ALMOST CLASSICAL SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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Abstract. We study the existence of everywhere differentiable functions which are almost everywhere solutions of certain Hamilton-Jacobi equations on manifolds of dimension ≥ 2. In particular, when M is a Riemannian manifold, we prove the existence of a differentiable function u on M which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on M.

1. Introduction

It has been proved by Z. Buczolich [4] that if $d \geq 2$, there exists $u : \mathbb{R}^d \to \mathbb{R}$, differentiable at every point, such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| \geq 1$ almost everywhere, thus giving a negative answer to the gradient problem of C. E. Weil [9]. Malý and Zelený [7] gave an elegant proof of this result using a new mathematical game. Then Deville and Matheron [6], refining the methods introduced by the above authors, proved that if Ω is a bounded open subset of $\mathbb{R}^d$ with $d \geq 2$, there exists a function $u : \overline{\Omega} \to \mathbb{R}$, continuous on $\overline{\Omega}$, differentiable at every point of Ω, such that $u(x) = 0$ for all $x \in \partial\Omega$, and such that $\|\nabla u(x)\| = 1$ almost everywhere on Ω. Notice that because of Rolle's theorem, there exists $x_0 \in \Omega$ such that $\nabla u(x_0) = 0$, so the function u cannot be C^1-smooth. We shall call u an almost-classical of the Eikonal equation $\|\nabla u\| = 1$. This equation has also a unique viscosity solution, which is the function $x \mapsto dist(x, \partial\Omega)$, where $dist(x, \partial\Omega) = \inf\{\|x - y\| : y \in \partial\Omega\}$. The viscosity solution is not everywhere differentiable on Ω. Therefore, an almost classical solution of the Eikonal equation is not equal to the viscosity solution of the Eikonal equation. Nevertheless, we should say that the viscosity solution is the "right" solution of the Eikonal equation (see, e.g. [2] of [5] for more information about viscosity solutions of Hamilton-Jacobi equations).

In particular, we will be interested here in the Eikonal equation $\|\nabla u(x)\|_x = 1$ on a Riemannian manifold. After reviewing, in Section 2, some technical results from [6] needed in this paper, we will study in Section 3 more geneal Hamilton-Jacobi equations on $\mathbb{R}^d$. Section 4 we will devoted to Hamilton-Jacobi equations on manifolds.

Now we introduce some terminology. Let Ω be an open subset of $\mathbb{R}^d$, and consider two continuous maps $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ and $u_0 : \partial\Omega \to \mathbb{R}$. We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a classical solution of $F(x, \nabla u(x)) = 0$ with Dirichlet condition $u\,|_{\partial\Omega} = u_0$ if $u(x) = u_0(x)$ for all $x \in \partial\Omega$, u is differentiable at each point of Ω, and u satisfies $F(x, \nabla u(x)) = 0$ for every $x \in \Omega$.

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Definition 1.1. We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is an almost classical solution of $F(x, \nabla u(x)) = 0$ with Dirichlet condition $u\,|_{\partial\Omega} = u_0$ if :

- $u(x) = u_0(x)$ for all $x \in \partial\Omega$,
- *at each point* x of Ω, u is differentiable and $F(x, \nabla u(x)) \le 0$,
- and u satisfies $F(x, \nabla u(x)) = 0$ for almost every $x \in \Omega$ (in the sense of Lebesgue measure on $\mathbb{R}^d$).

It is known that classical solutions of the Hamilton-Jacobi equation $F(x, \nabla u(x)) = 0$ exist only under very restrictive conditions on F. We prove the existence of almost classical solutions of $F(x, \nabla u(x)) = 0$ under general hypotheses on F. We obtain :

Theorem 1.2. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \ge 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a C^1 function. Suppose that the following conditions hold :*

(A) *There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$ such that u_0 is C^1-smooth on Ω, and $F(x, \nabla u_0(x)) \le 0$ for every $x \in \Omega$.*
(B) *For each $x \in \Omega$, the set $B(x) = \{p \in \mathbb{R}^d : F(x,p) \le 0\}$ is compact, the set $S(x) = \{p \in \mathbb{R}^d : F(x,p) = 0\}$ is connected, and the function $F(x, \cdot)$ has maximal rank on $S(x)$.*

Then there exists an almost classical solution of $F(x, \nabla u(x)) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = u_0$.

In the last section, we consider Hamilton-Jacobi equations defined on a smooth manifold M (of dimension $d \ge 2$). We denote T^*M the cotangent bundle of M. Under suitable hypotheses on $F : T^*M \to \mathbb{R}$, we show the existence of almost classical solutions $u : M \to \mathbb{R}$ of an equation of the form $F(x, du(x)) = 0$. More precisely, we obtain :

Theorem 1.3. *Let M be a smooth manifold of dimension $d \ge 2$, and let $F : T^*M \to \mathbb{R}$ be a C^1 function. Suppose that the following conditions hold :*

(A) *There exists a C^2 function $u_0 : M \to \mathbb{R}$ such that $F(x, du_0(x)) \le 0$, for every $x \in \Omega$.*
(B) *For each $x \in M$, the set $B(x) = \{\xi \in T_x^*M : F(x,\xi) \le 0\}$ is compact, the set $S(x) = \{\xi \in T_x^*M : F(x,\xi) = 0\}$ is connected, and the function $F(x, \cdot)$ has maximal rank on the set $S(x)$.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $F(x, du(x)) = 0$ for almost every $x \in \Omega$.

In particular, if (M, g) is a Riemannian manifold, we identify as usual the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product g_x. Thus if we consider the function $F : TM \to \mathbb{R}$ given by

$$F(x, v) = \|v\|_x^2 - 1 = g_x(v, v) - 1,$$

we obtain that there exists a differentiable function u on M which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on M. Whenever the manifold M is compact, there exists a point $x_0 \in M$ such that $\nabla u(x) = 0$. Therefore, there is no classical solution of this equation, and an almost classical solution u of this equation cannot be C^1-smooth. So almost classical solutions of Hamilton-Jacobi equations are often exotic.

2. Preliminary results

We recall three lemmas from [6] that we shall use here. The first lemma is a criterium of differentiability for the sum of a series of C^1-smooth functions.

Lemma 2.1. *Let $(u_n)_{n\geq 1}$ be a sequence of C^1 functions between two Banach spaces X and Y. Assume that :*

(a) *the series $\sum \nabla u_n(x)$ is pointwise convergent*
(b) *the sequence $\big(\nabla u_n\big)_{n\geq 1}$ converges uniformly to 0*
(c) $\|u_{n+1}\|_\infty = o\left(\|u_n\|_\infty\right)$ *for every $n \geq 1$*
(d) $\lim\limits_{n\to\infty} osc\left(\sum_{k=1}^n \nabla u_k;\, \|u_{n+1}\|_\infty\right) = 0$

Then the series $\sum u_n$ is uniformly convergent, the function $u := \sum_{n=1}^\infty u_n$ is everywhere differentiable, and $\nabla u(x) = \sum_{n=1}^\infty \nabla u_n(x)$ for all $x \in X$.

We say that a subset Q of $\mathbb{R}^d$ is a cube if $Q = \prod_{i=1}^d [a_i, b_i]$, where each $[a_i, b_i]$ is a closed and bounded interval of $\mathbb{R}$, all with the same length. A function v defined on a cube Q is said to be piecewise constant if there is a finite partition of Q into cubes such that v is constant of every cube of the partition. The following result gives the existence of a C^∞-smooth function $u : \mathbb{R}^d \to \mathbb{R}$, which vanishes in a neighbourhood of the exterior of a cube Q and such that its derivative is equal to a or $-a$ (where a is a given non zero vector in $\mathbb{R}^d$) on a subset of Q of measure almost equal to the measure of Q.

Lemma 2.2. *Let a be a non zero vector in $\mathbb{R}^d$, let Q be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, C^∞-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties :*

(a) *u vanishes in a neighbourhood of ∂Q and $\|u\|_\infty \leq \varepsilon$;*
(b) $\lambda_d\big(\{x \in Q_p;\ \nabla u(x) = -a \text{ or } \nabla u(x) = a\}\big) \geq (1-\varepsilon)\lambda_d(Q)$;
(c) *one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$, the set $\{v(x) : x \in Q\}$ is included in the segment $[-a, a]$, and the function v is piecewise constant on Q.*

The last lemma relies on ideas due to J. Maly and M. Zeleny [7], and is also from [6]. The mapping $\mathbf{t}$ is defined using that a suitable game has a winning strategy.

Lemma 2.3. *Let U be a bounded open subset of $\mathbb{R}^d$, and let B be a closed ball containing U. Then, there exists a map $\mathbf{t} : B \to \mathbb{R}^d$ such that the following property holds true : if (s_n) is a sequence in U and if there exists a sequence $(\sigma_n) \in B$ such that $s_n - \sigma_n$ converges and $\langle \mathbf{t}(\sigma_n), \sigma_{n+1} - \sigma_n\rangle \geq 0$ for all n, then (s_n) converges to some point $s \in \overline{U}$.*

3. Almost classical solutions on open subsets of $\mathbb{R}^d$

We first recall the definition of the *Hausdorff distance* between closed sets of a metric space. If X is a metric space, for each $A \subset X$ and $r > 0$ we denote $B(A, r) = \{x \in X : dist(x, A) < r\}$. We denote $\mathcal{C}(X)$ the set of all closed bounded subsets of X. If C and D are in $\mathcal{C}(X)$, the Hausdorff distance between them is

$$d_H(C, D) = \inf\{r \in (0, \infty] : C \subset B(D, r) \text{ and } D \subset B(C, r)\}.$$

If A is a subset of $\mathbb{R}^d$, we denote its complement by $A^c = \mathbb{R}^d \setminus A$. In the next theorem, we follow closely the proof of Theorem 4.2 by Deville and Matheron [6].

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$. For each $x \in \Omega$ let $U(x)$ be an open bounded subset of $\mathbb{R}^d$ containing* 0. *We assume that the set-valued mapping $x \mapsto \partial U(x)$ from Ω into $(\mathcal{C}(\mathbb{R}^d), d_H)$ is continuous on Ω.*
Then there exists a differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ such that :

(1) $u\mid_{\Omega^c} \equiv 0$ *and* $\nabla u\mid_{\Omega^c} \equiv 0$.
(2) $\nabla u(x) \in \overline{U(x)}$ *for every* $x \in \mathbb{R}^d$.
(3) $\nabla u(x) \in \partial U(x)$ *for almost every* $x \in \Omega$.

Proof. We will consider two cases.
First Case: Ω is bounded. Fix a cube Q_0 containing Ω, and choose a partition $\mathcal{Q}_0$ of Q_0 into cubes, such that some cube of $\mathcal{Q}_0$ is contained in Ω. We will construct a sequence $(\mathcal{Q}_n)_{n\geq 0}$ of partitions of Q_0 into cubes, where each $\mathcal{Q}_{n+1}$ is a refinement of $\mathcal{Q}_n$, and such that $\mathrm{diam}(\mathcal{Q}_n)$ tends to 0. For each $n \geq 0$ we will consider the family $\mathcal{R}_n$ of all cubes in $\mathcal{Q}_n$ which are contained in Ω, and the set

$$K_n = \bigcup \{Q : Q \in \mathcal{R}_n\}.$$

Then $(K_n)_{n\geq 0}$ is an increasing sequence of compact sets, $\Omega = \cup_{n\geq 0} K_n$, and $\lambda_d(\Omega) = \lim_n \lambda_d(K_n)$.

Fix a point $x_0 \in \Omega$ such that $x_0 \in \partial Q$ for some cube $Q \in \mathcal{Q}_0$. Fix a sequence $(\varepsilon_k)_{k\geq 1}$ of positive numbers, with $(\varepsilon_k) \downarrow 0$ and $0 < \varepsilon_k < \min\{1, dist(0, \partial U(x_0))\}$ for every $k \geq 1$. For each $x \in \Omega$ and $\varepsilon > 0$, we denote

$$\partial_\varepsilon U(x) = \{p \in \mathbb{R}^d : dist(p, \partial U(x)) < \varepsilon\}.$$

The function u will be given by a series $u = \sum_{n=1}^{\infty} u_n$, where each u_n is a C^∞ function on $\mathbb{R}^d$. For each n, we will write $\nabla u_n = v_n + w_n$, and we will denote $s_n = \sum_{k=1}^{n} \nabla u_k$ and $\sigma_n = \sum_{k=1}^{n} v_k$.

We will also define an increasing sequence of integers $(N_k)_{k\geq 0}$. The following conditions have to be fulfilled:

(o) $N_0 = 0$; u_0 is constant and nonzero, and $v_0 = w_0 = 0$.
(i) For each $n \geq 1$, $u_n\mid_{K_n^c} \equiv 0$; $v_n\mid_{K_n^c} \equiv 0$; and v_n is constant on each cube of $\mathcal{R}_n$.
(ii) For each $n \geq 1$, $\|w_n\|_\infty \leq 2^{-n}$.
(iii) For each $n \geq 1$, $s_n(x_0) = 0$ and for every $x \in \Omega$, $s_n(x) \in U(x)$.
(iv) For each $n \geq 1$ and $x \in \Omega$, if we denote $r_x = \sup\{\| p \| : p \in U(x)\}$, we have

$$\|\sigma_n(x)\| \leq 1 + r_x.$$

(v) For each $n \geq 1$ and $x \in \Omega$,

$$\langle \mathbf{t}(\sigma_n(x)) \, , \, \sigma_{n+1}(x) - \sigma_n(x) \rangle = 0,$$

where the map $\mathbf{t} : B(0, 1 + r_x) \to \mathbb{R}^d$ is given by Lemma 2.3.
(vi) For each $n \geq 1$, we have that $0 < \|u_{n+1}\|_\infty \leq 2^{-n}\|u_n\|_\infty$; and if $N_{k-1} < n \leq N_k$, then $\|v_n\|_\infty \leq \varepsilon_k/8$ and $osc(s_n, \|u_{n+1}\|_\infty) \leq \varepsilon_k/8$.
(vii) For each $n \geq 1$,

$$\lambda_d\{x \in \Omega : s_{N_k}(x) \notin \partial_{\varepsilon_k} U(x)\} \leq 2^{-k} \lambda_d(\Omega).$$

Inductive step: Fix $k \geq 1$, assume that N_{k-1} has been defined, and for some $n \geq N_{k-1}$ the partition $\mathcal{Q}_n$ and the function u_n have been constructed. By (i), σ_n is constant on each cube Q of $\mathcal{Q}_n$; denote $\sigma_n(Q)$ the value at this cube. Choose $a = a(Q) \in \mathbb{R}^d$ such that $\|a(Q)\| = \varepsilon_k/8$ and $\langle \mathtt{t}(\sigma_n(Q)), a\rangle = 0$ (this is possible since $d \geq 2$). Now choose a cube partition $\widehat{\mathcal{Q}}_n$ of Q_0 refining $\mathcal{Q}_n$, with $\mathrm{diam}(\widehat{\mathcal{Q}}_n) \leq \frac{1}{2}\,\mathrm{diam}\mathcal{Q}_n$, and such that:

∘ If we denote $\widehat{\mathcal{R}}_n$ the family of all cubes in $\widehat{\mathcal{Q}}_n$ which are contained in Ω, and $\widehat{K}_n = \cup\{\widehat{Q} : \widehat{Q} \in \widehat{\mathcal{R}}_n\}$, we have that

$$\lambda_d(\Omega \setminus \widehat{K}_n) < 2^{-(k+1)}\lambda_d(\Omega). \tag{3.1}$$

∘ For each $\widehat{Q} \in \widehat{\mathcal{R}}_n$, $osc(s_n, \widehat{Q}) < \varepsilon_k/8$.

∘ For each $\widehat{Q} \in \widehat{\mathcal{R}}_n$, if $x, y \in \widehat{Q}$ then $dist_H(\partial U(x), \partial U(y)) < \varepsilon_k/4$.

Now applying Lemma 2.2, for each cube $\widehat{Q} \in \widehat{\mathcal{R}}_n$ we obtain a C^∞ function $u_{\widehat{Q}}$ on $\mathbb{R}^d$, and a cube partition $\mathcal{Q}_{n+1}$ of Q_0 which is a refinement of $\widehat{\mathcal{Q}}_n$ (and therefore of $\mathcal{Q}_n$), such that:

• $u_{\widehat{Q}}$ vanishes on a neighborhood of $\partial\widehat{Q}$.

• $0 < \|u_{\widehat{Q}}\|_\infty \leq 2^{-n}\|u_n\|_\infty$ and $osc(s_n, \|u_{\widehat{Q}}\|_\infty) < \varepsilon/8$.

• $\lambda_d\{x \in \widehat{Q} : \nabla u_{\widehat{Q}}(x) = \pm a(Q)\} \geq (1 - 2^{-k})\lambda_d(\widehat{Q})$, where Q is the unique cube of $\mathcal{Q}_n$ containing $\widehat{Q}$.

• $\nabla u_{\widehat{Q}} = v_{\widehat{Q}} + w_{\widehat{Q}}$, where $\|w_{\widehat{Q}}\|_\infty \leq \varepsilon_k/2^{n+2}$, $v_{\widehat{Q}}$ is constant on each cube of $\mathcal{Q}_{n+1}$; and $v_{\widehat{Q}}(\widehat{Q}) \subset [-a(Q), a(Q)]$, where Q is the unique cube of $\mathcal{Q}_n$ containing $\widehat{Q}$.

Next we define the function u_{n+1} on $\mathbb{R}^d$. We first choose for each $\widehat{Q} \in \widehat{\mathcal{R}}_n$ a point $x_{\widehat{Q}} \in \widehat{Q}$, with the condition that $x_0 = x_{\widehat{Q}}$ for some $\widehat{Q}$. We define u_{n+1} on each cube of $\widehat{\mathcal{R}}_n$ in the following way:

(1) If $s_n(x_{\widehat{Q}}) \in \partial_{5\varepsilon_k/8}U(x_{\widehat{Q}})$, we set $u_{n+1} = 0$ on $\widehat{Q}$; and $v_{n+1} = w_{n+1} = 0$.

(2) If $s_n(x_{\widehat{Q}}) \notin \partial_{5\varepsilon_k/8}U(x_{\widehat{Q}})$, we set $u_{n+1} = u_{\widehat{Q}}$ on $\widehat{Q}$; and $v_{n+1} = v_{\widehat{Q}}; w_{n+1} = w_{\widehat{Q}}$. Note that, in this case, we have

$$\lambda_d\{x \in \widehat{Q} : \|\nabla u_{n+1}(x)\| = \varepsilon_k/8\} \geq (1 - 2^{-k})\lambda_d(\widehat{Q}).$$

(3) Finally, we set $u_{n+1} = 0$ on $(\widehat{K}_n)^c$; and $v_{n+1} = w_{n+1} = 0$.

In this way we obtain that u_{n+1} is a C^∞ function on $\mathbb{R}^d$, which vanish on a neighborhood of $\partial\widehat{Q}$ for every $\widehat{Q} \in \widehat{\mathcal{R}}_n$.

Next we are going to check conditions (i) to (vii) for $n + 1$. Conditions (i) and (ii) are clearly satisfied. On the other hand, since $x_0 \in \partial\widehat{Q}$ for some $\widehat{Q} \in \widehat{\mathcal{R}}_n$, we have that u_{n+1} vanishes on a neighborhood of x_0. Thus $s_{n+1}(x_0) = s_n(x_0) + \nabla u_{n+1}(x_0) = 0$. In order to complete condition (iii), let $x \in \Omega$:

- If $x \in (\widehat{K}_n)^c$, then $u_{n+1} = 0$ on a neighborhood of x and $s_{n+1}(x) = s_n(x) \in U(x)$.

- If $x \in \widehat{Q} \in \widehat{\mathcal{R}}_n$ with $s_n(x_{\widehat{Q}}) \in \partial_{5\varepsilon_k/8} U(x_{\widehat{Q}})$, then $u_{n+1} = 0$ on a neighborhood of $\widehat{Q}$ and $s_{n+1}(x) = s_n(x) \in U(x)$.

- Finally, if $x \in \widehat{Q} \in \widehat{\mathcal{R}}_n$ with $s_n(x_{\widehat{Q}}) \notin \partial_{5\varepsilon_k/8} U(x_{\widehat{Q}})$, since $osc(s_n, \widehat{Q}) < \varepsilon/8$ we have that

$$dist(s_n(x), \partial U(x_{\widehat{Q}}) \geq \frac{5\varepsilon_k}{8} - \frac{\varepsilon_k}{8} = \frac{\varepsilon_k}{2}.$$

Since $dist_H(\partial U(x_{\widehat{Q}}), \partial U(x)) < \varepsilon_k/4$, we have that $\partial U(x) \subset B(\partial U(x_{\widehat{Q}}), \varepsilon_k/4)$, and then

$$dist(s_n(x), \partial U(x)) \geq \frac{\varepsilon_k}{4}.$$

Since $s_{n+1}(x) = s_n(x) + \nabla u_{n+1}(x)$ and

$$\|\nabla u_{n+1}(x)\| \leq \|v_{n+1}\|_\infty + \|w_{n+1}\|_\infty < \frac{\varepsilon_k}{8} + \frac{\varepsilon_k}{8} = \frac{\varepsilon_k}{4},$$

we deduce that $s_{n+1}(x) \in U(x)$.

Condition (iv) follows from (iii) and

$$\|s_{n+1}(x) - \sigma_{n+1}(x)\| \leq \sum_{k=1}^{n+1} \|w_k(x)\| \leq 1.$$

In order to prove (v), let $x \in \Omega$. If $v_{n+1}(x) = 0$, then $\sigma_{n+1}(x) - \sigma_n(x) = 0$ and the condition is satisfied. On the other hand, if $v_{n+1}(x) \neq 0$, then $x \in \widehat{Q}$ and $v_{n+1} = v_{\widehat{Q}}$ for some $\widehat{Q} \in \tilde{\mathcal{R}}_n$. In this case $\sigma_n(x) = \sigma_n(Q)$, and $v_{n+1}(x) \in [-a(Q), a(Q)]$, where Q is the unique cube of $\mathcal{Q}_n$ containing $\widehat{Q}$. Thus $v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x)$ is proportional to $a(Q)$, and therefore orthogonal to $\mathbf{t}(\sigma_n(Q))$.

Now we are going to see that $u_{n+1} \neq 0$. Note that the point $x_0 = x_{\widehat{Q}}$ for some $\widehat{Q} \in \widehat{\mathcal{R}}_n$, and $0 = s_n(x_0) \notin \partial_{5\varepsilon_k/8} U(x_0)$. Therefore $u_{n+1} = u_{\widehat{Q}} \neq 0$ on $\widehat{Q}$. Then condition (vi) also holds, although we still have to define the integer N_k.

Finally, let us prove that (vii) is satisfied. Suppose, to the contrary, that for every $n > N_{k-1}$ we have:

$$\lambda_d\{x \in \Omega : s_n(x) \notin \partial_{\varepsilon_k} U(x)\} > 2^{-k} \lambda_d(\Omega).$$

By (3.1), we obtain that

$$\lambda_d\{x \in \widehat{K}_n : s_n(x) \notin \partial_{\varepsilon_k} U(x)\} > 2^{-(k+1)} \lambda_d(\widehat{K}_n).$$

Suppose now that $\widehat{Q} \in \widehat{\mathcal{R}}_n$ contains a point y which also belongs to the set

$$\{x \in \widehat{K}_n : s_n(x) \notin \partial_{\varepsilon_k} U(x)\}.$$

Since $osc(s_n, \widehat{Q}) < \varepsilon/8$ and $dist_H(\partial U(x_{\widehat{Q}}), \partial U(y)) < \varepsilon_k/4$, we have that

$$dist(s_n(x_{\widehat{Q}}), \partial U(x_{\widehat{Q}}) \geq \varepsilon_k - \frac{\varepsilon_x}{4} - \frac{\varepsilon_k}{8} = \frac{5\varepsilon_k}{8}.$$

As we have noticed, in this case

$$\lambda_d\{x \in \widehat{Q} : \|s_{n+1}(x) - s_n(x)\| \geq \varepsilon_k/8\} \geq$$
$$\lambda_d\{x \in \widehat{Q} : \|\nabla u_{n+1}(x)\| = \varepsilon_k/8\} \geq (1-2^{-k})\lambda_d(\widehat{Q}).$$

Now the proportion of cubes $\widehat{Q}$ in $\widehat{\mathcal{R}}_n$ satisfying this has to be at least $2^{-(k+1)}$. Therefore

$$\lambda_d\{x \in \widehat{K}_n : \|s_{n+1}(x) - s_n(x)\| \geq \varepsilon_k/8\} \geq (1-2^{-k})2^{-(k+1)} > 0.$$

This will be a contradiction with the fact that pointwise convergence implies convergence in probability, since we are going to prove that the sequence (s_n) is pointwise convergent. Indeed, for each $x \in \mathbb{R}^d$ it follows from (ii) that

$$s_n(x) - \sigma_n(x) = \sum_{k=1}^{n} w_k(x)$$

is convergent , so that conditions (iii), (iv) and (v) allow us to apply Lemma 2.3 to conclude that (s_n) is convergent. This contradiction shows that there exists an integer $N_k > N_{k+1}$ satisfying (vii). This concludes the inductive step.

The function u: We now define

$$u = \sum_{n=1}^{\infty} u_n.$$

By (vi) the series is uniformly convergent on $\mathbb{R}^d$, so that $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and it is clear that $u\,|_{\Omega^c} \equiv 0$. In order to see that u is differentiable on $\mathbb{R}^d$, we check the conditions of Lemma 2.1. For each $n \geq 1$ let k_n be an integer with $N_{k_n-1} < n \leq N_{k_n}$. From (ii) and (vi) we have that

- $\|\nabla u_n\|_\infty \leq \|v_n\|_\infty + \|w_n\|_\infty \leq \varepsilon_{k_n}/8 + 2^{-n} \to 0,$
- $\|\nabla u_{n+1}\|_\infty = o(\|\nabla u_n\|_\infty),$
- $osc(s_n\,, \|\nabla u_n\|_\infty) \leq \varepsilon_{k_n}/8 \to 0.$

Moreover, it follows as before that (s_n) is pointwise convergent, that is, $\sum_{n=1}^{\infty}(x)$ is convergent for every $x \in \mathbb{R}^d$. Then we have by Lemma 2.1 that u is everywhere differentiable and ∇u is the pointwise limit of s_n. Now from (iii) we obtain that $\nabla u(x) \in \overline{U(x)}$ for every $x \in \Omega$. To finish this case, consider $x \in \Omega$ such that $\nabla u(x) \notin \partial U(x)$. Taking into account that $\nabla u(x) = \lim_k s_{N_k}(x)$, we can find some integer k_0 such that $dist(s_{N_k}\,, \partial U(x)) > \varepsilon_{k_0} \geq \varepsilon_k$ for every $k \geq k_0$. Therefore

$$\lambda_d\{x \in \Omega : \nabla u(x) \notin \partial U(x)\} \leq \lambda_d\{x \in \Omega : s_{N_k}(x) \notin \partial_{\varepsilon_k} U(x)\} \leq 2^{-k}\lambda_d(\Omega) \to 0.$$

That means that $\nabla u(x) \in \partial U(x)$ for almost every $x \in \Omega$.

General Case: In the case that Ω is not bounded, we consider a decomposition

$$\mathbb{R}^d = \bigcup_{j=1}^{\infty} Q_j,$$

where $(Q_j)_{j\geq 1}$ is a locally finite family of cubes, and $int(Q_j)\cap int(Q_k)=\emptyset$ if $j\neq k$. For each $j\geq 1$, we denote $\Omega_j=\Omega\cap int(Q_j)$. By the previous case, if $\Omega_j\neq\emptyset$ we obtain a differentiable function $u_j:\mathbb{R}^d\to\mathbb{R}$ such that:

(1) $u_j\mid_{(\Omega_j)^c}\equiv 0$, and $\nabla u_j\mid_{(\Omega_j)^c}\equiv 0$.
(2) $\nabla u_j(x)\in\overline{U(x)}$ for every $x\in\Omega_j$.
(3) $\nabla u_j(x)\in\partial U(x)$ for almost every $x\in\Omega_j$.

Note that in particular $u_j\mid_{(Q_j)^c}\equiv 0$ and $\nabla u_j\mid_{(Q_j)^c}\equiv 0$. If $\Omega_j=\emptyset$, we set $u_j=0$. Then we define $u:\mathbb{R}^d\to\mathbb{R}$ by setting $u=u_j$ on each Q_j. Then it is easy to see that u is differentiable and satisfies the required conditions. □

Corollary 3.2. *Let Ω be an open subset of $\mathbb{R}^d$ with $d\geq 2$, and let $F:\Omega\times\mathbb{R}^d\to\mathbb{R}$ be a continuous function satisfying :*

(A) *There exists a continuous function $u_0:\overline{\Omega}\to\mathbb{R}$ such that u_0 is C^1-smooth on Ω, and $F(x,\nabla u_0(x))\leq 0$ for every $x\in\Omega$.*
(B) *For each $x\in\Omega$ the set $U(x)=\{p\in\mathbb{R}^d: F(x,p)<0\}$ is bounded and the set-valued mapping $x\mapsto\partial U(x)$ from Ω to $(\mathcal{C}(\mathbb{R}^d),d_H)$ is continuous.*

Then there exists an almost classical solution of $F(x,\nabla u(x))=0$, with Dirichlet condition $u\mid_{\partial\Omega}=u_0$.

Proof. Consider the open set $\Omega_0=\{x\in\Omega: F(x,\nabla u_0(x))<0\}$, which we may assume to be nonempty. For each $x\in\Omega_0$, we have that $\nabla u_0(x)\in U(x)$, and we consider the set $V(x)=U(x)-\nabla u_0(x)$. Since ∇u_0 is continuous on Ω, the sets $V(x)$ satisfy the continuity property required in Theorem 3.1. Then there exists a differentiable function $v:\mathbb{R}^d\to\mathbb{R}$ such that

(1) $v\mid_{\Omega_0^c}\equiv 0$ and $\nabla v\mid_{\Omega_0^c}\equiv 0$.
(2) $\nabla v(x)\in\overline{V(x)}$ for every $x\in\Omega_0$.
(3) $\nabla v(x)\in\partial V(x)$ for almost every $x\in\Omega_0$.

It is easy to see that the function u defined on $\overline{\Omega}$ by $u(x)=u_0(x)+v(x)$ is an almost classical solution of $F(x,\nabla u(x))=0$, with Dirichlet condition $u\mid_{\partial\Omega}=u_0$. □

We can now prove Theorem 1.2. We restate it for convenience.

Theorem 1.2 *Let Ω be an open subset of $\mathbb{R}^d$ with $d\geq 2$, and let $F:\Omega\times\mathbb{R}^d\to\mathbb{R}$ be a C^1 function. Suppose that the following conditions hold :*

(A) *There exists a continuous function $u_0:\overline{\Omega}\to\mathbb{R}$ such that u_0 is C^1-smooth on Ω, and $F(x,\nabla u_0(x))\leq 0$ for every $x\in\Omega$.*
(B) *For each $x\in\Omega$, the set $B(x)=\{p\in\mathbb{R}^d: F(x,p)\leq 0\}$ is compact, the set $S(x)=\{p\in\mathbb{R}^d: F(x,p)=0\}$ is connected, and the function $F(x,\cdot)$ has maximal rank on $S(x)$.*

Then there exists an almost classical solution of $F(x,\nabla u(x))=0$, with Dirichlet condition $u\mid_{\partial\Omega}=u_0$.

Proof. Consider the open set $\Omega_0=\{x\in\Omega: F(x,0)<0\}$, which we may assume to be nonempty. For each $x\in\Omega_0$ consider the set $U(x)=\{p\in\mathbb{R}^d: F(x,p)<0\}$, and $\partial U(x)=S(x)$.

Now fix $x_0\in\Omega_0$. We take into account that $S(x_0)$ is compact, and that F has maximal rank on $\{x_0\}\times S(x_0)$, and we apply the Implicit Function Theorem. Then we can find a neighborhood V^{x_0} and a finite family $V_1, \dots V_m$ of open subsets of $\mathbb{R}^d$ with compact closure such that $S(x_0) \subset V_1 \cup \dots \cup V_m$ and, for each $j = 1, \dots, m$, the set of points $(x, p) \in V^{x_0} \times V_j$ satisfying $F(x, p) = 0$ coincides, up to a permutation in the coordinates of p, with the graph of a mapping $g_j : V^{x_0} \times W_j \to \mathbb{R}$, where W_j is an open subset of $\mathbb{R}^{d-1}$ and g_j is C^1 and uniformly continuous on $V^{x_0} \times W_j$.

We claim that there exists a neighborhood $W^{x_0} \subset V^{x_0}$ such that, for every $x \in W^{x_0}$, we have that $S(x) \subset V_1 \cup \dots \cup V_m$. Indeed, if this is not the case, there exist a sequence $(x_n)_n \subset V^{x_0}$ converging to x_0 and a sequence $(p_n)_n \subset (V_1 \cup \dots \cup V_m)^c$ such that $F(x_n, p_n) = 0$ for every n. Since each $S(x_n)$ is connected and we have that $S(x_n) \cap (V_1 \cup \dots \cup V_m) \neq \emptyset$ for every n, we can assume that, in fact, $(p_n)_n \subset \partial(V_1 \cup \dots \cup V_m)$, which is a compact set. Then, taking a subsequence, we can assume that $(p_n)_n$ is convergent to some point $p_0 \in \partial(V_1 \cup \dots \cup V_m)$. Now $F(x_0, p_0) = \lim_n F(x_n, p_n) = 0$, that is, $p_0 \in S(x_0)$, and this contradicts the fact that $S(x_0) \subset V_1 \cup \ldots \cup V_m$.

Now we are going to check the continuity property for the Hausdorff distance of Corollary 3.2 (B), at the point x_0. For each $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x_0, \delta) \subset W^{x_0}$ and if $x \in B(x_0, \delta)$ and $\overline{p} \in W_j$, then

$$|g_j(x, \overline{p}) - g_j(x_0, \overline{p})| < \varepsilon$$

for every $j = 1, \dots, m$. Thus if $x \in B(x_0, \delta)$, since we have that

$$S(x_0) = \{p \in V_1 \cup \dots \cup V_m : F(x_0, p) = 0\}$$

and

$$S(x) = \{p \in V_1 \cup \dots \cup V_m : F(x, p) = 0\}$$

it is not difficult to see that

$$d_H(S(x), S(y)) < \varepsilon.$$

In this way we obtain from Corollary 3.2 that there exists a continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that :

(1) $u = u_0$ on $\partial\Omega$.
(2) For every $x \in \Omega$, u is differentiable at x and $F(x, \nabla u(x)) \leq 0$.
(3) For *almost* every $x \in \Omega$, $F(x, \nabla u(x)) = 0$.

□

Remarks. (1) In Theorem 1.2 above, if we replace condition (A) by :

(A') There exists a C^1 function $u_0 : \mathbb{R}^d \to \mathbb{R}$ such that $F(x, \nabla u_0(x)) \leq 0$, for every $x \in \Omega$,

then the same proof shows that there exists a differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $u = u_0$ on Ω^c and $\nabla u = \nabla u_0$ on Ω^c, and such that u is an almost classical solution of $F(x, \nabla u(x)) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = u_0$.

(2) It is possible to obtain a variant of Theorem 1.2, with an analogous proof, replacing condition (B) by the following alternative hypothesis :

(B') For each $x \in \Omega$, there exists a closed neighborhood W^x such that the set $\{(y, p) \in \Omega \times \mathbb{R}^d : y \in W^x ; F(y, p) \leq 0\}$ is compact, and the function $F(x, \cdot)$ has maximal rank on the set $S(x) = \{p \in \mathbb{R}^d : F(x, p) = 0\}$.

Corollary 3.3. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a C^1 function. Suppose that the following conditions hold :*

- *For each $x \in \Omega$, $F(x, 0) \leq 0$, the set $B(x) = \{p \in \mathbb{R}^d : F(x, p) \leq 0\}$ is compact, the set $S(x) = \{p \in \mathbb{R}^d : F(x, p) = 0\}$ is connected, and the function $F(x, \cdot)$ has maximal rank on $S(x)$.*

Then there exists a differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $u \equiv 0$ on Ω^c and $\nabla u \equiv 0$ on Ω^c, and such that u is an almost classical solution of $F(x, \nabla u(x)) = 0$, with Dirichlet condition $u \mid_{\partial\Omega} = 0$.

4. Almost classical solutions on smooth manifolds

In order to obtain our main result for smooth manifolds, we will use the concept of triangulation, as it is given by Whitney in [10] (see also [3]). In what follows we assume that every smooth manifold is Hausdorff and second countable. If M is a smooth d-dimensional manifold, a *triangulation* of M is a pair (K, π), where K is a simplicial complex and $\pi : K \to M$ is a homeomorphism, such that for each d-dimensional simplex S of K there exists a local chart (W, φ) of M, where W is a neighborhood of $\pi(S)$ and $\varphi \circ \pi$ is affine on S. According to Whitney [10], every smooth manifold admits a triangulation.

As usual, if M is a smooth manifold we denote by T^*M its cotangent bundle. We now establish the following generalization of Theorem 1.3.

Theorem 4.1. *Let M be a smooth manifold of dimension $d \geq 2$, consider an open subset Ω of M, and let $F : T^*\Omega \to \mathbb{R}$ be a C^1 function. Suppose that the following conditions hold:*

(A) *There exists a C^2 function $u_0 : M \to \mathbb{R}$ such that $F(x, du_0(x)) \leq 0$, for every $x \in \Omega$.*
(B) *For each $x \in \Omega$, the set $B(x) = \{\xi \in T^*_x M : F(x, \xi) \leq 0\}$ is compact, the set $S(x) = \{\xi \in T^*_x M : F(x, \xi) = 0\}$ is connected, and the function $F(x, \cdot)$ has maximal rank on the set $S(x)$.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that :

(1) *$u = u_0$ on Ω^c and $du = du_0$ on Ω^c.*
(2) *$F(x, du(x)) \leq 0$ for every $x \in \Omega$.*
(3) *$F(x, du(x)) = 0$ for almost every $x \in \Omega$.*

Proof. We will consider two cases.
First Case : Suppose first that $u_0 \equiv 0$ on M. Let (K, π) be a triangulation of M, where K is a simplicial complex and $\pi : K \to M$ is a homeomorphism, and consider the family $\{S_i\}_{i \in I}$ of all d-dimensional simplices of K. For each $i \in I$, denote $T_i = \pi(S_i)$. Then

$$M = \bigcup_{i \in I} T_i,$$

each ∂T_i has measure zero in M, and $int(T_i) \cap int(T_j) = \emptyset$ if $i \neq j$. Since M is locally compact and π is a homeomorphism, we have that the simplicial complex K is locally compact, and therefore locally finite. Thus the family $\{T_i\}_{i \in I}$ is locally finite. Since M is also σ-compact, we obtain that the index set I is countable. For each $i \in I$, denote $\Omega_i = \Omega \cap int(T_i)$. Then the set $\Omega \setminus (\cup_{i \in I} \Omega_i)$ has measure zero in M.

For each $i \in I$ there is a chart (W_i, φ_i) in M with $T_i \subset W_i$. Associated to this chart there is a natural diffeomorphism

$$\Phi_i : T^*W_i \to \varphi_i(W_i) \times \mathbb{R}^d$$

of the form $\Phi_i(x, \xi) = (\varphi_i(x), h_i(x, \xi))$, where $h_i(x, \xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h_i(x, \xi), p\rangle = \xi \circ d\varphi_i(x)^{-1}(p).$$

If $\varphi_i(\Omega_i) \neq \emptyset$, consider the function $G_i = F \circ \Phi_i^{-1} : \varphi_i(W_i) \times \mathbb{R}^d \to \mathbb{R}$. By applying Corollary 3.3 to the function G_i, we obtain that there exists a differentiable function $v_i : \varphi_i(W_i) \to \mathbb{R}$ such that :

1. $v_i \mid_{\varphi(\Omega_i)^c} \equiv 0$ and $\nabla v_i \mid_{\varphi(\Omega_i)^c} \equiv 0$.
2. $G_i(z, \nabla v_i(z)) \leq 0$ for every $z \in \varphi(\Omega_i)$.
3. $G_i(z, \nabla v_i(z)) = 0$ for almost every $z \in \varphi(\Omega_i)$.

Then the function $u_i = v_i \circ \varphi_i : W_i \to \mathbb{R}$ is differentiable on W_i, and for each $x \in W_i$ we have that

$$\begin{aligned} F(x, du_i(x)) &= F(x, dv_i(\varphi_i(x)) \circ d\varphi_i(x)) \\ &= F(\Phi_i^{-1}(\varphi_i(x), \nabla v_i(\varphi_i(x))) = G_i(\varphi_i(x), \nabla v_i(\varphi_i(x))). \end{aligned}$$

As a consequence, we obtain that

1. $u_i \mid_{\Omega_i^c} \equiv 0$ and $\nabla u_i \mid_{\Omega_i^c} \equiv 0$.
2. $F(x, du_i(x)) \leq 0$ for every $x \in \Omega_i$.
3. $F(x, du_i(x)) = 0$ for almost every $x \in \Omega_i$.

On the other hand, if If $\varphi_i(\Omega_i) = \emptyset$, we set $u_i = 0$. Now we define $u : M \to \mathbb{R}$ by setting $u = u_i$ on each T_i. Then u is well-defined, since $\partial T_i \subset \Omega_i^c$ for each $i \in I$. Taking into account that the family $\{T_i\}_{i \in I}$ is locally finite, we see that u is differentiable on M, and it satisfies the required conditions.

General Case : In general, we can consider the C^1 function $G : T^*\Omega \to \mathbb{R}$ defined by:

$$G(x, \eta) = F(x, \eta + du_0(x))).$$

It is clear that, for each $x \in \Omega$, the set

$$\{\eta \in T_x^*M : G(x, \eta) \leq 0\} = \{\xi \in T_x^*M : F(x, \xi) \leq 0\} - du_0(x),$$

is compact; and $G(x, \cdot)$ has maximal rank on the set

$$\{\eta \in T_x^*M : G(x, \eta) = 0\} = \{\xi \in T_x^*M : F(x, \xi) = 0\} - du_0(x).$$

Thus by the first case we obtain that there exists a differentiable function $v : M \to \mathbb{R}$ such that:

1. $v \mid_{\Omega_0^c} \equiv 0$ and $dv \mid_{\Omega_0^c} \equiv 0$.
2. $F(x, dv(x)) \leq 0$ for every $x \in \Omega$.
3. $F(x, dv(x)) = 0$ for almost every $x \in \Omega$.

Now it is easy to see that the function $u = u_0 + v$ satisfies the required properties. □

Corollary 4.2. *Let M be a Riemannian manifold (of dimension ≥ 2) and let Ω be an open subset of M. Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $u \mid_{\Omega^c} \equiv 0$ and $\|\nabla u(x)\|_x = 1$ for almost every $x \in \Omega$.*

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