

THE USEFULNESS OF COMPLETE LATTICES IN RELIABILITY THEORY

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Abstract : The main aim of this paper is to show how lattice theory in the very next future will be a useful tool in analysing complex real reliability problems, not properly modelled within classical reliability theory. The introduction of a complete lattice as a state space appears not only of theoretical importance that allows to understand several phenomena with respect to reliability theory better, but as a need claimed from practical engineering. Two important topics are discussed in this general framework: incomparability of component and system states and the duality principle. The strong relationship between the ideas of fuzzy set theory and the ideas that led to the introduction of the theory of multistate structure functions will become clear.

Keywords : Complete lattices, multistate structure functions, non-binary reliability theory.

1. INTRODUCTION

When developing a theory of structure functions defined on arbitrary complete lattices, one often meets some criticism against this choice. Some say that complete lattices provide a too abstract framework and that there are no obvious arguments

to introduce them. In this paper, we shall try to show that complete lattices apply to real life situations and that they make possible to study rather complex phenomena with respect to reliability theory and practice. Seminal ideas can be found in Montero et al. (1988) and Cappelle (1991).

Complete lattices have been successfully applied to other fields apart from reliability theory. We can at least trace two important applications in the past. First of all, recent developments in the theory of mathematical morphology show that the general framework of lattice theory is almost indispensable in explaining complex phenomena in an easy way. For an excellent overview and outline of this theory we refer to Ronse (1989). Second, the introduction of *L*-fuzzy sets (Goguen, 1967) provides a general framework for Zadeh's fuzzy set theory (Kerre, 1991).

In general, complete lattices apply whenever *ordinal* information must be represented. In the theory of mathematical morphology we must manipulate and classify images, while in fuzzy set theory we must check the fulfillment of some property (classical examples are the properties "young" or "beautiful"). Sentences such as "the aircraft is more similar to a Boeing 747 than to a Tupolev 159-M" or "our production quality is higher than the competitor's" make sense in pattern recognition and information representation. Analogous sentences are common among engineering practitioners.

Some claim that complete chains, i.e., totally ordered sets, can be applied in most cases. But very often this assumption is an oversimplification of reality, since we can not deal with incomparable elements. We encounter difficulties that are hard to solve when dealing with complex non-single criterion based problems, where the system state itself is evaluated according to various criteria (Montero et al., 1992).

Hence, when we have a closer look at real life problems, the claim for comparability is not so obvious. Let us suggest some concrete examples:

- (i) consider a genealogical tree and the set of John's ancestors; the *natural* order relation on the set of ancestors is $x \leq y$ if and only if y is an ancestor of x , x and y two arbitrary elements of the set of John's ancestors; the order relation on the set of John's ancestors is not linear, i.e., generally for each pair of ancestors neither (x, y) " x is an ancestor of y " nor " y is an ancestor of x " holds;
- (ii) suppose we must judge if one weather condition is more dangerous than another weather condition while driving a car; most of us will judge close and thick fog far more dangerous than ordinary rain, but will most of us be able to decide which weather condition is worse: thick fog or icy and snowy weather; some just will say that these weather circumstances are dangerous in a different way and, hence, somehow incomparable;
- (iii) consider a particular manufactured product; it can be very difficult to define a linear order when comparing the "quality" of products made by different people, since "quality" refers to many aspects of the manufactured product.

The state space representation in practical reliability has a deep link with preference relations in decision making and we all know that decision making relations are rarely linear. Hence, linearly ordered sets are not the most suitable tool to order objects. The application of partially ordered sets (posets for short) to reliability theory is not surprising from this point of view, since we must be able to sort the states from "bad" to "good" states. Even the applied uncertainty model to study the probability or possibility of failure or degradation, is mainly based upon an order

relation. Indeed, the uncertainty model allows to sort the system states from most probable (most possible) to least probable (least possible). Hence, the relationship between reliability theory and lattice theory is not artificial.

A quite different approach tries to introduce some kind of fuzzy uncertainty in the basic binary model, from the existence of non-random (possibilistic) uncertainties attached to real systems and the fuzzy nature of the idea of “performance” and “failure”. The term “safety” or “reliability” can then be modelled in a fuzzy framework (see Zimmermann (1983)). In the particular area of reliability, fuzziness can lead to at least two possible basic models:

- (i) characterize system reliability behaviour in a possibilistic context,
- (ii) assume that the system failure is defined in a fuzzy way.

Following Cai’s notation (see Cai, 1991 for a personal report on the research done at the Beijing University of Aeronautics and Astronautics), combination leads to at least three types of fuzzy reliability theories:

- (i) PROFUST reliability: a theory based upon PRObability theory and a FUZZY STATE assumption,
- (ii) POSBIST reliability: a theory based upon POSSibility theory and a BINARY STATE assumption,
- (iii) POSFUST reliability: a theory based upon POSSibility Theory and a FUZZY STATE assumption.

An alternative combined approach has been developed by Onisawa (see Onisawa, 1989 for a personal review of his research), by considering simultaneously equipment “failure possibility” and human “error possibility”, both derived from the estimation of failure/error probabilities based on a safety criterion.

Our paper deals with multistate structure functions. Hence, we consider the case that each component or system may assume one of many states. Since we assume that a state space can be any complete lattice, one notices the deep link with the L -fuzzy sets of Goguen (1967). Indeed, a structure function can be viewed as a kind of fuzzy set that models the relationship between the fuzzy notions failure, almost failure, almost functioning and perfect functioning between the components and the system. From that point of view, a state of a component is a kind of membership degree in the fuzzy set of "good states." The main aim of this paper is to show how complete lattices provide a general framework to explain complex phenomena in non-classical reliability theory. One will notice the deep link with the main ideas of fuzzy set theory.

2. COMPLETE LATTICES IN RELIABILITY THEORY

When studying classical reliability theory two parts can be distinguished: a general theory of binary structure functions and a probabilistic (time-dependent) uncertainty model. By means of a structure function the deterministic relationship between the states of the components and the system state is modelled, while probability theory provides a general framework to define the notion *reliability* properly: the reliability of an *item* (component or system) is the probability that the item functions properly. According to classical reliability theory, the items must always assume either one of two possible states: perfect functioning or total failure. The system reliability is then calculated from the (basic) component reliabilities.

Since many real life situations are simply not binary in nature, a dichotomous approximation, initially introduced by Birnbaum et al. (1961), sometimes is far too inaccurate. The model has been extended systematically from 1978 on in order

to allow intermediate states between perfect functioning and total failure¹. Among others we mention the finite models of El-Newehi et al. (1978), Griffith (1980) and Narvig (1982) and the infinite or continuous models introduced by Block and Savits (1984) and Baxter (1984, 1986). Some technical difficulties arise depending on the complexity of the state space, and a variety of solutions to particular problems can be found in the literature. We remind the reader, e.g., of the overwhelming number of notions of the coherence of structure functions, most of them reduce to the well-known coherence property when applied to binary structure functions (Ohi and Nishida (1984) mention at least fourteen different notions).

By assuming that the state space is a complete lattice, we are not just extending the notion of binary structure function to a multistate model, we are also modelling many real life systems where more restrictive approaches do not apply and, moreover, creating a general framework with notions that allow a better comprehension of the basic concepts initially defined for particular systems only.

In the sequel two problems are discussed in order to exemplify our approach: the possible *incomparability of states* and a general *duality principle*. The first problem requires a complete lattice as a state space, while the second one can be properly justified by considering the complete lattice of structure functions.

3. A STATE SPACE FOR COMPONENTS AND SYSTEMS

Let us assume that each component provides a set of *parameters* that can be evaluated. The evaluation of these parameters is called a *state*, that is, a characteristic for the behaviour of the component at a certain time. The performance level of a copy machine, e.g., can be tested by making a copy of a special grey chart; this grey chart copy can be an excellent parameter for the evaluation of the performance of

¹ The ideas and first attempts to formulate multistate structure functions are much older, cf., Premo (1963), and some notes in Gnedenko et al. (1972).

the copy machine. Hence, an *evaluation* is a mapping from the set of parameters into a set L , the set of all the possible states of the item. In order to be able to distinguish between worse and better states, L must be provided with an order relation \leq (Birkhoff, 1967). For any states a and b , $a \leq b$ implies that state b is better than state a or, equivalently, that state a is worse than state b .

The couple (L, \leq) is called a poset, and as pointed out above, in general the order relation is not linear. The state space must be a *complete lattice*, i.e., given any subset in L , at least its greatest lower bound, called infimum, and its smallest upper bound, called supremum, exist. Although an extensive justification of the completeness is far beyond the scope of this paper, this demand allows, e.g., the development of general reliability bounds. A *chain* or totally ordered set is a poset where any two elements are *comparable*, i.e.,

$$(\forall (a, b) \in L^2)(a \leq b \text{ or } b \leq a).$$

When $(\{0, 1\}, \leq)$ is the state space for both the system and its components, where \leq is the well-known order relation, we obtain the classical binary model, introduced by Birnbaum et al. (1961). When $L = \{0, 1, \dots, M\}$, with $M > 2$, the finite state models of El-Newehi et al. (1978) appear. When both the state space of components and system equal $([0, 1], \leq)$, Baxter's continuous infinite model is obtained (Baxter, 1984). It must be pointed out that Block and Savits (1984) also considered an infinite continuous model based on the chain (\mathbb{R}^+, \leq) . However, this chain is not complete and therefore this model has not been considered here². Obviously under our general approach, the state space for each component is not required to be always the same. Hence, the following definition applies (see Montero et al., 1988, and Cappelle, 1991):

²From our point of view, a greatest element, called infinity, must be added as the perfect functioning state to transform (\mathbb{R}^+, \leq) into a complete lattice.

Definition 1 Let (L_i, \leq) , $0 \leq i \leq n$, be n arbitrary complete lattices and assume that $(L_1 \times \cdots \times L_n, \leq)$ has been provided with its product ordering, i.e., for an arbitrary \mathbf{x} and \mathbf{y} in $L_1 \times \cdots \times L_n$

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow (\forall i \in \{1, \dots, n\})(x_i \leq y_i).$$

An $L_1 \times \cdots \times L_n \rightarrow L_o$ mapping ϕ is a structure function if and only if

(i) ϕ is isotone, i.e.,

$$(\forall (\mathbf{x}, \mathbf{y}) \in (L_1 \times \cdots \times L_n)^2)(\mathbf{x} \leq \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y})),$$

(ii) $\phi(0, \dots, 0) = 0$ and $\phi(1, \dots, 1) = 1$.

Since a system that assumes a worse state when every component assumes a better state is quite unusual or poorly designed (Barlow, 1975), the isotonicity is justified. The boundary condition is also widely accepted and applied. It reflects the fact that whenever each component assumes its worst (best) state, the system must assume its worst (best) state.

When $L_i = \{0, 1\}$, $0 \leq i \leq n$, the structure functions are called binary structure functions (Barlow and Proschan, 1975). An El-Newehi et al. (1978) structure function ϕ appears when $L_i = \{0, 1, \dots, M\}$, $0 \leq i \leq n$, and ϕ is idempotent, i.e.,

$$(\forall x \in \{0, 1, \dots, M\})(\phi(x, \dots, x) = x).$$

4. INCOMPARABLE STATES

In section 3 we have introduced arbitrary complete lattices as the state space for both components and systems. It has been pointed out that not every two states are comparable. For an arbitrary poset (P, \leq) , two elements a and b of P are *incomparable* if and only if $\neg(a \leq b)$ and $\neg(b \leq a)$ holds (Birkhoff, 1967), and it is denoted by $a \parallel b$.

As pointed out above, incomparable states are of particular interest in the theory of structure functions. Very often we cannot determine if state a is better or worse than another state b , or when both states are similar. When dealing with systems subject to n different kinds of failure, e.g., where a transition diagram indicates the possible transitions between all the states, the diagram often suggests a poset with $n + 1$ elements. Element 1 represents the perfect functioning state and the elements a_i , $1 \leq i \leq n$, represent the failure of type i (see for example Proctor (1976), Elsayed (1979) and Yamashiro (1980)). Hence, the order relation \leq imposed on the set $\{a_1, \dots, a_n, 1\}$ is obvious:

$$(\forall i \in \{1, \dots, n\})(a_i \leq 1) \text{ and } (\forall (i, j) \in \{1, \dots, n\}^2)(i \neq j \Leftrightarrow a_i \parallel a_j).$$

It is quite useless to consider a failure of type i better or worse than another different failure of type j , since different types of failure do not reflect a degradation in several stages. Adding a smallest element 0, i.e.,

$$(\forall x \in \{0, a_1, \dots, a_n, 1\})(0 \leq x),$$

transforms the given state space into the complete lattice $(\{0, a_1, \dots, a_n, 1\}, \leq)$. In fact, the structural model of a component subject to n different types of failure is more complex if only structural information had been taken into account. However, it has been simplified by considering some extra (probabilistic) information: having n binary failures a_i , $1 \leq i \leq n$, a binary space $\{0, 1\}$ is associated

to each failure, such that the initial state space for the system is the complete lattice $(\{0, 1\}^n, \leq)$, whereas 1 and 0, respectively, are the perfect functioning and the total failure state, respectively. If we additionally know that failure times are continuously distributed and that the system completely fails when the first failure is observed, all states within the lattice $(\{0, 1\}^n, \leq)$ except $n + 1$ will have zero probability. Therefore, only $n + 1$ states can be observed: no failure (1) and the *first* failure of type i , a_i , $1 \leq i \leq n$. Hence, the proposed state space simplifies the model, since many non-observable states are excluded.

Let us consider a motor opened valve, MOV for short. The MOV can fail to open when it is closed or fail to close when it is open. Hence, the MOV is subject to two kinds of failure which can hardly be compared. Practical considerations suggest a complete lattice with five elements as a state space:

- (i) state 1: the MOV is in optimal condition,
- (ii) state a: the MOV must be maintained but still functions,
- (iii) state b: the MOV can be opened but can no longer be closed,
- (iv) state c: the MOV can be closed but can no longer be opened,
- (v) state 0: a total failure of the MOV is observed.

Just like in the previous example, this state space is a simplified version of a more complex state space, by considering two failure types each having a three element evaluation set: $\{0, \frac{1}{2}, 1\}$, the intermediate state $\frac{1}{2}$ represents functioning with maintenance. One must notice that in the proposed state space, the associated order relation is not the product ordering, i.e., $(\{0, \frac{1}{2}, 1\}^2, \leq)$, since the engineer—according to the chosen state space—prefers functioning though any maintenance is needed of both opening and closing mechanisms to complete failure of one of the mechanisms, despite the other is perfectly functioning.

5. A GENERAL DUALITY PRINCIPLE

In the previous sections we have pointed out that the order relation on the set of possible states determines when states are similar, better, worse or incomparable and that the assumption of a linear ordering of the state space is not appropriate for many practical problems. In this section we shall exemplify how our general approach will gain insight into the standard concepts in reliability theory.

A duality principle deals with a *reversed* order on the set of states. The “better” states become the “worse” states, and *vice versa*. In fact, many systems subject to two dual failure types (Barlow and Proschan, 1975) can be represented as the conjunction of two dual systems. Structural duality, however, very often is defined without any special consideration. From our point of view, this particular property of the state space, i.e., the possibility of exchanging the good and the bad states, is a special case of the selfduality of a poset. An arbitrary poset (P, \leq) is selfdual if and only if there exists an order reversing permutation δ on P (Birkhoff, 1967), that is,

$$(\forall (a, b) \in P^2)(a \leq b \Leftrightarrow \delta(b) \leq \delta(a)).$$

Therefore, the following definition is proposed:

Definition 2 (Montero 1988, Cappelle 1991) Let δ_i be an order reversing permutation of (L_i, \leq) , $1 \leq i \leq n$, and δ_o be an order reversing permutation of (L_o, \leq) . The dual (w.r.t. $\delta_1, \dots, \delta_n$ and δ_o) of a structure function ϕ , denoted by $\Delta(\phi)$, is defined as

$$\Delta(\phi) : L_1 \times \dots \times L_n \rightarrow L_o : x \mapsto \delta_o(\phi(\delta_1(x_1), \dots, \delta_n(x_n))).$$

The mapping Δ on the set of structure functions is called a duality.

By $\mathcal{M}(L_1 \times \cdots \times L_n, L)$ we denote the set of all $L_1 \times \cdots \times L_n - L_0$ structure functions, and we provide it with a pointwise order relation \preceq , such that for any two arbitrary structure functions ϕ_1 and ϕ_2

$$\phi_1 \preceq \phi_2 \iff (\forall \mathbf{x} \in L_1 \times \cdots \times L_n)(\phi_1(\mathbf{x}) \leq \phi_2(\mathbf{x})).$$

It is easily verified that $(\mathcal{M}(L_1 \times \cdots \times L_n, L), \preceq)$ is a complete lattice (Cappelle 1991) and that Δ is an order reversing permutation of $(\mathcal{M}(L_1 \times \cdots \times L_n, L), \preceq)$.

From this point of view, structural duality is an order reversing permutation of $(\mathcal{M}(L_1 \times \cdots \times L_n, L), \preceq)$. As a special case we find that the classical duality in the binary case and linearly ordered finite multistate models is unique. Indeed, there exists a unique order reversing permutation δ on the chain $(\{0, \dots, M\}, \leq)$,

$$\delta : \{0, \dots, M\} \rightarrow \{0, \dots, M\} : x \mapsto M - x.$$

Hence, the duality Δ transforms any structure function ϕ into a structure function $\Delta(\phi)$,

$$\Delta(\phi) : \{0, \dots, M\}^n \rightarrow \{0, \dots, M\} : \mathbf{x} \mapsto M - \phi(M - x_1, \dots, M - x_n).$$

This is exactly the duality principle introduced in the past, for the linearly ordered finite state systems.

Problems arise when considering more general state spaces, e.g., when dealing with continuous systems, i.e., the complete chain $([0, 1], \leq)$ as the state space for both the system and its components. It is obvious that there is more than one order reversing permutation on $([0, 1], \leq)$. Hence, an enormous variety of order reversing permutations on $(\mathcal{M}([0, 1]^n, [0, 1]), \preceq)$ can be constructed, making Baxter's duality principle,

$$\Delta(\phi) : [0, 1]^n \rightarrow [0, 1] : \mathbf{x} \mapsto 1 - \phi(1 - x_1, \dots, 1 - x_n),$$

for any $[0, 1]^n \rightarrow [0, 1]$ structure function ϕ , not so evident. Still Montero et al. (1988) provide a very important argument to consider Baxter's choice: the order reversing permutation δ ,

$$\delta : [0, 1] \rightarrow [0, 1] : x \mapsto 1 - x,$$

is the only order reversing permutation that is measure preserving with respect to the Lebesgue measure³. However, when a possibilistic uncertainty model is applied, there is no reason to consider only Baxter's definition (cf. Cappelle, 1991).

6. CONCLUSION

On the one hand, the existence of components and systems with incomparable states requires a lattice-based reliability approach in order to capture all the essential information without artificial oversimplification of the mathematical model.

On the other hand, it has been pointed out that besides a real increase in potential applications, a general approach is useful in order to clarify basic concepts and notions. In particular, it has been shown that the idea of a duality must be related to the structure of $(\mathcal{M}(L_1 \times \dots \times L_n, L_o), \preceq)$, and although the standard duality principle for the finite chain model is unique, this result does not apply in general. Hence, in the near future, lattice theory must have a deep influence on system reliability, from a theoretical as well as from a practical point of view.

It must be pointed out that in this paper we have focused our attention strictly on the problem of the state space representation. We did not discuss the possibilistic uncertainty models (see, e.g., Cai, Chuan and Zhang (1991), and Cappelle (1991)).

³The result is more general, being the key assumption to the existence of an σ -finite measure associated to the space of states.

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