Squashed, magnetized black holes in $D = 5$ minimal gauged supergravity

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Abstract: We construct a new class of black hole solutions in five-dimensional Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant. These configurations are cohomogeneity-1, with two equal-magnitude angular momenta. In the generic case, they possess a non-vanishing magnetic potential at infinity with a boundary metric which is the product of time and a squashed three-dimensional sphere. Both extremal and non-extremal black holes are studied. The non-extremal black holes satisfying a certain relation between electric charge, angular momenta and magnitude of the magnetic potential at infinity do not trivialize in the limit of vanishing event horizon size, becoming particle-like (non-topological) solitonic configurations. Among the extremal black holes, we show the existence of a new one-parameter family of supersymmetric solutions, which bifurcate from a critical Gutowski-Reall configuration.

Contents

1 Introduction and motivation

The study of black objects in gravity models with a negative cosmological constant has attracted recently considerable interest, being fueled by studies of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1, 2]. This conjecture basically proposes a 'dictionary' between classical AdS bulk gravitational solutions (in $D-$dimensions) and field theory states at strong coupling (in $(D-1)-$dimensions).

Of particular interest in this context are the solutions of five dimensional $\mathcal{N} = 4$ SO(6) gauged supergravity, which is thought to be a consistent truncation of type IIB supergravity on $AdS_5 \times S^5$ [3, 4]. In its minimal version, the bosonic sector of this model is just Einstein-Maxwell (EM) theory with a negative cosmological constant and a Chern-Simons (CS) U(1) term (with a fixed value of the coupling constant). Despite its (apparent) simplicity, this theory possesses a variety of interesting solutions which have been investigated extensively over the last two decades.

Restricting to configurations possessing an event horizon, one remarks that most of the studies in the literature concentrate on two different classes of solutions. First, there are the black holes (BHs) with a spherical horizon topology[1] in a globally AdS$_5$ spacetime background, in which case the dual theory is formulated in a $D = 4$ Einstein universe. The Schwarzschild-AdS BH is the simplest example, while the most general such EMCS solutions rotate in two planes and possess four global charges: the mass, the electric charge, and two angular momenta [5]. A considerable simplification is obtained for an Ansatz with two equal-magnitude angular momenta, an assumption which factorizes the angular dependence of the problem. These BH solutions have been found in closed form by Cvetič, Lü and Pope (CLP) in [6] (see also [7–9]). Remarkably, their extremal limit contains a subset of solutions that preserves some amount of supersymmetry [10]. An extension of the CLP BHs which possesses an extra parameter Φ_m associated with a non-zero magnitude of the magnetic potential at infinity has been reported in the recent work [11].

Second, there are the black branes, which approach at infinity the Poincaré patch of the AdS spacetime. These solutions have a Ricci flat horizon, while their dual field theory states reside in a $D = 4$ Minkowski spacetime. The most general such configurations appear to be those reported in [12–14]; in addition to the mass and electric charge, they possess an extra parameter corresponding to the magnitude of the magnetic field at infinity.

However, it is worth remarking that the AdS/CFT correspondence does not constrain the way of approaching the boundary of spacetime, asymptotically *locally* AdS (AlAdS) solutions being also relevant. An interesting class of such configurations are the $D = 5$ AdS black strings[2] originally found by Copsey and Horowitz in [15]. These are natural AdS counterparts of the (better known) uniform black strings in a $D = 5$ Kaluza-Klein theory, the horizon topology being $S^2 \times S^1$. Also, the conformal boundary, where the dual theory lives, is the product of time and $S^2 \times S^1$.

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[1]Black rings with an $S^2 \times S^1$ event horizon topology exist as well, approaching at infinity a globally AdS$_5$ background. Such solutions have been constructed in [16] using approximate methods, and fully nonperturbatively in [17].

[2]These solutions have been generalized to higher dimensions and a more general topology of the event horizon in [18].
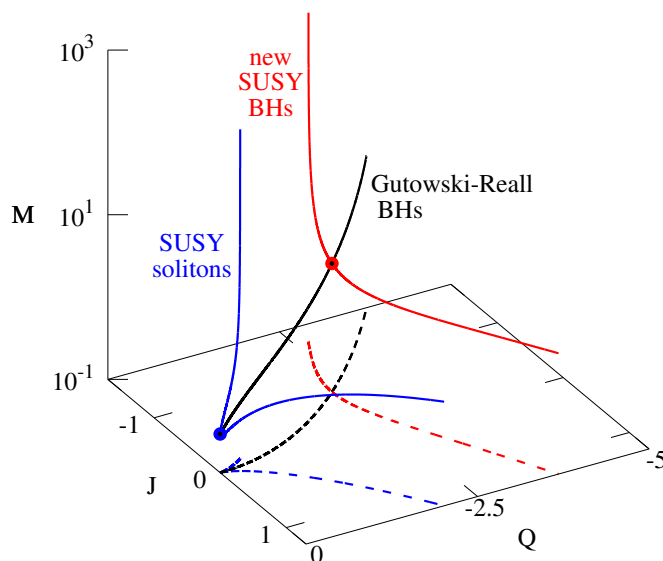
Figure 1. The *mass-angular momentum-electric charge* diagram is shown for the supersymmetric solitons in [19], the Gutowski-Reall BHs [10] and the new supersymmetric BHs in this work.
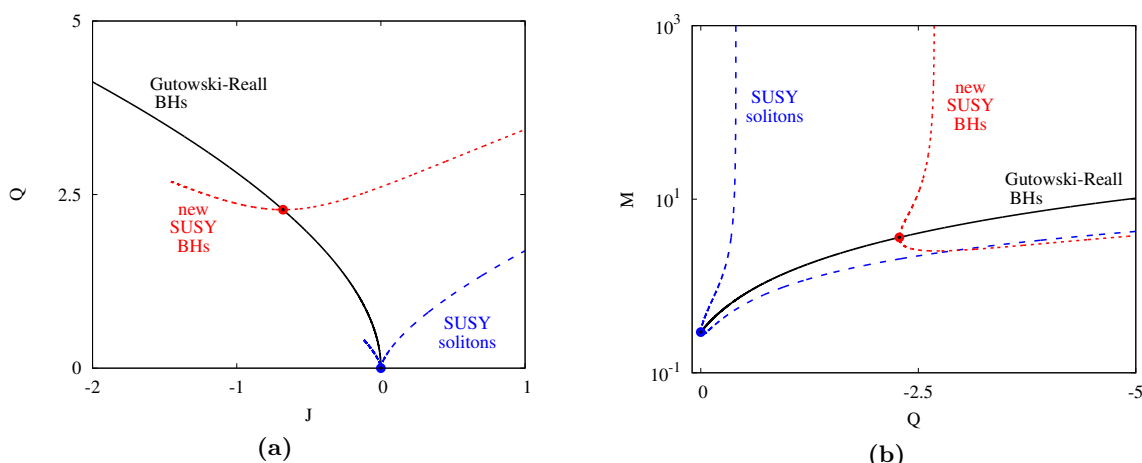
Figure 2. (a) The *electric charge-angular momentum* diagram is shown for the supersymmetric solitons in [19], the Gutowski-Reall BHs [10] and the new supersymmetric BHs in this work. (b) A similar figure with the *mass-electric charge* diagram for the three families of supersymmetric solutions.

The main purpose of this work is to report the existence of a new class of solutions of the $D = 5$ minimal gauged supergravity model. These solutions possess a squashed sphere in the boundary metric and can be viewed as interpolating between (some versions of) the three classes of black objects mentioned above.

Moreover, we find that a particular set of these configurations has special properties, forming a new one-parameter family of supersymmetric BHs.

A discussion of the basic properties of these solutions was given in the recent work [20], in a slightly different context.

1.1 Summary of results

In a convenient set of coordinates, the conformal boundary metric of the solutions in this work reads

$$ds^2_{(bdry)} = L^2 d\Omega^2_{(v)} - dt^2, \quad \text{where} \quad d\Omega^2_{(v)} = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\phi^2 + (d\bar{\psi} + v\cos\theta d\phi)^2\right), \quad (1.1)$$

with v a control parameter, $d\Omega^2_{(v)}$ the metric on a squashed S^3 sphere and $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, $0 \leq \bar{\psi} < 4\pi v$. The presence of a squashed sphere in the boundary geometry of some asymptotically *locally* AdS configurations has been found before in the literature, the $D = 4$ nutty instantons (reviewed in appendix A) being perhaps the best known case.

Clearly the sphere in (1.1) becomes a round one for $v = 1$, in which case the solutions approach a globally AdS background. Another case of interest is $v = 0$, the bulk solutions becoming AdS black strings and vortices, with a boundary which is the product of time and $S^2 \times S^1$ (with S^1 parametrized by $\bar{\psi}$, whose periodicity is arbitrary in this limit). Finally, for large values of v, one can show that, after a proper rescaling, the boundary geometry (1.1) is the product of time and a twisted R^3 part.

In this work we provide evidence for the existence of a family of black objects with a conformal boundary given by (1.1). The solutions are constructed numerically within the framework of the $D = 5$ minimal gauged supergravity model, and possess a gauge potential with both a magnetic and an electric part; they also rotate in the bulk, with equal-magnitude angular momenta.

The main properties of the generic nonextremal solutions can be summarized as follows:

(i) They possess an event horizon of spherical topology and are regular on and outside the horizon. Also, they do not present other pathologies (such as closed timelike curves (CTCs)).

(ii) In addition to the mass M, the electric charge Q and the angular momenta J, the new solutions possess an extra parameter c_m associated with a non-zero magnitude of the magnetic potential at infinity.

(iii) A particular set of BHs with $c_m \neq 0$ does not trivialize as the horizon size shrinks to zero, a limit which describes a one-parameter family of *squashed spinning charged solitons*. The angular momentum J and the electric charge Q of these solutions are determined by the magnetic flux at infinity Φ_m through the base space S^2 of the S^1-fibration, with $J = \Phi_m Q$.

(iv) The generic BH solutions possess an extremal limit, with a nonzero event horizon area. Moreover, supersymmetric BHs exist as well, forming a one-parameter family. These BHs bifurcate from a critical Gutowski-Reall [10] configuration, their mass, angular momenta and electric charge having relatively simple expressions in terms of the squashing parameter v only.

We note that supersymmetric solitons exist as well within the same framework, being investigated in the interesting work [19]. However, they do not correspond to a limit of the supersymmetric BHs.

The *mass − angular momentum − electric charge* diagrams summarizing the picture for these three different types of supersymmetric solutions are shown in figure 1 and in figure 2.

2 The general framework

2.1 The model and Ansatz

The action for $D = 5$ minimal gauged supergravity is given by

$$I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^5 \mathrm{x} \sqrt{-g} \left[R + \frac{12}{L^2} - F_{\mu\nu} F^{\mu\nu} - \frac{2\lambda}{3\sqrt{3}} \varepsilon^{\mu\nu\alpha\beta\gamma} A_\mu F_{\nu\alpha} F_{\beta\gamma} \right] - \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^4 \mathrm{x} \sqrt{-h} K, \tag{2.1}$$

where R is the curvature scalar, L is the AdS length scale (which is fixed by the cosmological constant $\Lambda = -6/L^2$) and A_μ is the gauge potential with the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Also λ is the CS coupling constant, with $\lambda = 1$ in the minimal gauged supergravity case. Since a number of basic results do not depend on the precise value of λ, we shall keep it general in all relations below, such that (2.1) will describe a generic Einstein-Maxwell-Chern-Simons (EMCS) model. However, the numerical results will cover the SUGRA case only. Finally, K is the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric of the boundary.

The field equations of this model consist of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{6}{L^2} g_{\mu\nu} = 2 \left(F_{\mu\rho} F^\rho{}_\nu - \frac{1}{4} F^2 \right), \tag{2.2}$$

together with the Maxwell-Chern-Simons (MCS) equations

$$\nabla_\nu F^{\mu\nu} + \frac{\lambda}{2\sqrt{3}} \varepsilon^{\mu\nu\alpha\beta\gamma} F_{\nu\alpha} F_{\beta\gamma} = 0. \tag{2.3}$$

A general parametrization of the metric Ansatz which covers both the generic and the supersymmetric configurations possesses a local $SU(2) \times U(1) \times U(1)$ symmetry and reads

$$ds^2 = -F_0(r)dt^2 + F_1(r)dr^2 + \frac{1}{4} F_2(r)(\sigma_1^2 + \sigma_2^2) + \frac{1}{4} F_3(r)\big(\sigma_3 - 2W(r)dt\big)^2, \tag{2.4}$$

with σ_i the left-invariant one-forms on S^3,

$$\sigma_1 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \quad \sigma_2 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \quad \sigma_3 = d\psi + \cos d\theta d\phi,$$

the coordinates θ, ϕ, ψ being the Euler angles on S^3, with the usual range (in particular, a periodicity 4π for ψ). Also, note the existence of gauge freedom degree in the line element (2.4)), which will be fixed by convenience.

A gauge field Ansatz compatible with the symmetries of (2.4) contains an electric potential and a magnetic one, with

$$A = a_0(r)dt + \frac{1}{2} a_k(r)\sigma_3. \tag{2.5}$$

The general configurations satisfy the following set of equations which follow from (2.2), (2.3):

$$\frac{2F_2''}{F_2} + \frac{F_3''}{F_3} - \frac{F_1'F_2'}{F_1F_2} - \frac{F_1'F_3'}{2F_1F_3} - \frac{1}{2}\left(\frac{F_2'}{F_2} - \frac{F_3'}{F_3}\right)^2 - \frac{8F_1}{F_2} + \frac{2F_1F_3}{F_2^2} + \frac{F_3W'^2}{2F_0}$$
$$- \frac{12F_1}{L^2} + 2\left(\frac{a_k'^2}{F_3} + \frac{4a_k^2F_1}{F_2^2}\right) + \frac{2}{F_0}(Wa_k' + a_0')^2 = 0,$$

$$\frac{F_2'^2}{4F_2^2} + \frac{F_0'F_2'}{2F_0F_2} + \frac{F_0'F_3'}{4F_0F_3} + \frac{F_2'F_3'}{2F_2F_3} - \frac{4F_1}{F_2} + \frac{F_1F_3}{F_2^2} + \frac{F_3W'^2}{4F_0}$$
$$- \frac{6F_1}{L^2} + \left(-\frac{a_k'^2}{F_3} + \frac{4a_k^2F_1}{F_2^2}\right) + \frac{1}{F_0}(Wa_k' + a_0')^2 = 0,$$

$$\frac{F_0''}{2F_0} + \frac{F_2''}{2F_2} + \frac{F_3''}{2F_3} - \frac{1}{4}\left(\frac{F_0'^2}{F_0^2} + \frac{F_3'^2}{F_3^2} + \frac{F_2'^2}{F_2^2}\right) - \frac{F_0'F_1'}{4F_0F_1} + \frac{F_0'F_2'}{4F_0F_2} - \frac{F_1'F_2'}{4F_1F_2} + \frac{F_0'F_3'}{4F_0F_3} - \frac{F_1'F_3'}{4F_1F_3} + \frac{F_2'F_3'}{4F_2F_3}$$
$$- \frac{F_1F_3}{F_2^2} - \frac{F_3W'^2}{4F_0} - \frac{6F_1}{L^2} + \left(\frac{a_k'^2}{F_3} - \frac{4a_k^2F_1}{F_2^2}\right) - \frac{1}{F_0}(Wa_k' + a_0')^2 = 0,$$

$$\frac{F_0''}{2F_0} + \frac{F_2''}{F_2} - \frac{1}{4}\left(\frac{F_0'^2}{F_0^2} + \frac{F_2'^2}{F_2^2}\right) - \frac{F_0'F_1'}{4F_0F_1} + \frac{F_0'F_2'}{2F_0F_2} - \frac{F_1'F_2'}{2F_1F_2} - \frac{4F_1}{F_2} + \frac{3F_1F_3}{F_2^2} \qquad (2.6)$$
$$- \frac{3F_3W'^2}{4F_0} - \frac{6}{L^2}F_1 + \left(-\frac{a_k'^2}{F_3} + \frac{4a_k^2F_1}{F_2^2}\right) - \frac{1}{F_0}(Wa_k' + a_0')^2 = 0,$$

$$W'' + \left(-\frac{F_0'}{2F_0} - \frac{F_1'}{2F_1} + \frac{F_2'}{F_2} + \frac{3F_3'}{2F_3}\right)W' - \frac{4a_k}{F_3}(Wa_k' + a_0') = 0, \quad (2.7)$$

$$a_0'' + Wa_k'' + a_k'W' + \left(-\frac{F_0'}{F_0} - \frac{F_1'}{F_1} + \frac{2F_2'}{F_2} + \frac{F_3'}{F_3}\right)\frac{1}{2}(Wa_k' + a_0') - \frac{8\lambda a_k'a_k}{\sqrt{3}F_2}\sqrt{\frac{F_0F_1}{F_3}} = 0,$$

$$Wa_0'' + \left(W^2 - \frac{F_0}{F_3}\right)a_k'' + \left(-\frac{F_0'}{F_0} + \frac{F_1'}{F_1} - \frac{2F_2'}{F_2} + \frac{F_3'}{F_3}\right)\frac{F_0}{2F_3}a_k' + \left(a_0'W' + 2WW'a_k' + \frac{4a_kF_0F_1}{F_2^2}\right)$$
$$+ \left(-\frac{F_0'}{F_0} - \frac{F_1'}{F_1} + \frac{2F_2'}{F_2} + \frac{F_3'}{F_3}\right)(Wa_k' + a_0')\frac{1}{2}W + \frac{8\lambda a_ka_0'}{\sqrt{3}F_2}\sqrt{\frac{F_0F_1}{F_3}} = 0,$$

where a prime denotes a derivative w.r.t. the radial coordinate r. Also, we notice the existence of the scaling symmetry

$$F_0 \to p^2F_0, \quad a_0 \to pa_0, \quad W \to pW, \qquad (2.8)$$

with p an arbitrary nonzero constant.

We remark that one cannot take $a_0 = 0$, unless the magnetic potential also vanishes, $a_k = 0$. Also, the equations of motion possess two first integrals[3]

$$\frac{1}{2}F_2\sqrt{\frac{F_3}{F_0F_1}}(Wa_k' + a_0') - \frac{2\lambda a_k^2}{\sqrt{3}} = c_t, \qquad (2.9)$$
$$\frac{1}{8}F_2F_3\sqrt{\frac{F_3}{F_0F_1}}W' - \left(a_kc_t + \frac{2}{9}\sqrt{3}\lambda a_k^3\right) = c_W,$$

with c_t, c_W two constants of integration.

The CLP BHs are a solution of the above equations, the corresponding expression of (F_i, W) and (a_0, a_k) being given e.g. in the appendix A of ref. [21]. In practice, the non-supersymmetric solutions are found for a reparametrization of (2.4) which fixes the metric

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[3]The origin of these first integrals can be traced back to the fact that the Einstein equation E_ψ^t and the MCS equations possess a total derivative structure.

gauge and enforces the far behaviour, with

$$F_0(r) = f(r)\left(1 + \frac{r^2}{L^2}\right), \ F_1(r) = \frac{m(r)}{f(r)}\frac{1}{1+\frac{r^2}{L^2}}, \ F_2(r) = \frac{m(r)}{f(r)}r^2, \ F_3(r) = \frac{n(r)}{f(r)}r^2, \ W(r) = \frac{\omega(r)}{r}.$$
(2.10)

The supersymmetric solutions are found for a more complicated parametrization of (2.4), which is discussed in section 4.

2.2 Asymptotics

2.2.1 The solutions in the far field

The far field expression of the solutions is found assuming that (i) they approach at infinity a locally AdS spacetime, with a conformal boundary metric given by[4] (1.1), and, (ii) they possess a boundary magnetic field. As such, as $r \to \infty$, the metric functions $1/F_1(r)$ and $F_0(r)$ behave as r^2/L^2, $F_2(r)$ and $F_3(r)/v^2$ as r^2, while $W(r)$ vanishes. Also we assume that $a_k(r) \to c_m$ in the same limit, (c_m, v) being input parameters. This implies the existence of a nonvanishing asymptotic magnetic field, $F_{\theta\phi} \to -\frac{1}{2}c_m\sin\theta$, such that the parameter c_m can be identified[5] with the magnetic flux at infinity through the base space S^2 of the S^1 fibration [11],

$$\Phi_m = \frac{1}{4\pi}\int_{S^2_\infty} F = -\frac{1}{2}c_m.$$
(2.11)

One should remark that the assumptions (i) and (ii) above are not related. There exist 'magnetized' solutions possessing a round sphere at infinity [11], and also vacuum BHs with a conformal boundary geometry (1.1). However, as we shall see in section 4, the existence of a Killing spinor imposes that both (i) and (ii) should hold, i.e. $c_m \neq 0$ *and* $v \neq 1$, with a special relation between these two constants.[6]

An expression of the solution compatible with above assumptions can be constructed in a systematic way, being shared by both (extremal and nonextremal) BHs and solitons. The first terms in the large-r expansion read[7]

$$f(r) = 1 + \frac{4}{9}(1-v^2)\left(\frac{L}{r}\right)^2 + \left[\frac{\hat\alpha}{L^4} - \frac{4}{15}\left(\frac{9c_m^2}{L^2} + (1-v^2)(4v^2-3)\right)\log\left(\frac{r}{L}\right)\right]\left(\frac{L}{r}\right)^4 + ...,$$

$$m(r) = 1 - \frac{1}{9}(1-v^2)\left(\frac{L}{r}\right)^2 + \left[\frac{\hat\beta}{L^4} - \frac{4}{15}\left(\frac{3c_m^2}{L^2} - (1-v^2)(2v^2+1)\log\left(\frac{r}{L}\right)\right)\right]\left(\frac{L}{r}\right)^4 + ...,$$

$$n(r) = v^2\left(1 + \frac{17}{9}(1-v^2)\left(\frac{L}{r}\right)^2 + \left[\frac{3(\hat\alpha - \hat\beta)}{L^4} + \frac{4c_m^2}{15L^2} + \frac{1}{405}(389 - 497v^2)(1-v^2)\right.\right. \tag{2.12}$$

$$\left.\left. + \frac{8}{5}\left(8 - \frac{3c_m^2 v^2}{L^2} - (1-v^2)(7-3v^2)\log\left(\frac{r}{L}\right)\right)\right]\left(\frac{L}{r}\right)^4\right) + ...,$$

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[4]In fact, the form (1.1) is found only after a suitable rescaling, see the discussion in section 3.2.

[5]Static magnetized squashed BHs in $D = 5$ Kaluza-Klein theory were constructed in refs. [22, 23]. However, the properties of those solutions are very different as compared to the AlAdS case.

[6]Here we exclude the supersymmetric Gutowski-Reall BHs, which have $c_m = 0$ and $v = 1$, being recovered as a limit of the new solutions in this work.

[7]The occurrence of log−terms in this asymptotic expansion makes the existence of an analytic solution unlikely. Moreover, this applies also in the supersymmetric case.

$$w(r) = \hat{j}\frac{1}{r^3}+..., \quad a_0(r) = -\frac{q}{r^2}+..., \quad a_k(r) = c_m + \left(\mu - 2c_m L^2 v^2 \log\left(\frac{r}{L}\right)\right)\frac{1}{r^2}+...,$$

containing, in addition to (v, c_m), the free parameters $\{\hat{\alpha}, \hat{\beta}, \hat{j}, q, \mu\}$. In principle, $f(\infty) = f_\infty$ is also a free parameter of the far field expansion, but we can always fix it to one by means of the scaling symmetry (2.8) We observe that the first integrals (2.9), evaluated for these asymptotics, imply the following relations:

$$c_t = -\frac{2c_m^2\lambda}{\sqrt{3}} + qv, \quad c_W = \frac{4c_m^3\lambda}{3\sqrt{3}} - c_m qv - \frac{1}{2}\hat{j}v^3. \tag{2.13}$$

The CLP BHs (as well as their $\lambda \neq 1$ generalizations in [21]) have $c_m = 0$, $v = 1$, in which case no $log-$terms are present in the far field asymptotics.

2.2.2 The near-horizon expansion

In this work we shall restrict our study of solutions to the region outside the (outer) BH horizon. For non-extremal solutions, this horizon resides at $r = r_H > 0$, where the function $f(r)$ vanishes. There the solutions possess the following expansion:

$$f(r) = f_2(r - r_H)^2 - f_2\left(\frac{1}{r_H} + \frac{3r_H}{L^2 + r_H^2}\right)(r - r_H)^3 + O\left(r - r_H\right)^4,$$

$$m(r) = m_2(r - r_H)^2 - 3m_2\left(\frac{1}{r_H} + \frac{r_H}{L^2 + r_H^2}\right)(r - r_H)^3 + O\left(r - r_H\right)^4,$$

$$n(r) = n_2(r - r_H)^2 - 3n_2\left(\frac{1}{r_H} + \frac{r_H}{L^2 + r_H^2}\right)(r - r_H)^3 + O\left(r - r_H\right)^4, \tag{2.14}$$

$$\omega(r) = \omega_0 + \frac{\omega_0}{r_H}(r - r_H) + O\left(r - r_H\right)^2,$$

$$a_0(r) = a_0^{(0)} + a_0^{(2)}(r - r_H)^2 + O\left(r - r_H\right)^3,$$

$$a_k(r) = a_k^{(0)} + a_k^{(2)}(r - r_H)^2 + O\left(r - r_H\right)^3,$$

where $\{f_2, m_2, n_2, \omega_0; a_0^{(0)}, a_0^{(2)}, a_k^{(0)}, a_k^{(2)}\}$ are free coefficients.

In the quasi-isotropic coordinates we are using, the horizon of extremal black holes is located at $r_H = 0$. As a result, the behavior of the functions near the horizon changes with respect to the non-extremal case, with the occurrence[8] of non-integer powers of r. The first terms in the near-horizon expression of the solutions are

$$f(r) = \bar{f}_4 r^{2k} + \bar{f}_4^{(s)} r^{3k} + ..., \qquad m(r) = \bar{m}_2 r^{2k-2} + \bar{m}_2^{(s)} r^{3k-2} + ...,$$

$$n(r) = \bar{n}_2 r^{2k-2} + \bar{n}_2^{(s)} r^{3k-2} + ..., \qquad \omega(r) = \omega_0^{(1)} r + \omega_0^{(2)} r^{k+1} + ..., \tag{2.15}$$

$$a_0(r) = a_0^{(0)} + \bar{a}_0^{(2)} r^k ..., \qquad a_k(r) = a_k^{(0)} + \bar{a}_k^{(2)} r^k ...,$$

with $k > 2$ a number fixed by numerics. The coefficients in near-horizon solutions are determined order by order by $\{\bar{f}_4, \bar{m}_2, \omega_0^{(1)}; a_0^{(0)}, a_0^{(2)}, a_k^{(0)}, a_k^{(2)}\}$, the corresponding expressions for $\{\bar{n}_2, \bar{n}_2^{(s)}, \bar{f}_4^{(s)}, \bar{m}_2^{(s)}, \omega_0^{(2)}, etc...\}$ being, however, very complicated. Let us also

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[8]This is a consequence of the metric gauge choice used in this work. It is worth to mention that this feature occurs already for the CLP solution, when written within the metric Ansatz (2.10).

notice that the near-horizon expression of the solutions takes a simpler form when written in terms of a new radial coordinate $x = r^k$. As such, the existence of squashed $AdS_2 \times S^3$ solutions (described by the leading order terms in (2.15)) becomes transparent. They form a particular class of the EMCS-*attractors* discussed in a more general context in ref. [21].

2.2.3 Solitons: the small-r expansion

As a new feature in contrast to the CLP case ($c_m = 0$, $v = 1$), the zero horizon size limit of the generic solutions is nontrivial. For a given v, this corresponds to a one-parameter family of spinning charged solitons with nonzero global charges. Such solutions possess no horizon, while the size of both parts of the S^3-sector of the metric shrinks to zero[9] as $r \to 0$.

A small-r approximate form of the solitonic solutions can be constructed as a power series in r, being compatible with the assumption of regularity at $r = 0$. The first terms in this expansion are

$$
f(r) = f_0 + \left(\frac{m_0 - f_0}{L^2} + \frac{4u^2 f_0^2}{3m_0} \right) r^2 + ..., \quad m(r) = m_0 + \tilde{m}_2 r^2 + ..., \quad \omega(r) = w_1 r + \frac{8u^3 f_0^{5/2} \lambda}{3\sqrt{3} m_0^2} r^3 + ...,
$$

$$
n(r) = m_0 + \left(\frac{3m_0(m_0 - f_0)}{f_0 L^2} - \tilde{m}_2 + \frac{4u^2 f_0}{3} \right) r^2 + ..., \quad a_0(r) = v_0 - \left(\frac{2u^2 f_0^{3/2} \lambda}{\sqrt{3} m_0} + u w_1 \right) r^2 + ...,
$$

$$
a_k(r) = u r^2 + \frac{u}{9 f_0 L^2 m_0} \left(4u^2 f_0^2 L^2 (1 + 2\lambda^2) + 3(4m_0^2 - 3f_0(2m_0 + L^2 M_2)) \right) r^4 + ..., \tag{2.16}
$$

with the free parameters $\{f_0, m_0, \tilde{m}_2, w_1; u, v_0\}$.

Finally, let us remark that after evaluating the first integrals (2.9) for the above asymptotics, one finds that the constants c_W, c_t vanish for solitons,

$$
c_W = c_t = 0. \tag{2.17}
$$

2.3 Physical quantities

2.3.1 Event-horizon quantities

The horizon is a squashed S^3 sphere, with different sizes for the S^1 and the round S^2 parts of it. There the Killing vector

$$
\zeta = \partial_t + 2\Omega_H \partial_\psi
$$

becomes null and orthogonal to the other Killing vectors on it. For non-extremal BHs, the induced horizon metric reads

$$
d\sigma_H^2 = \frac{r_H^2}{4f_2} \left[m_2(d\theta^2 + \sin^2\theta d\phi^2) + n_2(d\psi + \cos\theta d\phi)^2 \right], \tag{2.18}
$$

which leads us to define the horizon deformation parameter

$$
\varepsilon^2 = \left. \frac{n(r)}{m(r)} \right|_{r=r_H} = \frac{n_2}{m_2}, \tag{2.19}
$$

---

[9]This contrasts with the case of topological solitons which exist inside the general solution in [24].

with m_2, n_2 and f_2 the coefficients in (2.14). The area of the horizon A_H, the Hawking temperature T_H and the horizon angular velocity Ω_H of these solutions are given by

$$A_H = 2\pi^2 r_H^3 \frac{m_2}{f_2} \sqrt{\frac{n_2}{f_2}}, \quad T_H = \frac{1}{2\pi} \left(1 + \frac{r_H^2}{L^2}\right) \frac{f_2}{\sqrt{m_2}}, \quad \Omega_H = \frac{\omega_0}{r_H}. \quad (2.20)$$

The horizon electrostatic potential Φ_H as measured in a co-rotating frame on the horizon is

$$\Phi_H = a_0^{(0)} + \Omega_H a_k^{(0)}. \quad (2.21)$$

In the extremal case, the induced horizon metric is

$$d\sigma_H^2 = \frac{\bar{m}_2}{4\bar{f}_4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{\bar{n}_2}{4\bar{f}_4}(d\psi + \cos\theta d\phi)^2, \quad (2.22)$$

while the horizon quantities are

$$A_H = 2\pi^2 \frac{\bar{m}_2}{\bar{f}_4} \sqrt{\frac{\bar{n}_2}{\bar{f}_4}}, \quad \Omega_H = \omega_0^{(1)}, \quad \Phi_H = a_0^{(0)} + \Omega_H a_k^{(0)}, \quad (2.23)$$

in terms of the constants which enter the near-horizon expansion (2.15).

2.3.2 Holographic renormalization and global charges

The global charges of the solutions are encoded in the constants $\hat{\alpha}$, $\hat{\beta}$, $\hat{j}$ and q which enter the large$-r$ expansion of the solutions (2.12). In computing them, we use the holographic renormalization of the $D = 5$ EMCS system as discussed e.g. in ref. [25]. The first step is to rewrite the solution in the standard Graham-Fefferman coordinate system [26], by defining a new radial coordinate,

$$x = r + \frac{L^2}{18r}\left(7 - \frac{5}{2}v^2\right) + \left(\frac{\hat{\alpha} - \hat{\beta}}{8} - \frac{c_m^2 L^2}{20} + \frac{L^4}{2160}(83v^4 - 94v^2 - 124)\right. \quad (2.24)$$

$$\left. + \frac{1}{15}(3c_m^2 L^2 - L^4(1 - v^2)(1 - 3v^2))\log\left(\frac{L}{r}\right)\right)\frac{1}{r^3} + \dots .$$

This results in an equivalent asymptotic form of the line element

$$ds^2 = \frac{dx^2}{\frac{x^2}{L^2}} + \frac{x^2}{L^2}\left[g_{ab}^{(0)} + \frac{1}{x^2}g_{ab}^{(2)} + \frac{1}{x^4}\left(g_{ab}^{(4)} + h_{ab}^{(4)}\log\left(\frac{x^2}{L^2}\right)\right) + \dots\right]dy^a dy^b, \quad (2.25)$$

and of the gauge field

$$A_a = A_a^{(0)} + \frac{1}{x^2}\left(A_a^{(2)} + B_a^{(2)}\log\left(\frac{x^2}{L^2}\right)\right) + \dots . \quad (2.26)$$

The boundary metric tensor h_{ab} is found by taking $x = x_0$ in (2.25), with x_0 sent to infinity in the final relations. Also, $g^{(0)}$ and $A^{(0)}$ are imposed as boundary conditions, providing the background metric and the external gauge potential for the four dimensional dual theory. For the solutions in this work one takes

$$ds^2 = g_{ab}^{(0)} dy^a dy^b = -dt^2 + \frac{L^2}{4}(\sigma_1^2 + \sigma_2^2 + v^2\sigma_3^2), \quad A_a^{(0)} dy^a = \frac{1}{2}c_m\sigma_3. \quad (2.27)$$

The terms $g^{(2)}$, $h^{(4)}$ and $B^{(2)}$ are fixed by the equations of motion, while the terms $g^{(4)}$ and $A^{(2)}$ are not determined by the field equations. Their expression can easily be found from (2.12) together with (2.24), in practice they being extracted from the numerical output.

In the next step one defines a regularized total action $I_{\text{tot}} = I + I_{\text{ct}}$, which is the sum of (2.1) and a counterterm I_{ct}, with [19, 25, 27]

$$
I_{\text{ct}} = -\frac{1}{8\pi} \int_{\partial \mathcal{M}} d^4 \text{x} \sqrt{-h} \left[\frac{3}{L} + \frac{L}{4} \text{R} + \log\left(\frac{x}{L}\right) \frac{L^3}{8} \left(\text{R}_{ab} \text{R}^{ab} - \frac{1}{3} \text{R}^3 - \frac{4}{L^2} \text{F}_{ab}^{(0)} \text{F}^{(0)ab} \right) \right],
\tag{2.28}
$$

where R_{abcd}, R_{ab}, E_{ab} denote the Riemann, Ricci and Einstein tensors, respectively, R is the Ricci scalar for the boundary metric h and $\text{F}^{(0)} = d\text{A}^{(0)}$ is the boundary U(1) field. Note that in equation (2.28) we are only considering the counterterms that cancel the power-law and logarithmic divergences at the boundary, which provide finite expressions for the charges. However, additional counterterms can be added to the action [19] (see also the discussion in [28] for AlAdS solutions in supergravity). These additional counterterms in general introduce an ambiguity in the definition of the charges, but they can be useful in order to restore some lost symmetries at the boundary.

As usual in AdS/CFT, one defines the expectation value of the stress tensor and current in the dual theory as

$$
< T_{ab} > = -\frac{2}{\sqrt{-g^{(0)}}} \frac{\delta I_{\text{tot}}}{\delta g^{(0)ab}}, \qquad < J^a > = \frac{1}{\sqrt{-g^{(0)}}} \frac{\delta I_{\text{tot}}}{\delta A_a^{(0)}},
\tag{2.29}
$$

which results in

$$
\begin{aligned}
< T_{ab} > = & -\frac{1}{8\pi} \lim_{x \to \infty} \left(\frac{x}{L}\right)^4 \left\{ \left(K_{ab} - Kh_{ab} + \frac{3}{L} h_{ab} - \frac{L}{2} E_{ab} \right) \right. \\
& - \log\left(\frac{x}{L}\right) \frac{L^3}{2} \left[\frac{1}{12} h_{ab} \text{R}^2 - \frac{1}{4} h_{ab} \text{R}_{cd} \text{R}^{cd} - \frac{1}{3} \text{R} \text{R}_{ab} + \text{R}_{acbd} \text{R}^{cd} \right. \\
& \left. \left. + \frac{1}{2} \nabla^2 \text{R}_{ab} - \frac{1}{12} (h_{ab} \nabla^2 + 2\nabla_a \nabla_b) \text{R} - \frac{4}{L^2} \left(\text{F}_{ac}^{(0)} \text{F}_b^{(0)c} - \frac{1}{4} h_{ab} \text{F}^{(0)2} \right) \right] \right\},
\end{aligned}
\tag{2.30}
$$

and

$$
< J^a > = \frac{1}{8\pi} \left[g^{(0)ab}(A_b^{(2)} + B_b^{(2)}) - \lambda \epsilon^{abcd} A_b^{(0)} F_{cd}^{(0)} \right].
\tag{2.31}
$$

Then provided that the boundary geometry has an isometry generated by a Killing vector ξ, a conserved charge

$$
\mathfrak{Q}_\xi = \int_\Sigma d^3S \ u^a \xi^b < T_{ab} >,
\tag{2.32}
$$

can be associated with a closed surface Σ [29], with $u_a = \delta_a^t$ a unit timelike vector (in general the expression (2.32) will contain a contribution from the flux $< J^a >$ [28], but this extra-term is not relevant for the particular solutions we are considering). The mass/energy M is the conserved charge associated with $\xi = \partial/\partial t$; there is also an angular momentum J associated with the Killing vector $\partial/\partial\psi$. A similar expression holds for the electric charge Q

$$
Q = \int_\Sigma d^3S \ u^a < J_a >.
\tag{2.33}
$$

It is straightforward to apply this formalism to the solutions with the asymptotics (2.12) (together with (2.24)). The nonvanishing components of the boundary stress tensor (2.31) are

$$<T_\theta^\theta> \ = \ <T_\phi^\phi> = \frac{v}{8\pi L}\left(\frac{1}{8}+\frac{(1-v^2)(3479-11057v^2)}{3240}-\frac{5(\hat\alpha-\hat\beta)}{2L^4}-\frac{32c_m^2}{15L^2}\right),$$

$$<T_\psi^\psi> \ = \ \frac{1}{8\pi L}\left(\frac{1}{8}-\frac{(1-v^2)(2537-15911v^2)}{3240}+\frac{7\hat\alpha-11\hat\beta}{2L^4}+\frac{2c_m^2}{5L^2}\right), \qquad (2.34)$$

$$<T_\phi^\psi> \ = \ \cos\theta\big(<T_\psi^\psi>-<T_\phi^\phi>\big),\ \ <T_\psi^t>=\frac{1}{\cos\theta}<T_\phi^t>=-\frac{1}{4}L^2v^2<T_t^\psi>=\frac{\hat j}{8\pi L^3},$$

$$<T_t^t> \ = \ \frac{1}{8\pi L}\left(-\frac{3}{8}-\frac{(1-v^2)(101-1883v^2)}{3240}+\frac{3\hat\alpha+\hat\beta}{2L^4}-\frac{2c_m^2}{15L^2}\right).$$

while the boundary current is

$$<J^a> = -\frac{1}{2\pi v}\left(qv-\frac{4\lambda}{3\sqrt3}c_m^2\right)\delta_0^a. \qquad (2.35)$$

After replacing these expressions in (2.32) and (2.33) one finds the global charges of the solutions

$$M = \frac{\pi v}{8}\left(-\frac{(3\hat\alpha+\hat\beta)}{L^2}+\frac{4c_m^2}{15}+\frac{3L^2}{4}+\frac{(1-v^2)L^2}{1620}(101-1883v^2)\right),\ \ J=-\frac{\hat j\pi v^3}{4}, \quad (2.36)$$

$$Q = -\pi\left(qv-\frac{4\lambda}{3\sqrt3}c_m^2\right). \qquad (2.37)$$

Note also that the total derivative structure of the MCS equations implies the existence of a conserved *Page charge*

$$Q^{(Page)} = -\frac{1}{4\pi}\int_{S_\infty^3}\left(*_5F+\frac{2\lambda}{\sqrt3}A\wedge F\right) = \frac{1}{4\pi}\int_{S_\infty^3}d\Sigma_3\left(\sqrt{-g}F^{rt}-\frac{\lambda}{\sqrt3}\varepsilon^{abc}A_aF_{bc}\right)$$

$$= -\pi\left(qv-\frac{2\lambda}{\sqrt3}c_m^2\right). \qquad (2.38)$$

The Page charge is proportional to the conserved charge c_t of (2.13), with $Q^{(Page)}=-\pi c_t$. In the standard $c_m=0$ case, the (holographic) electric charge Q and $Q^{(Page)}$ are the same. However, (2.37) and (2.38) do not coincide for solutions with a boundary magnetic field (and, in fact, the Page-charge vanishes for solitons, while $Q\neq0$).

One also notices that the stress tensor (2.34) is not traceless,

$$<T_a^a> = \frac{(1-v^2)^2}{6\pi L}-\frac{c_m^2}{2L^3\pi}, \qquad (2.39)$$

its trace consisting of two parts. The first part is due to the conformal anomaly of the boundary CFT coming from the background curvature [30–33]

$$\mathcal{A}_{(g)} = -\frac{L^3}{8\pi}\left(-\frac{1}{8}\mathsf{R}_{ab}\mathsf{R}^{ab}+\frac{1}{24}\mathsf{R}^2\right). \qquad (2.40)$$

The part of (2.39) proportional to c_m^2 results from the coupling of the CFT to a background gauge field [34]

$$\mathcal{A}_{(em)} = -\frac{L}{16\pi} F_{ab} F^{ab}. \qquad (2.41)$$

Moreover, one can verify that the following Ward identities are satisfied [25]

$$\nabla_b < T^{ab} > = F^{(0)ab} < J_b > -\frac{\lambda}{16\pi\sqrt{-g^{(0)}}} \epsilon^{bcde} F_b^{(0)a} A_c^{(0)} F_{de}^{(0)}, \qquad (2.42)$$

$$\nabla_a < J^a > = \frac{\lambda}{64\pi\sqrt{-g^{(0)}}} \epsilon^{abcd} F_{ab}^{(0)} F_{cd}^{(0)}.$$

3 Nonsupersymmetric solutions

3.1 Numerical procedure

The equations (2.6) do not seem to possess closed-form solutions with $v \neq 1$ and/or $c_m \neq 0$. Therefore all new configurations reported in this work are found numerically. The methods we have used are similar to those used in [35–38] to find other numerical solutions with equal-magnitude angular momenta in $D = 5$ EM(CS) theory.

By making use of all the available symmetries, the set (2.6) of field equations can be reduced to a system of four second-order differential equations (ODEs) for the functions (f, m, n, a_k) together with two first-order ODEs for (ω, a_0). A relation between first-order derivatives of the functions f, n, m, ω and a_k can be used as a constraint, which the numerical solutions must satisfy with a given precision.

In our numerical scheme, the input parameters are: i) the AdS length scale L, ii) the magnetic parameter c_m, iii) the boundary squashing v, iv) the constants $\hat{j}, q$ in the far field asymptotics, and, for non-extremal BHs, v) the event horizon radius r_H. The event horizon data and the coefficients at infinity α, β and μ are read from the numerical output. In practice, we fix the AdS length scale $L = 1$ and construct families of solutions by varying the other input parameters.

The equations are solved by using a professional software package [39, 40] which employs a collocation method for boundary-value ordinary differential equations and a damped Newton method of quasi-linearization. The number of mesh points used in our calculation was around 10^4, distributed non-equidistantly on x, where $x = 1 - r_H/r$ is a compactified radial coordinate employed in the BH non-extremal case; for solitons and extremal BH solutions one takes $x = r/(1 + r)$ (with $0 \leq x \leq 1$ in both cases). One should remark that the computation of global charges for these solutions is a nontrivial problem which requires a very good numerical accuracy, since the coefficients $\hat{\alpha}$, $\hat{\beta}$, appear as subleading terms in the asymptotic expansion (2.12). The typical relative accuracy of the solutions here is around 10^{-10}.

Finally, let us mention that all solutions reported in this work are regular on the horizon or outside of it.[10] Also, since $g^{tt} = -f(r) < 0$ for any $r > r_H$, while the metric functions

---

[10]For example, the Ricci or Kretschmann scalars were monitored for most of the solutions and we did not find any sign of a singular behaviour.

$m(r)$ and $n(r)$ remain strictly positive (in particular $v^2 > 0$), the solutions are also free of closed timelike (or null) curves, t being a time function (see the general discussion in [24], which covers also the framework here). Similar to the well known CLP case, the generic BHs possess, however, an ergoregion located between the horizon and the ergosurface $r = r_c$ (with $g_{tt}(r_c) = 0$).

3.2 Black holes

In the generic case, given (v, c_m), the solutions possess three independent charges M, J and Q. Therefore finding their domain of existence is a considerable task which is beyond the purposes of this paper. Instead, we shall analyze several particular classes of solutions, hoping that they capture a part of the general pattern.

3.2.1 $v = 1$: a globally AdS background

Let us start with the simplest case of solutions possessing a round S^3-part in the boundary metric. For $c_m = 0$, these are the Cvetič, Lü and Pope (CLP) BHs [6]. However, as found in the recent work [11], they possess a generalization with a nonvanishing magnetic field in the far field, which can also be described within the framework in section 2.

The results in [11] show that the qualitative behaviour of the BHs with small $|c_m|$ resembles that of the unmagnetized CLP case. However, a different picture is found for large enough values of c_m, with a monotonic behaviour of mass and horizon area as a function of temperature (also the solutions do not appear to possess an upper bound on $|c_m|$). In contrast to the CLP case, one finds BHs which have $J = 0$ but still rotate in the bulk, with a nonvanishing angular momentum density, $T_\psi^t \neq 0$. Extremal BHs with $c_m \neq 0$ exist as well, possessing generically a nonzero horizon area. Moreover, the BH solutions satisfying a certain relation between J, Q and c_m do not trivialize as $r_H \to 0$, becoming solitonic deformations of the AdS background.

3.2.2 $v \neq 1$: static, vacuum configurations

As expected, the $v = 1$ solutions in [11] possess generalizations with a squashed sphere at infinity, and new qualitative features occur as well. To simplify the problem, let us consider first the static, vacuum configurations, in which case it is possible to perform a systematic study of the solutions together with their relevant limits. These BHs are found within a consistent truncation of the general Ansatz (2.4), (2.5) with $W = 0$, $a_k = a_0 = 0$. Our numerical results clearly indicate the existence of $v \neq 1$ (static, vacuum) BH solutions of the equations (2.6), smoothly interpolating between the asymptotics (2.14) and (2.12).

These solutions are most naturally interpreted as squashed[11] BHs, being in some sense the AdS counterparts of the ($\Lambda = 0$) Kaluza-Klein solutions in [41]. Such configurations exist for an arbitrary value of the horizon size, without an extremal limit. In fact, their thermodynamics is similar to the one of the Schwarzschild-AdS BHs [42]. For any v, their temperature is bounded from below, and one finds two branches consisting of small (unstable) and large (stable) BHs. As $r_H \to 0$, a singularity-free solitonic configuration is

---

[11]Their basic properties have been discussed in a different context in [42, 43].

approached, the size of both parts of the S^3-sector of the metric shrinking to zero. The properties of these solutions are discussed in the next subsection.

Let us now explore[12] the behaviour of the squashed BHs as a function of v. For a given value of r_H, the parameter v can take arbitrary values. Apparently, as $v \to 0$, the size in the far field of the U(1) fiber over S^2 shrinks to zero, such that this limit does not seem to be well defined. However, this is not the case. Following the discussion in the appendix A, we consider an equivalent form of (2.4) which absorbs the v^2 factor in the asymptotic value of $n(r)$ via a redefinition of ψ, with

$$ds^2 = \frac{1}{f(r)} \left[m(r) \left(\frac{dr^2}{1 + \frac{r^2}{L^2}} + \frac{1}{4} r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right) + \frac{1}{4} n(r) r^2 \big(d\bar\psi + v \cos\theta d\phi \big)^2 \right]$$
$$ - f(r) \left(1 + \frac{r^2}{L^2} \right) dt^2, \tag{3.1}$$

(where $n(r) \to v^2 n(r)$, $\psi \to \bar\psi/v$) in which case $n(r) \to 1$ as $r \to \infty$. Also, the numerics shows that for small v, the size of the S^1-circle in the horizon metric (2.18) becomes proportional to v [20]. As such, the limit $v \to 0$ is smooth for the $\bar\psi$-parametrization, and describes the AdS$_5$ black strings and vortices, originally found in [15] for a different metric Ansatz. The black strings' horizon metric reads

$$ds^2 = \frac{r_H^2}{4 f_2} \left[m_2 (d\theta^2 + \sin^2\theta d\phi^2) + n_2 d\bar\psi^2 \right], \tag{3.2}$$

the horizon topology being $S^2 \times S^1$, while the conformal boundary metric is the product of time and a line element of the form (3.2). Also, the solutions possess a nontrivial $r_H \to 0$ limit describing AdS vortices. We note that for both black strings and vortices, the range of $\bar\psi$ (usually denoted as z-coordinate in the literature) is not fixed a priori.[13]

No upper bound seems to exist for the value of v, although the numerical integration becomes more difficult as we increase this parameter, with the mass M diverging as $v \to \infty$. However, a careful analysis of this limit reveals the existence of a different solution of the field equations. Following again the discussion in the nutty instanton case (see appendix A), we define the scaled coordinates

$$r = \lambda \bar r, \quad \theta = \frac{\Theta}{\lambda}, \quad \psi = -\frac{\Psi}{v\lambda} - \phi, \tag{3.3}$$

together with

$$v = \lambda N . \tag{3.4}$$

Then as $\lambda \to \infty$ the line element (2.4) (with $W = 0$) becomes

$$ds^2 = \bar F_1(\bar r) d\bar r^2 + \bar F_2(\bar r) \frac{1}{4} (d\Theta^2 + \Theta^2 d\phi^2) + \bar F_3(\bar r) \frac{1}{4} \left(d\Psi + 2N \left(\frac{\Theta}{2} \right)^2 d\phi \right)^2 - \bar F_0(\bar r) dt^2, \tag{3.5}$$

---

[12]In understanding the limiting-v behaviour, some useful hints are provided by the nutty-instanton toy model in appendix A.

[13]However, for black strings, the Gregory-Laflamme instability [44] implies the existence of a critical periodicity of the $\bar\psi$-coordinate for a given value of the mass [45].
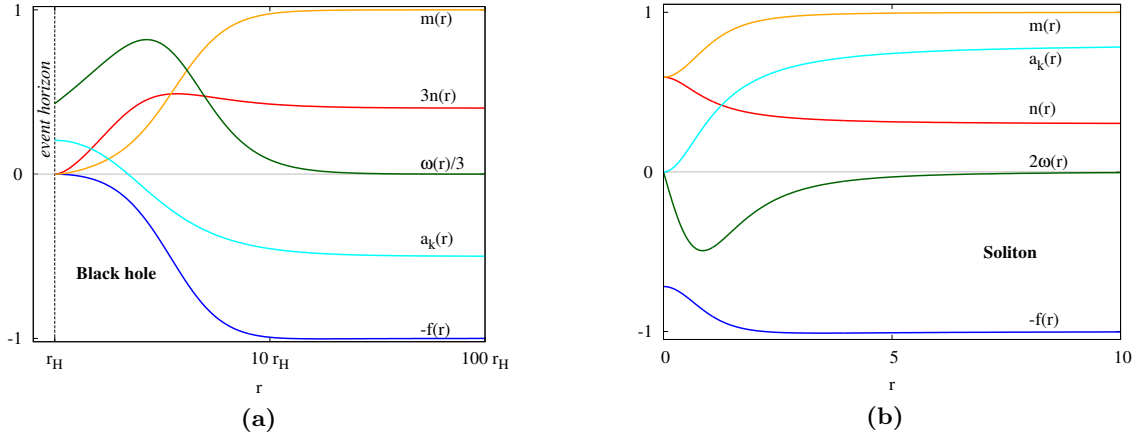
Figure 3. (a) Metric and gauge functions are shown for a typical squashed magnetized black hole. The input parameters here are $L = 1$, $r_H = 0.492$, $Q = -3.19$, $J = -1.33$, $c_m = -0.5$ and $v^2 = 0.133$. (b) Same functions for a squashed magnetized soliton with parameters $L = 1$, $c_m = 0.8$ and $v^2 = 0.3$.

which corresponds to a 'twisted' black brane configuration. The horizon is located again at some $\bar{r} = \bar{r}_H$, with an induced horizon geometry

$$ds^2 = \bar{F}_2(\bar{r}_H)\frac{1}{4}(d\Theta^2 + \Theta^2 d\phi^2) + \bar{F}_3(\bar{r}_H)\frac{1}{4}\left(d\Psi + 2N\left(\frac{\Theta}{2}\right)^2 d\phi\right)^2. \tag{3.6}$$

In the absence of an analytical solution, the expression of $F_i(\bar{r})$ is found numerically.[14] We also remark that they do not possess a solitonic limit. Their conformal boundary metric is[15]

$$ds^2 = \frac{1}{4}L^2\left[d\Theta^2 + \Theta^2 d\phi^2 + \left(d\Psi + 2N\left(\frac{\Theta}{2}\right)^2 d\phi\right)^2\right] - dt^2, \tag{3.7}$$

(with $0 \leq \Theta < \infty$, $0 \leq \phi < 2\pi$ and an arbitrary periodicity for Ψ). Although a $t = const.$ surface is topologically a direct product of Ψ and the (Θ, ϕ)- plane, the product is "twisted" (or warped), and the boundary is not flat (its Ricci scalar is proportional to N^2). More details of the limiting solutions can be found in appendix B.

3.2.3 $v \neq 1$: the generic case

Increasing the complexity of the solutions, we first notice that the (vacuum, static) BHs of the previous subsection possess spinning generalizations. They are interpreted as squashed rotating BHs, their thermodynamics being similar to that of the ($v = 1$) Myers-Perry-AdS BHs with two equal angular momenta.

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[14]Note that the limit $N = 0$ corresponds to a Schwarzschild black brane, with $F_0 = 1/F_1 = \bar{r}^2/L^2 - \bar{r}_H^2/L^2$, $F_2 = F_3 = \bar{r}^2$.

[15]It is interesting to note that (3.7) corresponds to an analytical continuation of the Som-Raychaudhuri spacetime [46].

Static, electrically charged BHs with $v \neq 1$ exist as well. These solutions are generalizations of the Reissner-Nordström-AdS BHs, with a squashed horizon (and a squashed sphere at infinity) and possess similar thermal properties.

What $J \neq 0$ or $Q \neq 0$ brings new is the absence of a smooth particle-like solitonic limit[16] (we recall $c_m = 0$). Instead, one notices the existence of extremal BHs with a nonzero horizon area, which are smooth on and outside the horizon.

Moreover, as expected, spinning $v \neq 1$ solutions with $Q \neq 0$ and $c_m = 0$ exist as well. They can be interpreted as squashed counterparts of the CLP BHs and appear to share all their basic properties. Again, these unmagnetized solutions do not possess a smooth solitonic limit. Also, their behaviour in terms of the squashing parameter v is similar to the one in the vacuum static case. In particular, the limit $v = 0$ describes a generalization of the AdS black strings in [15] with a nonzero electromagnetic field and a momentum along the ψ-direction.

However, these limits are in some sense less interesting, since, as we shall see in the next section, they do not allow for supersymmetric solutions. Therefore let us now consider the general case of spinning, magnetized solutions with a squashed sphere at infinity. As a general remark, our numerical results show that they share a number of basic properties of the $v = 1$ solutions with $c_m \neq 0$ in [11]. For example, for large enough J, Q, the magnetic field induces subleading effects only, and we recover the general pattern found for the CLP BHs. Also, these generic solutions do not allow for a smooth black string limit as $v \to 0$. However, the limit $v \to \infty$ is well defined, describing (after a suitable rescaling) a new family of twisted charged black branes. Although the asymptotics of these solutions is very similar to (2.14), (2.12) they possess a number of distinct properties, see the discussion in appendix B.

The typical profiles of the metric and gauge functions of squashed magnetized black holes are presented in figure 3a, for the a typical magnetized spinning BH (note the absence of nodes in the profile of the magnetic potential, a feature which holds for all configurations in this work, including the solitonic ones).

Let us now discuss some thermodynamical properties of these solutions, as shown in figure 4. The configurations there have a fixed value of the squashing parameter, $v = 1.56$, and of the magnetic parameter, $c_m = -1$. This choice implies that the trace (2.39) of the boundary stress tensor vanishes, $< T_a^a >= 0$, a condition which is a requirement for the existence of supersymmetric solutions (see the discussion in the next section). Several families of BHs are shown in that figure (with different color lines) as a function of the temperature T_H, each one possessing different values of the charges Q and J.

One can see that in general, the BHs possess an extremal limit, $T_H = 0$. However, the situation is different for BHs with a specific relation between J and Q as given by (3.9) (shown with the blue line in figure 4). For these solutions, the temperature diverges as the horizon size decreases, a smooth solitonic configuration being approached as the horizon size shrinks to zero. Anticipating the discussion in section 4, we mention that in general,

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[16]This feature can be understood from the results in section 2. The constants c_t and c_W necessarily vanish for solitons. Then the first integrals (2.9), imply that $W = 0$ and also $a_0 = 0$.
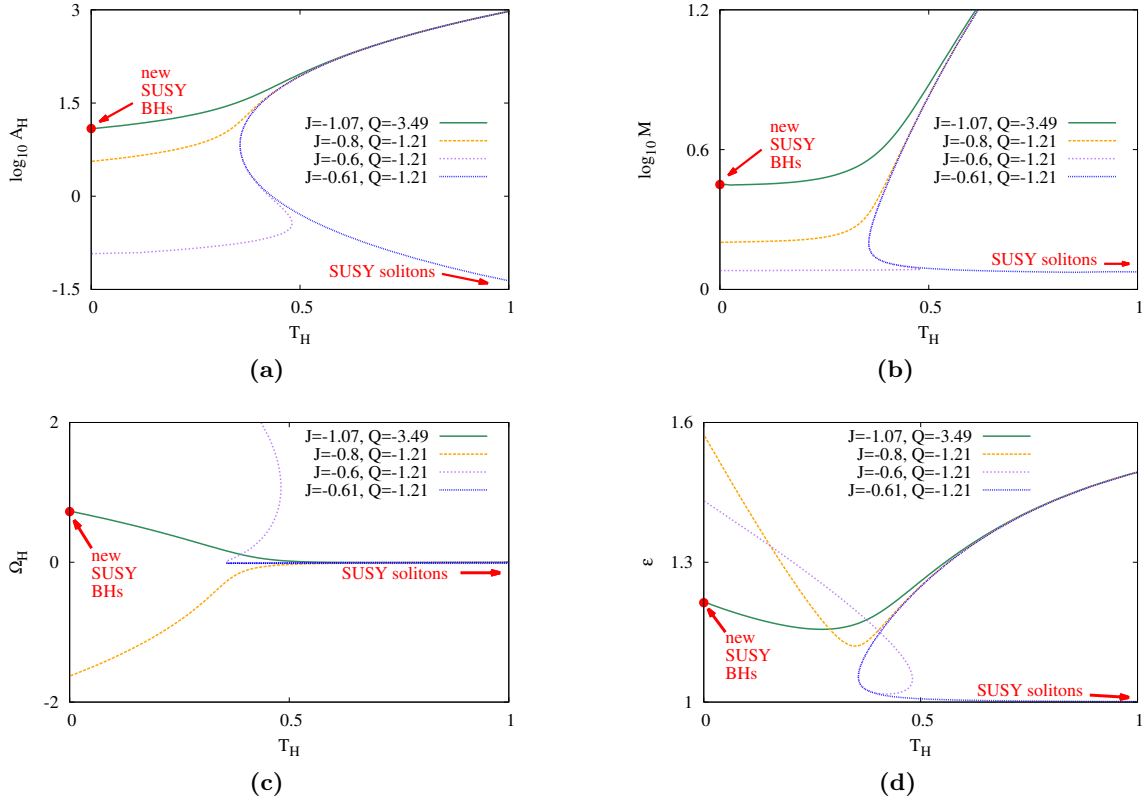
Figure 4. Several properties of squashed magnetized black holes are shown as a function of the horizon temperature. The configurations have $v = 1.65$, $c_m = -1$, $L = 1$ and different values of Q and J. The generic solutions possess an extremal (non-susy) limit (cyan, orange and purple lines). However, a particular set of solutions (red line in this figure) possess an extremal limit resulting in a supersymmetric black hole (red point). A different set of solutions (blue line) form a soliton in the limit of a vanishing horizon, $T_H \to \infty$, which corresponds to the susy configuration in ref. [19].

the extremal BHs are not supersymmetric. However, for particular values of the charges and the squashing, the limiting solutions satisfy the susy equations. This is the case of the extremal black hole marked with a red point (and also the solitonic limit of the blue line).

In figure 4a we show the event horizon area A_H vs. the temperature T_H. Depending on the values of Q and J, the area can be a monotonic function of T_H (red, cyan and orange lines); a more complicated picture is also allowed, with regions where A_H decreases with increasing temperature (purple and blue lines). In the solitonic limit of the blue line, the area vanishes as $T_H \to \infty$, although the limiting solitons are regular everywhere. Also note that this family of BHs is special, with a finite minimum temperature.

In figure 4b we present a similar figure for the mass M vs the temperature T_H. In general, the behaviour of M is similar to the area. However, in the solitonic limit the mass does not vanish, reaching a finite value, which, as we shall see, depends on the soliton charges and squashing.

The horizon angular velocity Ω_H is shown in figure 4c, again as a function of T_H. As an interesting feature, we note that the configurations can present a counter-rotating horizon,
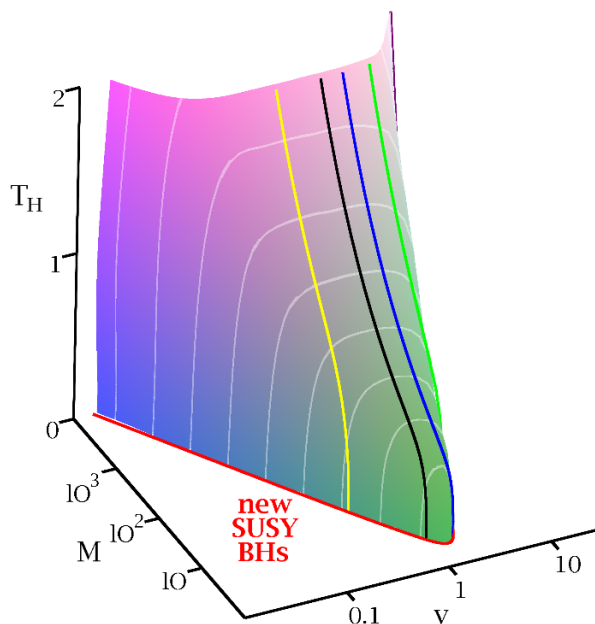
Figure 5. Mass M vs. squashing parameter v vs. temperature T_H for a particular set of black holes. Their magnetization parameter c_m, electric charge Q and angular momentum J are given functions of v such that the extremal limit, $T_H = 0$, corresponds to the susy black holes in section 4 (red line).
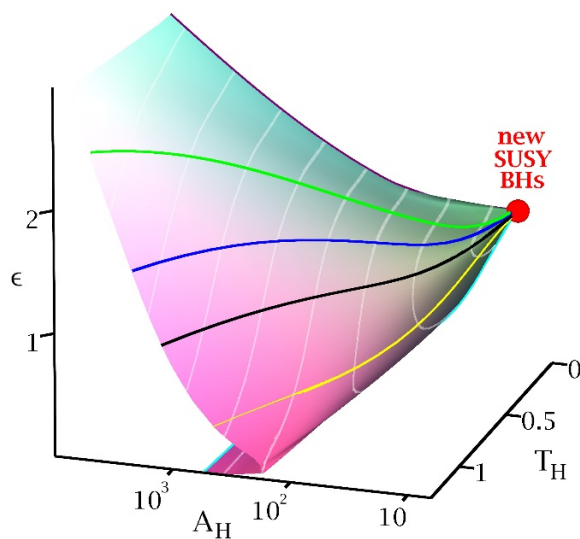
Figure 6. Horizon area A_H vs. horizon deformation ϵ vs. temperature T_H for the same set of solutions in figure 5. In the extremal limit, $T_H = 0$, the susy black holes in section 4 are obtained, all of them possessing the same horizon properties and being represented by a red point in this figure.
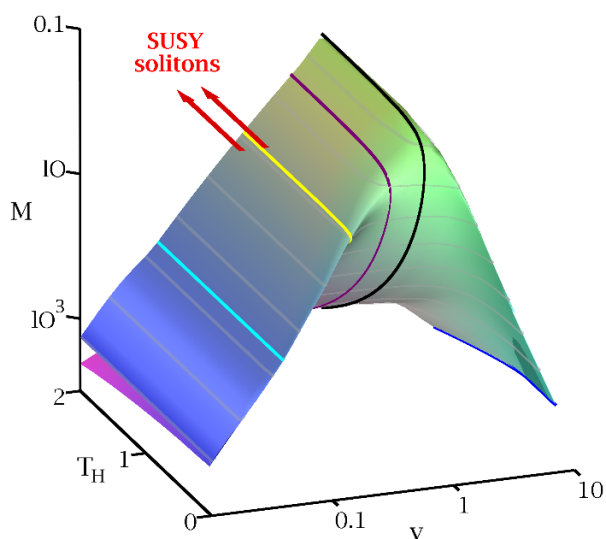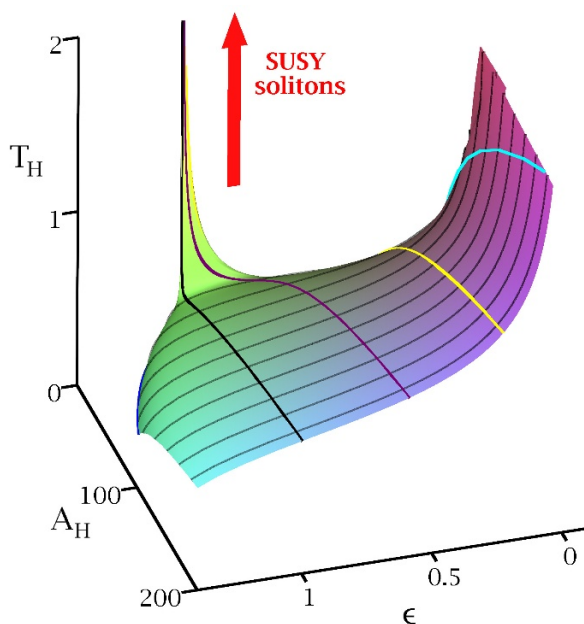
Figure 7. Mass M vs. squashing v vs temperature T_H for a particular set of solutions. Their magnetization parameter c_m, electric charge Q and angular momentum J are given functions of v such that no extremal black holes are obtained. Instead, the limit $T_H \to \infty$ describe susy solitons.

Figure 8. Area A_H vs. horizon deformation ϵ vs. temperature T_H for black holes is shown for the same solutions in figure 7. As the temperature diverges, the horizon area vanishes and the susy solitons are recovered.

depending on the specific combination of the charges. In the solitonic limit, the angular velocity vanishes since there is no horizon.

In figure 4d we present the horizon deformation ϵ (as given by rel. (2.19)) as a function of T_H. Although the squashing v of these particular sets is fixed to $v = 1.56$, the horizon

deforms depending on the black hole electric charge Q, and angular momentum J (and in general, on the magnetization parameter, which in these figures is also fixed). Although $\epsilon \geq 1$ for the solutions there, the horizon deformation can also be smaller than one for a different choice of the input parameters.

Other properties of the solutions are shown in figures 5, 6. The set of solutions presented there have the value of the magnetization parameter fixed by the squashing v, with $c_m = L(1 - v^2)/\sqrt{3}$, such that the trace (2.39) of the boundary stress tensor vanishes. Also, they have fixed values of the integration constants which enter the 1st integrals (2.9), $c_t = \frac{203\sqrt{3}}{484}L^2$ and $c_W = -\frac{2303}{5324}L^2$. As a result, one can see from (2.9) that the solutions' electric charge and angular momentum possess a dependence on the squashing v as given in the corresponding relations in (4.35). Therefore, for a given v, they form a one parameter family of solutions which are constructed by varying the value of the horizon radius r_H (note that other quantities of interest of the solutions (e.g. mass, horizon area and temperature) are unconstrained, being read from the numerical output).

The reason for this special choice of Q, J and c_m is that the extremal limit of these solutions possesses some special properties, being supersymmetric, as we will see in the next section. Here they are constructed directly, as solutions of the second-order equations of motion.

In figure 5 we show the mass M as a function of the squashing parameter v and the Hawking temperature T_H. We can see that M strongly increases as v decreases, with the existence of a minimal value for a given v. In figure 6 we show the event horizon area A_H as a function of the horizon deformation ϵ and the Hawking temperature T_H. Note that in the extremal limit, all solutions converge to a single point (in red), meaning that all the extremal BHs possess the same horizon properties. An explanation of this feature will be provided in the next section when studying the susy squashed BHs

Further properties of BHs are shown in figures 7, 8. There the choice of the magnetization parameter is the same as above, $c_m = L(1 - v^2)/\sqrt{3}$, and $c_t = c_W = 0$. This implies that Q and J possess a different dependence on v as given in relation (4.4) below, other quantities being determined by the value of r_H. As such, these configurations provide a different cut in the parameter space of solutions, the set of susy solitons being recovered in the vanishing horizon size limit (see the discussion below).

We mention that we have also studied families of solutions with $< T_a^a > \neq 0$ (which thus do not possess a supersymmetric limit). The picture we found here is similar to the generic case in figure 4 (red, cyan and orange lines), with the existence of an extremal limit possessing a nonzero horizon area.

3.3 Solitons

We start the discussion of the solitonic solutions of the model with the vacuum static case. These configurations naturally emerge as the zero horizon size limit of the corresponding families of BHs discussed in the section 3.2.2, representing deformations of the globally AdS$_5$ spacetime. As such, their most natural interpretation is as providing a *background* for models with given geometric parameter v. As expected their mass is nonvanishing, being shown in figure 9 as a function of v. These configurations can also be viewed as the
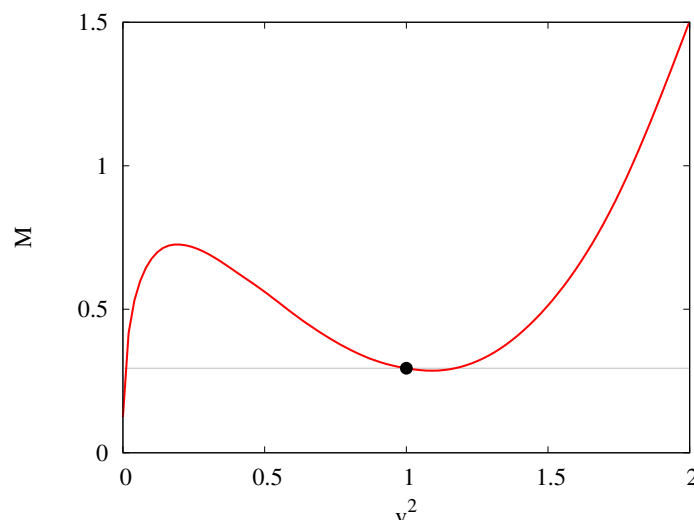
Figure 9. Mass M is shown *vs.* the deformation parameter v^2 for the squashed backgrounds (vacuum static spacetimes). The horizontal line and the dot mark the $v = 1$ mass, corresponding to the Casimir value $3\pi L^2/32$, with $L = 1$.

zero horizon size limit of families of charged and/or spinning BHs with $c_m = 0$. However, as already discussed above, both J and Q vanish as $r_H \to 0$.

One feature a nonzero boundary magnetic field brings new is the existence of a non-trivial limit of the solutions which describes spinning electrically charged solitons. Similar to the vacuum case, they possess no horizon, while the size of both parts of the S^3-sector of the metric shrinks to zero as $r \to 0$.

An interesting property of the solitons is that their electric charge and angular momentum are proportional. To prove it, we notice that since $c_t = c_W = 0$, they satisfy the simple relations[17]

$$qv = \frac{2c_m^2\lambda}{\sqrt{3}}, \quad \hat{j} = -\frac{2}{3v^2}c_m q. \tag{3.8}$$

Then, after expressing $\hat{j}$, c_m and q and in terms of the angular momentum J (as given by rel. (2.36)), the magnetic flux at infinity Φ_m (rel. (2.11)), and the electric charge Q (rel. (2.37)) one finds

$$J = \Phi_m Q, \quad \text{with} \quad Q = -\frac{8\pi\lambda}{2\sqrt{3}}\Phi_m^2 . \tag{3.9}$$

These relations are universal, being satisfied for any value of the CS coupling constant $\lambda \neq 0$.

As we have already seen in figure 4 the solitons appear as the $r_H \to 0$ limit of particular black hole solutions which satisfy (3.9). Further insight on the properties of soliton solutions can be found in figure 8. There the horizon area A_H is plotted as a function of the horizon deformation ϵ and the temperature T_H for a set of BH solutions with $c_t = c_W = 0$ and $c_m = L(1 - v^2)/\sqrt{3}$. In this case, the BHs only depend on the squashing parameter v and the temperature T_H. The lines in yellow, black, blue and purple represent sets of solutions

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[17]These relations can also be viewed as a consequence of the vanishing of the Page charge for solitons.

with constant values of v. The non-extremal solutions present a limit in which $T_H \to \infty$, the area vanishes and the horizon deformation goes to one. This limit comprises a whole family of solitons.

In figure 7 we represent the mass M of BHs as a function of the squashing parameter v and the temperature T_H for the same solutions as in figure 8. The solitons are approached in the limit of vanishing area and diverging temperature. This results in a family of solutions with finite mass and varying squashing parameter v (we recall that the magnetic parameter c_m is determined by v such that the boundary stress tensor (2.39) is traceless). We have verified that these solutions correspond in fact to the supersymmetric configurations in [19] (this provides another useful test of our numerical results). More details on these special solutions is presented in the next section.

Finally, let use remark that the considered choice $c_m^2 = L^2(1-v^2)^2/3$ is not a necessary condition for the existence of solitons (while (3.9) is mandatory). In fact, solitonic solutions in a globally AdS background ($v = 1$) have been already reported in ref. [11], most of their properties being recovered for other values of v.

4 Supersymmetric solutions

The only known supersymmetric BHs within the framework in section 2 are those found by Gutowski and Reall in ref. [10]. They possess a round sphere at infinity ($v = 1$) and no boundary magnetic field ($c_m = 0$), being a special limit of the CLP solution. These solutions contain a single parameter, $\alpha > 1/2$, the global charges being given by

$$
\begin{aligned}
M &= \frac{\pi L^2}{216}(3\alpha^2 - 1)(31 + 76\alpha^2 + 64\alpha^4) + \frac{3\pi L^2}{32}, \\
J &= -\frac{\pi L^3}{216}(1 - 4\alpha^2)^2(7 + 8\alpha^2), \\
Q &= \frac{\pi L^2}{12\sqrt{3}}(1 - 4\alpha^2)(5 + 4\alpha^2),
\end{aligned}
\tag{4.1}
$$

while the horizon area, horizon angular momentum, electrostatic potential and horizon angular velocity are

$$
\begin{aligned}
A_H &= \frac{\pi^2 L^3}{3\sqrt{3}}(4\alpha^2 - 1)\sqrt{(4\alpha^2 + 3)(4\alpha^2 - 1)}, \\
J_H &= \frac{\pi L^3}{384}\left(4\alpha^2 - 1\right)^2\left(4\alpha^2 + 3\right), \quad \Omega_H = \frac{1}{L}, \quad \Phi = \frac{1 + 2\alpha^2}{\sqrt{3}}.
\end{aligned}
\tag{4.2}
$$

It is natural to inquire if the more general squashed magnetized solutions in this work also possess a supersymmetric limit. A hint in this direction comes from the observation [47] that the two contributions in the trace $< T_a^a >$ of the stress tensor (2.39) exactly cancel for

$$
c_m = \pm \frac{L}{\sqrt{3}}(1 - v^2).
\tag{4.3}
$$

In the extremal BH case, this requirement leads to a two parameter family of solutions (the parameters can be taken as the squashing v and the electric charge Q). However,

the situation changes for solitons, the above condition together with the charge-angular momentum relation (3.9) leading to a single family of solutions which can be parametrized in terms of v. We have found numerically that these solutions correspond[18] in fact to the supersymmetric solitons in [19]. Although they cannot be written in closed form, the supersymmetry allows for an almost complete description of the solution in terms of the squashing parameter v. The mass, angular momentum and electric charge of the supersymmetric solitons are given by [19]

$$M = \pi L^2 \left(\frac{5}{288} + \frac{2}{27v^2} - \frac{7}{36}v^2 + \frac{89}{864}v^4 \right), \ J = \frac{\pi L^3}{27} \left(v^2 - 1 \right)^3, \ Q = -\frac{2\pi L^2}{9\sqrt{3}} \left(v^2 - 1 \right)^2 . \tag{4.4}$$

We have already shown in the previous section that these supersymmetric solitons can be connected with squashed magnetized black holes in the limit of vanishing size of the horizon. In what follows, we show that, in addition to these solitons, the EMCS equations possess as well a one-parameter family of supersymmetric BHs.

4.1 The formalism

In constructing the supersymmetric solutions which fit the framework in section 2, the most convenient approach is to use the general formalism proposed in [10]. Then such configurations have the following line element:[19]

$$ds^2 = -f^2(\rho) \left(dy + \Psi(\rho)\hat{\sigma}_3 \right)^2 + \frac{1}{f(\rho)} \left[d\rho^2 + a^2(\rho)(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + b^2(\rho)\hat{\sigma}_3^2 \right], \tag{4.5}$$

with

$$\hat{\sigma}_1 = -\sin\hat{\psi}d\theta + \cos\hat{\psi}\sin\theta d\phi, \ \ \hat{\sigma}_2 = \cos\hat{\psi}d\theta + \sin\hat{\psi}\sin\theta d\phi, \ \ \hat{\sigma}_3 = d\hat{\psi} + \cos\theta d\phi,$$

and the gauge potential

$$A = \frac{\sqrt{3}}{2} \left[f(\rho)\mathrm{d}y + \left(f(\rho)\Psi(\rho) + \frac{L}{3}p(\rho) \right) \hat{\sigma}_3 \right]. \tag{4.6}$$

Thus the framework contains five functions $\{a, b, p, f, \Psi\}$, instead of six as in the generic case. However, the expression of $\{b, p, f, \Psi\}$ is fixed by a, via the following relations (which are found from the corresponding Killing spinor equations) [10]:

$$\begin{aligned}
b &= 2aa', \\
p &= 4a'^2 + 2aa'' - 1, \\
f^{-1} &= \frac{L^2}{12a^2a'}[4(a')^3 + 7a\,a'a'' - a' + a^2a'''], \\
\Psi &= -\frac{La^2}{4} \left(\nabla^2 f^{-1} + 8L^{-2}f^{-2} - \frac{L^2g^2}{18} + f^{-1}g \right),
\end{aligned} \tag{4.7}$$

---

[18]Although one cannot exclude the existence of non-susy excitations of these solutions, so far we have no indication for that.

[19]To agree with the standard notation in the literature, in this section we use ρ for the radial coordinate, instead of r as in (4.5).

where we denote $g = -\frac{a'''}{a'} - 3\frac{a''}{a} - \frac{1}{a^2} + 4\frac{(a')^2}{a^2}$. The function a is the solution of a sixth order equation

$$\left(\nabla^2 f^{-1} + 8L^{-2}f^{-2} - \frac{L^2 g^2}{18} + f^{-1}g\right)' + \frac{4a'g}{af} = 0\,, \tag{4.8}$$

where $\nabla^2 = \frac{d^2}{d\rho^2} + (\frac{2a'}{a} + \frac{b'}{b})\frac{d}{d\rho}$. Any solution to this equation (together with (4.7), (4.6)) corresponds to a configuration which preserves at least one quarter of the supersymmetry. Let us also remark that the $a-$equation (4.8) is invariant under the transformation

$$\rho \to \lambda\rho,\ a \to a/\lambda, \tag{4.9}$$

with $\lambda > 0$.

The only (known) closed form solution of the eq. (4.8) which describes a BH has been found by Gutowski and Reall (GR) and has

$$a(\rho) = \alpha L \sinh\left(\frac{\rho}{L}\right), \tag{4.10}$$

with $\alpha > 1/2$ a real parameter (the value $\alpha = 1/2$ corresponding to the globally AdS background)).

Also, one notes that the line element (4.5) can be put into the form (2.4) by taking

$$F_1 = \frac{1}{f},\ \ F_2 = \frac{4a^2}{f},\ F_3 = \frac{4}{f}(b^2 - \Psi^2 f^3),\ F_0 = \frac{b^2 f^2}{b^2 - \Psi^2 f^3},\ W = \frac{1}{2}\left(\frac{f^3\Psi}{b^2 - f^3\Psi^2} - u\right), \tag{4.11}$$

with

$$t = y,\ \ \psi = \hat{\psi} - ut\,. \tag{4.12}$$

The reason we introduce the constant u in the above relations is that the supersymmetric solutions are found in a frame which rotates at infinity, with

$$\lim_{\rho\to\infty}\frac{f^3\Psi}{b^2 - f^3\Psi^2} = u \neq 0. \tag{4.13}$$

Then the transformation (4.12) brings the solution to a static frame at infinity, such that the solutions become a particular limit of the general case in sections 2, 3. For example, the corresponding expression of the U(1) potential (2.5) reads

$$a_0 = \frac{\sqrt{3}}{2}\left[f(\rho) + u\left(f(\rho)\Psi(\rho) + \frac{L}{3}p(\rho)\right)\right],\ \ a_k = \frac{\sqrt{3}}{2}\left(f(\rho)\Psi(\rho) + \frac{L}{3}p(\rho)\right). \tag{4.14}$$

4.2   The large-ρ expansion

Despite the absence of an exact solution in the general squashed magnetized case, it is still possible to find an approximate solution at the limits of the domain of integration. Keeping the notation of ref. [19], the first terms in the far field expansion of $a(\rho)$ read

$$\frac{a(\rho)}{L} = a_0 e^{\frac{\rho}{L}} + \left(a_2 + c\frac{\rho}{L}\right)\frac{e^{-\frac{\rho}{L}}}{a_0} + \left(a_4 + \frac{2 - 16a_2 - 5c}{12}c\frac{\rho}{L} - \frac{2}{3}c^2\left(\frac{\rho}{L}\right)^2\right)\frac{e^{-\frac{3\rho}{L}}}{a_0^3} \tag{4.15}$$

$$+ \left(a_6 + \frac{1}{972}(12 - 282a_2 + 1488a_2^2 - 1548a_4 - 54c + 537a_2c + 59c^2)c\frac{\rho}{L}\right.$$

$$\left. - \frac{90 - 840a_2 - 197c}{324}c^2\left(\frac{\rho}{L}\right)^2 + \frac{70}{81}c^3\left(\frac{\rho}{L}\right)^3\right)\frac{e^{-\frac{5\rho}{L}}}{a_0^5} + \mathcal{O}(e^{-6\rho/L}) + \ldots,$$

containing the free parameters $\{c, a_0, a_2, a_4, a_6\}$ (with $a_0 \neq 0$). Then it is straightforward to derive from (4.7) the asymptotic form of the other functions $\{b, p, f, \Psi\}$. However, these expressions are rather complicated and we shall not include them here.[20]

Instead, it is interesting to give the asymptotic expansion of the metric functions F_i, W which enter the metric Ansatz (2.4). The analysis is simplified by introducing a new radial coordinate

$$r = Le^{\frac{\rho}{L}}, \tag{4.16}$$

such that the far field expansion resembles (2.12)

$$F_0(r) = \frac{4a_0^2}{(1-4c)}\left(\frac{r}{L}\right)^2 + \frac{2}{3(1-4c)}\left(1 + 4a_2 + 4c + 4c\log\left(\frac{r}{L}\right)\right) + \dots,$$

$$F_1(r)\frac{r^2}{L^2} = 1 - \frac{1 + 16a_2 + 4c + 16c\log\left(\frac{r}{L}\right)}{12a_0^2}\left(\frac{L}{r}\right)^2 + \dots,$$

$$F_2(r) = 4a_0^2 r^2 + \frac{L^2}{3}\left(-1 + 8a_2 - 4c + 8c\log\left(\frac{r}{L}\right)\right) + \dots,$$

$$F_3(r) = 4a_0^2(1-4c)r^2 - \frac{(1-4c)L^2}{3}\left(1 - 8a_2 - 20c - 8c\log\left(\frac{r}{L}\right)\right) + \dots, \tag{4.17}$$

$$W(r) = \frac{1}{7776a_0^4(1-4c)^2 L}\Big(63 + 576a_2(3 + 12a_2 - 128a_2^2) - 1152a_4(40c + 516a_2 - 15)$$

$$- 373248a_6 + c(1860 - 5856a_2 - 51456a_2^2) + 8c^2(432a_2 + 3436c - 1935)\Big)\left(\frac{L}{r}\right)^4 + \dots,$$

while the asymptotic form of the gauge potentials is

$$a_k(r) = -\frac{2cL}{\sqrt{3}} + \frac{L}{96\sqrt{3}a_0^2}\Big(1 + 256a_2^2 + 384a_4 - 32a_2(7c - 1) \tag{4.18}$$

$$+ 8c(17c - 4) - 96c(4c - 1)\log\left(\frac{r}{L}\right)\Big)\left(\frac{L}{r}\right)^2 + \dots,$$

$$a_0(r) = \frac{4c - 3}{2\sqrt{3}(4c - 1)} + \frac{1}{48\sqrt{3}a_0^2(4c - 1)}\Big(-5 + 256a_2^2 + 384a_4$$

$$+ 32a_2(5c - 2) + 8c(29c - 4)\Big)\left(\frac{L}{r}\right)^2 + \dots.$$

One can see that $W \to 0$ as $r \to \infty$, such that the solution is indeed written in a nonrotating frame at infinity.[21]

Then, from (4.18), one finds that the constant c in the far field expansion corresponds to the magnetic flux parameter

$$c_m = \frac{c}{\sqrt{3}}. \tag{4.19}$$

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[20]The corresponding expression for supersymmetric solutions can be found in the appendix A of ref. [19]. Since the generic solitons and BHs possess the same far field expansion, the expressions there are valid also for the solutions in this work.

[21]Here we have used $u = \lim_{\rho \to \infty} \frac{f^3\Psi}{b^2 - f^3\Psi^2} = \frac{2}{4c-1}$, and replaced it in (4.12).

Also, one can see that the solution necessarily possesses a squashed sphere at infinity,[22] with

$$\lim_{r \to \infty} \frac{F_3}{F_2} = 1 - 4c = v^2 \,, \qquad (4.20)$$

the zero-trace condition (4.3) being satisfied. The squashing parameter v takes arbitrary values, the solutions with $v^2 < 0$ possessing CTCs.

An important observation here is that after evaluating the first integrals (2.9), for the asymptotic form (4.17), the constants a_4 and a_6 can be expressed in terms of c_t, c_W together with a_2, c:

$$a_4 = -\frac{13}{48}c^2 + (-5\,a_2 + 1)\frac{c}{12} - \frac{2}{3}\,a_2{}^2 + \frac{a_2}{6} + \frac{5}{384} + \frac{1}{32}\frac{\sqrt{3}c_t}{L^2}, \qquad (4.21)$$

$$a_6 = \frac{1105\,c^3}{11664} + \left(\frac{1913\,a_2}{3888} - \frac{125}{1944}\right)c^2 + \left(\frac{197\,a_2{}^2}{324} - \frac{61\,a_2}{324} + \frac{25}{3456} - \frac{19\,\sqrt{3}c_0}{2592\,L^2}\right)c \,,$$

$$+\frac{70\,a_2{}^3}{81} - \frac{5\,a_2{}^2}{18} - \frac{29\,a_2}{3456} - \frac{43\,\sqrt{3}a_2\,c_0}{864\,L^2} + \frac{1}{1296} + \frac{5\,\sqrt{3}c_t}{3456\,L^2} + \frac{c_W}{384\,L^3} \,. \qquad (4.22)$$

4.3 The near-horizon expansion

Without any loss of generality, the horizon is located at $\rho = 0$. There one assumes the existence of a power series expansion for the function $a(\rho)$, with

$$a(\rho) = L \sum_{k \geq 0} \alpha_k \left(\frac{\rho}{L}\right)^k \,. \qquad (4.23)$$

Then, after replacing the above expression in the sixth-order equation (4.8) and solving order by order in k, one finds that the problem possesses (at least) two independent solutions describing the near-horizon of a BH. The argument goes as follows. First, the existence of a horizon requires $\alpha_0 = 0$. Then, to lowest order the eq. (4.8) implies the algebraic relation

$$\frac{(8 + 13\alpha_1^2)\alpha_2}{\alpha_1^3} = 0, \qquad (4.24)$$

which implies $\alpha_2 = 0$ and $\alpha_1 \neq 0$. The next order relation reads

$$\frac{(-8 + 11\alpha_1^2)\alpha_4}{\alpha_1^3} = 0, \qquad (4.25)$$

which admits two independent solutions. The first one has $\alpha_4 = 0$, and leads to an expression for $a(\rho)$ containing odd powers of ρ only, with

$$\frac{a(\rho)}{L} = \alpha_1 \frac{\rho}{L} + \alpha_3 \left(\frac{\rho}{L}\right)^3 + \frac{3\alpha_3^2}{10\alpha_1}\left(\frac{\rho}{L}\right)^5 + \frac{3\alpha_3^3}{70\alpha_1^2}\left(\frac{\rho}{L}\right)^7 + O(\rho^9), \qquad (4.26)$$

---

[22]Note that given the asymptotics (4.15), (4.17), the conformal boundary metric is $ds^2_{(bdry)} = 4a_0^2 L^2(\sigma_1^2 + \sigma_2^2 + v^2\sigma_3^2) - \frac{4a_0^2}{v^2}dt^2$, which is slightly different from (1.1). However, one can always set $a_0 = 1/2$ by rescaling the radial coordinate (which implies a redefinition of other constants in (4.15)). Also, the scaling (2.8) can be used to dispose of the $1/v^2$ factor in the above expression of g_{tt}, such that the standard form (1.1) is recovered. Moreover, let us remark that although the $a-$equation (4.8) is invariant under the transformation (4.9), that symmetry is fixed by imposing the far field asymptotics (4.15).

in terms of two coefficients α_1, α_3. However, after using the scaling symmetry (4.9) (with $\lambda = \sqrt{\frac{\alpha_1}{6\alpha_3}}$), one finds that this corresponds in fact to the small-ρ expansion of the Gutowski-Reall solution (4.10) (where $\alpha = \alpha_1$).

The second choice to satisfy the eq. (4.25) is $\alpha_1 = 2\sqrt{\frac{2}{11}}$, which leads to a second consistent small-ρ expansion of $a(\rho)$ different from (4.26). One should remark that this possibility has been noticed in ref. [19], where the near-horizon expression of $f(\rho)$ has been already displayed.

The first few terms in this alternative expression of $a(\rho)$ are

$$\frac{a(\rho)}{L} = 2\sqrt{\frac{2}{11}}\frac{\rho}{L} + \alpha_3\left(\frac{\rho}{L}\right)^3 + \alpha_4\left(\frac{\rho}{L}\right)^4 + \frac{3}{20}\sqrt{\frac{11}{2}}\alpha_3^2\left(\frac{\rho}{L}\right)^5 + \frac{1}{8}\sqrt{\frac{11}{2}}\alpha_3\alpha_4\left(\frac{\rho}{L}\right)^6 + \dots, \quad (4.27)$$

in terms of two undetermined parameters α_3 and α_4. The near-horizon expansion of other functions which enter the line element (4.5) read

$$\frac{b(\rho)}{L} = \frac{16}{11}\frac{\rho}{L} + 16\sqrt{\frac{2}{11}}\alpha_3\left(\frac{\rho}{L}\right)^3 + 20\sqrt{\frac{2}{11}}\alpha_4\left(\frac{\rho}{L}\right)^4 + \frac{48\alpha_3^2}{5}\left(\frac{\rho}{L}\right)^5 + \dots, \quad (4.28)$$

$$f(\rho) = \frac{32}{7}\left(\frac{\rho}{L}\right)^2 - \frac{1424\sqrt{22}}{49}\alpha_3\left(\frac{\rho}{L}\right)^4 - \frac{176\sqrt{22}}{3}\alpha_4\left(\frac{\rho}{L}\right)^5 + \dots,$$

$$\frac{\Psi(\rho)}{L} = -\frac{147}{1408}\left(\frac{L}{\rho}\right)^2 - \frac{3885\alpha_3}{128\sqrt{22}} - \frac{1127}{128}\sqrt{\frac{11}{2}}\alpha_4\frac{\rho}{L} - \frac{92601}{1280}\alpha_3^2\left(\frac{\rho}{L}\right)^2 + \dots.$$

This solution translates into the following near-horizon expansion in terms of the metric Ansatz (2.4):

$$F_1(\rho) = \frac{7L^2}{32\rho^2} + \dots, \qquad F_2(\rho) = \frac{7L^2}{11} + \dots, \qquad F_3(\rho) = \frac{455L^2}{484} + \dots, \quad (4.29)$$

$$F_0(\rho) = \frac{131072}{3185L^4}\rho^4 + \dots, \qquad W(\rho) = -\frac{c}{2} + \dots,$$

while for the gauge potential one finds

$$a_0(\rho) = \frac{7\sqrt{3}}{44(4c-1)} + \left(\frac{12\sqrt{\frac{6}{11}}\alpha_3}{1-4c} + \frac{16\sqrt{3}}{7L^2}\right)\rho^2 + \dots, \quad (4.30)$$

$$a_k(\rho) = \frac{7\sqrt{3}L}{88} - 6\sqrt{\frac{6}{11}}\alpha_3 L\rho^2 - 175\sqrt{\frac{2}{33}}\alpha_4 L\rho^4 + \dots,$$

while

$$A_\mu A^\mu = -\frac{3}{32} - \frac{63\sqrt{\frac{11}{2}}}{32L^2}\alpha_3\rho^2 + \dots. \quad (4.31)$$

Also, this near-horizon expansion implies the following expressions for the constants c_t, c_W, which enter the first integrals (2.9):

$$c_t = \frac{203\sqrt{3}L^2}{484}, \quad c_W = -\frac{2303L^3}{5324}. \quad (4.32)$$

Then, after replacing in (4.21), one finds an expression of the far field coefficients a_4 and a_6 in terms of v and a_2.

4.4 The solutions

4.4.1 Numerical approach

We have not succeeded in solving analytically[23] the sixth-order eq. (4.8), to find a solution connecting the asymptotics (4.27) and (4.15). However, its numerical integration is straighforward. In our approach, we reformulate the problem in terms of a new function $\hat{a}(\rho) = e^{-\frac{\rho}{L}} a(\rho)$, which remains finite as $\rho \to \infty$. Similar to the treatment of the non-supersymmetric case, a new radial compactified radial coordinate x is introduced, with $0 \leq x \leq 1$ and $\rho = \frac{x}{1-x}$. The resulting differential equation for $\hat{a}(x)$ is written as a set of six first-order equations, which are solved with the following boundary conditions

$$\hat{a}\big|_{x=0} = 0, \quad \frac{d\hat{a}}{dx}\bigg|_{x=0} = 2\sqrt{\frac{2}{11}}, \quad \frac{d^2\hat{a}}{dx^2}\bigg|_{x=0} = 0, \quad \frac{d^4\hat{a}}{dx^4}\bigg|_{x=0} = 16\sqrt{\frac{2}{11}} + 8\frac{d^3\hat{a}}{dx^3}\bigg|_{x=0} + 24\xi,$$

$$\frac{d^5\hat{a}}{dx^5}\bigg| = 168\sqrt{\frac{2}{11}} + \sqrt{\frac{11}{8}}\left(\frac{d^3\hat{a}}{dx^3}\right)^2\bigg| + 56\frac{d^3\hat{a}}{dx^3}\bigg| + 360\xi \quad (\text{with} \ \xi \equiv \alpha_4), \tag{4.33}$$

and

$$\frac{d\hat{a}}{dx}\bigg|_{x=1} = 0 \ . \tag{4.34}$$

The first five algebraic relations above result directly from (4.27), while the condition at infinity is compatible with the asymptotics (4.15). Also note that the parameter $\alpha_3 = \frac{d^3\hat{a}}{dx^3}\big|_{x=0}$ is free, resulting from the numerical approach. Hence the boundary conditions present a single free parameter $\alpha_4 = \xi$, which we use as a control parameter to generate non-trivial solutions.

As an initial solution, we take $\hat{a} = 1$ and change the value of ξ in small steps. This setting works well, and the numerical iteration converges quickly (note that we have used the same solver [39, 40] as in the generic case). The solutions have around 2000 points in the mesh, with a numerical error of 10^{-8} or lower for $a(x)$ and its derivatives.

Once the profile of $a(\rho)$ is known, the full solution is reconstructed from (4.7) together with (4.11), (4.13). The coefficients $\{a_0, c, a_2, a_4, a_6\}$ in the far field expansion (4.15) are extracted from the numerical output.

In figure 10a we present several profiles of the function $\hat{a}(x)$. The corresponding profiles of the functions of the susy Ansatz (multiplied with suitable factors) are shown in figure 10b, 11 and 12a. The results in these plots are found for several values of ξ.

The relation between the squashing parameter v^2 of the far-field expansion and the parameter ξ of the near-horizon behaviour is presented in figure 12b. Note that the solutions are not invariant to changes in the sign of ξ. For example, the squashing v^2 grows for negative ξ, while for positive values it decreases, with $v \to 0$ for $\xi \simeq 0.03$. For larger values of ξ, a small branch of solutions with CTCs is found (dashed green line). Also note that the solution with $\xi = 0$ (red point) is a particular Gutowski-Reall black hole.

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[23]However, one cannot exclude the existence of a (partial) analytical solution. For example, a closed form (approximate) expression of the supersymmetric solitons has been reported in ref. [19]. The solution there has been found to first order in a perturbative expansion around the AdS background in terms of c_m. So far we did not succeed in finding a similar expression in the BH case.
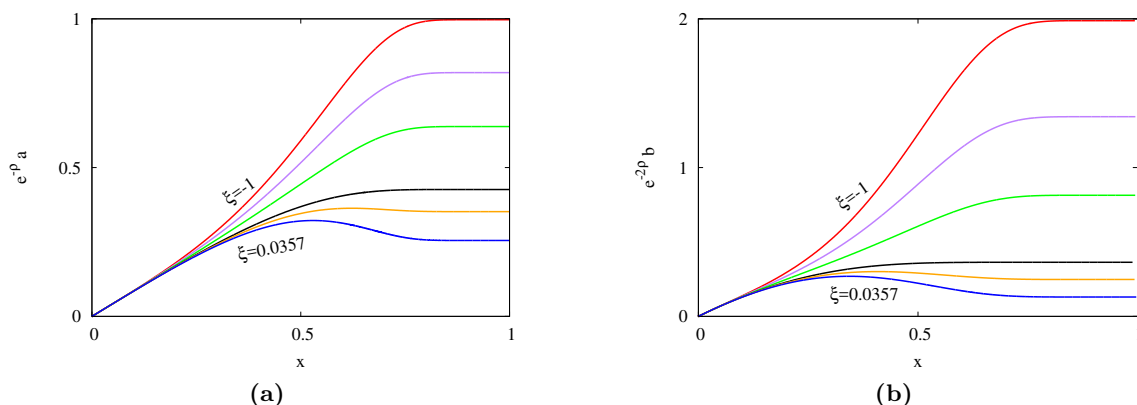
Figure 10. (a) Profiles for the function a (rescaled by a factor $e^{-\rho}$), as a function of the compactified radial coordinate $x = \rho/(\rho + 1)$. From top to bottom we show the profile for the values $\xi = -1, -0.5, -0.18, 0, 0.025$ and 0.0357. (b) Similar profiles for the rescaled metric function b. Here and in figures 11–12a the results are shown for the same values of the near-horizon parameter $\xi \equiv \alpha_4$.
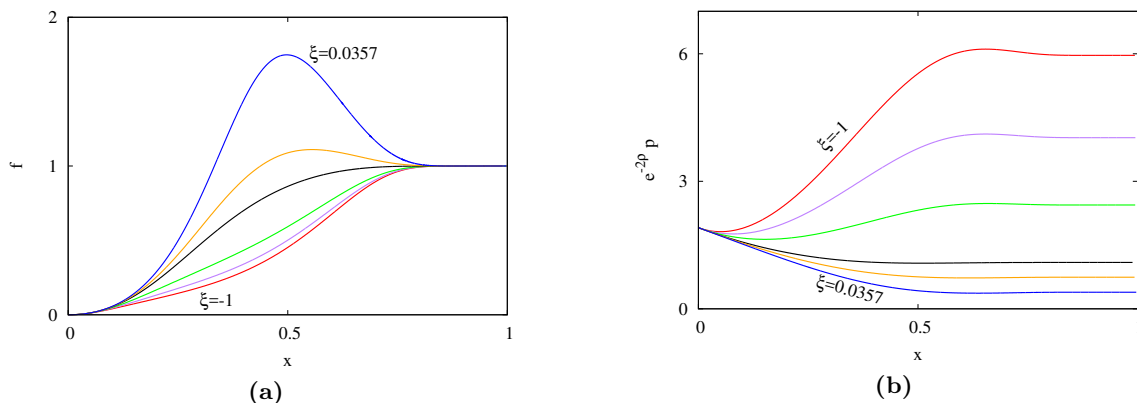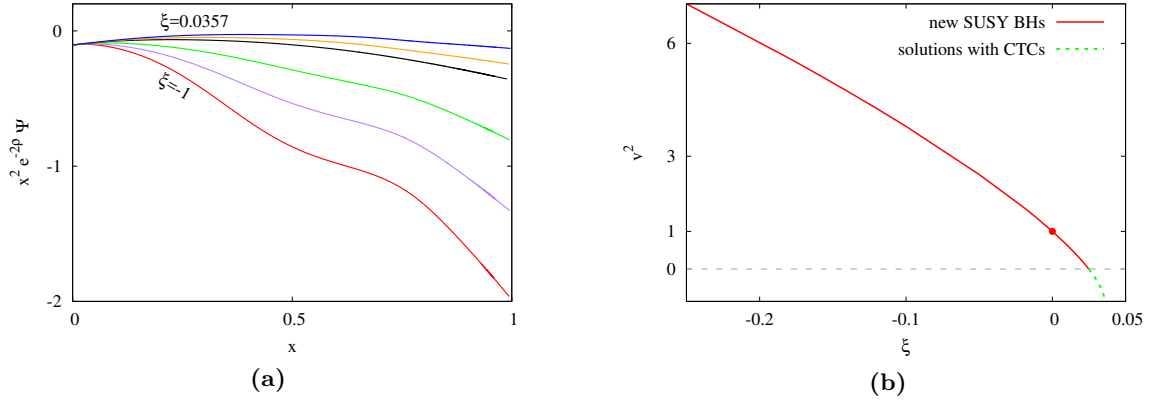

Figure 11. (a) Similar profiles for the function f and (b) the rescaled function p in terms of the compactified radial coordinate $x = \rho/(\rho + 1)$.

The relation of the remaining coefficients $\{a_0, a_2, a_4, a_6\}$ in the far field expansion (4.15) with the squashing parameter v is presented in figures 13 and 14, where we mark again the Gutowski-Real solution with a red point, and the sector with CTCs in green.

Finally, we also show the dependence of the parameter α_3 as a function of the squashing v^2 in figure 14b. We mark again the Gutowski-Real solution with a red point, and the sector with CTCs in green.

4.4.2 General properties

The computation of all quantities of interest of the solution is a direct application of the general formalism in section 2. This results in the following expression for the mass, angular

Figure 12. (a) Similar profiles for the function Ψ vs. the compactified radial coordinate $x = \rho/(\rho+1)$. (b) The squashing parameter v^2 as a function of the near-horizon parameter $\xi \equiv \alpha_4$. The red point marks the critical Gutowski-Reall solution from which the new susy black holes emerge. Note the existence a set of regular solutions with negative v^2 and closed timelike curves (dashed green line).
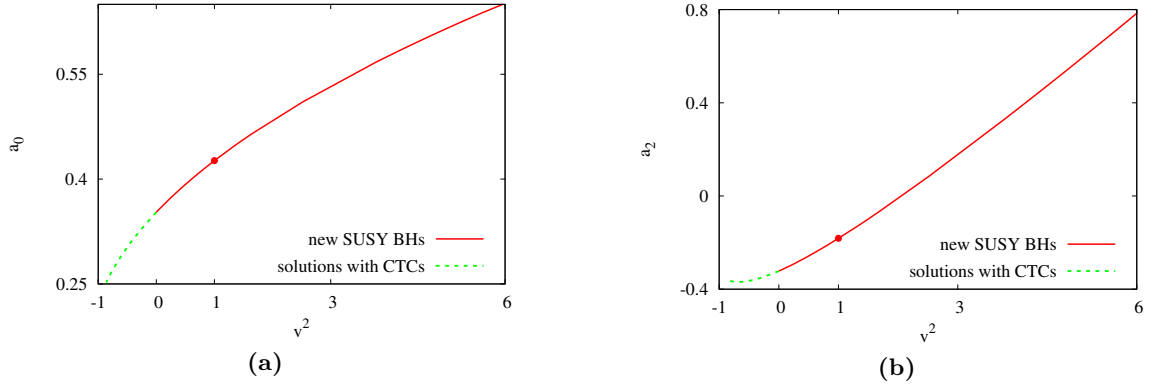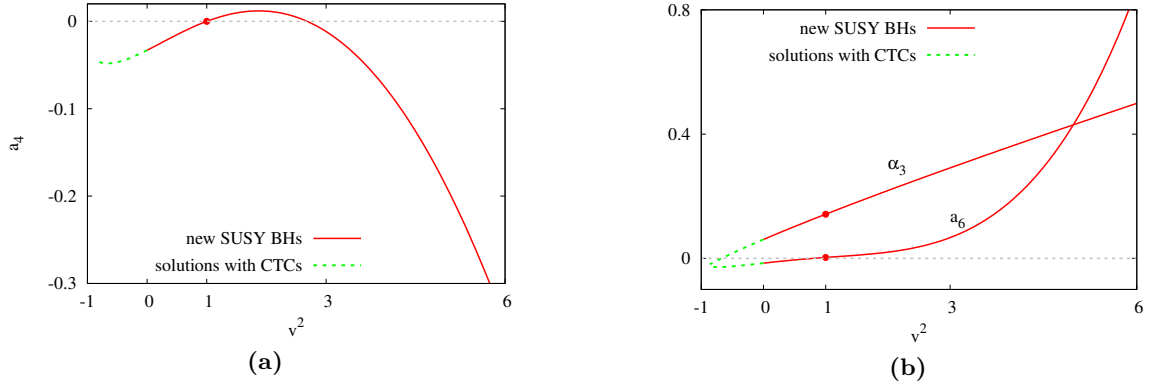

Figure 13. (a) The large-ρ expansion parameter a_0 as a function of the squashing v. (b) The same for the parameter a_2. Here and in figures 14 the red point marks the Gutowski-Reall solution from where the new susy black holes emerge, and the dashed green line the subset of solutions with closed timelike curves.

momentum and electric charge:

$$
\begin{aligned}
M &= \pi L^2 \left(\frac{7913}{34848} + \frac{33280}{35937 v^2} - \frac{7}{36} v^2 + \frac{89}{864} v^4 \right), \\
J &= -\pi L^3 \left(\frac{16640}{35937} - \frac{2795}{8712} v^2 + \frac{1}{9} v^4 - \frac{1}{27} v^6 \right), \\
Q &= -\pi \sqrt{3} L^2 \frac{1}{13068} \left(6449 - 1936\, v^2 + 968\, v^4 \right),
\end{aligned}
\tag{4.35}
$$

which depend on the squashing parameter v, only.

Figure 14. (a) The large-ρ expansion parameter a_4 as a function of the squashing v. (b) A similar figure for the parameters a_6 and α_3.

The supersymmetric BHs possess a nonzero horizon area

$$A_H = 7\pi^2 L^3 \frac{\sqrt{455}}{121},\qquad(4.36)$$

which does not depend on the parameter v. This is precisely what can be seen in the figure 6, where we fixed the conserved charges as in relation (4.32), and the magnetic parameter c_m such that equation (4.3) is satisfied. Then the extremal limit yields in fact the susy BHs, and all the horizon quantities converge to a single point (red), although the global charges are different (see figure 5, red line). Their horizon angular momentum, electrostatic potential, angular velocity, horizon mass and horizon deformation are

$$J_H = \frac{9555\,\pi L^3}{170368}, \quad \Phi_H = \frac{\sqrt{3}}{2}, \quad \Omega_H = \frac{2}{Lv^2}, \quad \epsilon_H = \sqrt{\frac{65}{44}}, \quad M_H = \frac{28665\pi L^2}{85184v^2}. \qquad (4.37)$$

Now we can return to figure 1, where we display a *(mass-angular momentum-electric charge)* diagram summarizing the picture for three different classes of supersymmetric solutions: (*i*) the Gutowski-Reall BHs, (*ii*) the supersymmetric solitons in [19] and (*iii*) the new BHs in this work. One can notice that the curves for the last two types of solutions do never intersect. However, one can see that the susy squashed magnetized BHs bifurcate from a critical Gutowski-Reall solution. That is where the two BH curves meet at a critical configuration with

$$M^{(c)} = \frac{49213L^2\pi}{42592}, \quad J^{(c)} = -\frac{2303L^3\pi}{10648}, \quad Q^{(c)} = -\frac{203\sqrt{3}L^2\pi}{484}. \qquad (4.38)$$

The critical Gutowski-Reall BH has a control parameter (see (4.1))

$$\alpha = 2\sqrt{\frac{2}{11}},\qquad(4.39)$$

while on the squashed BHs side, this corresponds to the limit[24] $v \to 1$ (i.e. a round S^3 sphere at infinity and no boundary magnetic field).

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[24] The supersymmetric solitons in [19] approach instead the globally AdS background as $v \to 1$.

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Also, note the solutions with $v \simeq 3.61691$ possess a vanishing total angular momentum, although they still rotate in the bulk ($T_\psi^t \neq 0$ and $\Omega_H \neq 0$), while their mass and electric charge are positive.

The limit $v = 0$ (i.e. $c = 1/4$ in the far field expansion (4.15)) is special. While the horizon geometry does not change, the asymptotics are different in this case, the conformal boundary structure being lost. One finds e.g.

$$F_0 = q_0 \left(\frac{r}{L}\right)^2 + \dots, \quad F_1(r) = 1 - \frac{1 + 8a_2 + 2\log(\frac{r}{L})}{6a_0^2}\left(\frac{L}{r}\right)^2 + \dots, \quad (4.40)$$

$$F_2(r) = 4a_0^2 r^2 + \frac{2}{3}L^2\left(4a_2 - 1 + \log\left(\frac{r}{L}\right)\right) + \dots,$$

$$F_3(r) = q_3 \left(\frac{L}{r}\right)^2 + \dots, \quad W(r) = q_w \left(\frac{r}{L}\right)^4 + \dots,$$

with q_0, q_3 and q_w possessing a complicated dependence on a_0, a_2, a_4 and a_6. Thus, similar to the extremal case in section 3, the $v \to 0$ limit does not result in a black string configuration. Moreover, this limiting solution is not asymptotically (locally) AdS (despite the absence of (obvious) pathologies). We hope to return elsewhere with a detailed study of this interesting limiting solution.

The infinite squashing limit $v \to \infty$ is taken again together with the rescaling (3.3), (3.4). This results in a BH solution with a different topology (e.g. the horizon geometry is of the form (3.6)), whose spacetime asymptotics are again non-standard, the AdS conformal boundary structure being lost. This limit is described by an exact solution, which is discussed in appendix B.

Finally, let us mention that we have found numerical evidence for the existence of solutions with $v^2 < 0$. However, such configurations possess closed timelike curves (in the bulk and on the boundary) and are less interesting. Non-supersymmetric solutions with this behaviour exist as well.

5 Further remarks. Conclusions

The solutions of the $D = 5$ gauged supergravity models play a central role in the AdS/CFT correspondence, providing a dual description of strongly-coupled CFT on the four-dimensional AdS boundary. The main purpose of this work was to report the existence of a new class of BH solutions of the minimal gauged supergravity model and to provide a discussion of their basic properties (see also [20]) . They are built within the same general framework as the well-known Cvetič-Lü-Pope BHs, sharing some of their basic properties; for example, the horizon has a spherical topology and the two angular momenta have equal magnitude. However, the conformal boundary of the new BHs in this work possesses a squashed sphere, such that the solutions could become in certain limits black strings or black branes. Moreover, new features occur as one allows for a nonvanishing value of the magnetic field at infinity. For example, as discussed in section 3.3, this supports the existence of smooth particle-like solitons, which satisfy a universal relation between angular momentum and electric charge. Moreover, a particular set of extremal configurations corresponds to a new one-parameter family of supersymmetric black holes, which were reported

in section 4. They satisfy a certain relation between the squashing parameter and the magnetic parameter and bifurcate from a critical Gutowski-Reall configuration.

There are many open questions and avenues for future investigation. For example, the general framework in this paper provides a ground for further study of the properties of these solutions, such as a systematic investigation of their domain of existence, thermodynamics and stability. Moreover, various limits of the solutions briefly mentioned in section 3 certainly deserve a systematic study. Also, our results in the generic non-susy case were found for the supersymmetric value of the CS coupling, $\lambda = 1$. However, it would be interesting to consider other values as well. Here we remark that the results in [21] (valid EMCS BHs with $v = 1$ and no boundary magnetic field) show the existence of new qualitative features (e.g. the existence of excited solutions) once the CS coupling constant exceeds a critical value. As yet another possible direction, we note that it is straightforward to extend the framework introduced in section 2 to other (odd) spacetime dimensions $D > 5$ and the same matter content. Therefore we predict the existence of AlAdS solutions also in that case, which would share some basic properties of the solutions in this work.

Another interesting question is the relevance of such configurations in an AdS/CFT context. Here one remarks that the supersymmetric solitons have been interpreted in [19] as providing the gravity supersymmetric dual of an $\mathcal{N} = 1$ supersymmetric gauge theory on a squashed Einstein universe background, with a nontrivial background gauge field coupling to the R-symmetry current. We expect that the interpretation of the supersymmetric BHs in this work would be similar, describing different phases of the same $D = 4$ model.

Finally, let us remark that the presence of a nontrivial magnetic field on the boundary can be viewed in a larger context as a consequence of the '*box-type*' behaviour of the AdS spacetime. As realized in [48–50], for the $D = 4$ case, the AdS asymptotics supports the existence of everywhere regular Maxwell-field multipole solutions, which survive when including the backreaction. That is, the U(1) field mode, which in the (asymptotically) flat case is divergent at infinity, gets regularized, leading to new families of Einstein-Maxwell solutions (which include both BHs and solitons). Although further study is necessary, this feature appears to be universal, some partial results being reported in [51] for $D = 2k+1 \geq$ 5 dimensions (see also [52, 53]).

This observation leads us to predict the existence of a variety of other A(l)AdS solutions of the $D = 5$ minimal gauged supergravity model. First, we remark that for the solutions in this work, the asymptotics of the electric potential are standard, while the magnetic part can be interpreted as an AdS dipolar field. However, this is just the simplest type of non-standard boundary conditions for the U(1) field (supported by AdS asymptotics), which has the advantage to lead to a codimension-1 numerical problem. More general solutions should exist as well. For example, our preliminary numerical results indicate the existence of generalizations of the $D = 5$ Reissner-Nordström BHs with a vanishing magnetic field and an asymptotic value of the electric potential $a_0 = c_e \cos(2\theta)$ (with c_e a control parameter). These configurations are static and not spherically symmetric, without being possible to factorize the θ-dependence on both metric and gauge sectors. Therefore the numerical treatment of this problem is much more complicated, the solutions being found by solving partial differential equations. In the absence of an electric charge, they

possess a nontrivial solitonic limit and can be interpreted as AdS electric dipoles. More general solutions of the $D = 5$ minimal gauged supergravity model possessing higher-order multipoles for both the electric and magnetic potentials should also exist. We hope to return elsewhere with a systematic discussion of these aspects.

Acknowledgments

A A squashed S^3 boundary: limiting behaviour of nutty instantons in AdS_4

A simple model, which helps to understand the limiting behaviour of the solutions in this work in terms of the squashing parameter v is provided by the AdS_4 nut-charged instantons. These solutions solve the vacuum Einstein equations on the Euclidean section and can be written in a form resembling (2.4), with

$$ds^2 = F_1(r)dr^2 + \frac{1}{4}F_2(r)(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}F_3(r)\sigma_3^2, \tag{A.1}$$

(σ_i being the one-forms on S^3 as given by (2.1)), where

$$\frac{1}{F_1(r)} = \frac{r^2 + n^2}{r^2 - n^2} + \frac{-2Mr + \frac{1}{L^2}(r^4 - 6n^2r^2 - 3n^4)}{r^2 - n^2}, \ F_2(r) = 4(r^2 - n^2), \ F_3(r) = 16n^2\frac{1}{F_1(r)}. \tag{A.2}$$

This line element can be put into the standard form given in the literature via the coordinate transformation

$$\psi = -\left(\phi + \frac{\hat{\psi}}{2n}\right), \tag{A.3}$$

which results in

$$ds^2 = \frac{dr^2}{V(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2) + V(r)\left(d\hat{\psi} + 4n\sin^2\left(\frac{\theta}{2}\right)d\phi\right)^2, \ \text{with } V(r) = \frac{1}{F_1(r)}. \tag{A.4}$$

The nutty instantons possess two constants: M, which is a mass parameter, and n — the nut parameter. Also, r is a radial coordinate, while $\hat{\psi}$ parameterizes a circle S^1, which is fibred over the two sphere S^2, with coordinates θ and ϕ. As a result, the metric (A.4) is only locally asymptotically AdS, while its boundary is a squashed three sphere as $r \to \infty$. As discussed in [54], this becomes a round S^3 for $M = 0$, $n = L/2$, such that (A.4) corresponds

to AdS_4. In the generic case, the absence of a conical singularity imposes some constraints on the value of M, the solutions possessing a variety of interesting features. However, these aspects are of no interest in the context of this work (for a detailed analysis, we refer the reader to refs. [54–59]). Instead we shall simply only consider their small/large n limits.

For $n = 0$, one recovers the Schwarzschild-AdS Euclideanized solution,

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r} + \frac{r^2}{L^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{2M}{r} + \frac{r^2}{L^2}\right) d\hat\psi^2 \,, \qquad \text{(A.5)}$$

with a $S^2 \times S^1$ topology of an $r = const.$ surface. Note that, for the parametrization (A.1), the limit $n \to 0$ should be taken with a rescaled ψ-coordinate, which, however has a period fixed by M, L.

Another case of interest is $n \to \infty$, in which a different solution is recovered. To understand this limit, one starts again with the line element (A.4) and defines the scaled coordinates

$$r = \lambda\bar{r}, \;\; \theta = \frac{\Theta}{\lambda}, \;\; \hat\psi = \frac{\Psi}{\lambda}, \qquad \text{(A.6)}$$

together with

$$n = \lambda N, \;\; M = \bar{M}\lambda^3. \qquad \text{(A.7)}$$

Let us now consider the limit $\lambda \to \infty$. Then the line element (A.4) becomes

$$ds^2 = f_1(\bar{r})d\bar{r}^2 + f_2(\bar{r})(d\Theta^2 + \Theta^2 d\phi^2) + f_0(\bar{r})\left(d\Psi + 4N\left(\frac{\Theta}{2}\right)^2 d\phi\right)^2, \qquad \text{(A.8)}$$

where

$$f_0(\bar{r}) = \frac{1}{f_1(\bar{r})} = \frac{-2\bar{M}\bar{r} + \frac{1}{L^2}(\bar{r}^4 - 6N^2\bar{r}^2 - 3N^4)}{\bar{r}^2 - N^2}, \;\; f_2(\bar{r}) = \bar{r}^2 - N^2, \qquad \text{(A.9)}$$

which is the planar version of the Taub-NUT-AdS spacetime [54]. This limit possesses a number of interesting properties; here we mention only that an $r = const.$ surface has an R^3 topology, with a warped product $R \times R^2$. Also, they possess no Misner string singularity (i.e. the periodicity of Ψ is arbitrary) with a breakdown of the entropy/area relationship [57, 58].

B   The black branes

B.1   The generic case

The $D = 5$ Schwarzschild-AdS BH possesses a well-known generalization with a planar horizon topology, which approaches a Poincaré patch at infinity. Adding extra charges (and also a boundary magnetic field) results in generalizations of the Schwarzschild black brane, which possess a variety of interesting properties (see e.g. [12–14]).

We have found that the infinite squashing limit of the BHs in this work leads to a new set of solutions which can be viewed as a generalization of the configurations in [12–14]. Their horizon (and the spacelike part of the boundary metric) is R^3, being topologically a

direct product of a flat direction and the R^2-plane. However, this product is 'twisted' (or warped), and the induced geometry is not flat.

These solutions have a line element

$$ds^2 = -f(r)\frac{r^2}{L^2}dt^2 + \frac{1}{f(r)}\left[m(r)L^2\frac{dr^2}{r^2} + \frac{1}{4}r^2\left(m(r)(d\Theta^2 + \Theta^2 d\phi^2)\right.\right. \tag{B.1}$$
$$\left.\left. + n(r)\left(d\Psi + 2v\left(\frac{\Theta}{2}\right)^2 d\phi - \frac{2\omega(r)}{r}dt\right)^2\right)\right],$$

their gauge potential being

$$A = a_0(r)dt + \frac{1}{2}a_k(r)\left(d\Psi + 2v\left(\frac{\Theta}{2}\right)^2 d\phi\right). \tag{B.2}$$

Again, we assume that the configurations are AlAdS, with a conformal boundary metric given by (3.7), while $a_k \to c_m$ as $r \to \infty$. Then a far field solution can be constructed in a systematic way, the leading order terms in the asymptotics being

$$f(r) = 1 - \frac{4v^2}{9}\left(\frac{L}{r}\right)^2 + \left[\frac{\hat{\alpha}}{L^4} - \frac{4v^2}{15}\left(\frac{9c_m^2}{L^2} - 4v^2\right)\log\left(\frac{r}{L}\right)\right]\left(\frac{L}{r}\right)^4 + \dots,$$
$$m(r) = 1 + \frac{v^2}{9}\left(\frac{L}{r}\right)^2 + \left[\frac{\hat{\beta}}{L^4} + \frac{4v^2}{15}\left(\frac{3c_m^2}{L^2} + 2v^2\right)\log\left(\frac{r}{L}\right)\right]\left(\frac{L}{r}\right)^4 + \dots,$$
$$n(r) = 1 - \frac{17v^2}{9}\left(\frac{L}{r}\right)^2 + \left[\frac{3(\hat{\alpha} - \hat{\beta})}{L^4} + \frac{4c_m^2 v^2}{15L^2} + \frac{497v^4}{405}\right. \tag{B.3}$$
$$\left. + \frac{8v^2}{5}\left(\frac{3c_m^2}{L^2} - 3v^2\right)\log\left(\frac{r}{L}\right)\right]\left(\frac{L}{r}\right)^4 + \dots,$$
$$w(r) = \hat{j}\frac{1}{r^3} + \dots, \quad a_0(r) = -\frac{q}{r^2} + \dots, \quad a_k(r) = c_m + \left[\mu - 2c_m L^2 v^2 \log\left(\frac{r}{L}\right)\right]\frac{1}{r^2} + \dots.$$

Restricting to the non-extremal case, the black branes possess a horizon at $r = r_H > 0$, with approximate solution very similar to (2.14), with $f(r) = f_2(r - r_H) + \dots$, $m(r) = f_2(r - r_H) + \dots$, $n(r) = f_2(r - r_H) + \dots$, and nonvanishing ω, a_0 and a_k. This leads to an induced metric on the horizon with a form very similar to (3.6). The horizon area density and Hawking temperature are

$$A_H = \pi\Delta\Psi r_H^3 \frac{m_2}{32f_2}\sqrt{\frac{n_2}{f_2}}, \quad T_H = \frac{1}{2\pi}\frac{r_H^2}{L^2}\frac{f_2}{\sqrt{m_2}}, \tag{B.4}$$

with $\Delta\Psi$ the (arbitrary) periodicity of the Ψ-coordinate.

The global charges for the black branes are computed by using the approach described in section 2. A major difference in this case is that one deals with densities of relevant charges, since Θ has an infinite range. One finds

$$M = \frac{1}{8}\left(-\frac{(3\hat{\alpha} + \hat{\beta})}{8L^2} + \frac{c_m^2 v^2}{30} + \frac{1883L^2 v^4}{12960}\right)\Delta\Psi, \quad J = -\frac{\hat{j}v^3}{64\pi}\Delta\Psi, \quad Q = -\pi\left(qv - \frac{4\lambda}{3\sqrt{3}}c_m^2\right)\Delta\Psi. \tag{B.5}$$
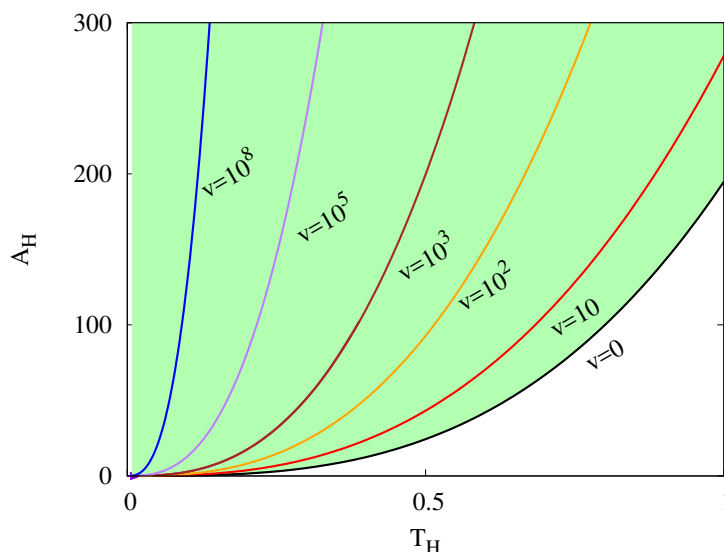
Figure 15. Area vs. temperature of static twisted black branes without electric charge. The lines represent families of black branes with constant twisting parameter v.

The study of these configurations can be performed in a similar way to the (spherical horizon topology) BHs in the paper. So far the only case we have investigated more systematically corresponds to vacuum, static black branes. The horizon area-temperature diagram of these solutions is shown in figure 15. One can see that these twisted branes present similar properties to the standard $v = 0$ Schwarzschild black brane (represented with a black line in this figure). We have also found numerical evidence for the existence of regular configurations in the more general case of twisted black branes in EMCS theory, with (in principle) arbitrary values of the J, Q, c_m and v. However, a systematic study of their properties is beyond the purposes of this work.

B.2 A supersymmetric solution

An interesting question here concerns the possible existence of a supersymmetric limit of the black brane solutions discussed above. To address it, we use a slightly modified version of the framework employed for black holes with a spherical horizon topology. These solutions are a particular member of the timelike case in [60] (see also [61] for a related study). Then we consider the usual framework with

$$ds^2 = -f^2(\rho)(dy + w)^2 + \frac{1}{f}ds_B^2, \quad A = \frac{\sqrt{3}}{2}\left(f(dy + w) + \frac{L}{3}P\right), \qquad \text{(B.6)}$$

with a Kähler metric ds_B^2 on a four-dimensional base space (characterized by a one-form X^1, while P is the Ricci one-form potential) and w a transverse one-form. The solutions of interest are found for the following choice of the base space

$$ds_B^2 = d\rho^2 + a^2(\rho)(\sigma_1^2 + \sigma_2^2) + b^2(\rho)\sigma_3^2, \qquad \text{(B.7)}$$

with the one forms

$$\sigma_1 = dx, \quad \sigma_2 = dy, \quad \sigma_3 = dz + \frac{1}{2}(xdy - ydx),$$

x, y, z possessing the usual range.

Then a similar reasoning as in the case above of a Bergmann manifold as the base space, leads to a formulation of the problem in terms of a single function $a(\rho)$, with

$$w = \Psi(\rho)\sigma_3, \quad P = p(\rho)\sigma_3, \quad X = -a^2(\rho)\sigma_1 \wedge \sigma_2 + b(\rho)\sigma_2 \wedge d\rho, \tag{B.8}$$

and the gauge potential

$$A = \frac{\sqrt{3}}{2}\left[f(\rho)\mathrm{d}y + \left(f(\rho)\Psi(\rho) + \frac{L}{3}p(\rho)\right)\sigma_3\right]. \tag{B.9}$$

The function $a(\rho)$ satisfies again a sixth-order equation which can formally be written in the compact form (4.8). However, the expressions for p, f and g are (slightly) different, with some terms which are absent here:

$$p = 4a'^2 + 2aa'', \quad f^{-1} = \frac{L^2}{12a^2a'}[4(a')^3 + 7a\,a'a'' + a^2a'''], \quad g = -\frac{a'''}{a'} - 3\frac{a''}{a} + 4\frac{(a')^2}{a^2}, \tag{B.10}$$

while b, Ψ and the operator ∇^2 are the same. We would like to emphasize that the explicit a-equation here does not coincide with the one resulting from (4.8). Despite that,

$$a(\rho) = \alpha L \sinh\left(\frac{\rho}{L}\right), \tag{B.11}$$

is still a solution. After defining

$$x = \Theta\cos\phi, \quad y = \Theta\sin\phi, \tag{B.12}$$

together with a new radial coordinate

$$\rho = \frac{L}{2}\mathrm{ArcCosh}\left(\frac{1}{3}(2r^2 + 1)\right), \tag{B.13}$$

this supersymmetric configuration can be written in a compact form, with

$$ds^2 = -\left(1 - \frac{L^2}{r^2}\right)^2\left[dy + \frac{2\alpha^2 L}{3}\left(\frac{r^2}{L^2} + 2 + \frac{3}{2(\frac{r^2}{L^2} - 1)}\right)\left(d\psi + \frac{\Theta^2}{2}d\phi\right)\right]^2 \tag{B.14}$$

$$+ \frac{dr^2}{(\frac{L^2}{r^2} - 1)^2(\frac{r^2}{L^2} + 2)} + \frac{\alpha^2 r^2}{3}\left[d\Theta^2 + \Theta^2 d\phi^2 + \frac{4}{3}\alpha^2\left(2 + \frac{r^2}{L^2}\right)\left(d\psi + \frac{\Theta^2}{2}d\phi\right)^2\right],$$

$$A = \frac{\sqrt{3}}{2}\left(\frac{L^2}{r^2} - 1\right)dt - \frac{\alpha^2 L^3}{2\sqrt{3}r^2}\left(d\psi + \frac{\Theta^2}{2}d\phi\right). \tag{B.15}$$

This corresponds, in fact, to the solution obtained by Gutowski and Reall in ref. [10], as the large size limit of the configuration (4.10). It describes a black brane, the above line element possessing a horizon at $r = L$, whose properties are functions of the parameter α

(e.g. the event horizon area density is $A_H = \alpha^4 L^3/(3\sqrt{3})$). However, as $r \to \infty$, a non-asymptotically AdS geometry is approached, which corresponds to a plane-fronted wave with a vanishing magnetic field [10].

Based on the results in section 4, one may expect the existence of other solutions of the sixth-order a-equation, different from (B.13). However, we have failed to find any, the freedom in the choice of the boundary conditions at $\rho = 0$ (as implied by (4.25)) being absent in this case. Thus one finds a single possible form of the solutions at $\rho \to 0$, corresponding to (4.26). Then the numerical integration of the a-equation leads always to the solution (B.11).

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