

An integro-differential equation arising as a limit of individual cell-based models.

M. BODNAR^{†‡} & J. J. L. VELAZQUEZ[†]

[†]*Departamento de Matemática Aplicada
Facultad de Ciencias Matemáticas
Universidad Complutense,
Avd. Complutense, 28040 Madrid, Spain*

[‡]*Institute of Applied Mathematics and Mechanics,
Faculty of Mathematics, Informatics & Mechanics,
Warsaw University,
ul. Banacha 2, 02-097 Warszawa, Poland*

Abstract — In this paper we study mathematical properties of an integro-differential equation that arises as a particular limit case in the study of individual cell based model. We obtain global well-posedness for some classes of interaction potentials and finite blow-up for others. We also discuss steady states and long time asymptotics for the solutions of the problem.

1 Introduction

The aim of the present paper is to study some basic mathematical properties of the following integro-differential equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} \left(u(x, t) \int_{\mathbb{R}} V(x - y) \frac{\partial}{\partial y} u(y, t) dy \right) \quad (1.1a)$$

with the initial condition

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \quad (1.1b)$$

In this paper, besides proving local well-posedness theorems we find a class of functions V for which the solutions of (1.1) are globally defined in time. We also describe some potentials for which the corresponding solutions of (1.1) blow-up in a finite time. Other questions like long time asymptotics and steady states are also considered.

The model (1.1) arises as a continuum limit for a system of particles evolving by means of the equations

$$\frac{dX_k(t)}{dt} = - \sum_{\substack{i=1 \\ i \neq k}} \nabla V(X_k(t) - X_i(t)), \quad k = 0, \pm 1, \dots, \quad (1.2)$$

where we suppose that the potential V produces relevant interactions between particles placed at some characteristic distance d and it decays fast enough for larger distances.

The problem of deriving macroscopic limit equations for the cell density in the limit of many particles has been extensively studied by different authors (cf. [10, 12–14, 17]). Particular applications to specific biological problems of models arising as limits of individual cell based models can be found in [5, 6, 9, 11]. In particular, a detailed discussion of the mathematical difficulties that can be found in the rigorous derivation of continuous models starting with individual cell based models is contained in [10, 11]. A problem that is well understood is the

study of the limit continuum equations for some classes of repulsive potentials V having a range of interaction much larger than the average distance between particles. In that case, we can "guess" the asymptotics of the limit "continuum" density with the following simple heuristic argument. Suppose that the range of interaction of V is some distance d . Let us denote as $u(x, t)$ the cell density. The average distance between particles might be then estimated as $\frac{1}{\rho}$. If $\frac{1}{\rho} \ll d$ we can compute the speed of the particles placed at a point x approximating the sum in (1.2) as:

$$v = - \sum_{\substack{i=1 \\ i \neq k}} \nabla V(X_k(t) - X_i(t)) \approx - \nabla \int V(\xi - X_i(t)) u(\xi, t) d\xi \quad (1.3)$$

The evolution of the macroscopic cell density is given by the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \quad (1.4)$$

where the flux of particles j is given by:

$$j = \rho v \quad (1.5)$$

Combining Eqs (1.3), (1.4) and (1.5) Eq. (1.1a) would follow. More detailed derivation of (1.1a) as well as heuristic derivation of the limit equation for large class of different potentials V can be found in [4, 10].

Suppose that the initial data $u_0(x)$ as well as the solution $u(x, t)$ changes in a spatial scale much larger than the range of interaction of the potential V . In that case, if V decays fast enough over distances larger than d , it would be possible to approximate $V(x - y)$ by means of a Dirac mass centered at $x = y$, i.e.:

$$\int V(x - y) u_y(y, t) dy \approx \left(\int V(\xi) d\xi \right) u_x(x, t)$$

With this approximation, Eq. (1.1a) becomes:

$$u_t = \frac{\left(\int V(\xi) d\xi \right)}{2} (u^2)_{xx} \quad (1.6)$$

If $\int V(\xi) d\xi > 0$, the equation (1.6) is just a particular case of the well known porous medium equation that has been extensively studied (cf. [1–3]). On the contrary, if $\int V(\xi) d\xi < 0$, equation (1.6) is a backward parabolic equation, that is an ill-posed mathematical problem.

The fact that for some particular classes of repulsive potentials satisfying $\int V(\xi) d\xi > 0$ the solutions of (1.1a) converge, in the limit of many particles to the solutions of the porous medium equation, if $d \ll 1$ has been rigorously proved by Oeshläger (cf. [12–14]). On the other hand, in the cases in which $\int V(\xi) d\xi < 0$ the ill-posedness of the formal limit problem indicates that it is not possible to approximate neither the solutions of (1.1a) or (1.2) by means of a simple PDE even if the range of interaction of the potential is very small compared with the natural scale for the variation of the initial data $u_0(x)$. Indeed, the ill-posedness of (1.6) is just a reminiscence of the fact that, for attractive potentials, the solutions of (1.1a) tend to aggregate in "clusters" with characteristic length d . Therefore, the solution of (1.1a) cannot be approximated by means of PDEs like (1.6) even if the initial datum has a characteristic length scale much larger than d .

The goal of this paper is to establish some basic mathematical properties of (1.1a). In particular we will address questions like local and global well-posedness, blow-up, steady states, and long time asymptotics.

The paper is organized as follows. In Section 2 we state the main assumptions and study basic properties of Problem (1.1): in Subsection 2.2 we prove theorems ensuring local existence and uniqueness of the solutions of (1.1a) and in Subsection 2.3 we discuss the nonnegativity and mass conservation property. In Section 3.1 we consider a global boundness of the solutions under suitable assumption on the potential as well as the existence and a profile of blow-up when the potential V satisfies other conditions. In Section 4 we study the existence of non-trivial steady states (in Subsection 4.1 and discuss the cases when only trivial steady states exist (see Subsection 4.2. In Section 5 long time asymptotics are discussed.

Section 3.2 a blow-up phenomena is discussed. In Section 4 we study the existence of steady states of Eq. (1.1a) and its form.

2 Basic properties

2.1 Basic definitions and assumptions

In this Subsection we obtain classical solvability for (1.1). We remark that several examples of nonuniqueness for nonlocal problems analogous to (1.1) with nonsmooth initial data has been obtained in [7] In this paper we will restrict our attention to smooth solutions. We begin formulating the main assumptions we will make on the potential V as well as a suitable functional framework.

We will assume that potential V satisfies the following:

- (A1) $\int_{\mathbb{R}} |V(x)| (1 + |x|) dx < \infty$;
- (A2) $\int_{|x|>1} (|V'(x)| + |V''(x)|) dx < \infty$;
- (A3) $V(x) = V_{\text{reg}}(x) + K(x)_+$, where $(x)_+ = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$;
- (A4) $V_{\text{reg}} \in C^3(\mathbb{R})$ and $|V'_{\text{reg}}(x)| + |V''_{\text{reg}}(x)| + |V'''_{\text{reg}}(x)| < \infty$.

The first two assumptions just provide suitable decay condition for V . The reason for assuming the form (A2) for the potential V is because we are interesting in handling potentials having jump in its first derivative at the origin.

We introduce the following functional norms

$$\|f\|_{C^k} = \sup_{x \in \mathbb{R}} \{|f(x)| + |f'(x)| + \dots + |f^{(k)}(x)|\}$$

in the space $C^k(\mathbb{R})$ and also

$$\|u\|_{X_T} = \sup_{t \in [0, T]} (\|u(\cdot, t)\|_{C^1}),$$

$$X_T \equiv \{u \in C((0, T); C^1(\mathbb{R})) : \|u\|_{X_T} < \infty\}$$

where $T > 0$. By $\|\cdot\|_p$, for $p \in [1; \infty]$, we denote a standard norm in $L^p(\mathbb{R})$.

2.2 Local existence and uniqueness of the solutions

The main result that we will prove in this subsection is the following

Theorem 2.1 (local existence and uniqueness). *Suppose that V satisfies assumptions (A1)–(A4). Then, for any given $u_0 \in C^1(\mathbb{R})$ there exists $T > 0$ and $u \in X_T$ which solves Problem (1.1) in the classical sense. Moreover T depends only on the $\|u_0\|_{C^1}$ and u is unique in the class of functions X_T .*

For the sake of clarity we state in formal manner the main heuristic idea used in the proof of Theorem 2.1. Notice that Assumption (A1) implies that equation (1.1a) might be rewritten as

$$u_t = \left(Ku \int_{-\infty}^x u(y, t) dy + u \int_{\mathbb{R}} V'_{\text{reg}}(x - y) u(y, t) dy \right)_x \quad (2.1)$$

Suppose that the initial data u_0 as well as $u(\cdot, t)$ are compactly supported. We then define ψ as

$$\psi(x, t) = \int_{-\infty}^x u(y, t) dy, \quad (2.2)$$

whence $\psi_x = u$. Then after integrating (2.1) with the respect to the variable x and using the fact that $u(\cdot, t)$ is compactly supported we obtain

$$\psi_t = K\psi\psi_x + \psi_x \int_{\mathbb{R}} V'_{\text{reg}}(x - y)\psi_x(y, t) dy. \quad (2.3)$$

In the absence of the last non-local term, (2.3) becomes well-known Burgers equation (cf. [18]). Then it might be easily solved integrating by characteristics. We will deal with the non-local term in (2.3) treating it as a perturbation, using a fixed point argument. Finally to solve the problem with non-compactly supported initial data we will use a density argument.

Notice that the rationale using (2.2) is to transform the non-local term $u \int_{-\infty}^x u(y, t) dy$ in (2.1) into the local term $\psi\psi_x$ in (2.3).

Lemma 2.2. *Suppose that the potential V satisfies assumptions (A1)–(A4). For any given $u_0 \in C^2(\mathbb{R})$ compactly supported, there exists a unique solution ψ of (2.3) (with the initial function $\psi_0 = \int_{-\infty}^x u_0(y) dy$) in $t \in [0, T]$ with $T > 0$ small enough and $\psi(\cdot, t) \in C^2(\mathbb{R})$.*

Proof: Let Y_T be the functional space

$$Y_T = \{\psi(\cdot, t) \in C^2(\mathbb{R}), 0 \leq t \leq T : \|\Phi\|_{Y_T} \equiv \sup_{t \in [0, T]} \|\psi(\cdot, t)\|_{C^2} < +\infty\}.$$

endowed with its natural norm $\|\cdot\|_{Y_T}$.

We now define an operator $S: Y_T \rightarrow Y_T$. As a preliminary step, for any $\bar{\psi} \in Y_T$ and for any $\eta \in \mathbb{R}$ we define $x(t, \eta)$ and $\psi(t, \eta)$ as the solutions of the following set of differential equations:

$$\frac{dx}{dt} = - \left(K\bar{\psi} + \int_{\mathbb{R}} V'_{\text{reg}}(x - y)\bar{\psi}_x(y, t) dy \right) \quad \frac{d\psi}{dt} = 0, \quad (2.4)$$

$$x(0, \eta) = \eta, \quad \psi(0, \eta) = \psi_0(\eta). \quad (2.5)$$

The functions $x(t, \eta)$ and $\psi(t, \eta)$ are uniquely defined for $t \in [0, T]$ due to assumption (A4). Moreover the equation for ψ can be integrated trivially and Eq. (2.4) transformed into

$$\frac{dx}{dt} = - \left(K\psi_0(\eta) + \int_R V'_{\text{reg}}(x(t, \eta) - y) \bar{\psi}_x(y, t) dy \right). \quad (2.6)$$

Since $\bar{\psi} \in Y_T$, the function $x(t, \eta)$ is differentiable with respect to η and its derivative can be computed by means of the solutions to the equation

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial x}{\partial \eta} \right) &= -K\psi'_0(\eta) - \left(\int_R V'_{\text{reg}}(x(t, \eta) - y) \bar{\psi}_{xx}(y, t) dy \right) \frac{\partial x}{\partial \eta}, \\ \frac{\partial x}{\partial \eta}(0, \eta) &= 1. \end{aligned} \quad (2.7)$$

whose solution is given by

$$\begin{aligned} \frac{\partial x}{\partial \eta}(t, \eta) &= \exp \left(- \int_0^t \int_R V'_{\text{reg}}(x(s, \eta) - y) \bar{\psi}_{xx}(y, s) dy ds \right) - \\ &K\psi'_0(\eta) \int_0^t \exp \left(- \int_s^t \int_R V'_{\text{reg}}(x(\xi, \eta) - y) \bar{\psi}_{xx}(y, \xi) dy d\xi \right) ds \end{aligned} \quad (2.8)$$

Henceforth

$$\left| \frac{\partial x}{\partial \eta}(t, \eta) - 1 \right| \leq CT \quad (2.9)$$

for some constant $C = C(\|\psi_0\|_{C^2}, \|\bar{\psi}\|_{Y_T})$. Therefore if T is assumed to be small enough the function $x(t, \cdot)$ is invertible if $0 \leq t \leq T$, due to implicit function theorem. Let us define a function $\eta(x, t)$ for $0 \leq t \leq T$ and $x \in \mathbb{R}$ by means of

$$x(t, \eta(x, t)) = x. \quad (2.10)$$

Notice that the function η depends on the choice of $\bar{\psi}$ but will not write this dependence explicitly unless it is needed. Then for any $\bar{\psi} \in Y_T$ we define the operator S as follows

$$(S\bar{\psi})(x, t) = \psi_0(\eta(x, t)). \quad (2.11)$$

It can be readily seen that the solutions of the fixed point equation $\psi = S(\psi)$ are exactly the solutions of Eq (2.3) with initial data $\psi(x, 0) = \psi_0(x)$. This follows from the fact that the ODEs (2.4), (2.5) are the characteristics equations associated to (2.3) if $\bar{\psi} = \psi$.

In order to show that this operator brings the space Y_T to Y_T we must show that the function η is twice differentiable with respect to x . This just follows from classical results for solutions of ODEs because $\frac{\partial^2 x}{\partial \eta^2}$ solves the equation

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial^2 x}{\partial \eta^2} \right) &= -K\psi''_0(\eta) - \left(\int_{\mathbb{R}} V''_{\text{reg}}(x(t, \eta) - y) \bar{\psi}_{xx}(y, t) dy \right) \left(\frac{\partial x}{\partial \eta} \right)^2 \\ &- \left(\int_{\mathbb{R}} V'_{\text{reg}}(x(t, \eta) - y) \bar{\psi}_{xxx}(y, t) dy \right) \frac{\partial^2 x}{\partial \eta^2} \end{aligned} \quad (2.12)$$

with initial condition $\frac{\partial^2 x}{\partial \eta^2}(0, \eta) = 0$. Arguing as in derivation of (2.9) we obtain

$$\left| \frac{\partial^2 x}{\partial \eta^2} \right| < CT,$$

where we are using assumption (A1)–(A4), as well as the regularity on ψ_0 and $\bar{\psi}$ and C depends on the same variables as in (2.9).

Moreover, using Gronwall's Lemma, as well as (2.8) and (2.12) we obtain

$$\|S\psi - \psi_0\|_{Y_T} \leq CT,$$

where $C = C(\|\psi_0\|_{C^2}, \|\bar{\psi}\|_{Y_T})$ is bounded for $\|\bar{\psi}\|_{Y_T}$ bounded. Therefore, S transforms the ball $\|\bar{\psi} - \psi_0\|_{Y_T} \leq 1$ into itself for $T > 0$ small enough.

In order to apply Banach fixed point Theorem it only remains to show that the operator S is contractive in this ball for $T > 0$ small enough. Let us assume that $\bar{\psi}_1, \bar{\psi}_2$ are in the ball $\|\bar{\psi} - \psi_0\|_{Y_T} \leq 1$ and η_1, η_2 are the corresponding functions defined by (2.10), with x_1, x_2 being the corresponding solutions of (2.4). Then

$$S(\bar{\psi}_i) = \psi_0(\eta_i(x, t)) \quad i = 1, 2. \quad (2.13)$$

Therefore we need to obtain

$$\|S(\bar{\psi}_1) - S(\bar{\psi}_2)\|_{Y_T} \leq \lambda(T) \|\bar{\psi}_1 - \bar{\psi}_2\|_{Y_T}, \quad \lambda(T) < 1 \text{ for } T \text{ small enough.}$$

To this end notice that using (2.13) one obtains

$$\|S(\bar{\psi}_1) - S(\bar{\psi}_2)\|_{Y_T} \leq C \|\psi_0\|_{C^3} \sup_{t \in [0, T]} \|\eta_1(\cdot, t) - \eta_2(\cdot, t)\|_{C^2}. \quad (2.14)$$

In order to estimate $\|\eta_1(\cdot, t) - \eta_2(\cdot, t)\|_{C^2}$ notice, that (2.10) implies

$$\int_{\eta_2}^{\eta_1} \frac{\partial x_1}{\partial \eta}(t, z) dz = x_1(\eta_1, t) - x_1(\eta_2, t) = x_2(\eta_2, t) - x_1(\eta_2, t).$$

Due to (2.9) we have

$$\left| \frac{\partial x_1}{\partial \eta}(\eta, t) \right| \geq \frac{1}{2}$$

for T small enough. Therefore

$$|\eta_1(x, t) - \eta_2(x, t)| \leq 2 |x_1(\eta_2, t) - x_2(\eta_2, t)|. \quad (2.15)$$

The right-hand side of (2.15) can be estimated using the fact that x_1, x_2 solves (2.6). Then Gronwall's like argument yields

$$|x_1(\eta_2, t) - x_2(\eta_2, t)| \leq CT \|\bar{\psi}_1 - \bar{\psi}_2\|_{Y_T}.$$

Plugging this formula into (2.15) we obtain

$$|\eta_1(x, t) - \eta_2(x, t)| \leq CT \|\bar{\psi}_1 - \bar{\psi}_2\|_{Y_T}$$

for $0 \leq t \leq T$ and $x \in \mathbb{R}$. The terms $\eta_{1,x} - \eta_{2,x}$ and $\eta_{1,xx} - \eta_{2,xx}$ can be estimated in an analogous manner. Indeed using (2.10) we arrive at

$$\eta_{1,x} - \eta_{2,x} = -\frac{(x_{1,\eta} - x_{2,\eta})\eta_{2,xx}}{x_{1,\eta}} \quad (2.16)$$

$$\eta_{1,xx} - \eta_{2,xx} = \frac{(x_{2,\eta} - x_{1,\eta})\eta_{2,xx}}{x_{1,\eta}} + \frac{(x_{2,\eta\eta} - x_{1,\eta\eta})(\eta_{2,xx})^2}{x_{1,\eta}} + \frac{x_{1,\eta\eta}}{x_{1,\eta}}((\eta_{1,x})^2 - (\eta_{2,x})^2), \quad (2.17)$$

where $x_{i,\eta} = \frac{\partial}{\partial \eta} x_i(t, \eta_i)$ and the same for higher order derivatives. Using again (2.15), as well as the fact that $\eta_{i,x} = \frac{1}{x_{i,\eta}}$, we can reduce the problem of estimating $\eta_{1,x} - \eta_{2,x}$, $\eta_{1,xx} - \eta_{2,xx}$ to the one of estimating differences $|x_{1,\eta} - x_{2,\eta}|$, $|x_{1,\eta\eta} - x_{2,\eta\eta}|$. This can be made using the fact that $x_{i,\eta}$ and $x_{i,\eta\eta}$, $i = 1, 2$ solve (2.7) and (2.12) respectively. Subtracting these equations for different values of i we can bound the differences $|x_{1,\eta}(t, \eta) - x_{2,\eta}(t, \eta)|$ by means of $CT \|\bar{\psi}_1 - \bar{\psi}_2\|$. Notice however, that these differences are computed for a given value η from the ones in (2.16) and (2.17). Therefore, more a careful argument is needed. To illustrate it we estimate the term $x_{1,\eta\eta} - x_{2,\eta\eta}$ because this one is the term containing more derivatives and henceforth is the hardest to estimate. We write

$$x_{1,\eta\eta}(t, \eta_1) - x_{2,\eta\eta}(t, \eta_2) = (x_{1,\eta\eta}(t, \eta_1) - x_{1,\eta\eta}(t, \eta_2)) + (x_{1,\eta\eta}(t, \eta_2) - x_{2,\eta\eta}(t, \eta_2)) \quad (2.18)$$

In the last parenthesis the argument η is the same and then it can be estimated as $CT \|\bar{\psi}_1 - \bar{\psi}_2\|$ as indicated above. On the other hand to estimate the first parenthesis on the right-hand side of (2.18) we need to control $x_{1,\eta\eta\eta}$. This can be made using the regularity assumptions on ψ_0 and V_{reg}''' made in (A4). Indeed classical regularity theory for ODEs implies that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial^3 x_i}{\partial \eta^3} \right) &= -K\psi_0'''(\eta) - \left(\int_R V_{\text{reg}}'''(x_i(t, \eta) - y) \bar{\psi}_{xx}(y, t) dy \right) \left(\frac{\partial x_i}{\partial \eta} \right)^3 \\ &\quad - 3 \left(\int_R V_{\text{reg}}''(x_i(t, \eta) - y) \bar{\psi}_{xx}(y, t) dy \right) \frac{\partial x_i}{\partial \eta} \cdot \frac{\partial^2 x_i}{\partial \eta^2} \\ &\quad - \left(\int_R V_{\text{reg}}'(x_i(t, \eta) - y) \bar{\psi}_{xx}(y, t) dy \right) \frac{\partial^3 x_i}{\partial \eta^3} \end{aligned}$$

with initial condition $\frac{\partial^3 x_i}{\partial \eta^3}(0, \eta) = 0$ henceforth, $\frac{\partial^3 x_i}{\partial \eta^3}(0, \eta) \leq CT$. The first derivative can be estimated similarly. Therefore,

$$\sup_{t \in [0, T]} \|\eta_1(\cdot, t) - \eta_2(\cdot, t)\|_{C^2} \leq CT \|\bar{\psi}_1 - \bar{\psi}_2\|_{Y_T}$$

Plugging this inequality into (2.14) we obtain

$$\|S(\bar{\psi}_1) - S(\bar{\psi}_2)\|_{Y_T} \leq CT \|\bar{\psi}_1 - \bar{\psi}_2\|_{Y_T},$$

where C is independent on T , $\bar{\psi}_i$, $i = 1, 2$. Thus the operator S is contractive for T small enough and Lemma 2.2 follows. \blacksquare

Lemma 2.3. *Suppose that potential V satisfies assumptions (A1)–(A4). For any given compactly supported $u_0 \in C^2(\mathbb{R})$ there exists a unique solution u to (1.1) for $t \in [0, T]$ with $T > 0$ small enough and $u(\cdot, t) \in C^1(\mathbb{R})$.*

Proof: This result is a reformulation of Lemma 2.2. Indeed given u solution of (1.1) with u_0 compactly supported it follows that $u(\cdot, t)$ is compactly supported for $0 \leq t \leq T$ because of boundedness of u as well as assumptions (A1)–(A4) for V imply that the speed of characteristics associated to (1.1) is bounded. Therefore given such u solution of (1.1) we obtain a solution of (2.3) by means of the transformation (2.2). Indeed combining (2.1) and (2.2) we obtain, after integrating once on x that

$$\psi_t - K\psi\psi_x - \psi_x \int_{\mathbb{R}} V'_{\text{reg}}(x-y)\psi(y, t)dy = f(t).$$

Since the support of ψ is contained in the half-line $[-R, +\infty)$ for some R it follows that $f(t) \equiv 0$. Therefore ψ solves (2.3).

Reciprocally, given a solution of (2.3) with initial data $\psi_0(x) = \int_{-\infty}^x u_0(y)dy$ we immediately obtain a solution to (1.1) by means $u = \psi_x$. ■

Lemma 2.4. *Suppose that potential V satisfies assumptions (A1)–(A4). Moreover, let us assume that derivatives $|V'_{\text{reg}}(x)|, \dots, |V_{\text{reg}}^{(k+1)}(x)|$ are bounded, where $k = 2, 3, 4, \dots$ and also that*

$$\int_R |V_{\text{reg}}^{(l)}(x)| dx < +\infty \quad \text{where } l = 2, 3, \dots, k+1.$$

For any given compactly supported $u_0 \in C^k(\mathbb{R})$ there exists a unique solution u to (1.1) for $t \in [0, T]$ with $T > 0$ small enough and $u(\cdot, t) \in C^{k-1}(\mathbb{R})$.

Proof: It is just a slight modification of Lemmas 2.2 and 2.3. The only novelty that additional derivatives of the characteristics curves must be estimated but this might be made with the same arguments as in proof of those lemmas using the additional regularity requested on u_0 and V_{reg} . ■

Actually, it turns out that it is possible to obtain better regularity estimates for the solutions of the Problem (1.1).

Lemma 2.5. *Suppose that the assumptions of Lemma 2.4 are fulfilled. Then the corresponding solution u is in $C^k(\mathbb{R})$ for $k = 2, 3, \dots$ and $0 \leq t \leq T$.*

Proof: Notice that we can rewrite Eq. (2.1) in the form

$$u_t = a(x, t)u_x + b(x, t)u + Ku^2, \tag{2.19}$$

where

$$a(x, t) = K \int_{-\infty}^x u(y, t)dy + \int_{\mathbb{R}} V'_{\text{reg}}(x-y)u(y, t)dy \tag{2.20}$$

$$b(x, t) = \int_{\mathbb{R}} V''_{\text{reg}}(x-y)u(y, t)dy \tag{2.21}$$

Notice that we can obtain representation of the solution to (2.19) using the characteristics curves $(x(t, \eta), u(t, \eta))$ defined by means of

$$\frac{dx}{dt} = -a(x, t), \quad \frac{du}{dt} = Ku^2 + b(x, t)u, \quad (2.22a)$$

$$x(0, \eta) = \eta, \quad u(0, \eta) = u_0(\eta). \quad (2.22b)$$

Due to the fact $u \in C^{k-1}(\mathbb{R})$ it follows that $a(\cdot, t)$ and $b(\cdot, t)$ are bounded in $C^k(\mathbb{R})$. Therefore classical regularity theory for ODEs implies that the function $\eta(\cdot, t)$ defined as in (2.10) is bounded in $C^k(\mathbb{R})$. Therefore, since $u(\cdot, t)$ solution of (2.1) is given by $u(x, t) = u_0(\eta(x, t))$ the claimed regularity for u follows from the assumptions of the lemma. \blacksquare

Proof of Theorem 2.1:

Existence. Given an initial datum $u_0 \in C^1(\mathbb{R})$ we construct the sequence of approximations $u_{0,n} \in C^2(\mathbb{R})$ compactly supported, uniformly bounded in C^1 -norm and such that $\lim_{n \rightarrow \infty} u_{0,n} = u_0$ uniformly in compact set of \mathbb{R} . Using Lemmas 2.2–2.5 it follows that there is a sequence of solutions $u_n(\cdot, t) \in C^2(\mathbb{R})$ to Problem (1.1) defined in $0 \leq t \leq T_n$. We will obtain uniform regularity estimates for the sequence u_n and also lower estimate for their time of existence T_n .

To this end notice that the assumptions for the potential V ((A1)–(A4)) imply that the function b defined by (2.21) satisfies

$$|b(\cdot, t)| \leq C \|u(\cdot, t)\|_\infty,$$

where the constant C depends only on the potential V . Using (2.22b) we obtain

$$\|u_n(\cdot, t)\|_\infty \leq \|u_{0,n}\|_\infty + C \int_0^t \|u_n(\cdot, s)\|_\infty^2 ds. \quad (2.23)$$

Therefore, a Gronwall like argument implies that there exists $T > 0$ independent on n such that $\|u_n(\cdot, t)\|_\infty \leq C$ for $0 \leq t \leq T$ as long as solution is defined.

We can now use this boundedness to derive uniform estimate for the function a, b defined by (2.20) and (2.21). Indeed if we denote as a_n the corresponding functions a, b associated to u_n

$$\|a_n(\cdot, t)\|_\infty + \left\| \frac{\partial a_n}{\partial x}(\cdot, t) \right\|_\infty \leq C \|u_n(\cdot, t)\|_\infty, \quad (2.24)$$

$$\|b_n(\cdot, t)\|_\infty + \left\| \frac{\partial b_n}{\partial x}(\cdot, t) \right\|_\infty \leq C \|u_n(\cdot, t)\|_\infty. \quad (2.25)$$

Standard ODE theory yields that $\eta(\cdot, t)$ defined by means of the system of equations (2.22) as well as (2.10) is uniformly bounded in $C^1(\mathbb{R})$. This implies the estimate

$$\|u_{x,n}(\cdot, t)\|_\infty \leq C \quad \text{for } 0 \leq t \leq \min\{T_n, T\}. \quad (2.26)$$

Finally, we can obtain a lower estimate for T_n arguing as in derivation for (2.24) and (2.25) and using the fact that $u_n(\cdot, t)$ is uniformly bounded in C^1 it follows that $\|a_{xx,n}(\cdot, t)\|_\infty$ and $\|b_{xx,n}(\cdot, t)\|_\infty$ are uniformly bounded. Using again the regularity properties for the solution of the characteristic equations (2.22) it follows that $\|u_{xx,n}(\cdot, t)\|_\infty \leq C \|u_{xx,0,n}(\cdot, t)\|_\infty$ for $0 \leq$

$t \leq \min\{T_n, T\}$. If $T_n < T$ we might use the Lemmas 2.3 and 2.4 to extend the solution for later times. Moreover using the differential equation (2.19) it follows that $\|u_{t,n}(x, \cdot)\|_\infty$ is also uniformly bounded in the same time interval.

A classical compactness argument using (2.23) and (2.26) yields then the desired existence result. Notice that Eq. (1.1a) implies estimate for the time derivative after having estimates for space derivative.

Uniqueness. Suppose that there exist two solutions u_1 and u_2 to (1.1) with the same initial datum u_0 with regularity stated in Theorem 2.1. Therefore, $u_{i,t}$, $i = 1, 2$ are uniformly bounded in \mathbb{R} for small t , whence

$$\|u_1(\cdot, t_\epsilon) - u_2(\cdot, t_\epsilon)\| \leq \epsilon,$$

for some $t_\epsilon \rightarrow 0$. Subtracting those two solutions and plugging them into (1.1a) we obtain

$$\begin{aligned} \frac{\partial(u_1 - u_2)}{\partial t} &= (u_1 - u_2)_x \int_R V'(x - y) u_1(y, t) dy + K u_1(u_1 - u_2) \\ &\quad + (u_1 - u_2) \int_R V''_{\text{reg}}(x - y) u_1(y, t) dy + (u_2)_x \int_R V'(x - y) (u_1 - u_2)(y, t) dy \\ &\quad + K u_2(u_1 - u_2) + u_2 \int_R V''_{\text{reg}}(x - y) (u_1 - u_2)(y, t) dy. \end{aligned}$$

Arguing as in derivation of (2.23) and (2.26) it will follow $\|u_1(\cdot, t) - u_2(\cdot, t)\|_\infty \leq C\epsilon$ for $0 \leq t \leq T$ whence the uniqueness result follows as $\epsilon \rightarrow 0$. \blacksquare

2.3 Nonnegativity and mass conservation

In this Subsection we derive some properties of the solution of Problem (1.1) that might be obtained using elementary arguments.

Proposition 2.6 (Nonnegativity and mass conservation). *Suppose that the assumptions of Theorem 2.1 hold. Then:*

- (i) *If $u_0 \geq 0$ the corresponding solution of Problem (1.1) $u(x, t) \geq 0$ for $x \in \mathbb{R}$ and $t \in [0, T]$.*
- (ii) *Suppose that $u_0 \in L^1(\mathbb{R})$. Then the corresponding solution of (1.1) $u(\cdot, t)$ belongs to $L^1(\mathbb{R})$. Moreover*

$$\int_{\mathbb{R}} u_0(x) dx = \int_{\mathbb{R}} u(x, t) dx \qquad \int_{\mathbb{R}} |u_0(x)| dx = \int_{\mathbb{R}} |u(x, t)| dx \quad (2.27)$$

for $t \in [0, T]$.

Proof: The nonnegativity property of the solution u is just a consequence of the fact that Eq. (2.22b) preserves the nonnegativity of initial data and the part (i) is proved.

Concerning the mass conservation property, multiplying (1.1a) by $\text{sgn}(u)$ we obtain that in the sense of measures

$$\frac{\partial |u(x, t)|}{\partial t} = \frac{\partial}{\partial x} \left(|u(x, t)| \int_{\mathbb{R}} V'(x - y) u(y, t) \right). \quad (2.28)$$

Suppose first, that $u(\cdot, t) \in L^1(\mathbb{R})$. Integrating this equation, as well as Eq. (1.1a), on \mathbb{R} and using assumption (A1)–(A4) on V we arrive at (2.27).

In order to show that the hypothesis $u(\cdot, t) \in L^1(\mathbb{R})$ holds, we use the fact that

$$u(x, t) = U(t, \eta(x, t)),$$

where (X, U) are the solutions of the characteristics equations (2.22). Notice that the boundedness of b implies that $|U(t, \eta)| \leq C |u(\eta)|$. Then

$$\int_R |u(x, t)| dx \leq C \int_R |u_0(\eta(x, t))| dx.$$

Using then the fact that $\eta(x)$ is bounded below (2.9) it then follows that

$$\int_R |u(x, t)| dx \leq C \int_R |u_0(\eta)| d\eta.$$

This shows that $u \in L^1(\mathbb{R})$ and the result follows. ■

Remark. Notice that the same argument shows that the amount of mass of u contained between two zeros of u remains constant.

3 Global existence and blow-up.

In this Section we describe conditions that ensure that the solution obtained in Theorem 2.1 is defined for arbitrary long times. We will also obtain an initial data that blows-up in finite time. Roughly speaking, our results state that for potential that are repulsive near the origin the solution of (1.1) are globally defined. On the contrary, if the potential V is an attractive one near the origin then the corresponding solutions might exhibit blow-up in finite time.

3.1 Global existence.

Theorem 3.1 (global existence). *Suppose that assumption of Theorem 2.1 are satisfied and also that $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R})$. If $K \leq 0$, where K is defined in assumption (A3), then the solution of Problem (1.1) is globally defined, i.e. $T = +\infty$. Moreover if $K < 0$ then*

$$\|u(\cdot, t)\|_\infty \leq \max \left\{ \|u_0\|_\infty, \frac{\|V''_{reg}\|_\infty \|u_0\|_1}{|K|} \right\}, \quad t \geq 0. \quad (3.1)$$

On the other hand, if $K = 0$ there exists $\beta > 0$ depending only V such that

$$\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty e^{\beta t}, \quad t \geq 0.$$

Proof: Our assumptions on u_0 imply that $|b(x, t)| \leq \|V''_{\text{reg}}\|_{\infty} \|u_0\|_1 \equiv \beta$, where b is defined by (2.21). Therefore, since $K \leq 0$ it follows from second equation of (2.22) that

$$\|u(\cdot, t)\|_{\infty} \leq \|u_0\|_{\infty} e^{\beta t}.$$

The fact that a solution satisfying this estimate might be extended to arbitrarily long times can be deduced as a proof of Lemma 2.5.

Indeed, in all arguments in Lemma 2.5 and previous ones it was crucial to have $\frac{\partial x}{\partial \eta} \neq 0$. Differentiating (2.22a) and using the definition (2.20) it follows that

$$\frac{d}{dt} \left(\frac{\partial x}{\partial \eta} \right) = - \left(Ku + \int_{\mathbb{R}} V''_{\text{reg}}(x - y) u(y, t) dy \right) \frac{\partial x}{\partial \eta}, \quad \frac{\partial x}{\partial \eta}(0, \eta) = 1. \quad (3.2)$$

Therefore, $0 < C_1 \leq \frac{\partial x}{\partial \eta} \leq C_2$ as long as u remains bounded, with C_1, C_2 depending on $\|u_0\|_{\infty}$ and T .

The estimate (3.1) for the case $K < 0$ can be also obtained using (2.22) that combined with the previous bounds yields inequality

$$u_t \leq -|K| u^2 + \beta u.$$

■

3.2 Blow-up

We will say that the solution u to Problem (1.1) exhibits a blow-up at time $t^* < +\infty$ if and only if there exists a sequence (t_n) such that $t_n \rightarrow t^*$ and $\lim_{n \rightarrow \infty} \|u(\cdot, t_n)\|_{L^{\infty}} = +\infty$. We will say that $x_0^* \in \mathbb{R}$ is a blow-up point if there are sequences $x_n \rightarrow x_*$ such that $\lim_{n \rightarrow \infty} |u(x_n, t_n)| = +\infty$.

Theorem 3.2. *Suppose that the assumptions of Theorem 2.1 are satisfied. Let us assume also that $u_0 \geq 0$ and $u_0 \in L^1(\mathbb{R})$ and let $K > 0$ with K as in assumption (A3). Let $\beta = \|u_0\|_1 \|V''_{\text{reg}}\|_{\infty}$. Then if there exists $x_0 \in \mathbb{R}$ such that $u_0(x_0) > \frac{\beta}{K}$ then the solution of Problem (1.1) blows-up in a finite time.*

Proof: The second equation in (2.22) as well as (2.21) imply that along characteristics

$$\frac{du}{dt} \geq Ku^2 - \beta u.$$

Therefore, if at particular point $x = x_0$ the $u_0(x_0) > \frac{\beta}{K}$ then the solution must become unbounded in a finite time. It only remains to show that solution of (1.1a) does not become singular due to blow-up in its derivatives before becoming unbounded, but this is a consequence of the fact that arguments in the proof of Lemma 2.5 yield bounds for the derivatives of u as long as u is bounded. ■

Remark. Notice that the argument showing Theorem 3.2 is a local one, i.e. it depends only on the properties of u_0 at a given point.

3.3 Blow-up profile.

In this subsection we will show that asymptotic behaviour of the solutions to (1.1) can be computed in a manner analogous to the asymptotic of the solutions of Burger's equation at the time of generation of a shock wave. By simplicity, we restrict our attention to compactly supported initial data. In that case the change of variables (2.2) transforms the original equation (1.1a) into (2.3).

The solutions of (2.3) might generate shock waves in a finite time, even for a smooth initial datum. Due to the regularity assumptions that we have made on V_{reg} it is possible to approximate (2.3) locally near the point and time of generation of a shock wave as:

$$\psi_t = K\psi\psi_x + c\psi_x, \quad (3.3)$$

where c is a constant. The asymptotic behaviour of solutions ψ for (3.3) can be readily computed integrating by characteristics, and this yields the asymptotic behaviour of u by means of (2.2). In the next theorem we make this argument precise and rigorous.

Theorem 3.3 (blow-up profile). *Suppose that the assumptions of Theorem 2.1 are satisfied. Let us assume also that $u_0 \geq 0$ and $u_0 \in L^1(\mathbb{R}) \cap C^2(\mathbb{R})$ and let $K > 0$ with K as in assumption (A3). Suppose that u , solution of (1.1) blows up at time t^* . Then u blows up in a bounded region, i.e. $|u(x, t)| < \infty$ for $0 \leq t < t^*$ $|x| > L$ and some $L > 0$ large enough. Moreover for any blow up point $x_0^* \in \mathbb{R}$*

$$u(x, t) \sim A(t^* - t)^{-1} \xi_y \left(\frac{x - x_0^*}{(t^* - t)^{\frac{3}{2}}} \right) + o((t^* - t)^{-1}), \quad \text{as } t \rightarrow (t^*)^- \quad (3.4)$$

uniformly on $|x - x_0| \leq C(t^* - t)^{\frac{3}{2}}$, where the differentiable function $\xi(y)$ is defined by means of

$$y = -\Gamma_1 \xi + \Gamma_2 \xi^3, \quad \xi(0) = 0 \quad (3.5)$$

and where the constants $\Gamma_1 > 0$, $\Gamma_2 \geq 0$ depend on the initial data u_0 .

Proof: Under the assumptions of the Theorem, problem (1.1) is equivalent to (2.3). Notice that the solution of (2.3) cannot be continued in a classical sense if the characteristic associated to this equation cross with each other, or equivalently if the function $\eta(\cdot, t)$ cannot be defined in a smooth manner by means of (2.10). Arguing as in derivation of (2.6) we have that the characteristics associated to (2.3) reduced just to

$$x_t = -\psi_0(\eta) - R(x, t) \quad (3.6a)$$

$$x(0, \eta) = \eta, \quad (3.6b)$$

where $R(x, t) = \int_{\mathbb{R}} V'_{\text{reg}}(x - y) \psi_x(y, t) dy$. Notice that characteristic associated to (1.1) do not cross as long as $\frac{\partial x}{\partial \eta} \neq 0$, where $x(t, \eta)$ is the solution of (3.6). Moreover, arguing as in a proof of Theorem 3.1 it follows that u remains bounded as long as $\frac{\partial x}{\partial \eta}$ does not vanish.

Differentiating (3.6) with respect to η we obtain

$$\frac{\partial x}{\partial \eta} = \exp \left(- \int_0^t R_x(x(\eta, s), s) ds \right) - u_0(\eta) \int_0^t \exp \left(- \int_s^t R_x(x(\eta, \tau), \zeta) d\zeta \right) ds. \quad (3.7)$$

The integrability assumption on u_0 combined with (2.27) imply that R_x is uniformly bounded as long as the solution u is defined. Therefore, since $u_0(\eta)$ approaches to 0 as $|\eta| \rightarrow 0$, it follows

that $\frac{\partial x}{\partial \eta}$ does not vanish for $|x| > L$, L large enough. This means that blow-up might take place only in bounded regions.

Suppose that u blows up at (x_0^*, t^*) . Then

$$\inf_{\eta \in \mathbb{R}} \frac{\partial x}{\partial \eta}(\eta, t) > 0 \quad \text{for } t < t^* \quad \text{and} \quad \inf_{\eta \in \mathbb{R}} \frac{\partial x}{\partial \eta}(\eta, t^*) = 0 \quad (3.8)$$

Let us denote as η^* the initial value for a characteristic, that reaches a blow-up point, i.e. $\frac{\partial x}{\partial \eta}(\eta^*, t^*) = 0$, $x(\eta^*, t^*) = x_0^*$.

We rewrite (3.7) by shortness as

$$\frac{\partial x}{\partial \eta} = \kappa(\eta, t) e^{-\int_0^t G(\eta, \zeta) d\zeta}, \quad (3.9)$$

where $G(x, t) = R_x(x(\eta, t), t)$ and

$$\kappa(\eta, t) = 1 - u_0(\eta) \int_0^t e^{\int_0^s G(\eta, \zeta) d\zeta}$$

In order to study the asymptotics of u near (x_0^*, t^*) we use a Taylor expansion in (3.9). Let us denote as

$$X = \eta - \eta^* \quad T = t - t^*.$$

Then,

$$\begin{aligned} \frac{\partial x}{\partial \eta} = & B \left(\kappa(\eta^*, t^*) + \kappa_\eta(\eta^*, t^*)X + \kappa_t(\eta^*, t^*)T + \kappa_{\eta\eta}(\eta^*, t^*)\frac{X^2}{2} + \kappa_{tt}(\eta^*, t^*)\frac{T^2}{2} \right. \\ & \left. + \kappa_{t\eta}(\eta^*, t^*)TX + o(T^2 + X^2) \right), \end{aligned} \quad (3.10)$$

where $(X, T) \rightarrow 0$ and $B = e^{-\int_0^{t^*} G(\eta^*, s) ds}$. Due to (3.8) $\kappa_\eta(\eta^*, t^*) = 0$. Keeping only the leading terms in (3.10) we obtain

$$\frac{\partial x}{\partial \eta} = B \left(\kappa_t(\eta^*, t^*)T + \kappa_{\eta\eta}(\eta^*, t^*)\frac{X^2}{2} + o(T + X^2) \right) \quad \text{as } (X, T) \rightarrow (0, 0).$$

Integration with respect to η leads to

$$x(t) - x_0^* = \Gamma_1 TX + \Gamma_2 X^3 + o((T + X^2)X) \quad \text{as } (X, T) \rightarrow (0, 0), \quad (3.11a)$$

where $\Gamma_1 = B\kappa_t(\eta^*, t^*) < 0$ and $\Gamma_2 = B\frac{\kappa_{\eta\eta}(\eta^*, t^*)}{6} \geq 0$. On the other hand ψ is constant along the characteristics and using the Taylor series we arrive at

$$\psi = \psi_0(\eta^*) + \psi'_0(\eta^*)X = \psi_0(\eta^*) + u_0(\eta^*)X + o(X) \quad \text{as } X \rightarrow 0. \quad (3.11b)$$

Let us make the following change of variables

$$X = (-T)^{\frac{1}{2}}\xi, \quad x - x_0^* = (-T)^{\frac{3}{2}}y.$$

Using this change of variables (3.11) become

$$\psi = \psi_0(\eta^*) + u_0(\eta^*)(t^* - t)^{\frac{1}{2}}\xi \left(\frac{x - x_0^*}{(t^* - t)^{\frac{3}{2}}} \right) + o\left((t^* - t)^{\frac{1}{2}}\right) \quad \text{as } t \rightarrow t^*. \quad (3.12)$$

Differentiating (3.12) with respect to x we obtain (3.4). This differentiation is possible by a standard regularity results for the solution of ODE as well as the smoothness assumed for u_0 in Theorem 3.3. \blacksquare

Remark. The constant Γ_2 in (3.5) seems to be positive for generic initial data.

4 Steady states

In this Section we will study the steady states of Eq. (1.1a). Notice that in the Eq. (1.1a) for repulsive potentials V can be thought as some kind of regularized version of the porous medium equation. Therefore, for initial data with a bounded mass the long time asymptotics that one might expect for this equation is convergence to 0. On the contrary for general class of potentials V , a richer class of asymptotic behaviours is possible, as we will show in this and next section. In this Section we will prove that for some general potentials V there exist many non-trivial steady states.

The form of the equation (1.1a) implies that the u is its steady state if and only if for any $x \in \mathbb{R}$ the following condition is fulfilled

$$u(x) \int_{\mathbb{R}} V'(x - y)u(y)dy = J \in \mathbb{R}, \quad u(x) \geq 0. \quad (4.1)$$

For $J \neq 0$ there would be a flux of particles moving from one side of space to another. Therefore, we will concentrate here in genuinely steady state, where the particles are at rest, whence $J = 0$. From now we would denote as steady states only the solutions of (4.1) with $J = 0$.

4.1 Non-trivial steady states: pattern formation

In this subsection we will discuss some class of potentials V yielding steady states that are not homogenous in space. Our first example gives a class of potentials for which steady states are piecewise constant. Nevertheless, this example shows that for non purely repulsive (or attractive) potentials the steady state solutions of (1.1a) might exhibit some “clustering”

Theorem 4.1. *Let the potential V be defined as follows*

$$V(x) = \begin{cases} 0 & \text{for } |x| > r \\ \alpha(x + r) & \text{for } -2r < x < 0 \\ \alpha(x - r) & \text{for } 0 < x < 2r, \end{cases} \quad (4.2)$$

where α is a compactly supported function satisfying

$$\alpha(x) = \alpha(-x) \quad \alpha(-r) = \alpha(r) = 0. \quad (4.3)$$

Let us define

$$u(x) = \sum_i c_i \chi_{I_i}, \quad (4.4)$$

where the sum is extended over finite or countable set of index, c_i are positive constants, I_i are interval each of them having the length $2r$ and the distance between any of two intervals I_i and I_j is greater than $4r$. Then u solves (4.1) with $J = 0$.

Proof: It is basically an explicit computation. Notice that since the distance between intervals are greater or equal than $4r$ the different intervals do not interact with each other. On the other hand, for a given interval I_i that we might assume to be interval $[0, 2r]$ due to invariance of the problem under translations. Hence, for $x \in [0, 2r]$ we have

$$\begin{aligned} \int_{\mathbb{R}} V'(x-y)u(y)dy &= c_i \int_0^{2r} V'(x-y)dy \\ &= c_i \int_{x-2r}^x V'(t)dt = c_i \int_{-r}^r V'(t)dt = 0. \end{aligned}$$

where we have used that the definition of V implies that for any $x \in [0, 2r]$

$$\int_{-r}^{x-2r} V'(t)dt = \int_r^x V'(t)dt.$$

■

The following example shows that even for purely repulsive potentials the steady states are not necessarily homogenous in the space.

Theorem 4.2. Suppose that the potential $V \in C^3(\mathbb{R})$ satisfying $V(x) = -V(x-r) + C$, for some constant C , $x \in [0, r]$ and $V(x) = 0$ outside the interval $[-r, r]$. Then any continuous positive periodic function with period r is a solution of (4.1) with $J = 0$.

Proof: Using a periodicity of u we obtain

$$\begin{aligned} \int_{\mathbb{R}} V(x-y)u(y)dy &= \int_{-r}^r V(y)u(x-y)dy = \int_{-r}^0 V(y)u(x-(y+r))dy + \int_0^r V(y)u(x-y)dy \\ &= \int_0^r (V(y-r) + V(y))u(x-y)dy = 0, \end{aligned}$$

due to assumptions of the theorem. ■

Remark. Notice, that there exists large class of the potentials satisfying Theorem 4.2. For example we may construct a function $\varphi : [0, r] \rightarrow \mathbb{R}$ which is a skew-symmetric with respect to $\frac{r}{2}$ then we extend it for the interval $[-r, 0]$ in the following manner $\varphi(x) = -\varphi(x+r)$ for $x \in [-r, 0]$. Finally we construct the potential $V(x) = \varphi(x) - \varphi(r)$. In this case, the potential V is symmetric with respect to 0. However it is possible to construct also potentials satisfying Theorem 4.2, which are non-symmetric.

Notice, that the regularity assumptions made in Theorem 4.2 were made only the potential V to satisfy regularity assumption (A4). It is not difficult to construct potentials which are not smooth but for which there exist periodic steady states. For example

Proposition 4.3. *Suppose that $V(x) = (r - |x|)_+$. Then any continuous positive periodic function with period r is a solution of (4.1) with $J = 0$.*

The following theorem provides a large class of potentials yielding non-trivial steady states.

Theorem 4.4. *Suppose that V is a potential satisfying assumptions (A1)–(A4) as well as the symmetry assumption $V(x) = V(-x)$. Suppose also that $K < 0$ in (A3). Let us suppose also that there exists $\beta > 0$ such that $V(0) < V(2\beta)$. and that $V''(x) > 0$ for $x \in (0, 2\beta]$. Then, there exists an interval $I \subset \mathbb{R}$ and the function $\bar{u} \in L^2(I)$ solving (4.1) with $J = 0$.*

Remark. Notice that, the assumption $V(x) = V(-x)$ implies that $V'(x) = -V'(-x)$. Hence, $u \equiv \text{const.}$ satisfies the condition (4.1) with $J = 0$.

Proof: We may assume without loss of generality that $\|u\|_{L^2} = 1$.

Let $I \subset \mathbb{R}$ be an interval $[-\alpha, \alpha]$, where $\alpha > 0$. Define

$$F_u(x; I) = \int_I V'(x - y)u(y)dy.$$

With this notation the condition (4.1) is equivalent to

$$F_u(x; I) = 0 \quad \text{for } x \in I. \quad (4.5)$$

Differentiating (4.5) we obtain

$$F'_u(x; I) = \int_I V''_{\text{reg}}(x - y)u(y)dy + Ku(x).$$

Then (4.5) implies

$$-Ku(x) = \int_I V''_{\text{reg}}(x - y)u(y) \equiv (T_I(u))(x), \quad x \in I. \quad (4.6)$$

We reformulate the problem (4.6) in a convenient variational form. To this end we introduce the following problem: maximize

$$\frac{\langle u, T_I u \rangle}{\langle u, u \rangle}$$

in the class of non-negative functions $u \in L^2_s(I)$,

$$L^2_s(I) = \{u \in L^2(I) : u(x) = u(-x) \text{ a.e. in } I\}.$$

It is not hard to see that a function $\bar{u} \in L^2_s(I)$, $\bar{u} \geq 0$ exists due to the fact that operator T_I is compact in $L^2(I)$ (cf. [15]). Our goal is to show that under the conditions in Theorem 4.4 \bar{u} solves (4.5) for suitable choice of I . To this end we define

$$\lambda(I) = \sup_{u \in L^2_s(I) \text{ } u \geq 0} \frac{\langle u, T_I u \rangle}{\langle u, u \rangle} = \frac{\langle \bar{u}, T_I \bar{u} \rangle}{\langle \bar{u}, \bar{u} \rangle}, \quad (4.7)$$

where $\langle \cdot, \cdot \rangle$ is a classical inner-product in $L^2(I)$.

Notice that \bar{u} , solution of (4.7) solves the variational inequality

$$\bar{u}(x) (T_{I_0}(\bar{u})(x) - \lambda(I)\bar{u}(x)) = 0 \quad \text{a.e. } x \in I \quad (4.8a)$$

$$T_{I_0}(\bar{u})(x) - \lambda(I)\bar{u}(x) \leq 0 \quad \text{a.e. } x \in I. \quad (4.8b)$$

In order to show (4.8) we use a standard arguments. Indeed, due to (4.7) it follows that

$$\langle (\bar{u} + \varphi), T_I(\bar{u} + \varphi) \rangle \leq \lambda(I) \langle \bar{u}, (\bar{u} + \varphi), (\bar{u} + \varphi) \rangle. \quad (4.9)$$

for any test function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi + \bar{u} \geq 0$. The inequality (4.9) implies (4.8).

We now proceed to estimate $\lambda(I)$ for large small and large intervals I . We first remark, that $\lim_{\alpha \rightarrow 0} \lambda(I) = 0$. This follows from (4.7) as well as the inequality

$$|(T_I u)(x)| \leq \int_I |V''_{\text{reg}}(x-y)| |u(y)| dy \leq \sup_{x \in \mathbb{R}} |V''_{\text{reg}}(x)| \sqrt{|I|} \|u\|_{L^2(\mathbb{R})}.$$

On the other hand plugging a test function

$$u = \begin{cases} 1 & x \in (-\beta, \beta) \\ 0 & x \notin (-\beta, \beta), \end{cases}$$

where $2\beta < |I|$ we obtain

$$\lambda(I) \geq \frac{\langle u, T_I u \rangle}{\langle u, u \rangle} = -K + \frac{V(2\beta) - V(0)}{\beta} > -K.$$

We have used the fact that

$$\langle u, T_I u \rangle = V_{\text{reg}}(-2\beta) + V_{\text{reg}}(2\beta) - V_{\text{reg}}(0).$$

Hence, using the assumption (A3) we get

$$\langle u, T_I u \rangle = 2(V(2\beta) - V(0)) - 2\beta K.$$

Therefore, using a continuity of $\lambda(I)$ it follows that there exists an interval $I_0 \subset [-\beta, \beta]$, such that $\lambda(I_0) = -K$. It then follows, due to (4.8), that the (4.7) holds in the regions where $\bar{u}(x) > 0$. As a next step we will show that actually $\bar{u} > 0$ in the interval I_0 . This is just a consequence of the fact, that convexity of V on the interval $(0, 2\beta]$ yields

$$T_{I_0} \bar{u} = \int_{I_0} V''_{\text{reg}}(x-y) \bar{u}(y) dy > 0,$$

whence, due to (4.8b), $\bar{u} > 0$ in I_0 .

Let us summarize. We have obtained that the function \bar{u} associated to the interval I_0 satisfies (4.6) on the interval I_0 , or in equivalent manner $F'_u(x; I_0) = 0$ for any $x \in I_0$. On the other hand, since \bar{u} is symmetric with respect to origin it follows, that $F'_u(0, I_0) = 0$. henceforth, $F_{\bar{u}}(x, I_0) = 0$ for any $x \in I_0$ and the theorem follows. \blacksquare

Remark. Notice that, the steady state obtained in Theorem 4.4 is in general no homogenous in the interval I_0 . Indeed, $T_{I_0}(x) = -V'_{\text{reg}}(x - \alpha) + V'_{\text{reg}}(x + \alpha)$, whence in general $u = \chi_{I_0}$ does not satisfies (4.6).

Remark. Notice that for potentials V compactly supported we may construct a complicated steady state of (1.1a) putting together many solutions as in Theorem 4.4 separated far enough.

4.2 Trivial steady states

In this subsection we provide some sufficient condition on the potentials V that ensure that the steady states are homogenous.

We will show now, that if the potential V is an exponential one or a small perturbation of it, then there is no steady state of Eq. (1.1a) except the constant functions.

In the following result we show that for potentials V that decay fast enough the nontrivial steady state of (1.1a) must necessarily vanish in some region.

Theorem 4.5. *Suppose that the potential V in (A1)–(A4) satisfies $|V(x)| + |V'(x)| < Ce^{-ax}$, $x \in \mathbb{R}$, $V \not\equiv 0$ for some positive constants a, C . Then are not solutions $u \in L^1(\mathbb{R})$, $u \not\equiv 0$ of*

$$u(x) \int_{\mathbb{R}} V'(x-y)u(y)dy = 0, \quad x \in \mathbb{R}, \quad (4.10)$$

such that $u(x) > 0$ for any $x \in \mathbb{R}$.

Proof: Suppose that there exists $u \in L^1(\mathbb{R})$ satisfying $u(x) > 0$ for $x \in \mathbb{R}$. Then, $\int_{\mathbb{R}} V'(x-y)u(y)dy = 0$ for $x \in \mathbb{R}$. Taking the Fourier transform of these formula we obtain

$$\xi \hat{V}(\xi) \hat{u}(\xi) = 0, \quad (4.11)$$

where the Fourier transform are understood in the sense tempered distributions (cf. [16]). The exponential decay of V implies that \hat{V} is analytic in the strip $\text{Im } \xi < a$. On the other hand the assumptions on u imply that $\hat{u}(\xi) \neq 0$ in a subset of the real line having at least one accumulation point. Therefore, (4.11) implies that $\hat{V} \equiv 0$, whence $V \equiv 0$. This is a contradiction, whence the theorem follows. ■

The following results is, in some sense, complementary to the Theorem 4.4, and it provides a large class of monotonic potentials whose only steady states are the constant ones.

Theorem 4.6. *Suppose that V is a potential satisfying assumptions (A1)–(A4) as well as the symmetry assumption $V(x) = V(-x)$. Suppose also that $K < 0$ in (A3). Let us suppose also that $V''(x) \geq 0$ for any $x > 0$. Then, the only bounded solutions of (4.1) with $J = 0$ are the constant ones.*

Proof: Arguing as in proof of Theorem 4.4 it follows that the stationary, bounded solutions of (4.1) with $J = 0$ solves (4.6). Therefore, the Theorem would follow proving that under the assumptions of Theorem 4.6 and for any solution of the integral equation $T_I u = au$, where I is the support of u with $a \neq 0$ the following inequality holds

$$|a| < |K|. \quad (4.12)$$

Indeed, since $V''_{\text{reg}} \geq 0$ we have $T_I(u) \leq T_I(|u|)$. For any fixed $\epsilon > 0$ let choose $x_{0,\epsilon}$ such that $|u(x_{0,\epsilon})| \geq \|u\|_{\infty} - \epsilon$. Then

$$a(\|u\|_{\infty} - \epsilon) \leq a|u(x_{0,\epsilon})| \leq T_I(\|u\|_{\infty}) = \|V''_{\text{reg}}\|_{L^1} \|u\|_{\infty} = K \|u\|_{\infty},$$

whence taking the limit $\epsilon \rightarrow 0$ we obtain $|a| \leq |K|$.

It only remains to show that $|a| \neq |K|$. Suppose then, that $|a| = |K|$. Then

$$u(x) = \frac{(T_I u)(x)}{|K|}. \quad (4.13)$$

Due to (A1)–(A3) it follows that $u(\cdot)$ is a continuous function on I . Let define the set

$$U = \{x \in I : u(x) = M = \|u\|_\infty\}.$$

This set is closed in I . Let us denote as $[-\rho, \rho]$ the connected component of the support of $V''_{\text{reg}}(x)$ that contains $x = 0$. Given $x_0 \in U$ it follows from the representation formula (4.13) that the interval $(x_0 - \rho, x_0 + \rho) \subset U$. Therefore, U is open and closed set in I . Hence, it fills each connected component of U with nonempty intersection with U . Moreover, the same argument shows that $I = \mathbb{R}$ and therefore, either $U = \mathbb{R}$ or $U = \emptyset$. In the first case u would be a constant.

If, on the contrary, $U = \emptyset$ we argue as follows. Let us denote $m = \inf_{x \in \mathbb{R}} u(x)$. The representation formula (4.13) implies, in the analogous manner, that u does not reach the value of m at any point $x_0 \in \mathbb{R}$. Therefore $m < u(x) < M$ for any $x \in \mathbb{R}$. Moreover, suppose that there exists a sequence $x_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} u(x_n) = M$. Then, the representation formula (4.13) implies that there exists a sequence of numbers L_n satisfying $\lim_{n \rightarrow \infty} L_n = \infty$ and such that

$$\lim_{n \rightarrow \infty} (\inf\{u(x) : x \in [x_n - L_n, x_n + L_n]\}) = M. \quad (4.14)$$

Indeed, (4.13) implies that

$$\lim_{n \rightarrow \infty} (\inf\{u(x) : x \in [x_n - \rho, x_n + \rho]\}) = M.$$

whence (4.14) follows by an interaction.

Moreover, similar estimates hold for $u'(x)$. To check this we rewrite (4.13) as

$$u(x) = \int_{\mathbb{R}} \Phi(x - y)u(y)dy,$$

where $\Phi(x) \geq 0$, $\int_{\mathbb{R}} \Phi(x)dx = 1$. Notice, that due to the assumption (A3) Φ is differentiable and $\int_{\mathbb{R}} \Phi'(y)dy < +\infty$. Thus, differentiating we obtain

$$u'(x) = \int_{\mathbb{R}} \Phi'(x - y)u(y)dy = \int_{\mathbb{R}} \Phi(x - y)u'(y)dy,$$

where u' solves (4.13) and $|u'(x)|$ is bounded. Therefore, similar estimates hold for u and u' . It then follow that $\lim_{|x| \rightarrow \infty} u'(x) = 0$ since otherwise (4.14) applied to u' would imply the existence of arbitrary long intervals where $u'(x) = a \neq 0$, but this contradicts the boundness of u . Therefore, $\lim_{|x| \rightarrow \infty} u'(x) = 0$ and since u' cannot reach its maximum or minimum at any interior point it follows that $u'(x) \equiv 0$ and then $u(x) \equiv \text{constant}$. \blacksquare

Remark. Notice that the assumption $K > 0$ in Theorem 4.4 can be easily replaced by $K < 0$ since the steady states for potentials V and $-V$ are the same.

We now study a class of potentials V for which the steady states can be computed in a completely explicit manner.

Theorem 4.7. *Suppose $V(x) = \alpha \exp(-\lambda|x|)$ for some $\alpha \neq 0$ and $\lambda > 0$. Then the only bounded steady states of (1.1a) are the constant functions.*

Proof: We may assume, without loss of generality, that $\alpha = 1$. Let us define

$$f(x) = \int_{\mathbb{R}} V(x-y)u(y)dy. \quad (4.15)$$

Notice that the equation satisfied by the steady states is equivalent to $u(x)f'(x) = 0$.

Differentiating (4.15) twice we obtain

$$u(x) = \frac{1}{\lambda} (\lambda^2 f(x) - f''(x)). \quad (4.16)$$

Multiplying both sides of (4.16) by $f'(x)$ and using the fact $u(x)f'(x) = 0$ we obtain

$$0 = f'(x) (f''(x) - \lambda^2 f(x)).$$

Now, there are two possibilities: either $f'(x) = 0$ for any $x \in \mathbb{R}$ or alternatively f is not globally constant. In the first case it follows from (4.16) that u is identically constant.

Suppose then f is not identically constant. There exists some $x_0 \in \mathbb{R}$ such that $f'(x_0) = 0$. Indeed, otherwise f will satisfy the equation $f''(x) = \lambda^2 f(x)$, for $x \in \mathbb{R}$. Since the definition of f (cf. (4.15)) implies that f is bounded it follows that $f \equiv 0$. Again it follows from (4.16) that $u \equiv 0$.

Therefore, we can assume that there exists x_0 such that $f'(x_0) = 0$ and some x_1 such that $f'(x_1) \neq 0$. Due to symmetry of the problem we might assume that $x_1 > x_0$. Moreover, due to continuity of f' that the set of points in which $f'(x) \neq 0$ is an open set. Let us denote as

$$x_0^* = \sup\{x < x_1 : f'(x) = 0\}.$$

Notice that $x_0 \leq x_0^* < x_1$ and also that $f'(x) \neq 0$ for $x \in (x_0^*, x_1]$. Therefore,

$$f''(x) = \lambda^2 f(x) \quad \text{for } x \in (x_0^*, x_1].$$

Solving this differential equation we obtain $f(x) = f(x_0^*) \cosh(\lambda(x - x_0^*))$ for $x > x_0^*$. In particular $f'(x)$ would be different from 0 and $f(x)$ would be unbounded unless $f(x_0^*) = 0$. But this implies that $f(x) = 0$ for $x > x_0^*$. Then it is not hard to see, using the similar argument for negative values of f that $f(x) \equiv 0$, whence $u(x) \equiv 0$ and the theorem follows. ■

As the next goal prove a results analogous to Theorem 4.7 for potentials that are close to the ones considered there by means of perturbative arguments. To this end will use the following technical lemma

Lemma 4.8. *Let $g \in W^{2,\infty}(\mathbb{R})$, $\nu \in L^\infty(\mathbb{R})$ satisfying*

$$g'(x)(g''(x) - g(x) + \nu(x)) = 0, \quad \text{a.e. in } \mathbb{R}. \quad (4.17)$$

Then either $\|g\|_{L^\infty} + \|g''\|_{L^\infty} \leq 2(1 + \lambda^2) \|\nu\|_{L^\infty}$ or $g' \equiv 0$ a.e in \mathbb{R} .

Proof: Suppose that there exists $x_0 \in \mathbb{R}$ such that $|g(x_0)| > 2 \|\nu\|_\infty$. Without loss of generality we might assume $g(x_0) > 2 \|\nu\|_\infty$. Suppose first that $g'(x_0) \neq 0$. Then due to symmetry of the problem with respect to reflections on the x -coordinate we can assume that $g'(x_0) > 0$. Whence (4.17) gives

$$g''(x) \geq \frac{1}{4}g(x)$$

for $x \in [x_0, x_0 + \delta]$, for some $\delta > 0$. Then, by comparison, we get

$$g(x) \geq g(x_0) \cosh\left(\frac{x - x_0}{2}\right), \text{ for } x \in [x_0, x_0 + \delta].$$

Since g is concave, $g'(x) > 0$ in the same interval. A continuation argument then shows that $g(x) \geq g(x_0) \cosh\left(\frac{x - x_0}{2}\right)$ holds for any $x > x_0$. Therefore, g would be unbounded and this gives a contradiction.

It then follows that $g'(x_0) = 0$. Let us denote as I the largest closed interval containing x_0 such that $g'(x) = 0$ for any $x \in I$. If $I = \mathbb{R}$ the lemma would follow. Otherwise, there would exist $x_1 \in \mathbb{R}$ such that $g(x_0) > \frac{2}{3} \|\nu\|_\infty$ and $g'(x_1) \neq 0$. Arguing exactly as before it then follows that g is unbounded and the contradiction concludes the $\|g\|_{L^\infty} \leq 2 \|\nu\|_{L^\infty}$.

In order to estimate g'' we argue as follows. Let us define

$$I = \{x \in \mathbb{R} : g'(x) = 0\}.$$

Due to continuity of g' the set I is closed one and I^c is an open one. On the other hand g' is a Lipschitz continuous function and therefore, Rademacher's Theorem (cf. [8]) implies g' is differentiable almost everywhere. Suppose first, that $x_0 \in I^c$. Then, since I^c is open (4.17) implies that $g'' = \lambda^2 - \nu$. Hence, $\|g''\|_{L^\infty} \leq (1 + 2\lambda^2) \|\nu\|_{L^\infty}$.

On the other hand, if $x \in I$ we must distinguish two cases. Suppose first, that x_0 is an accumulation point of I such that $g''(x_0)$ exists. Then there exists a sequence of points $x_n \rightarrow x_0$ such that $g'(x_n) = 0$. Whence, $g''(x_0) = \lim_{x_n \rightarrow x_0} \frac{g'(x_n) - g'(x_0)}{x_n - x_0} = 0$.

On the other hand, there exists only a countable number of points $x_0 \in I$ that are not accumulation points of I , and since this is a set of zero measure the theorem follows. \blacksquare

Theorem 4.9 (perturbation). *Let $R: \mathbb{R} \rightarrow \mathbb{R}$ be in $C^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$. Suppose that*

$$V(x) = \alpha e^{-\lambda|x|} + \epsilon R(x), \quad (4.18)$$

$\alpha \neq 0$, $\lambda > 0$. Then there exists $\epsilon_0 > 0$ such that for $|\epsilon| < \epsilon_0 \equiv \frac{1}{2C_R}$ where $C_R = \|R\|_{W^{1,2}}$, any bounded steady state for (1.1a) with potential (4.18) is constant.

Proof: Without loss of generality we may assume that $\alpha = 1$ and $\epsilon > 0$. We define f by means of

$$f(x) = \int_{\mathbb{R}} (e^{-\lambda|x-y|} + \epsilon R(x-y)) u(y) dy.$$

Differentiating $f(x)$ twice we obtain

$$\lambda u(x) = \lambda^2 f(x) - f''(x) + \nu(x), \quad (4.19)$$

where $\nu(x) = \epsilon \int_{\mathbb{R}} (R(x-y) - R''(x-y)) u(y) dy$ and multiplying both sides of Eq. (4.19) by $f'(x)$ and using the fact that for a steady states $f'(x)u(x) = 0$ we obtain

$$0 = f'(x) (f''(x) - \lambda^2 f(x) + \nu(x)).$$

Lemma 4.8 implies then that either $f'(x) \equiv 0$ or $\|f\|_{L^\infty} + \|f''\|_{L^\infty} < 2(1 + \lambda^2) \|\nu\|_{L^\infty} \leq 2(1 + \lambda^2)\epsilon \|R\|_{W^{2,1}} \|u\|_\infty$, whence using (4.19) $\|u\|_{L^\infty} < C\epsilon \|u\|_{L^\infty}$, where C depends on λ and on $\|R\|_{W^{2,1}}$ but not on ϵ .

Suppose then that $f'' \equiv 0$. Then, $f(x) = \alpha$ where α is a constant and (4.19) becomes

$$\lambda u(x) = \lambda^2 \alpha + \nu(x). \quad (4.20)$$

A particular solution of Eq. (4.20) is the constant one $u(x) = \frac{\lambda^2 \alpha}{1 - \epsilon \int_{\mathbb{R}} R(y) dy}$. Moreover, there exists a unique bounded solution of (4.20). Indeed, given two such solutions u_1, u_2 it follows from (4.20)

$$\|u_1 - u_2\|_{L^\infty} < \frac{\epsilon}{\lambda} \|R\|_{W^{2,1}} \|u_1 - u_2\|_{L^\infty}.$$

Henceforth, $u_1 = u_2$, and since u is a constant, the Theorem follows. ■

5 Long time asymptotics

In this Section we discuss the possible asymptotic behaviour for the solutions to (1.1a) as $t \rightarrow +\infty$ that are globally defined in time. Notice, that as indicated in Section 3 there are solutions that are not globally defined in time because they exhibit blow-ups in a finite time.

Notice also, that the long time asymptotic of the solution to (1.1a) depends very sensitively on the choice of the initial data for general potentials V as it might be seen from many different steady states that exist for some classes of potentials (cf. Subsection 4.1).

In this section we show that, besides approximation to steady states as $t \rightarrow +\infty$ solution of (1.1a) might yield also Dirac mass formation as $t \rightarrow +\infty$.

5.1 Stationary long time asymptotics

In this section we assume a symmetry of the potential V (i.e. $V(x) = V(-x)$). Under this assumption the equation (1.1a) has a gradient flow structure, that is inherited from the gradient flow structure of underline particle model. Therefore, there exists a Lyapunov functional that might be used to prove that solution to (1.1) behaves as a steady state as $t \rightarrow \infty$.

Define a functional $E: L^1(\mathbb{R}) \rightarrow \mathbb{R}$ as

$$E_u = \int_{\mathbb{R}} \int_{\mathbb{R}} V(x - y) u(x) u(y) dx dy. \quad (5.1)$$

Then the following results holds.

Theorem 5.1. *Assume that the potential V is symmetric. Let u be a solution of Problem (1.1) with $u_0 \in L^1(\mathbb{R})$. Let us define $E(t) = E_{u(\cdot, t)}$. Then the function $E(t)$ is decreasing as long as the function $u(\cdot, t)$ is defined. Moreover, if $u(\cdot, t)$ is globally defined in time then there exists $\lim_{t \rightarrow \infty} E(t) > -\infty$ and*

$$\int_t^{t+1} \left(\int_{\mathbb{R}} u(x, s) \left(\int_{\mathbb{R}} V'(x - y) u(y, s) dy \right)^2 dx \right) ds \longrightarrow 0, \text{ as } t \rightarrow +\infty. \quad (5.2)$$

Remark. Sufficient conditions for a global existence in time are presented in Theorem 3.1.

Remark. The formula (5.2) provides approximation of $u(\cdot, t)$ in a very weak, average sense. In order to prove the actual approximation of $u(\cdot, t)$ to a steady state as state of infintum a more careful local analysis near stationary solutions should be required. Nevertheless, the structure of the steady states might be very complicated for a given potential V as explained in section 4. Therefore, such a detailed study will not be attempted in this paper.

Proof: The argument is a standard one:

$$\frac{d}{dt}E_f(t) = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} V(x-y) u_t(x, t) u(y, t) dx dy$$

Using Eq. (1.1a) and integrating by parts we obtain

$$\frac{d}{dt}E(t) = -2 \int_{\mathbb{R}} u(x, t) \left(\int_{\mathbb{R}} V'(x-y) u(y, t) dy \right)^2 dx \leq 0 \quad (5.3)$$

This implies that $E(t)$ is decreasing. On the other hand

$$E_u(t) \geq - \int_{\mathbb{R}} \int_{\mathbb{R}} \|V\|_{L^\infty} u(x) u(y) dx dy = - \|V\|_{L^\infty} \|u\|_{L^1}^2 = - \|V\|_{L^\infty} \|u_0\|_{L^1}^2 .$$

Therefore, there exists $\lim_{t \rightarrow +\infty} E(t) > -\infty$ (if u is globally defined). This follows integrating (5.3) between t and $t+1$ and taking the limit as $t \rightarrow +\infty$. ■

5.2 Concentration to Dirac masses

In this subsection we describe the asymptotic of some solutions of (1.1) yielding concentration to Dirac masses as $t \rightarrow +\infty$. Suppose that $K = 0$ in (A3) to avoid blow-up in a finite time.

Suppose that $K = 0$ in (A3) to avoid blow-up in a finite time. We begin proving the following auxiliary result

Proposition 5.2. *Suppose that the potential V satisfies the assumptions (A1)–(A4). Let us assume also that V is symmetric (i.e. $V(x) = V(-x)$ and $V'(x) \geq 0$ for $x > 0$). Let u be a the solution of (1.1) with the initial datum $u_0(x) \geq 0$ compactly supported, satisfying the hypothesis of Theorem 2.1. Then the solution u to (1.1) is also compactly supported. If the smallest interval containing support of $u(\cdot, t)$ at time $t \geq 0$ is $[x_-(t), x_+(t)]$ we have*

$$\frac{dx_-(t)}{dt} > 0 \quad \frac{dx_+(t)}{dt} < 0. \quad (5.4)$$

Proof: This result is just a consequence of the fact that the equation for the evolution of $x_-(t)$ and $x_+(t)$ are the same as the evolution of characteristics associated to (1.1) starting at $x_-(0)$, $x_+(0)$, respectively, whence

$$\begin{aligned} \frac{dx_-(t)}{dt} &= - \int_{\mathbb{R}} V'(x_-(t) - y) u(y, t) dy, \\ \frac{dx_+(t)}{dt} &= - \int_{\mathbb{R}} V'(x_+(t) - y) u(y, t) dy. \end{aligned}$$

Thus, the result follows from the assumptions on the potential V . ■

Theorem 5.3. *Suppose that the potential V satisfies the assumptions (A1)–(A4) with $K = 0$. Let us suppose also that $V(x) = V(-x)$, $V'(x) \geq 0$ for $x > 0$. Assume that the $V(x) < 0$ in the interval $[-L, L]$. Let u be the solution to (1.1) with initial datum $u_0(x) \geq 0$, compactly supported with the support contained in the interval $(-L/2, L/2)$, satisfying the assumptions in Theorem 2.1 as well as $u_0(x) = u_0(-x)$. Then*

$$\lim_{t \rightarrow +\infty} u(x, t) = M\delta(x), \quad \text{where } M = \int_{\mathbb{R}} u_0(x) dx, \quad (5.5)$$

where the convergence take place in the sense of distributions.

Moreover, if we assume also that $V \in C^3(\mathbb{R})$ and $V''(0) > 0$ then

$$u(0, t) \sim C_1 e^{V''(0)Mt}, \quad \text{as } t \rightarrow +\infty \quad (5.6)$$

and the support of $u(\cdot, t)$ is contained in the interval of the form

$$|x| \leq C_2 e^{-V''(0)Mt}, \quad (5.7)$$

where C_1, C_2 are suitable positive constants depending on the initial data.

Proof: Let us denote as $[x_-(t), x_+(t)]$ the support of $u(\cdot, t)$. Due to our assumptions we have $x_-(t) = -x_+(t)$. Proposition 5.2 yields $\frac{dx_-(t)}{dt} > 0$. Thus, there exists $\lim_{t \rightarrow +\infty} x_+(t) = \alpha \in [0, \frac{L}{2})$.

Suppose that $\alpha > 0$. Notice that

$$\frac{dx_+}{dt} = - \int_{\mathbb{R}} V'(x - y) u(y, t) dy. \quad (5.8)$$

We now claim that the right hand-side of (5.8) is smaller then a strictly negative number $-\epsilon < 0$ for t long enough. Indeed, given our assumptions on V as well as the mass conservation property (Proposition 2.6), the only way in which this could not happen is with $u(\cdot, t)$ concentrated near $x = x_+$. But this is not possible due to symmetry of function u with respect to x . Then $\frac{dx_+}{dt} < -\epsilon < 0$ which yields a contradiction.

Hence, $\alpha = 0$ and due to the mass conservation property (Proposition 2.6) (5.5) follows.

In order to derive the more precise asymptotic (5.6) and (5.7) we linearize the equations for the characteristics associated to (1.1a) that due to the assumption $K = 0$ take the form

$$\frac{dx}{dt} = - \int_{\mathbb{R}} V'(x - y) u(y, t) dy, \quad (5.9a)$$

$$\frac{du}{dt} = u \int_{\mathbb{R}} V''(x - y) u(y, t) dy \quad (5.9b)$$

Linearizing these equations near $x = 0$ we obtain

$$\begin{aligned} \frac{dx}{dt} &= -MV''(0)x, \\ \frac{du}{dt} &= MV''(0)u. \end{aligned}$$

The error terms made in these linearizations are exponentially small and can be easily estimated using standard ODEs arguments. Nevertheless, Eqs. (5.9) imply (5.6) and (5.7). ■

Remark. We have assumed that $V(x) = V(-x)$, $u_0(x) = u_0(-x)$ by simplicity. Otherwise the statement would be similar with a little modification. Notice however, that if $V'(0) \neq 0$ the solution would translate asymptotically with a constant speed as $t \rightarrow \infty$.

Remark. Notice, that the same argument used in the proof of Theorem 5.3 might be used to show that for compactly supported attractive potentials with support in the interval $[-L, L]$, as in the statement of Theorem 5.3, and initial data u_0 whose support is a family of intervals separated more than $2L$ the long time asymptotics of (1.1) is a set of Dirac masses. More precisely each of the intervals that make the support of u_0 aggregates precisely to one of the Dirac masses, because, due to the compactly supported character of the potential V , they do not interact with each other.

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