapsto d_K(x) = d(x, K)$. Now, fixed $\varepsilon > 0$ we can find a C^∞ function $\delta_K : M \to [0, +\infty)$ such that $d_K \leq \delta_K \leq d_K + \varepsilon/3$ (see [67, Exercise 36, p. 152]). We choose ε in such a way that $\varepsilon \in d_K(M)$. As a consequence, $\delta_K(M) \supset [\varepsilon/3, 2\varepsilon/3)$ and by Sard's Theorem [59, Corollary, p.11] there exists a regular value $a \in (\varepsilon/3, 2\varepsilon/3)$. Then, $\delta_K^{-1}((-\infty, a])$ is a 2-manifold with boundary [59, Lemma 3, p.12] which, by the local compactness of M, is compact if we choose ε small enough. It is clear that K is contained in the interior of $\delta_K^{-1}((-\infty, a])$ since, if $x \in K$, $\delta_K(x) \leq \varepsilon/3 < a$. Therefore, choosing N as the component of $\delta_K^{-1}((-\infty, a])$ containing K we have found the desired neighborhood. Since ε can be chosen arbitrarily small, the claim follows.

On the other hand, since N can be chosen as close to K as desired, we can choose it to be an isolating neighborhood of K. Let $\widehat{N}$ be the closed surface obtained by capping each boundary component of N with a disk. By the Keesling reformulation of Beck's Theorem [50, Theorem 2] we can obtain a flow φ' on M such that φ' is topologically equivalent to φ in $\mathring{N}$ and is stationary in ∂N. Then, the restriction flow $\varphi'|_N$ can be extended to a flow $\widehat{\varphi}$ on $\widehat{N}$ by keeping all the points in $\widehat{N} \backslash N$ fixed. Besides, the flow $\widehat{\varphi}$ is topologically equivalent to a C^1 flow ψ

by Gutiérrez' Theorem and, as a consequence, $\varphi'|_{\mathring{N}}$ is topologically equivalent to $\psi|_{h(\mathring{N})}$, where $h : \widehat{N} \to \widehat{N}$ is the homeomorphism which realizes the equivalence. This proves the first part of the statement.

The remaining part follows from the fact that results from [22] and the connectivity of K ensure the existence of a basis of connected isolating block manifolds of K for ψ and, hence, for φ. □

Theorem 1.1.4. *Let K be an isolated invariant continuum of a flow on a surface. Then, K has the shape of a finite polyhedron. Moreover, if N is a connected isolating block manifold of K,*

$$\beta_1(K) \leq \beta_1(N)$$

Proof. Let N be a connected isolating block manifold of K. By Alexander duality

$$\check{H}^2(N, K) \cong H_0(N \setminus K, \partial N),$$

and the latter group must be zero since, if not, there would be a component U of $N \setminus K$ not meeting ∂N, which means that, given $x \in U$, the trajectory $\gamma(x)$ must be contained in N since it only can leave N through ∂N. This fact contradicts N to be an isolating neighborhood of K.

Consider the initial segment of the long exact sequence of reduced Čech cohomology of the pair (N, K)

$$0 \to \check{H}^1(N, K) \to \check{H}^1(N) \to \check{H}^1(K) \to \check{H}^2(N, K) = 0$$

Therefore, the homomorphism $i^* : \check{H}^1(N) \to \check{H}^1(K)$ is surjective and, since $\check{H}^1(N)$ is finitely generated, being N a compact manifold, so is $\check{H}^1(K)$. Thus, K has the shape of a wedge of $\beta_1(K)$ circumferences by Corollary 1 and $\beta_1(K) \leq \beta_1(N)$. □

Corollary 1.1.5. *Let K be an isolated invariant continuum of a flow on a surface. Suppose that K admits an isolating block which is a disk, then K has trivial shape and contains a fixed point.*

Proof. Since $\beta_1(N) = 0$, Theorem 1.1.4 guarantees that $\beta_1(K) = 0$ and, hence, Corollary 1 ensures that K has trivial shape. Let us see that K must contain a fixed point. Since K admits an isolating block N which is a disk, this disk can be embedded into $\mathbb{R}^2$ and, by the arguments presented in the proof of Lemma 1.1.3, we may assume, without loss of generality, that the flow restricted to $\mathring{N}$ can be extended to a C^1 flow on the whole plane. This fact allows us to use the Poincaré-Bendixson Theorem. Choose a point $x \in K$, hence $\emptyset \neq \omega(x) \subset K$ and either it contains a fixed point or it is a limit cycle. If $\omega(x)$ is a limit cycle, it must decompose $\mathbb{R}^2$ into two connected components, and, since $\mathring{N}$ is an open disk, the bounded component U must be contained in $\mathring{N}$. Thus, $\overline{U}$ is an invariant disk contained in $\mathring{N}$ and, hence, in K. As a consequence, for each $t \in \mathbb{R}$, the correspondence $x \to \varphi(x,t)$ defines by restriction a map $\varphi_t|_{\overline{U}} : \overline{U} \to \overline{U}$ and, by Brouwer's fixed point theorem, there exists a sequence of points $x_n \in \overline{U}$ and a sequence of numbers $t_n \in \mathbb{R}$, $t_n \to 0$ such that $\varphi(x_n, t_n) = x_n$. By the compactnes of $\overline{U}$ there is a convergent subsequence x_{n_i} whose limit $x \in \overline{U} \subset K$ is a fixed point of the flow. $\square$

Remark 1.1.6. Theorem 1.1.4 does not hold for flows on higher-dimensional manifolds. For instance, consider on $\mathbb{R}^3$ the vector field

$$X(x,y,z) = \Phi(x,y,z)\vec{e}_3,$$

where $\vec{e}_3 = (0,0,1)$ and $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is a C^∞ function which takes the value 0 exactly in those points which belong to the subset

$$H = \bigcup_{n\in\mathbb{N}} \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2},\ z = 0 \right\},$$

and it takes the value 1 outside a neighborhood of H. The flow induced by X is depicted in figure 1.1 and it has the set H, which is known as the Hawaaian earring, as an isolated invariant set. It is clear that H admits an isolating block which is a ball but, in spite of it, $\beta_1(H) = \infty$. In particular, H does not have polyhedral shape.

This example is a particular instance of a general result from [32] which states that any finite dimensional compactum can be an isolated invariant set of a flow

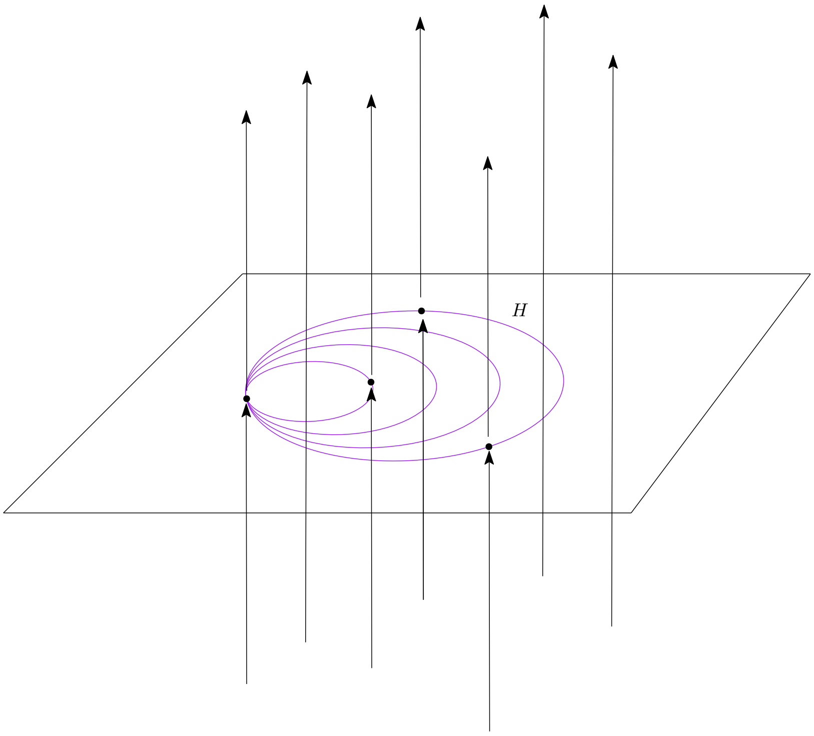

Figure 1.1: Flow on $\mathbb{R}^3$ having the Hawaaian earring as an isolated invariant set.

on some $\mathbb{R}^n$. This example also shows that in higher-dimensional manifolds, given a connected isolating block manifold N of an isolated invariant continuum K, $\beta_1(K)$ may be greater than $\beta_1(N)$. In [79] some conditions involving $\beta_1(N)$ are used to find lower bounds of $\beta_1(K)$ for flows on 3-manifolds.

1.2 On the structure of the unstable manifold

In this section we study the general case of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a locally compact metric space M, and we consider an isolated invariant compactum K of the flow. Our aim is to understand the dynamics in $W^u(K)$, the unstable manifold of K. The set $W^u(K) \setminus K$ is called the *truncated unstable manifold of* K (we remark that this terminology has been used with other meaning in [88]). If we consider the restriction $\varphi_0 = \varphi|_{W^u(K)}$ of the flow to $W^u(K)$ then, in general, K is not negatively stable and, therefore, it is not a repeller of φ_0. Moreover, the flow restricted to the truncated unstable manifold $W^u(K) \setminus K$ is not, in general, parallelizable. However, we shall prove in this section that if we restrict ourselves to an initial part of the truncated unstable manifold (in a sense that will be precised) then we obtain a parallelizable structure.

We start by studying an important particular case in which the flow on the truncated unstable manifold is, indeed, parallelizable. This result is similar to [88, Theorem 6], however we give here a more direct proof. Following [88] we recall that an isolating block N is *non-return* if every orbit leaving N (in positive time) never returns to N. In Example 1.2.7 we shall show that this result does not hold in the absence of non-return isolating blocks.

Theorem 1.2.1. *Let K be an isolated invariant compactum and suppose that K has a non-return isolating block N. Then K is a repeller for the flow $\varphi_0 = \varphi|_{W^u(K)}$ and, as a consequence, for every compact section S of $W^u(K) \setminus K$ the map $h : S \times \mathbb{R} \to W^u(K) \setminus K$ defined by $(x, t) \mapsto xt$ is a homeomorphism, i.e. the truncated unstable manifold is parallelizable.*

Proof. By the definition of unstable manifold, K is a repelling set for $\varphi_0 = \varphi|_{W^u(K)}$. In order to qualify as a repeller K must also be negatively stable. In order to prove this, we remark that the fact that N is non-return implies that $W^u(K) \cap N = N^-$. Now, if K is not negatively stable, then there exist a neighborhood U of K, a sequence $x_n \in W^u(K)$, $x_n \to x_0 \in K$ and a sequence $t_n \to -\infty$, $t_n < 0$, such that $x_n t_n \notin U$. Since $W^u(K) \cap N = N^-$ we may assume that $x_n \in N^-$ for every n and, since N^- is negatively invariant, $x_n t_n \in N^-$. By the compactness of N^- we may also assume that $x_n t_n \to y \in N^-$. Since $x_n t_n \notin U$ for every n we have that $y \in N^- \setminus K$. Moreover for every $t \in \mathbb{R}$ we have that $t_n + t$ is negative and $x_n(t_n + t) \in N^-$ for almost all n, hence $yt \in N^-$. Thus the trajectory $\gamma(y) \subset N^- \setminus K$, which is in contradiction with the fact that N is isolating. This completes the proof of the theorem. □

If K does not have a non-return isolating block then $W^u(K) \setminus K$ is not, in general, parallelizable. We postpone the proof of this fact to Example 1.2.7 since we must establish first some results. Our aim now is to study the general situation and prove that, in spite of this negative feature, certain parts of the truncated unstable manifold admit a parallelizable structure. We start by introducing a definition.

Definition 1.2.2. Let K be an isolated invariant compactum and let S be a compact section of the truncated unstable manifold $W^u(K) \setminus K$. Then, S is said to be an *initial section* provided that $\omega^*(S) \subset K$.

It is easy to see that if N is an isolating block of K then n^- is an example of initial section. If S is an initial section we define $I^u_S(K) = S(-\infty, 0]$ and we say that $I^u_S(K)$ is an *initial part of the truncated unstable manifold.* Obviously $I^u_S(K) = \{x \in W^u(K) \setminus K : xt \in S \text{ with } t \geq 0\}$. We see that, although $I^u_S(K)$ depends on S, all the initial parts have basically the same structure. In accordance with this terminology we say that $I^u_S(K) \cup K$ is an *initial part of the unstable manifold of* K and we denote it by $W^u_S(K)$.

Theorem 1.2.3. *Let K be an isolated invariant compactum and suppose that S is a compact section of the truncated unstable manifold $W^u(K) \setminus K$. If S is initial then the map $h : S \times (-\infty, 0] \to I^u_S(K)$ defined by $(x,t) \mapsto xt$ is a homeomorphism. Conversely, if h is a homeomorphism then S is initial.*

Proof. The map h is, obviously, a continuous bijection, hence we have to prove only that if $x_n t_n \to x_0 t_0$, with x_n,$x_0 \in S$ and t_n,$t_0 \in (-\infty, 0]$ then $x_n \to x_0$ and $t_n \to t_0$. We remark that the sequence t_n is bounded since, otherwise, there exists a subsequence $t_{n_k} \to -\infty$ and, thus, $x_{n_k} t_{n_k} \to x_0 t_0 \in \omega^*(S)$ with $x_0 t_0 \notin K$, in contradiction with the hypothesis that S is an initial section. Now consider a subsequence x_{n_m} of x_n. Suppose that $x_{n_m} \to y \in S$. Since t_{n_m} is also bounded, it has a convergent subsequence as well, say $t_{n_{m_l}} \to s \in (-\infty, 0]$. Hence $x_{n_{m_l}} t_{n_{m_l}} \to ys \in I^u_S(K)$. But $x_{n_{m_l}} t_{n_{m_l}} \to x_0 t_0$ and, as a consequence, $x_0 t_0 = ys$ and, being S a section, $y = x_0$. This proves that every convergent subsequence of x_n converges to x_0 and, since S is compact, $x_n \to x_0$. On the other hand, using that the sequence t_n is bounded, a similar argument shows that t_n converges to t_0.

Suppose now that the map $h : S \times (-\infty, 0] \to I^u_S(K)$ defined by $(x,t) \mapsto xt$ is a homeomorphism. We consider an isolating block N of K such that $N \cap S = \emptyset$. This implies that $N^- \subset I^u_S(K)$. Suppose, to get a contradiction, that there exists $y \in \omega^*(S)$, $y \notin K$. Then, by definition, there exist $x_n \in S$, $t_n \to -\infty$ such that $x_n t_n \to y$. We may assume that $t_n < 0$ for every n. Now, if there is a subsequence $(x_{n_k} t_{n_k}) \subset N^-$ then $x_{n_k} t_{n_k} \to y$ and, hence, $y \in N^-$. But, since $N^- \subset I^u_S(K)$, we have that $y = xt_0$ with $x \in S$ and $t_0 < 0$ and this is in contradiction with the fact that h is a homemorphism. Then, necessarily, $x_n t_n \notin N^-$ for almost every n and, hence, there is a sequence s_n such that $s_n < t_n$ and $x_n s_n \in n^-$ for almost every n. By the compactness of n^- there is a subsequence $x_{n_k} s_{n_k} \to z \in n^-$ with $s_{n_k} \to -\infty$ and the same argument as before leads to contradiction. ∎

In the next result we establish a topological property of $I_S^u(K)$.

Proposition 1.2.4. *If S is an initial section of the truncated unstable manifold then the closure of $I_S^u(K)$ in M is contained in $I_S^u(K) \cup K$. As a consequence $W_S^u(K) = I_S^u(K) \cup K$ is closed in M. In fact, $W_S^u(K)$ is compact.*

Proof. If y is in the closure of $I_S^u(K)$ then $x_n t_n \to y$ with $x_n \in S$, $t_n \leq 0$. We may assume that $x_n \to x \in S$. If t_n is bounded then there exists a convergent subsequence $t_{n_m} \to t$. Hence $x_{n_m} t_{n_m} \to xt = y \in I_S^u(K)$. If t_n is unbounded, then there exists a subsequence $t_{n_k} \to -\infty$ and $x_{n_k} t_{n_k} \to y \in \omega^*(S) \subset K$. This proves the inclusion. Since K is compact, it is obvious that $W_S^u(K)$ is closed in M. Moreover, if N is an isolating block, the fact that S is initial implies the existence of a $t_0 < 0$ such that $S(-\infty, t_0] \subset N^-$. Hence $W_S^u(K) = (W_S^u(K) \cap N) \cup S[0, t_0]$ is compact. □

We see now that all initial sections are homeomorphic and that the homeomorphism can be defined in a very natural way.

Theorem 1.2.5. *Let K be an isolated invariant compactum and suppose that S and T are initial sections of the truncated unstable manifold $W^u(K) \setminus K$. Then the map $h : S \to T$ defined by $h(x) = \gamma(x) \cap T$ is a homeomorphism.*

Proof. As we said before, if N is an isolating block of K then n^- is an initial section and there is a $t_0 < 0$ such that $S(-\infty, t_0] \subset N^-$. Now, the exit map of N^- (i.e. the map which assigns to each $x \in N^- \setminus K$ the point $\gamma(x) \cap n^-$) can be used to define a homeomorphism $e : St_0 \to n^-$ and, as a consequence, the map $S \to n^-$ defined by $x \to \gamma(x) \cap n^-$ is also a homeomorphism. The map h in the statement of the theorem is a composition of this homeomorphism and the inverse of the analogous homeomorphism $T \to n^-$. □

All our considerations so far are relative to the unstable manifold of K. It is clear, however, that they can be dualized for the stable manifold $W^s(K)$ so that they are valid for the dual notions of *final section* and *final part of the truncated stable manifold* $W^s(K) \setminus K$, which are defined in the obvious way. We shall use the notations $F_S^s(K)$ and $W_S^s(K)$ for the final part of the truncated stable manifold and final part of the stable manifold respectively, corresponding to the final section S. All the previous results hold for this dual situation and, in particular, Theorem 1.2.3 takes the following nice form.

Theorem 1.2.6. *Let K be an isolated invariant compactum and suppose that S is a compact section of the truncated stable manifold $W^s(K) \setminus K$. If S is final then the map $h : S \times [0, \infty) \to F^s_S(K)$ defined by $(x, t) \mapsto xt$ is a homeomorphism. Moreover, the restriction $\varphi_0 = \varphi|_{W^s_S(K)} : W^s_S(K) \times [0, \infty) \to W^s_S(K)$ of the flow to the final part of the stable manifold $W^s_S(K)$ defines a semi-dynamical system and K is a global attractor of φ_0.*

We remark that it is not in general true that K is an attractor for the flow considered in the whole stable manifold $W^s(K)$. This a consequence of the following example.

Example 1.2.7. The flow defined by Mendelson in [58] (see Figure 1.2) provides an example of an isolated invariant continuum $K = \{p_2\}$ which is an unstable attracting set of $\mathbb{R}^2$ with $W^s(K) = \mathbb{R}^2 \setminus \{p_1\}$ (we remind that the lack of stability means that K does not qualify as an attractor according to our definition). Here the final section S is homeomorphic to a segment (we can take, for instance, a semicircle with centre p_2 and radius $r = d(p_1, p_2)/2$ in the lower semiplane) while the truncated stable manifold $W^s(K) \setminus K$ is $\mathbb{R}^2 \setminus \{p_1, p_2\}$. Then $W^s(K) \setminus K$ is not parallelizable since, otherwise, $\mathbb{R}^2 \setminus \{p_1, p_2\}$ would be homeomorphic to $S \times \mathbb{R}$, which is not the case. This proves that K is not an attractor in $W^s(K)$. This example can be dualized to show that, in general, the truncated unstable manifold $W^u(K) \setminus K$ is not parallelizable.

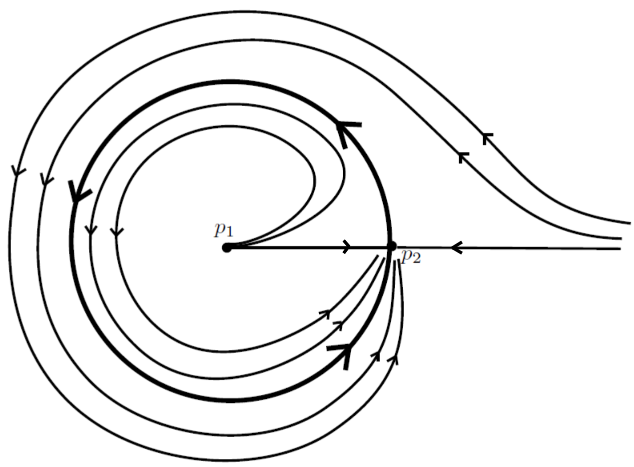

Figure 1.2: Mendelson flow

Example 1.2.8. The flow described by Figure 1.3 provides an example of a compact section of a continuum $K = \{p\}$ which is not initial. The section is marked in red.

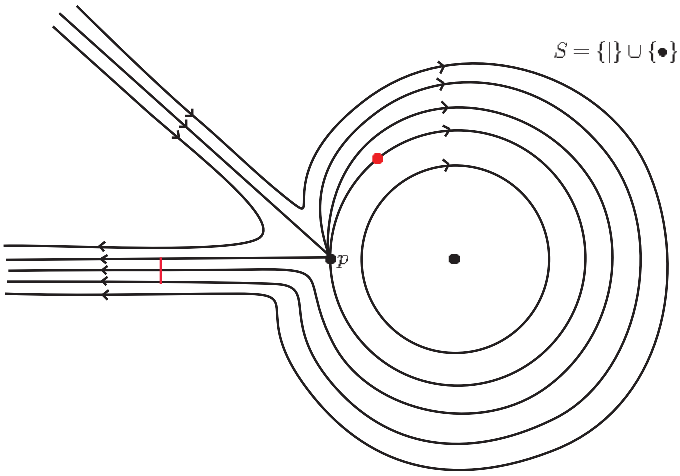

Figure 1.3: Non-initial compact section

Example 1.2.9. The following remarkable example (Figure 1.4), presented by Campos, Ortega and Tineo in [20], describes a flow on a disk where all points in the boundary are stationary and such that the whole boundary is the ω-limit and the ω^*-limit of every interior point. The boundary K is not isolated and its truncated unstable manifold does not have compact sections. This example shows that the condition of K being isolated is necessary in Theorem 1.2.3.

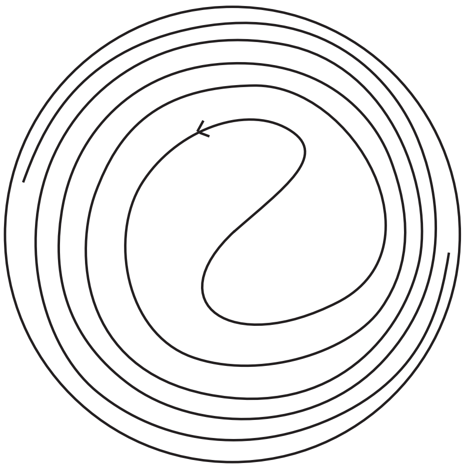

Figure 1.4: Flow on a Disk

The following proposition gives a topological characterization of the initial sections of the truncated unstable manifold of an isolated invariant continuum of a flow on a surface. As a consequence, it also characterizes the topology of the S-initial part of the truncated unstable manifold.

Proposition 1.2.10. *Let $\varphi : M \times \mathbb{R} \to M$ be a flow defined on a surface, K be an isolated invariant continuum and S an initial section of the truncated unstable manifold $W^u(K) \setminus K$. Then, S has a finite number of connected components and each one is either an interval (possibly degenerate) or a circle. Moreover, $I^u_S(K)$ is homeomorphic to a finite disjoint union of half-open rays, strips and cylinders.*

Proof. By Lemma 1.1.3 we can find a connected isolating block manifold N of K. Besides, S is homeomorphic to n^-. Hence, Lemma 1.1.1 guarantees that it has a finite number of components. Moreover, N^o consists of a disjoint union of finite many circumferences and closed intervals. Then, since n^- is a compact subset of this disjoint union, it must be a finite union of points, closed intervals and circumferences as we wanted to prove. Therefore, the result follows $I^u_S(K)$ being homeomorphic to $S \times (-\infty, 0]$. □

1.3 Regular isolating blocks and the Conley index

In this section we see that the knowledge of the first Betti number of an isolated invariant continuum of a flow on a surface and the topology of an initial section of its truncated unstable manifold allow us to compute its Conley index. For this purpose we make use of a special kind of isolating blocks, the so-called *regular isolating blocks*. This kind of blocks was first introduced and studied by Easton in [27] and subsequently studied by Gierzkiewicz and Wójcik [30] and J.J. Sánchez-Gabites [79, 81]. Most of the known results are referred to the 3-dimensional case and some more general results, which appear in [30], do not apply to the kind of isolating blocks considered here since we are dealing with a more restrictive definition of isolating block. Although the arguments from [79, 81] can be translated almost directly to the case of surfaces, ensuring the existence of basis of neighborhoods comprised of regular blocks for isolated invariant continua in surfaces, we provide here a direct proof.

Definition 1.3.1. A connected isolating block manifold N of an isolated invariant continuum K is said to be *regular*, provided that the inclusion $i : K \hookrightarrow N$ is a shape equivalence.

Remark 1.3.2. Notice that the condition for an isolating block to be regular in Definition 1.3.1 differs from the one introduced and studied in [27, 30]. However, it follows from Theorem 2 that for connected isolating block manifolds in surfaces both definitions agree. In addition, it follows from [81, Teorema B.7] that all regular isolating blocks of the same isolated invariant continuum must be homeomorphic. This facts also hold in 3-manifolds [81].

Theorem 1.3.3. *Suppose K is an isolated invariant continuum of a flow on a surface. Then, K admits a basis of neighborhoods comprised of regular isolating blocks.*

Proof. Let N be a connected isolating block manifold of K. From the proof of Theorem 1.1.4 we have that the sequence

$$0 \to \check{H}^1(N, K) \to \check{H}^1(N) \to \check{H}^1(K) \to 0,$$

is exact and, as a consequence, from Theorem 2, the obstruction for N to be a regular block is the existence of non-trivial elements in $\check{H}^1(N, K)$. On the other hand, as we have seen in the proof of Lemma 1.1.1, $\check{H}^1(N, K) \cong \check{H}^1(N^o, n^-)$ and, by Alexander duality, we get

$$\check{H}^1(N^o, n^-) \cong H_0(N^o \setminus n^-, \partial N^o).$$

Notice that $H_0(N^o \setminus n^-, \partial N^o)$ is finitely generated. We construct the desired block from N by cutting from it the leftover information in the following way:

Assume that C is a circular component of N^o not contained in n^-. Each component of $C \setminus n^-$ represents a generator of $H_0(N^o \setminus n^-, \partial N^o)$ since it does not contain points of ∂N^o. Choose a point $x_0 \in (C \setminus n^-)$ and a compact and connected neighborhood U of x_0 in C disjoint from n^-. Notice that U, being a proper nondegerate subcontinuum of the circle must be homeomorphic to the

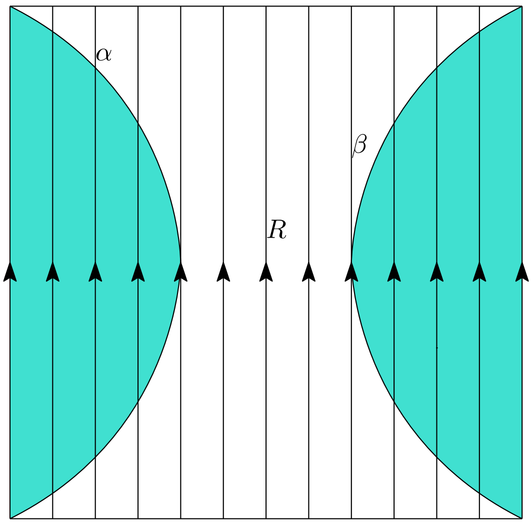

Figure 1.5: The curves α and β and the region R in $[0,1]\times[0,1]$.

unit interval $[0,1]$. Thus, Lemma 1.1.1 guarantees that the set

$$W = \bigcup_{x\in U} x[t^i(x), 0],$$

is homeomorphic to the unit square $[0,1]\times[0,1]$ via a homeomorphism $h : W \to [0,1]\times[0,1]$ which carries each segment of trajectory $x[t^i(x), 0]$ to $\{g(x)\}\times[0,1]$, where $g : U \to [0,1]$ is a homeomorphism. Now we perform the following operation: choose in $[0,1]\times[0,1]$ the parabolic segments α and β depicted in figure 1.5 and let R be the open region between these curves in $[0,1]\times[0,1]$. Then, if we consider $N_{(1)} = N \setminus h^{-1}(R)$, it is clear from the construction that it is a connected isolating block manifold. Notice that this operation keeps n^- unaltered. Moreover, the number of boundary components has been reduced by 1 since the component C has been joined with a component of N^i, which lies in a different component of ∂N. As a consequence, C becomes an interval, say J, and $J \setminus n^-_{(1)}$ has one more component than $C \setminus n^-$. However, J must contain two points of ∂N^o, each one lying in a different component of $J \setminus n^-_{(1)}$ and, thus, the homology group $H_0(N^o_{(1)} \setminus n^-_{(1)}, \partial N^o_{(1)})$ has exactly one generator less than $H_0(N^o \setminus n^-, \partial N^o)$. After performing this operation to each circular component of N^o not contained in n^- we obtain a connected isolating block manifold $N_{(r)}$ such that, all the circular components of $N^o_{(r)}$ are contained in $n^-_{(r)}$.

We denote $N_{(r)}$ by N since it should not lead to confusion. Choose a component J of N^o which contains more than one component of n^-. Then, J must be an interval. Thus, each component of $J \setminus n^-$ not containing one of the endpoints represents a generator of $H_0(N^o \setminus n^-, \partial N^o)$. Choose an orientation in J and let n_1^- and n_2^- be the first and the second components of n^- appeared regarding the chosen orientation. Choose a point in the interval J lying between n_1^- and n_2^- and perform the previously described operation. We obtain in this way a new isolating block manifold $N_{(1)}$ in which the component J has been splitted into two disjoint exit intervals, one of them containing n_1^- and the other containing remaining components of n^- which were contained in the original J. Notice that $N_{(1)}$ is also connected since, if not, K and one of the chosen components of n^- should lie in different components of $N_{(1)}$ and this cannot happen. If we perform this operation until we separate all the components of n^- (i.e. a finite number of times) we get the desired block. □

Definition 1.3.4. A non-empty continuum K contained in a surface is said to be *orientable* if it admits a basis of neighborhoods comprised of orientable 2-manifolds (with or without boundary). Otherwise K is said to be *nonorientable*.

Remark 1.3.5. Notice that, since an orientable 2-manifold (with or without boundary) cannot contain a nonorientable one it follows:

i) Every continuum contained in an orientable surface must be orientable.

ii) An orientable continuum K cannot possess a basis of neighborhoods comprised of nonorientable 2-manifolds.

iii) A nonorientable continuum K must admit a basis of neighborhoods comprised of nonorientable 2-manifolds.

However, as the next example points out, nonorientable surfaces contain both orientable and nonorientable continua.

Example 1.3.6. Consider M as the surface obtained as a connected sum of the torus $S^1 \times S^1$ with the Klein bottle $\mathcal{K}$ (which is homeomorphic to a connected sum of four projective planes [54]). In this surface we can find two copies of $S^1 \vee S^1$ as the 1-skeleton of the torus and the Klein bottle summands respectively. It is clear that the one contained in the torus summand is orientable while the other is not.

Now we are ready to prove the main result of this section.

Theorem 1.3.7. *Suppose K is an isolated invariant continuum of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a surface. Let u be the number of components of an initial section S of the truncated unstable manifold and u_c the number of contractible components of S. Then,*

i) If K is neither an attractor, nor a repeller then $u \neq 0$ and the Conley index of K is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,k} S_i^1, *\right)$, where $k = \beta_1(K) + u_c - 1$ and S_i^1 is a pointed 1-sphere based on $*$ for $i = 1, \dots, k$.

ii) If K is an attractor, $u = 0$ and its Conley index is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,\beta_1(K)} S_i^1 \cup \{\bullet\}, \bullet\right)$, where the S_i^1 are pointed 1-spheres based on $*$ and $\bullet$ denotes a point not belonging to $\bigvee_{i=1,\dots,\beta_1(K)} S_i^1$.

iii) If K is a repeller $u \neq 0$ and:

(a) If K is orientable its Conley index is the pointed homotopy type of $\left(\Sigma_g \bigvee \left(\bigvee_{i=1,\dots,u-1} S_i^1\right), *\right)$, where Σ_g is a closed orientable surface of genus $g = \frac{1+\beta_1(K)-u}{2}$. The surface Σ_g and all the S_i^1 are pointed and based on $*$.

(b) If K is nonorientable its Conley index is the pointed homotopy type of $\left(N_g \bigvee \left(\bigvee_{i=1,\dots,u-1} S_i^1\right), *\right)$, where N_g is a closed nonorientable surface of genus $g = 1 + \beta_1(K) - u$. The surface N_g and all the S_i^1 are pointed and based on $*$.

Proof. Let N be a regular isolating block of K. Then, given an initial section S of the truncated unstable manifold $W^u(K) \setminus K$, S is homotopy equivalent to N^o. Indeed, since the inclusion $i : K \hookrightarrow N$ is a shape equivalence, the cohomology groups $\check{H}^k(N, K) = 0$. But, as we have seen before $\check{H}^k(N, K) \cong \check{H}^k(N^o, n^-)$ and, hence, $i : n^- \hookrightarrow N^o$ induces isomorphisms in Čech cohomology. It easily follows that n^- and N^o have the same homotopy type and the claim follows n^- being homeomorphic to S.

From this observation we get that N^o has u_c components which are intervals and $u - u_c$ circular components.

Suppose that K is neither an attractor nor a repeller. It is clear that $u \neq 0$. Let N be a regular isolating block of K. The block N is a compact 2-manifold with boundary and, since it has the same shape as K it must have the pointed homotopy type of a wedge of $\beta_1(K)$ circumferences. Collapsing to a point an interval component of N^o does not change the homotopy type of N. Therefore, the topological space obtained by collapsing all the interval components to a single point is pointed homotopy equivalent to the wedge sum of N with u_c-1 copies of S^1. On the other hand, collapsing a circular component C of N^o produces the same effect on N as capping the boundary component C with a disk. Then, the topological space obtained by collapsing to a point all the circle components is pointed homotopy equivalent to a wedge sum of $(u - u_c - 1)$ circumferences with the manifold obtained after capping $(u - u_c)$ boundary components with disks. Thus, since N^o is neither empty nor the whole ∂N the Conley index of K must be the pointed homotopy type of a wedge sum of a compact and connected 2-manifold with boundary with some circumferences. Hence, it must be pointed homotopy equivalent to a wedge of circumferences. To determine the number of circumferences on the wedge we compute the Euler characteristic of $h(K)$. Since $\chi(h(K))$ agrees with $\chi(N, N^o)$ and N^o is a union of u_c intervals and $u - u_c$ circumferences it follows

$$\chi(h(K)) = \chi(N) - \chi(N^o) = 1 - \beta_1(N) - u_c,$$

and, hence, $\operatorname{rk} CH^1(K) = \beta_1(N) + u_c - 1$. This proves i).

If K is an attractor it admits a positively invariant isolating neighborhood and, hence, $u = 0$. Thus, if N is a regular isolating block it must have empty exit set. As a consequence, the effect of collapsing its exit set N^o to a point is the same as making the disjoint union of N with a singleton not contained in N. This proves ii).

Suppose that K is a repeller. Then, $u \neq 0$ and given a regular isolating block N of K, N^o must be the whole boundary ∂N which must be comprised of u connected components. The space obtained after collapsing the whole boundary of N to a point is pointed homotopy equivalent to the wedge sum of $u-1$ circumferences with the surface obtained after capping all the boundary components of N with disks. This surface is orientable if and only if K is orientable. Indeed, if K is orientable it admits a basis of neighborhoods comprised of orientable 2-manifolds. As a consequence, K admits an orientable regular isolating block. If K is nonorientable an analogous argument shows that K admits a nonorientable regular block. Let us compute the genus of S_g, the closed surface obtained after capping with a disk each boundary component of ∂N. Since, ∂N has exactly u components, it easily follows that

$$\chi(S_g) = 1 - \beta_1(N) + u.$$

On the other hand,

$$\chi(S_g) = \begin{cases} 2-2g & \text{if } S_g \text{ is orientable} \\ 2-g & \text{otherwise} \end{cases}$$

This proves iii). □

Remark 1.3.8. Notice that in the item iii) of Theorem 1.3.7 the genus of the surface which appears as a direct summand must be less than or equal to than the genus of the phase space M. This can be easily seen using the Mayer-Vietoris sequence.

Theorem 1.3.7 takes a very nice form if we assume that the surface M is $\mathbb{R}^2$ or the 2-sphere S^2.

Corollary 1.3.9. *Let K be an isolated invariant continuum of a flow $\varphi : M \times \mathbb{R} \to M$, where $M = \mathbb{R}^2$ or S^2, and S an initial section of its truncated unstable manifold. If we denote by n the number of components of $M \setminus K$, by u the number of components of S and by u_c the number of contractible components of S, then $u - u_c \leq n$ and*

i) If $u \neq 0$ and $u - u_c < n$ then the Conley index of K is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,k} S^1_i, *\right)$, where $k = n + u_c - 2$ and S^1_i is a pointed 1-sphere based on $*$ for $i = 1, \dots, k$.

ii) If $u - u_c = n$ then K is a repeller and its Conley index is the pointed homotopy type of $\left(S^2 \bigvee \left(\bigvee_{i=1,\dots,n-1} S^1_i\right), *\right)$, where the 2-sphere S^2 and all the S^1_i are pointed and based on $*$.

iii) If $u = 0$ then K is an attractor and its Conley index is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,n-1} S^1_i \cup \{\bullet\}, \bullet\right)$, where the S^1_i are pointed 1-spheres based on $*$ and $\bullet$ denotes a point not belonging to $\bigvee_{i=1,\dots,n-1} S^1_i$.

Proof. Notice that Alexander duality ensures that $n = \beta_1(K) + 1$. Let N be a regular block of K, then N must be a disk with $n-1$ holes. It can be easily seen, using the long exact sequence of reduced homology of the pair $(N, \partial N)$, Lefschetz duality and Alexander duality, that each component of $M \setminus K$ contains exactly a boundary component of N and, hence, from the proof of Theorem 1.3.7, it follows that $u - u_c \leq n$ and the equality holds if and only if K is a repeller. On the other hand, K is an attractor if and only if $W^u(K) \setminus K = \emptyset$ or equivalently $u = 0$. The result follows from these considerations and Theorem 1.3.7. □

1.4 The cohomology index

The aim of this section is to study the cohomology index of an isolated invariant continuum of a flow on a surface and its relations with the Conley index. Since cohomology groups are easier to compute than homotopy type it is interesting to study to what extent the cohomology index determines the Conley index.

Example 1.4.1. Let M be an orientable surface of genus greater than or equal to 1 and consider two flows φ and φ' on M having isolated invariant sets K_1 and K_2 respectively whose local dynamics are depicted in figures 1.6 and 1.7. The Conley indices of K_1 and K_2 are the pointed homotopy type of $(S^2 \vee S^1_1 \vee S^1_2, *)$ and $(S^1 \times S^1, *)$. Then, their cohomology indices agree being

$$CH^i(K_j) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if} \quad i = 1 \\ \mathbb{Z}_2 & \text{if} \quad i = 2 \\ 0 & \text{otherwise} \end{cases}$$

However, these spaces are not homotopy equivalent. This can be seen using the ring structure of $CH^*(K_1)$ and $CH^*(K_2)$. As rings

$$\begin{aligned} CH^*(K_1) &\cong \widetilde{H}^*(S^2) \oplus \widetilde{H}^*(S^1_1) \oplus \widetilde{H}^*(S^1_2) \\ CH^*(K_2) &\cong \widetilde{H}^*(S^1 \times S^1). \end{aligned}$$

Let σ_1, σ_2 be elements of $CH^1(K_1)$. Then $\sigma_i = a_i\gamma_1 + b_i\gamma_2$, where γ_i is the

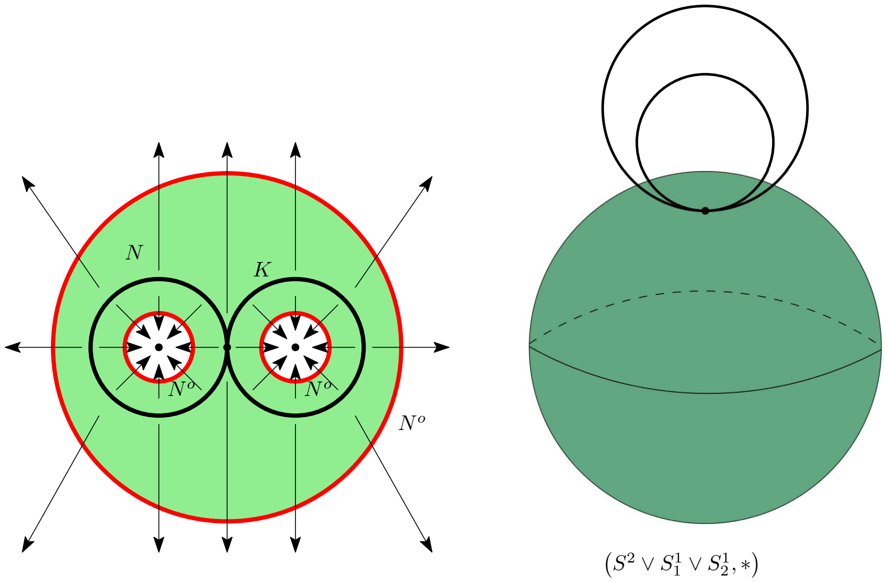

Figure 1.6: Flow having $S^1 \vee S^1$ as a repeller whose Conley index is the pointed homotopy type of $(S^2 \vee S_1^1 \vee S_2^1, *)$.

generator of $H^1(S_i^1)$, $i = 1, 2$. As a consequence, $\sigma_1 \smile \sigma_j = 0$ since $\gamma_1 \smile \gamma_2 = 0$ by the direct sum structure of $CH^*(K_1)$ and $\gamma_i \smile \gamma_i \in H^2(S_i^1) = 0$, $i = 1, 2$.

On the other hand, if α, β are the standard generators of $CH^1(K_2)$, $\alpha \smile \beta$ generates $CH^2(K_2) \cong \mathbb{Z}_2$. Therefore, the rings $CH^*(K_1)$ and $CH^*(K_2)$ are not isomorphic and $h(K_1) \neq h(K_2)$.

The previous example shows that the knowledge of the groups which conform the cohomology index is not enough to know the Conley index. We see that in spite of it, the cohomology ring $CH^*(K)$ determines the Conley index.

Given a topological space M with $H^2(M) = \mathbb{Z}_2$ it is possible to define a bilinear form

$$I : H^1(M) \times H^1(M) \to \mathbb{Z}_2,$$

given by $I(\alpha_1, \alpha_2) = \alpha_1 \smile \alpha_2$. This form determines the cohomology ring $H^*(M)$ when M is a closed surface. The rank of I is defined as the rank of any matrix representing I. This number is well defined since two matrices representing I must be congruent.

Theorem 1.4.2. *Suppose that K is an isolated invariant continuum of a flow on a surface. Then, the cohomology ring $CH^*(K)$ determines its Conley index. In particular,*

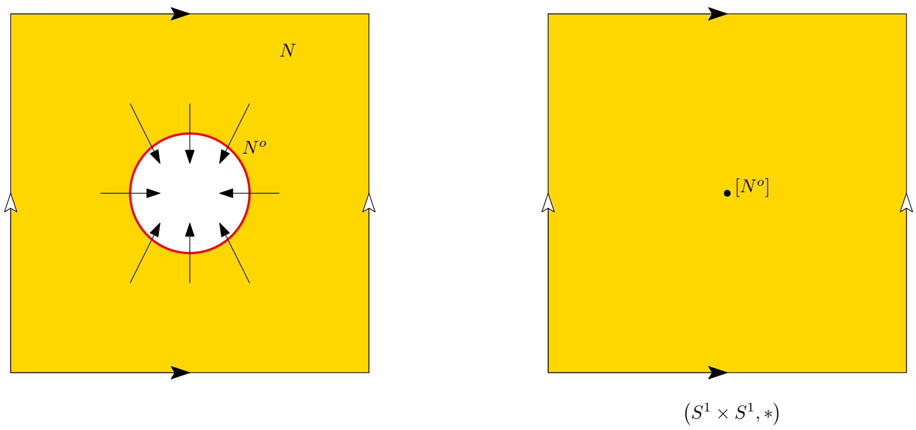

Figure 1.7: Flow having $S^1 \vee S^1$ as a repeller whose Conley index is the pointed homotopy type of $(S^1 \times S^1, *)$.

i) If $CH^0(K) = CH^2(K) = 0$, then K is neither an attractor nor a repeller and its Conley index is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,s} S^1_i, *\right)$, where s agrees with $\operatorname{rk} CH^1(K)$.

ii) If $CH^0(K) \neq 0$ then K is an attractor and its Conley index is the pointed homotopy type of $\left(\bigvee_{i=1,\dots,s} S^1_i \cup \{\bullet\}, \bullet\right)$ where s agrees with $\operatorname{rk} CH^1(K)$. In particular, K has the shape of s circumferences.

iii) If $CH^2(K) \neq 0$ then K is a repeller and:

(a) If $\alpha \smile \alpha = 0$ for each $\alpha \in CH^1(K)$ the Conley index of K is the pointed homotopy type of $\left(\Sigma_g \bigvee \left(\bigvee_{i=1,\dots,r} S^1_i\right), *\right)$, where $g = \frac{\operatorname{rk} I}{2}$ and $r = \operatorname{rk} CH^1(K) - 2g$.

(b) If there exists $\alpha \in CH^1(K)$ such that $\alpha \smile \alpha \neq 0$ the Conley index of K is the pointed homotopy type of $\left(N_g \bigvee \left(\bigvee_{i=1,\dots,r} S^1_i\right), *\right)$, where $g = \operatorname{rk} I$ and $r = \operatorname{rk} CH^1(K) - g$.

In both cases the number of components of an initial section S of the truncated unstable manifold is $r + 1$ and K has the shape of $\operatorname{rk} CH^1(K)$ circumferences.

Proof. Suppose that $CH^0(K) = CH^2(K) = 0$. Thus, Theorem 1.3.7 ensures that K cannot be an attractor or a repeller and $h(K)$ must be the pointed homotopy type of a wedge of circumferences. It is clear that the number of circumferences in the wedge is determined by $\operatorname{rk} CH^1(K)$. This proves i).

Let us assume that $CH^0(K) \neq 0$. Then $h(K)$ is the pointed homotopy type of a non connected space and by Theorem 1.3.7 K must be an attractor. Moreover, $h(K)$ must be the pointed homotopy type of the disjoint union of a wedge of circumferences and a point. As before $\operatorname{rk} CH^1(K)$ determines the number of circumferences in the wedge.

To prove iii) assume that $CH^2(K) \neq 0$, then Theorem 1.3.7 guarantees that K is a repeller. Moreover, $h(K)$ must be the pointed homotopy type of a wedge sum of closed surface with some circumferences. This surface is orientable (and hence K is orientable) if and only if, given any element $\alpha \in CH^1(K)$, $\alpha \smile \alpha = 0$. This is a straightforward consequence of the cohomology ring structure of closed surfaces (See Examples 3.7 and 3.8 in [42]).

Suppose that K is orientable. Then $h(K)$ is the pointed homotopy type of $\left(\Sigma_g \bigvee \left(\bigvee_{i=1,\dots,r} S^1_i\right), *\right)$. Let us show that g is exactly $\operatorname{rk} I/2$. By [42, pg.202] we have that

$$CH^*(K) \cong \widetilde{H}^*(\Sigma_g) \oplus \left(\bigoplus_{i=1}^{r} \widetilde{H}^*(S^1_i)\right), \tag{1.1}$$

as rings. Choose the basis $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots \beta_g, \gamma_1, \dots, \gamma_r\}$ of $CH^1(K)$ where $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ is the standard basis of $H^1(\Sigma_g)$ and each γ_i is the generator of $H^1(S^1_i)$ for each i. Let σ be the generator of $CH^2(K) \cong \mathbb{Z}_2$, then

$$\alpha_i \smile \beta_j = \begin{cases} \sigma & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

and $\alpha_i \smile \alpha_j = 0$, $\beta_i \smile \beta_j = 0$ for each i, j. Besides, (1.1) ensures that $\gamma_i \smile \omega = 0$ for each $i = 1, \dots, r$ and each $\omega \in CH^1(K)$. Therefore, the matrix associated to the bilinear form I with respect to the chosen basis takes the form

$$\left(\begin{array}{cc|c} O_g & I_g & \\ I_g & O_g & O_{s\times r} \\ \multicolumn{2}{c|}{O_{r\times 2g}} & \end{array}\right)$$

where I_g denotes the order g identity matrix, O denotes the zero matrix of the corresponding order and $s = \operatorname{rk} CH^1(K)$. Hence, the rank of I is $2g$ and the result follows.

Suppose that K is nonorientable. In this case $h(K)$ is the pointed homotopy type of $\left(N_g \bigvee \left(\bigvee_{i=1,\ldots,r} S_i^1\right), *\right)$. We see that g agrees with the rank of I. Consider the basis $\{a_1, \ldots, a_g, \gamma_1, \ldots, \gamma_r\}$ of $CH^1(K)$ where $\{a_1, \ldots, a_g,\}$ is the standard basis of $H^1(N_g)$ and each γ_i is the generator of $H^1(S_i^1)$ for each i. Let σ be the generator of $CH^2(K) \cong \mathbb{Z}_2$, then

$$a_i \smile a_j = \begin{cases} \sigma & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

and, reasoning as before, $\gamma_i \smile \omega = 0$ for each $i = 1, \ldots r$ and $\omega \in CH^1(K)$. Therefore, the matrix associated to the bilinear form I with respect to the chosen basis takes the form

$$\left(\begin{array}{c|c} I_g & O_{g\times r} \\ \hline O_{r\times g} & O_{r\times r} \end{array}\right)$$

Thus, the rank of I is g and the result follows.

Notice that from this discussion it also follows that the cohomology ring $CH^*(K)$ determines $h(K)$ as we wanted to prove. □

1.5 Dynamics of continua in surfaces

In this section we present several results about the dynamics of continua (or near continua) in surfaces. In many of them we make use of the structure of the unstable manifold studied in Section 1.2. We start by discussing to what extent the numbers u and u_c determine the dynamics.

The next result shows a duality property of those isolated invariant continua which are neither attractors nor repellers for flows on surfaces.

Proposition 1.5.1. *Suppose that K is an isolated invariant continuum of a flow on a surface. Then, the number of contractible components of an initial section S of the truncated unstable manifold of K, u_c, agrees with the number of contractible components s_c of a final section S^* of the truncated stable manifold.*

As a consequence, if K is neither an attractor nor a repeller, the Conley index $h(K)$ agrees with the Conley index for the reverse flow $h^*(K)$. Besides, the initial sections of the truncated unstable manifold and the final sections of the truncated stable manifold have the same homotopy type if and only if they have the same number of connected components (i.e. if and only if $u = s$).

Proof. Consider a regular isolating block N of K. As we have seen in the proof of Theorem 1.3.7, N^o possesses exactly u_c interval components and, working with the reverse flow, it also follows that N^i has exactly s_c interval components. Since ∂N is a disjoint union of circumferences, it is clear that the number of components of N^o and N^i contained in a component C of ∂N not contained neither in n^- nor in n^+ must be the same and, hence, $u_c = s_c$. The second part of the statement follows straightforward from Theorem 1.3.7.

Concerning the last part of the statement, since $u_c = s_c$, then $u = s$ if and only if the number of non-contractible components of the initial section agrees with the number of components of the final one and from this thes result follows. □

In the next result we see that the vanishing of u_c is related to the dynamical property of being *non-saddle* introduced by Bhatia in [14].

Definition 1.5.2. A compact invariant set K is said to be *saddle* if it admits a neighborhood U such that every neighborhood V of K contains a point $x \in V$ with $\gamma^+(x) \nsubseteq U$ and $\gamma^-(x) \nsubseteq U$. Otherwise we say that K is *non-saddle*.

Non-saddle sets are the main topic of Chapter 2 in this dissertation and some instances of them are attractors, repellers and unstable attractors with no external explosions (see [3, 63, 82]).

Proposition 1.5.3. *An isolated invariant continuum K of a flow on a surface is non-saddle if and only if $u_c = 0$.*

Proof. Suppose, arguing by contradiction, that K is non-saddle and $u_c \neq 0$. Let N be a regular isolating block of K. Since $u_c \neq 0$, N^o must have at least one component J which is an interval. Besides, J must contain in its interior a component of n^-. Let (x_n) be a sequence in $J \setminus n^-$ convergent to $x \in n^-$. The trajectory of each x_n must leave N in the past and in the future but, since $x_n \to x \in n^-$, fixed any neighborhood U of K, there exists n such that the trajectory of x_n meets U before it leaves N in the past. This is in contradiction with K being non-saddle.

Conversely, assume that $u_c = 0$. Then, given a regular isolating block N of K, N^o must agree with n^-. We see that given any $x \in N$, either $\gamma^+(x)$ or $\gamma^-(x)$ is contained in N. Suppose, arguing by contradiction, that there exists a point x whose trajectory leaves N in the past and in the future. Thus, the exit time function t^o is defined in x and, $xt^o(x) \in N^o = n^-$. As a consequence, $x \in N^-$ which is in contradiction with the trajectory of x leaving N in the past. Therefore, K is non-saddle since K admits a basis of neighborhoods comprised of regular isolating blocks. □

Remark 1.5.4. In [32, Theorem 4] it was proved that isolated non-saddle sets (possibly not connected) in manifolds have the shape of finite polyhedra. Then, they have a finite number of components, each one of them isolated and non-saddle. As a consequence, Proposition 1.5.3 also holds if K has a finite number of components.

In the next result we show that very strong dynamical consequences are derived from the topological property of connectedness of the initial sections for planar flows.

Theorem 1.5.5. *Let K be an isolated invariant continuum of a flow in $\mathbb{R}^2$ and let S be an initial section of the truncated unstable manifold $W^u(K)\setminus K$. Suppose S is connected and denote by A the component of $\mathbb{R}^2\setminus K$ which contains S. Then, in every bounded component $B \neq A$ of $\mathbb{R}^2 \setminus K$ there is a repeller $R \subset B$ whose basin of repulsion is B. Moreover, the repeller R contains a critical point of the flow.*

Proof. Suppose B is a bounded component of $\mathbb{R}^2 \setminus K$ different from the component A which contains S. If N is an isolating block of K as described in the proof of Theorem 1.3.7 then $N^o \subset A$ since, otherwise, S would meet other components of $\mathbb{R}^2 \setminus K$ and would not be connected. Hence, the component C of ∂N lying in B is totally contained in N^i. The circle C is also the boundary of a disk D contained in B and, since every orbit through C enters N (in the future) and remains there, the disk D is negatively invariant by the flow. As a consequence, in the interior of D there is a repeller R which repels the whole disk. Moreover, since N is isolating, every point of $N \cap B$ goes to D in the past (and remains there), which implies that the basin of repulsion of R is all B. Since D is an isolating block of R, Corollary 1.1.5 ensures that R must contain a fixed point. □

The next results are concerned with the topological characterization of those isolated invariant continua in surfaces which do not have fixed points. For this purpose we make use of an index introduced by Srzednicki [94] which generalizes the degree of C^1 vector fields on $\mathbb{R}^n$.

Let $\varphi : M \times \mathbb{R} \to M$ be a flow defined on an ENR and $U \subset M$ an open set with compact closure which does not have fixed points in ∂U. The index $i(\varphi, U)$ is defined as

$$i(\varphi, U) = \lim_{t \to 0} \operatorname{ind}(\varphi_t, U),$$

where, for each $t \in \mathbb{R}$, $\varphi_t = \varphi(\cdot, t)$ and ind denotes the classical fixed point index.

This index is a well-defined integer number which measures, in some extent, the number of fixed points φ has in U. In particular, φ must have fixed points in U if $i(\varphi, U) \neq 0$. It easily follows from [94, Theorem 4.4] that if K is an isolated invariant compactum which admits an isolating block that is an ENR and whose exit set is also an ENR then, if N is an isolating neighborhood of K

$$i(\varphi, \mathring{N}) = \chi(h(K)).$$

A straightforward consequence of this fact and Theorem 1.3.7 is the following result, which establishes a relation between this index, the number of contractible components of the initial sections on the unstable manifold and the topology of K.

Proposition 1.5.6. *Let K be an isolated invariant continuum of a flow φ defined on a surface and N an isolating neighborhood of K. Then,*

$$i(\varphi, \mathring{N}) = 1 - \beta_1(K) - u_c.$$

If $M = \mathbb{R}^2$ or the 2-sphere S^2 there is a nice interpretation of Proposition 1.5.6 in terms of the Brouwer degree since, in this case, [94, Theorem 5.1] ensures that if the flow φ is generated by $\dot{x} = -X(x)$, where X is C^1, (see also [46, 55]) then

$$i(\varphi, U) = \deg(X, U).$$

The following result is a consequence of this fact and Proposition 1.5.6.

Corollary 1.5.7. *Let X be a C^1 vector field on $M = \mathbb{R}^2$ or S^2 and suppose that the flow φ is generated by $\dot{x} = -X(x)$. Let K be an isolated invariant continuum of φ and N an isolating block for K. Then $\deg(X, \mathring{N}) = 2 - n - u_c$, where n denotes the number of components of $M \setminus K$.*

Remark 1.5.8. Corollary 1.5.7 can be obtained as a particular instance of [74, Main Theorem 1]. This result gives a formula for the fixed point index of an isolated invariant continuum of a planar local homeomorphism.

The property of being non-saddle turns out to be related to the non existence of fixed points. In fact, we have the following result, which gives necessary conditions for the non-existence of fixed points contained in isolated continua.

Theorem 1.5.9. *Suppose that K is an isolated invariant continuum of a flow on a surface M and that K does not contain fixed points. Then, K is non-saddle and it is either a limit cycle, a closed annulus bounded by two limit cycles, or a Möbius strip bounded by a limit cycle.*

Proof. Let N be a regular isolating block of K. Since K does not have fixed points, applying Proposition 1.5.6 we get

$$0 = i(\varphi, \mathring{N}) = 1 - \beta_1(K) - u_c.$$

Hence, $\beta_1(K) + u_c = 1$ and we have two possibilities. The first one is that $\beta_1(K) = 0$ and $u_c = 1$, which must be excluded since N would be a disk [70, Theorem 3] and K would contain a fixed point by Corollary 1.1.5. The remaining possibility is $\beta_1(K) = 1$ and $u_c = 0$. In this case K would be non-saddle and it would have the shape of a circle. Moreover, $\beta_1(N) = 1$ being N a regular isolating block. Therefore N is either an annulus or a Möbius strip depending on its orientability. Indeed, capping each component of N with a disk we get a closed surface $\widehat{N}$ and

$$\chi(\widehat{N}) = \chi(N) + c,$$

where c is the number of boundary components of N. But, since $\beta_1(N) = 1$ and N has non-empty boundary it follows that $\chi(N) = 0$ and, hence, $\chi(\widehat{N}) = c > 0$. If N is orientable, so is $\widehat{N}$ and, hence, $c = 2$ and $\widehat{N}$ must be a sphere. Therefore, N is a sphere with two open disks removed, i.e., an annulus, as we wanted to prove. On the other hand, if N is nonorientable so is $\widehat{N}$ and, as a consequence, $c = 1$ and $\widehat{N}$ must be a projective plane. Then, N is a projective plane with an open disk removed, i.e., a Möbius strip.

Suppose that N is orientable, i.e., an annulus. Then, N can be embedded in $\mathbb{R}^2$ and, by the arguments presented in the proof of Lemma 1.1.3, we may assume, without loss of generality, that the flow restricted to $\mathring{N}$ can be extended to the whole $\mathbb{R}^2$. Since K has the shape of a circle it must decompose the plane into two components A and B. Suppose A is the unbounded component, since K is isolated non-saddle every point $x \in A$ point sufficiently close to K satisfies that either $\emptyset \neq \omega(x) \subset K$ or $\emptyset \neq \omega^*(x) \subset K$. Suppose that $\emptyset \neq \omega(x) \subset K$ (the argument is analogous if $\emptyset \neq \omega(x) \subset K$). Then, the Poincaré-Bendixson Theorem ensures that $\omega(x)$ is a periodic orbit contained in K, that we denote by γ. Moreover, B is contained in the interior of γ (otherwise we would have a fixed point in K). By the same argument, there is a point $y \in B$ whose ω or ω^*-limit is a periodic orbit γ' contained in K. If $\gamma \neq \gamma'$ the orbits γ and γ' bound a plane region C homeomorphic to an annulus. C is contained in K since, otherwise K would disconnect the plane in more than two components. On the other hand, we prove now that there are no points $z \in K \setminus C$. Suppose, to get a contradiction, that $z \in K$ is in the unbounded component of $\mathbb{R}^2 \setminus C$ (the other case is only slightly different). Then $\omega(z)$ is a periodic orbit, γ'', containing γ in its interior since, otherwise, the interior of γ'' would be entirely contained in K and, thus, it would contain a fixed point of K. Since γ is in the interior of γ'', γ cannot be a limit orbit of points of A. This contradiction establishes that $C = K$. If $\gamma = \gamma'$ an easier argument proves that $K = \gamma = \gamma'$.

Assume, on the other hand, that N is nonorientable, i.e., a Möbius strip. Since K is non-saddle and N has only one boundary component it must be either an attractor or a repeller. Consider another copy N^* of N and the flow $\varphi^* = \varphi(\cdot, -t)$ on M. We obtain a flow without fixed points on the Klein Bottle by identifying the boundaries of N and N^* via the identity map and considering the flow $\widehat{\varphi}$ which agrees with φ in N and with φ^* in N^*. Now, choosing a point in ∂N, either its ω- or its ω^*-limit is a limit cycle contained in K [25, Theorem 3.7].

This limit cycle cannot bound a disk in N since K does not contain fixed points and, as a consequence, it either does not bound any region in N and, hence, it agrees with K or it bounds a Möbius strip contained in $\mathring{N}$. In this case K must agree with this Möbius strip and the result follows. □

Remark 1.5.10. According to Theorem 1.5.9 every isolated periodic orbit γ of a flow on a surface is a non-saddle set. If γ is neither an attractor nor a repeller, it follows from our previous discussion that $W^u(\gamma)$ is homeomorphic to a punctured disk, while every initial part of its unstable manifold $W^u_S(\gamma)$ is homeomorphic to an annulus with γ as one of the boundary components. On the other hand, if p is an isolated equilibrium which is neither an attractor nor a repeller then $u = u_c$, and it follows from Proposition 1.5.1 that the initial parts of the truncated unstable manifold, $I^u_S(p)$, and the final parts of the truncated stable manifold, $I^s_S(\gamma)$ have the same homotopy type. As a matter of fact, it can be readily seen that the unstable manifold $W^u(p)$ is the bijective continuous image (*although not necessarily the homeomorphic image*) of a set of $\mathbb{R}^2$ composed of a finite union of rays from 0 plus a finite union of closed plane sectors with vertex at 0.

The following results are consequences of Theorem 1.5.9.

Corollary 1.5.11. *Suppose φ is a flow defined on a surface and K an isolated invariant continuum which is minimal. Then, K is either a fixed point or a limit cycle.*

Remark 1.5.12. If M is compact and the flow is C^2, Corollary 1.5.11 holds even if we drop the assumption about the isolation as it has been seen in [92].

Corollary 1.5.13. *If φ is a flow defined on a compact surface and every minimal set of φ is isolated, then φ is topologically equivalent to a C^∞ flow.*

Proof. It readily follows from Corollary 1.5.11 and Gutiérrez' Theorem. □

1.6 Fixed points, bounded orbits and attractors of planar flows

In this section we are concerned with the study of planar flows $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. In particular, we provide a dynamical characterization of isolated invariant continua which are global attractors for planar dissipative flows. This characterization is inspired by a result of Alarcón, Guíñez and Gutiérrez about dissipative planar embeddings with only one fixed point (see [2]). Moreover, we will derive a sufficient condition for a planar continuum to be an attractor or a repeller provided that it contains all the fixed points of φ.

We start by recalling the definition of *dissipative flow*. Let M be a locally compact metric space and $\varphi : M \times \mathbb{R} \to M$ a flow on M. The flow φ is said to be *dissipative* if $\omega(x) \neq \emptyset$ for every $x \in M$ and $\bigcup_{x \in M} \omega(x)$ has compact closure. If the phase space M is not compact, dissipativeness is equivalent to $\{\infty\}$ being a repeller of the extended flow $\widehat{\varphi} : (M \cup \{\infty\}) \times \mathbb{R} \to M \cup \{\infty\}$ to the Alexandrov compactification of M leaving ∞ fixed (See [29, 39, 91]), and therefore to the existence of a global attractor for φ.

The following result gives a relation between global asymptotic stability of a fixed point and the non-existence of additional fixed points in the case of discrete dynamical systems.

Theorem 1.6.1 (Alarcón-Guíñez-Gutiérrez [2], Ortega-Ruiz del Portal [66]). *Assume that $h \in \mathcal{H}_+$ (orientation preserving homeomorphisms of $\mathbb{R}^2$) is dissipative and p is an asymptotically stable fixed point of h. The following conditions are equivalent:*

1. *p is globally asymptotically stable;*

2. $\mathrm{Fix}(h) = p$ *and there exists an arc $\gamma \subset S^2$ with end points at p and ∞ such that $h(\gamma) = \gamma$.*

The proof in [2] is based on Brouwer's theory of fixed point free homeomorphisms of the plane. Ortega and Ruiz del Portal give in [66] an alternative proof based on the theory of prime ends.

We prove in our following theorem that in the continuous case this result holds even if the asymptotically stable fixed point p is substituted by a connected isolated invariant set K which contains every fixed point of the flow. We obtain in this way a simple characterization of global attractors of dissipative planar flows.

Theorem 1.6.2. *Let K be an isolated invariant continuum of a dissipative flow φ in $\mathbb{R}^2$. The following conditions are equivalent:*

1. *K is a global attractor;*

2. *There are no fixed points in $\mathbb{R}^2 \setminus K$ and there exists an orbit γ connecting ∞ and K (i.e. such that $\|\gamma(t)\| \to \infty$ when $t \to -\infty$ and $\omega(\gamma) \subset K$).*

Proof. Gutiérrez Theorem allows us to assume that the flow φ is smooth. Since φ is dissipative, given $x \in \mathbb{R}^2$ its ω-limit is non-empty and compact. Moreover, by the Poincaré-Bendixson Theorem either $\omega(x)$ contains fixed points and, hence, $\omega(x) \cap K \neq \emptyset$ or $\omega(x)$ is a periodic orbit. If $\omega(x)$ is a periodic orbit then K is not contained in its interior since, in that case, γ would meet $\omega(x)$, which is impossible. Therefore, if $\omega(x)$ is not contained in K, then K is in the exterior of $\omega(x)$ and, moreover, $\omega(x)$ being a periodic orbit, there must exist a fixed point in its interior. Hence this point belongs to K, which is a contradiction. We conclude that if $\omega(x)$ is a periodic orbit then $\omega(x) \subset K$.

If $\omega(x)$ is not a periodic orbit then $\omega(x) \cap K \neq \emptyset$ and we shall prove that, in fact, $\omega(x) \subset K$. We suppose, to get a contradiction, that there exists $y \in \omega(x) \setminus K$. By hypothesis y is not a fixed point and, thus, we can take a local interval section I containing y and meeting transversally the trajectory of y. Since $y \notin K$ we can assume that $I \cap K = \emptyset$. It is a well-known fact that the trajectory of x meets I infinitely many times. We consider two consecutive points of intersection $x_1 = xt_1$ and $x_2 = xt_2$ with x_1,$x_2 \in I$, $0 < t_1 < t_2$ and $x[t_1, t_2] \cap I = \{x_1,x_2\}$. Then the set $C = x[t_1, t_2] \cup J$, where J is the subinterval of I bounded by x_1 and x_2, is a simple closed curve which, by the Jordan Theorem, decomposes $\mathbb{R}^2$ into two connected components U and V. If U is the bounded component then U is either positively or negatively invariant. Then, a simple argument involving again the Poincaré-Bendixson Theorem, leads to the existence of a fixed point in U which, by hypothesis, belongs to K. Now, the intersection of K with C is empty, which implies that $K \subset U \cup V$ and, K being connected, that $K \subset U$. If U is negatively invariant, the trajectory γ linking ∞ with K cannot enter in U since the only possibility would be through J, which is an exit set. This makes it impossible that $\omega(\gamma) \subset K$ and we get a contradiction with the hypothesis. If U is positively invariant then an easy argument shows that $y \in \omega(\gamma)$ in contradiction with the assumption. This proves that $\omega(x) \subset K$ for every $x \in \mathbb{R}^2$ and, as a consequence, K is a globally attracting set. Since K is isolated, by Theorem 3 K must be stable, i.e. a global attractor. This establishes the implication 2. $\Rightarrow$ 1. The converse implication is straightforward. $\square$

The following result, which is a consequence of Theorem 1.6.2 and [15, Theorem 4.1] by Bhatia, Lazer and Szego, gives a nice characterization of globally attracting fixed points.

Corollary 1.6.3. *Let K be a minimal attractor of a dissipative flow in $\mathbb{R}^2$. The following conditions are equivalent:*

1. *K is a globally attracting fixed point*
2. *There are no fixed points in $\mathbb{R}^2 \setminus K$ and there exists an orbit connecting ∞ and K.*

Proof. It is a consequence of Theorem 1.6.2 and Bhatia, Lazer and Szego's Theorem 4.1 in [15] according which minimal global attractors in $\mathbb{R}^n$ are fixed points. $\square$

The following result establishes a relation between the homoclinic orbits of a plane continuum and the existence of fixed points in its complement.

Lemma 1.6.4. *Suppose that K is an isolated invariant continuum of a plane flow and a component U of $\mathbb{R}^2 \setminus K$ contains a trajectory γ such that $\omega(\gamma) \cap K \neq \emptyset$ and $\omega^*(\gamma) \cap K \neq \emptyset$ then there exists a fixed point in U.*

Proof. Suppose, to get a contradiction, that there exists a trajectory γ in a component U of $\mathbb{R}^2 \setminus K$ such that U does not contain fixed points and $\omega(\gamma) \cap K \neq \emptyset$ and $\omega^*(\gamma) \cap K \neq \emptyset$. Let N be a regular isolating block of K. Hence, N must be a topological closed disk with i holes, one for every bounded component of $\mathbb{R}^2 \setminus K$. We suppose that U is the unbounded component (the argument being only slightly diferent in the other case) and consider the only circle $C \subset \partial N$ contained in U. Then there exists a point $x \in C \cap \gamma$ leaving N and returning to N after a time $t \neq 0$, i.e. such that $xt \in C$ and $x(0,t) \cap N = \emptyset$. The possibility that the time t be positive or negative is irrelevant in this construction. Consider the arc A in C with endpoints x and xt such that the topological circle $x[0,t] \cup A$ does not contain K in its interior. This arc can be mapped to the unit interval $I = [0,1]$ of the real line by a homeomorphism $h : A \to I$. If we take the point $x_1 \in \mathring{A}$ corresponding to the center of I then x_1 must leave N (in the past or in the future) and return again at a time $t_1 \neq 0$ since, otherwise, the Theorem of Poincaré-Bendixson would imply the existence of a fixed point in the disk limited by $x[0,t] \cup A$. Hence we can repeat the operation with $x_1[0,t_1] \cup A_1$, where A_1 is an arc in A with endpoints x_1 and x_1t_1 and the topological circle $x_1[0,t_1] \cup A_1$ does not contain K in its interior. Now take $x_2 \in \mathring{A}_1$ corresponding to the middle point of $h(A_1)$ and repeat the construction. In this way we obtain a sequence $A \supset A_1 \supset A_2 \supset \cdots$ of arcs whose intersection $\bigcap_{i=1}^{\infty} A_i$ consists of one point $p \in \partial N$. The orbit of p defines an internal tangency to ∂N, which is in contradiction with the properties of isolating blocks. This contradiction proves that if $\omega(\gamma) \cap K \neq \emptyset$ and $\omega^*(\gamma) \cap K \neq \emptyset$ then there exists a fixed point in U. □

As a consequence of the last proposition, a lower bound for the number of fixed points in the complement of an isolated invariant plane continuum is obtained.

Corollary 1.6.5. *Let K be an isolated invariant continuum of a plane flow. Suppose that $\mathbb{R}^2 \setminus K$ has i connected components. Then, there are at least $i-1$ fixed points in $\mathbb{R}^2 \setminus K$.*

Proof. We see that, in fact, there is at least one fixed point in every bounded component U of $\mathbb{R}^2 \setminus K$. Otherwise, if γ is a trajectory in the bounded component U, where U does not contain fixed points, then by Lemma 1.6.4 either $\omega(\gamma) \cap K = \emptyset$ or $\omega^*(\gamma) \cap K = \emptyset$. Hence, the Poincaré-Bendixson Theorem ensures the existence of a periodic orbit contained in U, and thus the existence of a fixed point in its interior. This leads to a contradiction with the assumption, since the interior of any periodic orbit contained in U is also contained in U. The number of bounded components of $\mathbb{R}^2 \setminus K$ is exactly $i-1$, so this contradiction establishes the corollary. □

Theorem 1.6.6. *Let K be an isolated invariant continuum of a flow φ in $\mathbb{R}^2$. Suppose that there is a closed disk D containing K in its interior such that there are no fixed points in $D \setminus K$ and that there is an orbit γ completely contained in $D \setminus K$. Then, K is either an attractor or a repeller. Moreover, K has trivial shape.*

Proof. We can assume again that φ is smooth. Since $\overline{\gamma} \subset D$ we have that $\emptyset \neq \omega(\gamma) \subset D$ and $\emptyset \neq \omega^*(\gamma) \subset D$. We start by proving that there exists an orbit

Γ in $D \backslash K$ satisfying the additional condition that either $\omega(\Gamma) \subset K$ or $\omega^*(\Gamma) \subset K$. As a consequence of the Poincaré-Bendixson Theorem and the hypothesis of the present theorem we have that either $\omega(\gamma) \cap K \neq \emptyset$ or $\omega(\gamma)$ is a periodic orbit not meeting K, and the same can be said for $\omega^*(\gamma)$. If $\omega(\gamma)$ is a periodic orbit not meeting K then K is in its interior and, by the Ura-Kimura Theorem, there exists a point x, also in the interior of $\omega(\gamma)$, with either $\emptyset \neq \omega(x) \subset K$ or $\emptyset \neq \omega^*(x) \subset K$, and the same happens if $\omega^*(\gamma)$ is a periodic orbit not meeting K. Hence, in both cases Γ can be taken as the trajectory of x. On the other hand, Lemma 1.6.4 ensures that the possibility that both intersections, $\omega(\gamma) \cap K$ and $\omega^*(\gamma) \cap K$, are non-empty can never happen. Therefore, it follows from the previous remarks that there exists an orbit Γ in $D \setminus K$ satisfying the additional condition that either $\emptyset \neq \omega(\Gamma) \subset K$ or $\emptyset \neq \omega^*(\Gamma) \subset K$.

Suppose that $\omega(\Gamma) \subset K$. Then, $\omega^*(\Gamma)$ is a periodic orbit containing K in its interior. Let V be the interior of $\omega^*(\Gamma)$ and consider the flow restricted to $\overline{V}$. An elementary argument involving local sections again shows that $\omega^*(\Gamma)$ is a repeller for $\varphi|_{\overline{V}}$ and, as a consequence, the restriction of φ to V is a dissipative flow. Then, using an arbitrary homeomorphism between V and $\mathbb{R}^2$ we can define a dissipative flow in $\mathbb{R}^2$ conjugated to $\varphi|_V$ and satisfying the conditions of Theorem 1.6.2. We deduce from that theorem that K is an attractor of φ whose basin of attraction, V, is an open topological disk. Hence, K has trivial shape by [49, Theorem 3.6]. In the dual situation (when $\omega^*(\Gamma) \subset K$ and $\omega(\Gamma)$ is a periodic orbit containing K in its interior), which could be discussed analogously using the reverse flow, it follows that K is a repeller with trivial shape. □

From Theorem 1.6.6 it follows:

Corollary 1.6.7. *Let K be an isolated invariant continuum of a flow φ in $\mathbb{R}^2$. Suppose that K contains all the fixed points of φ and that there exists a bounded orbit γ in $\mathbb{R}^2 \setminus K$. Then K is either an attractor or a repeller. Moreover, K has trivial shape.*

Proof. The set $K \cup \overline{\gamma}$ is compact and as a consequence there exists a closed disk D such that $K \cup \overline{\gamma} \subset \mathring{D}$. Then, Theorem 1.6.6 applies since the bounded orbit $\gamma \subset D \setminus K$, and $D \setminus K$ does not contain fixed points by assumption.

□

Remark 1.6.8. The assumptions about the existence of a disk D such that there is an entire orbit contained in $D \setminus K$ in Theorem 1.6.6 and the existence of a bounded orbit in $\mathbb{R}^2 \setminus K$ in Corollary 1.6.7 are unavoidable. For instance, consider the flow φ induced by the linear system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}$$

The origin $(0,0)$ is a fixed point which is isolated as an invariant set and there are neither fixed points nor other bounded orbits in $\mathbb{R}^2 \setminus \{(0,0)\}$. In this case, $\{(0,0)\}$ is a saddle and hence, it is neither an attractor nor a repeller.

As a consequence of Corollary 1.6.7 and Theorem 1.5.9 we obtain the following dichotomy for dissipative flows:

Corollary 1.6.9. *Let K be an isolated invariant continuum of a dissipative flow φ in $\mathbb{R}^2$. Suppose that K contains all the fixed points of φ, then K has trivial shape and it is either an attractor or a repeller. Moreover, if K is a repeller then there exists an attractor $K^* \subset \mathbb{R}^2 \setminus K$ which is either a limit cycle or a closed annulus bounded by two limit cycles.*

Proof. The dissipativeness of φ guarantees the existence of a global attractor K' and as a consequence $K \subset K'$. Suppose $K' \neq K$, since otherwise we have nothing to prove. Let $x \in K' \setminus K$, the orbit $\gamma(x)$ is a bounded orbit being contained in the invariant compactum K'. Then, Corollary 1.6.7 ensures that K is either an attractor or a repeller. This proves the first part of the statement.

Suppose that K is a repeller and consider the flow $\varphi|_{K'}$, i.e. the restriction of φ to the global attractor. The continuum K is also a repeller for $\varphi|_{K'}$ and then there exists an invariant compactum $K^* \subset K'$ such that the pair (K^*, K) is an attractor-repeller decomposition of $\varphi|_{K'}$. Besides, the invariant compactum K^* is an attractor for φ since K^* is an attractor for $\varphi|_{K'}$ and K' is an attractor. The region of attraction of K^* agrees with $\mathbb{R}^2 \setminus K$ since K is a repeller and (K^*, K) is an attractor-repeller decomposition of the restriction of φ to the global attractor K'. Moreover, $\mathbb{R}^2 \setminus K$ is connected K being of trivial shape and hence so is K^* by [49, Theorem 3.6] and Borsuk's Theorem. We have proved that K^* is a connected attractor which does not contain fixed points, thus by Theorem 1.5.9 it must be either a limit cycle or a closed annulus bounded by two limit cycles. □

1.7 Parametrized families of flows on surfaces

In this section we shall discuss some matters using the point of view of continuation, a central notion in the Conley index theory. We point out that when talking about continuations it is implicit that we only consider continuations by non-empty compacta. Notice that the Conley index prevents the existence of a continuation between an isolated invariant continuum $\emptyset \neq K \subsetneq M$ and the total space M when M is a surface.

In figure 1.8 we show an example from [36], in which it is shown that some dynamical and topological properties of the original isolated invariant continuum are not preserved by its continuations. For instance, connectedness, shape or non-saddleness are some properties not preserved by continuation. We see that in spite of this fact, for flows on surfaces we can have a good understanding of how continuations work.

In the following result we show that if the shape is not preserved then the global complexity of isolated invariant continua can only decrease through small perturbations, i.e. the shape of each one of the components of the continuation K_λ is dominated by the shape of the initial continuum K_0 for small values of λ.

Theorem 1.7.1. *Let $(\varphi_\lambda)_{\lambda\in I}$ be a parametrized family of flows defined on a surface M and K_0 an isolated invariant continuum for φ_0. Suppose that the family $(K_\lambda)_{\lambda\in I}$ is a continuation of K_0. Then, there exists $\lambda_0 > 0$ such that*

$$\beta_1(K_\lambda) \leq \beta_1(K_0), \quad \textit{if} \quad 0 < \lambda < \lambda_0.$$

In particular, if $0 < \lambda < \lambda_0$ and K_λ^α is a component of K_λ then, $\mathrm{Sh}(K_0) \geq \mathrm{Sh}(K_\lambda^\alpha)$.

Proof. Let N be a regular isolating block of K_0. Then, there exists $0 < \lambda_0 \leq 1$ such that N is an isolating neighborhood of K_λ for $0 < \lambda < \lambda_0$. Hence, reasoning

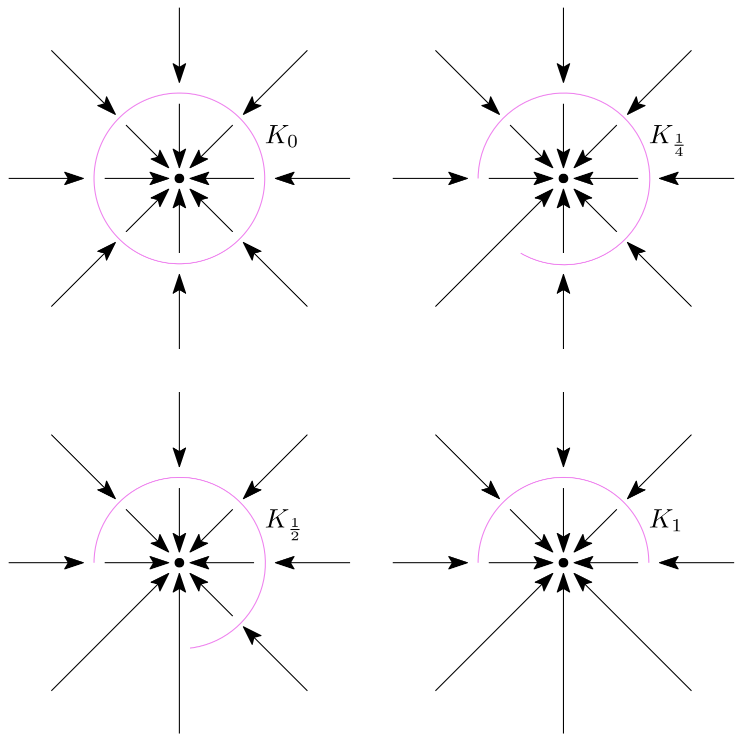

Figure 1.8: Continuation of an isolated non-saddle circumference by a family of saddle sets with the shape of a point.

as in the proof of Theorem 1.1.4 we get that

$$i_\lambda^* : \check{H}^1(N) \to \check{H}^1(K_\lambda)$$

is surjective. Therefore, $\beta_1(K_\lambda) \leq \beta_1(N)$ and the result follows since N is a regular block of K_0. □

We study continuations of isolated invariant continua regarding their dynamical nature. For this purpose we make use of the next proposition.

Proposition 1.7.2. *Let $(\varphi_\lambda)_{\lambda\in I}$ be a parametrized family of flows defined on a surface M and K_0 an isolated invariant continuum for φ_0 which is neither an attractor nor a repeller. Suppose that the family $(K_\lambda)_{\lambda\in I}$ is a continuation of K_0 such that K_λ consists of a finite number of connected components. Then*

$$(\beta_1(K_0) - \beta_1(K_\lambda)) + \left(u_c - u_c^\lambda\right) = 1 - n_\lambda,$$

where u_c^λ is the number of contractible components of an initial section of the truncated unstable manifold $W^u(K_\lambda) \setminus K_\lambda$ and n_λ is the number of components of K_λ. As a consequence, if $M = \mathbb{R}^2$ or S^2 and $\mathrm{Sh}(K_0) = \mathrm{Sh}(K_\lambda)$ for some $\lambda \in I$, the initial parts of the truncated unstable manifolds of K_0 and K_λ have the same homotopy type.

Proof. Since K_λ has a finite number of components $K_\lambda^1, \ldots, K_\lambda^{n_\lambda}$, all of them are isolated and

$$h(K_\lambda) = \bigvee_{i=1}^{n_\lambda} h(K_\lambda^i) \tag{1.2}$$

Moreover, $h(K_0) = h(K_\lambda)$ and, hence, $h(K_\lambda)$ is the pointed homotopy type of a wedge of $\beta_1(K_0) + u_c - 1$ circumferences. The result follows from (1.2) and Theorem 1.3.7.

Suppose that $M = \mathbb{R}^2$ or S^2 and $Sh(K_0) = Sh(K_\lambda)$ for some $\lambda \in I$. Then K_λ is a continuum and $M \setminus K_0$ and $M \setminus K_\lambda$ have the same number of components, say i. We discuss the case $i = 1$ since the general case can be reduced to this one. By the preservation of the Conley index by continuation, the numbers u and u_c remain the same for all $\lambda \in I$. This means that the initial sections of K_0 and K_λ, and also the initial parts of their truncated unstable manifolds, have the same homotopy type. □

Theorem 1.7.3. *Let $(\varphi_\lambda)_{\lambda \in I}$ be a parametrized family of flows defined on a surface M and K_0 an isolated invariant continuum for φ_0. Suppose that the family $(K_\lambda)_{\lambda \in I}$ is a continuation of K_0. Then,*

i) If K_0 is an attractor (repeller), K_λ has a component K_λ^1 which is also an attractor (repeller) and $\mathrm{Sh}(K_\lambda^1) = \mathrm{Sh}(K_0)$.

ii) If K_0 is neither an attractor nor a repeller then K_λ is neither an attractor nor a repeller and:

(a) If K_0 is saddle there exists $\lambda_0 > 0$ such that K_λ is also saddle for every λ with $0 < \lambda < \lambda_0$.

(b) If K_0 is non-saddle and K_λ is a continuum for each λ, then K_λ is non-saddle if and only if $\mathrm{Sh}(K_0) = \mathrm{Sh}(K_\lambda)$.

Proof. Suppose that K_0 is an attractor. The case of K_0 being a repeller is completely analogous reasoning with the reverse flow. Since $h(K_0) = h(K_\lambda)$, this means that $h(K_\lambda)$ is the pointed homotopy type of $\left(\bigvee_{i=1,\ldots,\beta_1(K_0)} S_i^1 \cup \{\bullet\}, \bullet\right)$. A consequence of this fact is that either K_λ is an attractor or it is the disjoint union of an attractor K_λ^1 and an isolated invariant set K_λ^2 with trivial Conley index. Indeed, if K_λ is connected then by Theorem 1.3.7 it must be an attractor and its first Betti number must agree with $\beta_1(K_0)$. Hence, $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K_0)$. Suppose, on the other hand, that K_λ is not connected. Then, given an isolating block N of K_λ it cannot be connected and it must have a connected component with empty exit set. Let K_λ^1 be the maximal invariant compactum contained in that component. It follows that K_λ^1 is an attractor and, by the additive property of the Conley index, the index of $K_\lambda^2 = K_\lambda \setminus K_\lambda^1$ must be trivial. Besides, K_λ^1 must be connected, since if not it would be the disjoint union of $i > 1$ attractors and its Conley index would be the pointed homotopy type of a space with $i + 1$ components. This is not possible since $h(K_\lambda^1)$ must agree with $h(K_0)$ which is the pointed homotopy type of a 2-component space. Therefore, Theorem 1.3.7 ensures that $\beta_1(K_\lambda^1)$ agrees with $\beta_1(K_0)$ and, as a consequence, $\mathrm{Sh}(K_\lambda^1) = \mathrm{Sh}(K_0)$.

Suppose that K_0 is neither an attractor, nor a repeller. Then, since $h(K_\lambda)$ agrees with $h(K_0)$, the former must be the pointed homotopy type of a connected space and it cannot be the pointed homotopy type of a space having a closed surface as a wedge summand. Hence, it is easy to conclude that K_λ is neither an attractor nor a repeller. Suppose that K_0 is saddle. If K_λ has an infinite number of components it must be saddle since non-saddle sets have the shape of finite polyhedra [32, Theorem 4]. Let us assume that K_λ has a finite number of components. From Theorem 1.7.1 we get that there exists $\lambda_0 > 0$ such that $\beta_1(K_\lambda) \leq \beta_1(K_0)$ if $0 < \lambda < \lambda_0$. Then, using this and Proposition 1.7.2 we obtain that $u_c^\lambda \geq u_c > 0$ for $\lambda < \lambda_0$ and, thus, K_λ is saddle for $0 < \lambda < \lambda_0$.

Let us assume that K_0 is non-saddle, i.e. $u_c = 0$, and K_λ is connected for every λ. We see that K_λ is non-saddle if and only if $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K_0)$. Indeed, suppose that K_λ is also non-saddle, i.e. $u_c^\lambda = 0$. Thus, Proposition 1.7.2 ensures that $\beta_1(K_0)$ must agree with $\beta_1(K_\lambda)$. Therefore, K_0 and K_λ must have the same shape. Conversely, if $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K_0)$ then $\beta_1(K_0) = \beta_1(K_\lambda)$ and, by Proposition 1.7.2, u_c^λ must be zero. Then, K_λ is non-saddle. □

Remark 1.7.4. We would like to point out the following things regarding Theorem 1.7.3:

1. If K_0 is an attractor (repeller) it was proven in [87] that for small values of λ, K_λ is actually connected and, hence, an attractor with the shape of K_0 even for flows defined on more general spaces than surfaces such as locally compact ANR's. A version for more general phase spaces of point i) in the statement of Theorem 1.7.3 was proven in [35].

2. It was seen in [36] that if $(\varphi_\lambda)_{\lambda \in I}$ is a differentiable family of flows on a differentiable manifold and K_0 is non-saddle, then, if K_λ is non-saddle for small values of λ, K_0 and K_λ must have the same shape, i.e. the hypotheses about the connectedness of K_λ can be dropped for small values of λ. We study the converse statement in Chapter 2. Notice that Theorem 1.7.3 shows that this converse statement is true for any surface and for any parameter value.

We shall be concerned now with bifurcations at critical points of the flow. Suppose that we have a parametrized family of flows $\varphi_\lambda : M \times \mathbb{R} \to M$, defined on a surface M, with $\lambda \in I$, such that $p \in M$ is a fixed point for every λ. There are several non-equivalent definitions of bifurcation at p when $\{p\}$ is an attractor for φ_0. We adopt the following one, which conveys the idea that a new continuum, evolving from p, is created in the bifurcation.

Definition 1.7.5. Let $\varphi_\lambda : M \times \mathbb{R} \to M$, with $\lambda \in I$, be a continuous family of flows. Suppose that p is a fixed point for every φ_λ and $\{p\}$ is an attractor for φ_0. Suppose also that $(M^\lambda)_{\lambda \in I}$, with $M^0 = \{p\}$, is a continuation of $\{p\}$. If there is a $\lambda_0 \in (0, 1]$ and a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ of M^λ into two continua, where one of them is $\{p\}$ for every λ with $0 < \lambda < \lambda_0$, we say that a bifurcation takes place at p.

Concerning the former definition, we remark that the order is essential in the Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ and that we admit the two possibilities $M_a^\lambda = \{p\}$ for every λ with $0 < \lambda < \lambda_0$, or $M_b^\lambda = \{p\}$ for every λ with $0 < \lambda < \lambda_0$. Since $\{p\}$ is an attractor for φ_0 we can select λ_0 so small that M^λ is an attractor of trivial shape for φ_λ with $0 < \lambda < \lambda_0$. Since M_a^λ is an attractor for the restriction flow $\varphi_\lambda|_{M^\lambda}$, then M_a^λ is also an attractor for the flow φ_λ. The most notorious particular case is when $M_b^\lambda = \{p\}$ is a repeller for φ_λ with $0 < \lambda < \lambda_0$ and M_a^λ is a periodic orbit. In this case we say that a *Hopf bifurcation* takes place at p.

The bifurcation may be embedded in a more complex process of continuation of an isolated invariant continuum. Suppose we have a continuum $K = K_0$ which is invariant and isolated for φ_0, endowed with a Morse decomposition $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ with $M_1 = \{p\}$. Assume that K continues to a family of continua $(K_\lambda)_{\lambda \in I}$. Then, $\mathcal{M}$ also continues to Morse decompositions $\mathcal{M}^\lambda = \{M_1^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$ of the K_λ. Suppose that simultaneously a bifurcation takes place at p according to the previous definition, i.e. that M_1^λ has itself a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ as in Definition 1.7.5. Then,

$$\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$$

is also a Morse decomposition of K_λ which embodies information about the bifurcation and about the continuation. We call $\hat{\mathcal{M}}^\lambda$ the Morse decomposition associated to the bifurcation. We write the Morse equation of $\hat{\mathcal{M}}^\lambda$ in the usual form $P^\lambda(t) = R^\lambda(t) + (1+t)Q^\lambda(t)$, where $Q^\lambda(t)$ is a polynomial whose coefficients are non-negative integers.

Theorem 1.7.6. *Let $\varphi_\lambda : M \times \mathbb{R} \to M$ a parametrized family of flows on a surface and K_0 be an isolated invariant continuum of φ_0. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be a Morse decomposition of K with $M_1 = \{p\}$. Suppose that a Hopf bifurcation takes place at p and denote by $\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, \ldots, M_k^\lambda\}$ the associated Morse decomposition. Then $P^\lambda - P = t^2 + t$, where P corresponds to the Morse equation of $\mathcal{M}$.*

Proof. The main difference of $\hat{\mathcal{M}}^\lambda$ with the initial Morse decomposition $\mathcal{M}$ is that the point p becomes repelling and an attracting periodic orbit M_a^λ evolves from p. The repelling point is responsible for the term t^2 and the attracting orbit adds the term t to the Morse equation. The contribution of the rest of the Morse sets remains the same, since they are continuations of the Morse sets of the initial decomposition. □

We shall see now that the relation $P^\lambda - P = t^2 + t$ captures some of the topology involved in the Hopf bifurcation, although not the whole of the dynamics: if we have a bifurcation (not necessarily Hopf) whose Morse equation satisfies this particular relation, then we shall show that a new attractor with the shape of S^1 (although not necessarily a periodic orbit) is created in the bifurcation. The following result enumerates all the possible types of bifurcations. We see that the rest of the bifurcations have no effect on the Morse equation.

Theorem 1.7.7. *Let $\varphi_\lambda : M \times \mathbb{R} \to M$ be a parametrized family of flows on a surface and K_0 be an isolated invariant continuum of φ_0. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be a Morse decomposition of K_0 with $M_1 = \{p\}$. Suppose that a bifurcation (not necessarily Hopf) takes place at p and denote by $\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, \ldots, M_k^\lambda\}$ the associated Morse decomposition. Then, there exists $\delta > 0$ such that for every λ with $0 < \lambda < \delta$ one of the following possibilities holds:*

i) $M_a^\lambda = \{p\}$ is an attractor and M_b^λ is a non-saddle set with the shape of S^1,

ii) $M_a^\lambda = \{p\}$ is an attractor and M_b^λ a saddle-set with trivial shape,

iii) M_a^λ is an attractor of trivial shape and $M_b^\lambda = \{p\}$ is a saddle-set, and

iv) M_a^λ is an attractor with the shape of S^1 and $M_b^\lambda = \{p\}$ is a repeller.

In case iv), we have the relation $P^\lambda - P = t^2 + t$ for the Morse equation and in cases i),ii) and iii) the Morse equation remains unaltered.

Proof. By the properties of continuations we have that, choosing $0 < \delta < \lambda_0$ small enough, M_1^λ lies in a topological disk $D \subset M$ for each $0 < \lambda < \delta$. The Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ of M_1^λ consists of two sets, one of them, for instance M_b^λ, is equal to $\{p\}$ and the other, M_a^λ, is a continuum. This continuum cannot separate D into more than two components since, being M_1^λ of trivial shape, all the bounded components of $D \setminus M_a^\lambda$ must be contained in M_1^λ and, thus, each of them must contain a Morse set of the decomposition of M_1^λ other than M_a^λ, and there is only one. We get to an analogous conclusion with M_b^λ if we assume that $M_a^\lambda = \{p\}$. As a consequence, we have the following possibilities: i) $M_a^\lambda = \{p\}$ and M_b^λ a continuum with the shape of S^1, ii) $M_a^\lambda = \{p\}$ and M_b^λ a continuum with trivial shape, iii) M_a^λ a continuum of trivial shape and $M_b^\lambda = \{p\}$, iv) M_a^λ a continuum with the shape of S^1 and $M_b^\lambda = \{p\}$. We discuss first the case iv). As we remarked before, since M_1^λ is an attractor and M_a^λ is an attractor of the restriction flow $\varphi_\lambda|_{M_1^\lambda}$, then M_a^λ is, in fact, an attractor of φ_λ. The bounded component of $D \setminus M_a^\lambda$ must be contained in M_1^λ and p must lie there. As a consequence, the bounded component of $D \setminus M_a^\lambda$ is the basin of repulsion of $\{p\}$, which means that $\{p\}$ is a repeller for φ_λ (and not only for the restriction $\varphi_\lambda|_{M_1^\lambda}$). If we calculate now the Morse equation of the associated Morse decomposition we see that the repeller $\{p\}$ contributes with the term t^2 and the evolving attractor M_a^λ contributes with a new t. The rest of the Morse sets have the same contribution to the Morse equation as in P since they are continuations of those of the decomposition $\mathcal{M}$. Hence, $P^\lambda - P = t^2 + t$. The rest of the cases are similarly discussed. Case i) is very similar to case iv). Cases ii) and iii) have in common the fact that M_1^λ has a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ into two sets of trivial shape. The Conley index of M_a^λ is the index of an attractor of trivial shape and the Conley index of M_b^λ can be easily calculated from the long exact sequence of the Morse decomposition of M_1^λ, from which it results a trivial Conley index. A consequence of this is that M_b^λ is a saddle-set and the Morse equation P^λ is not changed afer the bifurcation. □

For a discussion of generalized Poincaré-Andronov-Hopf bifurcations we refer the reader to the paper [89].

CHAPTER 2

TOPOLOGY AND DYNAMICS OF NON-SADDLE SETS

In this chapter we study the structure of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a locally compact metric space M, having an isolated non-saddle set (see Definition 1.5.2) K. We recall that, to avoid trivial cases, when we consider an isolated invariant set K, we always assume that it is a proper subset of M, i.e., $\emptyset \neq K \subsetneq M$ unless otherwise specified. The theory of non-saddle sets can be considered as a general theory of stability and attraction, which extends the classical one and encompasses recent developments such as the theory of unstable attractors with no external explosions [3,63,82]. Saddle and non-saddle sets were first studied by N.P. Bhatia [14] and by T. Ura [96] but, according to Ura, they were introduced before by P. Seibert in an oral communication.

In this chapter, in contrast with the previous one, we do not focus only on local aspects of the flow. In particular, we are interested in how the global dynamics is affected by the existence of an isolated non-saddle set. Near the isolated non-saddle set the flow exhibits a nice structure since it is organized into a purely attracted part and a purely repelled part, where the properties of attraction and repulsion are uniform. In other words, isolated non-saddle sets have isolating blocks consisting of the asymptotic (positive and negative) parts only. Notice that this is the local structure exhibited by a flow near attractors, repellers and isolated unstable attractors without external explosions which are examples of non-saddle sets. However, the global dynamics which may appear in the case of a flow having a non-saddle set not being neither an attractor, nor a repeller nor an unstable attractor, can be much more complicated than the simpler ones showed in those cases.

Through this chapter we consider homology and cohomology with coefficients in G which is either $\mathbb{Z}$ or $\mathbb{Z}_2$.

All the results of this chapter are contained in [6, 9, 12].

2.1 Topological aspects of non-saddle sets

In this section we study some topological aspects of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a locally compact metric space having an isolated non-saddle set. For instance, we give a characterization of non-saddleness in terms of influence-like properties and we also characterize the shape of those non-separating isolating non-saddle sets which are neither attractors nor repellers in the torus.

We start by stating a well-known result about isolating blocks of isolated non-saddle sets whose proof we include here for the sake of completeness.

Proposition 2.1.1. *Every isolated non-saddle set admits a basis of isolating blocks of the form $N = N^+ \cup N^-$.*

Proof. Since any isolated invariant compactum admits a neighborhood basis comprised of isolating blocks, it would be sufficient to prove that given an arbitrary isolating block N of an isolated non-saddle set K, it contains an isolating block $N_0 = N_0^+ \cup N_0^-$.

Let N be an isolating block of the isolated non-saddle set K. Then, K being non-saddle, there exists a neighborhood $V \subset N$ of K such that for each $x \in V$, either $\gamma^+(x) \subset N$ or $\gamma^-(x) \subset N$. As a consequence, the compact subset $N_0 = N^+ \cup N^-$ is an isolating neighborhood of K since $V \subset N_0$ and it is contained in the isolating block N. Moreover, N_0 is an isolating block. To see this we prove that $\partial N_0 \subset \partial N$. Suppose that there exists $x \in \partial N_0 \setminus \partial N$. Then, $x \in \mathring{N}$ and there exists a sequence x_n in $N \setminus (N^+ \cup N^-)$ such that $x_n \to x$. Suppose that $x \in N^+$, otherwise the argument is analogous. Let t_n be a sequence of positive times , $t_n \to \infty$. Then, $xt_n \to K$ as $n \to \infty$ and, maybe after passing to a subsequence, $x_n t_n \to K$ as $n \to \infty$. Besides, the choice of x_n ensures that neither $\gamma^+(x_n t_n)$

nor $\gamma^-(x_nt_n)$ are contained in N leading to a contradiction with K being non-saddle. Indeed, x_nt_n cannot be in N^- since x_n leaves N in negative time, being $x_n \in N \setminus (N^+ \cup N^-)$ and, since $t_n > 0$ so does x_nt_n. We see that x_nt_n cannot be in N^+. Suppose, to get a contradiction, that $x_nt_n \in N^+$ for almost all n. Then, there exists a sequence $0 < s_n < t_n$ such that $x_ns_n \in N^o$ for all n. Moreover, the sequence s_n must be bounded. If not, x_ns_n would have a subsequence convergent to a point $z \in N^-$ and, as a consequence, x_n would have a subsequence such that the positive semi-trajectory of each one of its elements gets arbitrarily close to K before leaving N in contradiction with its non-saddleness. Thus, we may assume that $s_n \to s_0 \geq 0$ and, hence, $x_ns_n \to xs_0$. Since $x_ns_n \in N^o$ for each n, so does xs_0. However, $xs_0 \in N^+$ which has empty intersection with N^o and we get a contradiction. Therefore, ∂N_0 agrees with $\partial N \cap N_0$ and the compact subsets $N_0^i = N^+ \cap \partial N$ and $N_0^o = N^- \cap \partial N$ are respectively an entrance and an exit set ensuring that N_0 is an isolating block. □

Remark 2.1.2. It is not difficult to see that the isolating block N_0 defined in the proof of Proposition 2.1.1 agrees with the union of those components of N which contain some component of K. It follows from this fact that every connected isolating block of a connected non-saddle set is of the form $N^+ \cup N^-$.

Remark 2.1.3. Notice that the existence of an isolating block of the form $N = N^+ \cup N^-$ of K is sufficient to ensure that K is non-saddle. Indeed, given a neighborhood U of K, the compactness and the positive and negative invariance of N^+ and N^- respectively ensure the existence of $T > 0$ such that $N^+[T, +\infty)$ and $N^-(-\infty, -T]$ are contained in U. Since $V = N^+T \cup N^-(-T) = NT \cap N(-T)$ is a neighborhood of K the result follows.

A nice consequence of Proposition 2.1.1 is that for flows defined on locally compact ANR's, isolated non-saddle sets have the shape of finite polyhedra and hence, finitely generated Čech homology and cohomology. Moreover, it can be seen that if N is an isolating block of the form $N^+ \cup N^-$ of K the inclusion $i : K \hookrightarrow N$ is a shape equivalence. These results were obtained in [32]. Using the same kind of arguments it is easy to see that N, N^o and N^i must be ANR.

We see that, in analogy with the basin of attraction of an attracting set (attractors and unstable attractors), the region of influence of an isolated non-saddle set is an open set.

Proposition 2.1.4. *If K is an isolated non-saddle compactum then $\mathcal{I}(K)$ is an open neighborhood of K.*

Proof. Given an isolating neighborhood of K of the form $N = N^+ \cup N^-$, it must be contained in $\mathcal{I}(K)$, hence $\mathcal{I}(K)$ is a neighborhood of K. On the other hand, if $x \in \mathcal{I}(K)$ and, say, $\emptyset \neq \omega(x) \subset K$, then there is a neighborhood U of x and a $t_0 \geq 0$ such that Ut_0 is contained in N^+. Hence for every $y \in Ut_0$ we have that $\emptyset \neq \omega(y) \subset K$ and, as a consequence, the same thing happens for every $z \in U$. Therefore $\mathcal{I}(K)$ is open. □

However, in contrast with the case of attracting sets, the converse does not necessarily hold.

Remark 2.1.5. There are isolated saddle continua K such that $\mathcal{I}(K)$ is an open neighborhood of K, hence this property does not characterize non-saddleness. For instance, consider the Mendelson's flow on the plane [58], see Example 1.2.7 and Figure 1.2. Then $K = \{p_2\}$ is a saddle set (in fact, an unstable attractor) and its region of influence (region of attraction in this case) is $\mathbb{R}^2 \setminus \{p_1\}$.

The following result gives a sufficient condition for a non-saddle set to be either an attractor or a repeller.

Proposition 2.1.6. *Let K be an isolated non-saddle continuum of a flow on a locally compact metric space M. Suppose that K has arbitrarily small neighborhoods in M which are not disconnected by K. Then K is either an attractor or a repeller. In particular, if M is a G-orientable n-dimensional manifold with $n > 1$ and K has trivial shape or, more generally, if $\check{H}^{n-1}(K) = 0$, then K is either an attractor or a repeller.*

Proof. Let N be a connected isolating block of K of the form $N = N^+ \cup N^-$, which is contained in $\mathcal{I}(K)$, and let W be a neighborhood of K such that $W \subset N$ and W is not disconnected by K. Then either $W \cap (N^+ \setminus K)$ or $W \cap (N^- \setminus K)$ is empty. In the first case K is a repeller and in the second case an attractor. If M is a G-orientable n-manifold then, by Alexander duality, $H_1(M, M \setminus K) \cong \check{H}^{n-1}(K)$ and, by excision, $H_1(\mathring{N}, \mathring{N} \setminus K) \cong H_1(M, M \setminus K)$. Thus, if K has trivial shape or more generally $\check{H}^{n-1}(K) = 0$ from the terminal part of the exact homology sequence of the pair $(\mathring{N}, \mathring{N} \setminus K)$

$$\cdots \to H_1(\mathring{N}, \mathring{N} \setminus K) = 0 \to \widetilde{H}_0(\mathring{N} \setminus K) \to \widetilde{H}_0(\mathring{N}) = 0$$

we get that $\widetilde{H}_0(\mathring{N} \setminus K) = 0$. As a consequence $\mathring{N}$ is not disconnected by K. Since N can be taken arbitrarily small, this is a particular case of the situation considered before. $\square$

Throughout this chapter we often use the torus as a relevant phase space for flows which illustrate the main notions we introduce. The previous proposition, together with some classical results in Algebraic Topology, can be used to describe the topological structure of an important class of non-saddle sets in the torus.

Theorem 2.1.7. *Suppose that K is an isolated non-saddle continuum of a flow on the torus, T, such that K does not separate T and K is neither an attractor nor a repeller. Then, K has the shape of a circle.*

Proof. We take coefficients in $\mathbb{Z}_2$. Since K does not separate T we have that $\widetilde{H}_0(T \setminus K) = 0$. By using the exact homology sequence of the pair $(T, T \setminus K)$

$$\cdots \to H_1(T) \to H_1(T, T \setminus K) \to \widetilde{H}_0(T \setminus K) = 0 \to \cdots$$

and Alexander duality we get that $\beta_1(K) = \beta_1(T, T \setminus K) \leq 2$. We prove, arguing by contradiction, that, in fact, $\beta_1(K) \neq 2$. Suppose $\beta_1(K) = 2$ and let $N = N^+ \cup N^-$ be an isolating block of K. Consider the terminal part of the reduced homology long exact sequence of the pair (T, N)

$$0 = H_2(N) \to H_2(T) = \mathbb{Z}_2 \to H_2(T, N) \to H_1(N) \to H_1(T) \to$$
$$H_1(T, N) \to \widetilde{H}_0(N) = 0$$

The homomorphism $H_2(T) \to H_2(T, N)$ is an isomorphism since $H_2(T, N) = H^2(T, N)$ by the Universal Coefficients Theorem and by Alexander duality $H^2(T, N) = \check{H}^2(T, K) \cong H_0(T \setminus K)$, which is $\mathbb{Z}_2$ since K does not separate T. As a consequence, the homomorphism $H_1(N) \to H_1(T)$ is injective and hence an isomorphism. Therefore, $H_1(T, N) = 0$ and by excising K, $H_1(T \setminus K, N \setminus K) = 0$. Then, from the homology long exact sequence of the pair $(T \setminus K, N \setminus K)$ we get $H_0(N \setminus K) \cong H_0(T \setminus K) = \mathbb{Z}_2$, i.e. K does not separate N. Besides, N can be chosen arbitrarily small, N being an isolating block, and hence Proposition 2.1.6

ensures that it has to be an attractor or a repeller in contradiction with the assumption. As a consequence, $\beta_1(K)$ is either 0 or 1 and by Corollary 1 or a concatenation of results from [18, 26, 56, 57] K has either trivial shape or the shape of a circle respectively. The first case is excluded by Proposition 2.1.6 and hence K has the shape of a circle. □

Remark 2.1.5 shows that unstable attractors are not necessarily non-saddle. As a matter of fact we have the following characterization, whose proof is given in [90].

Proposition 2.1.8. *Let K be an unstable attractor of a flow. Then, K has no external explosions if and only if it is non-saddle.*

In order to characterize non-saddle sets by influence-like properties we introduce the following notion first.

Definition 2.1.9. A point p is *strongly influenced* by a compact invariant set K if it has a neighborhood U_p with the following property: for every neighborhood V of K there is a $T \geq 0$ such that for every $x \in U_p$ we have $x[T, \infty) \subset V$ or $x(-\infty, -T] \subset V$. We also say that the neighborhood U_p is strongly influenced by K. There are, with obvious changes, similar definitions for the notions of a point and a neighborhood *strongly attracted* or *strongly repelled* by K.

If p is strongly influenced by K then $p \in \mathcal{I}(K)$. Moreover, it is clear that if p is strongly influenced by K then all points in $\gamma(p)$ are strongly influenced by K.

Definition 2.1.9 provides all we need to characterize non-saddleness.

Proposition 2.1.10. *The following are equivalent for an isolated invariant compactum K:*

i) K is non-saddle;

ii) All points of K are strongly influenced by K;

iii) K has a neighborhood U all whose points are strongly influenced by K.

Moreover, if K is non-saddle then $\mathcal{I}(K)$ agrees with the set of all points strongly influenced by K, all points in $W^s(K) \setminus K$ are strongly attracted by K and all points in $W^u(K) \setminus K$ are strongly repelled by K.

Proof. Conditions ii) and iii) are clearly equivalent since, by definition, strong influence on a point p requires strong influence on all points of a neighborhood U_p. Moreover, if K is non-saddle then every point in the interior of an isolating neighborhood of K of the form $N = N^+ \cup N^-$ is strongly influenced by K and, as a consequence, we have that i) implies iii). On the other hand suppose that all points of K are strongly influenced by K. We claim that for every isolating neighborhood N of K there is an $\varepsilon > 0$ such that if a point $x \in N$ abandons N in the past and in the future then $d(x, K) > \varepsilon$. If not, there is and isolating neighborhood N and a sequence of points x_n contained in N with $x_n \to x \in K$ such that every x_n abandons N in the past and in the future in times, say $T_n < 0$ and $T'_n > 0$. Since $x_n \to x \in K$ and K is invariant we must have that $T_n \to -\infty$ and $T'_n \to \infty$. However, this is in contradiction with the fact that x is strongly influenced by K. Hence such an $\varepsilon > 0$ exists. As a consequence, if we define $N_0 = N^+ \cup N^-$ we obtain another isolating neighborhood $N_0 \subset N$ with $N_0^+ = N^+$ and $N_0^- = N^-$ and K is non-saddle. Thus ii) implies i). Concerning the last assertion in the statement of the proposition, if K is non-saddle and $x \in \mathcal{I}(K)$ then $\gamma(x)$ enters every isolating neighborhood of the form $N = N^+ \cup N^-$, all whose points are strongly influenced by K. Hence x is strongly influenced by K. If $x \in W^s(K) \setminus K$ then $\gamma^+(x)$ enters N^+ and, hence x is strongly attracted and similarly, if $x \in W^u(K) \setminus K$ then x is strongly repelled. □

2.2 Some situations in which non-saddle sets naturally arise

The aim of this section is to present some dynamical situations in which isolated non-saddle sets arise in a natural way. In particular, we see that isolated non-saddle sets appear whenever we have an attractor A and a repeller R satisfying some conditions or in bifurcations of asymptotically stable fixed points.

The following result shows a situation in which isolated non-saddle arise in a natural way.

Theorem 2.2.1. *Let φ be a flow on a compact and connected metric space M and A and R an attractor and a repeller of φ with $\mathcal{A}(A) \cap \mathcal{R}(R) = \emptyset$. Then, $K = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R))$ is an isolated non-saddle set. More generally, if A and*

R are an attractor and a repeller of φ such that $A \cup R \cup (\mathcal{A}(A) \cap \mathcal{R}(R))$ is closed in M, then $K = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R))$ is an isolated non-saddle set.

Proof. Suppose that $A \cup R \cup (\mathcal{A}(A) \cap \mathcal{R}(R))$ is closed in M and assume that K is non-empty since otherwise there is nothing to prove. Since A is an attractor there exists a Lyapunov function $\Phi : \overline{\mathcal{A}(A)} \to [0, \infty]$ such that Φ is strictly decreasing on orbits contained in $\mathcal{A}(A) \setminus A$, $\Phi|_A = 0$ and $\Phi|_{\partial(\mathcal{A}(A))} = \infty$. Similarly, since R is a repeller, there exists a Lyapunov function $\Psi : \overline{\mathcal{R}(R)} \to [0, \infty]$ such that Ψ is strictily increasing on orbits contained in $\mathcal{R}(R) \setminus R$, $\Psi|_R = 0$ and $\Psi_{\partial(\mathcal{R}(R))} = \infty$. By combining these two functions we can define a new one $H : M \to [0, \infty]$ by

$$H(x) = \begin{cases} \Phi(x) & \text{if } x \in \mathcal{A}(A) \setminus \mathcal{R}(R) \\ \Psi(x) & \text{if } x \in \mathcal{R}(R) \setminus \mathcal{A}(A) \\ 0 & \text{if } x \in \mathcal{A}(A) \cap \mathcal{R}(R) \\ \infty & \text{otherwise} \end{cases}$$

Since $A \cup R \cup (\mathcal{A}(A) \cap \mathcal{R}(R))$ is closed in M, H is continuous. Moreover,

$$K = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R)) = H^{-1}(\infty).$$

We can consider for every $a > 0$ the set $N_a = H^{-1}([a, \infty])$ and we obtain in this way a basis of neighborhoods of K in M. If $x \in N_a \cap \mathcal{A}(A)$ then for every $t \leq 0$ we have

$$H(\varphi(x,t)) = \Phi(\varphi(x,t)) \geq \Phi(x) = H(x) \geq a.$$

Hence $x \in N_a^-$. Similarly, if $x \in N_a \cap \mathcal{R}(R)$ we can prove that $x \in N_a^+$. From this it is clear that N_a is an isolating block of the form $N_a^+ \cup N_a^-$ and, thus, that K is non-saddle. □

The situation is particularly interesting when M is a manifold. Then, the cohomology of the non-saddle set K defined in the first part of the previous theorem is related to the homology of A, R and M via a long exact sequence which we introduce in the following result.

Proposition 2.2.2. *Let φ be a flow on a closed, connected and G-orientable n-dimensional manifold M and A and R an attractor and a repeller of φ with $\mathcal{A}(A) \cap \mathcal{R}(R) = \emptyset$. Let $K = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R))$. Then, there exists a long exact sequence involving the homologies of A, R and M and the cohomology of K which takes the form*

$$\cdots \to \check{H}_r(A) \oplus \check{H}_r(R) \to H_r(M) \to \check{H}^{n-r}(K) \to \check{H}_{r-1}(A) \oplus \check{H}_{r-1}(R) \to \cdots$$

Proof. Since $M \setminus K = \mathcal{A}(A) \cup \mathcal{R}(R)$ and the inclusion $i : A \cup R \to \mathcal{A}(A) \cup \mathcal{R}(R)$ is a shape equivalence, which preserves Čech homology, then $H_r(M \setminus K) \cong \check{H}_r(A) \oplus \check{H}_r(R)$. Moreover, by Alexander duality $\check{H}^{n-r}(K) \cong H_r(M, M \setminus K)$. Therefore the sequence referred to in the statement of the proposition is nothing else that the form which takes the long homology sequence of the pair $(M, M \setminus K)$ after making the appropiate substitutions. □

Non-saddle sets are also involved in situations related to the generalized Poincaré-Andronov-Hopf bifurcations as we illustrate in the next example.

Example 2.2.3. Consider the family of ordinary differential equations defined on the plane in polar coordinates

$$\begin{cases} \dot{r} = -r(r-\lambda)^2, & \lambda \in [0,1] \\ \dot{\theta} = 1 \end{cases} \tag{2.1}$$

The picture on the left in figure 2.1 describes the phase portrait of equation (2.1) when the parameter $\lambda = 0$. We see that in this case the origin is a globally asymptotically stable fixed point and the orbit of any other point spirals towards it. The picture on the right describes the phase portrait of equation (2.1) when $\lambda > 0$. In this case we see that the origin is still an asymptotically stable fixed point but it is not a global attractor anymore since, for each $\lambda > 0$, the circle centered at the origin and radius λ is a periodic trajectory which attracts the unbounded component of its complement and repels the bounded one. It is clear

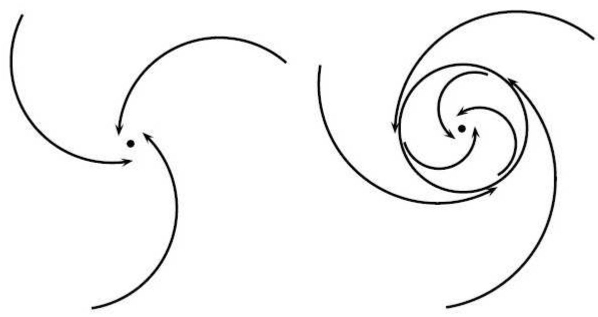

Figure 2.1: Phase portrait of the family of equations (2.1).

from these remarks that this periodic trajectory is an isolated non-saddle set which is neither an attractor nor a repeller. Moreover, the region of attraction of the origin appears to be, for each $\lambda > 0$ the open disk of radius λ. This phenomenon motivates the next definition.

Definition 2.2.4. Suppose that $\varphi_\lambda : M \times \mathbb{R} \to M$, with $\lambda \in [0,1]$, is a parametrized family of flows on a locally compact metric space M such that A is an attractor for φ_λ for every λ. Suppose that $\mathcal{A}_\lambda$ denotes the basin of attraction of A for the flow φ_λ. We say that $\mathcal{A}_\lambda$ *implodes* at $\lambda = 0$ provided that, for every neighborhood U of A contained in $\mathcal{A}_0$, there exists $\lambda_0 > 0$ such that $\mathcal{A}_\lambda \subset U$ for $0 < \lambda < \lambda_0$.

The next result shows that an implosion of the basin of attraction of an asymptotically stable fixed point on the Euclidean space produces a bifurcation where a family of isolated non-saddle sets with a nice topological structure is created.

Theorem 2.2.5. *Let φ_λ, with $\lambda \in [0,1]$, be a continuous parametrized family of flows on $\mathbb{R}^n$ such that the point $p \in \mathbb{R}^n$ is an attractor for every λ. Suppose that the basin of attraction $\mathcal{A}_\lambda$ of p for φ_λ implodes at $\lambda = 0$. Then, there exists a $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ there exists an isolated non-saddle set K_λ of φ_λ with the shape of S^{n-1}. Moreover, the family K_λ shrinks to p as $\lambda \to 0$ (i.e. for every neighborhood U of p there exists a $\lambda_0' > 0$ such that $K_\lambda \subset U$ for $0 < \lambda < \lambda_0'$).*

Proof. Let $\mathcal{A}_0$ be the basin of attraction of p for the flow φ_0. Consider a continuation A_λ ot the attractor p, i.e. with $A_0 = \{p\}$. It is known that all A_λ are attractors for $\lambda > 0$ sufficiently small and that for every compact set $C \subset \mathcal{A}_0$ there exists a λ_0 such that C is attracted by A_λ for every $\lambda < \lambda_0$. A proof of this fact can be found in [89] or [87]. Consider the particular case when C is a closed ball centered at p. We have that $A_\lambda \subset C$ for $0 < \lambda < \lambda_0'$, where we can assume for simplicity that $\lambda_0' = \lambda_0$. Obviously $p \in A_\lambda$ and $\{p\} \neq A_\lambda$ for $\lambda \neq 0$. Moreover, maybe after replacing λ_0 with a smaller $\lambda_1 > 0$, $\overline{\mathcal{A}_\lambda}$ (the closure of the basin of attraction of p for φ_λ) is a compact invariant set attracted by A_λ, which is possible only if $\overline{\mathcal{A}_\lambda} \subset A_\lambda$. We define $K_\lambda = A_\lambda \setminus \mathcal{A}_\lambda$. If we consider the restriction flow $\varphi_\lambda|_{A_\lambda}$ then K_λ is the complementary repeller of p for this flow. Then K_λ is an isolated non-saddle set for φ_λ with $W^s(K_\lambda) = \mathcal{A}(A_\lambda) \setminus \mathcal{A}_\lambda$ and $W^u(K_\lambda) = A_\lambda \setminus \{p\}$ and, thus, $\mathcal{I}(K_\lambda) = \mathcal{A}(A_\lambda) \setminus \{p\}$.

We prove now that K has spherical shape. Consider a small open ball B centered at p and contained in $A_\lambda \setminus K_\lambda$. By using the attracting character of K_λ in the exterior of A_λ and the repelling character in the interior of A_λ it is possible to define a family of maps $h_n : C \setminus B \to \mathbb{R}^n$ such that

1) for every neighborhood U of K_λ in $\mathbb{R}^n$ the image of h_n is contained in U for almost all n, and

2) $h_n \simeq h_{n+1}$ in U for almost all n.

We can even achieve that $h_n|_{K_\lambda}$ is the identity for every n and the homotopies $h_n \simeq h_{n+1}$ leave all points of K_λ fixed. This sequence of maps is called by Borsuk an approximative sequence from $C \setminus B$ to K_λ (see [19]) and it defines a shape domination of K_λ by $C \setminus B$. Since $C \setminus B$ is homotopy equivalent to an $(n-1)$-sphere we deduce that $\mathrm{Sh}(S^{n-1}) \geq \mathrm{Sh}(K_\lambda)$. On the other hand K_λ separates $\mathbb{R}^n$ since $\mathbb{R}^n \setminus K_\lambda = \mathcal{A}_\lambda \cup (\mathbb{R}^n \setminus A_\lambda)$, which implies that K_λ does not have trivial shape. Then, by [19] $\mathrm{Sh}(S^{n-1}) = \mathrm{Sh}(K_\lambda)$. □

2.3 On the structure of a flow having a non-saddle set

This section is devoted to the study of the structure of a flow on a locally compact metric space having a non-saddle set. For this purpose we use the following notation:

- $\mathcal{H}(K) = W^s(K) \cap W^u(K)$, the set of all points x such that $\omega(x) \neq \emptyset$, $\omega^*(x) \neq \emptyset$ and $\omega(x) \cup \omega^*(x) \subset K$. If $x \in \mathcal{H}(K) \setminus K$ we say that the point x and the trajectory $\gamma(x)$ are *homoclinic.*

- $\mathcal{A}^*(K) = W^s(K) \setminus W^u(K)$, the set of all points x such that $\emptyset \neq \omega(x) \subset K$ but $\omega^*(x) \nsubseteq K$ or $\omega^*(x) = \emptyset$.

- $\mathcal{R}^*(K) = W^u(K) \setminus W^s(K)$, the set of all points x such that $\emptyset \neq \omega^*(x) \subset K$ but $\omega(x) \nsubseteq K$ or $\omega(x) = \emptyset$.

Proposition 2.3.1. *Let K be an isolated non-saddle set. Then:*
i) $\mathcal{H}(K) \setminus K$ is an open set in M.
ii) $\mathcal{A}^*(K) \cup K$ and $\mathcal{R}^*(K) \cup K$ are closed in $\mathcal{I}(K)$.

Proof. If $x \in \mathcal{H}(K) \setminus K$ then x is both strongly attracted and strongly repelled by K, which means that it has a neighborhood U_x contained in $W^s(K) \cap \mathcal{R}(K)$ and not meeting K. Hence $\mathcal{H}(K) \setminus K$ is open.

To prove ii) let us argue by contradiction. If $\mathcal{A}^*(K) \cup K$ is not closed in $\mathcal{I}(K)$ then there exists a sequence $x_n \to x \in \mathcal{I}(K)$ with $x_n \in \mathcal{A}^*(K) \cup K$ and $x \notin \mathcal{A}^*(K) \cup K$. Since $x \notin \mathcal{A}^*(K) \cup K$ we must have that $x \in \mathcal{R}(K) \setminus K$ and, as a consequence, x is strongly repelled by K. But this implies that x_n is repelled by K for almost all n, which is in contradiction with the choice of the sequence x_n. □

In the sequel we are concerned with the study of the region of influence $\mathcal{I}(K)$ and, in particular, with the structure of $\mathcal{I}(K) \setminus K$. By the previous results, $\mathcal{I}(K) \setminus K$ is the disjoint union of the sets $\mathcal{H}(K) \setminus K$, $\mathcal{A}^*(K)$ and $\mathcal{R}^*(K)$ where $\mathcal{H}(K) \setminus K$ is open and $\mathcal{A}^*(K)$ and $\mathcal{R}^*(K)$ are closed in $\mathcal{I}(K) \setminus K$. However, $\mathcal{H}(K) \setminus K$ is not necessarily closed in $\mathcal{I}(K) \setminus K$ as the following example shows.

Example 2.3.2. This example shows an isolated non-saddle set K of a flow on the torus T (which is represented in Figure 2.2 as a square with opposite sides identified). All the points of K are stationary and, in addition, there is a fixed point $p \notin K$. The orbits of all points $x \in T - K$ are homoclinic except the equilibrium p, the orbit finishing in p and the orbit starting in p. The region of influence of K is $\mathcal{I}(K) = T \setminus \{p\}$. The set $\mathcal{H}(K) \setminus K$ is not closed in $\mathcal{I}(K) \setminus K$.

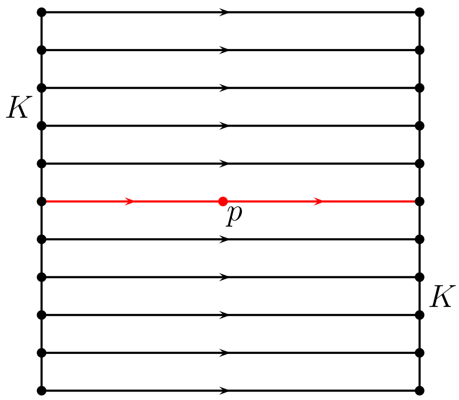

Figure 2.2: Flow on the torus

The fact that $\mathcal{H}(K) \setminus K$ is not necessarily closed in $\mathcal{I}(K) \setminus K$ (and, hence, $\mathcal{H}(K)$ is not necessarily closed in $\mathcal{I}(K)$) accounts for much of the complexity of the structure of $\mathcal{I}(K)$, specially when compared, for example, with the more simple case of unstable attractors without external explosions, where $\mathcal{H}(K)$ is indeed closed in $\mathcal{I}(K) = \mathcal{A}(K)$. In spite of this, some properties of that family of unstable attractors are shared by non-saddle sets. However, if we want to have some understanding of the structure of the region of influence of non-saddle sets we must acknowledge the existence of a special kind of orbits which are responsible for both the topological and the dynamical complexity. This we do in Definition 2.3.4 .

In the sequel we often use the prolongational limits J^+, J^- and J^*. The following result, whose proof is left to the reader, provides a useful characterization of attracting, repelling and homoclinic points lying outside K.

Proposition 2.3.3. *Let K be an isolated non-saddle set of a flow on M and $x \in M \setminus K$. Then*

i) $x \in W^u(K)$ if and only if $J^+(x) \neq \emptyset$ and $J^+(x) \subset K$.

ii) $x \in W^u(K)$ if and only if $J^-(x) \neq \emptyset$ and $J^-(x) \subset K$.

iii) If $x \in \mathcal{H}(K)$ then $J^*(x) \neq \emptyset$ and $J^*(x) \subset K \times K$. The converse holds if M is compact.

We stress that the previous proposition refers to points $x \in M \setminus K$ only. These properties do not generally hold for points in K.

In the following definition we introduce a kind of points which play an essential role in our discussion.

Definition 2.3.4. A point $x \in \mathcal{I}(K)$ is said to be *positively dissonant* if $x \notin W^s(K)$ (in which case $x \in W^u(K)$) but $J^+(x) \cap K \neq \emptyset$. We also say that the orbit $\gamma(x)$ is positively dissonant. There is a similar definition for *negatively dissonant* points and orbits. A point $x \notin \mathcal{I}(K)$ and its orbit are said to be *externally dissonant* if $J^*(x) \cap (K \times K) \neq \emptyset$. We denote by $\mathcal{D}$ the set of all dissonant points .

This definition conveys the idea that positively dissonant points are not attracted by K but, nevertheless, K has a kind of attractive influence on some points close to them. Externally dissonant points do not belong to the region of influence of K (therefore they are neither atracted nor repelled) but K has *simultaneously* a kind of attractive and repulsive influence on some points close to them. We remark that a flow on M migth have an isolated non-saddle set K and points x in $M \setminus \mathcal{I}(K)$ with $J^+(x) \cap K \neq \emptyset$ and $J^-(x) \cap K \neq \emptyset$ but $J^*(x) \cap (K \times K) = \emptyset$. Obviously, such points are not externally dissonant.

By using dissonant points we can give a nice dynamical characterization of the closure of the set of homoclinic points.

Proposition 2.3.5. *Let x be a point not contained in K. Then, x is dissonant if and only if it is in the boundary of $\mathcal{H}(K)$. As a consequence, $\overline{\mathcal{H}(K)} = \mathcal{H}(K) \cup \mathcal{D}$ i.e. the closure or $\mathcal{H}(K)$ consists of K and its homoclinic points together with the dissonant points of K.*

Proof. Suppose x is in the boundary of $\mathcal{H}(K)$. Then x is the limit of a sequence of points $x_n \in \mathcal{H}(K)$, and hence there exist a subsequence x_{n_k}, sequences $t_k \to \infty$ and $s_k \to -\infty$ and points $y, z \in K$ with $x_{n_k} t_k \to y$, $x_{n_k} s_k \to z$. As a consequence, if $x \notin \mathcal{I}(K)$ then x is an externally dissonant point. If $x \in \mathcal{I}(K) \setminus K$ and $x \in W^s(K)$ then $x \notin W^u(K)$ since, otherwise, x would be in $\mathcal{H}(K) \setminus K$, which is an open set of M, and this is in contradiction with x being in the boundary of $\mathcal{H}(K)$. Since $x \notin W^u(K)$, $x_{n_k} \to x$ and $x_{n_k} s_k \to z \in K$ we have that x is a negatively dissonant point. An analogous argument applies when $x \in W^u(K)$. This proves that x is dissonant whenever x is in the boundary of $\mathcal{H}(K)$ and $x \notin K$. On the other hand if $x \notin \mathcal{I}(K)$ is an externally dissonant point, there exist $x_n \to x$, $t_n \to \infty$, $s_n \to -\infty$ with $x_n t_n \to y$ and $x_n s_n \to z$ and $(y, z) \in K \times K$.

Then the points $x_n \in \mathcal{H}(K)$ for all n sufficiently large and, thus, x is in the boundary of $\mathcal{H}(K)$. Positively and negatively dissonant points are readily seen to belong to the boundary of $\mathcal{H}(K)$. □

Remark 2.3.6. Observe that if K does not have dissonant points and $\mathcal{H}(K) \setminus K \neq \emptyset$, then $\mathcal{H}(K)$ is compact and hence an isolated non-saddle set. Moreover, $\mathcal{H}(K)$ is the smallest non-saddle set without homoclinic trajectories containing K and its region of influence agrees with $\mathcal{I}(K)$.

The next result deepens into the topological structure of $\mathcal{I}(K) \setminus K$.

Proposition 2.3.7. *Let K be an isolated non-saddle set of a flow φ defined on a locally compact ANR M and $N = N^+ \cup N^-$ an isolating block of K. Then, $\mathcal{I}(K)\setminus K$ has a finite number of connected components. Moreover, if a component C does not contain dissonant points the restriction flow $\varphi|_C$ is parallelizable with section a component of ∂N. Hence, the flow provides a strong deformation retraction of C onto a component of $\partial N \cap C$. Conversely, if the restriction flow $\varphi|_C$ to a component C of $\mathcal{I}(K) \setminus K$ is parallelizable, then C does not contain dissonant points.*

Proof. Since N is an ANR so is ∂N and hence, it has a finite number of components. Since every component of $\mathcal{I}(K) \setminus K$ contains at least one component of ∂N there must be a finite number of them.

Suppose that C is a component of $\mathcal{I}(K)\setminus K$ which does not contain dissonant points. Then, by Proposition 2.3.1 and Proposition 2.3.5, C is contained in one of the subsets $\mathcal{A}^*(K)$, $\mathcal{R}^*(K)$ or $\mathcal{H}(K)\setminus K$ and, hence, K is either a global attractor or a global repeller or a global unstable attractor with no external explosions respectively for the restriction flow $\varphi|_{C\cup K}$. Then, by [16, Exercise 3.14.2, p. 83] and [82, Corollary 4] the flow in C is parallelizable with section a component of ∂N.

Conversely, assume that $\varphi|_C$ is parallelizable. We see that C does not contain dissonant points. Choose an isolating block $N = N^+ \cup N^-$ of K. Suppose, arguing by contradiction, that $x \in C$ is a positively dissonant point the other case being analogous. Then there exists a sequence of homoclinic points $x_n \in C$ such that $x_n \to x$. For each n choose t_n to be the unique time for which $x_n t_n \in N^i \cap C$. Since $N^i \cap C$ is compact we may assume that $x_n t_n \to y \in N^i \cap C$. Besides, since x is positively dissonant then $x \notin W^s(K)$ and, as a consequence, $t_n \to +\infty$ in contradiction with $\varphi|_C$ being parallelizable. □

Remark 2.3.8. For the second part of Proposition 2.3.7 there is no need of M to be an ANR. It also holds if M is a locally compact metric space.

The following result shows that in fact, in absence of dissonant points, the study of the region of influence of an isolated non-saddle set can be reduced to the study of the basin of attraction of an isolated attractor without external explosions.

Theorem 2.3.9. *Let K be an isolated non-saddle set of a flow φ on a locally compact separable metric space M. If K does not have dissonant points, then it is possible to define another flow ψ on M such that:*

i) K is an attractor without external explosions for ψ

ii) The region of attraction $\mathcal{A}_\psi(K)$ for the flow ψ agrees with the region of influence $\mathcal{I}_\varphi(K)$ for φ

iii) The homoclinic orbits of ψ in $\mathcal{A}_\psi(K) \setminus K$ agree with the homoclinic orbits of φ in $\mathcal{I}_\varphi(K) \setminus K$.

Proof. We shall make use of a classical result by Whitney [99] according to which it is possible to define a flow on M having a prescribed set of trajectories C provided that the family C satisfies certain conditions of regularity. We consider first the family D of curves given by the trajectories of φ in $(\mathcal{H}_\varphi(K) \setminus K) \cup \mathcal{A}^*_\varphi(K) \cup \mathcal{R}^*_\varphi(K)$. All these curves are homeomorphic images of the real line $\mathbb{R}$ and we consider them as oriented curves. The orientation we take is the one given by the flow φ for the curves in $(\mathcal{H}_\varphi(K) \setminus K) \cup \mathcal{A}^*_\varphi(K)$ and the opposite one for the curves in $\mathcal{R}^*_\varphi(K)$. This family of curves is an oriented regular family of curves in the sense of Whitney [99] because $(\mathcal{H}_\varphi(K) \setminus K)$, $\mathcal{A}^*_\varphi(K)$ and $\mathcal{R}^*_\varphi(K)$ are disjoint open sets (since K has no dissonant points). We enlarge the family D to a partition C of M by adding the singletons $\{\{q\} | q \in (M \setminus \mathcal{I}(K)) \cup K\}$, obtainig in this way what Whitney calls a family of paths. The flow ψ, whose trajectories are the previously described ones, is the flow we are looking for. ☐

Remark 2.3.10. From Theorem 2.3.9 and [49, Theorem 3.6], it follows that, in absence of homoclinic trajectories, the inclusion $i : K \hookrightarrow \mathcal{I}(K)$, of an isolated non-saddle set into its region of influence, is a shape equivalence.

The existence of positively, negatively and externally dissonant points is mutually related as the following result shows.

Proposition 2.3.11. *If an isolated non-saddle set K has externally dissonant points, then it has also positively and negatively dissonant points. Conversely, if M is compact and K has either positively or negatively dissonant points in $\mathcal{I}(K)$ then K has externally dissonant points.*

Proof. Suppose that x is an externally dissonant point. Then there exists a sequence x_n in $\mathcal{H}(K) \setminus K$ such that $x_n \to x$. Let $N = N^+ \cup N^-$ be an isolating block of K contained in $\mathcal{I}(K)$. For almost all n there exists $t_n > 0$ such that $x_n t_n \in N^i$. By compactness we may assume that $x_n t_n \to y \in N^i \subset W^s(K) \setminus K$. The sequence $t_n \to +\infty$ since, otherwise it would have a bounded subsequence t_{n_k} such that $t_{n_k} \to t_0$. Then, $x_{n_k} t_{n_k} \to y = xt_0$ which is a contradiction since $xt_0 \notin \mathcal{I}(K)$. As a consequence $y \in J^+(x)$ and by [16, Theorem 4.9, p. 29] $x \in J^-(y)$. Then $J^-(y) \cap K \neq \emptyset$ but $J^-(y) \nsubseteq K$ and, hence, y is a negatively dissonant point. An analogous argument leads us to find a positively dissonant point.

For the converse, suppose that M is compact and there exists a negatively dissonant point x. Then there exists a sequence x_n in $\mathcal{H}(K) \setminus K$ such that $x_n \to x$. By assumption, $\emptyset \neq \omega^*(x) \subset M \setminus \mathcal{I}(K)$. Let $y \in \omega^*(x)$. Then there exists a sequence $t_n \to -\infty$ such that $xt_n \to y$. Given $\varepsilon > 0$, it is possible to choose subsequences x_{n_k} and t_{n_k} such that $d(x_{n_k} t_{n_k}, xt_{n_k}) < \varepsilon/2$ for k larger than a certain k_0. Moreover, there exists k_1 such that if $k \geq k_1$, then $d(xt_{n_k}, y) < \varepsilon/2$. As a consequence $d(x_{n_k} t_{n_k}, y) \leq d(x_{n_k} t_{n_k}, xt_{n_k}) + d(xt_{n_k}, y) < \varepsilon$, for $k \geq \max\{k_0, k_1\}$. This proves that the point y is externally dissonant. If the point x is chosen to be positively dissonant the argument is analogous. ☐

Remark 2.3.12. In the previous proposition it is proved that points $x \notin \mathcal{I}(K)$ which are ω^*-limits of negatively dissonant points are externally dissonant, and the same is true for ω-limits of positively dissonant points, but it can be easily shown that the converse does not hold. In fact, there are externally dissonant points which are neither ω-limits nor ω^*-limits of points of $\mathcal{I}(K)$. However, in the same proposition it is proved the weaker property that externally dissonant points lie in the positive prolongational limit (J^+) of positively dissonant points and in the negative prolongational limit (J^-) of negatively dissonant points.

Example 2.3.13. Our previous Example 2.3.2 can be presented in a more general way. Consider the vector field in the torus induced by the differential equation in the square $I^2 = [0,1]^2 \subset \mathbb{R}^2$ defined by

$$\begin{cases} \dot{x} = \psi(x)(x^2 + y^2 - \frac{1}{4}) \\ \dot{y} = 0 \end{cases}$$

Where $\psi : [-1,1] \to [0,1]$ is a smooth function satisfying that $\psi(x) = 1$ if $x \in [-1/2, 1/2]$ and $\psi(x) = 0$ if and only if $x = \pm 1$.

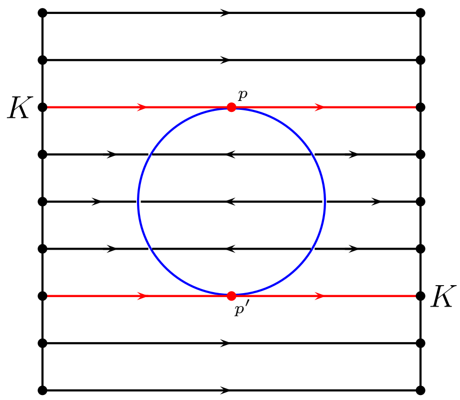

Figure 2.3: Saddle-node bifurcations at p and p'

The phase diagram of this flow can be understood as a collective phase diagram of a saddle-node bifurcation at the point p (and symetrically at the point p'). All points contained in the blue circle are fixed. The bifurcation points agree in this example with the externally dissonant points. The boundary of the region of influence of K separates the torus into two connected components.

We see that the previous example is only a particular instance of a much more general situation.

Sánchez-Gabites proved in [82] a topological result which he used to construct families of unstable attractors in manifolds and which is useful as well in the present context. The proof of the result makes use of a classical theorem by Thom [95] about representation of homology classes. We reproduce its statement here because, originally, this result is contained in the proof of a different proposition and it is not formulated separately.

Proposition 2.3.14. *(Sánchez-Gabites). Let M be a connected oriented closed smooth n-manifold with $H^1(M;\mathbb{Z}) \neq 0$ (or, equivalently, with $H_{n-1}(M;\mathbb{Z}) \neq 0$). Then, there exists a connected oriented closed smooth hypersurface $Z \subset M$ satisfying the following conditions:*

i) Z has a product neighborhood in M, i.e. there exists an open neighborhood U and a homeomorphism $h : Z \times \mathbb{R} \to U$ with $h(z, 0) = z$ for every $z \in Z$.

ii) $M \setminus U$ is connected.

It turns out that Proposition 2.3.14 is useful to construct non-saddle sets. The next result illustrates that non-saddle sets with dissonant points are indeed abundant. Its proof is, simply, an adaptation of the argument given in [82] for unstable attractors without external explosions.

Theorem 2.3.15. *Let M be a connected closed oriented smooth manifold. If $H^1(M;\mathbb{Z}) \neq 0$ then, there exists a flow on M having a connected isolated non-saddle set with dissonant points.*

Proof. We use the notation of Proposition 2.3.14. Construct a flow on $Z \times \mathbb{R}$ such that $Z \times (-\infty, 0]$ and $Z \times [1, \infty)$ consist of fixed points and points in $Z \times (0, 1)$ move from $Z \times \{0\}$ to $Z \times \{1\}$ except for a distinguished point z_0 for which the interval $\{z_0\} \times (-1, 1)$ is broken in three different trajectories covering $\{z_0\} \times (-1, 0)$, $\{z_0\} \times \{0\}$ and $\{z_0\} \times (0, 1)$ respectively. Now carry this flow to U via h and extend it to M by leaving fixed all points in $M \setminus U$. It is easy to see that $K = M \setminus U$ is an isolated non-saddle set, $h(\{z_0\} \times (-1, 0))$ and $h(\{z_0\} \times (0, 1))$ are positively and negatively dissonant orbits and $h(z_0)$ is an externally dissonant point. □

Theorem 2.3.15 applies to manifolds with $H^1(M;\mathbb{Z}) \neq 0$. However, if we consider M to be a connected locally compact ANR with $H^1(M) = 0$ the situation is much simpler.

Theorem 2.3.16. *Let M be a connected, locally compact ANR with $H^1(M) = 0$ and suppose that K is a connected isolated non-saddle set of a flow on M. Then, K does not have dissonant points. Moreover, if U is a component of $M \setminus K$, then the flow restricted to U is either locally attracted by K (i.e. all points lying in U near K are attracted by K) or locally repelled by K. Furthermore, if N is an* *isolating block of K of the form $N = N^+ \cup N^-$ then each component of $M \setminus K$ contains exactly one component of ∂N.*

Proof. Consider an isolating block N of K such that $N = N^+ \cup N^-$. Notice that, since ∂N has a finite number of components, it easily follows that so have $N \setminus K$ and $M \setminus K$.

Consider the initial part of the long exact cohomology sequence of the pair (M, N)

$$0 \to H^0(M, N) \to H^0(M) \to H^0(N) \to H^1(M, N) \to H^1(M) = 0$$

and, since M and N are connected, the homomorphism $H^0(M) \to H^0(N)$ is an isomorphism ensuring that $H^i(M, N) = 0$ for $i = 0, 1$.

On the other hand, by excision we get

$$H^i(M, N) \cong H^i(M \setminus K, N \setminus K)$$

and, as a consequence, $H^i(M \setminus K, N \setminus K) = 0$ for $i = 0, 1$.

Taking this into account in the cohomology long exact sequence of the pair $(M \setminus K, N \setminus K)$ we get that

$$H^0(M \setminus K) \cong H^0(N \setminus K).$$

This proves that each component of $M \setminus K$ contains exactly one component of $N \setminus K$. Besides, since $N^+ \cap N^- = K$ it easily follows that every component of $N \setminus K$ must be either contained in $N^+ \setminus N^-$ or $N^- \setminus N^+$. This shows that each component of $M \setminus K$ is either locally attracted or locally repelled by K, which prevents K of having homoclinic trajectories and, hence, dissonant points. The remaining part of the statement follows easily from the fact that the flow provides a deformation retraction from $N \setminus K$ onto ∂N. □

A nice consequence in the case of unstable attractors is the following result from [82].

Corollary 2.3.17. *All connected isolated unstable attractors of flows on manifolds M with $H^1(M) = 0$ have external explosions.*

The next theorem gives a local sufficient condition for an isolated invariant continuum to be non-saddle in terms of its Conley index.

Theorem 2.3.18. *Let M be a connected and locally compact metric space and suppose that K is an isolated invariant continuum of a flow φ on M such that K disconnects a connected neighborhood W of K in M into two components. If $C\check{H}^1(K)$ and $C\check{H}^1_-(K)$ are trivial then K is non-saddle and it is neither an attractor nor a repeller. Moreover, if M is manifold such that K does not disconnect M then $H^1(M) \neq 0$.*

Proof. Let W be a connected neighborhood of K in M such that $W - K$ consists of two different connected components, C_1 and C_2. Consider a connected isolating block N of K contained in W with entrance and exist sets N^i and N^o respectively. We remark that if N^o is empty then N^i is necessarily non-empty and consisting of at least two components (one at least in each C_i). Suppose that N^o is non-empty. Since $C\check{H}^1(K) = 0$ we have that $\check{H}^1(N, N^o)$ is trivial and from the cohomology exact sequence of the pair (N, N^o)

$$\cdots \leftarrow \check{H}^1(N, N^o) = 0 \leftarrow \widetilde{\check{H}^0}(N^o) \leftarrow \widetilde{\check{H}^0}(N) = 0$$

we get that $\widetilde{\check{H}^0}(N^o) = 0$ and, as a consequence N^o is connected. A similar argument using the fact that $C\check{H}^1_-(K)$ is trivial would establish that if N^i is non-empty then it is also connected. From this, we get that if both N^i and N^o are non-empty then the boundary ∂N consists of exactly two connected components N^o and N^i with one of them N^o contained in, say, C_1, and the other, N^i, contained in C_2. The previous argument can be used to exclude the fact that one of the sets N^i or N^o is empty since, in that case, the other would be non-empty and consisting of more than one component and we would have a contradiction. Then K can be neither an attractor nor a repeller and it readily follows that N is an isolating block of K with the structure $N = N^+ \cup N^-$ and, thus, that K is non-saddle. On the other and, if M is a manifold not disconnected by K and $H^1(M) = 0$ then, by Theorem 2.3.16, K attracts or repells all the points of one of its neighborhoods in M and this is a contradiction with the fact just proved that K is neither an attractor nor a repeller. □

The following example is a modification of [63, Example 2] which shows how to produce plenty of examples of flows having an isolated invariant continuum in the conditions of Theorem 2.3.18.

Example 2.3.19. Let K be a compact and connected manifold (without boundary) endowed with a flow φ_1 and consider the unit interval $[0,1]$ together with a dynamical system φ_2 which has 0 and 1 as fixed points and otherwise moves points away from 0 and towards 1. The product flow $\varphi(x,s,t) := (\varphi_1(x,t), \varphi_2(s,t))$ in the phase space $K \times [0,1]$ restricts to φ_1 on $K \times \{0\}$ and $K \times \{1\}$, hence these can be identified to get a flow on the quotient space $K \times S^1$ (with the obvious identifications). Observe that given a closed product neighborhood W of K, i.e. a neighborhood mapped onto $K \times [-1,1]$ by a homeomorphism $c : W \to K \times [-1,1]$ such that $c(K) = K \times \{0\}$, W is disconnected by K into two components. Moreover, W can be chosen to be an isolating block with entrance and exit set corresponding to $K \times \{-1\}$ and $K \times \{1\}$ respectively. Then, $C\check{H}^1(K) = C\check{H}^1_-(K) = 0$ and Theorem 2.3.18 ensures that K is non-saddle. A flow $\widehat{\varphi}$ in the conditions of Theorem 2.3.18 having dissonant points can be obtained modifying φ. Indeed, let S be a proper closed subset of the exit set of W. By using a theorem of Beck [13] φ can be modified to a new flow in such a way that all the orbits of φ not meeting a point of S are preserved while the orbits containing a point of S are decomposed in two orbits together with that point of S. After this modification W is no longer an isolating block but it is possible to build isolating blocks of K as before in the interior of W. It is straightforward to see that S contains externally dissonant points.

Notice that more general examples than Example 2.3.19 may be constructed using the mapping torus.

We study now the general structure of a flow on a compact ANR having an isolated non-saddle set. The next result gives an overall picture of the situation.

Theorem 2.3.20. *Let K be a connected isolated non-saddle set of a flow on a compact and connected ANR, M, (in particular on a compact and connected manifold) and let L be the complement $M \setminus \mathcal{I}(K)$ of its region of influence. Then, L is an isolated invariant compactum which is non-empty if and only if K is not a global unstable attractor. The saddle components of L are exactly those containing externally dissonant points of K. The union of these components is* *an isolated invariant (saddle) compactum* L_s *and* $L \setminus L_s$ *is an isolated invariant non-saddle compactum that we denote by* L_n*. Moreover, if* x *is a non-homoclinic point in* $\mathcal{I}(K) \setminus K$ *and the component of* $\mathcal{I}(K) \setminus K$ *containing* x *contains also homoclinic points then either* $\omega(x) \subset L_s$ *or* $\omega^*(x) \subset L_s$*.*

Proof. Obviously, L is compact and invariant, being the complement in M of the open invariant set $\mathcal{I}(K)$ and it is non-empty if and only if K is not an unstable global attractor. Moreover, if U is a closed neighborhood of L with $U \cap K = \emptyset$ then the trajectory of every point in $U \setminus L$ is contained in $\mathcal{I}(K)$ and hence either its ω-limit or its ω^*-limit is in K, which implies that this trajectory is not entirely contained in U. Hence L is isolated.

Suppose that C is a component of L. We see that C contains externally dissonant points if and only if C is saddle. Suppose that C is saddle. Then there exists a neighborhood U of C disjoint from K and a sequence of points $x_n \in U$, $x_n \to C$ such that $\gamma^+(x_n) \nsubseteq U$ and $\gamma^-(x_n) \nsubseteq U$. Choose an isolating block N of L such that the component N_C of N containing C is contained in U. Since for each n, x_n belongs to $\mathcal{I}(K)$ we may assume without loss of generality that either $\omega(x_n) \subset K$ for each n or $\omega^*(x_n) \subset K$ for each n. We consider the first situation. The trajectory of x_n abandons N_C in positive and negative time since $N_C \subset U$. Let y_n be the sequence corresponding to the exit points of $\gamma^+(x_n)$. We may assume that $y_n \to y \in \partial N_C$ and a simple argument shows that $\omega^*(y) \subset L$. As a consequence $\omega^*(x_n) \subset K$ for almost all n since if not, there is a contradiction with the fact that $\mathcal{A}^*(K)$ is closed in $\mathcal{I}(K) \setminus K$. This proves that C contains externally dissonant points. The converse statement is straightforward and it is left to the reader.

We see that L_s is a compactum. Otherwise there are points x_n in L_s converging to a point $x \in L_n$ and hence x belongs to a non-saddle component C of $M \setminus \mathcal{I}(K)$. Consider an arbitrary neighborhood U of C not meeting K. Then there is an arbitrarily small neighborhood $U' \subset U$ of C such that if a component of L meets U' then it is entirely contained in U'. Hence U' contains the components of some x_n and, as a consequence, dissonant and, thus, homoclinic points whose orbit leaves U in the past and in the future. Hence C cannot be non-saddle.

We see now that every component C of L_n is isolated. Consider a closed neighborhood U of L such that the component U_0 containing C does not meet

$L_s \cup K$. Then $U_0 \cap L$ is a compactum and U_0 can be chosen in such a way that $U_{\cap}L$ consists entirely of non-saddle components. Hence $U_0 \cap L$ is non-saddle. Moreover, $U_0 \cap L$ is isolated, hence it consists of a finite number of components all isolated non-saddle. On the other hand, we prove that L_n itself consists of a finite number of components. Consider an isolating block N of K in M. We know that its boundary, ∂N, consists of a finite number of components. Let C be a component of L_n and S the set of points of ∂N which are attracted or repelled by C. We see that S is open-closed in ∂N and, hence, S attracts or repells complete components of ∂N. We prove that S is a closed subset of ∂N. Suppose $x_n \in S$ are attracted by C and $x_n \to x$. All the points x_n and x itself belong to $\mathcal{R}^*(K)$. If x is not attracted by C then $\omega(x)$ is contained in a component $C' \subset L_s$. Consider an isolating block N of L such that C and C' lie in different components N_C and $N_{C'}$ of N. We have points $x_n s_n \in N^o_{C'}$ with $s_n \to \infty$. The points $x_n s_n$ converge to $y \in N^o_{C'}$ with $ys \in N_{C'}$ for every negative s. Hence $\omega^*(y) \subset N_{C'}$ and $y \notin W^u(K)$. Hence $\mathcal{R}^*(K)$ is not closed in $\mathcal{I}(K) \setminus K$. This contradiction proves that S is open-closed in ∂N and hence C attracts or repells complete components. Since every C must attract or repell some component we deduce that there is a finite number of non-saddle components and, thus, L_n is isolated non-saddle.

If the component of $\mathcal{I}(K) \setminus K$ containing x contains homoclinic points then the component of ∂N containing its orbit contains homoclinic points and by the previous discussion it can be neither attracted nor reppelled by L_n. Then, either $\omega(x) \subset L_s$ or $\omega^*(x) \subset L_s$. □

Corollary 2.3.21. *$M \setminus \mathcal{I}(K)$ is non-saddle if and only if the flow does not have dissonant points.*

We say that an orbit $\gamma \subset \mathcal{I}(K) \setminus K$ is *deviant* if either $\omega(\gamma)$ or $\omega^*(\gamma)$ contains an externally dissonant point.

All the externally dissonant points are contained in L_s. In the important particular case in which all the points of L_s are dissonant we have

Corollary 2.3.22. *If L_s consists entirely of dissonant points, then $\hat{K} = \mathcal{H}(K) \cup \mathcal{I}(L_s)$ is the smallest non-saddle set without homoclinic trajectories containing K, i.e. $\hat{K}$ is obtained from K by adding all the homoclinic orbits plus all the deviant orbits.*

2.4 Dynamical complexity of non-saddle sets

So far we have seen that almost all the complexity of the dynamics in the region of influence of an isolated non-saddle set K is due to the existence of homoclinic trajectories. In this section, we see that the way in which K lies in the phase space gives some constraints on the number of components the region of influence of K containing homoclinic trajectories. These results generalize to the context of isolated non-saddle sets some results about isolated attractors without external explosions from [82, 83].

Definition 2.4.1. Let M be a connected, locally compact ANR and suppose that K is a connected isolated non-saddle set of a flow on M. We define the *dynamical complexity* $\mathrm{dc}(\mathcal{I}(K))$ to be the number of components of $\mathcal{I}(K) \setminus K$ containing homoclinic trajectories.

Notice that $\mathrm{dc}(\mathcal{I}(K))$ is finite as a consequence of Proposition 2.3.7.

Lemma 2.4.2. *Let K be an isolated non-saddle continuum of a flow defined on a connected, locally compact ANR M. If N is an isolating block of the form $N^+ \cup N^-$, then*

$$\mathrm{dc}(\mathcal{I}(K)) \leq \beta_0(N \setminus K) - \beta_0(M \setminus K)$$

Proof. Notice that $\beta_0(N \setminus K)$ is finite since $N \setminus K$ has the homotopy type of the ANR ∂N. Besides, it is easy to see that $\beta_0(N \setminus K) \geq \beta_0(\mathcal{I}(K) \setminus K)$.

On the other hand, since $\mathcal{I}(K) \setminus K$ has $\mathrm{dc}(\mathcal{I}(K))$ components having homoclinic trajectories, it means that $\mathcal{I}(K) \setminus K$ has $\mathrm{dc}(\mathcal{I}(K))$ components which contain more than one component of $N \setminus K$. Hence,

$$\beta_0(\mathcal{I}(K) \setminus K) + \mathrm{dc}(\mathcal{I}(K)) \leq \beta_0(N \setminus K).$$

The result follows from this inequality and the fact that $\beta_0(M \setminus K) \leq \beta_0(\mathcal{I}(K) \setminus K)$. □

The following result provides some readily computable upper bounds on $\mathrm{dc}(\mathcal{I}(K))$ involving only information about how K lies in M.

Theorem 2.4.3. *Let K be an isolated non-saddle continuum of a flow defined on a connected, locally compact ANR M. Then,*

$$\mathrm{dc}(\mathcal{I}(K)) \leq \mathrm{rk}(\ker i_1^*).$$

Moreover, if we assume that M is a closed, connected and G-orientable n-manifold, then we also have that

$$\mathrm{dc}(\mathcal{I}(K)) \leq \mathrm{rk}(\mathrm{im}\, i_{n-1}^*).$$

Here i_1^* and i_{n-1}^* denote the homomorphisms induced in Čech cohomology by the inclusion $i : K \hookrightarrow M$ in dimensions 1 and $n-1$ respectively.

Proof. Since it should be clear from the context, through this proof we use the notation i^* for both homomorphisms i_1^* and i_{n-1}^*. Let us start by proving i). If $\mathrm{rk}(\ker i^*) = \infty$ there is nothing to prove, so we assume that $\mathrm{rk}(\ker i^*) < \infty$. Let $N = N^+ \cup N^-$ be an isolating block of K. The homomorphism $j^* : \check{H}^1(M) \to \check{H}^1(N)$ induced by the inclusion $j : N \hookrightarrow M$ satisfies that $\ker j^* = \ker i^*$. Indeed, it follows from the fact that $i^* = k^* j^*$, where k^* is the homomorphism induced in Čech cohomology by the inclusion $k : K \hookrightarrow N$, which is an isomorphism being K a shape deformation retract of N.

Consider the initial part of the long exact sequence of Čech cohomology of the pair (M, N),

$$0 \to \check{H}^0(M,N) \to \check{H}^0(M) \to \check{H}^0(N) \to \check{H}^1(M,N) \to \check{H}^1(M) \to \check{H}^1(N) \to \cdots$$

Since M and N are connected, the second homomorphism is an isomorphism and, hence, $\check{H}^0(M,N) = 0$ and $\check{H}^1(M,N) \cong \ker i^*$. Then, by excising K, we obtain that $\check{H}^0(M \setminus K, N \setminus K) = 0$ and $H^1(M \setminus K, N \setminus K) \cong \ker i^*$. Therefore, the initial part of the long exact sequence of Čech cohomology of the pair $(M \setminus K, N \setminus K)$ takes the form

$$0 \to \check{H}^0(M \setminus K) \to \check{H}^0(N \setminus K) \to \ker i^* \to \cdots$$

and, as a consequence,

$$\beta_0(N \setminus K) \leq \mathrm{rk}(\ker i^*) + \beta_0(M \setminus K).$$

Then, i) follows from the last inequality and Proposition 2.4.2.

Let us prove *ii*). Observe that, by Alexander duality, $\check{H}^n(K) \cong H_0(M, M \setminus K) = 0$ and consider terminal part of the long exact sequence of Čech cohomology of the pair (M, K),

$$\cdots \to \check{H}^{n-1}(M) \to \check{H}^{n-1}(K) \to \check{H}^n(M, K) \to \check{H}^n(M) \to \check{H}^n(K) = 0.$$

This long exact sequence breaks into the short exact sequence

$$0 \to \operatorname{coker} i^* \to \check{H}^n(M, K) \to \check{H}^n(M) \to 0,$$

Then, $\check{H}^n(M, K) \cong \operatorname{coker} i^* \oplus G$, and by Alexander duality theorem we get $H_0(M \setminus K) \cong \operatorname{coker} i^* \oplus G$.

On the other hand, if $N = N^+ \cup N^-$ another application of Alexander duality ensures that $H_1(\mathring{N}, \mathring{N} \setminus K) \cong \check{H}^{n-1}(K)$ and from this, together with the homology long exact sequence of the pair of ANR's $(\mathring{N}, \mathring{N} \setminus K)$, we get

$$\beta_0(\mathring{N} \setminus K) \leq \beta_{n-1}(K) + 1.$$

Besides, since $\beta_{n-1}(K) = \mathrm{rk}(\operatorname{coker} i^*) + \mathrm{rk}(\operatorname{im} i^*)$ the previous discussion guarantees that

$$\beta_0(\mathring{N} \setminus K) \leq \beta_0(M \setminus K) + \mathrm{rk}(\operatorname{im} i^*),$$

and the result follows from Proposition 2.4.2 after observing that $\beta_0(\mathring{N} \setminus K) = \beta_0(N \setminus K)$ □

From Theorem 2.4.3, it follows the next result which generalizes [83, Theorem 4.6].

Corollary 2.4.4. *Suppose that M is an ANR and K is an isolated non-saddle continuum. Then,* $\mathrm{dc}(\mathcal{I}(K)) = 0$ *if and only if the homomorphism* i^* :

$\check{H}^1(\mathcal{I}(K)) \to \check{H}^1(K)$, induced by the inclusion, is injective. Equivalently, $\mathrm{dc}(\mathcal{I}(K)) = 0$ if and only if, given an isolating block N of the form $N^+ \cup N^-$, each component of $\mathcal{I}(K) \setminus K$ contains exactly one component of $N \setminus K$.

The following results are some applications of Theorem 2.4.3.

Proposition 2.4.5. *Let K be an isolated non-saddle continuum of a flow on a closed surface M. If $\beta_1(K) = \beta_1(M)$ and K does not disconnect M, then $\mathrm{dc}(\mathcal{I}(K)) = 0$ and K must be either an attractor or a repeller.*

Proof. We consider through this proof $\mathbb{Z}_2$ coefficients. Since K is a non-separating continuum, Alexander duality ensures that

$$\check{H}^2(M, K) \cong H_0(M \setminus K) \cong \mathbb{Z}_2.$$

Let us consider the long exact sequence of reduced Čech cohomology of the pair (M, K),

$$0 \to \check{H}^1(M, K) \to \check{H}^1(M) \to \check{H}^1(K) \to \check{H}^2(M, K) \to \check{H}^2(M) \to 0.$$

The previous observation guarantees that the last homomorphism must be an isomorphism. As a consequence, the homomorphism $i^* : \check{H}^1(M) \to \check{H}^1(K)$ is surjective and, since $\beta_1(K) = \beta_1(M)$, it must be an isomorphism. Therefore, it follows from Theorem 2.4.3 that $\mathrm{dc}(\mathcal{I}(K)) = 0$.

Let us see that K must be either an attractor or a repeller. Let N be an isolating block of the form $N^+ \cup N^-$. Since the inclusion $i : K \hookrightarrow N$ is a shape equivalence, it easily follows that $\check{H}^i(M, N) \cong \check{H}^i(M, K) = 0$ for $i = 0, 1$. Hence, using excision and the long exact sequence of homology of the pair $(M \setminus K, N \setminus K)$, it follows that, since M is not disconnected by K, neither is N. As a consequence, K must be either an attractor or a repeller. □

Proposition 2.4.6. *Assume that K is a connected isolated non-saddle set of a flow defined on a closed, orientable surface M of genus g. Then, $\mathrm{dc}(\mathcal{I}(K)) \leq g$. Moreover, there exists a flow on M having an isolated non-saddle continuum K such that $\mathrm{dc}(\mathcal{I}(K)) = g$.*

Proof. Since $H^1(M) = \ker i^* \oplus \operatorname{im} i^*$ and $\beta_1(M) = 2g$, we get that either the rank of $\ker i^*$ or the rank of $\operatorname{im} i^*$ is at most g. Then, the result follows from Theorem 2.4.3.

To construct a flow for which the equality holds, observe that M can be built from the closure of a 2-sphere with $2g$ closed disks removed by attaching g closed cylinders, each one connecting two different holes. Consider each cylinder $S^1 \times [0,1]$ endowed with a flow which is stationary in the boundary $S^1 \times \{0,1\}$ and such that the trajectories of points in $S^1 \times (0,1)$ move from $S^1 \times \{0\}$ to $S^1 \times \{1\}$. The desired flow is defined by extending these flows on the cylinders to he whole surface by keeping all the points of the perforated sphere fixed. Observe that we can make this flow to have dissonant points in each cylinder by imposing that, for a distinguished point $z \in S^1$ the interval $\{z\} \times (0,1)$ is broken into three orbits, covering $\{z\} \times (0,1/2)$, $\{z\} \times \{1/2\}$ and $\{z\} \times (1/2,1)$ respectively. □

Proposition 2.4.7. *Suppose K is an isolated non-saddle continuum in the n-dimensional torus T^n, $n \geq 2$. Then,* $\operatorname{dc}(\mathcal{I}(K)) \leq 1$.

Proof. To prove this we exploit that the cohomology ring $H^*(T^n)$ is the exterior algebra with n generators $\omega_1, \ldots, \omega_n \in H^1(T^n)$. In particular, we use this ring structure to show that if $\operatorname{rk}(\ker i^*_{n-1}) > 1$, then $i^*_{n-1} = 0$. Indeed, suppose that there exist $\alpha_1, \alpha_2 \in \ker i^*_1$, with α_1 and α_2 linearly independent. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $H^1(T^n)$ containing α_1, α_2. It is straightforward to see that any element $\beta \in H^{n-1}(T^n)$ is of the form $\sum_{i=1}^n m_i(\alpha_1 \smile \ldots \smile \widehat{\alpha}_i \smile \ldots \smile \alpha_n)$, where the *hat* symbol $\widehat{\ }$ over α_i denotes that this cohomology class is removed from the cup product. Then, by the properties of the cup product we have

$$i^*_{n-1}(\beta) = \sum_{i=1}^n m_i(i^*_1(\alpha_1) \smile \ldots \smile \widehat{i^*_1(\alpha_i)} \smile \ldots \smile i^*_1(\alpha_n)) = 0,$$

since in each summand it has to appear at least one of the $i^*_1(\alpha_j)$ with $j = 1, 2$. The result follows from Theorem 2.4.3. □

We end this section by giving some bounds for the dynamical complexity of the region of influence of an isolated non-saddle continuum in terms of its Conley index.

Proposition 2.4.8. *Suppose K is an isolated non-saddle continuum of a flow defined on a connected, locally compact ANR. If K is neither an attractor nor a repeller, then*

$$\mathrm{dc}(\mathcal{I}(K)) \leq 1 + \min\{\mathrm{rk}\, C\check{H}^1(K), \mathrm{rk}\, C\check{H}^1_-(K)\}.$$

Proof. Let $N = N^+ \cup N^-$ be an isolating block of K. Consider the terminal part of the reduced long exact sequence of Čech cohomology of the pair (N, N^o)

$$\cdots \leftarrow \check{H}^1(N) \leftarrow \check{H}^1(N, N^o) \leftarrow \widetilde{\check{H}^0}(N^o) \leftarrow \widetilde{\check{H}^0}(N) = 0.$$

From this sequence we infer that

$$\beta_0(N^o) \leq 1 + \beta_1(N, N^o) = 1 + \mathrm{rk}\, C\check{H}^1(K).$$

An analogous argument with the reduced long exact sequence of Čech cohomology of the pair (N, N^i) shows that

$$\beta_0(N^i) \leq 1 + \beta_1(N, N^i) \leq 1 + \mathrm{rk}\, C\check{H}^1_-(K).$$

The result follows since each homoclinic trajectory must join a component of N^o with a component of N^i and, hence, $\mathrm{dc}(\mathcal{I}(K))$ must be less or equal than the number of components of both N^o and N^i. □

2.5 Non-saddle sets and Morse decompositions

In this section we study Morse decompositions and non-saddle sets. In particular, we prove a necessary and sufficient condition for a Morse set to be non-saddle. Besides, we see that if a flow on compact G-orientable manifold with trivial first cohomology group having an isolated non-saddle continuum admits a natural Morse decomposition and we calculate the Morse equation of this decomposition. We recall that G is either $\mathbb{Z}$ or $\mathbb{Z}_2$.

In the following result we see that the dynamical property of a Morse set being non-saddle turns out to be a topological property of its dual.

Theorem 2.5.1. *Let φ be a flow on a compact metric space M and $\{M_1, \dots, M_n\}$ a Morse decomposition of M. Then, M_k is non-saddle if and only if its dual M_k^* is closed or, equivalently, if and only if $\mathcal{I}(M_k)$ is open. In this case M_k^* is also non-saddle. As a consequence, if M is a manifold (or, more generally, an ANR) and M_k^* is closed then M_k and M_k^* have polyhedral shape.*

Proof. A point x is in $\mathcal{I}(M_k)$ if and only if $\omega(x) \subset M_k$ or $\omega^*(x) \subset M_k$ and this happens if and only if $x \notin M_k^*$. Hence $\mathcal{I}(M_k)$ is open if and only if M_k^* is closed.

Suppose that M_k^* is closed. Then we define A as the union of all Morse sets M_i with index $i < k$ together with all the trajectories connecting them and R as the union of all Morse sets M_i with index $i > k$ together with all the trajectories connecting them. By the properties of Morse decompositions it is clear that A is an attractor and R is a repeller. Moreover, it is easy to see that

$$M_k^* = A \cup R \cup (\mathcal{A}(A) \cap \mathcal{R}(R))$$

and

$$M_k = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R)).$$

Since M_k^* is closed we are in a position to apply Theorem 2.2.1 and we conclude that M_k is non-saddle.

Suppose now that M_k is non-saddle and consider a neighborhood N of M_k of the form $N = N^+ \cup N^-$. If x is a point in ∂M_k^* not contained in M_k^* then either $\omega(x) \subset M_k$ or $\omega^*(x) \subset M_k$. In the first case there is a positive t with $xt \in N^+$ and in the second case a negative t with $xt \in N^-$ and, as a consequence, there is a point $y \in \partial M_k^* \cap \mathring{N}$. This implies that there are points $z \in \mathring{N}$ close to y belonging to M_k^*. These points have their ω-limits and their ω^*-limits contained in Morse sets other than M_k, which is in contradiction with the fact that they belong to N. This contradiction shows that M_k^* is closed.

On the other hand, in [32] it has been studied the notion of dual of a non-saddle set and if M_k is non-saddle then M_k^* turns out to be the dual non-saddle set of M_k. Hence, if M_k^* is closed then both M_k and M_k^* are isolated and non-saddle.

Since isolated non-saddle sets in manifolds (and in ANR's) have polyhedral shape our result is proved. □

As a consequence, we can obtain a more general result. If M_{kl} is the union of all Morse sets M_i with $k \leq i \leq l$ together with all the trajectories connecting them we define

$$M_{kl}^* = \{x \in M \mid \omega(x) \nsubseteq M_{kl} \text{ and } \omega^*(x) \nsubseteq M_{kl}\}.$$

Corollary 2.5.2. *Let φ be a flow on a compact metric space M and $\{M_1, \ldots, M_n\}$ a Morse decomposition of M. Then, M_{kl} is non-saddle if and only if M_{kl}^* is closed.*

Proof. The proof is a consequence of Theorem 2.5.1 and of the fact that

$$\{M_1, \ldots, M_{kl}, \ldots, M_n\},$$

where all sets M_i with $k \leq i \leq l$ are excluded, is a Morse decomposition of M. □

Suppose that M is a compact and connected d-manifold with $H^1(M) = 0$. From Theorem 2.3.16 it follows that if K is an isolated non-saddle continuum, given a component U of $M \setminus K$, it happens that K has either an attracting behaviour or a repelling behaviour towards the points of U which are close to K. In fact, K is either an attractor or a repeller of the restriction flow $\varphi|_{U \cup K}$. The first kind of components, which are the components of $M \setminus K$ having empty intersection with $W^u(K)$, will be called *a-components* and the second kind, i.e. those with empty intersection with $W^s(K)$ will be called *r-components.* A consequence of the previous remark is that if U is an a-component, it contains a dual repeller R_U of the flow $\varphi|_{U \cup K}$ whose basin of repulsion is U. This dual repeller is the largest compact invariant set contained in U. An easy consequence of this fact is that R_U does not disconnect M. Similarly, if V is an r-component contains an attractor A_V which does not disconnect M and whose basin of attraction is the whole component.

If K is neither an attractor, nor a repeller, then the family $\mathcal{M} = \{A, K, R\}$, where A is the union of the attractors A_V and R the union of all the repellers

R_U, is a Morse decomposition of M. This Morse decomposition is called the *natural Morse decomposition* associated to K.

We start by computing the Morse equation of the natural Morse decomposition of M associated to an isolated non-saddle continuum K in the 2-sphere S^2.

Theorem 2.5.3. *Suppose K is an isolated non-saddle continuum of a flow φ on S^2 which is neither an attractor nor a repeller. Suppose that the number of r-components of $S^2 \setminus K$ is k. Then the Morse equation of the natural Morse decomposition $\mathcal{M}$ of M associated to K with coefficients in G is:*

$$k + (n-2)t + (n-k)t^2 = 1 + t^2 + (1+t)Q(t)$$

where n is the number of components of $S^2 \setminus K$ and the coefficients of $Q(t)$ are non-negative integers.

Hence, the Morse equation completely determines the shape of K and the dynamical structure near K.

Proof. None of the attractors and repellers involved in the Morse decomposition disconnect S^2. On the other hand, K is a non-saddle set disconnecting S^2 into n components. With these data, we can calculate the Conley index of all the elements of the Morse decomposition by using Corollary 1.3.9. In particular, the Conley index of K is the pointed homotopy type of a wedge of $n-2$ circles. Since the coefficients of the Morse equation are obtained from the Betti numbers of the homological Conley indices, we readily get the equation in the statement of the theorem. In particular, K is responsible for the term $(n-2)t$, the attractor A gives the term k and the repeller R contributes with the term $(n-k)t^2$. □

In the case of a d-dimensional manifold with $d > 2$ and trivial first cohomology group, it is still possible to get an analogous Morse equation associated to an isolated non-saddle continuum. However, to do that, we need to impose an additional condition.

Theorem 2.5.4. *Let M be a connected, closed and G-orientable d-dimensional manifold with $d > 2$ and $H^1(M) = 0$. Suppose that K is an isolated non-saddle continuum of a flow φ on M such that K is neither an attractor nor a repeller* *and all components of $M \setminus K$ are contractible. Then, the Morse equation of the natural Morse decomposition $\{A, K, R\}$ associated to K with coefficients in G is*

$$k + (k-1)t + \sum_{i=2}^{d-2} \beta_i(M)t^i + (n-1-k)t^{d-1} + (n-k)t^d$$
$$= 1 + \sum_{i=2}^{d-2} \beta_i(M)t^i + t^d + (1+t)Q(t) \qquad (2.2)$$

where n denotes the number of connected components of $M \setminus K$, k denotes the number r-components and $Q(t)$ is a polynomial whose coefficients are non-negative integers. In particular, the ranks of the homological Conley indices of K are $\operatorname{rk} CH^1(K) = k-1$, $\operatorname{rk} CH^{d-1}(K) = (n-1-k)$, $\operatorname{rk} CH^i(K) = \beta_i(M)$ for $i \neq 0, 1, (d-1), d$ and $\operatorname{rk} CH^i(K) = 0$ otherwise.

Proof. Let U be an r-component and A_U the component of A contained in U. Since the contractible set U is the region of attraction of A_U [49, Theorem 3.6] ensures that the homological Conley index of A_U is trivial for every i except for $i = 0$ which is G and, as a consequence, A contributes to the Morse equation with the term k. By time duality [64] and a similar argument, R contributes with the term $(n-k)t^d$. Consider now an isolating block $N = N^+ \cup N^-$ of K, a component C of $M \setminus K$ containing an attractor A_C and the set $N^o_C = N^o \cap C$. By Alexander duality

$$H_{d-i}(C, C - A_C) \cong \check{H}^i(A_C)$$

for every i and, using the long homology sequence of $(C, C \setminus A_C)$, we get that

$$\check{H}^i(A_C) \cong \tilde{H}_{d-i-1}(C \setminus A_C) \cong \tilde{H}_{d-i-1}(N^o_C),$$

where the last isomorphism is a consequence of the fact that the flow is parallelizable in $C \setminus A_C$ and N^o_C is a section. Hence N^o_C is a strong deformation retract of $C \setminus A_C$ and their homologies agree. Therefore N^o_C has the homology of a $(d-1)$-sphere and, since only the components C containing attractors contribute to N^o, then N^o has the homology of a disjoint union of k copies of S^{d-1}.

From the homology sequence of (N, N^o) we get that

$$H_i(N, N^o) \cong H_i(N) \text{ if } i \neq 0, 1, (d-1), d.$$

Moreover $H_i(N, N^o) = 0$ for $i = 0, d$ since K is neither an attractor nor a repeller. On the other hand, using the initial and the terminal part of the sequence, we get that

$$\beta_{d-1}(N, N^o) = \beta_{d-1}(N) - k$$

$$\beta_1(N, N^o) = \beta_1(N) + k - 1.$$

We remark that the hypotesis that $d > 2$ was used to obtain the second equality. If we consider now the Čech cohomology sequence of the pair (M, K)

$$\cdots \to \check{H}^i(M, K) \to \check{H}^i(M) \to \check{H}^i(K) \to \check{H}^{i+1}(M, K) \to \cdots$$

and Alexander duality in the form $\check{H}^i(M, K) \cong H_{d-i}(M \setminus K)$ then, since $M \setminus K$ is a union of contractible components, we get that

$$H^i(M) \cong \check{H}^i(K) \cong H^i(N) \text{ for } i \neq 0, (d-1), d.$$

For $i = d - 1$ we have

$$\beta_{d-1}(M) = 1 + \beta_{d-1}(K) - n = 1 + \beta_{d-1}(N) - n$$

and $\beta_{d-1}(M) = 0$ by the hypothesis that $H^1(M) = 0$ and Poincaré duality. From this we get the equation. □

Next corollary shows the form taken by the equation (2.2) when M is the d-sphere S^d, $d > 2$, or the complex projective space $\mathbb{C}P^r$, $r > 1$.

Corollary 2.5.5. *If $M = S^d$, $d > 2$, the equation (2.2) takes the form*

$$k + (k-1)t + (n-1-k)t^{d-1} + (n-k)t^d = 1 + t^d + (1+t)Q(t)$$

and, if $M = \mathbb{C}P^r$*, they are*

$$k + (k-1)t + \sum_{i=1}^{r-1} t^{2i} + (n-1-k)t^{d-1} + (n-k)t^d = \sum_{i=0}^{r} t^{2i} + (1+t)Q(t)$$

with $d = 2r$*,* $r > 1$*.*

Corollary 2.5.5 applied to the one-point compactification of $\mathbb{R}^d$ (i.e. S^d), $d > 2$, allows us to compute the ranks of the cohomological indices of non-saddle sets for certain flows on $\mathbb{R}^d$.

Corollary 2.5.6. *Suppose that* K *is an isolated non-saddle continuum of a flow* φ *on* $\mathbb{R}^d$*,* $d > 2$*, such that* K *is neither an attractor nor a repeller, all the bounded components of* $\mathbb{R}^d \setminus K$ *are contractible and* K *either attracts or repels the whole unbounded component* U*. Then, the ranks of the homological Conley indices of* K *are* $\operatorname{rk} CH^1(K) = k-1$*,* $\operatorname{rk} CH^{d-1}(K) = n-1-k$ *and* $\operatorname{rk} CH^i(K) = 0$ *for* $i \neq 1, d-1$*. Here,* n *denotes the number of connected components of* $\mathbb{R}^d \setminus K$ *and* k *denotes the number of* r*-components.*

The situation studied in Theorem 2.5.4 appears whenever there are equilibria satisfying some conditions described in the following result.

Corollary 2.5.7. *Let* φ *be a flow on a connected closed and* G*-orientable* d*-dimensional manifold* M *with* $d > 2$ *and* $H^1(M) = 0$*. Suppose that* φ *has* $n > 1$ *isolated fixed points* $p_1, \ldots, p_n$ *with no connecting orbits between them and that there is a* k*,* $0 < k < n$*, such that* $CH^0(\{p_i\}) \neq 0$ *for* $i \leq k$ *and* $CH^d(\{p_i\}) \neq 0$ *for* $i > k$*. Then, there exists an isolated non-saddle continuum* K *in the conditions of Theorem 2.5.4. Besides, if* $A = \{p_1, \ldots, p_k\}$ *and* $R = \{p_{k+1}, \ldots, p_n\}$*, the collection* $\{A, K, R\}$ *is the natural Morse decomposition associated to* K *and, by Theorem 2.5.4, its corresponding Morse equation is of the form (2.2).*

Proof. The condition on the Conley index means that every p_i with $i \leq k$ is an attractor and every p_i with $i > k$ is a repeller. Moreover their basins are contractible and the non-existence of connecting orbits implies that they are disjoint. Then, $A = \{p_1, \ldots, p_k\}$ is an attractor and $R = \{p_{k+1}, \ldots, p_n\}$ is a repeller which satisfy the hypothesis of Theorem 2.2.1. Therefore, the set $K = M \setminus (\mathcal{A}(A) \cup \mathcal{R}(R))$ is isolated non-saddle and clearly it is neither an attractor nor a repeller. Moreover, K is connected since it has the same shape as $\mathcal{I}(K)$ (Remark 2.3.10), which is connected being the complement of the points $p_1, \ldots, p_n$. Then, K satisfies the conditions of Theorem 2.5.4, $\{A, K, R\}$ is the natural Morse decomposition associated to K and its Morse equation is (2.2). $\square$

2.6 Non-saddle sets in 2-dimensional flows

In this section, we study some dynamical properties of non-saddle sets of flows defined on surfaces.

The following result shows that if an isolated invariant continuum of a planar flow has global region of influence, then it must be non-saddle. In particular, it must be either an attractor or a repeller.

Lemma 2.6.1. *Let K be an isolated invariant continuum of a flow on $\mathbb{R}^2$ and suppose that $\mathcal{I}(K) = \mathbb{R}^2$. Then, K is non-saddle and, as a consequence, a global attractor or a global repeller.*

Proof. Suppose, to get a contradiction, that K is saddle. Let N be a regular isolating block K, which must be a topological closed disk, and a sequence of points $x_n \to K$ whose trajectories leave N in the future and in the past. Denote by y_n and z_n the corrresponding exit points (in the future and in the past respectively). We may assume that $y_n \to y_0 \in \partial N$ and $z_n \to z_0 \in \partial N$. A simple argument proves that $\emptyset \neq \omega^*(y_0) \subset K$ and $\emptyset \neq \omega(z_0) \subset K$. Since $\mathcal{I}(K) = \mathbb{R}^2$ then we may suppose that either $\emptyset \neq \omega(y_n) \subset K$ for almost all n or $\emptyset \neq \omega^*(y_n) \subset K$ for almost all n; we consider the first situation and suppose, for the sake of simplicity in notation, that $\emptyset \neq \omega(y_n) \subset K$ for $n \geq 1$. We may also assume that all points y_n and y_0 are contained in an arc $A \subset N^o$ with no tangency points. Consider now an arbitrary isolating block $N_1 \subset \mathring{N}$ which is also a disk. We can define a topological circle C, not having K in its interior, consisting of the union of the following sets: a) the trajectory $\gamma^-(y_0)$ until it reaches ∂N_1 in a point a, b) an arc $J \subset A \subset N^o$, linking y_0 to y_1, c) the trajectory $\gamma^+(y_1)$ until it reaches ∂N_1 in a point b and d) an arc in ∂N_1 linking a to b. Denote by D the disk bounded by C.

Then, $\mathring{J}$ is either an exit set or an entrance set for D. If $\mathring{J}$ is an exit set then either $\gamma^-(y_1)$ is contained in D or there exists a point of $\gamma^-(y_1)$ contained

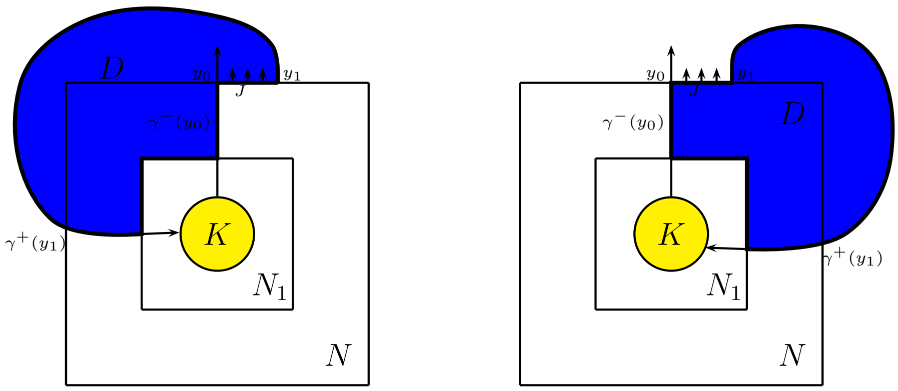

Figure 2.4: The region D

in ∂N_1. It is not difficult to see, using the Poincaré Bendixson Theorem, that the first case is impossible. If $\mathring{J}$ is an entrance set for D then either $\gamma^+(y_0)$ is contained in D or there exists a point of $\gamma^+(y_0)$ contained in ∂N_1. The first case is impossible as before. We conclude that either there exists a point of $\gamma^-(y_1)$ contained in ∂N_1 or there exists a point of $\gamma^+(y_0)$ contained in ∂N_1. If we repeat this construction for a sequence of isolating neighborhoods N_i shrinking to K, then we get that there exists a point of K in the ω-limit of y_0 or a point of K in the ω^*-limit of y_1. Then, as a consequence of Proposition 1.6.4, there would exist a fixed point in $\mathbb{R}^2 \setminus K$, which cannot be in the region of influence of K. It follows from this contradiction that K is non-saddle and, as a consequence, a global atractor or a global repeller. □

A direct consequence of Theorem 2.6.1 is the following result from [63].

Corollary 2.6.2 (Morón, Sánchez-Gabites and Sanjurjo [63])**.** *Let K be an isolated invariant continuum of a flow on $\mathbb{R}^2$ and suppose that $\mathcal{A}(K) = \mathbb{R}^2$. Then, K is stable and, thus, a global attractor.*

Now we are in position to characterize in topological terms those isolated invariant continua of planar flows which are non-saddle.

Theorem 2.6.3. *Let K be an isolated invariant continuum of a planar flow. Then, K is non-saddle if and only if $\mathcal{I}(K)$ is open and $\chi(K) = \chi(\mathcal{I}(K))$.*

Proof. If K is an isolated non-saddle continuum, then $\mathcal{I}(K)$ is open and, by Remark 2.3.10, the shapes of K and $\mathcal{I}(K)$ are the same. Hence, their Euler characteristics agree.

Conversely, suppose that $\mathcal{I}(K)$ is open and $\chi(K) = \chi(\mathcal{I}(K))$ (we note that $\chi(K)$ is well-defined and it is finite since K is a planar isolated invariant set). Extend the flow to the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$ by leaving fixed the point at infinity. By Borsuk's Theorem (Theorem 1), the continuum K disconnects $\mathbb{R}^2 \cup \{\infty\}$ into $2 - \chi(K)$ components and, since all components contain fixed points of the extended flow by Corollary 1.6.5, none of them is entirely contained in $\mathcal{I}(K)$. On the other hand, since $\mathcal{I}(K)$ is an open subset of $\mathbb{R}^2 \cup \{\infty\}$ and its Euler characteristic is $\chi(K)$, then $\mathcal{I}(K)$ is homeomorphic to a sphere with $2 - \chi(K)$ punctures (see [70, Theorem 3]).

By Alexander duality applied to the compactum $(\mathbb{R}^2 \cup \{\infty\}) \setminus \mathcal{I}(K)$, it consists also of $2 - \chi(K)$ components and, hence, each component of $(\mathbb{R}^2 \cup \{\infty\}) \setminus \mathcal{I}(K)$ contains exactly one component of $(\mathbb{R}^2 \cup \{\infty\}) \setminus K$. By using Alexander duality again we have that every component of $(\mathbb{R}^2 \cup \{\infty\}) \setminus \mathcal{I}(K)$ has trivial one-dimensional Čech cohomology and, thus, trivial shape. If C is one of such components and $\hat{C}$ is the component of $(\mathbb{R}^2 \cup \{\infty\}) \setminus K$ containing C, then by Borsuk's Theorem $(\mathbb{R}^2 \cup \{\infty\}) \setminus C$ is homeomorphic to $\mathbb{R}^2$ and $(\mathbb{R}^2 \cup \{\infty\}) \setminus \hat{C}$ is an isolated invariant continuum of the flow restricted to $(\mathbb{R}^2 \cup \{\infty\}) \setminus C$ (which can be seen as a flow in the plane) with global region of influence. Now, we are in a position to apply , which characterizes plane continua with global region of influence, to deduce that $(\mathbb{R}^2 \cup \{\infty\}) \setminus \hat{C}$ is either a global attractor or a global repeller for the flow defined in $(\mathbb{R}^2 \cup \{\infty\}) \setminus C$. This implies that K acts either as an attractor or as a repeller for points of $\hat{C}$. Since this can be repeated for all the components of $(\mathbb{R}^2 \cup \{\infty\}) \setminus K$ we have that K is non-saddle. □

Corollary 2.6.4 (Sánchez-Gabites [82])**.** *If K is a connected attracting set in the plane, then K is an attractor if and only if $\chi(K) = \chi(\mathcal{A}(K))$.*

In contrast with the planar case, there exist isolated non-saddle continua on surfaces whose Euler characteristic does not agree with the Euler characteristic of its region of influence (see Example 2.3.2). In fact, the following result shows that the coincidence of these Euler characteristics is closely related to the existence of dissonant points.

Theorem 2.6.5. *An isolated non-saddle continuum K of a flow on a surface M does not have dissonant points if and only if $\chi(K) = \chi(\mathcal{I}(K))$.*

Proof. We consider homology and cohomology with $\mathbb{Z}_2$ coefficients. Suppose that K has no dissonant points. It follows from Alexander duality theorem that

$$H_k(\mathcal{I}(K), \mathcal{I}(K) \setminus K) \cong \check{H}^{2-k}(K)$$

and, since K has finitely generated Čech homology and cohomology, $\chi(\mathcal{I}(K), \mathcal{I}(K) \setminus K)$ is defined and agrees with $\chi(K)$. Moreover, if K does not have dissonant points then $\mathcal{I}(K)$ and $\mathcal{H}(K)$ are of the same shape, which implies, since $\mathcal{H}(K)$ is non-saddle, that $\chi(\mathcal{I}(K))$ is well defined. As a consequence, $\chi(\mathcal{I}(K) \setminus K)$ is also defined and:

$$\begin{aligned}\chi(\mathcal{I}(K)) &= \chi(\mathcal{I}(K), \mathcal{I}(K) \setminus K) + \chi(\mathcal{I}(K) \setminus K) \\ &= \chi(K) + \chi(\mathcal{I}(K) \setminus K).\end{aligned}$$

Consider now an isolating block manifold N of K in M of the form $N = N^+ \cup N^-$. Notice that its boundary ∂N consists of a finite union of circles. Then, there exists a strong deformation retraction of $\mathcal{I}(K) \setminus K$ onto some of those circles (especifically onto the union of N^o with $N^i \setminus \mathcal{H}(K)$) and it follows that $\chi(\mathcal{I}(K) \setminus K) = 0$. As a consequence, $\chi(\mathcal{I}(K)) = \chi(K)$.

Conversely, suppose that the Euler characteristic of $\mathcal{I}(K)$ is defined and $\chi(K) = \chi(\mathcal{I}(K))$. Then $\chi(\mathcal{I}(K) \setminus K) = 0$. Moreover, $\mathcal{I}(K) \setminus K$ is a disjoint union of connected surfaces $S_1, \ldots, S_n$, all of them proper open subsets of the surface M. If S_i does not contain dissonant points then there exists a strong deformation retraction of S_i onto a component of $\partial N \cap S_i$ (a circle) and, thus, $\chi(S_i) = 0$. Now suppose, to get a contradiction, that S_i contains dissonant points. Since it is a proper subset of M then $\chi(S_i) \leq 1$. We see that the possibilities $\chi(S_i) = 0$ or 1 are excluded. If $\chi(S_i) = 0$ then by [70, Theorem 3] S_i is homeomorphic either to a punctured open disk $D \setminus \{p\}$ or an open Möbius strip. We treat the case in which S_i is homeomorphic to a punctured open disk being the other case easier. If C is a component of $\partial N \cap S_i$ then C is a circle not contractible in S_i (otherwise the disc bounded by C would be positively or negatively invariant by the flow and would contain an invariant set not influenced by K). The intersection $\partial N \cap S_i$ consists of more than one component since, otherwise, there would be no homoclinic orbits in S_i and, thus, no dissonant points. The external and the internal components of $\partial N \cap S_i$ limit a region R in $D \setminus \{p\}$ homeomorphic to a planar ring. All the trajectories of points of R abandon R in the past and in the future (otherwise there would exist an invariant set in R not influenced by K). This implies that all the trajectories in S_i are homoclinic and, hence, there are no dissonant points. A similar but easier argument excludes the possibilities that $\chi(S_i) = 1$. Hence, $\chi(S_i) < 0$ for every surface containing dissonant points. Since $\chi(\mathcal{I}(K) \setminus K) = \sum_{i=1}^{n} \chi(S_i) = 0$ and all the surfaces S_i without dissonant points are of zero Euler characteristic, we conclude that there are no dissonant points in $\mathcal{I}(K) \setminus K$. □

2.7 Robustness and splittings of non-saddle sets

It was shown in [36] (see figure 1.8) that the property of being non-saddle is not robust, i.e. it is not preserved by continuation of isolated invariant sets. However, it turns out that there exist some relations between the preservation of certain topological properties by continuation and the preservation of the dynamical property of non-saddleness. As a matter of fact, in some situations both properties are equivalent as it has been seen in Theorem 1.7.3. In this section we see more situations in which the preservation of some topological properties is equivalent to the preservation of non-saddleness. We start by establishing the necessary definitions.

Definition 2.7.1. Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a parametrized family of flows (parametrized by $\lambda \in I$, the unit interval) on a locally compact ANR, M, and suppose that K_0 is an isolated non-saddle set for φ_0. We say that K_0 is *dynamically robust* if for every isolating neighborhood N of K_0 there exists $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, the isolated invariant subset K_λ of N (with respect to the flow φ_λ) which has N as an isolating neighborhood is a (non-empty) non-saddle set.

By [78, Lemma 6.2], we have that K_0 is dynamically robust if and only if there exist an isolating neighborhood N of K_0 and a $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, the isolated invariant subset K_λ of N (with respect to the flow φ_λ) which has N as an isolating neighborhood is a (non-empty) non-saddle set.

Definition 2.7.2. Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a parametrized family of flows (parametrized by $\lambda \in I$, the unit interval) on a locally compact ANR, M, and suppose that K_0 is an isolated invariant set for φ_0. We say that K_0 is *topologically robust* if for every isolating neighborhood N of K_0 there exists $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, the isolated invariant subset K_λ of N (with respect to the flow φ_λ) which has N as an isolating neighborhood has the same shape as K_0.

By [78, Lemma 6.2], we have that K_0 is topologically robust if and only if there exist an isolating neighborhood N of K_0 and a $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, the isolated invariant subset K_λ of N (with respect to the flow φ_λ) which has N as an isolating neighborhood has the same shape as K_0.

Note that when a non-saddle set is dynamically robust, this fact implies the existence of a (local) continuation made of non-saddle sets. On the other hand, if an isolated invariant set is topologically robust, then it has a (local) continuation whose members have the same shape.

The following result gives a characterization of dynamical robustness in the case of differentiable manifolds.

Proposition 2.7.3. *Let φ_λ, with $\lambda \in [0, 1]$, be a differentiable parametrized family of flows on a connected differentiable n-manifold M and K_0 be a connected isolated non-saddle set of φ_0. Then, K_0 is dynamically robust if and only if there exists a connected differentiable isolating block manifold N of K_0 and a $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, each component of $N \setminus K_\lambda$ contains exactly a component of ∂N. Here, K_λ denotes the isolated invariant subset of N (with respect to the flow φ_λ) which has N as an isolating neighborhood.*

Proof. Suppose that K_0 is dynamically robust. Then, from the proof of [36, Theorem 5] it follows that, if N is a connected differentiable isolating block manifold of K_0, there exists $\lambda_0 > 0$ such that, for $\lambda \in [0, \lambda_0)$, N is an isolating block of the form $N = N^+ \cup N^-$ for the (non-empty) isolated invariant set K_λ which has N as an isolating neighborhood. The necessity follows from the fact that ∂N is a deformation retract of $N \setminus K_\lambda$.

Conversely, suppose that there exists a connected differentiable isolating block manifold N of K_0 and a $\lambda_0 > 0$ such that, for every $\lambda \in [0, \lambda_0)$, each component of $N \setminus K_\lambda$ contains exactly a component of ∂N. We make use of the fact, proven in [36], that there exists $\lambda_0' > 0$ such that N is an isolating block of K_λ for $\lambda \in [0, \lambda_0')$ satisfying that the entrance and exit sets for φ_λ are strict (i.e. without tangencies) and that they agree with those for φ_0. We may assume that in fact $\lambda_0' = \lambda_0$. Suppose that K_λ is saddle for some $\lambda \in (0, \lambda_0)$. Then, there exists $x_0 \in N \setminus (N^+ \cup N^-)$ and, hence, if we consider the entrance and exit times $t_\lambda^i(x_0)$ and $t_\lambda^o(x_0)$ we have that $x_0[t_\lambda^i(x_0), t_\lambda^o(x_0)]$ is a path in $N \setminus K_\lambda$ joining a component of N^i with a component of N^o which, by the previous discussion must be different components of ∂N. This contradiction proves the converse statement. □

Corollary 2.7.4. *Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a differentiable parametrized family of flows defined on a connected n-dimensional differentiable manifold M. Let K_0 be an isolated non-saddle continuum for φ_0 and K_λ a continuation of K_0. Suppose that there exists $\lambda_0 > 0$ such that, for $\lambda \in [0, \lambda_0)$, $K_0 \subset K_\lambda$. Then, K_0 is dynamically robust.*

We have already seen in Chapter 1 (Theorem 1.7.3) that, for families of flows defined on surfaces, the topological robustness of an isolated non-saddle continuum implies its dynamical robustness and, hence, by [36, Theorem 5], if the family is differentiable the equivalence holds. Now, we are in position to see that this holds for a large class of higher dimensional manifolds.

Theorem 2.7.5. *Let φ_λ, with $\lambda \in [0, 1]$, be a differentiable parametrized family of flows on a connected G-orientable differentiable n-manifold M with $H^1(M) = 0$ and K_0 be a connected isolated non-saddle set. Then, K_0 is dynamically robust if and only if it is topologically robust.*

Proof. It has been proved in [36] that dynamical robustness implies topological robustness. We prove the converse statement using the characterization given in Proposition 2.7.3. Let N be a connected differentiable isolated block manifold of K_0. Notice that N must be of the form $N^+ \cup N^-$. By assumption, there exists $\lambda_0 > 0$ such that K_λ, the isolated invariant set isolated by N for $\lambda \in [0, \lambda_0)$, has the same shape as K_0.

Reasoning as in the proof of Theorem 2.3.16, it follows that for every $\lambda \in [0, \lambda_0)$, each component of $M \setminus K_\lambda$ contains exactly a component of $N \setminus K_\lambda$. Besides, since $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K_0)$ and $H^1(M) = 0$, we have that $M \setminus K_\lambda$ and $M \setminus K_0$ must have the same number of components. Indeed, if we consider, for

$\lambda \in [0, \lambda_0)$, the initial part of the reduced cohomology long exact sequence of the pair $(M, M \setminus K_\lambda)$, we get

$$0 \to \widetilde{H}^0(M \setminus K_\lambda) \to H^1(M, M \setminus K_\lambda) \to H^1(M) = 0,$$

and hence, $H^1(M, M \setminus K_\lambda) \cong \widetilde{H}^0(M \setminus K_\lambda)$. On the other hand, Alexander duality ensures that $H_1(M, M \setminus K_\lambda) \cong \check{H}^{n-1}(K_\lambda)$. The claim follows easily from this discussion together with the Universal Coefficient Theorem and the shape invariance of Čech cohomology.

As a consequence, we have that for each $\lambda \in [0, \lambda_0)$, $N \setminus K_\lambda$ and $N \setminus K_0$ have the same number of components. Moreover, N being an isolated neighborhood of K_λ must satisfy that each component of $N \setminus K_\lambda$ must contain a component of ∂N and, since ∂N is a deformation retract of $N \setminus K_0$ it follows that ∂N and $N \setminus K_\lambda$ must also have the same number of components. Therefore, each component of $N \setminus K_\lambda$ must contain exactly one component of ∂N and Proposition 2.7.3 ensures the dynamical robustness of K_0. □

So far we are unable to establish the equivalence between the topological and dynamical robustness, for isolated non-saddle continua, without further assumptions. However, we can prove the equivalence between the dynamical robustness of an isolated non-saddle continuum and a strong form of topological robustness which we introduce in the following definition.

Definition 2.7.6. Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a parametrized family of flows (parametrized by $\lambda \in I$, the unit interval) on a locally compact ANR, M, and suppose that K_0 is an isolated non-saddle set for φ_0. We say that K_0 is *strongly topologically robust*, if for each isolating neighborhood N of K_0 there exist a $\lambda_0 > 0$ and an isolating block $N' = N'^{+} \cup N'^{-}$ of K_0 such that, for each $\lambda \in [0, \lambda_0)$, N' is an isolating neighborhood of the isolated invariant subset K_λ of N having N as an isolating neighborhood (with respect to the flow φ_λ) and the inclusion $i_\lambda : K_\lambda \hookrightarrow N'$ is a shape equivalence.

Theorem 2.7.7. *Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a differentiable parametrized family of flows defined on a connected G-orientable differentiable n-dimensional manifold M. Let K_0 be an isolated non-saddle continuum for φ_0. Then, K_0 is dynamically robust if and only if it is strongly topologically robust.*

Proof. We only prove that strong topological robustness implies dynamical robustness, since the other case follows from the proof of [36, Theorem 5]. Let K_λ be the continuation of K_0 in some isolating neighborhood and $N' = N'^+ \cup N'^-$ be the isolating block of K_0 satisfying that N' isolates K_λ for $\lambda \in [0, \lambda_0)$ and $i'_\lambda : K_\lambda \hookrightarrow N'$ is a shape equivalence. Let N be a connected differentiable isolating block manifold for K_0 contained in N'. Notice that N is also of the form $N^+ \cup N^-$ and, hence, the flow φ_0 provides a deformation retraction from N' onto N_0. We may assume that N isolates K_λ for $\lambda \in [0, \lambda_0)$ (otherwise we only have to choose a smaller λ_0). Since the inclusion $i'_\lambda : K_\lambda \hookrightarrow N'$ is the composition of the inclusions $i_\lambda : K_\lambda \hookrightarrow N$ and $j : N \hookrightarrow N'$ and i'_λ and j are a shape and a homotopy equivalence respectively, it follows that i_λ must be a shape equivalence. The isolating block N of K_0 is also an isolating block of K_λ, for $\lambda \in [0, \lambda_0)$, with the same entrance and exit sets as it was shown in [36, Theorem 5]. Moreover, $\check{H}^k(N, K_\lambda) = 0$ for $k \geq 0$ and $\lambda \in [0, \lambda_0)$ by the previous discussion. Then, Alexander duality ensures that $H_{n-k}(N \setminus K_\lambda, \partial N) = 0$ for each k. As a consequence, from the terminal part of the homology long exact sequence of the pair $(N \setminus K_\lambda, \partial N)$, we deduce that the homomorphism $i^\lambda_* : H_0(\partial N) \to H_0(N \setminus K_\lambda)$, induced by the inclusion $i^\lambda : \partial N \hookrightarrow N \setminus K_\lambda$, is an isomorphism for each $\lambda \in [0, \lambda_0)$. Therefore, each component of $N \setminus K_\lambda$ contains exactly one component of ∂N and the result follows from Proposition 2.7.3. □

The remaining part of the chapter is devoted to study a special kind of continuations of non-saddle sets. In particular, the following result studies the important case when all points of the non-saddle set are stationary. It formulates in a precise way the intuitive idea that non-saddle sets produce attractor-repeller splittings.

Theorem 2.7.8. *Let K be an isolated non-saddle set of a flow φ on a locally compact metric space M such that K is neither an attractor nor a repeller and all points of K are stationary. Then, there exists a continuous parametrized family of flows, φ_λ, $\lambda \in [0, \varepsilon]$, with $\varphi_0 = \varphi$ such that K continues to a family of non-saddle sets K_λ with the following properties:*

a) K_λ has an attractor-repeller decomposition (A_λ, R_λ) with $R_\lambda = K$ for every $\lambda > 0$, where A_λ and R_λ are an attractor and a repeller for the flow φ_λ (not merely for the restricted flow $\varphi_\lambda|_{K_\lambda}$), and *b)* $\partial K_\lambda \subset A_\lambda \cup R_\lambda$.

Proof. Consider an isolating block of K in M of the form $N = N^+ \cup N^-$. Since all points of K are stationary we can perform in N^+ the following operation: consider a section S_λ of the flow with $S_\lambda \subset N^+$ and transform the flow in N^+ in such a way that all the points in S_λ are stationary and the flow is reversed for all points between S_λ and K. The flow does not experiment any change outside N^+. The fact that this operation can be performed obtaining, in fact, another flow is justified by the results in [13] where it is shown how to modify a flow by making it stationary in all the points of an arbitrary compact set (in our case S_λ). We can do this for S_λ as close to K as we wish and, in this way, we obtain a continuous family of flows φ_λ, $\lambda \in [0, \varepsilon]$ with $\varphi_0 = \varphi$. It is obvious that K is now a repeller and S_λ is an attractor for every φ_λ with $\lambda > 0$, which we denote by R_λ and A_λ respectively. Moreover the family of sets $K_\lambda = K \cup \{\varphi_0(x,t) | x \in S_\lambda, t \geq 0\}$ defines a continuation of K. Obviously the pair (A_λ, R_λ) is an attractor-repeller decomposition for the flow $\varphi_\lambda|_{K_\lambda}$, meaning by this that $\omega(x) \subset A_\lambda$ and $\omega^*(x) \subset R_\lambda$ for every point $x \in K_\lambda \setminus (A_\lambda \cup R_\lambda)$. Since all the points $\varphi_0(x,t)$ with $t > 0$, $x \in S_\lambda$ are in the interior of K_λ, we have that $\partial K_\lambda \subset A_\lambda \cup R_\lambda$. The non-saddleness of K_λ is straightforward. □

Our following result shows that the property described in Theorem 2.7.8 is characteristic for isolated non-saddle continua of flows defined on a large class of spaces.

Theorem 2.7.9. *Suppose that we have a continuous parametrized family of flows,* φ_λ, $\lambda \in [0,1]$ *on a manifold* M *with* $H^1(M) = 0$ *and a continuation* K_λ, $\lambda \in [0,1]$, *of an isolated invariant continuum* $K = K_0$. *Suppose, additionally, that the sets* K_λ *have the following properties for* $\lambda > 0$:

a) K_λ *has a non-trivial attractor-repeller decomposition* (A_λ, R_λ) *with* $R_\lambda = K$ *for every* $\lambda > 0$, *where* A_λ *and* R_λ *are an attractor and a repeller for the flow* φ_λ *(not merely for the restricted flow* $\varphi_\lambda|_{K_\lambda}$*), and*

b) $\partial K_\lambda \subset A_\lambda \cup R_\lambda$.

Then, $K = K_0$ *is non-saddle.*

Proof. By non-trivial we mean that both A_λ and R_λ are non-empty. Suppose, to get a contradiction, that K is saddle. Then there is a component C of $M \setminus K$ such that K is saddle for the flow φ_0 restricted to $K \cup C$. Now $(K_\lambda \cap C) \setminus A_\lambda$ is open in $K \cup C$ for $\lambda > 0$ since $\partial K_\lambda \subset A_\lambda \cup R_\lambda$ and from this it is easy to see that $K_\lambda \cap (K \cup C)$ is either a repeller or an attractor for the flow φ_λ, with $\lambda > 0$, restricted to $K \cup C$. But $K_\lambda \cap (K \cup C)$ is a continuation of K in $K \cup C$ and the Conley index of a saddle set is different from that of an attractor or a repeller. We get from this contradiction that, in fact, K is non-saddle. □

The following are consequences of Theorem 2.7.8.

Corollary 2.7.10. *Suppose that K is an isolated non-saddle continuum of a flow on a closed, connected and G-orientable n-manifold M such that K is neither an attractor nor a repeller and $CH^{n-1}(K) = 0$. Then,*

$$1 \leq \beta_{n-1}(K) \leq 1 + \operatorname{rk} CH_1(K).$$

Proof. Let $N = N^+ \cup N^-$ be an isolating block of K. Since $CH^{n-1}(K) = 0$ it follows, by time-duality [64], that $H_1(N, N^i) = CH_1^-(K) = 0$ and, from the terminal part of the long exact sequence of homology the pair (N, N^i)

$$\cdots \to 0 = H_1(N, N^i) \to H_0(N^i) \to H_0(N) = \mathbb{Z} \to H_0(N, N^i) = 0$$

it follows that N^i is connected.

On the other hand, we may assume without loss of generality that K is made of fixed points since, if not, we can construct another flow using Whitney Theorem [99] about oriented regular families of curves preserving the dynamics in $N \setminus K$ and keeping fixed all the points in K. Hence, from the proof of Theorem 2.7.8 it follows that K can be continued to an isolated invariant non-saddle continuum K_λ which admits an attractor-repeller decomposition (A_λ, R_λ) where $R_\lambda = K$ and A_λ is a section of N^+, and hence, homeomorphic to N^i (see Chapter 1). Moreover, R_λ is a repeller and A_λ is an attractor for the flow. Consider the terminal part of the attractor-repeller sequence of the pair (A_λ, R_λ)

$$\cdots \to CH_1(A_\lambda) \to CH_1(K_\lambda) \to CH_1(R_\lambda) \to CH_0(A_\lambda) \cong \mathbb{Z} \to CH_0(K_\lambda) = 0.$$

Then

$$1 \leq \operatorname{rk} CH_1(R_\lambda) \leq 1 + \operatorname{rk} CH_1(K_\lambda).$$

Since K_λ is a continuation of K we have that $\operatorname{rk} CH_1(K) = \operatorname{rk} CH_1(K_\lambda)$ and, on the other hand, by time duality, $\operatorname{rk} CH_1(R_\lambda)$ must agree with $\operatorname{rk} CH_-^{n-1}(R_\lambda)$. Since R_λ becomes an attractor when the reversed flow is considered and $R_\lambda = K$ we have that $\operatorname{rk} CH_-^{n-1}(R_\lambda) = \beta_{n-1}(K)$, which establishes the inequality. □

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