UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Análisis matemático de un modelo de ecuaciones en derivadas parciales con términos quimiotácticos

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Antonio Manuel Vargas Ureña

Directores

Mihaela Negreanu Pruna
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Programa de Doctorado en Ingeniería Matemática Estadística e Investigación Operativa por la Universidad Complutense de Madrid y la Universidad Politécnica de Madrid

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Tesis doctoral

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Directores

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Entréme donde no supe,
y quedéme no sabiendo,
toda ciencia transcendiendo.
(San Juan de la Cruz)

Un conocimiento va entrando a ser científico conforme se hace más preciso y organizado, conforme va pasando de la precisión cualitativa a la cuantitativa. En un tiempo la verdadera ciencia científica era la matemática; la física ha entrado en el período realmente científico cuando, subordinándose a la mecánica racional, se ha hecho matemática, y se ha pasado de la alquimia a la química al reducir la previsión cualitativa de cambios químicos a previsión cuantitativa según peso, número y medida.
(Miguel de Unamuno, En torno al casticismo)

Agradecimientos y dedicatoria

Obligadamente, debo agradecer a mis directores de tesis, Mihaela y José Ignacio, en primer lugar. Mi relación con ellos nació en 2017 cuando me propusieron como trabajo de fin de grado un modelo de EDP con quimiotaxis y desde entonces venimos manteniendo una amplia y fructífera colaboración académica (pero no solamente), siempre bajo su tutela, casi ininterrumpidamente. La culminación del doctorado que supone su defensa no es óbice para que esta estrecha relación continúe por mucho tiempo. Así lo deseo.

No puedo dejar de mencionar después a mi tío Paco. Él me ha enseñado desde siempre a estudiar con gusto y a disfrutar aprendiendo y trabajando. Con toda seguridad, a nadie más que a él debo mi formación (sin excluir a otra mucha gente) y, qué duda cabe, el optar al grado de doctor. Que esta memoria conste de dos partes distintas (y no sea un rimero de capítulos por ello se lo vuelvo a agradecer a mis directores).

Por supuesto, agradezco todo el apoyo y ayuda que me ha prestado su grupo de trabajo: Ángel, Juanjo, Luis, y Miguel, cuya amistad y colaboración espero conservar siempre. No resultaría justo por mi parte dejar de mencionar a todos los departamentos de matemáticas a los que he pertenecido estos dos años. Estos han pagado mi inexperiencia y descuidos siempre con compresión y apoyo. Ha sido una enriquecedora experiencia la mía en los institutos de enseñanza secundaria.

En último lugar, quizás sean los más importantes por su apoyo, me gustaría dedicar esta memoria a mi familia: a mis padres, que me han dado todo; a mis abuelos, a los dos, por fortuna, que aún conservo y a la memoria de los otros dos, que se fueron tan triste y rápidamente, que lamentablemente no me verán acabar este proyecto que emprendí con ellos. A mi hermana, Nazaret. Y a Olga, que ha aparecido en último lugar pero cuyo cariño me ha ayudado en todo. A todos vosotros, gracias.

Con esta tesis trato de comenzar la que espero sea una larga vida académica e investigadora. Humildemente, espero aportar con ella un pequeño grano de arena, aunque temo no añadir más que ruido al ruido. Pero, ¿qué es todo esto sino *a tale, told by an idiot, full of sound and fury, signifying nothing*?

A todos vosotros, gracias.

Abstract

In this thesis we analyze the properties of a set of systems of Partial Differential Equations which model various phenomena in biology, both from the point of view of mathematical analysis and numerical analysis.

The models present terms of logistic growth with periodic behavior or tend towards a periodic behavior. Parabolic-elliptic, parabolic-parabolic and parabolic-ordinary systems are addressed. In the thesis, the results are obtained on the global existence of solutions, their uniform boundedness and their periodic uniform asymptotic behavior. The systems of chemotaxis present a non-linearity of second order (in the derivatives) where the main difficulty of the problem lies.

We begin, in Chapter II, with the analysis of the parabolic-elliptic problem, where a comparative method, known as the "rectangle method" is used to test the overall existence of solutions and their asymptotic behaviour. This solution converges to a solution that is homogeneous in space and periodic in time, under the appropriate hypothesis in the data of the problem. The problem is generalized to a system of 3 equations where the two biological species compete with each other and both show chemotactic movement.

In Chapter III, we analyze the parabolic-parabolic model and the Alikakos-Moser method based on iterations in the exponent "p" of the integral of u to the power p is used. Under the same hypotheses in the logistic term as in Chapter II, the same asymptotic behavior is obtained.

Chapter IV studies the parabolic-parabolic problem with non-local terms, and a relationship between the exponents of the problem is imposed. The different cases are studied in detail, for a wide range of parameters.

In Chapter V, we consider the parabolic-ordinary problem, as there is no diffusion in the second variable, there is no regularising effect as in the previous chapters. In this chapter we obtain global existence and study successfully the asymptotic behavior.

Chapters VI-XI discuss the problem from a numerical point of view. First, the fundamentals of the "Generalized Finite Difference" method (GFDM) applied to partial derivative equations are introduced.

In Chapter VII the parabolic-elliptic problem is considered numerically, the convergence of the method is studied and it is applied to different data, obtaining results on blow-up, convergence and periodicity, depending on the values of the parameters and the data of the problem.

In Chapter VIII, the GFDM is applied to the parabolic-parabolic problem and its convergence is studied and applied to several examples with known functions.

Chapters IX and X are devoted to the treatment of the parabolic-parabolic problem with integral terms and the parabolic-ordinary problem with periodic coefficients respectively. The convergence of the method is analysed and several examples with different behaviors

are presented.

To finish the memory, in chapter XI the method is extended to a problem of 3 parabolic-parabolic-ordinary equations, known in the literature for its applications to medicine.

Resumen

En esta tesis analizamos las propiedades de un conjunto de sistemas de Ecuaciones en Derivadas Parciales que modelan varios fenómenos de la biología, tanto desde el punto de vista del análisis matemático como del análisis numérico.

Los modelos presentan términos de crecimiento logístico con un comportamiento periódico o tienden a un comportamiento periódico. Se abordan los sistemas parabólico-elíptico, parabólico-parabólico y parabólico-ordinario. En la tesis se obtienen resultados sobre la existencia global de las soluciones, su límite uniforme y su comportamiento asintótico periódico uniforme. Los sistemas de quimiotaxis presentan una no linealidad de segundo orden (en las derivadas) donde radica la principal dificultad del problema.

Comenzamos, en el Capítulo II, con el análisis del problema parabólico-elíptico, donde se utiliza un método comparativo, conocido como "método del rectángulo", para comprobar la existencia global de las soluciones y su comportamiento asintótico. Esta solución converge a una solución homogénea en el espacio y periódica en el tiempo, bajo la hipótesis adecuada en los datos del problema. El problema se generaliza a un sistema de 3 ecuaciones en el que dos especies biológicas compiten entre sí y ambas muestran un movimiento quimiotáctico.

En el Capítulo III, analizamos el modelo parabólico-parabólico y se utiliza el método de Alikakos-Moser basado en iteraciones en el exponente "p" de la integral de u a la potencia p. Bajo las mismas hipótesis en el término logístico que en el capítulo II, se obtiene el mismo comportamiento asintótico.

En el Capítulo IV se estudia el problema parabólico-parabólico con términos no locales, y se impone una relación entre los exponentes del problema. Se estudian detalladamente los diferentes casos, para una amplia gama de parámetros.

En el Capítulo V, consideramos el problema parabólico-ordinario, ya que no hay difusión en la segunda variable, no hay efecto regularizador como en los capítulos anteriores. En este obtenemos la existencia global y estudiamos con éxito el comportamiento asintótico.

Los Capítulos VI-XI tratan el problema desde un punto de vista numérico. En primer lugar, se introducen los fundamentos del método de la "diferencias finita generalizadas" aplicado a las ecuaciones en derivadas parciales.

En el Capítulo VII se considera numéricamente el problema parabólico-elíptico, se estudia la convergencia del método y se aplica a diferentes datos, obteniendo resultados sobre el hinchamiento, la convergencia y la periodicidad, en función de los valores de los parámetros y los datos del problema.

En el Capítulo VIII se aplica el método de las diferencias finitas generalizadas al problema parabólico-parabólico, se estudia su convergencia y se aplica a varios ejemplos con funciones conocidas.

Los Capítulos IX y X están dedicados al tratamiento del problema parabólico-parabólico

con términos integrales y el problema parabólico-ordinario con coeficientes periódicos, respectivamente. Se analiza la convergencia del método y se presentan varios ejemplos con comportamientos diferentes.

Para terminar la memoria, en el Capítulo XI se amplía el método a un problema de 3 ecuaciones parabólico-ordinarias, conocido en la literatura por sus aplicaciones a la medicina.

Contents

Introduction

1 Introduction

Chemotaxis is the process under which some living organisms (such as bacterias, cells of the immune system, cells of the endothelium etc.) direct their movement in the direction of a chemical gradient. The individuals of a biological species are able to recognize a chemical signal, to measure its concentration and to move towards the higher concentrations of the substance (positive taxis) or away from it (negative taxis). The phenomena was discovered at the end of the XIX century and since then it has been widely studied.
Since 1970s, chemotaxis has been studied from the mathematical point of view, starting with the first models of PDEs suggested by Keller and Segel, [62], [63], after Patlak [93] who introduced a model based in reinforced random walks. The mathematical bibliography is extensive and a wide summary of results in the area during the last decades can be found in [52] and [53].
From the point of view of applications mathematical models with chemotactic terms have been applied to model *Angiogenesis*, a key process in Tumor Growth, whereby endothelial cells move towards the tumor following a chemical gradient, creating new blood vessels and providing extra supply to the tumor. The process has been largely studied and mathematical models of PDEs have been used to describe the creation of new blood vessels (more details can be found in Anderson and Chaplain [5], Levine, Sleeman and Nilsen-Hamilton [70] and Holmes and Sleeman [51]). Chemotaxis terms also appear in *Astrophysics* to describe gravitational interaction of particles on the gravitational equilibrium of polytropic stars (in Biler [13]), in *Ecology*, to describe the attraction of predators to certain chemical signals (pheromons) of the prey (appearing in Tello and Wrzosek [111]), in *Morphogenesis*, the creation of shapes and organs in embryonic development, as in Bollenbach *et al* [15] among others.

Derivation of the models

In what follows (unless otherwise stated), let "u" be the density of population of a biological species and "v" the density of the chemical substance, secreted by the individuals of the species, responsible of the chemotactic movement. Both of these quantities depend on the space and time, so we may write $u(x,t)$, $v(x,t)$. Also, consider a bounded domain $\Omega \subset \mathbb{R}^d$, for any $d \in \mathbb{N}$ (restrictions on the dimension d will appear later depending on the studied model), and $t \in [0,\infty)$. We use the notation $\Omega_t = \Omega \times (0,t)$, for $t \in (0,\infty]$ throughout this document.
We discuss briefly the governing equation for the density u. Consider some subdomain $\omega \subset \Omega$. We call $u(x,t)$ and $u(x,t_0)$ the density of population at times t and $t_0 < t$. Assume, as usual in fluid mechanics, that the mass of the species (the amount of individuals)

at some time t in the subdomain ω is equal to the mass at a previous time, t_0, plus the flux of individuals at the border, $\partial\omega$ for any time $s \in [t_0, t]$. It is natural that, since the density u represents a biological species, we shall also consider the number of births and deaths during that period of time. In other words,

$$\int_\omega u(x,t)dx = \int_\omega u(x,t_0)dx - \int_{t_0}^t \int_{\partial\omega} \Phi \cdot \nu d\sigma_x ds + \int_{t_0}^t \int_\omega a(u,x,s)dxds,$$

where Φ is the flux at the border, ν is the unit normal vector and $a(u,x,s)$ is the growth rate of "u", which may depend on the space, time and u. Because of the motility of the species, we assume some random movement occurs (*diffusion*) where the individuals move from regions with high densities of its own species towards regions with lower densities. This assumption is represented by *Fick's first law* (depending on the field, it is also called -or analogous to- *Darcy's law*, *Ohm's law* or *Fourier's law of heat conduction*),

$$\Phi = -D\nabla u.$$

Here D is the diffusion coefficient (with dimensions of area per unit of time) and we shall consider $D = 1$. Not only this, the individuals of the biological species are also able to move in the direction of some stimulus, in this case some chemical substance whose density is represented by v. This chemotactic term is modeled as

$$\chi u \nabla v,$$

meaning that the individuals of the species move, proportionally to their own density, towards the higher concentrations of the chemical whenever the chemotactic coefficient or chemosensitivity, χ, is positive (then we talk of positive taxis and the chemical is said to be *chemoattractant*), or towards the lower concentrations of "v" when $\chi < 0$ (the chemical is *chemorepellent*). Our analysis in the following chapters are devoted to the case where χ is a strictly positive constant. By adding this new term to the flux tensor, the governing equation of u reads

$$\int_\omega u(x,t)dx - \int_\omega u(x,t_0)dx = -\int_{t_0}^t \int_{\partial\omega} \nabla(-\nabla u + \chi u \nabla v)dxds + \int_{t_0}^t \int_\omega a(u,x,s)dxds,$$

where we have applied the Divergence theorem. We look now at the growth term. It is expected that the resources decrease as the population grows and the mortality increases if the population is higher than a threshold value in limited resources scenarios. One of the most widely used terms to describe the population growth is a logistic type function, i.e.,

$$a(u,x,t) = \mu u(N(x,t) - u), \tag{1}$$

where N represents this threshold value -*carrying capacity*- of the system (more details are presented in Murray [75]) and μ is the ratio of growth of the population. If the density of population exceeds this value N, it starts to decrease because of the shortage of resources. Hence, the u-equation is

$$\int_{t_0}^t \int_\omega \frac{\partial u}{\partial t} dxds = \int_{t_0}^t \int_\omega \Delta u dxds - \int_{t_0}^t \int_\omega \chi \nabla(u\nabla v)dxds + \int_{t_0}^t \int_\omega \mu u(N(x,s) - u)dxds.$$

Finally, since ω, t and t_0 are arbitrary, we obtain

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla(u \nabla v) + \mu u(N(x,t) - u). \tag{2}$$

We consider two different cases in this document:

- In the largest part of this document we work under the assumption that these resources depend on time and present some "kind of periodicity" which we shall explain below, and consequently the population presents some seasonal behavior. Several examples of this kind of periodicity are common in the literature: in the movement of the amebas *Dictyostelium discoideum* towards its center of aggregation, the medium velocity is periodic, for instance in Steinbock, Hashimoto and Müller [96]. In Dunn and Zicha [31], it is also observed the periodicity in the chemotaxis of the human neutrophils. For the sake of simplicity we denote

$$N(x,t) = 1 + f(x,t),$$

where the function $f(x,t)$ has some periodic asymptotic behavior which we shall discuss later. We shall consider this the general scenario of this thesis.

- We also consider the effect of nonlocality. In (1) individuals compete locally for the resources of the environment. Now, we can assume that the total mass of the species (the total amount of individuals) affects the growth of the population, that is to say, we assume some kind of nonlocal term. We represent this growth term as

$$a(u,x,t) = u\left(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx\right), \qquad (3)$$

for some constants α, a_0, a_1 and a_2 (with $a_1 > 0$). In this case, if $a_2 < 0$, individuals compete locally and globally (nonlocally) for the resources and if $a_2 > 0$ they compete locally but cooperate globally. This case is discussed theoretical and numerically in Chapters IV and IX, respectively.

To obtain the governing equation for the chemical substance we follow similar arguments to those employed for the equation of u. Depending on the nature of the chemical, we divide the models into two forms:

- Dominant diffusion of the chemical: by assuming the diffusion of the chemical we proceed as before and use Fick's first law. Also, we consider some function $g(u,v)$ representing the production and degradation of the chemical, so we have

$$\tau \frac{\partial v}{\partial t} = \Delta v + g(u,v),$$

for some constant τ. We take a linear expression for the function g such as:

$$g(u,v) = g_0 u - g_1 v,$$

where g_0 is the rate of production of the chemical (that is secreted by the individuals of the species and therefore, it is proportional to u) and g_1 is the rate of degradation of the chemical (so the degradation process is proportional to v). For simplicity we take $g_0 = g_1 = 1$, so we have

$$\tau \frac{\partial v}{\partial t} = \Delta v + u - v.$$

Here, we find two subcases:

- If we assume that the chemical diffuses faster than the biological species, we take $\tau = 0$. Hence, the governing equation of v becomes elliptic:

$$0 = \Delta v + u - v. \tag{4}$$

The parabolic-elliptic system (2)-(4) (together with the boundary conditions and the initial data for u) is studied in Chapter II.

- If we assume that the diffusion of the species is comparable with the diffusion of the chemical, we take τ to be a positive constant. More precisely, we make $\tau = 1$:

$$\frac{\partial v}{\partial t} = \Delta v + u - v. \tag{5}$$

We dedicate Chapter III to the study of the fully parabolic system (2)-(5).

- Non-diffusive chemical substance: if we assume now that once the chemical is secreted no diffusion may occur, the governing equation does not have any spatial variation. Hence, we have some ordinary differential equation of the form

$$\frac{\partial v}{\partial t} = h(u, v), \tag{6}$$

with some function h representing the production and degradation of the chemical substance. The parabolic-ODE system (2)-(6) is analyzed in Chapter V.

Literature review

In this subsection we review some well-known results from the mathematical literature concerning chemotaxis. The first mathematical model of chemotaxis was proposed by Keller and Segel in [62], [63] and reads as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v), & x \in \Omega, \quad t > 0, \\ v_t = k_c \Delta v - k_3(v)v + uf(v), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Keller and Segel study the stability of the uniform state (u_0, v_0), finding that the steady state is instable under the condition

$$\frac{k_2(u_0,v_0)v_0}{k_1(u_0,v_0)u_0} + \frac{u_0 f'(v_0)}{k_3(v_0 + v_0 k_3'(v_0))} > 1.$$

The Keller-Segel models, named after the above system appeared, have demonstrated their success being a consequence of its intuitive simplicity, analytical tractability and capacity to replicate key behavior of chemotactic populations, as stated in [50]. This minimal model has rich and interesting properties including globally existing solutions, finite time blow-up and spatial pattern formation. Detailed reviews can be found in the survey of Horstmann [52].
One of the properties of the populations under the effects of chemotaxis is the *auto-aggregation*. This phenomenon has been shown to lead to finite-time blow-up under certain

formulations of the model, and a large body of work has been devoted to determining when blow-up occurs or whether globally existing solutions exist. In this sense, many remarkable results have been achieved in the models with linear chemotactical sensitivity. In 1992, using the transformation

$$U(x,t) := \frac{|\Omega| u(x,t)}{\int_\Omega u(x,t)dx}, \quad \text{and } V(x,t) := v(x,t) - \frac{1}{|\Omega|}\int_\Omega v(x,t)dx,$$

Jäger and Luckhaus in [60] introduce the parabolic-elliptic model

$$\begin{cases} \dfrac{\partial U}{\partial t} = \nabla(\nabla U - \chi U \nabla V), & x \in \Omega, \quad t > 0, \\ 0 = \Delta V - \alpha(U-1), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ U(x,0) = U_0(x), & x \in \Omega. \end{cases}$$

For this system, the authors find, in the 2-dimensional setting, that there exists a critical value $c(\Omega)$ such that a unique, smooth positive solution exists globally in time if $\alpha\chi U_0(x) < c(\Omega)$. A few years later, Herrero and Velázquez in [44] used the asymptotic expansion methods and proved that there exists radially symmetric initial data such that the solution of the previous system forms, in the center of a disk Ω, a δ-function singularity (in the sense of the Dirac measure) at finite time. In [42] and [43] Herrero, Medina and Velázquez found that no radial, self-similar solutions such that $\int_{|x|<r} U(T,s)ds < \infty$ as $r \to 0$ exist. The first author also proved that the blow-up occurs in $\mathbb{R}^3$ in [46].
The blow-up phenomenon was later studied by Nagai, who in [76] addressed the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(\nabla u - \chi u \nabla v), & x \in \Omega, \quad t > 0, \\ 0 = k_c \Delta v - \gamma v + \tilde{\alpha} u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

He proved that the classical solution exists globally in time and it is uniformly bounded under the condition $\tilde{\alpha}\chi \int_\Omega u_0(x)dx < 4\pi$. In [77], the author demonstrated that the solutions to the minimal Keller-Segel system blow-up in finite time if $\tilde{\alpha}\chi \int_\Omega u_0(x)dx > 8\pi$. We use this condition in the numerical chapters of this thesis.
In the case we do not consider a linear chemotactical sensitivity, we may consider it to be a function of the density u, $f(u)$. In general, this function depends on the particle density and the external signal. The above cases are related to $f(u) = u$ and, as stated, blow-up occurs under certain conditions for $d = 2, 3$. There have been several attempts to introduce certain reasonable effects into the Keller-Segel equations that might prevent blow-up like volume-filling and quorum sensing aspects. The volume filling aspect is reflected as a certain dependence of the chemotactic sensitivity function on the particle density u, which leads to bounded global-in-time solutions of the system. This has been done for example by Hillen and Painter in [50]. In [54], Horstmann and Winkler study the more general case $f(u) = u^\alpha$, for $\alpha > 0$. They found that $\alpha = 2/d$, for $d \geq 2$ is a critical blow-up exponent.

In [94], Potapov and Hillen chose a chemotactical sensitivity of the form $f(u) = \chi(1-u)$. The solutions show spatio-temporal patterns which allow for ultra-long transients and merging or coarsening. The authors study the underlying bifurcation structure and show that the existence time for the pseudostructures exponentially grows with the size of the system. Rascle and Ziti in [103] consider a chemotactical sensitivity that depends on the concentration v. They investigate the possibility that in finite time the population of predators aggregates to form a delta-function.
A different approach is followed in order to prevent the blow-up of solutions. It is well-known that the solutions of the minimal-chemotaxis-logistic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + au - bu^2, & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$, doesn't present blow-up for any $a \in \mathbb{R}$, $\tau \geq 0$, $\chi > 0$ and $b > 0$ (see Tello and Winkler [109] and Winkler [117]).
In [109], Winkler and Tello obtained unique global bounded classical solutions under the assumptions $d \leq 2$ or $\mu > \frac{d-2}{2}\chi$ for any initial data for some logistic term of the form $g(s) \leq c_1 - \mu s^2$. Also, if $\frac{d}{ds}\frac{g(s)}{s} < -2\chi$, they prove that the solution to the parabolic-elliptic problem has a globally asymptotically stable equilibrium in L^∞ in (1,1). Similar results are obtained in [39] by Galakhov, Salieva and Tello. More recently, in [120], Winkler has obtained that the solutions to the parabolic-elliptic system with logistic term $\lambda u - \mu u^\kappa$ do not blow-up provided

$$\kappa < \begin{cases} \dfrac{7}{6}, & \text{if } d = 3,\ 4, \\ 1 + \dfrac{1}{2(d-1)}, & \text{if } d \geq 5. \end{cases}$$

In Xiang [122], the author replaces the logistic source $au - bu^2$ with a kinetic term $g(u)$ fulfilling $g(0) \geq 0$,

$$\lim \inf_{s\to\infty} \{-g(s)\frac{\ln s}{s^2}\} = \mu_1 \in (0, \infty],$$

as well as

$$(\chi - \mu_1)^+ M < \frac{1}{2C_{GN}^4},$$

where $c^+ = \max\{c, 0\}$, C_{GN} is the Gagliardo-Nirenberg constant and

$$M = ||u_0||_{L^1(\Omega)} + |\Omega| \inf_{\eta>0} \frac{\sup\{g(s) + \eta s : s > 0\}}{\eta}.$$

In this setup, it is shown that this problem doesn't have any blow-up by ensuring all the solutions are global-in-time and uniformly bounded. Clearly, g covers the sub-logistic sources like $g(s) = as - bs^\theta$, with $b > 0$ and $\theta \geq 2$.
In [80] Negreanu and Tello considered the parabolic-elliptic system

$$\begin{cases} u_t - \Delta u = -\chi \nabla \cdot (u(N-u)\nabla v) + \lambda u(1-u), & x \in \Omega, \quad t > 0, \\ -\Delta v - v = u, & x \in \Omega, \quad t > 0, \end{cases}$$

and obtained the convergence of the solutions to the steady state $(1,1)$ applying a comparison argument. This method is known as *rectangle method* and was introduced by Pao

in 1980 [92] for reaction-diffusion systems. It consists in using an auxiliary ODE system and compare its solutions, which are easier to analyse, with those of the original system. We also employ this method to establish an asymptotic behavior result for the parabolic-elliptic problem.
Results concerning the existence of the solutions and their asymptotic dynamics for competitive systems of two biological species and a chemical are also widely studied in the bibliography. In [34], Espejo, Stevens and Velázquez considered different diffusivities for u and v, no logistic term and a constant degradation term for the chemical. Motivated by the question whether multi-species chemotaxis mechanisms can be responsible for processes of cell sorting, their work focuses on the occurrence of the blow-up phenomena and the asymptotic behavior of such unbounded solutions (also in [35] and [36] can be seen the case when the chemotactical sensitivities have different signs). Multi-species chemotaxis systems with non-constant coefficients are also found in Issa and Shen [58], [59], in Tello and Winkler [110] (the case $f_1 = f_2 = 0$ and constant coefficients), where the stability of the homogeneous steady states is obtained (as in Stinner, Tello and Winkler [100], Bai and Winkler [6], Black, Lankeit and Mizukami [14] and Cruz, Negreanu and Tello [25]). Recently, in Negreanu and Tello [84], for a predator-prey interaction system with periodic functions in time as coefficients, the global existence and asymptotic behavior are obtained for positive and bounded initial data under suitable conditions.
Due to the non-diffusivity nature of some chemical substances, some models require considering an ordinary differential equation for their modelling. Parabolic-ODE systems with chemotactic terms have been considered from the $90'$, after the pionering works of Levine and Sleeman [69] and Anderson and Chaplain [5] modeling tumor angiogenesis, a considerable number of authors have analyzed such models. In Othmer and Stevens [91] and Stevens [97], the authors obtain a Parabolic-ODE system of chemotaxis passing to the limit from a discrete to a continuous system of equations. Concerning Angiogenesis, the model has been considered in Kubo and Suzuki [65] and Kubo, H. Hoshino and K. Kimura [64]. Mathematical analysis of these models with two equations can be found in Fontelos, Friedman and Hu [37], Friedman and Tello [38] and Negreanu and Tello [83] among others. Systems with 3 or more equations involving chemotaxis and diffusive or non-diffusive processes also appear in ecology and other biological applications (also in [78]). In [79] the authors study a similar Parabolic-Parabolic-ODE system

$$\begin{cases} u_t = \Delta u - div(\chi_1(w)u\nabla w) + \mu_1 u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - div(\chi_2(w)v\nabla w) + \mu_2 v(1-v), & x \in \Omega, \quad t > 0, \\ w_t = h(u, v, w), & x \in \Omega, \quad t > 0, \end{cases}$$

where the chemosensitivities χ_1, χ_2 are non-constant. The global existence and convergence of the solution to a steady-state $(1, 1, \tilde{w})$ satisfying $h(1, 1, \tilde{w}) = 0$ are presented under suitable assumptions on the coefficients and the spatial dimension of the domain. The results in [79] have been improved in Mizukami and Yokota [74] for a larger range of parameters.
The non-locality also appears in the literature as a realistic scenario. For instance, in [102] integral terms were used to describe the competition between the cancer cell density and the extracellular matrix density. A wide summary of the existing results concerning nonlocal terms as (3) for the case $a_2 = 0$ is the one given in [29]. For $a_2 = 0$ there exists a unique global solution which tends to $((a_0/a_1)^{1/\alpha}, (a_0/a_1)^{\gamma/\alpha})$ as t goes to infinity, assumed $\alpha + 1 > m + \gamma$ or $\alpha + 1 = m + \gamma$ and a_1 large enough.

In [81], a parabolic-elliptic PDE system with non-local terms is considered, the global existence of the solutions and its asymptotic behavior are studied, provided $a_1 > 2\chi + |a_2|$. Recently, in [57], a parabolic-parabolic-elliptic system with non-local terms of the form

$$\begin{cases} u_t = d_1\Delta u - \chi_1 div(u\nabla w) + u(a_0 - a_1 u + a_2 v - a_3 \int_\Omega u dx - a_4 \int_\Omega v dx), \\ v_t = d_2\Delta u - \chi_2 div(v\nabla w) + v(b_0 - b_1 u + b_2 v - b_3 \int_\Omega u dx - b_4 \int_\Omega v dx), \\ 0 = d_3\Delta w + ku + lv - \lambda w, \end{cases}$$

is analyzed. The authors obtained asymptotic convergence to a unique constant steady state by using a comparison argument.
A related process to chemotaxis is haptotaxis. Chemotaxis, as stated earlier, is defined as targeted cellular locomotion in response to a concentration gradient of a chemical factor in solution. The cells perceive the chemical and migrate to higher concentrations of this substance until they reach the source that secretes it. On the other hand, gradients do not have to be in solution. An adhesive molecule could be present in increasing amounts along an extracellular matrix. A cell that was constantly making and breaking adhesions with such a molecule would move from a region of low concentration to an area where that adhesive molecule was more highly concentrated. Such a phenomenon is called haptotaxis (see [19], [26]). Due to the key role that invasive processes play in biological phenomena such as wound healing, morphogenesis or tumor invasion, there are a large number of mathematical studies about chemotaxis-haptotaxis. It is well known that the growth of solid tumors proceeds through two distinct phases: the avascular and the vascular phase. Tumor cells find a variety of substratum-bound factors that can influence their migration directed to different stages in the process of tumor invasion and metastasis. Such factors can promote the targeted movement of tumor cells by at least two mechanisms, called chemotaxis and haptotaxis.
Initially, Chaplain and Lolas in [20]–[21] developed a mathematical model consisting of three partial differential equations describing the evolution in time and space of the system variables. It is assumed that the key physical variables are tumor cell density (denoted by u), protein density of the extracellular matrix (denoted by w) and the concentration of urokinase plasminogen activator (denoted by v) each of them considered at $x \in \Omega$ and time $t > 0$. The model is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) - div(\xi u \nabla w) + \mu u(1 - u - w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial t} = -vw, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{7}$$

It is natural to assume that there is no-flux of tumor cells or protease across the boundary of the domain, $\partial\Omega$ due to the vitro experimental protocol in which invasion takes place within an isolated system (more details are given in [21])

$$-\frac{\partial u}{\partial \nu} + \chi u \frac{\partial v}{\partial \nu} + \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{8}$$

The three-component chemotaxis-haptotaxis model (XI.1) is an extension of the two-component Keller-Segel model by taking $w = 0$.
It is well known that in the case $\mu = 0$ and $w = 0$ the two-component subsystem possesses solutions which blow up in finite time in spatially two- or higher-dimensional settings as it was proved for instance in [45] and [119]. Moreover, it admits global bounded solutions when either $d = 2$ and $\mu > 0$ is arbitrary [105], or $d = 3$ and μ is sufficiently large [18]. For $d = 3$ and $\mu > 0$ is small enough, only certain global weak solutions are known to exist (as in [67] and the references therein).
For the subproblem of (XI.1), the haptotaxis-only system by letting $\chi = 0$ there have been studied the global existence [116] and asymptotic behavior [71] of the solutions. The global solvability for the full system has been analyzed in the two and higher-dimensional setting in [104] and [106]. Under the assumptions that either $d = 2$ and $\mu > 0$ is arbitrary in [105] or $d = 3$ and μ is sufficiently large in [18], there has been proved the existence of the global bounded solutions to the full chemotaxis-haptotaxis system. In a smoothly bounded domain $\Omega \in \mathbb{R}^d$, $d \leq 3$, with zero-flux boundary conditions, for every χ, ξ, and μ given positive parameters, Tao and Winkler in [107] have demonstrated that whenever the initial data (u_0, v_0, w_0) are regular fulfilling $u_0 > 0$ and $w_0 \leq 1$, the solution w decays asymptotically in $L^\infty(\Omega)$. Moreover, if $\mu > \frac{\chi^2}{8}$, the authors in [107] obtained the exponential stabilization of the solution (u, v, w) to the constant stationary solution $(1, 1, 0)$ in $L^\infty(\Omega)$ as $t \to \infty$.

These are only some of the most important results connected to the thesis, there are many achievements that show the importance of the investigation of such patterns, but our intention is to relate our results to those already known.

Numerical methods for chemotaxis systems

Recently, numerical solutions of chemotaxis systems are being investigated. For instance, MacDonald *et al.* used a moving mesh finite element method [73] and the same method was applied to solve a two species system with no logistic term in [24]. In [28] Dehghan *et al.* used Radial Basis Collocation method for solving similar systems. Also, finite element methods have been applied in [101] as well as [55]. Nonstandard finite difference schemes have been implemented for the one dimensional case with no diffusion in [22].
In [32], Epshteyn and Kurganov developed a family of new interior penalty discontinuous Galerkin methods for solving the Keller-Segel chemotaxis model. Authors of [27] investigated nonnegativity of exact and numerical solutions to a generalized Keller-Segel model. The main aim of [33] is to develop a novel upwind-difference potentials method for the Patlak-Keller-Segel chemotaxis model that can be used to approximate problems in complex geometries. A fractional step numerical method is developed by Tyson, Stern and LeVeque in [112] for the nonlinear partial differential equations arising in chemotaxis models, which include density-dependent diffusion terms for chemotaxis, as well as reaction and Fickian diffusion terms. They took the novel approach of viewing the chemotaxis term as an advection term which is possible in the context of fractional steps. The work of Chertock and Kurganov in [23] is concerned with the development of a new finite-volume method for a class of chemotaxis models and for a closely related haptotaxis model. Authors of [123] proposed an efficient and stable lattice Boltzmann method (LBM) for solving the generalized Keller-Segel chemotaxis model.
Concerning the method that we use, the *Generalized Finite Difference Method* has been

applied succesfully for solving highly nonlinear PDEs such as the Fokker-Planck equation [114] and the Telegraph equation [115] in both 2D and 3D. A few more words about the method are stated in Chapter VI.

Main topics of the thesis

This thesis consists of two different but not separated parts. The first part is related to the analytical study of the chemotactic models introduced previously. We understand for this "analytical study" the rigurous proofs of the existence, uniqueness and boundedness of solutions to these systems and their asymptotic behavior. This study takes place in Chapters II-V. The second part of the thesis is devoted to the analysis of the numerical solutions of the discrete versions (by means of meshless finite difference formulations). We perform this analysis in Chapters VII-XI. The common line of the following chapters (except from the nonlocal model -Chapters IV and IX- and the chemotaxis-haptotaxis model -Chapter XI-) is the consideration of some kind of periodic or seasonal behavior of the resources. More precisely, we consider that the environment (the carrying capacity) presents periodicity in the sense of

$$\lim_{t\to\infty} \|f(x,t) - f^*(t)\|_{L^\infty(\Omega)} = 0, \tag{9}$$

for some $T-$periodic in time and homogeneous in space function f^*. By assuming such condition, we are able to find a non-constant steady state for the u-equation as the solution to the Bernoulli ODE

$$u^*_t = \mu u^*(1 - u^* + f^*).$$

We can solve this ODE and find the T-periodic solution, $u^*(t)$, explicitly,

$$u^*(t) = \frac{u^*_0 e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u^*_0 \displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}, \tag{10}$$

with

$$u^*_0 := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}.$$

The asymptotic solution for v varies depending on the model and will be discussed in each chapter.

Main results

To end this section we summarize the main achievements of each chapter of the thesis.

Chapter I: Notations and previous known mathematical results
We state notations and previous mathematical results that we use throughout this thesis.

Chapter II: Parabolic-elliptic model

We present the parabolic-elliptic system given by (2)-(4), that is,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ -\Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{11}$$

under the assumption (9). We prove, by using fixed point arguments and compact embeddings the existence of a unique classical global solution. This result is enclosed in Theorem II.1. Later we introduce a system of auxiliary ODEs and use comparison arguments (the so-called *Rectangle method* introduced by Pao in [92]) to prove that the solution of (11) converges to the periodic function u^* given by (10). In other words, we prove that the solution inherits the periodic environmental conditions. For v^* we obtain the same asymptotic behavior. We state this result in Theorem II.4.
In addition, we achieve a generalization of the problem considering two biological species competing for a chemical substance in a periodic context, described by the following model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi_1 \nabla(u \nabla w) + \mu_1 u \left(1 - a_1 v - u + f_1(x,t)\right), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - \chi_2 \nabla(v \nabla w) + \mu_2 v \left(1 - v - a_2 u + f_2(x,t)\right), & x \in \Omega, \ t > 0, \\ -\Delta w + w = \alpha u + \beta v & x \in \Omega, \ t > 0, \end{cases} \tag{12}$$

The novelty of this model is that we add a second species that competes with the first one, as well as another periodic function in the new corresponding logistic term. We try to restore the periodical behavior of the solutions in long periods of time. On this occasion, the periodic limit solution is not explicitly expressed, instead it is the solution of an ODEs system with periodic coefficients. For this more general case, we also obtain existence and uniqueness of the solutions and the periodic asymptotic behavior, results presented in Theorem II.5. In Chapter VII we study the convergence of the method *Generalized Finite Difference Method* to the analytical solution of both systems and provide several examples illustrating their asymptotic behavior.
The results of this first part (one species and one chemical substance) have been published in [85] and the second part (two species and one chemical substance) in [89].

Chapter III: Parabolic-parabolic model
We analyze the parabolic-parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{13}$$

with f as in (9). A different and more complicated approach must be followed. First, we obtain local existence using standard arguments and then we extend it globally using the *Moser-Alikakos iteration method*. This result is presented in Theorem III.1. The asymptotic behavior is now studied using Lyapunov-type functionals in a two step process: first we obtain convergence of the solutions towards their averages,

$$(u,v) \xrightarrow{L^2(\Omega)} \left(\int_\Omega u dx, \int_\Omega v dx \right), \text{ as } t \to \infty,$$

and, later, the convergence of the averages towards the periodic functions (u^*, v^*), where $u^*(t)$ is given by (10) and $v^*(t)$ is the solution of

$$\tau \frac{dv^*}{dt} = u^* - v^*. \tag{14}$$

This result is proved in Theorem III.2. The results obtained in Chapter III are published in [86].
The numerical solution of this fully parabolic system is studied in Chapter VIII.

Chapter IV: Nonlocal model
We study the existence and boundedness of solutions of the nonlocal model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u^m \nabla v) + u(a_0 - a_1 u^\alpha + a_2 \displaystyle\int_\Omega u^\alpha dx), & x \in \Omega \ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u^\gamma, & x \in \Omega \ \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega \ \ t > 0. \end{cases} \tag{15}$$

The asymptotic behavior of the solutions is obtained for both competitive and cooperative cases. In particular, convergence towards

$$(u^*, v^*) = \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2}}, \sqrt[\alpha]{\left(\frac{a_0}{a_1 - a_2} \right)^\gamma} \right)$$

under some hypotheses on a_0 and a_2 or an exponential decay to 0 are found. The discrete version, using the GFD formulae, is presented and analyzed in Chapter IX.
The contents of this chapter are submitted to a journal [88].

Chapter V: Parabolic-ODE model
We assume a non-diffusive chemical substance, so we consider the system

$$\begin{cases} u_t = \Delta u - div(\chi u \nabla v) + \mu u(1 - u + f(x,t)), & x \in \Omega, \ \ t > 0, \\ v_t = h(u,v), & x \in \Omega, \ \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} - u\chi \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0, \end{cases} \tag{16}$$

for some function f as in (9). We prove in this chapter two main results under some assumptions on the function h. The first one, Theorem V.1, is devoted to the proof of the existence of the global classical solutions, using the Alikakos' method. The second one, Theorem V.2, states the asymptotic behavior of the solutions using, as in the previous chapter, Lyapunov's functionals. Convergence towards (u^*, v^*), where now v^* is the solution of

$$\frac{dv^*}{dt} = h(u^*, v^*),$$

is obtained. This chapter is related to Chapter X, where we propound an explicit scheme for solving numerically system (16) and prove the convergence of the discrete solution to the continuous one.
The results of Chapter V are partially published in [87].

Chapter VI: Fundamentals of the Generalized Finite Difference Method
We start the numerical part of this thesis by explaining the fundamentals and the derivation of the Generalized Finite Difference Method. This section is introduced for the sake of completeness.

Chapter VII: Numerical solution of the parabolic-elliptic model
We find the solutions obtained by the numerical scheme which approximates (11). We prove the conditional convergence of the GFD explicit scheme to the analytical solution of the system and provide several numerical examples. In particular, for a different range of parameters we find *blow-up* solutions (unbounded solutions at finite time), solutions converging to the constant steady state $(1, 1)$ when $f = 0$, and periodic solutions. Finally, we also solve numerically the generalized version of the parabolic-elliptic system. We give examples of admissible functions f, f_1 and f_2.
All the contents of this chapter are published in [7] or submitted in [89].

Chapter VIII: Numerical solution of the parabolic-parabolic model
We obtain the GFD scheme for the fully parabolic system and find explicitly the condition on the time step, Δt, for the convergence of the scheme. We give examples where the solutions converge to the asymptotic limits u^* and v^*. We also study the influence of many factors as number of nodes and time increment on the numerical solution. Another examples of possible functions f are given.
We have recently published the results of this chapter in [8].

Chapter IX.2: Numerical solution of the nonlocal model
The numerical discretization of the system (15) is obtained and the convergence of the method is proved under some smallness (but not restrictive) assumptions on the time step. Several examples which validate the discrete version of the nonlocal terms are given.
The content of this chapter is being reviewed in [11].

Chapter X: Numerical solution of the parabolic-ODE model
We propound an explicit scheme for solving the parabolic-ODE system studied in Chapter V. We prove the conditional convergence of the method, using assumptions on the function h of the continuous model. In our numerical tests we provide some examples of the functions $h(u, v)$ fulfilling the assumptions of Chapter V.
Recently, the paper [10] has appeared containing the results of this chapter.

Chapter XI: Chemotaxis-haptotaxis model

We consider a well-known model of chemotaxis-haptotaxis:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) - div(\xi u \nabla w) + \mu u(1-u-w), & x \in \Omega,\ t>0,\\
\dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t>0,\\
\dfrac{\partial w}{\partial t} = -vw, & x \in \Omega,\ t>0,\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega.\\
-\dfrac{\partial u}{\partial \nu} + \chi u \dfrac{\partial v}{\partial \nu} + \xi u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{17}
$$

We study its linealization to obtain the stability of the steady-states and solve numerically the parabolic-parabolic-ODE system using the GFDM. This results have been published in [9].

2 Introducción

La quimiotaxis es el proceso por el cual algunos organismos (como bacterias, células del sistema inmune, células del endotelio, etc.) dirigen su movimiento en la dirección de una sustancia química. Los individuos de una cierta especie biológica son capaces de reconocer la señal de la sustancia, de medir su concentración y de moverse hacia sus altas concentraciones (hablamos de taxis positiva) o en el sentido contrario (taxis negativa). El fenómeno fue descubierto a finales de siglo XIX y ha sido ampliamente estudiado desde entonces.

Desde 1970, el proceso de la quimiotaxis ha sido estudiado desde el punto de vista matemático, empezando por los primeros modelos de ecuaciones en derivadas parciales (EDPs) propuestos por Keller y Segel en [62] y [63]. Antes, en los años 50, Patlak empleó *reinforced random walks* para obtener un modelo quimiotáctico en [93]. La bibliografía matemática referente a este fenómeno es extensa, aunque un buen compendio de resultados en el área durante las últimas décadas son los realizados por Horstmann [52] y [53].

Los modelos matemáticos con términos quimiotácticos han sido aplicados para modelizar la *Angiogénesis*, un proceso clave en el crecimiento de tumores en el que las células del endotelio se mueven hacia el tumor, siguiendo una sustancia, creando nuevas venas e irrigando el tumor. El proceso ha sido muy estudiado y diversos modelos de EDPs se han usado para describir la creación de venas (véase, por ejemplo, Anderson y Chaplain [5], Levine, Sleeman y Nilsen-Hamilton [70] y Holmes y Sleeman [51]). También aparecen modelos matemáticos con quimiotaxis en Astrofísica para describir la interacción de partículas en el equilibrio gravitacional de estrellas politrópicas (en Biler [13]); en Ecología, para modelizar la atracción de depredadores a ciertas señales químicas (feromonas) producidas por las presas (en Tello and Wrzosek [111]). Además, es un fenómeno que interviene en la creación de órganos en el desarrollo embrionario (en Stinner, Tello and Winkler [99] entre otros).

Obtención de los modelos

En lo que sigue (a menos que se diga lo contrario) denotaremos por "u" a la densidad de población de una cierta especie biológica y por "v" a la densidad de una sustancia química, que es segregada por los individuos de la especie biológica, responsable del movimiento quimiotáctico. Supondremos que ambas cantidades dependen del lugar y el instante en que se encuentren, por lo que escribiremos $u(x,t)$, $v(x,t)$. También consideraremos un dominio acotado $\Omega \subset \mathbb{R}^d$, para cualquier $d \in \mathbb{N}$ (las restricciones que se impongan a d se harán específicas más adelante en función de las particularidades de cada modelo), y $t \in [0,\infty]$. Usaremos la notación $\Omega_t = \Omega \times (0,t)$, con $t \in (0,\infty]$ en toda la memoria.
Explicaremos brevemente, pues aunque importante no es el propósito de esta tesis, cómo obtenemos la ecuación por la que se rige la densidad u. Sea $\omega \subset \Omega$ un subdominio y $u(x,t)$ y $u(x,t_0)$ las densidades de población en ciertos instantes t y $t_0 < t$. Asumimos que la masa de la especie (la cantidad total de individuos) en el instante t en el subdominio ω es igual a la masa en un instante anterior, t_0, más el flujo de individuos en la frontera del subdominio durante ese intervalo de tiempo. Es natural tener en cuenta, ya que tratamos con individuos de una especie biológica, un término que presente la cantidad de muertes

y nacimientos en ese periodo. Formulado matemáticamente,

$$\int_\omega u(x,t)dx = \int_\omega u(x,t_0)dx - \int_{t_0}^t \int_{\partial\omega} \Phi \cdot \nu d\sigma_x ds + \int_{t_0}^t \int_\omega a(u,x,s)dxds,$$

donde Φ es el flujo en el borde, ν es el vector normal unitario y $a(u,x,t)$ es la tasa de crecimiento poblacional, que puede depender del espacio, del tiempo y de la propia densidad u. Ya que los individuos de la especie pueden moverse, asumimos que parte de este movimiento es aleatorio (*difusivo*, pues tiende a ocupar todo el dominio), en el que los individuos van de zonas con altas densidades de su propia especie hacia zonas con densidades más bajas. Esta hipótesis se representa por la *primera ley de Fick* (que, dependiendo de la materia, recibe otros nombres como *ley de Darcy*, *ley de Ohm* o *ley de Fourier de la conducción del calor*),

$$\Phi = -D\nabla u.$$

Aquí D es el coeficiente de difusión -o difusividad- (con dimensiones de área por unidad de tiempo) y que consideraremos igual a 1. Además, los individuos de la especie también son capaces de moverse en la dirección de algún estímulo, en este caso alguna sustancia química cuya densidad (*concentración*) denotamos por v. Este término, que llamamos quimiotáctico, lo expresamos mediante

$$\chi u \nabla v,$$

que modeliza cómo la densidad de la especie crece, proporcionalmente a su propio valor, en la dirección de las mayores concentraciones de la sustancia si el coeficiente quimiotáctico, χ, es positivo (se habla entonces de taxis positiva) o en la dirección de las menores concentraciones (cuando la sustancia repele, y se habla de taxis negativa) cuando $\chi < 0$. En esta memoria asumiremos siempre que χ es una constante positiva. Añadiendo este término al tensor de flujo, la ecuación que rige la evolución de la densidad u es

$$\int_\omega u(x,t)dx - \int_\omega u(x,t_0)dx = -\int_{t_0}^t \int_{\partial\omega} \nabla(-\nabla u + \chi u \nabla v)dxds + \int_{t_0}^t \int_\omega a(u,x,s)dxds,$$

donde hemos aplicado el teorema de la divergencia. Nos centramos ahora en el término del crecimiento poblacional. Resulta esperable en un modelo realista que los recursos de que disponen los individuos decrezcan a medida que el tamaño de la población aumente; de este modo, la mortalidad crece una vez que se ha superado un cierto valor umbral. Comúnmente, para describir el crecimiento de la población se usa una función de tipo logístico, esto es,

$$a(u,x,t) = \mu u(N(x,t) - u), \tag{18}$$

donde N representa ese valor umbral -*capacidad máxima de carga*- del sistema (véase el libro de Murray [75] para más detalles) y μ es el ratio de crecimiento de la población. Cuando la densidad de la población supera este valor umbral N comienza a decrecer debido a la falta de recursos.
Por tanto, la ecuación queda:

$$\int_{t_0}^t \int_\omega \frac{\partial u}{\partial t} dxds = \int_{t_0}^t \int_\omega \Delta u dxds - \int_{t_0}^t \int_\omega \chi \nabla(u\nabla v)dxds + \int_{t_0}^t \int_\omega \mu u(N(x,t) - u)dxds.$$

Finalmente, ya que ω, t y t_0 son arbitrarios, llegamos a

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla(u \nabla v) + \mu u(N(x,t) - u). \tag{19}$$

En esta memoria vamos a considerar dos casos:

- En la mayor parte del documento trabajaremos bajo la hipótesis de que los recursos dependen del tiempo y el lugar y presenta algún tipo de periodicidad (mostraremos de qué modo en la siguiente subsección). Podemos encontrar varios ejemplos de periodicidad en la literatura: en el movimiento de las amebas *Dictyostelium discoideum* hacia su centro de agregación, la velocidad media es periódica (véase Steinbock, Hashimoto y Müller [96]); también, Dunn y Zicha observan periodicidad en la quimiotaxis de los neutrófilos de humanos en [31]. Por simplicidad escribiremos

$$N(x,t) = 1 + f(x,t),$$

  donde $f(x,t)$ presenta algún tipo de comportamiento asintótico periódico que haremos explícito más adelante. Este es el caso general de la tesis.

- También consideraremos el efecto de la no localidad. En el término logístico (18) modelizamos que los individuos compiten localmente por los recursos del medio ambiente. Podemos asumir además que la cantidad total de población afecta a su crecimiento, lo que supone añadir un término no local. Este término puede ser de la forma

$$a(u,x,t) = u\left(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx\right), \tag{20}$$

  para ciertas constantes α, a_0, a_1 y a_2 (with $a_1 > 0$). Bajo esta hipótesis, si $a_2 < 0$ los individuos compiten localmente y globalmente (no localmente) por los recursos y si $a_2 > 0$ compiten localmente pero cooperan globalmente. Este caso lo estudiamos teórica y numéricamente en los Capítulos IV y IX, respectivamente.

Para obtener ahora la ecuación que rige la evolución de la concentración de la sustancia química seguimos un razonamiento idéntico. Dependiendo de la naturaleza de la sustancia, dividimos los modelos en dos:

- Difusión de la sustancia química dominante: si asumimos que la sustancia tiene la capacidad de difundirse por el medio emplearemos como antes la primera ley de Fick. Incorporaremos también una función $g(u,v)$ que represente la producción y degradación de la sustancia, de modo que tendremos

$$\tau \frac{\partial v}{\partial t} = \Delta v + g(u,v),$$

  para alguna constante τ. Supondremos que el término g es una expresión lineal de la forma:

$$g(u,v) = g_0 u - g_1 v,$$

  donde g_0 es la tasa de producción de la sustancia (que al ser secretada por los individuos de la especie es proporcional a su densidad) y g_1 es su tasa de degradación (de modo que la cantidad degradada será proporcional a la densidad de la sustancia que había). Ya que no supone ninguna restricción, pondremos $g_0 = g_1 = 1$, luego es

$$\tau \frac{\partial v}{\partial t} = \Delta v + u - v.$$

Aparecen aquí dos posibles subcasos:

- Si asumimos que la difusión de la sustancia se produce más rápidamente que la difusión de la densidad de la especie, consideremos $\tau = 0$. Por lo tanto, la ecuación para v será elíptica:

$$0 = \Delta v + u - v. \tag{21}$$

  El sistema parabólico-elíptico (19)-(21) (junto con las correspondiente condiciones de contorno y el dato inicial para u) es el tema estudiado en el Capítulo II.

- Si, por contra, suponemos que ambas difusividades son comparables, elegiremos $\tau = 1$, y con ello,

$$\frac{\partial v}{\partial t} = \Delta v + u - v. \tag{22}$$

  El Capítulo III está dedicado al estudio del sistema parabólico-parabólico (19)-(22).

- Sustancia química no difusiva: si nuestro modelo simula una sustancia química que después de secretada no se difunde por el dominio, no habrá ningún término que represente variación espacial en la ecuación de v. Esto significa que será una ecuación diferencial ordinaria de la forma

$$\frac{\partial v}{\partial t} = h(u, v), \tag{23}$$

  para alguna función h que represente la producción y degradación de la sustancia. Analizamos el sistema parabólico-ordinario (19)-(23) en el Capítulo V.

Revisión de la bibliografía

En esta subsección revisamos algunos de los resultados más conocidos de la bibliografía matemática concerniente al fenónemo de la quimiotaxis. El primer modelo de quimiotaxis fue propuesto por Keller y Segel en [62], [63] y es como sigue

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v), & x \in \Omega, \quad t > 0, \\ v_t = k_c \Delta v - k_3(v)v + uf(v), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

En sus artículos, Keller y Segel estudian la estabilidad del estado estacionario (u_0, v_0), encontrando que es inestable bajo la condición

$$\frac{k_2(u_0,v_0)v_0}{k_1(u_0,v_0)u_0} + \frac{u_0 f'(v_0)}{k_3(v_0 + v_0 k_3'(v_0))} > 1.$$

Los siguientes resultados que reseñamos se refieren a modelos con una sensitividad quimiotáctica lineal. En 1992, usando la transformación

$$U(x,t) := \frac{|\Omega| u(x,t)}{\int_\Omega u(x,t)dx}, \quad \text{and } V(x,t) := v(x,t) - \frac{1}{|\Omega|}\int_\Omega v(x,t)dx,$$

Jäger y Luckhaus en [60] presentaron (tras asumir algunas condiciones) el modelo parabólico-elíptico

$$\begin{cases} \dfrac{\partial U}{\partial t} = \nabla(\nabla U - \chi U \nabla V), & x \in \Omega, \quad t > 0, \\ 0 = \Delta V - \alpha(U-1), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ U(x,0) = U_0(x), & x \in \Omega. \end{cases}$$

Para este sistema los autores encuentran, en el caso de 2 dimensiones espaciales, que existe un valor crítico $c(\Omega)$ tal que existe una única solución suave y positiva global en el tiempo si $\alpha\chi U_0(x) < c(\Omega)$. Unos años después, Herrero y Velázquez en [44] prueban usando métodos de expansión asintóticos que existen datos iniciales radialmente simétricos tales que la solución del sistema anterior forma una singularidad en el centro de un disco Ω en tiempo finito. En [42], [43] Herrero, Medina y Velázquez demostraron que no existe ninguna solución radial y *auto-similar* tal que $\int_{|x|<r} U(T,s)ds < \infty$ cuando $r \to 0$. El primer autor también demuestra que se produce *blow-up* en $\mathbb{R}^3$ en [46].
El fenónemo del *blow-up* (explosión de las soluciones o, mejor, soluciones no acotadas) fue a su vez estudiado por Nagai, quien en [76] estudia el siguiente sistema

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(\nabla u - \chi u \nabla v), & x \in \Omega, \quad t > 0, \\ 0 = k_c \Delta v - \gamma v + \tilde{\alpha} u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

El autor demuestra que la solución clásica existe globalmente en el tiempo y está uniformemente acotada si se cumple que $\tilde{\alpha}\chi \int_\Omega u_0(x)dx < 4\pi$. En [77], muestra que las soluciones al sistema Keller-Segel sufren *blow-up* en tiempo finito si $\tilde{\alpha}\chi \int_\Omega u_0(x)dx > 8\pi$. Usaremos estos resultados en los capítulos de análisis numérico de esta memoria.
Es un hecho conocido que las soluciones del sistema

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + au - bu^2, & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \end{cases}$$

en un dominio suave y acotado $\Omega \subset \mathbb{R}^2$, no presentan *blow-up* para ningún $a \in \mathbb{R}$, $\tau \geq 0$, $\chi > 0$ y $b > 0$ (más detalles pueden encontrarse en Tello and Winkler [109] and Winkler [117]).
En [122], el autor reemplaza el término logístico $au - bu^2$ por otro de la forma $g(u)$ cumpliendo $g(0) \geq 0$,

$$\lim \inf_{s \to \infty} \{-g(s)\frac{\ln s}{s^2}\} = \mu_1 \in (0, \infty],$$

y

$$(\chi - \mu_1)^+ M < \frac{1}{2C_{GN}^4},$$

donde $c^+ = \max\{c, 0\}$, C_{GN} es la constante de la desigualdad de Gagliardo-Nirenberg y

$$M = ||u_0||_{L^1(\Omega)} + |\Omega| \inf_{\eta>0} \frac{\sup\{g(s) + \eta s : s > 0\}}{\eta}.$$

En este escenario se evita la aparación de *blow-up* asegurando que todas las soluciones son globales en el tiempo y acotadas uniformemente. Claramente, g cubre los casos $g(s) = as - bs^\theta$ con $b > 0$ y $\theta \geq 2$.
En [109] Tello y Winkler obtuvieron soluciones clásicas globales y acotadas asumiendo $d \leq 2$ o $\mu > \frac{d-2}{2}\chi$ para cualquier dato inicial para un término de la forma $g(s) \leq c_1 - \mu s^2$. También, bajo la hipótesis $\frac{d}{ds}\frac{g(s)}{s} < -2\chi$, probaron que la solución del problema parabólico-elíptico tiene un equilibio asintóticamente estable en L^∞ en (1,1). Resultados similares son los de Galakhov, Salieva y Tello en [39]. Más recientemente, en [120], Winkler ha obtenido acotación uniforme de soluciones para el modelo parabólico-elíptico con un término de la forma $\lambda u - \mu u^\kappa$

$$\kappa < \begin{cases} \frac{7}{6}, & \text{si } d = 3,\, 4, \\ 1 + \frac{1}{2(d-1)}, & \text{si } d \geq 5. \end{cases}$$

En [80] Negreanu y Tello demostraron el siguiente teorema para un sistema similar al nuestro:

$$\begin{cases} u_t - \Delta u = -\chi \nabla \cdot (u(N-u)\nabla v) + \lambda u(1-u), & x \in \Omega, \quad t > 0, \\ -\Delta v - v = u, & x \in \Omega, \quad t > 0. \end{cases}$$

Los autores probaron la convergencia de las soluciones al estado estacionario $(1, 1)$ usando un argumento de comparación como el arriba citado. Este consiste en usar las soluciones de un sistema de EDOs asociado como sub- y supersoluciones del sistema de EDPs. Entonces, una vez visto que ambas soluciones del sistema EDO comparten un mismo límite, se demuestra que la solución del sistema EDP también tiene ese mismo comportamiento. Usaremos este método para demostrar el comportamiento asintótico de las soluciones de nuestro problema.
En la bibliografía aparecen también varios resultados de existencia de soluciones y su comportamiento asintótico para modelos competitivos de dos especies biológicas y una sustancia química. En [34], Espejo, Stevens y Velázquez consideraron diferentes difusividades para las especies u y v, degradación constante de la sustancia y ausencia de término logístico. Movidos por la pregunta de si los mecanismos de quimiotaxis pueden ser responsables de los procesos de clasificación celular, sus trabajos se centran en la presencia de *blow-up* y comportamiento asintótico de esas soluciones no acotadas (también en [35] y [36] estudiaron el caso en el que las sensitividades quimiotácticas tienen signos distintos). También se encuentran en la bibliografía sistemas de varias especies con quimiotaxis y coeficientes no constantes, como en Issa y Shen [58], [59], en Tello y Winkler [110] (el caso $f_1 = f_2 = 0$ y coeficientes constantes), donde se obtiene la estabilidad de los estados estacionarios constantes (como en Stinner, Tello y Winkler [100], Bai y Winkler [6], Black, Lankeit y Mizukami [14] y Cruz, Negreanu y Tello [25]). Más recientemente, en Negreanu

y Tello [84] se obtiene la existencia global y comportamiento asintótico de un modelo de interacciones de tipo depredador-presa en el que los coeficientes son funciones periódicas.

En ocasiones, debido a la naturaleza de la sustancia, es necesario considerar una ecuación diferencial ordinaria. Los sistemas parabólico-EDO con términos quimiotácticos se han estudiado desde los años 90 tras las obras pioneras de Levine y Sleeman [69] y Anderson and Chaplain [5] que modelizan la angiogénesis. En Othmer y Stevens [91] y Stevens [97], los autores consideran un sistema parabólico-ordinario de quimiotaxis pasando de un sistema discreto de ecuaciones a uno continuo. También modelos de angiogénesis han sido considerados por Kubo y Suzuki [65] y Kubo, Hoshino y Kimura [64]. El analísis matemático de estos modelos con dos ecuaciones puede encontrarse en Fontelos, Friedman y Hu [37], Friedman y Tello [38] y Negreanu y Tello [83] entre otros.
En [79] los autores estudian el modelo, similar al nuestro,

$$\begin{cases} u_t = \Delta u - div(\chi_1(w)u\nabla w) + \mu_1 u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - div(\chi_2(w)v\nabla w) + \mu_2 v(1-v), & x \in \Omega, \quad t > 0, \\ w_t = h(u,v,w), & x \in \Omega, \quad t > 0, \end{cases}$$

donde χ_1, χ_2 no son constantes. Se demuestra la existencia global y la convergencia de la solución al punto $(1,1,\tilde{w})$ donde $h(1,1,\tilde{w}) = 0$. Los resultados de [79] fueron mejorados en Mizukami and Yokota [74] para un mayor rango de parámetros.
Los términos no locales también han sido introducidos en los sistemas quimiotácticos como modelos realistas. Por ejemplo, en [102] se usan términos integrales para describir la competición entre células cancerígenas y la densidad de la matriz extracelular. Un amplio resumen de resultados, referentes al sistema (20) para el caso $a_2 = 0$, es el de [29]. Para $a_2 = 0$ existe una única solución global que tiende a $((a_0/a_1)^{1/\alpha}, (a_0/a_1)^{\gamma/\alpha})$ cuando t tiende a infinito, siempre que $\alpha + 1 > m + \gamma$ o bien $\alpha + 1 = m + \gamma$ y a_1 suficientemente grande. En [81] se considera un sistema de ecuaciones en derivadas parciales parabólico-elíptico con términos no locales. En este artículo se estudia la existencia de soluciones y su comportamiento asintótico para el caso $a_1 > 2\chi + |a_2|$. Recientemente, en [57], se estudia un sistema parabólico-elíptico de la forma

$$\begin{cases} u_t = d_1\Delta u - \chi_1 div(u\nabla w) + u(a_0 - a_1 u + a_2 v - a_3 \int_\Omega u dx - a_4 \int_\Omega v dx), \\ v_t = d_2\Delta u - \chi_2 div(v\nabla w) + v(b_0 - b_1 u + b_2 v - b_3 \int_\Omega u dx - b_4 \int_\Omega v dx), \\ 0 = d_3\Delta w + ku + lv - \lambda w. \end{cases}$$

Los autores obtienen convergencia asintótica a un estado estacionario constante usando un método de comparación.
Mención aparte requieren los modelos de quimiotaxis-haptotaxis, que se usan para describir el proceso de invasión tumoral en los alrededores de la matriz extracelular. Debido a la importancia que los procesos invasivos tienen en fenómenos biológicos como la curación de heridas, la morfogénesis o la invasión tumoral, hay una amplia bibliografía matemática referente a ellos. Nos centraremos en estos últimos. Es un hecho conocido que el crecimiento de tumores sólidos tiene lugar en dos fases: la avascular y la vascular. La quimiotaxis-haptotaxis aparece en la segunda.
Las células tumorales encuentran una gran de variedad de factores que influyen en su migración durante los procesos invasivos y de metástasis. Estos factores pueden producir

el movimiento dirigido de las células por al menos dos mecanismos: la quimiotaxis y la haptotaxis. Del primero ya hemos dicho unas palabras. El segundo es el fenómeno que ocurre cuando los gradientes no se encuentran en una solución, sino que las moléculas involucradas deben crear y romper adhesiones en la matriz extracelular para moverse de las regiones con bajas concentraciones de la sustancia responsable del movimiento hacia regiones con más altas concentraciones. Unas explicaciones más detallada son las de [19] y [26].

Inicialmente, Chaplain y Lolas en [20]–[21] desarrollaron un modelo matemático de tres ecuaciones en derivadas parciales para describir la evolución de la densidad de células tumorales, denotada por u, la densidad de proteínas de la matriz extracelular, w, y la concentración del activador plasminógeno urocinasa, v. El modelo es el siguiente:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) - div(\xi u \nabla w) + \mu u(1-u-w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial t} = -vw, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{24}$$

para $\Omega \subset \mathbb{R}^d$ Es natural asumir que no hay flujo de células a través de la frontera del dominio ya que el modelo se estudia para un sistema cerrado *in vitro*, por lo que

$$-\frac{\partial u}{\partial \nu} + \chi u \frac{\partial v}{\partial \nu} + \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{25}$$

El modelo de tres componentes de quimiotaxis-haptotaxis (XI.1) es un extensión del modelo propuesto en 1970 por Keller y Segel en [63] tomando $w = 0$. Hemos mencionado que para el caso $\mu = 0$ y $w = 0$ el sistema posee *blow-up* de soluciones en tiempo finito para dos o más dimensiones, como se probó en [45] y [119]. Además, el sistema admite soluciones globales y acotadas si $d = 2$ y $\mu > 0$ ([105]) o $d = 3$ y μ suficientemente grande ([18]). Para $d = 3$ y μ suficientemente pequeño solo algunas soluciones globales débiles existen ([67]).

Para el subproblema de (XI.1) en el que solo consideramos haptotaxis haciendo $\chi = 0$ se ha estudiado la existencia global [116] y el comportamiento asintótico de [71] de las soluciones. El sistema completo ha sido analizado en dos y más dimensiones en [104] y [106]. Para un dominio suave y acotado de $\mathbb{R}^d$, con $d \leq 3$, para todo χ, ξ, y μ positivos, Tao y Winkler han demostrado que, siempre que los datos iniciales sean regulares y cumplan $u_0 > 0$ y $w_0 \leq 1$, la solución decae asintóticamente en $L^\infty(\Omega)$. Además, si $\mu > \frac{\chi^2}{8}$, los autores en [107] obtienen la estabilización exponencial de las soluciones al estado estacionario constante $(1, 1, 0)$.

Métodos numéricos para sistemas con quimiotaxis

Dos razones hacen patente la necesidad de implementar métodos numéricos para resolver los sistemas antes mencionados. Por una parte, la importancia de estos modelos en medicina, biología, química y una gran variedad de ciencias aplicadas; por otra, la gran dificultad que supone no poder encontrar explícitamente soluciones clásicas debido a la no linealidad

(y la no localidad).
Es por ello que recientemente se ha incrementado el número de investigaciones numéricas de dichos modelos quimiotácticos. Entre muchos otros, MacDonald *et al.* proponen usar elementos finitos con un mallado no fijo en [73], y el mismo método se ha aplicado con éxito a un sistema que describe la evolución de dos especies sin término logístico en [24]. En [28] Dehghan *et al.* utiliza el método sin malla de las funciones de bases radiales (cuyo fundamento es similar al de las diferencias finitas generalizadas que proponemos nosotros en esta tesis) para resolver sistemas parecidos. Por último, se han usado esquemas en diferencias finitas no estándar para discretizar sistemas en una dimensión sin difusión en [22].
En [32], Epshteyn y Kurganov desarrollaron una familia de métodos discontinuos de Galerkin de penalización interior para resolver el sistema de Keller-Segel. Los autores de [27] investigaron la no negatividad de las soluciones exacta y numérica de un modelo de Keller-Segel generalizado. En [33] se implementa un nuevo método de *upwind-difference potentials* en dominios de geometría complicada. Tyson, Stern y LeVeque en [112] proponen un método numérico de paso fraccionario para resolver ecuaciones en derivadas parciales con quimiotaxis, que incluyen términos difusivos dependientes de la densidad (la propia solución). Su método de paso fraccionario les permite tratar el término quimiotáctico como un término advectivo. El trabajo de Chertock y Kurganov en [23] tiene que ver con un nuevo procedimiento de volúmenes finitos para modelos de quimiotaxis y haptotaxis. En [123], los autores proponen el método de *lattice Boltzmann*, que se demuestra estable y eficiente, para resolver un modelo de Keller-Segel generalizado.

En cuanto al método que usaremos en esta tesis, el de las *Diferencias finitas generalizadas*, citaremos los siguientes. Este método sin malla ha sido aplicado recientemente con éxito para resolver ecuaciones en derivadas parciales no lineales como la de Fokker-Planck [114] y la del Telégrafo [115], ambas en dos y tres dimensiones. En el Capítulo VI expondremos la obtención y desarrollo del método de diferencias finitas generalizadas.

Contenido general de la tesis

Hemos dividido esta memoria en dos partes bien diferenciadas pero relacionadas. La primera parte está dedicada al estudio analítico de los cuatro modelos anteriores; a saber, parabólico-elíptico, parabólico-parabólico, no local (que también es parabólico-parabólico) y parabólico-ordinario. Por este estudio analítico entendemos la demostración de la existencia, unicidad y acotación de las soluciones de estos sistemas así como su comportamiento asintótico. Este estudio tiene lugar en los Capítulos II-V. En la segunda parte de esta tesis llevamos a cabo el análisis numérico de las versiones discretas de estos modelos y de un conocido modelo de quimiotaxis-haptotaxis (discretizados por medio de diferencias finitas generalizadas). Esta segunda parte ocupa los Capítulos VII-XI.
Sin embargo, la memoria no trata de una colección de capítulos inconexos. La línea común que atraviesa todos ellos (a excepción de los referentes al modelo no local -Capítulos IV y IX- y el referente al modelo de quimiotaxis-haptotaxis -Capítulo XI-) es la introducción de la hipótesis de los recursos periódicos en el medio. Más concretamente, consideraremos que el medio ambiente (la capacidad de carga) presenta el siguiente tipo de periodicidad:

$$\lim_{t\to\infty} \|f(x,t) - f^*(t)\|_{L^\infty(\Omega)} = 0, \tag{26}$$

para una cierta función $T-$periódica y homogénea en el espacio f^*. Si asumimos esto, encontraremos como límite asintótico no constante (en el tiempo) para la ecuación de u la solución de la ecuación diferencial ordinaria de Bernoulli

$$u_t^* = \mu u^*(1 - u^* + f^*).$$

Podemos resolver esta EDO y encontrar explícitamente -valga la redundancia- la solución $T-$periódica, $u^*(t)$,

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}, \tag{27}$$

con

$$u_0^* := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}.$$

Resultados principales

Para acabar esta introducción exponemos brevemente los principales contenidos de cada capítulo.

Capítulo I: Notaciones y resultados matemáticos previos
Mostramos notaciones y resultados matemáticos previos que después usaremos a lo largo de este documento.

Capítulo II: Modelo parabólico-elíptico
Estudiamos el sistema parabólico-elíptico dado por las ecuaciones (19)-(21), esto es,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ -\Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\nu} = \dfrac{\partial v}{\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{28}$$

Probaremos, usando razonamientos de punto fijo y *compact embeddings*, la existencia de una única solución global, lo que recogemos en el Teorema II.1. Después presentaremos un sistema de EDOs auxiliar para usar un método de comparación (también llamado *método de los rectángulos*), descrito por Pao en [92], con el que demostraremos que la solución de (28) converge a la función periódica u^* de (27). Dicho de otro modo, se prueba que la solución hereda el comportamiento estacional del entorno. Este resultado se enuncia en el Teorema II.4.
Finalmente presentamos una generalización en la que se consideran dos especies biológicas que compiten por una sustancia química en un contexto periódico,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi_1 \nabla(u \nabla w) + \mu_1 u \left(1 - a_1 v - u + f_1(x,t)\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - \chi_2 \nabla(v \nabla w) + \mu_2 v \left(1 - v - a_2 u + f_2(x,t)\right), & x \in \Omega,\ t > 0, \\ -\Delta w + w = \alpha u + \beta v & x \in \Omega,\ t > 0, \end{cases} \tag{29}$$

donde f_1 y f_2 cumplen relaciones del tipo (26). La novedad que supone este modelo es que añadimos una segunda especie al modelo que compite con la primera, así como otra función periódica en el nuevo término logístico. Tratamos de recuperar el comportamiento periódico para largos periodos de tiempo. En esta ocasión, el límite periódico de la solución no se expresa explícitamente; en cambio, se expresa como la solución de un sistema EDO con coeficientes periódicos. Para este caso más general obtenemos igualmente existencia y unicidad de soluciones y comportamiento periódico de las soluciones para tiempos grandes, resultados que incluimos en el Teorema II.5. Este capítulo está relacionado con el Capítulo VII, ya que en este estudiamos la convergencia del Método de las Diferencias Finitas Generalizadas a la solución continua de ambos sistemas y mostramos varios ejemplos en los que queda claro el comportamiento asintótico periódico.
Los resultados de este capítulo han sido publicados en [85] y en [89].

Capítulo III: Modelo parabólico-parabólico
Consideramos el sistema dado por (19) y (22),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{30}$$

para su estudio, que en este caso debe ser distinto debido a la naturaleza del sistema. Primero, obtendremos existencia local para luego extender las soluciones usando el método iterativo de Moser-Alikakos, lo que mostramos en el Teorema III.1. Para este sistema demostraremos el comportamiento asintótico usando funcionales de *Lyapunov* en dos pasos: en primer lugar, mostraremos cómo las soluciones convergen a sus respectivas medias (o centros de masas)

$$(u, v) \xrightarrow{L^2(\Omega)} \left(\int_\Omega u dx, \int_\Omega v dx \right),$$

para después probar que estos valores convergen a las funciones periódicas (u^*, v^*), donde $u^*(t)$ viene dada por (27) y $v^*(t)$ es la solución de

$$\tau \frac{dv^*}{dt} = u^* - v^*. \tag{31}$$

El resultado se demuestra en el Teorema III.2, y la solución numérica de este modelo la estudiamos en el Capítulo VIII.
El artículo [86] recoge los resultados de este capítulo.

Capítulo IV: Modelo no local

Demostraremos la existencia y acotación de las soluciones del modelo no local

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u^m \nabla v) + u(a_0 - a_1 u^\alpha + a_2 \displaystyle\int_\Omega u^\alpha dx), & x \in \Omega \;\; t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u^\gamma, & x \in \Omega \;\; t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0, & c \in \partial\Omega \;\; t > 0. \end{cases} \tag{32}$$

Obtendremos el comportamiento asintótico de las soluciones para los casos competitivo y cooperativo. En particular, encontraremos convergencia hacia

$$(u^*, v^*) = \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2}}, \sqrt[\alpha]{\left(\frac{a_0}{a_1 - a_2}\right)^\gamma} \right)$$

bajo ciertas combinaciones de a_0 y a_2, o convergencia exponencial hacia 0 en otros casos. Presentamos y analizamos la versión discreta, usando las fórmulas de las diferencias finitas generalizadas, en el Capítulo IX.
El contenido de este capítulo se encuentra siendo revisado en [88].

Capítulo V: Modelo parabólico-EDO
Asumimos en este apartado que la sustancia química no se difunde, por lo que consideramos el sistema

$$\begin{cases} u_t = \Delta u - div(\chi u \nabla v) + \mu u(1 - u + f(x,t)), & x \in \Omega, \;\; t > 0, \\ v_t = h(u,v), & x \in \Omega, \;\; t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} - u\chi \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \;\; t > 0. \end{cases} \tag{33}$$

Probamos en este capítulo dos resultados bajo ciertas hipótesis sobre la función h. El primero, el Teorema V.1, está dedicado a la prueba de la existencia de soluciones clásicas globales, usando de nuevo el método de Alikakos. En el segundo, Teorema V.2, enunciamos el comportamiento asintótico de las soluciones usando, como en el capítulo anterior, funcionales de Lyapunov. Obtendremos convergencia hacia las funciones (u^*, v^*), donde en este caso v^* es la solución de

$$\frac{dv^*}{dt} = h(u^*, v^*).$$

Este capítulo está relacionado con el Capítulo X, en el que proponemos un esquema explícito para resolver numéricamente el sistema (33) y donde probaremos la convergencia de la solución discreta a la continua.
Los resultados de este capítulo han sido parcialmente publicados en [87].

Capítulo VI: Fundamentos del método de las diferencias finitas generalizadas
Comenzamos el estudio numérico de esta memoria mostrando los fundamentos del método

y cómo se obtiene la discretización de las derivadas parciales mediante las fórmulas en diferencias finitas generalizadas. Introducimos esta sección por completitud.

Capítulo VII: Solución numérica del modelo parabólico-elíptico
Mostramos el esquema numérico que aproxima (28). Probaremos la convergencia condicional del esquema en diferencias a la solución analítica del sistema y presentaremos varios ejemplos numéricos. Concretamente, encontraremos *blow-up* de soluciones (soluciones no acotadas en tiempo finito), soluciones que convergen al estado estacionario constante $(1, 1)$ si tomamos $f = 0$ y soluciones que convergen a funciones periódicas. Finalmente, resolveremos numéricamente la versión generalizada del sistema parabólico-elíptico. Daremos ejemplos de funciones f, f_1 y f_2 que estén en consonancia con las hipótesis.
Hemos publicado el contenido de este capítulo en [7].

Capítulo VIII: Solución numérica del modelo parabólico-parabólico
Obtendremos el esquema dado por las diferencias finitas generalizadas para el sistema de EDPs parabólicas y daremos explícitamente la condición que debe cumplir el paso temporal para que el esquema sea convergente. Daremos ejemplos en los que las soluciones convergen a los límites asintóticos (de nuevo una redundancia) u^* y v^*. También estudiaremos la influencia de factores como el número de nodos y el incremento temporal en la solución numérica. Se proponen más ejemplos de funciones $f(x, t)$.
Recientemente ha aparecido publicado el artículo [8], donde recogemos los resultados de este capítulo.

Capítulo IX.2: Solución numérica del modelo no local
Se implementa la discretización del sistema (32) y se prueba la convergencia bajo condiciones en el paso temporal (que no suponen restricción debido a la flexibilidad del método). Se muestran varios ejemplos con los que la validez del modelo no local queda patente.

Capítulo X: Solución numérica del modelo parabólico-EDO
Proponemos un esquema explícito para resolver el sistema parabólico-ordinario del Capítulo V. Demostramos la convergencia condicional del método, usando una condición sobre h del modelo continuo. En nuestros ejemplos numéricos damos posibles funciones h que cumplen las hipótesis requeridas.
El artículo [10] está dedicado a este capítulo.

Capítulo XI: Modelo de quimiotaxis-haptotaxis
Para finalizar esta memoria, consideramos un modelo, ya conocido, de quimiotaxis-haptotaxis

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) - div(\xi u \nabla w) + \mu u(1-u-w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial t} = -vw, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega. \\ -\dfrac{\partial u}{\partial \nu} + \chi u \dfrac{\partial v}{\partial \nu} + \xi u \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{34}$$

Estudiamos su linealización para obtener la estabilidad de los estados estacionarios y lo resolvemos numéricamente usando diferencias finitas generalizadas.
Como resultado del estudio de este capítulo se obtuvo el artículo [9].

3 Publications

1. On a Parabolic-Elliptic chemotaxis system with periodic asymptotic behavior, Mathematical Methods in Applied Sciences, 2019.
2. Continuous and discrete periodic asymptotic behavior of solutions to a competitive chemotaxis PDEs system, Communications in Nonlinear Science and Numerical Simulation, 2020.
3. On a fully Parabolic chemotaxis system with source term and periodic asymptotic behavior, Zeitschrift für angewandte Mathematik und Physik, 2020.
4. On a fully parabolic chemotaxis system with nonlocal growth term, submitted.
5. A note on a periodic Parabolic-ODE chemotaxis system, Applied Mathematics Letters, 2020.
6. On the numerical solution to a parabolic-elliptic system with chemotactic and periodic terms using Generalized Finite Differences, Engineering Analysis with Boundary Elements, 2020.
7. Solving a fully parabolic chemotaxis system with periodic asymptotic behavior using Generalized Finite Difference Method, Applied Numerical Mathematics, 2020.
8. On the convergence of the generalized finite difference method for solving a chemotaxis system with no chemical diffusion, Computational Particle Mechanics, 2020.
9. Solving a chemotaxis-haptotaxis system in 2D using Generalized Finite Difference Method, Computers & Mathematics with Applications, 2020.
10. Solving a reaction-diffusion system with chemotaxis and non-local terms using Generalized Finite Difference Method. Study of the convergence, submitted.

Chapter I

Notations and previous known mathematical results

In this chapter we present well-known results, which are very common in specialised literature, and concentrate them to simplify the reading of the text and to keep a certain homogeneity in the notations throughout the work. Several definitions are taken from the handbook of Brézis [16].

Notation

- We denote by $\Omega \subset \mathbb{R}^d$ an open and bounded domain $\mathbb{R}^d$ with regular boundary, i.e. $\mathcal{C}^2$.
- $\partial\Omega$ represents the boundary of Ω.
- ν is exterior normal vector of $\partial\Omega$.
- We denote by $|\Omega|$ the volume of Ω, that is,

$$|\Omega| = \int_\Omega 1dx.$$

Definition I.1. *Recall the definition of the spaces*

$$\mathcal{C}^0(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is continuous in } \Omega\},$$

$$\mathcal{C}^k(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ continuous and differentiable up to the order } k \text{ in } \Omega\}.$$

Definition I.2. *A function f is Hölder-continuous when there exist real and nonnegative constants C, α such that*

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha$$

for all x, y in the domain of f. Define the spaces

$$\mathcal{C}^{k,\alpha}(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} : f \in \mathcal{C}^k(\Omega) \text{ and its } k\text{-derivative is } (\alpha)\text{-Hölder continuous}\}.$$

Definition I.3. *Define the space L^p, $p < \infty$ as*

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : \int_\Omega |f|^p < \infty\}.$$

Let $f \in L^p(\Omega)$. The norm $\|\cdot\|_p$ is given by

$$\|f\|_p = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}.$$

Similarly, we use

$$L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : f|_K \in L^p, \forall K \subset \Omega \ compact\}$$

and, for $p = \infty$,

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : \exists M > 0 \ such \ that \ |f| \leq M < \infty \ a.e. \ in \ \mathbb{R}\}.$$

The corresponding norm is defined by

$$\|f\|_\infty = ess \ sup \ |f| = min\{M \geq 0 : |f| \leq M \ a.e.\}.$$

Definition I.4. *Let $x \in \mathbb{R}^d$. We define the norm $\|\cdot\|_\infty$ as*

$$\|x\|_\infty = \max\{|x_1|, |x_2|, ..., |x_d|\}$$

Definition I.5. *For the Sobolev' spaces we use the classical definitions*

$$H^1(\Omega) = \{u : \Omega \to \mathbb{R} : \int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leq C < \infty\};$$
$$H^1_0(\Omega) = \{u \in H^1(\Omega) \ such \ that \ u = 0 \ en \ \partial\Omega\};$$
$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p \leq C < \infty\};$$
$$W^{1,p}_0(\Omega) = \{u \in W^{1,p}(\Omega) : u = 0 \ in \ \partial\Omega\};$$
$$W^{q,p}(\Omega) = \{D^s u \in L^p(\Omega) : \forall s \quad 0 \leq s \leq q\};$$
$$W^{q,p}_0(\Omega) = \{u \in W^{q,p}(\Omega) : u = 0 \ in \ \partial\Omega\}.$$

Previous mathematical results

Maximum principle

Theorem I.1 (Maximum principle of Hopf for parabolic equations)**.** *Let $u \in C(\overline{\Omega} \times [0,T]) \cap C^2(\Omega \times (0,T))$ be a function defined in $\mathbb{R}^n \times [0,T]$ satisfying*

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + a_0(x,t)u - \frac{\partial u}{\partial t} = f \tag{I.1}$$

in the open domain $\Omega \times (0,T)$ where the coefficients a_{ij} constitute a symmetric matrix locally definite positive in $\Omega \times (0,T)$, with $a_i(x,t)$ locally bounded by $i = 0, 1, ..., n$ and $f \geq 0$ in $\Omega \times (0,T)$.

If there exists $x_0 \in \Omega \times (0,T)$ such that $u(x_0) = \min_{x \in \overline{\Omega} \times [0,T]} u$ and if $u(x_0) \leq 0$, then u is constant in $\Omega \times (0,T)$.

Corollary I.1 (to the Maximum principle)**.** *Let $u \in C(\overline{\Omega} \times [0,T]) \cap C^2(\Omega \times (0,T))$ be a function satisfying (I.1) with $f \geq 0$ in $\Omega \times (0,T)$, under the same assumption of the previous theorem. If $u \geq 0$ in the boundary of $\Omega \times (0,T)$,*

1. *or $u > 0$ in $\Omega \times (0,T)$,*
2. *or $u \equiv 0$ in $\Omega \times (0,T)$.*

Aubin-Lions' lemma

Theorem I.2 (Aubin-Lions' lemma)**.** *Let X_0, X and X_1 be Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p, q \leq \infty$, let*

$$W = \{u \in L^p([0,T]; X_0) | \frac{\partial u}{\partial t} \in L^q([0,T]; X_1)\}$$

then,

- *If $p < \infty$, then the embdding of W into $L^p([0,T]; X)$ is compact.*
- *If $p = \infty$ and $q > 1$, then the embedding W into $\mathcal{C}([0,T]; X)$ is compact.*

Schauder's fixed point

Definition I.6. *A mapping $f : X \to X$ is compact if the closure of $f(Y)$ is relatively compact whenever $Y \subset X$ is bounded.*

Theorem I.3 (Schauder's fixed point theorem)**.** *Let C be a closed convex subset of the Banach space X. Suppose $f : C \to C$ and f is compact mapping. Then, f has a fixed point in C.*

Gronwall's lemma

Theorem I.4 (Generalized Gronwall's lemma)**.** *Let $b(t) : [0,T] \to \mathbb{R}_+$ and $u, a : [0,T] \to \mathbb{R}$ be continuous functions. If u satisfies*

$$u(t) \leq a(t) + \int_0^t b(s)u(s)ds, \qquad \forall t \in [0,T],$$

then

$$u(t) \leq a(t) + \int_0^t b(s)a(s)\exp(\int_0^t b(z)dz)ds$$

Frequently, we use the version

Lemma I.1. *Let $a \in \mathbb{R}_+$ and $u : [t_0, t_1] \to \mathbb{R}$ be a differentiable function such that*

$$\frac{d}{dt}u(t) \leq au(t),$$

then,

$$u(t) \leq e^{a(t-t_0)}u(t_0).$$

Lemmas

We have the following widely known lemma, [[54], Lemma 4.1], [[41] Lemma 3.4], for instance.

Lemma I.2. *For $p \geq 1$ we consider*

$$\begin{cases} q \in [1, \frac{dp}{d-p}), & \text{if} \quad p \leq d, \\ q \in [1, \infty], & \text{if} \quad p > d. \end{cases} \tag{I.2}$$

Then there exists $C = C(p, q, v_0, \Omega) > 0$ such that the unique global-in-time classical solution (u, v) to (III.1) satisfies

$$||v(t)||_{W^{1,q}} \leq C(1 + \sup_{s \in (0,t)} ||u(s)||_{L^p}). \tag{I.3}$$

The following auxiliary statement is applied to obtain the existence of solutions and the asymptotic behavior. Since the proof is standard and similar to the proof of Lemma 2.3 in [98], we omit the details.

Lemma I.3. *Let $T \leq \infty$ and α be positive constants, suppose that y is a nonnegative absolutely continuous function on $[0, T)$ satisfying*

$$\begin{cases} y' + \alpha y \leq g(t), & \text{for a.e. } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

for g, a nonnegative function satisfying

$$\int_t^{t+t_0} g(s)ds \leq C, \quad \text{for all } t \in [0, T - t_0)$$

and $t_0 > 0$. Then,

$$y \leq \max\left\{y(0) + C, \frac{C}{\alpha} + 2C\right\}, \quad \textit{for all } t \in (0, T).$$

Brezis-Strauss, 1974

Lemma I.4. *[Lemma 23 from Brezis-Strauss, 1974 [17]] Let u be a weak solution of the problem*

$$Lu = f \text{ in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega,$$

where L is an elliptic operator. Then we have $u \in W^{1,q}(\Omega)$ for all $1 \leq q < d/(d-1)$ and $\|u\|_{1,q} \leq C_q(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)})$

We use the following modification on the previous lemma:

Lemma I.5. *Let $\Omega \subset \mathbb{R}^d$ for $p \in \mathbb{N}$, $p \in (\max[\frac{d}{2}, 1], \infty)$ and v be the solution of*

$$\begin{cases} -\Delta v + v = u, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{I.4}$$

for $u \in L^p(\Omega)$. Then, for any $q \leq \infty$ the following inequality holds:

$$\|v\|_{L^q(\Omega)} \leq C(\Omega, p, q)\|u\|_{L^p(\Omega)}$$

Proof. Multiplying the equation (II.2) by v^{q-1}, integrating by parts and applying the Sobolev inequality, one obtains the result (the complete proof is given with details in [80]). □

Gagliardo-Nirenberg inequality

Theorem I.5. *Let $u : \mathbb{R}^d \to \mathbb{R}$. Fix $1 \leq q, r \leq \infty$ and a natural m. Consider $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$ such that*

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}, \qquad \frac{j}{m} \leq \alpha \leq 1. \tag{I.5}$$

Then:

1. *if $u \in L^q(\mathbb{R}^d)$ y $D^m u \in L^r(\mathbb{R}^d) \Rightarrow D^j u \in L^p(\mathbb{R}^d)$,*
2. *there exists a constant $C = C(m, d, j, q, r, \alpha)$ such that*

$$\|D^j u\|_p \leq C\|D^m u\|_r^\alpha \|u\|_q^{1-\alpha}. \tag{I.6}$$

Moreover if $u : \Omega \to \mathbb{R}$ is defined in a bounded and regular domain $\Omega \subset \mathbb{R}^d$ then for some arbirtrary $s > 0$ there exists constants $C_1, C_2 \geq 0$ such that

$$\|D^j u\|_p \leq C\|D^m u\|_r^\alpha \|u\|_q^{1-\alpha} + C_2\|u\|_s. \tag{I.7}$$

Theorems from Álvarez and Lazer

Theorem I.6 (Theorem 1 from Álvarez and Lazer [3])**.** *Let $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$ and $f(t)$ be positive and $T-$periodic functions such that*

$$\frac{b^L}{e^M} > \frac{a^M}{d^L}, \qquad \frac{f^L}{c^M} > \frac{d^M}{a^L}, \tag{I.8}$$

where $\phi^L := \inf_t \phi(t)$, and $\phi^M := \sup_t \phi(t)$. Then, the system

$$\begin{cases} u_t(t) = u(t)(a(t) - b(t)x(t) - c(t)y(t)), \\ v_t(t) = v(t)(d(t) - e(t)x(t) - f(t)y(t)), \end{cases} \tag{I.9}$$

possesses a unique solution $(u^*, v^*) = (u^*(t), v^*(t))$ which is T-periodic and positive. Moreover, the solution is asymptotically stable,

$$(u(t) - u^*, v(t) - v^*) \to (0, 0), \qquad \text{as } t \to \infty \tag{I.10}$$

for all positive solution (u, v) of (I.9).

Theorem I.7 (Theorem 2 from Álvarez and Lazer [3])**.** *If $(u^*(t), v^*(t))$ is the unique periodic solution of system (I.9), whose existence have been established in the previous theorem, then for all $t \in (-\infty, \infty)$*

$$\frac{a^L f^L - c^M d^M}{b^M f^L - c^M e^L} \leq u^*(t) \leq \frac{a^M f^M - c^L d^L}{b^L f^M - c^L e^M}, \tag{I.11}$$

$$\frac{b^L d^L - e^M a^M}{b^L f^M - c^L e^M} \leq u^*(t) \leq \frac{b^M d^M - e^L a^L}{b^M f^L - c^M e^L}. \tag{I.12}$$

Comparison lemma

Lemma I.6. *Let us consider the scalar differential equation*

$$\begin{cases} \dfrac{du}{dt} = f(t,u), \\ \\ u(0) = u_0, \end{cases} \tag{I.13}$$

where $f(t,u)$ is continuous in the variable t and locally Lipschitz in u for all $t \geq 0$ and all $u \in J \subset \mathbb{R}$. Let $[0, T_{max})$ be the maximal existence interval of the solution $u(t) \in J$. Let $v(t) \in J$ be a continuous function with its right hand derivative D^+v satisfying the differential inequality,

$$D^+v(t) \leq f(t,v), \quad v(0) \leq u_0, \tag{I.14}$$

for all $t \in [0, T_{max})$. Then, $v(t) \leq u(t)$ for all $t \in [0, T_{max})$.

Lemma I.7. *[Friedman-Tello [38]] Let $k : [0, \infty) \to \mathbb{R}$ be a C^1 function such that*

i. $k(t) \geq 0$ *and* $k(t) \leq C_0 < \infty$ *for some constant* $C_0 > 0$ *in* $[0, \infty)$*;*

ii. $\displaystyle\int_0^\infty k(t)dt \leq C_1 < \infty$*;*

iii. $|k'| \leq C_2 < \infty$ *for some constant* $C_2 > 0$ *in* $[0, \infty)$*.*

Then $k(t) \to 0$, as $t \to \infty$.

The following lemma is a weaker version. Here, the boundedness of k' is replaced by a weaker assumption given in (iii).

Lemma I.8. *Let $k : \mathbb{R}_+ \to \mathbb{R}$ a function satisfying*

(i) $k(t) \geq 0$ *for any* $t \geq 0$,

(ii) $\displaystyle\int_0^\infty k(s)ds \leq c < \infty$,

(iii) $k' \leq c$ *for any* $t \geq 0$,

then, $k(t) \to 0$ as $t \to \infty$.

Proof. By contradiction, we assume that there exists a sequence t_n such that $t_n \to \infty$ and $\limsup_{n\to\infty} k(t_n) \geq \epsilon > 0$. Then, there exist a subsequence t'_n such that $t'_n \geq t'_{n-1} + 1$ and

$$k(t'_n) \geq \frac{\epsilon}{2} > 0.$$

Then, $k \geq \frac{\epsilon}{4}$ in the interval $[t'_n - a, t'_n]$ for $a := \min\{1, c\epsilon/4\}$. So

$$\int_0^{t'_n} k(s)ds \geq \frac{n\epsilon}{4},$$

and taking limits when $n \to \infty$ we reach the contradiction. ☐

Nonlocal lemmas

Lemma I.9. *(***Hieber, Prüss [49]***) Let Ω be a bounded domain in $\mathbb{R}^d$, $\partial\Omega \in C^2$, $T < \infty$, $f \in L^p(0,T : L^q(\Omega))$ for $1 < p, q < \infty$ and $v_0 = 0$. Then, the solution v of the problem*

$$\begin{cases} v_t - \Delta v + v = f, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = 0, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

satisfies

$$v \in W^{1,p}(0,T : L^q(\Omega)) \cap L^p(0,T : W^{2,q}(\Omega)),$$

moreover for $p = q$ there exists a contant $c > 0$ such that

$$\int_0^T \int_\Omega v_t^q dxdt + \int_0^T \int_\Omega |\Delta v|^q dxdt \leq c\|f\|^q_{L^q(0,T:L^q(\Omega))}.$$

Corollary I.2. *Let $f \in L^p(0,T : L^q(\Omega))$ for $1 < p, q < \infty$ and $v_0 \in W^{2,q}(\Omega)$, such that*

$$\frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

then, the solution v to

$$\begin{cases} v_t - \Delta v + v = f, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

satisfies

$$v \in W^{1,p}(0,T : L^q(\Omega)) \cap L^p(0,T : W^{2,q}(\Omega)),$$

moreover for $p = q$ there exists a constant $c > 0$ such that

$$\int_0^T \int_\Omega v_t^q dxdt + \int_0^T \int_\Omega |\Delta v|^q dxdt \leq c\left[\|f\|^q_{L^q(0,T:L^q(\Omega))} + \|v_0\|^q_{W^{2,q}(\Omega)}\right].$$

Proof. We consider $v = V + e^{-\lambda t} v_0(x)$ for $\lambda > 0$, then V satisfies

$$\begin{cases} V_t - \Delta V + V = f + e^{-\lambda t}((\lambda - 1)v_0 + \Delta v_0), & (x,t) \in \Omega \times (0,T), \\ V(x,0) = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

We apply Lemma I.9 to V to conclude the result. □

Corollary I.3. *For $f \in L^p(0,T : L^q(\Omega))$ for $1 < p, q < \infty$ and $v_0 \in W^{2,q}(\Omega)$, such that*

$$\frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

the solution v to

$$\begin{cases} v_t - \Delta v + v = f, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{I.15}$$

satisfies

$$v \in W^{1,p}(0,T : L^q(\Omega)) \cap L^p(0,T : W^{2,q}(\Omega)),$$

moreover for $p = q$ there exists a contant $c_{cor} > 0$ such that, v satisfies

$$\int_0^T e^t \int_\Omega |\Delta v|^q dxdt \leq c_{cor} \left[\int_0^T e^t \int_\Omega f^q dxdt + \|v_0\|^q_{W^{2,q}(\Omega)} \right].$$

Proof. We consider $V = e^{\frac{1}{q}t} v$ then, V satisfies

$$\begin{cases} V_t - \Delta V + \left(1 - \frac{1}{q}\right) V = e^{\frac{t}{q}} f, & (x,t) \in \Omega \times (0,T), \\ V(x,0) = v_0(x), & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

We now apply Corollary I.2 to V to obtain

$$\int_0^T \int_\Omega |\Delta V|^q dxdt \leq c \left[\|e^{\frac{1}{q}t} f\|^q_{L^q(0,T:L^q(\Omega))} + \|v_0\|^q_{W^{2,q}(\Omega)} \right],$$

then, in terms of v, the previous inequality reads

$$\int_0^T e^t \int_\Omega |\Delta v|^q dxdt \leq c \left[\int_0^T e^t \int_\Omega f^q dxdt + \|v_0\|^q_{W^{2,q}(\Omega)} \right],$$

which ends the proof. □

Lemma I.10. *There exists $\epsilon_2 > 0$ such that, for any $x \geq 0$ and $y > 0$ satisfying $x < \frac{y}{2}$, and $\alpha > 1$ we have*

$$y^\alpha - x^\alpha = \alpha \xi^{\alpha-1}(y - x),$$

where ξ satisfies

$$\xi \geq \epsilon_2 y \tag{I.16}$$

for

$$\epsilon_2 := \frac{(2^\alpha - 1)^{\frac{1}{\alpha-1}}}{2^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{1}{\alpha-1}}}. \tag{I.17}$$

Proof. The proof is a direct application of the Mean Value Theorem, where ξ satisfies

$$\xi^{\alpha-1} = \frac{y^\alpha - x^\alpha}{y - x} \geq \frac{(2^\alpha - 1) y^\alpha}{\alpha 2^\alpha y} = \frac{2^\alpha - 1}{2^\alpha \alpha} y^{\alpha-1},$$

i.e.,

$$\xi \geq \frac{(2^\alpha - 1)^{\frac{1}{\alpha-1}}}{2^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{1}{\alpha-1}}} y.$$

□

Chapter II

Parabolic-elliptic model

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain connected open set whose boundary is regular enough. We will use the notation $\Omega_t := \Omega \times (0,t)$ and $\sigma_t := \partial\Omega \times (0,t)$. The aim of this chapter is to analyze the existence and asymptotic behavior of the solutions of the parabolic-elliptic model (11). In order to state the problem, we introduce the system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ -\Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\nu} = \dfrac{\partial v}{\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{II.1}$$

Throughout the chapter we work under the following hypotheses:

- The initial data u_0 satisfies

$$u_0 \in C^{0,\alpha}(\bar{\Omega}), \qquad \text{for some } \alpha \in (0,1), \tag{II.2}$$

  and

$$\frac{\partial u_0}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{II.3}$$

- There exist positive constants $\overline{u}_0$ and $\underline{u}_0$ such that

$$0 < \underline{u}_0 \le u_0(x) \le \overline{u}_0 < \infty. \tag{II.4}$$

- Function $f \in C(0, \infty : L^\infty(\Omega))$ satisfies

$$\int_0^\infty |\sup_{x\in\Omega}\{f(x,t)\} - \inf_{x\in\Omega}\{f(x,t)\}|dt \le C < \infty; \tag{II.5}$$

  there exists $\epsilon_1 > 0$ such that

$$f(x,t) > -1 + \epsilon_1, \tag{II.6}$$

$$|\sup_{x\in\Omega}\{f(x,t)\} - \inf_{x\in\Omega}\{f(x,t)\}| \to 0, \qquad \text{as} \quad t \to \infty. \tag{II.7}$$

  and

$$\sup_{t>0} \|f(x,t)\|_{L^\infty(\Omega)} := \|f(x,t)\|_{L^\infty(\Omega_\infty)} < \infty. \tag{II.8}$$

- There exist functions $\overline{f}(t), \underline{f}(t)$ (independent of x) such that

$$\underline{f}(t) \leq f(x,t) \leq \overline{f}(t), \tag{II.9}$$

and a periodic function, of period $T > 0$,

$$f^* : \mathbb{R}_+ \to \mathbb{R} \tag{II.10}$$

such that

$$\underline{f} < f^*(\cdot) < \overline{f}. \tag{II.11}$$

The chapter is organized as follows: in Section 1 we study the existence and uniqueness of the solutions to (II.1) under assumption

$$\chi < \mu. \tag{II.12}$$

In Section 2 we introduce a system of ODEs associated to the PDEs system. The solutions of the ODEs are taken as sub and super-solutions of (II.1). The asymptotic behavior of the ODEs system is studied for

$$2\chi < \mu \tag{II.13}$$

and we show that the solution to the PDEs satisfies

$$\lim_{t\to\infty} \left(\|u(\cdot,t) - u^*(t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - u^*(t)\|_{L^\infty(\Omega)} \right) = 0,$$

where $u^*(t)$ is the known periodic function given in the Introduction, i.e.,

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}.$$

Finally, in Section 4 we proceed in the same manner for a generalized parabolic-parabolic-elliptic problem modelling two biological species iteracting to reach a certain chemical substance.

1 Existence and uniqueness of solutions

In this section we study the existence and uniqueness of global-in-time solutions to (II.1). The results are enclosed in the following theorem.

Theorem II.1. *Under assumptions (II.2)-(II.4), (II.8) and (II.12), there exists a unique classical solution (u,v) to (II.1) such that*

$$u, v \in C^{2,1}_{loc}(\bar{\Omega} \times [0,\infty)),$$

and fulfills

$$0 \leq u \leq \tilde{k}, \qquad 0 \leq v \leq \tilde{k}$$

for

$$\tilde{k} = \max\left\{ \|u_0\|_{L^\infty(\Omega)}, \mu \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\mu - \chi} \right\}.$$

We first notice that, for any solution of (II.1), the total mass is uniformly bounded, i.e.,

$$\int_\Omega u(x,t)dx \le C < \infty. \tag{II.14}$$

The inequality is obtained after integration in (II.1), applying the Cauchy-Schwarz inequality and Gronwall's lemma.

The cross diffusion term in (II.1) is also expressed in the following way

$$\chi div \cdot (u\nabla v) = \chi \nabla u \cdot \nabla v + \chi u \Delta v = \chi \nabla u \cdot \nabla v - \chi u(u-v). \tag{II.15}$$

Hence, the equation for u becomes

$$u_t - \Delta u = -\chi \nabla u \nabla v + \chi u(u-v) + \mu u(1-u+f(x,t)), \quad x \in \Omega, \quad t > 0. \tag{II.16}$$

We now proceed to prove Theorem II.1.

Proof of Theorem II.1.
Let $\tilde{u}(t)$ be the solution to the ordinary differential equation

$$\tilde{u}_t(t) = (\chi - \mu)\tilde{u}^2(t) + a_1 \tilde{u}(t) \quad \text{in } (0,T), \qquad \tilde{u}(0) = \tilde{u}_0 := \|u_0(x)\|_{L^\infty(\Omega)}, \tag{II.17}$$

where

$$a_1 := 1 + \|f(x,t)\|_{L^\infty(\Omega_\infty)}.$$

Equation (II.17) is a Bernoulli differential equation, and its solution is given by

$$\tilde{u}(t) = \left[\tilde{u}_0^{-1} e^{-a_1 t} + \int_0^t (\mu - \chi) e^{a_1(s-t)} ds\right]^{-1},$$

which satisfies

$$\tilde{u}(t) \le \max\left\{\|u_0\|_{L^\infty(\Omega)}, \mu \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\mu - \chi}\right\}, \qquad \forall t > 0.$$

We now construct a fixed point argument. Consider $T_1 < \infty$, and $p \in (d, \infty)$, and the set

$$G := \Big\{u \in L^p((0,T_1); C^0(\bar{\Omega})) \quad \text{such that} \quad 0 \le u(x,t) \le \tilde{u}(t) \text{ a.e. in } \Omega \times [0,T_1)\Big\},$$

and the function $J : G \to L^p((0,T_1); C^0(\bar{\Omega}))$ defined by $J(u) := U$, where U is the solution to the differential equation

$$\begin{cases} U_t = \Delta U - \chi \nabla U \cdot \nabla V + U g(V,f) + (\chi - \mu) U u, & \text{in } \Omega \times (0,T_1), \\ \dfrac{\partial U}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,T_1) \\ U(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{II.18}$$

where V is the unique solution to

$$\begin{cases} -\Delta V + V = u, & \text{in } \Omega \times (0,T_1), \\ \dfrac{\partial V}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,T_1) \end{cases} \tag{II.19}$$

and

$$g(V,f) := -\chi V + \mu(1 + f(x,t)). \tag{II.20}$$

Since $u \in G$, we have that $V \in L^p([0,T_1); W^{2,q}(\Omega))$ (for $q < \infty$) and due to the Maximum Principle, we obtain

$$0 \le V(x,t) \le \tilde{u}(t). \tag{II.21}$$

Furthermore, since $u \le \tilde{u}$ and $u \ge 0$ it results

$$\nabla V \in [L^\infty(\Omega_{T_1})]^d$$

and

$$g(V,f) \in L^\infty(\bar{\Omega} \times [0,T_1)).$$

Thanks to the Maximum Principle we get

$$0 \le U \tag{II.22}$$

as far as the solution is defined.
The proof of the existence of solutions of (II.18) is obtained by using a compactness method for the equation

$$U_t = \Delta U - \chi \nabla U \cdot \nabla V + U g(V,f) + (\chi - \mu) U \tilde{u}, \quad x \in \Omega, \quad t \in (0,T_1), \tag{II.23}$$

for a given V and $\tilde{u} \in G$. We notice that (II.23) has a unique solution satisfying

$$U \in L^p((0,T_1); W^{2,q}(\Omega)) \cap W^{1,2}((0,T_1); L^q(\Omega)),$$

see for instance Quittner and Souplet [95] Example 51.4 and Remark 51.5. The Maximum principle gives us

$$0 \le U \le \tilde{u}.$$

Since $W^{2,q}(\Omega) \subset C^0(\bar{\Omega})$ is a compact embedding and $U_t \in L^2(\Omega \times (0,T_1))$, thanks to the Aubin-Lions' lemma, we have that G is a relatively compact subset of $L^p((0,T_1); C^0(\bar{\Omega}))$. Applying the Schauder's fixed point theorem, we obtain the existence of, at least, a solution in G. This fixed point is a solution to (II.18).
Since $\tilde{u} \in C^1(0,T_1)$ and it is independent of x, $\tilde{u}$ satisfies the equation

$$\tilde{u}_t = \Delta \tilde{u} - \chi \nabla \tilde{u} \nabla V + \mu \tilde{u}(1 + \|f\|_{L^\infty(\Omega_{T_1})}) + (\chi - \mu)\tilde{u}^2.$$

Therefore $U - \tilde{u}$ satisfies

$$\begin{aligned}
&(U - \tilde{u})_t - \Delta (U - \tilde{u}) + \chi \nabla (U - \tilde{u}) \cdot \nabla V \\
&= U(-\chi V + \mu(1 + f(x,t))) - \mu \tilde{u}(1 + \|f\|_{L^\infty(\Omega_{T_1})}) + (\chi - \mu)(U^2 - \tilde{u}^2) \\
&= (U - \tilde{u}) \left[-\chi V + \mu(1 + f(x,t)) + (\chi - \mu)(U + \tilde{u}) \right] \\
&+ \tilde{u}(-\chi V + \mu(f(x,t) - \|f\|_{L^\infty(\Omega_{T_1})})).
\end{aligned}$$

Since $V \ge 0$ we deduce

$$-\tilde{u}\chi V \le 0,$$

and

$$\mu \tilde{u}(f(x,t) - \|f\|_{L^\infty(\Omega_{T_1})}) \le 0.$$

Then, we have the inequality

$$(U-\tilde{u})_t - \Delta(U-\tilde{u}) + \chi\nabla(U-\tilde{u})\cdot\nabla V \leq (U-\tilde{u})\left[-\chi V + \mu(1-u+f(x,t)) + \mu(U+\tilde{u})\right].$$

Applying the Maximum Principle to $U-\tilde{u}$, in view of $U_0 \leq \tilde{u}_0$, it follows that

$$U \leq \tilde{u}.$$

Hence, U is uniformly bounded in $(0,T_1)$. As before, by application of the Aubin-Lions' lemma and the Schauder's fixed point theorem, we conclude that J has a fixed point in $L^p((0,T_1);C^0(\bar{\Omega}))$, which is a solution to the system (II.1).
The uniqueness of solutions is obtained by contradiction: assume that there exist two pairs of solutions (u_1,v_1) y (u_2,v_2), and call

$$w := u_1 - u_2,$$

which satisfies

$$w_t - \Delta w + \chi\nabla w\nabla v_1 + \chi\nabla u_2\cdot\nabla(v_1-v_2) + u_1 g(u_1,v_1) - u_2 g(u_2,v_2), \tag{II.24}$$

for

$$g(u,v) := \chi(u-v) + \mu(1-u+f).$$

Since $g \in W^{1,\infty}_{loc}(\mathbb{R}^2)$, taking w as test function (II.24), and having in mind

- $|\chi(\nabla w\cdot\nabla v_1)w| \leq \frac{1}{2}|\nabla w|^2 + \frac{\chi^2}{2}\|v_1\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}w^2;$
- $|\chi(\nabla u_2\cdot\nabla(v_1-v_2))w| \leq \frac{\chi}{2}|\nabla(v_1-v_2)|^2 + \frac{\chi}{2}\|u_2\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}w^2,$

it follows

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w^2 + \frac{1}{2}\int_\Omega|\nabla w|^2 \leq k\left(\int_\Omega w^2 + \int_\Omega|v_1-v_2|^2 + \int_\Omega|\nabla(v_1-v_2)|^2\right). \tag{II.25}$$

As $v_1 - v_2$ satisfies the equation

$$-\Delta(v_1-v_2) + (v_1-v_2) = u_1 - u_2 = w,$$

we take $v_1 - v_2$ as test function to obtain

$$\int_\Omega|\nabla(v_1-v_2)|^2 + \int_\Omega(v_1-v_2)^2 = \int_\Omega(v_1-v_2)w \leq \frac{1}{2}\int_\Omega(v_1-v_2)^2 + \frac{1}{2}\int_\Omega w^2,$$

and therefore

$$\int_\Omega|\nabla(v_1-v_2)|^2 + \int_\Omega(v_1-v_2)^2 \leq \frac{1}{2}\int_\Omega w^2. \tag{II.26}$$

By (II.25), (II.26) and the Gronwall's lemma, we conclude $w \equiv 0$ which implies uniqueness of solutions. To end the proof of the theorem we take limits when $T_1 \to \infty$. □

Remark II.1. *The global existence of solutions is proved under assumption (II.12) for any dimension d, nevertheless (II.12) can be relaxed to*

$$\begin{cases} \mu \geq 0, & d = 1, \\ \mu > \frac{d-2}{d}\chi, & d \geq 2. \end{cases}$$

The proof is similar to the proof of Theorem 2.5 in Tello and Winkler [109], see also Galakhov, Salieva and Tello [39].

2 Qualitative properties

In this section we consider a system of ordinary differential equations associated to the nonlinear system of PDEs (II.1) to obtain the asymptotic behavior of its solutions. After it, we relate and compare the properties of the solutions of the initial PDEs system (II.1) and the ODEs system (II.27). We use the solutions to the ODE system as bounds for the solution of the original PDE system and, then, prove that both converge to the same limit, u^*.

2.1 Associated ODE system

Recall that we need a relation of order to bound the solution of problem (II.1), (u, v), between the solutions of the new ODEs system. In Lemma II.1 and Lemma II.2 we prove that both solutions of the ODEs system have the same limit and, hence, so does any function bounded between them. We obtain an explicit expression of such limit in terms of f^*.
For the sake of simplicity, let us introduce the following notation that we use in the remainder of the section

$$\overline{f}(t) := \sup_{x\in\Omega}\{f(x,t)\}, \qquad \underline{f}(t) := \inf_{x\in\Omega}\{f(x,t)\}.$$

In order to prove the convergence and the stability of solutions of system (II.1), we introduce two auxiliary functions, $(\overline{u}, \underline{u}) = (\overline{u}(t), \underline{u}(t))$, defined as solutions of the initial value problem

$$\begin{cases} \overline{u}_t(t) = \chi\overline{u}(t)(\overline{u}(t) - \underline{u}(t)) + \mu\overline{u}(t)\left(1 - \overline{u}(t) + \overline{f}(t)\right), & t > 0, \\ \underline{u}_t(t) = \chi\underline{u}(t)(\underline{u}(t) - \overline{u}(t)) + \mu\underline{u}(t)\left(1 - \underline{u}(t) + \underline{f}(t)\right), & t > 0, \\ \overline{u}(0) = \overline{u}_0 \quad \underline{u}(0) = \underline{u}_0, \end{cases} \tag{II.27}$$

where the initial data satisfy

$$0 < \underline{u}_0 < \overline{u}_0 < \infty. \tag{II.28}$$

Now we study the properties of the solutions of the above system, i.e., we find a relationship between solutions $(\overline{u}(t), \underline{u}(t))$ when the initial data $(\overline{u}_0, \underline{u}_0)$ satisfies (II.28) and then we show that the initial ordering is inherited by the solution. Furthermore we prove that $(\overline{u}(t), \underline{u}(t))$ are actually global in time and bounded.

Lemma II.1. *The solution to the system (II.27) exists in* $(0, \infty)$*. Moreover, for every positive bounded initial data* $\overline{u}_0$ *and* $\underline{u}_0$ *verifying (II.28), the solution* $(\overline{u}(t), \underline{u}(t))$ *satisfies the order relation*

$$0 < \underline{u}(t) < \overline{u}(t) \leq \max\{\overline{u}_0, 1 + \|\overline{f}(t)\|_{L^\infty(\Omega_\infty)}, \} \quad \text{for any} \quad t \in (0, \infty). \tag{II.29}$$

Proof. First we notice that the right-hand side terms in (II.27), i.e., $\chi\overline{u}(t)(\overline{u}(t) - \underline{u}(t)) + \mu\overline{u}(t)(1 - \overline{u}(t) + \overline{f}(t))$ and $\chi\underline{u}(t)(\underline{u}(t) - \overline{u}(t)) + \mu\underline{u}(t)(1 - \underline{u}(t) + \underline{f}(t))$, are continuous and locally Lipschitz in $\overline{u}(t)$ and $\underline{u}(t)$. Furthermore, we claim that (II.27) is locally well-posed and, therefore, there exists an unique solution for $t \in (0, T_{max})$ such that, if $T_{max} < \infty$, we have $\limsup_{t\to T_{max}} |\overline{u}(t)| + |\underline{u}(t)| = \infty$.
Notice that the first equation of (II.27) can be written in the form

$$\overline{u}_t(t) = \overline{u}(t)h(\overline{u}(t), \underline{u}(t), \overline{f}(t)),$$

for some regular C^1 function h. Taking into account the positivity of the initial data $\overline{u}_0$, it follows that $\overline{u}(t) > 0$ for all $t > 0$. In the same way, we have $\underline{u}(t) > 0$.
In order to see $\underline{u}(t) < \overline{u}(t)$, we proceed by contradiction. If $\underline{u}(t) < \overline{u}(t)$ was false, then it would exist some $0 < t_0 < T_{max}$, such that

$$\underline{u}(t_0) = \overline{u}(t_0), \qquad \underline{u}(t) < \overline{u}(t), \quad \text{if} \quad t < t_0. \tag{II.30}$$

The solution of (II.27) with initial data $\underline{u}(t_0) = \overline{u}(t_0)$ is extended to obtain that $\underline{u}(t) \geq \overline{u}(t)$ in $(t_0 - \epsilon, t_0)$, which contradicts (II.30) and proves

$$\underline{u}(t) < \overline{u}(t) \quad t \in (0, T_{max}). \tag{II.31}$$

Since $\max\{\overline{u}_0, 1 + \|\overline{f}(t)\|_{L^\infty(\Omega_\infty)}\}$ is a supersolution of the first equation in (II.27) and $\underline{u}(t) = 0$ is a subsolution of the second equation of (II.27), due to the uniqueness of solutions, it follows

$$0 < \underline{u}(t) \qquad \text{and} \qquad \overline{u}(t) \leq \max\{\overline{u}_0, 1 + \|\overline{f}(t)\|_{L^\infty(\Omega_\infty)}\}. \tag{II.32}$$

Relations (II.31) and (II.32) show that $T_{max} = \infty$, which ends the proof. □

Lemma II.2. *Under assumptions (II.5) and (II.13), i.e.,*

$$\int_0^\infty |\overline{f}(t) - \underline{f}(t)| dt \leq C < \infty, \qquad \textit{and} \qquad 2\chi < \mu,$$

there exists a positive constant K, such that,

$$\overline{u}(t) \leq K \underline{u}(t). \tag{II.33}$$

Proof. Dividing the first equation in (II.27) by $\overline{u}(t)$ and the second one by $\underline{u}(t)$ we have

$$\frac{\overline{u}_t(t)}{\overline{u}(t)} = \chi(\overline{u}(t) - \underline{u}(t)) + \mu(1 - \overline{u}(t) + \overline{f}(t)), \tag{II.34}$$

$$\frac{\underline{u}_t(t)}{\underline{u}(t)} = \chi(\underline{u}(t) - \overline{u}(t)) + \mu(1 - \underline{u}(t) + \underline{f}(t)). \tag{II.35}$$

By subtracting (II.34) and (II.35), it results:

$$\frac{\overline{u}_t(t)}{\overline{u}(t)} - \frac{\underline{u}_t(t)}{\underline{u}(t)} = (2\chi - \mu)(\overline{u}(t) - \underline{u}(t)) + \mu(\overline{f}(t) - \underline{f}(t)),$$

which is equivalent to

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{\underline{u}(t)} = (2\chi - \mu)(\overline{u}(t) - \underline{u}(t)) + \mu(\overline{f}(t) - \underline{f}(t)). \tag{II.36}$$

Note that by (II.13) and Lemma II.1 we get

$$\frac{d}{dt} \ln \frac{\overline{u}(t)}{\underline{u}(t)} \leq \mu(\overline{f}(t) - \underline{f}(t)). \tag{II.37}$$

Integrating (II.37), we obtain

$$\ln \frac{\overline{u}(t)}{\underline{u}(t)} \leq \mu \int_0^t |\overline{f}(s) - \underline{f}(s)| ds + \ln \frac{\overline{u}_0}{\underline{u}_0}.$$

Undoing the logarithm,

$$\frac{\overline{u}(t)}{\underline{u}(t)} \leq \frac{\overline{u}_0}{\underline{u}_0} \exp\left(\int_0^t |\overline{f}(s) - \underline{f}(s)| ds \right) \leq K,$$

where $K := \overline{u}_0/\underline{u}_0 \exp(C)$ for C defined in (II.5). □

Our next aim is to prove that the difference between $\overline{u}(t)$ and $\underline{u}(t)$ tends to 0 as t tends to infinity. We then state the following lemma:

Lemma II.3. *If hypothesis (II.5), (II.6) and (II.13) are verified, i.e.,*

$$\int_0^\infty |\overline{f}(t) - \underline{f}(t)| dt \leq C < \infty, \qquad \underline{f}(t) \geq -1 + \epsilon_1, \quad \textit{and} \quad 2\chi - \mu < 0,$$

for initial data satisfying

$$0 < \underline{u}_0 < \overline{u}_0 < \infty,$$

the following chain of inequalities holds

$$0 < \epsilon_0 \leq \underline{u}(t) \leq \overline{u}(t) \leq K_0, \qquad \text{for} \qquad t > 0$$

where

$$\epsilon_0 := \min\left\{ \underline{u}_0, \frac{\mu\epsilon_1}{\chi K + \mu - \chi} \right\}$$

and

$$K_0 := \max\left\{ \overline{u}_0, \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\chi/K + \mu - \chi} \right\}.$$

Proof. Thanks to Lemma II.2, we have that $-\overline{u}(t) \geq -K\underline{u}(t)$ and (II.35) becomes

$$\begin{aligned}\frac{\underline{u}_t(t)}{\underline{u}(t)} &\geq \chi(\underline{u}(t) - K\underline{u}(t)) + \mu(1 - \underline{u}(t) + \underline{f}(t)) = (\chi - \chi K - \mu)\underline{u}(t) + \mu(1 + \underline{f}(t)) \\ &\geq (\chi - \chi K - \mu)\underline{u}(t) + \mu\epsilon_1.\end{aligned}$$

Hence,

$$\underline{u}(t) \geq \min\left\{ \underline{u}_0, \frac{\mu\epsilon_1}{\chi K + \mu - \chi} \right\}.$$

In the same way, we show

$$\begin{aligned}\frac{\overline{u}_t(t)}{\overline{u}(t)} &\leq \chi(\overline{u}(t) - \frac{1}{K}\overline{u}(t)) + \mu(1 - \overline{u}(t) + \overline{f}(t)) = (\chi - \chi\frac{1}{K} - \mu)\overline{u}(t) + \mu(1 + \overline{f}(t)) \\ &\leq (\chi - \chi\frac{1}{K} - \mu)\overline{u}(t) + 1 + \|f\|_{L^\infty(\Omega_\infty)}.\end{aligned}$$

Then,

$$\overline{u}(t) \leq \max\left\{ \overline{u}_0, \frac{1 + \|f\|_{L^\infty(\Omega_\infty)}}{\chi/K + \mu - \chi} \right\}$$

and the proof ends. □

In the following lemma we apply assumption (II.7) to prove the asymptotic behavior of the solution of the auxiliary problem. We first notice that assumption (II.7) is equivalent to

$$\lim_{t\to\infty}\int_0^t e^{\epsilon(\tau-t)}\left(\sup_{x\in\Omega}\{f(x,\tau)\}-\inf_{x\in\Omega}\{f(x,\tau)\}\right)d\tau=0 \tag{II.38}$$

for any $\epsilon>0$.

Remark II.2. *Under assumption (II.7) we have (II.38).*

Proof. We denote by $\overline{b}$ a non increasing function satisfying

$$\left(\sup_{x\in\Omega}\{f(\cdot,t\}-\inf_{x\in\Omega}\{f(\cdot,t)\}\right)\le\overline{b}(t)\quad\text{ and }\quad\overline{b}(t)\to 0.$$

We have

$$\begin{aligned}\int_0^t & e^{\epsilon(\tau-t)}\left(\sup_{x\in\Omega}\{f(x,\tau)\}-\inf_{x\in\Omega}\{f(x,\tau)\}\right)d\tau\\ &\le\int_0^t e^{\epsilon(\tau-t)}\left(\sup_{x\in\Omega}\{f(x,\tau)\}-\inf_{x\in\Omega}\{f(x,\tau)\}\right)d\tau\\ &\le\overline{b}(t/2)e^{-\epsilon t}\int_{t/2}^t e^{\epsilon\tau}d\tau+\int_0^{t/2}e^{\epsilon(\tau-t)}\overline{b}(\tau)d\tau\\ &\le\overline{b}(t/2)e^{-\epsilon t}\left(e^{\epsilon t}-e^{\epsilon t/2}\right)+\overline{b}(0)e^{-\epsilon t/2}\to 0.\end{aligned}$$

□

Lemma II.4. *For every positive constants χ and μ as in (II.7) and every functions $\overline{f}$, $\underline{f}$ verifying (II.13), i.e.,*

$$2\chi<\mu\qquad\text{and}\qquad|\overline{f}(t)-\underline{f}(t)|\to 0,\qquad\text{when}\qquad t\to\infty,$$

the solutions of system (II.27) satisfy:

$$\frac{\overline{u}(t)}{\underline{u}(t)}\to 1,\qquad\text{when}\qquad t\to\infty. \tag{II.39}$$

Proof. We simplify expression (II.36) introducing $b(t)$ as

$$b(t):=\mu(\overline{f}(t)-\underline{f}(t)).$$

Then, (II.36) becomes

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{\underline{u}(t)}=(2\chi-\mu)\left(\frac{\overline{u}(t)}{\underline{u}(t)}-1\right)\underline{u}(t)+b(t).$$

We define v given by $v(t):=\ln(\overline{u}(t)/\underline{u}(t))$ and substituting in the previous expression it results

$$\frac{d}{dt}v(t)=(2\chi-\mu)(e^{v(t)}-1)\underline{u}(t)+b(t).$$

Thanks to Lemma II.1 and II.3, we obtain:

$$\frac{d}{dt}v(t)\le\epsilon_0(2\chi-\mu)v(t)+b(t).$$

Now, taking $\epsilon_2 := (\mu - 2\chi)\epsilon_0$, the above inequality is reduced to

$$\frac{d}{dt}v(t) + \epsilon_2 v(t) \leq b(t).$$

After integration we get

$$v(t) \leq e^{-\epsilon_2 t}\left(\int_0^t b(s)e^{\epsilon_2 s}ds + v_0\right).$$

Using now Remark II.2 we conclude

$$v(t) \to 0, \qquad \text{as} \qquad t \to \infty.$$

Finally, undoing the change, we have

$$\ln\frac{\overline{u}(t)}{\underline{u}(t)} \to 0, \qquad \text{as} \qquad t \to \infty,$$

which implies (II.39). □

The importance of this intermediate result lies in obtaining $(\overline{u}(t) - \underline{u}(t)) \to 0$ when $t \to \infty$, which implies that any other function bounded between them will inherit their asymptotic behaviour.
Let us consider now $u^* = u^*(t)$, the solution of the initial value problem

$$u_t^*(t) = \mu u^*(t)(1 - u^*(t) + f^*(t)) \tag{II.40}$$

with

$$u^*(0) = u_0^* := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds}d\tau} > 0, \tag{II.41}$$

and $f^*(t)$ is a periodic function of period T.

Lemma II.5. *Let f^* be defined in (II.10) satisfying (II.11), and let u_0^* be a positive value given by (II.41). Then, the solution $u^*(t)$ of (II.40) with initial data u_0^* is computed explicitly*

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^*\displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds}d\tau} \tag{II.42}$$

and it is a periodic function.

Proof. Let us note that (II.40) is a Bernoulli differential equation. Using the change $w(t) = (u^*(t))^{-1} = (u^*(t))^{-1}$, (II.40) is equivalent to

$$w_t(t) + \mu(1 + f^*(t))w(t) = \mu, \tag{II.43}$$

with the integrating factor $e^{\mu(1+f^*(t))}$ we obtain an explicit expression for w,

$$w(t) = \frac{w_0 + \int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds}d\tau}{e^{\int_0^t \mu(1+f^*(s))ds}}.$$

We impose here the periodic condition on w (taking into account the expression of w, for u periodic), that is, $w_0 = w(0) = w(T)$,

$$w_0 = \frac{w_0 + \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}{e^{\int_0^T \mu(1+f^*(s))ds}}$$

which is equivalent to

$$w_0(e^{\int_0^T \mu(1+f^*(s))ds} - 1) = \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau.$$

Computating, we then have

$$w_0 = \frac{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}{e^{\int_0^T \mu(1+f^*(s))ds} - 1} > 0,$$

which is positive under the assumption

$$e^{\int_0^T \mu(1+f^*(s))ds} - 1 > 0.$$

Furthermore, it is necessary that $w'(t) = w'(t+T)$, i.e., $f^*(t)$ is a time periodic function of period T. Now, returning to the initial equation with $u_0^* = {w_0}^{-1}$, we obtain

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}$$

where

$$u_0^* = \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} > 0.$$

To check the periodicity of u^* we compute $u^*(T)$ which is given by

$$\begin{aligned} u^*(T) &= \frac{u_0^* e^{\int_0^T \mu(1+f^*(s))ds}}{1 + u_0^* \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} \\ &= \frac{e^{\int_0^T \mu(1+f^*(s))ds}}{(u_0^*)^{-1} + \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} = \frac{e^{\int_0^T \mu(1+f^*(s))ds}(e^{\int_0^T \mu(1+f^*(s))ds} - 1)}{e^{\int_0^T \mu(1+f^*(s))ds} \int_0^T \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau} = u_0^* \end{aligned}$$

and thanks to (II.40) and periodicity of f^* we conclude the lemma.

□

The following lemma will be used to prove that the solution $u^*(t)$ of (II.40) is bounded between $\underline{u}(t)$ and $\overline{u}(t)$, solutions of the system (II.27):

Lemma II.6. *Let f^* be defined in (II.10) satisfying (II.11), and let u_0^* be a positive value (II.41) verifying*

$$\underline{u}_0 < u_0^* < \overline{u}_0,$$

then, the solution $u^*(t)$ of (II.40) fulfills

$$\underline{u}(t) \leq u^*(t) \leq \overline{u}(t). \tag{II.44}$$

Proof. We prove first $u^*(t) \leq \overline{u}(t)$. Subtracting (II.34) and (II.40), it is obtained:

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{u^*(t)} = \chi(\overline{u}(t) - \underline{u}(t)) + \mu(u^*(t) - \overline{u}(t) + \overline{f}(t) - f^*(t)). \tag{II.45}$$

Now, by assumption (II.11) and Lemma II.1, i.e., $\overline{f}(t) - f^*(t) > 0$ and $\overline{u}(t) - \underline{u}(t) > 0$, we can find an upper bound for the right-hand side term of the equation (II.45) such that

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{u^*(t)} \geq \mu(u^*(t) - \overline{u}(t)). \tag{II.46}$$

By the Mean Value Theorem, it results

$$\begin{aligned}\mu(u^*(t) - \overline{u}(t)) &= \mu(e^{\ln u^*(t)} - e^{\ln \overline{u}(t)}) = \mu\xi^*\left(\ln u^*(t) - \ln \overline{u}(t)\right)\\ &= \mu\xi^*\left(\ln\frac{u^*(t)}{\overline{u}(t)}\right) = -\mu\xi^*\left(\ln\frac{\overline{u}(t)}{u^*(t)}\right)\end{aligned}$$

where $0 < u^*(t) \leq \xi^* \leq \overline{u}(t)$ if $u^*(t) < \overline{u}(t)$ and $\epsilon_0 \leq \overline{u}(t) \leq \xi^* \leq u^*(t)$ otherwise. Returning to the equation (II.46), we obtain

$$\frac{d}{dt}\ln\frac{\overline{u}(t)}{u^*(t)} \geq -\mu\xi^*\left(\ln\frac{\overline{u}(t)}{u^*(t)}\right), \qquad \text{with} \qquad \ln\frac{\overline{u}_0}{u_0^*} > 0.$$

Applying now the Gronwall's lemma, we deduce

$$\ln\frac{\overline{u}(t)}{u^*(t)} \geq \ln\frac{\overline{u}_0}{u_0^*}e^{-\mu\int_0^t \xi^*} > 0$$

which implies

$$\frac{\overline{u}(t)}{u^*(t)} \geq 1.$$

In the same way we get $\underline{u}(t) \leq u^*(t)$ and that ends the proof. □

The following theorem gives a precise description of the asymptotic behavior under certain assumptions.

Theorem II.2. *Let $(\underline{u}(t), \overline{u}(t))$ be the solution of system (II.27) and $u^*(t)$ the solution of (II.40) for f^* defined in (II.10) and satisfying (II.11), then, the following limits hold*

$$|\overline{u}(t) - u^*(t)| \to 0 \qquad \text{and} \qquad |\underline{u}(t) - u^*(t)| \to 0, \tag{II.47}$$

as $t \to \infty$.

Proof. Using now the result obtained in Lemma II.6 we claim

$$0 < \overline{u}(t) - \underline{u}(t) = |\overline{u}(t) - u^*(t)| + |u^*(t) - \underline{u}(t)| \leq |\overline{u}(t) - \underline{u}(t)| + |\overline{u}(t) - \underline{u}(t)| = 2|\overline{u}(t) - \underline{u}(t)| \to 0.$$

Lemma II.4 ends the proof. □

3 Comparison principle and asymptotic behavior of solutions

The aim of this section is to find the relation between the solution $(\underline{u}(t), \overline{u}(t))$ of the ODEs system (II.27) and the solution (u, v) of the PDEs system (II.1). Under some order relation between the initial conditions, we prove that such order is preserved. Taking into account the results of the previous sections, i.e. the functions $\underline{u}(t)$ and $\overline{u}(t)$ converge to $u^*(t)$ as $t \to \infty$, where $u^*(t)$ is the periodic function defined in (II.42), we bound the solution of (II.1) between $\underline{u}(t)$ (lower bound) and $\overline{u}(t)$ (upper bound) to obtain the same qualitative behavior than $\underline{u}(t)$ and $\overline{u}(t)$. The proof follows the *rectangle method* used in Pao [92] for reaction-diffusion systems, see also Negreanu and Tello [82] where the method is applied to Parabolic-Elliptic systems with chemotactic terms.

Notice first that, since $u_0^* > 0$, it is possible to find positive numbers $\underline{u}_0$ and $\overline{u}_0$ satisfying assumption (II.4), such that the inequalities

$$0 < \underline{u}_0 < u_0^* < \overline{u}_0 \tag{II.48}$$

hold. The main result of this section is as follows

Theorem II.3. *Let $u_0 \in L^\infty(\Omega)$ and ϵ_0 be a positive number such that, for all positive initial data $(\underline{u}_0, \overline{u}_0)$ verifying*

$$\epsilon_0 \le \underline{u}_0 \le u_0 \le \overline{u}_0, \qquad in\ \Omega,$$

and (II.48) the solution (u, v) of (II.1) is bounded and satisfies

$$\underline{u}(t) \le u(x,t) \le \overline{u}(t), \qquad \underline{u}(t) \le v(x,t) \le \overline{u}(t), \quad (x,t) \in \Omega \times (0,\infty).$$

Let us introduce the following notations in order to prove it:

$$\begin{aligned} \overline{U}(x,t) &:= u(x,t) - \overline{u}(t), \qquad & \underline{U}(x,t) &:= u(x,t) - \underline{u}(t) \\ \overline{V}(x,t) &:= v(x,t) - \overline{u}(t), \qquad & \underline{V}(x,t) &:= v(x,t) - \underline{u}(t) \end{aligned} \tag{II.49}$$

where (u, v) is the solution of (II.1) and $(\underline{u}(t), \overline{u}(t))$ is the solution of (II.27). We consider the standard positive and negative part functions defined by

$$(s)_+ = \begin{cases} s & \text{if } s \le 0, \\ 0 & \text{in other case,} \end{cases} \qquad (s)_- = (-s)_+.$$

Now, our purpose is to prove that the positive and negative parts $\overline{U}_+$ and $\underline{U}_-$ are 0. To obtain the partial differential equation which is satisfied by $\overline{U}$, we subtract (II.1) and the first equation in (II.27) to obtain

$$\begin{aligned} \frac{\partial}{\partial t}(u - \overline{u}(t)) - \Delta(u - \overline{u}(t)) = &- \chi\nabla(u - \overline{u}(t))\nabla v + \chi(u^2 - \overline{u}(t)^2 - uv + \overline{u}(t)\underline{u}(t)) \\ &+ g(u) - g(\overline{u}(t)) + \mu(fu - \overline{f}(t)\overline{u}(t)), \end{aligned}$$

where $g(u) := \mu u(1-u)$. Now, substituting $\overline{U}$, and adding $\pm\chi\overline{u}(t)v$ and $\pm\mu\overline{f}(t)u$

$$\begin{aligned} \overline{U}_t - \Delta\overline{U} = &- \chi\nabla\overline{U}\nabla v + \chi\overline{U}(u - v + \overline{u}) + \chi\overline{u}(t)(\underline{u}(t) - v) \\ &+ \mu u(f - \overline{f}(t)) + \mu\overline{f}(t)\,\overline{U} + g(u) - g(\overline{u}(t)). \end{aligned}$$

Applying the Mean Value Theorem to g for some $\xi(x,t) \in (u(x,t), \overline{u}(t))$ if $u \leq \overline{u}(t)$ and $\xi(x,t) \in (\overline{u}(t), u(x,t))$ otherwise, the previous equation becomes

$$\begin{aligned} \overline{U}_t - \Delta \overline{U} = & -\chi \nabla \overline{U} \nabla v + \overline{U}[\chi(u - v + \overline{u}(t)) + g'(\xi)] + \chi \overline{u}(t)(\underline{u}(t) - v) \\ & + \mu u(f - \overline{f}(t)) + \mu \overline{f}(t)\, \overline{U}. \end{aligned} \tag{II.50}$$

We multiply now by the test function $\overline{U}_+$ and integrate by parts over Ω. Hence, after some rutinary computations, (II.50) remains

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \overline{U}_+^2 dx + \int_\Omega |\nabla \overline{U}_+|^2 dx = -\chi \int_\Omega \overline{U}_+ \nabla \overline{U} \nabla v dx + \chi \int_\Omega \overline{U}_+ \overline{u}(\underline{u}(t) - v) dx \\ + \int_\Omega (\chi(u - v + \overline{u}(t)) + g'(\xi))\overline{U}_+^2 dx + \mu \int_\Omega \overline{U}_+ u(f - \overline{f}(t)) dx + \mu \int_\Omega \overline{f}(t)\, \overline{U}_+^2 dx. \end{aligned}$$

We notice that

$$-\chi \int_\Omega \overline{U}_+ \nabla \overline{U} \nabla v dx = -\frac{1}{2}\chi \int_\Omega \nabla \overline{U}_+^2 \nabla v dx = \frac{\chi}{2}\int_\Omega \overline{U}_+^2 \Delta v dx = \frac{\chi}{2}\int_\Omega \overline{U}_+^2 (v - u) dx.$$

By the definition of $\overline{f}$, the term $\mu \int_\Omega \overline{U}_+ u(f - \overline{f}) dx < 0$, so we find the bound

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (\overline{U}_+)^2 dx + \int_\Omega |\nabla(\overline{U}_+)|^2 dx \leq \int_\Omega \Big(\frac{\chi}{2}(u - v + 2\overline{u}(t)) + g'(\xi)\Big)\overline{U}_+^2 dx \\ + \chi \int_\Omega \overline{U}_+ \overline{u}(t)(\underline{u}(t) - v) dx + \mu \int_\Omega \overline{f}(t)\, \overline{U}_+^2 dx. \end{aligned} \tag{II.51}$$

We notice that

$$\frac{\chi}{2}(u - v + 2\overline{u}(t)) + g'(\xi) \leq K_1 < \infty,$$

and

$$\chi \int_\Omega \overline{u}(t)(\underline{u}(t) - v)\overline{U}_+ dx \leq \chi \overline{u}_0 \int_\Omega (\underline{u}(t) - v)_+ \overline{U}_+ dx.$$

We can bound the terms of (II.51) as follows

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t} & \int_\Omega (\overline{U}_+)^2 dx + \int_\Omega |\nabla(\overline{U}_+)|^2 dx \\ & \leq K_1 \int_\Omega \overline{U}_+^2 dx + \chi \overline{u}_0 \int_\Omega (\underline{u}(t) - v)_+ \overline{U}_+ dx + \mu ||\overline{f}||_{L^\infty(\Omega_\infty)} \int_\Omega \overline{U}_+^2 dx \\ & \leq \left(K_1 + \mu ||\overline{f}||_{L^\infty(\Omega_\infty)} + \frac{\chi \overline{u}_0}{2}\right) \int_\Omega \overline{U}_+^2 dx + \frac{\chi \overline{u}_0}{2} \int_\Omega (\underline{u}(t) - v)_+^2 dx. \end{aligned}$$

Now, thanks to Lemma I.5, it results

$$\frac{\partial}{\partial t}\int_\Omega (\overline{U}_+)^2 dx + \int_\Omega |\nabla(\overline{U}_+)|^2 dx \leq C_+ \left(\int_\Omega \overline{U}_+^2 dx + \int_\Omega \overline{U}_-^2 dx\right), \qquad C_+ > 0.$$

In the same way we get

$$\frac{\partial}{\partial t}\int_\Omega (\underline{U}_-)^2 dx + \int_\Omega |\nabla(\underline{U}_-)|^2 dx \leq C_- \left(\int_\Omega \underline{U}_-^2 dx + \int_\Omega \underline{U}_+^2 dx\right), \qquad C_- > 0.$$

Adding both last equations, we have that

$$\frac{d}{dt}\left(\int_\Omega (\overline{U}_+^2 + \underline{U}_-^2)\right) dx \leq (C_+ + C_-) \int_\Omega \left(\overline{U}_+^2 + \underline{U}_-^2\right) dx.$$

Applying the Gronwall's lemma, we obtain that $\overline{U}_+ = \underline{U}_- = 0$ (for more details, see for instance, [78] or [81]). Hence we have $\underline{u}(t) \leq u(x,t) \leq \overline{u}(t)$, for all $(x,t) \in \Omega \times (0,\infty)$. By Lemma I.5, we also obtain $\overline{V}_+ = \underline{V}_- = 0$ and therefore $\underline{u}(t) \leq v \leq \overline{u}(t)$ and the proof of the theorem ends. □

The following theorem gives a precise description of the asymptotic behavior of (u,v) under certain assumptions, using the bounds obtained. That is to say, the solution (u,v) behaves in the limit like $u^*(t)$, whose expression and periodic behavior we already know.

Theorem II.4. *For any nonnegative initial data $u_0 > 0$, $u_0 \in L^\infty(\Omega)$ the solution (u,v) of (II.1) fulfills*

$$\lim_{t\to\infty} \|u(x,t) - u^*(t)\|_{L^\infty(\Omega)} + \|v(x,t) - u^*(t)\|_{L^\infty(\Omega)} = 0,$$

where $u^*(t)$ is given by (II.42).

Proof. The proof is immediate by Theorem II.2 and Theorem II.3. □

All results of this three first sections have been published in [85].

Remark II.3. *Until this point we have expressed explicitly the time and space dependence for all functions appearing in our model. From now on, since no confussion is possible, we shall omit it.*

4 Generalization with two species

In this section we consider the reaction-diffusion system, which is a generalization for two biological species of system (II.1),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi_1 \nabla(u\nabla w) + \mu_1 u\left(1 - a_1 v - u + f_1(x,t)\right), & x\in\Omega,\ t>0, \\ \dfrac{\partial v}{\partial t} = \Delta v - \chi_2 \nabla(v\nabla w) + \mu_2 v\left(1 - v - a_2 u + f_2(x,t)\right), & x\in\Omega,\ t>0, \\ -\Delta w + w = \alpha u + \beta v, & x\in\Omega,\ t>0, \end{cases} \tag{II.52}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with regular boundary, where the chemotactic sensitivity χ_1, χ_2 are real constants, as well as α and β. The individuals of the biological species "u" and "v" are able to recognize the chemical signal "w", to measure its concentration and to move towards the higher concentrations of the substance (positive taxis) or away from it (negative taxis). In other words, the populations densities "u" and "v" follow a chemical gradient of a substance "w", the rate of substance production is proportional to "u" and "v" while the rate of degradation is proportional to "w".
The interaction between the species is described by the classical logistic terms of Lotka Volterra competitive system, $\mu_1 u(1 - a_1 v - u + f_1(x,t))$ and $\mu_2 v(1 - a_2 u - v + f_2(x,t))$, where the coefficients a_i and μ_i (for $i = 1,2$) are positive given data, assumed constants.

Functions $f_i = f_i(x,t)$, $i = 1,2$, describing the resources of the systems, present a periodic asymptotic behavior in the sense,

$$\lim_{t\to\infty} \sup_{x\in\Omega} |f_i(x,t) - f_i^*(t)| = 0, \qquad i = 1,2,$$

where f_i^* are independent of the space variable "x" and periodic in time, with the same period "T".
We complete the model with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{II.53}$$

and bounded initial data

$$u(0,x) = u^0(x), \qquad v(0,x) = v^0(x), \quad x \in \Omega, \tag{II.54}$$

satisfying

$$u^0(x),\ v^0(x) \in C^{2+\gamma}(\Omega), \qquad \frac{\partial u^0}{\partial \nu} = 0, \qquad \frac{\partial v^0}{\partial \nu} = 0, \ \text{ in } \partial\Omega, \tag{II.55}$$

$$\underline{u}^0 \le u^0(x) \le \overline{u}^0 < \infty, \qquad \underline{v}^0 \le v^0(x) \le \overline{v}^0 < \infty,\ x \in \Omega, \tag{II.56}$$

for some $\gamma > 0$ and positive constants $\underline{u}^0$ and $\underline{v}^0$, i.e., $0 < \underline{u}^0 \le \overline{u}^0$, $0 < \underline{v}^0 \le \overline{v}^0$. If $v = 0$, (II.52) is reduced to a parabolic-elliptic chemotactic PDEs system which describes the evolution of a biological population "u" and a chemical substance "w". The periodic behavior of the analytical solutions has been studied in the previous sections; thus, system (II.73) can be understood as a generalization of them.
Unless specified otherwise, we set

$$\overline{f}_i(t) := \sup_{x\in\Omega}\{f_i(x,t)\}, \qquad \underline{f}_i(t) := \inf_{x\in\Omega}\{f_i(x,t)\}; \tag{II.57}$$

$$f_i^L := \inf_{t\in\mathbb{R}} \underline{f}_i(t) = \min_{t\in[0,T]} \underline{f}_i(t), \qquad f_i^M := \sup_{t\in\mathbb{R}} \overline{f}_i(t) = \max_{t\in[0,T]} \overline{f}_i(t), \tag{II.58}$$

i.e., for a given function $\varphi : \mathbb{R} \to \mathbb{R}$, we wrote $\varphi^L = \inf \varphi$, $\varphi^M = \sup \varphi$.
For convenience and technical reasons, we assume:

1. Functions $f_1(x,t)$ and $f_2(x,t)$ are both smooth in $\mathcal{C}\left(0,\infty; L^\infty(\Omega)\right)$ and have the following properties:

   - There are positive constants C_1 and C_2 such that, for some $\varepsilon \ge 0$

   $$\int_0^\infty e^{\varepsilon t}\left(\overline{f}_i(s) - \underline{f}_i(s)\right) ds \le C_i < \infty, \qquad i = 1,2. \tag{II.59}$$

   - There exist positive constants ϵ_1 and ϵ_2 such that

   $$f_1(x,t) \ge -1 + \epsilon_1, \qquad f_2(x,t) \ge -1 + \epsilon_2, \qquad \text{for all } (x,t) \in \Omega_t, \tag{II.60}$$

   or its equivalent

   $$-1 < f_1^L, \qquad\qquad -1 < f_2^L. \tag{II.61}$$

- There exist $\overline{f}_i$ and $\underline{f}_i$ such that

$$|\overline{f}_i(t) - \underline{f}_i(t)| \to 0 \quad \text{for } t \to \infty, \quad \text{with } i = 1, 2. \tag{II.62}$$

-

$$\sup_{t>0} \|f_i\|_{L^\infty(\Omega)} := \|f_i\|_{L^\infty(\Omega_\infty)} < \infty. \tag{II.63}$$

- There are two periodic functions $f_1^*(t)$, $f_2^*(t) : \mathbb{R}_+ \to \mathbb{R}$ with the same period T that satisfy

$$\inf_{x\in\Omega} f_i(x,t) \le f_i^*(t) \le \sup_{x\in\Omega} f_i(x,t), \qquad i = 1, 2. \tag{II.64}$$

- Functions f_1 and f_2 have a periodic asymptotic behavior in the following sense

$$\lim_{t\to\infty} \sup_{x\in\Omega} |f_i(x,t) - f_i^*(t)| = 0, \qquad \text{i=1,2}. \tag{II.65}$$

2. Constants a_1, a_2, μ_1, μ_2, χ_1, χ_2, α and β are such that

$$0 < a_1 \le 1, \qquad 0 < a_2 \le 1, \qquad a_1 a_2 < 1, \tag{II.66}$$

$$\mu_1, \mu_2 > 0, \qquad \chi_1, \chi_2 \in \mathbb{R}, \qquad \alpha, \beta \in \mathbb{R}, \tag{II.67}$$

$$\mu_1 - a_2\mu_2 - 2\alpha(|\chi_1| + |\chi_2|) > 0, \qquad \mu_2 - a_1\mu_1 - 2\beta(|\chi_1| + |\chi_2|) > 0. \tag{II.68}$$

3. Given the functions f_1, f_2 and the constants a_1, a_2 of the previous assumptions, we have that

$$\left(1 + f_1^L\right) > a_1 \left(1 + f_2^M\right), \qquad \left(1 + f_2^L\right) > a_2 \left(1 + f_1^M\right). \tag{II.69}$$

All the hypotheses above appear in the previous sections for the corresponding case of one species and will lead us to obtain the uniform (in time) boundedness and the uniform asymptotic behavior of (u, v, w) given by Theorem II.5.
The main result of this section is enclosed in the following theorem.

Theorem II.5. *Under assumptions (II.59)–(II.69), for any nonnegative initial data* (u^0, v^0) *verifying (II.55)–(II.56), there exists a unique solution to (II.52)–(II.54) satisfying*

$$u, v, w \in C_{x,t}^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega_\infty),$$

moreover, the solution (u, v, w) *of (II.52) fulfills*

$$\lim_{t\to\infty} \left(\|u - u^*\|_{L^\infty(\Omega)} + \|v - v^*\|_{L^\infty(\Omega)} + \|w - \alpha u^* - \beta v^*\|_{L^\infty(\Omega)}\right) = 0, \tag{II.70}$$

where (u^*, v^*) *is the unique positive T-periodic solution of the system*

$$\begin{cases} \widetilde{u}_t(t) = \mu_1 \widetilde{u}(t) \left(1 - a_1 \widetilde{v}(t) - \widetilde{u}(t) + f_1^*(t)\right), \\ \widetilde{v}_t(t) = \mu_2 \widetilde{v}(t) \left(1 - a_2 \widetilde{u}(t) - \widetilde{v}(t) + f_2^*(t)\right), \end{cases} \tag{II.71}$$

for all $t \in (0, \infty)$, *with initial data*

$$\widetilde{u}(0) = u^0, \quad \widetilde{v}(0) = v^0. \tag{II.72}$$

In the absence of any chemotaxis ($\chi_1 = \chi_2 = 0$), the system becomes a two parabolic system already studied in [1], [47], [48] where the solutions have the same asymptotic behavior that the ODEs system (II.71).
The remaining of the chapter is organized as follows. In subsection 4.1 we give the statement of the problem and, in particular, we introduce an auxiliary system of ODEs and study the convergence of the solutions of the auxiliary system to a periodic in time state. In subsection 4.2 we prove, by using a comparison method (the *rectangle method*), that the solutions of the PDEs system (II.52) converge to the periodic in time solutions of the ODEs system. Subsection 4.3 is devoted to the proof of the existence, uniqueness, boundedness and asymptotic behavior of solutions of the problem (II.52) using the previous results, i.e., the complete proof of Theorem II.5.

4.1 Qualitative properties of an associated ODE system

In this subsection we consider a system of ordinary differential equations associated to the system of partial differential equations (II.52) in order to construct some kind of super- and sub- solutions. Our purpose is to find properties of the system of ordinary differential equations and to compare the solutions of the two systems at sufficiently large times. The tools we use to prove the uniform (in time) boundedness of (u, v, w) and the asymptotic behavior of the solutions of (II.52), are, as in the previous sections, certain a priori estimates on the solutions of an ODE associated system. Expressing the cross diffusion terms as in (II.15) we get

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi_1 \nabla u \nabla w + \chi_1 u(\alpha u + \beta v - w) + \mu_1 u(1 - a_1 v - u + f_1(x,t)), \\ \dfrac{\partial v}{\partial t} = \Delta v - \chi_2 \nabla v \nabla w + \chi_2 v(\alpha u + \beta v - w) + \mu_2 v(1 - v - a_2 u + f_2(x,t)), \\ -\Delta w + w = \alpha u + \beta v. \end{cases} \tag{II.73}$$

By recalling (II.57), we introduce the auxiliary functions

$$(\overline{u}, \underline{u}, \overline{v}, \underline{v}) := (\overline{u}(t), \underline{u}(t), \overline{v}(t), \underline{v}(t))$$

as solutions to the following system of ordinary differential equations

$$\begin{cases} \overline{u}' = \overline{u}|\chi_1|(\alpha\overline{u} + \beta\overline{v} - \alpha\underline{u} - \beta\underline{v}) + \mu_1\overline{u}\left(1 - a_1\underline{v} - \overline{u} + \overline{f}_1(t)\right), \\ \underline{u}' = \underline{u}|\chi_1|(\alpha\underline{u} + \beta\underline{v} - \alpha\overline{u} - \beta\overline{v}) + \mu_1\underline{u}\left(1 - a_1\overline{v} - \underline{u} + \underline{f}_1(t)\right), \\ \overline{v}' = \overline{v}|\chi_2|(\alpha\overline{u} + \beta\overline{v} - \alpha\underline{u} - \beta\underline{v}) + \mu_2\overline{v}\left(1 - \overline{v} - a_2\underline{u} + \overline{f}_2(t)\right), \\ \underline{v}' = \underline{v}|\chi_2|(\alpha\underline{u} + \beta\underline{v} - \alpha\overline{u} - \beta\overline{v}) + \mu_2\underline{v}\left(1 - \underline{v} - a_2\overline{u} + \underline{f}_2(t)\right), \end{cases} \tag{II.74}$$

for $t \in (0, \infty)$, complemented with the non-negative initial conditions

$$\overline{u}(0) = \overline{u}^0, \qquad \underline{u}(0) = \underline{u}^0, \qquad \overline{v}(0) = \overline{v}^0, \qquad \underline{v}(0) = \underline{v}^0, \tag{II.75}$$

that fulfill order relations of the type

$$0 < \underline{u}^0 < \overline{u}^0 < \infty, \qquad 0 < \underline{v}^0 < \overline{v}^0 < \infty. \tag{II.76}$$

We are going to prove some a priori estimates for the solution to the above problem, assuming suitable conditions on the data.

Lemma II.7. *The solution* $(\overline{u}, \underline{u}, \overline{v}, \underline{v})$ *of system (II.74)–(II.76) exists locally and satisfies* $[C^1(0, T_{max})]^4$ *for* T_{max} *defined as follows*

$$\limsup_{t \to T_{max}} (|\overline{u}(t)| + |\underline{u}(t)| + |\overline{v}(t)| + |\underline{v}(t)| + t) = \infty.$$

Moreover the solution satisfies

$$0 < \underline{u} < \overline{u}, \quad 0 < \underline{v} < \overline{v}, \quad for \;\; any \; t \in (0, T_{max}). \tag{II.77}$$

Proof. It should be noted that the terms on the right-hand side of (II.74) form a second order polynomial and due to the continuity of the coefficients, these are regular functions of class C^1. Thus, the standard ODE theory gives us the local existence and uniqueness of solutions of system (II.74)–(II.76) in the range $(0, T_{max})$. In order to study the positivity and the relationship between the components of the solution, we proceed as follows: we rewrite the second and fourth equations in (II.74)

$$\begin{cases} \underline{u}' = F(\underline{u}, \overline{u}, \underline{v}, \overline{v})\underline{u}, \\ \underline{v}' = G(\underline{u}, \overline{u}, \underline{v}, \overline{v})\underline{v}, \end{cases} \tag{II.78}$$

where F y G are C^1 functions. Now, $(\underline{u}, \underline{v}) = (0, 0)$ is a solution of (II.78), because of the existence and uniqueness of solutions and $\underline{u}^0 > 0$ and $\underline{v}^0 > 0$, we get $\underline{u}(t) > 0$ and $\underline{v}(t) > 0$, for all $t > 0$. By an argument of contradiction, it yields $\underline{u} < \overline{u}$ and $\underline{v} < \overline{v}$. Assume that there exists $t_0 \in (0, T_{max}]$ such that $\underline{u}(t) < \overline{u}(t)$ and $\underline{v}(t) < \overline{v}(t)$, for all $t \in (0, t_0)$ and one of the following relation holds:

$$\underline{u}(t_0) = \overline{u}(t_0) \quad \text{ó} \quad \underline{v}(t_0) = \overline{v}(t_0). \tag{II.79}$$

Suppose without loss of generality $\underline{u}(t_0) = \overline{u}(t_0)$. Then, by (II.79) we have

$$(\overline{u} - \underline{u})'(t_0) \leq 0. \tag{II.80}$$

Next, we introduce the functions ξ and η:

$$\xi = \overline{u} - \underline{u}, \qquad \eta = \overline{v} - \underline{v}.$$

We compute the derivative of ξ

$$\begin{aligned} \xi' = \overline{u}' - \underline{u}' = {} & [(|\chi_1|\alpha - \mu_1)(\overline{u} + \underline{u}) + \mu_1](\overline{u} - \underline{u}) - \mu_1 a_1(\overline{u}\underline{v} - \underline{u}\overline{v}) + \\ & + |\chi_1|\beta(\overline{u} + \underline{u})(\overline{v} - \underline{v}) + \mu_1(\overline{u}\overline{f}_1 - \underline{u}\underline{f}_1). \end{aligned} \tag{II.81}$$

Using

$$\overline{u}\underline{v} - \underline{u}\overline{v} = \frac{1}{2}(\overline{u} - \underline{u})(\overline{v} + \underline{v}) - \frac{1}{2}(\overline{u} + \underline{u})(\overline{v} - \underline{v}),$$

and

$$\overline{u}\overline{f}_1 - \underline{u}\underline{f}_1 = \frac{1}{2}(\overline{u} - \underline{u})(\overline{f}_1 + \underline{f}_1) + \frac{1}{2}(\overline{u} + \underline{u})(\overline{f}_1 - \underline{f}_1),$$

we can rewrite (II.81) in the sense

$$\begin{aligned} \xi' = {} & \left[(|\chi_1|\alpha - \mu_1)(\overline{u} + \underline{u}) + \mu_1 - \frac{\mu_1 a_1}{2}(\overline{v} + \underline{v}) + \frac{1}{2}(\overline{f}_1 + \underline{f}_1) \right] (\overline{u} - \underline{u}) + \\ & + \left[\left(\frac{\mu_1 a_1}{2} + |\chi_1|\beta \right) (\overline{u} + \underline{u}) \right] (\overline{v} - \underline{v}) + \frac{\mu_1}{2}(\overline{u} + \underline{u})(\overline{f}_1 - \underline{f}_1). \end{aligned} \tag{II.82}$$

Hence, we get

$$(\overline{u}-\underline{u})'(t_0)=\xi'(t_0)=\frac{\mu_1}{2}\left[\overline{u}(t_0)+\underline{u}(t_0)\right]\left[\overline{f}_1(t_0)-\underline{f}_1(t_0)\right]>0$$

which contradicts (II.80). The other relation is analogous. □

In the following lemma we establish some properties of the solutions to (II.74). We show that they are bounded and there are more order relations between them. All the properties that we obtain are useful to see that the sub- and super- solutions have the same limit when $t\to\infty$.

Lemma II.8. *There exist some positive constants c_i and δ_j with $i=1,5$ and $j=1,2$, respectively, such that the solution of system (II.74), with initial data (II.75) verifying (II.76), for any $t\in(0,\infty)$, satisfies*

1.
$$\overline{u}\,\overline{v}\le c_1, \tag{II.83}$$

2.
$$\overline{u}\le c_2 \quad\text{and}\quad \overline{v}\le c_3, \tag{II.84}$$

3.
$$\overline{u}\le c_4\underline{u} \quad\text{and}\quad \overline{v}\le c_5\underline{v}, \tag{II.85}$$

4.
$$0<\delta_1\le\underline{u} \quad\text{and}\quad 0<\delta_2\le\underline{v}, \tag{II.86}$$

Proof. 1. By denoting $\varphi=\varphi(t):\mathbb{R}_+\to\mathbb{R}$, $\varphi(t)=\ln\overline{u}(t)+\ln\overline{v}(t)$, dividing the first equation of (II.74) by $\overline{u}$ and the third one by $\overline{v}$ and adding them up, we get

$$\varphi'\le\mu_1(1+\overline{f_1})+\mu_2(1+\overline{f_2})-[\mu_1-\alpha(|\chi_1|+|\chi_2|)]\overline{u}-[\mu_2-\beta(|\chi_1|+|\chi_2|)]\overline{v}. \tag{II.87}$$

Moreover, $\mu_1(1+\overline{f_1})+\mu_2(1+\overline{f_2})<\mu_1\left(1+f_1^M\right)+\mu_2\left(1+f_2^M\right)$, then φ verifies the inequality

$$\varphi'\le K_1-2K_2e^{\frac{\varphi}{2}},$$

with two positive constants $K_1=\mu_1\left(1+f_1^M\right)+\mu_2\left(1+f_2^M\right)$ and, due to (II.68), $0<K_2=\min\left\{\mu_1-\alpha(|\chi_1|+|\chi_2|),\mu_2-\beta(|\chi_1|+|\chi_2|)\right\}$. Hence,

$$\varphi\le\max\{\ln\overline{u}^0+\ln\overline{v}^0,2\ln\frac{K_1}{2K_2}\}.$$

Consequently, (II.83) is done.

2. We should see only one of the two inequalities, since the other is similar. From the first equation of (II.74) and using the previous point (II.83), we have,

$$\begin{aligned}\overline{u}'&=\mu_1(1+\overline{f}_1)\overline{u}-(\mu_1-|\chi_1|\alpha)\overline{u}^2-|\chi_1|\alpha\underline{u}\overline{u}-(|\chi_1|\beta+\mu_1a_1)\underline{v}\overline{u}+|\chi_1|\beta\overline{v}\,\overline{u}\\&\le|\chi_1|\beta C+\mu_1\left(1+f_1^M\right)\overline{u}-(\mu_1-|\chi_1|\alpha)\overline{u}^2.\end{aligned}$$

By a comparison argument and (II.68), it holds

$$\overline{u}\le c_2:=\max\left\{\overline{u}^0,\frac{\mu_1\left(1+f_1^M\right)+\sqrt{\left[\mu_1\left(1+f_1^M\right)\right]^2+4|\chi_1|\beta c_1(\mu_1-|\chi_1|\alpha)}}{2(\mu_1-|\chi_1|\alpha)}\right\}.$$

3. In order to prove that the super- and sub- solutions are comparable as in (II.85), we proceed as follows

$$\begin{aligned}\frac{d}{dt}\left(\ln\frac{\overline{u}}{\underline{u}}+\ln\frac{\overline{v}}{\underline{v}}\right) &\leq -\epsilon[(\overline{u}-\underline{u})+(\overline{v}-\underline{v})]+\mu_1(\overline{f}_1-\underline{f}_1)+\mu_2(\overline{f}_2-\underline{f}_2)\\ &\leq \mu_1(\overline{f}_1-\underline{f}_1)+\mu_2(\overline{f}_2-\underline{f}_2),\end{aligned} \tag{II.88}$$

with
$0<\epsilon=\min\{\mu_1-\mu_2 a_2-2\alpha(|\chi_1|+|\chi_2|),\mu_2-\mu_1 a_1-2\beta(|\chi_1|+|\chi_2|)\}$, by hypothesis (II.68). By integrating (II.88) in time, thanks to (II.59) it holds

$$\ln\frac{\overline{u}}{\underline{u}}+\ln\frac{\overline{v}}{\underline{v}}-\ln\frac{\overline{u}^0}{\underline{u}^0}-\ln\frac{\overline{v}^0}{\underline{v}^0}\leq \mu_1 C_1+\mu_2 C_2, \tag{II.89}$$

with C_i as in (II.59). Thus, we easily obtain $\overline{u}\leq c_4\underline{u}$, with $c_4=\frac{\overline{u}^0\overline{v}^0}{\underline{u}^0\underline{v}^0}e^{(\mu_1 C_1+\mu_2 C_2)}$. In an analogous way we find a positive constant c_5 such that $\overline{v}\leq c_5\underline{v}$.

4. The solution $\underline{u}$ of system (II.74) verifies

$$\underline{u}'\geq -c_4\underline{u}(|\chi_1|\alpha+|\chi_1|\beta+\mu_1 a_1\underline{u}+\mu_1)-(|\chi_1|\alpha-\mu_1)\underline{u}^2$$

thus, $\underline{u}$ is a super solution of the ordinary differential equation $y'=-Ay-By^2$, $A>0$, $B>0$ given in the above inequality. The application leads us to the searched result. The details of the proof including the application of the comparison principle for ODE is similar to the case where the coefficients are constant or the predator-prey case (more details can be found in [84] and [25]). Through the symmetry of the problem the other inequality for the solution $\underline{v}$ is obtained in an analogous way. □

The following lemma is a standard result of the comparison method and it is fundamental since we prove that the two pairs of solutions of the ODE's system (II.74), i.e., $(\underline{u},\overline{u})$ and $(\underline{v},\overline{v})$ have the same limit respectively and, hence, also any function between them.

Lemma II.9. *Under hypotheses (II.59)–(II.68), the solutions of (II.74) fulfill*

$$\frac{\overline{u}}{\underline{u}}\to 1 \quad \textit{and} \quad \frac{\overline{v}}{\underline{v}}\to 1, \quad \textit{as} \quad t\to\infty. \tag{II.90}$$

Proof. Recover the expression:

$$\begin{aligned}\frac{d}{dt}\left(\ln\frac{\overline{u}}{\underline{u}}+\ln\frac{\overline{v}}{\underline{v}}\right) &= -[\mu_1-\mu_2 a_2-2\alpha(|\chi_1|+|\chi_2|)](\overline{u}-\underline{u})\\ &-[\mu_2-\mu_1 a_1-2\beta(|\chi_1|+|\chi_2|)](\overline{v}-\underline{v})+\mu_1(\overline{f}_1-\underline{f}_1)+\mu_2(\overline{f}_2-\underline{f}_2).\end{aligned} \tag{II.91}$$

We introduce the functions

$$\Psi_1(t)=\ln\frac{\overline{u}}{\underline{u}}, \qquad \Psi_2(t)=\ln\frac{\overline{v}}{\underline{v}}, \qquad F(t)=\mu_1(\overline{f}_1-\underline{f}_1)+\mu_2(\overline{f}_2-\underline{f}_2), \tag{II.92}$$

we can rewrite (II.91) as below:

$$\begin{aligned}\frac{d}{dt}(\Psi_1+\Psi_2) &= -[\mu_1-\mu_2 a_2 - 2\alpha(|\chi_1|+|\chi_2|)](e^{\Psi_1}-1)\underline{u} \\ &\quad - [\mu_2-\mu_1 a_1 - 2\beta(|\chi_1|+|\chi_2|)](e^{\Psi_2}-1)\underline{v} + F(t).\end{aligned} \tag{II.93}$$

Using Lemma II.8, hypothesis (II.68) and $e^{\Psi_1} \geq \Psi_1 + 1$, we can ensure that there is a positive constant δ such that

$$\frac{d}{dt}(\Psi_1+\Psi_2) \leq -\delta(\Psi_1+\Psi_2) + F(t), \tag{II.94}$$

and by applying the Gronwall's lemma one has

$$(\Psi_1+\Psi_2) \leq e^{-\delta t}\left(\Psi_1(0)+\Psi_2(0)+\int_0^t e^{\delta s}F(s)ds\right). \tag{II.95}$$

Then,

$$\Psi_1+\Psi_2 = \ln\frac{\overline{u}}{\underline{u}} + \ln\frac{\overline{v}}{\underline{v}} \to 0 \text{ as } t \to \infty, \tag{II.96}$$

which means that

$$\frac{\overline{u}}{\underline{u}} \to 1 \quad \text{y} \quad \frac{\overline{v}}{\underline{v}} \to 1, \tag{II.97}$$

when $t \to \infty$. □

Notice that as a consequence of the previous lemmas, it is $T_{max} = \infty$, which implies the global existence of the solutions.
Let $f_i^*(t)$ be the functions entered in (II.65), a_1, a_2 positive constants such that $a_1 a_2 < 1$ and $(\widetilde{u}(t), \widetilde{v}(t))$ the solution of (II.71), i.e.,

$$\begin{cases} \widetilde{u}_t(t) = \mu_1\widetilde{u}(t)(1-a_1\widetilde{v}(t)-\widetilde{u}(t)+f_1^*(t)), \\ \widetilde{v}_t(t) = \mu_2\widetilde{v}(t)(1-a_2\widetilde{u}(t)-\widetilde{v}(t)+f_2^*(t)), \end{cases}$$

for $t \in (0,\infty)$, with initial data (II.72), $\widetilde{u}(0) = \widetilde{u}^0$ and $\widetilde{v}(0) = \widetilde{v}^0$.
It is well known that this system models the interaction between two competitive species in a T-periodic context. Below we give a known result, adapted to our problem, concerning the behavior of its solutions.

Lemma II.10. *Suppose that $-1 < f_i^*$ are T-periodic functions for $i = 1, 2$, and*

$$\frac{1}{a_2} > \frac{(1+f_1^*)^M}{(1+f_2^*)^L}, \qquad \frac{1}{a_1} > \frac{(1+f_2^*)^M}{(1+f_1^*)^L}.$$

Then, system (II.71) has a unique positive T-periodic positive solution $(u^*, v^*) = (u^*(t), v^*(t))$. Furthermore, this solution is asymptotically stable,

$$(\widetilde{u}(t)-u^*, \widetilde{v}(t)-v^*) \to (0,0), \qquad \textit{as} \quad t \to \infty, \tag{II.98}$$

for any positive solution $(\widetilde{u}, \widetilde{v})$ of (II.71). In addition, the solution is delimited above and below as follows:

$$0 < \frac{(1+f_1^L) - a_1(1+f_2^M)}{1-a_1a_2} \leq u^*(t) \leq \frac{(1+f_1^M) - a_1(1+f_2^L)}{1-a_1a_2}, \tag{II.99}$$

$$0 < \frac{(1+f_2^L) - a_2(1+f_1^M)}{1-a_1a_2} \leq v^*(t) \leq \frac{(1+f_2^M) - a_2(1+f_1^L)}{1-a_1a_2}. \tag{II.100}$$

Proof. Theorem I.6 and Theorem I.7 give the complete proof for a general case of ODEs systems with periodic coefficients, we have only customized the general coefficients for our system by transcribing the known result in order to apply it to our system (II.74). □

We proceed to prove the asymptotic behavior of the solutions of (II.74), the result is enclosed in the following lemma.

Lemma II.11. *If the initial conditions (II.75) of system (II.74) and the initial conditions (II.72) of (II.71) satisfy the following relationships of order:*

$$0 < \underline{u}^0 < \widetilde{u}_1^0 < \overline{u}^0, \qquad 0 < \underline{v}^0 < \widetilde{v}^0 < \overline{v}^0, \tag{II.101}$$

the pairs of solutions $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are super and sub solutions of (II.74), i.e., we have the ordering

$$\underline{u}(t) \leq \widetilde{u}(t) \leq \overline{u}(t), \quad \underline{v}(t) \leq \widetilde{v}(t) \leq \overline{v}(t), \tag{II.102}$$

for all $t \in (0, \infty)$.

Proof. Taking into account Lemma II.7, by (II.74), it follows

$$\begin{cases} \overline{u}' \geq \mu_1 \overline{u}(1 + f_1^* - \overline{u} - a_1 \overline{v}), \\ \overline{v}' \geq \mu_2 \overline{v}(1 + f_2^* - a_2 \overline{u} - \overline{v}), \end{cases} \tag{II.103}$$

and

$$\begin{cases} \underline{u}' \leq \mu_1 \underline{u}(1 + f_1^* - \underline{u} - a_1 \underline{v}), \\ \underline{v}' \leq \mu_2 \underline{v}(1 + f_2^* - \underline{u} - a_2 \underline{v}), \end{cases} \tag{II.104}$$

for $t \in (0, \infty)$, with initial data

$$\overline{u}(0) = \overline{u}^0, \;\; \underline{u}(0) = \underline{u}^0, \;\; \overline{v}(0) = \overline{v}^0, \;\; \underline{v}(0) = \underline{v}^0,$$

satisfying

$$0 < \underline{u}^0 < \overline{u}^0, \qquad 0 < \underline{v}^0 < \overline{v}^0,$$

for $t \in (0, \infty)$. By contradiction, we assume that there exists t_0, $t_0 \in (0, \infty)$ such that

$$\begin{cases} [\overline{u}(t_0) - \widetilde{u}(t_0)][\widetilde{u}(t_0) - \underline{u}(t_0)][\overline{v}(t_0) - \widetilde{v}(t_0)][\widetilde{v}(t_0) - \underline{v}(t_0)] = 0, \\ \overline{u}(t) > \widetilde{u}(t), \quad \widetilde{u}(t) > \underline{u}(t), \quad \overline{v}(t) > \widetilde{v}(t), \quad \widetilde{v}(t) > \underline{v}(t) \quad \forall t \leq t_0. \end{cases} \tag{II.105}$$

If $\overline{u}(t_0) = \widetilde{u}(t_0)$, then

$$\frac{d}{dt} \ln \frac{\overline{u}}{\widetilde{u}} \geq -\mu_1 \eta \ln \frac{\overline{u}}{\widetilde{u}}, \quad \forall t \leq t_0, \quad \eta > 0. \tag{II.106}$$

If we integrate in time, we have $\ln \frac{\overline{u}}{\widetilde{u}} > 0 \;\; \forall t \leq t_0$ and in particular this implies that $\overline{u}(t_0) > \widetilde{u}(t_0)$. Performing the same calculations for the other cases we conclude that (II.105) is not possible and therefore we have the result. □

The main theorem of this section concerning the asymptotic behavior of the solutions of the ODE's system (II.74) is

Theorem II.6. *Let $(\overline{u}, \underline{u}, \overline{v}, \underline{v})$ be the solution of system (II.74) and (u^*, v^*) the unique T-periodic solution of (II.71). Under hypothesis (II.64) the following limits hold*

$$\lim_{t \to \infty} \left(|\overline{u}(t) - u^*(t)| + |\underline{u}(t) - u^*(t)| + |\overline{v}(t) - v^*(t)| + |\underline{v}(t) - v^*(t)| \right) = 0.$$

Proof. Theorem II.6 is a direct consequence of the proved properties of the solutions in the previous lemmas,

$$0 < |\overline{u}(t) - \underline{u}(t)| \leq |\overline{u}(t) - \widetilde{u}(t)| + |\widetilde{u}(t) - \underline{u}(t)| \leq 2|\overline{u}(t) - \underline{u}(t)| \to 0, \tag{II.107}$$

for $t \to \infty$, then we get

$$|\overline{u}(t) - u^*(t)| \to 0 \quad \text{and} \quad |u^*(t) - \underline{u}(t)| \to 0. \tag{II.108}$$

The test for $\overline{v}$, $\underline{v}$ and v^* is analogous so it is omitted. □

4.2 Comparison principle and asymptotic behavior of solutions

The aim of this subsection is to use again the principle of comparison introduced by Pao in [92] for reaction systems, the so-called *rectangle method* (see also [81], [82], [85] where the method is applied to Parabolic-Elliptic systems with chemotactic terms). In Theorem II.7, which is the main result of this subsection, we see that a relationship of order that we impose in the initial conditions of the ODEs and PDEs systems is also preserved for their solutions.
We have obtained that the functions $(\underline{u}, \overline{u}, \underline{v}, \overline{v}) - (u^*, u^*, v^*, v^*)$ vanish as $t \to \infty$, where (u^*, v^*) is the unique T-periodic solution of (II.71), under different restrictions on the coefficients. In short, if

$$0 < \underline{u}^0 \leq u^0(x) \leq \overline{u}^0 < \infty, \;\; 0 < \underline{v}^0 \leq v^0(x) \leq \overline{v}^0 < \infty, \; x \in \Omega,$$

our goal is to reach the following relationship between the solutions

$$\underline{u}(t) \leq u(x,t) \leq \overline{u}(t), \quad \underline{v}(t) \leq v(x,t) \leq \overline{v}(t).$$

Theorem II.7. *Let $(u^0, v^0) \in (L^\infty(\Omega))^2$. The solution of (II.52)–(II.53) with initial data verifying (II.54)–(II.56) is bounded and satisfies*

$$\begin{aligned} &\underline{u}(t) \leq u(x,t) \leq \overline{u}(t), \quad \underline{v}(t) \leq v(x,t) \leq \overline{v}(t), \quad (x,t) \in \Omega \times (0,\infty), \\ &\alpha\underline{u}(t) + \beta\underline{v}(t) \leq w \leq \alpha\overline{u}(t) + \beta\overline{v}(t), \qquad\qquad (x,t) \in \Omega \times (0,\infty). \end{aligned} \tag{II.109}$$

where $(\underline{u}, \overline{u}, \underline{v}, \overline{v})$ is the solution of the ODE system (II.74).

In order to prove Theorem II.7, we define the functions

$$\begin{aligned} &\overline{\mathcal{U}}(x,t) := u(x,t) - \overline{u}(t), && \underline{\mathcal{U}}(x,t) := u(x,t) - \underline{u}(t), \\ &\overline{\mathcal{V}}(x,t) := v(x,t) - \overline{v}(t), && \underline{\mathcal{V}}(x,t) := v(x,t) - \underline{v}(t), \\ &\overline{\mathcal{W}}(x,t) := w(x,t) - \alpha\overline{u}(t) - \beta\overline{v}(t), \\ &\underline{\mathcal{W}}(x,t :) = w(x,t) - \alpha\underline{u}(t) - \beta\underline{v}(t), \end{aligned} \tag{II.110}$$

where (u, v, w) is the solution to (II.52) and $(\overline{u}, \underline{u}, \overline{v}, \underline{v})$ is the solution to (II.74).

We aim to prove that the positive and negative parts $\overline{\mathcal{U}}_+ = \underline{\mathcal{U}}_- = \overline{\mathcal{V}}_+ = \underline{\mathcal{V}}_- = 0$ are identically zero and therefore the solutions verify (II.109).

Remark II.4. *First we observe that given any $\tilde{T} \in (0, T_{max})$, since (u, v, w) are continuous and differentiable in $\Omega \times (0, \tilde{T})$ there exists a positive constant $c_1(\tilde{T})$ such that, $\forall (x,t) \in \Omega \times (0, \tilde{T})$.*

$$u(x,t) \leq c_1(\tilde{T}), \qquad v(x,t) \leq c_1(\tilde{T}), \qquad w(x,t) \leq c_1(\tilde{T}), \tag{II.111}$$

Proof. We consider $0 < \tilde{T} < \infty$. To begin with, we want to see what partial differential equation satisfies $\overline{\mathcal{U}}$. Taking into account that $\overline{u}$ has no spatial dependence, we can say that $\Delta\overline{\mathcal{U}} = \Delta u$ and $\nabla\overline{\mathcal{U}} = \nabla u$, the same applies to $\underline{\mathcal{U}}$, $\underline{\mathcal{V}}$, $\overline{\mathcal{V}}$, $\underline{\mathcal{W}}$, $\overline{\mathcal{W}}$. By operating with (II.73) and (II.74), we get

$$\overline{\mathcal{U}}' - \Delta\overline{\mathcal{U}} + |\chi_1|\nabla\overline{\mathcal{U}}\nabla w = \overline{\mathcal{U}}[(\alpha|\chi_1| - \mu_1)(u+\overline{u}) + (\beta|\chi_1| - \mu_1 a_1)v - |\chi_1|w$$
$$+\mu_1(1+f_1)] + [\beta|\chi_1|\overline{u}\overline{\mathcal{V}} - \mu_1 a_1\overline{u}\underline{\mathcal{V}} + |\chi_1|\overline{u}(\alpha\underline{u} + \beta\underline{v} - w) + \mu_1\overline{u}(f_1 - \overline{f}_1)].$$

Define $b(x,t)$ as:

$$b(x,t) = (\alpha|\chi_1| - \mu_1)(u+\overline{u}) + (\beta|\chi_1| - \mu_1 a_1)v - |\chi_1|w + \mu_1(1+f_1).$$

Then, we multiply by $\overline{\mathcal{U}}_+$ e integrate by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{\mathcal{U}}_+^2 + \int_\Omega |\nabla\overline{\mathcal{U}}_+|^2 = -\frac{|\chi_1|}{2}\int_\Omega \nabla\left(\overline{\mathcal{U}}_+^2\right)\nabla w + \int_\Omega b(x,t)\overline{\mathcal{U}}_+^2 + \\ + \beta|\chi_1|\int_\Omega \overline{u}\overline{\mathcal{U}}_+\overline{\mathcal{V}} - \mu_1 a_1\int_\Omega \overline{u}\overline{\mathcal{U}}_+\underline{\mathcal{V}} + |\chi_1|\int_\Omega \overline{u}(\alpha\underline{u} + \beta\underline{v} - w)\overline{\mathcal{U}}_+ \\ + \mu_1\int_\Omega \overline{u}\overline{\mathcal{U}}_+(f_1 - \overline{f_1}).\end{aligned} \tag{II.112}$$

For the first term, we have

$$\begin{aligned}-\frac{|\chi_1|}{2}\int_\Omega \nabla\left(\overline{\mathcal{U}}_+^2\right)\nabla w = \frac{|\chi_1|}{2}\int_\Omega \overline{\mathcal{U}}_+^2\Delta w = \frac{|\chi_1|}{2}\int_\Omega \overline{\mathcal{U}}_+^2(w - \alpha u - \beta v) \leq \\ \leq \frac{(1+\alpha+\beta)c_1(T)|\chi_1|}{2}\int_\Omega \overline{\mathcal{U}}_+^2.\end{aligned} \tag{II.113}$$

For the second one we notice

$$|b(x,t)| \leq |\chi_1|(2\alpha + \beta + 1)c_1(T) + \mu_1(2 + a_1)c_1(T) + \mu_1\left(1 + f_1^M\right). \tag{II.114}$$

We call, for simplicity, $c_2(T) = |\chi_1|(2\alpha+\beta+1)c_1(T) + \mu_1(2+a_1)c_1(T) + \mu_1\left(1+f_1^M\right)$ so $b(x,t) \leq c_2(T)$. Therefore,

$$\int_\Omega b(x,t)\overline{\mathcal{U}}_+^2 \leq c_2(T)\int_\Omega \overline{\mathcal{U}}_+^2. \tag{II.115}$$

By the Young's inequaity,

$$\begin{aligned}\beta|\chi_1|\int_\Omega \overline{u}\overline{\mathcal{U}}_+\overline{\mathcal{V}} \leq \beta|\chi_1|\int_\Omega \overline{u}\overline{\mathcal{U}}_+\overline{\mathcal{V}}_+ \leq \beta|\chi_1|c_1(T)\int_\Omega \overline{\mathcal{U}}_+\overline{\mathcal{V}} \leq \\ \leq \frac{\beta|\chi_1|c_1(T)}{2}\left(\int_\Omega \overline{\mathcal{U}}_+^2 + \int_\Omega \overline{\mathcal{V}}_+^2\right),\end{aligned} \tag{II.116}$$

$$
\begin{aligned}
-\mu_1 a_1 \int_\Omega \overline{u}\,\overline{\mathcal{U}}_+ \underline{\mathcal{V}} \leq \mu_1 a_1 \int_\Omega \overline{u}\,\overline{\mathcal{U}}_+ \underline{\mathcal{V}}_- \leq & \mu_1 a_1 c_1(T) \int_\Omega \overline{\mathcal{U}}_+ \underline{\mathcal{V}}_- \leq \\
& \leq \frac{\mu_1 a_1 c_1(T)}{2} \left(\int_\Omega \overline{\mathcal{U}}_+^2 + \int_\Omega \underline{\mathcal{V}}_-^2 \right).
\end{aligned} \tag{II.117}
$$

and

$$
\begin{aligned}
|\chi_1| \int_\Omega \overline{u}(\alpha \underline{u} + \beta \underline{v} - w) \overline{\mathcal{U}}_+ & \leq |\chi_1| \int_\Omega \overline{u}(\alpha \underline{u} + \beta \underline{v} - w)_+ \overline{\mathcal{U}}_+ \leq \\
& \leq |\chi_1| c_1(T) \int_\Omega (\alpha \underline{u} + \beta \underline{v} - w)_+ \overline{\mathcal{U}}_+ \leq \\
& \frac{|\chi_1| c_1(T)}{2} \left(\int_\Omega (\alpha \underline{u} + \beta \underline{v} - w)_+^2 + \int_\Omega \overline{\mathcal{U}}_+^2 \right) \leq \\
& \leq \frac{|\chi_1| c_1(T)}{2} \left[2 \int_\Omega (\alpha^2 \underline{\mathcal{U}}_-^2 + \beta^2 \underline{\mathcal{V}}_-^2) + \int_\Omega \overline{\mathcal{U}}_+^2 \right].
\end{aligned} \tag{II.118}
$$

Let us define

$$
\begin{aligned}
k_1(T) = \max \Bigg\{ & \frac{(1+\alpha+\beta)|\chi_1| c_1(T)}{2} + c_2(T) + \frac{\beta |\chi_1| c_1(T)}{2} \\
& + \frac{\mu_1 a_1 c_1(T)}{2} + \frac{|\chi_1| c_1(T)}{2}, \frac{\mu_1 a_1 c_1(T)}{2} + \chi_1 c_1(T)(\alpha^2 + \beta^2) \Bigg\},
\end{aligned} \tag{II.119}
$$

so (II.112) is reduced to:

$$
\frac{1}{2}\frac{d}{dt} \int_\Omega (\overline{\mathcal{U}})_+^2 + \int_\Omega |\nabla (\overline{\mathcal{U}})_+|^2 \leq k_1(\tilde{T}) \left(\int_\Omega (\overline{\mathcal{U}})_+^2 + \int_\Omega (\overline{\mathcal{V}})_+^2 + \int_\Omega (\underline{\mathcal{V}})_-^2 \right),
$$

for all $t \in (0, \tilde{T})$. Doing the same for the equation that satisfies $\underline{\mathcal{U}}$, multiplying by the test function $(\underline{\mathcal{U}})_-$ and integrating by parts, we reach

$$
\frac{1}{2}\frac{d}{dt} \int_\Omega (\underline{\mathcal{U}})_-^2 + \int_\Omega |\nabla (\underline{\mathcal{U}})_-|^2 \leq k_2(\tilde{T}) \left(\int_\Omega (\underline{\mathcal{U}})_-^2 + \int_\Omega (\overline{\mathcal{V}})_+^2 + \int_\Omega (\underline{\mathcal{V}})_-^2 \right),
$$

for some positive $k_2(\tilde{T})$. Proceeding in the same way with $\overline{\mathcal{V}}$, $\underline{\mathcal{V}}$, we find constants $k_3(\tilde{T})$ and $k_4(\tilde{T})$ such that

$$
\frac{1}{2}\frac{d}{dt} \int_\Omega (\overline{\mathcal{V}})_+^2 + \int_\Omega |\nabla (\overline{\mathcal{V}})_+|^2 \leq k_3(\tilde{T}) \left(\int_\Omega (\overline{\mathcal{V}})_+^2 + \int_\Omega (\overline{\mathcal{U}})_+^2 + \int_\Omega (\underline{\mathcal{U}})_-^2 \right),
$$

$$
\frac{1}{2}\frac{d}{dt} \int_\Omega (\underline{\mathcal{V}})_-^2 + \int_\Omega |\nabla (\underline{\mathcal{V}})_-|^2 \leq k_4(\tilde{T}) \left(\int_\Omega (\underline{\mathcal{V}})_-^2 + \int_\Omega (\overline{\mathcal{U}})_+^2 + \int_\Omega (\underline{\mathcal{U}})_-^2 \right).
$$

Now we add the above four equations to get

$$
\begin{aligned}
& \frac{d}{dt} \left(\int_\Omega (\overline{\mathcal{U}})_+^2 + (\underline{\mathcal{U}})_-^2 + (\overline{\mathcal{V}})_+^2 + (\underline{\mathcal{V}})_-^2 \right) \\
& \leq k(\tilde{T}) \left(\int_\Omega (\overline{\mathcal{U}})_+^2 + (\underline{\mathcal{U}})_-^2 + (\overline{\mathcal{V}})_+^2 + (\underline{\mathcal{V}})_-^2 \right),
\end{aligned}
$$

with $k(\tilde{T}) = \max\{k_i(\tilde{T}) : i = 1, ..., 4\}$. As in $t = 0$ we have $\overline{\mathcal{U}}_+(0) = \overline{\mathcal{V}}_+(0) = \underline{\mathcal{U}}_-(0) = \underline{\mathcal{V}}_-(0) = 0$, then, by applying the Gronwall's lemma, the result follows, i.e., $\overline{\mathcal{U}}_+(t) =$

$\overline{\mathcal{V}}_+(t) = \underline{\mathcal{U}}_-(t) = \underline{\mathcal{V}}_-(t) = 0$, for all $t \in (0, \tilde{T})$, (similar results can be found in [78] or [81]). Hence we have

$$\underline{u}(t) \le u(x,t) \le \overline{u}(t), \qquad \underline{v}(t) \le v(x,t) \le \overline{v}(t), \qquad (x,t) \in \Omega \times (0, \tilde{T}).$$

By Lemma I.5 or Lemma 3.2 in [79], we obtain $\overline{\mathcal{W}}_+ = \underline{\mathcal{W}}_- = 0$, $i = 1, 2$, and therefore

$$\alpha \underline{u}(t) + \beta \underline{v}(t) \le w \le \alpha \overline{u}(t) + \beta \overline{v}(t), \quad (x,t) \in \Omega \times (0, \tilde{T}).$$

As $\tilde{T} > 0$ is arbitrary, we take limits as $\tilde{T} \to \infty$ and the proof of the Theorem II.7 ends. □

4.3 Existence of the solution and asymptotic behavior

This final subsection is dedicated to the proof of Theorem II.5. The a priori estimates in the previous sections allow us to prove it.

Proof of Theorem II.5. First we give the known local existence result and later the global existence and uniqueness, i.e., we prove that under assumptions (II.59)–(II.69), there exists a unique solution (u, v, w) to (II.52)–(II.54) in $(0, \infty)$ satisfying

$$u, v, w \in C^{2+\gamma, 1+\frac{\gamma}{2}}_{x,t}(\Omega_\infty), \qquad \text{for any } t < \infty.$$

Moreover,

$$u(x,t) \ge 0, \quad v(x,t) \ge 0, \quad w(x,t) \ge 0 \qquad x \in \Omega, \quad t < \infty. \tag{II.120}$$

To reach this result, we take T_{max} such that

$$\limsup_{t \to T_{max}} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + t \right) = \infty.$$

In order to get the local existence of solutions in $L^2(0 : T, H^1(\Omega)) \cap L^\infty(\Omega_t)$, for any $t < T_{max}$, we apply standard fixed point theory (Biler [13], Horstmann and Winkler [54], or Negreanu and Tello [82], [84] obtained similar results). Moreover, due to Theorem II.7 and Lemmas II.8 and II.11, where we have obtained the boundedness of u, v, and by the Maximum Principle,

$$\|w\|_{L^\infty(\Omega)} \le \alpha \|u\|_{L^\infty(\Omega)} + \beta \|v\|_{L^\infty(\Omega)}, \tag{II.121}$$

which implies that $T_{max} = \infty$.

The regularity of the solutions u and v is a consequence of the parabolic and elliptic regularity of the equations, the regularity of the coefficients and the boundedness of u, v and w. We get the uniqueness of them by applying standard contradiction arguments, as in the one species case.

The second part of the theorem, the asymptotic behavior of the solutions (II.70), is a consequence of Theorems II.7 and II.6. The proof finishes. □

The results for the 2-species system, (II.52), are have been recently published in [89].

Remark II.5. *The two systems of this chapter (parabolic-elliptic and parabolic-parabolic-elliptic) are solved numerically in Chapter VII. Also, examples of functions f, f_1 and f_2 are given.*

Chapter III

Parabolic-parabolic model

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. We consider now the case where the diffusion of the biological species is comparable to the diffusion of the chemical substance the following parabolic-parabolic system (13) which describes the evolution of a biological species "u" and a chemical substance "v". The system reads as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(x,t) - u), & x \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, \quad t > 0, \\ u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{III.1}$$

where $f(x,t)$ converges to a homogeneous in space and periodic in time function $f^*(t)$. With this periodic function we are able to construct the asymptotic limit

$$\lim_{t\to\infty} \|u - u^*\|_{L^\infty(\Omega)} + \|v - v^*\|_{L^\infty(\Omega)} = 0,$$

where u^* is the periodic in time function defined in the same way as before, i.e.,

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \mu \int_0^t e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}, \tag{III.2}$$

where

$$u_0^* := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\mu \int_0^T e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}.$$

Notice that u^* is the solution of the equation

$$\frac{du^*}{dt} = \mu u^*(1 + f^* - u^*). \tag{III.3}$$

and v^* is the solution of

$$\tau \frac{dv^*}{dt} = u^* - v^*. \tag{III.4}$$

In this chapter we work under the following assumptions:

- The initial data (u_0, v_0) satisfies

$$(u_0, v_0) \in \left[C^{2+\beta}(\bar{\Omega})\right]^2, \tag{III.5}$$

for some $\beta > 0$.

$$\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{III.6}$$

$$u_0 \geq 0, \qquad v_0 \geq 0. \tag{III.7}$$

$$\int_\Omega ln(u_0)dx \geq -L > -\infty. \tag{III.8}$$

which implies, in particular that

$$\int_\Omega u_0 dx > 0, \qquad \int_\Omega v_0 dx > 0. \tag{III.9}$$

- There exists a positive constant $\epsilon_1 > 0$ such that

$$f(t,x) > -1 + \epsilon_1. \tag{III.10}$$

- Function f satisfies

$$\sup_{t>0} \|f\|_{L^\infty(\Omega)} := \|f\|_{L^\infty(\Omega_\infty)} < \infty, \tag{III.11}$$

$$\int_0^\infty \int_\Omega |\nabla f|^2 dxdt \leq c < \infty, \tag{III.12}$$

$$\int_0^\infty \|f - f^*\|_{L^1(\Omega)} dt \leq c < \infty, \tag{III.13}$$

where $f^* = f^*(t)$ is independent of x and periodic in time of period T.

- The coefficients χ and μ fulfill

$$\mu > \frac{\chi^2}{16} \max\left\{\int_\Omega u_0 dx, \frac{1}{\mu}(1 + \|f\|_{L^\infty(\Omega)})\right\}. \tag{III.14}$$

- For simplicity and without loss of generality we assume

$$|\Omega| = 1. \tag{III.15}$$

Notice that assumptions (III.3), (III.10) and (III.11) are the same assumptions of the previous chapter and assumption (III.14) is imposed for problem (III.1).
The chapter is organized as follows: in Section 1 we study the existence and uniqueness of solutions of the system (III.1) using the Moser-Alikakos iteration method [2] under hypothesis (III.5)–(III.13) and (III.15). The existence result is standard and similar to the proof in Winkler [117] (it can be also consulted Xiang [121]), nevertheless for completeness of the thesis we present the details of the proof. In Section 2, we also assume (III.14) to prove that the solution to the PDEs system satisfies

$$\lim_{t \to \infty} \|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)} = 0,$$

in a two-step process. First, we obtain that the solution (u, v) goes to $(\tilde{u}, \tilde{v})$ as t goes to ∞, where $\tilde{u}, \tilde{v}$ are given by

$$\tilde{u} = \frac{1}{|\Omega|}\int_\Omega u dx, \qquad \tilde{v} = \frac{1}{|\Omega|}\int_\Omega v dx,$$

respectively. Secondly, we prove that $(\tilde{u}, \tilde{v})$ converges to (u^*, v^*) where the known function u^* is defined in (III.2) and v^* in (III.4)

1 Existence and uniqueness of solutions

In this section we study the global existence of solutions of (III.1). The main result of this section is the following.

Theorem III.1. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Let be $\tau > 0$ and $\chi \in \mathbb{R}$. Then, for all $\mu > 0$, for any nonnegative u_0 and v_0 fulfilling assumptions (III.5)–(III.9) and f satisfying (III.10)-(III.13), (III.1) possesses a uniquely determined global solution (u, v) for which both u and v are nonnegative and bounded in $\Omega \times (0, \infty)$.*

The proof follows a "Moser-Alikakos iteration method". The local existence presented in Lemma III.1, the uniqueness and extendibility of classical solutions of system (III.1) are obtained applying the well-known results of Amann [4].

Lemma III.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with regular boundary. Assume that the initial data (u_0, v_0) is nonnegative, satisfying (III.5)-(III.13) such that $0 < u_0 \in C^0(\bar{\Omega})$ and $0 < v_0 \in W^{1,\infty}(\bar{\Omega})$. Then, there exist $T_{max} \in (0, \infty]$ and a unique pair of nonnegative functions (u, v),*

$$u \in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})),$$

$$v \in C(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,s}(\Omega)),$$

for $s > 2$

$$(u, v) \in \left[C^{2+\beta, 1+\frac{\beta}{2}}_{x,t}(\bar{\Omega} \times [0, T_{max}))\right]^2,$$

which is the classical maximal solution of (III.1) on $\Omega \times [0, T_{max})$. Furthermore, or $T_{\max} = \infty$ or, $T_{max} < \infty$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \text{ if } t \nearrow T_{max}.$$

The proof is similar to the proof of [Lemma 3.3, [78]] or [Lemma 1.1, [117]], therefore we only provide a sketch of the proof of Lemma III.1.

Proof. We consider the system (6.2) of [4] where

$$u_1 = u, \quad u_2 = v, \quad \mathcal{A}_1 u = -\Delta u, \quad \mathcal{A}_2(u, v)v = div(u\chi\nabla v),$$

$$f_1(\cdot, t, u, v) = \mu u(1 - u + f), \quad f_2(\cdot, t, u, v) = \Delta v - v + u$$

and

$$\mathcal{B}_1 u = \frac{\partial u}{\partial \nu}, \quad \mathcal{B}_2 v = -u\chi \frac{\partial v}{\partial \nu}.$$

We can rewrite then (III.1) as follows

$$\begin{cases} u_t + \mathcal{A}_1 u + \mathcal{A}_2 v = f_1(\cdot, t, u, v), & x \in \Omega, \quad (0, T_{max}), \\ v_t = f_2(\cdot, t, u, v), & x \in \Omega, \quad (0, T_{max}), \\ \mathcal{B}_1 u + \mathcal{B}_2 v = 0, & x \in \partial\Omega, \quad (0, T_{max}). \end{cases}$$

To end the proof we apply Theorem 6.4 in [4] and consider the maximal interval of existence. Since the result is standard, we omit the details. □

We control the $W^{1,q}$-bounds of v in terms of L^p-norms of u in order to get higher-order regularity of u. For this purpose, we shall utilize the widely known smoothing $L^p - L^q$ properties of the Neumann heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ in Ω, more details can be found in [118].

1.1 Basic a priori bounds for u and v

According to Theorem III.1, in order to prove the global existence of (u, v) over $\Omega \times (0, \infty)$, we establish the uniform boundedness of (u, v) in $L^\infty(\Omega)$. First, we present the estimates of u in $L^p(\Omega)$, and then we give basic properties concerning the total mass of the population and the boundedness assertions of the chemical.

Lemma III.2. *Suppose that (u, v) is the solution to (III.1), then, under assumptions (III.5),(III.6) and (III.13), the solution (u, v) satisfies*

$$u, v \geq 0$$

$$\int_\Omega u(x,t)dx \leq c_1 := c_1(u_0, ||f||_{L^\infty(\Omega)}, \mu, |\Omega|), \quad \forall t \in [0, T_{max}), \tag{III.16}$$

$$\int_\Omega v(x,t)dx \leq c_2 := c_2(||u||_{L^1(\Omega)}), \quad \forall t \in [0, T_{max}), \tag{III.17}$$

$$\int_t^{t+t_0} \int_\Omega u^2(x,t)dxds \leq \max\{1, t_0\} c_3, \quad \forall t \in [0, T_{max} - t_0), \tag{III.18}$$

$$\int_\Omega |\nabla v(x,t)|^2 dx \leq c_4, \quad \forall t \in [0, T_{max}), \tag{III.19}$$

$$\int_t^{t+t_0} \int_\Omega |\nabla v(x,t)|^2 dxds \leq \max\{1, t_0\} c_4, \quad \forall t \in [0, T_{max} - t_0), \tag{III.20}$$

$$\int_t^{t+t_0} \int_\Omega |\Delta v(x,t)|^2 dxds \leq \max\{1, t_0\} c_5, \quad \forall t \in [0, T_{max} - t_0), \tag{III.21}$$

with $t_0 = \min\{1, \frac{1}{6} T_{max}\}$.

Proof. The nonnegativity of u and v follows from the Maximum Principle. (III.16) is obtained, as in the previous chapter, directly by integrating in the first equation of (III.1). Relation (III.19) can be obtained by multiplying the second equation of (III.1) by $-\Delta v$ and integrate. Inequality (III.20) follows directly from (III.19). To get (III.17), we integrate the second equation of (III.1) and by means of an ODE comparison arguments together with (III.16) we get (III.17).
Integrating (III.1) over Ω and in time over $(t, t+t_0)$, thanks to (III.16) we have

$$\int_t^{t+t_0} \int_\Omega u^2(x,s)dxds \leq (1 + ||f||_\infty)c_1 t_0 + \frac{t_0 c_1}{\mu} \leq \max\{1, t_0\}c_3,$$

for any $t \in (0, T_{\max} - t_0)$, which implies (III.18). Next, we multiply the second equation of (III.1) by $-\Delta v$, integrate over Ω and apply the Young's inequality to get

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega |\nabla v(x,t)|^2 dx + \int_\Omega |\nabla v(x,t)|^2 dx + \frac{1}{2}\int_\Omega |\Delta v(x,t)|^2 dx \leq \int_\Omega u^2(x,t)dx, \quad \text{(III.22)}$$

$\forall t \in (0, T_{max})$. Integrating over $(t, t+t_0)$ in (III.22) and taking into account the previous estimate, we obtain (III.21) and the proof ends. □

Based on the previous boundedness, there are many common methods to obtain L^2 boundedness of u, as in [90], [108], [121] among others.

Lemma III.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that the initial data satisfy the conditions in Theorem III.1 and (u, v) is the local solution to (III.1) over $\Omega \times (0, T_{max})$. Then, there exists a positive constant $\mathcal{C}_1 = \mathcal{C}_1(u_0, \tau v_0, |\Omega|, \chi, f)$ independent of T_{max} such that*

$$\|u \ln u\|_{L^1(\Omega)} < \mathcal{C}_1, \qquad \forall t \in (0, T_{max}). \quad \text{(III.23)}$$

Proof. We multiply the first equation of (III.1) by $(1 + \ln u)$, integrate by parts and apply the Young's inequality to obtain

$$\int_\Omega (1+\ln u)\left(u_t - \Delta u - \chi\nabla\cdot(u\nabla v)\right)dx =$$
$$= \frac{d}{dt}\int_\Omega u\ln u dx + \int_\Omega \frac{|\nabla u|^2}{u}dx - \chi\int_\Omega u\Delta v dx = \mu\int_\Omega u(1+\ln u)(1-u+f)dx.$$

Taking into account the inequality $-u^2\ln u \leq -u\ln u \leq \frac{1}{2e}$, we get

$$\frac{d}{dt}\int_\Omega u\ln u dx + \int_\Omega \frac{|\nabla u|^2}{u}dx \leq \chi\int_\Omega u\Delta v dx + \mu(1+\|f\|_{L^\infty(\Omega)})\int_\Omega u|1+\ln u|dx + \frac{\mu|\Omega|}{2e},$$

equivalent to

$$\frac{d}{dt}\int_\Omega u\ln u dx + \int_\Omega \frac{|\nabla u|^2}{u}dx \leq \left(\frac{\chi}{2} + \mu(1+\|f\|_{L^\infty(\Omega)})\right)\int_\Omega u^2 dx + \frac{\chi}{2}\int_\Omega |\Delta v|^2 dx$$
$$+ \mu(1+\|f\|_{L^\infty(\Omega)})\int_\Omega u dx + C \quad \text{(III.24)}$$

for all $t \in (0, T_{\max})$, for some $C > 0$. By the Gagliardo-Nirenberg inequality and the boundedness of u in L^1, since $n \leq 2$, there exists a positive constant $c_6 > 0$ verifying

$$\int_\Omega u^2 dx \leq c_2 \int_\Omega |\nabla \sqrt{u}|^2 dx + c_6, \qquad \forall t \in (0, T_{max})$$

and on the other hand it is fulfilled

$$|\nabla \sqrt{u}|^2 = \left(\frac{|\nabla u|}{2\sqrt{u}} \right)^2 = \frac{1}{4} \frac{|\nabla u|^2}{u}.$$

Therefore, by the positivity of u, we can delimit the second term of (III.24) of the form

$$\frac{4}{c_2} \int_\Omega u \ln u dx \leq \int_\Omega \frac{|\nabla u|^2}{u} dx + 4, \qquad \forall t \in (0, T_{max})$$

and in (III.24) we reach

$$\begin{aligned} \frac{d}{dt} \int_\Omega u \ln u dx + \frac{4}{c_2} \int_\Omega u \ln u dx \leq \left(\frac{\chi}{2} + \mu(1 + \|f\|_{L^\infty(\Omega)}) \right) \int_\Omega u^2 dx + \frac{\chi}{2} \int_\Omega |\Delta v|^2 dx \\ + \mu(1 + \|f\|_{L^\infty(\Omega)}) \int_\Omega u dx + c_7, \end{aligned} \tag{III.25}$$

with $c_7 > 0$. Let us denote by $y(t) := \int_\Omega u \ln u dx$ and

$$g(t) := \left(\frac{\chi}{2} + \mu(1 + \|f\|_{L^\infty(\Omega)}) \right) \int_\Omega u^2 dx + \frac{\chi}{2} \int_\Omega |\Delta v|^2 dx + \mu(1 + \|f\|_{L^\infty(\Omega)}) \int_\Omega u dx + c_7.$$

Applying (III.16)–(III.21) and Lemma I.3, the classical theory of ordinary differential inequalities proves (III.23). □

Lemma III.4. *Under assumptions (III.5)-(III.13), we have*

$$\int_\Omega u^2(x,t) dx \leq c_9, \quad \forall t \in (0, T_{max}) \tag{III.26}$$

for some uniform constant c_9 *independent of* t*, i.e.,*

$$c_9 = c_9(u_0, v_0, ||f||_{L^\infty(\Omega)}, \mu, |\Omega|, C_{GN}),$$

where C_{GN} *relates to the Gagliardo-Nirenberg constant.*

Proof. Here, in $2-D$ setting, we use the Gagliardo-Nirenberg interpolation inequality to derive an ordinary differential equation satisfied by $||u||^2_{L^2(\Omega)}$, which enables us to deduce an estimate for $||u||_{L^2(\Omega)}$. This is the key point for us to derive qualitative bounds for $||u||_{L^2(\Omega)}$ and $||v||_{W^{1,\infty}(\Omega)}$, later on. We have from the integration by parts that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx &= - \int_\Omega |\nabla u|^2 dx - \frac{\chi}{2} \int_\Omega u^2 \Delta v dx + \mu \int_\Omega u^2 (1 - u + f) dx \\ &= - \int_\Omega |\nabla u|^2 dx + \frac{\chi}{2} \|u\|^2_{L^3(\Omega)} \|\Delta v\|_{L^3(\Omega)} + \mu \int_\Omega u^2 (1 - u + f) dx. \end{aligned} \tag{III.27}$$

We apply Lemma 3.5 of [77] with $p = 3$: for any $\varepsilon > 0$, there exists a constant $c_{10} > 0$ such that

$$\|u\|_{L^3(\Omega)} \leq \varepsilon \left(\|\nabla u\|^{\frac{2}{3}}_{L^2(\Omega)} \|u \ln u\|^{\frac{1}{3}}_{L^1(\Omega)} \right) + c_{10}(\varepsilon) \left(\| \; u \ln u\|_{L^1(\Omega)} + \|u\|^{\frac{1}{3}}_{L^1(\Omega)} \right).$$

By Lemma III.3 we get

$$\|u\|^2_{L^3(\Omega)} \leq \left(\varepsilon \|\nabla u\|^2_{L^2(\Omega)} + c_{10}(\varepsilon) \right)^{\frac{2}{3}}. \tag{III.28}$$

Thanks to the standard Gagliardo-Nirenberg inequality and Lemma III.3, we have

$$\|\Delta v\|_{L^3(\Omega)} \leq c_{10}(\varepsilon) \left(\|\nabla \Delta v\|^{\frac{2}{3}}_{L^2(\Omega)} \|\nabla v\|^{\frac{1}{3}}_{L^2(\Omega)} + \| \; \nabla v\|_{L^2(\Omega)} \right) \leq c_{10}(\varepsilon) \left(\|\nabla \Delta v\|^{\frac{2}{3}}_{L^2(\Omega)} + 1 \right). \tag{III.29}$$

Combining (III.28) and (III.29) and Young's inequality we obtain

$$||u||^2_{L^3(\Omega)} \|\Delta v\|_{L^3(\Omega)} \leq \frac{2\varepsilon}{3} \|\nabla u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{3} \|\nabla \Delta v\|^2_{L^2(\Omega)} + c_{11}$$

and in (III.27),

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \left(1 - \frac{2\varepsilon}{3}\right) \int_\Omega |\nabla u|^2 dx \\ &\quad \leq \frac{\chi \varepsilon}{6} \int_\Omega |\nabla \Delta v|^2 dx + \mu \int_\Omega u^2 (1 - u + f) dx + c_{12}, \end{aligned} \tag{III.30}$$

where we pick ε smaller than $\min\{3/2\chi, 3/4\}$. We take gradients in the second equation in (III.1) and it results

$$\tau \nabla v_t - \nabla \cdot \Delta v = \nabla u - \nabla v. \tag{III.31}$$

Then, multiplying (III.31) by $-\nabla \Delta v$, and after integration by parts and using the Young's inequality, it results

$$\begin{aligned} &\frac{\tau}{2} \frac{d}{dt} \int_\Omega |\Delta v|^2 dx = \tau \int_\Omega \Delta v \Delta v_t dx = -\tau \int_\Omega \nabla \Delta v \nabla v_t dx \\ &= -\int_\Omega \nabla \Delta v \nabla (\Delta v + u - v) dx \\ &\leq -\frac{1}{2} \int_\Omega |\nabla \Delta v|^2 dx - \int_\Omega |\nabla v|^2 dx + \frac{1}{2} \int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{III.32}$$

Adding (III.30) and (III.32), since $u^2(1 - u + f) \leq -u^2 + c_{13}$, we conclude

$$\frac{d}{dt} \left(\int_\Omega u^2 dx + \frac{\tau}{2} \int_\Omega |\Delta v|^2 dx \right) + \frac{1}{4} \int_\Omega |\nabla \Delta v|^2 dx + \int_\Omega |\Delta v|^2 dx + \int_\Omega u^2 dx \leq c_{14}. \tag{III.33}$$

Relation (III.26) follows from (III.33) thanks to the Gronwall's inequality. □

Lemma III.5. *The u-component of the unique global-in-time classical solution to the model (III.1) satisfies the uniform estimate*

$$||u(t)||_{L^3(\Omega)} \leq c_{15}, \tag{III.34}$$

for all $t \in (0, \infty)$ and for some c_{15} depending on u_0, v_0, μ, $||f||_{L^\infty(\Omega)}$ and Ω.

Proof. Taking into account the uniform L^2-bound of u, it follows from Lemma I.2 and Brezis and Strauss [17] with $d = 2$ and $p = 2$, for any $1 < q < \infty$, that

$$||v(t)||_{W^{1,q}(\Omega)} \leq c_{16}, \tag{III.35}$$

thanks to I.4. We multiply the first equation in (III.1) by u^2 and integrate by parts in Ω

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 dx + 2\int_\Omega u|\nabla u|^2 dx = 2\chi\int_\Omega u^2\nabla u\nabla v dx + \int_\Omega \mu(1+f)u^3 - \mu u^4 dx$$
$$\leq 2\int_\Omega u|\nabla u|^2 dx + \frac{\chi^2}{2}\int_\Omega u^3|\nabla v|^2 dx + \int_\Omega \mu(1+f)u^3 - \mu u^4 dx$$
$$\leq 2\int_\Omega u|\nabla u|^2 dx + \frac{\mu}{2}\int_\Omega u^4 dx + \frac{3^3\chi^8}{2\cdot 4^4\mu^3}\int_\Omega |\nabla v|^8 dx + \int_\Omega \left(\mu(1+f)u^3 - \mu u^4\right) dx$$
$$\leq 2\int_\Omega u|\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega u^4 dx + \frac{3^3\chi^8}{2\cdot 4^4\mu^3}\int_\Omega |\nabla v|^8 dx + \mu(1+\|f\|_{L^\infty(\Omega)})\int_\Omega u^3 dx.$$

Using the algebraic inequality

$$au^3 - bu^4 \leq -\frac{1}{3}u^3 + \frac{3^3(a+\frac{1}{3})^4}{2^8 b^3}$$

with $a = \mu(1+\|f\|_{L^\infty(\Omega)})$ and $b = \mu/2$, the above inequality gives rises to

$$\frac{d}{dt}\int_\Omega u^3 dx + \int_\Omega u^3 dx \leq \frac{3^4\chi^8}{2\cdot 4^4\mu^3}||\nabla v||^8_{L^8(\Omega)} + \frac{3^4\left(\mu(1+||f||_{L^\infty(\Omega)}) + \frac{1}{3}\right)^4}{2^5\mu^3}|\Omega|.$$

Applying the Gronwall's inequality we have

$$||u^3||^3_{L^3} \leq ||u_0^3||^3_{L^3(\Omega)} + \frac{3^4\chi^8}{2\cdot 4^4\mu^3}\sup_{t\in(0,\infty)}||\nabla v||^8_{L^8(\Omega)} + \frac{3^4\left(\mu(1+||f||_{L^\infty(\Omega)}) + \frac{1}{3}\right)^4}{2^5\mu^3}|\Omega|$$

which for $q = 8$ in (III.35) allows us to (III.34). □

Lemma III.6. *The v-component of the solution to (III.1) satisfies the uniform estimate*

$$\|\nabla v\|_{L^\infty(\Omega)} \leq c$$

for some c independent of t.

Proof. The explicit $W^{1,\infty}$-bound of v follows directly from the uniform L^3-estimate of u and Lemma I.2 with $(d,p,q) = (2,3,\infty)$ by the cited result from Brezis and Strauss [17]. □

Lemma III.7. *Under hypothesis (III.5)-(III.6), for each $q \in [2,\infty)$ there exists a positive constant $c_q > 0$, such that the solution (u,v) of (III.1) satisfies:*

$$\|u(\cdot,t)\|_{L^q(\Omega)} \leq c_q. \tag{III.36}$$

Proof. We multiply the first equation of (III.1) by u^{q-1} and integrate by parts

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega u^q dx &= -(q-1)\int_\Omega u^{q-2}|\nabla u|^2 dx \\ &+ \chi(q-1)\int_\Omega u^{q-1}\nabla u\nabla v dx + \mu\int_\Omega u^q(1-u+f)dx.\end{aligned} \tag{III.37}$$

Taking into account that

$$u^{q-2}|\nabla u|^2 = \frac{4}{q^2}|\nabla u^{q/2}|^2,$$

it follows

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega u^q dx \le& -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \frac{2\chi(q-1)}{q}\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\nabla v dx\\ &+ \mu\int_\Omega u^q(1-u+f)dx\\ &= -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \frac{2\chi(q-1)}{p}\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\nabla v dx\\ &- \frac{\mu}{2}\int_\Omega u^{q+1}dx + c_{17},\end{aligned} \tag{III.38}$$

where $c_{17} = C(q)$ such that $\mu u^q(1-u+f) \le -\mu u^{q+1} + \mu(1+||f||_{L^\infty(\Omega)})u^q \le -\frac{\mu}{2}u^{q+1} + c_{17}$. Now, the following holds

$$\begin{aligned}\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\nabla v dx \le& \|u^{\frac{q}{2}}\|_{L^{\frac{2(q+1)}{q}}(\Omega)}\|\nabla u^{\frac{q}{2}}\|_{L^2(\Omega)}\|\nabla v\|_{L^{2(q+1)}(\Omega)}\\ &\le \frac{2}{q\chi}\|\nabla u^{\frac{q}{2}}\|^2_{L^2(\Omega)} + c_{18}\|u\|^q_{L^{q+1}(\Omega)}.\end{aligned} \tag{III.39}$$

Using (III.39) in (III.38) we obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q dx \le c_{19}\|u\|^q_{L^{q+1}(\Omega)} - \frac{\mu}{2}\|u\|^{q+1}_{L^{q+1}(\Omega)} + c_{17}. \tag{III.40}$$

By setting $y_p(t) := \int_\Omega u^q dx$, we have by the Hölder inequality

$$y_q'(t) \le -c_{20}y_q^{\frac{q+1}{q}}(t) + c_{21}, \quad y_q(0) = \|u_0\|^q_{L^q(\Omega)}.$$

By solving the last inequality we get $y_q(t) \le c_q$, $\forall t \in (0,\infty)$. □

In order to prove the $L^\infty(\Omega)$ boundedness of u we have the following result.

Lemma III.8. *Under hypothesis (III.5)-(III.6), there exists a positive constant $C > 0$, independent of t, such that the solution of (III.1) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \forall t > 0. \tag{III.41}$$

Proof. We multiply by $q-1$ the first equation of (III.1)

$$
\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega u^q dx \le & -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \frac{2\chi(q-1)}{q}\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\nabla v dx \\
& + \mu(1+\|f\|_{L^\infty(\Omega)})\int_\Omega u^q dx - \mu\int_\Omega u^{q+1}dx \\
& \le -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \frac{2\chi(q-1)C_{\|\nabla v\|_\infty}}{q}\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}} dx \\
& + \mu(1+\|f\|_{L^\infty(\Omega)})\int_\Omega u^q dx - \mu\int_\Omega u^{q+1}dx \\
& \le -\frac{4(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \frac{\chi_1(q-1)}{q}\int_\Omega \left(\frac{q\chi_1}{2}u^q + \frac{2}{q\chi_1}|\nabla u^{\frac{q}{2}}|^2\right)dx \\
& + \mu_1\int_\Omega u^q dx - \mu\int_\Omega u^{q+1}dx \\
& \le -\frac{2(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \left(\frac{\chi_1^2(q-1)}{2} + \mu_1\right)\int_\Omega u^q dx - \mu\int_\Omega u^{q+1}dx,
\end{aligned}
\tag{III.42}
$$

where $\chi_1 := \chi C(\|\nabla v\|_{L^\infty(\Omega)})$ and $\mu_1 = \mu(1+\|f\|_{L^\infty(\Omega)})$.
Now, by using the Gagliardo-Nirenberg together with the Young's inequality, we have that

$$
\|u^{\frac{q}{2}}\|^2_{L^2(\Omega)} \le \varepsilon\|\nabla u^{\frac{q}{2}}\|^2_{L^2(\Omega)} + c_0(1+\varepsilon^{-1/2})\|u^{\frac{q}{2}}\|^2_{L^1(\Omega)}. \tag{III.43}
$$

We choose

$$
\varepsilon = \frac{2(q-1)}{q^2((q-1)\chi_1^2 + 2\mu_1)} \tag{III.44}
$$

such that

$$
\frac{q-1}{q^2\varepsilon} = \frac{(q-1)\chi_1^2}{2} + \mu_1. \tag{III.45}
$$

Notice that we also have

$$
\begin{aligned}
\left(\frac{(q-1)\chi_1^2}{2} + \mu_1\right)\int_\Omega u^q dx = & \frac{2(q-1)}{q^2\varepsilon}\int_\Omega u^q dx - \left(\frac{(q-1)\chi_1^2}{2} + \mu_1\right)\int_\Omega u^q dx \\
\le & \frac{2(q-1)}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 dx + \left(\frac{2(q-1)c_0(1+\varepsilon^{-\frac{1}{2}})}{q^2\varepsilon}\right)\left(\int_\Omega u^{\frac{q}{2}}dx\right)^2 \\
& - \left(\frac{(q-1)\chi_1^2}{2} + \mu_1\right)\int_\Omega u^q dx.
\end{aligned}
\tag{III.46}
$$

By using (III.46) in (III.42) we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^q dx \le & -\left(\frac{q(q-1)\chi_1^2}{2} + \mu_1 q\right)\int_\Omega u^q dx \\
& + \frac{2(q-1)c_0(1+\varepsilon^{-1/2})}{q^2\varepsilon}\left(\int_\Omega u^{\frac{q}{2}}dx\right)^2 - \mu q\int_\Omega u^{q+1}dx.
\end{aligned}
\tag{III.47}
$$

Now we pick q_0 large such that $q \ge q_0$,

$$
\varepsilon = \frac{2(q-1)}{q^2((q-1)\chi_1^2 + 2\mu_1)} > \left(\frac{1}{q\chi_1}\right)^2
$$

and

$$\frac{1+\varepsilon^{-1/2}}{\varepsilon} < \frac{1+q\chi_1}{(1/q^2\chi_1^2)},$$

then

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^q dx \leq & -\left(\frac{q(q-1)\chi_1^2}{2}+\mu q\right)\int_\Omega u^q dx \\ & + 2q(q-1)c_0\chi_1^2(1+q\chi_1)\left(\int_\Omega u^{\frac{q}{2}}dx\right)^2 - \mu q\int_\Omega u^{q+1}dx.\end{aligned} \tag{III.48}$$

Let us denote $m := \frac{q(q-1)\chi_1^2}{2} + q\mu_1$, $\forall q \geq q_0$. Then, after integration in (III.48) we have

$$\begin{aligned}\int_\Omega u^q dx \leq & e^{-mt}\int_\Omega u_0^q dx + 4c_0(1+q\chi_1)\int_0^t e^{-m(t-s)}\left(\int_\Omega u^{\frac{q}{2}}dx\right)^2 ds \\ & \leq e^{-mt}\int_\Omega u_0^q dx + 4c_0(1+q\chi_1)\sup_{t\in(0,T)}\left(\int_\Omega u^{\frac{q}{2}}\right)^2.\end{aligned} \tag{III.49}$$

We define $M(q) = \max\{\|u_0\|_{L^\infty(\Omega)}, \sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^q(\Omega)}\}$. Therefore, it verifies

$$M(q) \leq (c_{22}+c_{22}q\chi_1)^{1/q}M(q/2), \quad \forall q \geq q_0, \tag{III.50}$$

where c_{22} is positive and depending on c_0 and Ω. We set $q = 2^i$, $q_0 = 2^{i_0}$ (where i_0 is large fixed). Then (III.50) reads

$$\begin{aligned}M(2^k) & \leq (c_{22}+c_{22}\chi_1 2^k)^{2^{-k}}M(2^{k-1}) \\ & \leq (c_{22}+c_{22}\chi_1 2^k)^{2^{-k}}(c_{22}+c_{22}\chi_1 2^{k-1})^{2^{-(k-1)}}M(2^{k-2}) \\ & \leq M(2^{k_0})\prod_{i=k_0+1}^{k}(c_{22}+c_{22}\chi_1 2^i)^{2^{-i}} \\ & \leq M(2^{k_0})\prod_{i=k_0+1}^{k}(c_{22}2^i+c_{22}\chi_1 2^i)^{2^{-i}} \\ & \leq M(2^{k_0})(c_{22}+c_{22}\chi_1)\prod_{i=k_0+1}^{k}(2^i)^{2^{-i}} \\ & \leq M(2^{k_0})(c_{22}+c_{22}\chi_1)2^{\sum_{i=k_0+1}^{k} i2^{-i}} \\ & \leq c(1+\chi_1)M(2^{k_0}), \quad \forall t\in(0,\infty),\end{aligned} \tag{III.51}$$

where $c = c(\Omega, u_0)$ and $M(2^{k_0})$ is bounded by the previous lemma. For $k\to\infty$ in the last inequality, since $\sum_{i=k_0}^{\infty} i2^{-i} \leq C$ and thanks to Lemma III.7, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \lim_{k\to\infty} M(2^k) \leq CM(2^{k_0}),\ \forall t\geq 0, \tag{III.52}$$

which completes the proof of Lemma III.8. □

2 Asymptotic behavior

In this section we address our study to the asymptotic behavior of the solutions of the problem (III.1). We obtain that such solution converges to a homogeneous in space and periodic in time function u^* defined in (III.2) which satisfies the equation (III.3)

$$\frac{du^*}{dt} = \mu u^*(1 - u^* + f^*).$$

Recall that we have assumed $|\Omega| = 1$. The result is enclosed in the following theorem.

Theorem III.2. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $\tau > 0$ and $\chi \in \mathbb{R}$. Then, for all $\mu > 0$, for any nonnegative u_0 and v_0 fulfilling assumptions (III.5)–(III.13), the solution (u, v) to problem (III.1) fulfills*

$$\|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)} \to 0, \qquad \text{as } t \to \infty, \tag{III.53}$$

where v^* is the solution to

$$\tau \frac{dv^*}{dt} = u^* - v^*, \qquad v(0) = \int_\Omega v_0(x).$$

As stated in the introduction, we divide the proof into two steps. We start with the convergence of (u, v) to $(\tilde{u}, \tilde{v})$.

Lemma III.9. *For $u \in L^\infty(\Omega_\infty)$ and $|\nabla v| \in L^\infty(0, T_{max} : L^2(\Omega))$, there exists a positive constant $c_{\frac{2}{3}} > 0$ such that*

$$\int_\Omega u dx \geq c_{\frac{2}{3}}.$$

Proof. We proceed as in Mizukami-Yokota [[74] Lemma 4.2.], we first take β satisfying

$$0 < \beta < \min\{1, \alpha, 2(m - 1)\} \tag{III.54}$$

and consider the energy

$$\int_\Omega u^{-\beta} dx. \tag{III.55}$$

We shall prove that (III.55) has an upper bound for any $t > 0$.

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^{-\beta} dx = -\beta \int_\Omega u^{-\beta-1} u_t dx \\
&= -\beta \int_\Omega u^{-\beta-1}\left(\Delta u - div\chi(u\nabla v) + \mu u(1 - u + f)\right) dx \\
&= -\beta(\beta + 1)\int_\Omega u^{-\beta-2}\left(|\nabla u|^2 - \chi u \nabla v \nabla u\right) dx - \beta\mu \int_\Omega u^{-\beta}(1 - u + f) dx.
\end{aligned}$$

By applying the Young's inequality to the term involving cross gradients, we have:

$$\begin{aligned}
\chi\beta(\beta + 1)\int_\Omega u^{-\beta-2}\nabla v \nabla u dx \leq{}& \beta(\beta + 1)\int_\Omega u^{-2-\beta}|\nabla u|^2 dx \\
&+ \frac{1}{4}\beta(\beta + 1)\chi^2 \int_\Omega u^{-2-\beta}|\nabla v|^2 dx.
\end{aligned}$$

In view of (III.54), the uniform boundedness of u in $L^\infty(\Omega)$ and, as a consequence the boundedness of ∇v in $L^2(\Omega)$ (see Lemma I.2) we arrive at

$$\begin{aligned}\chi\beta(\beta+1)\int_\Omega u^{-\beta-2}\nabla v\nabla u dx &\le \beta(\beta+1)\int_\Omega u^{-2-\beta}|\nabla u|^2dx \\ &\quad + \frac{1}{4}\beta(\beta+1)\chi^2\|u\|_{L^\infty(\Omega)}^{-2-\beta}\|v\|_{H^1(\Omega)}^2 \\ &\le \beta(\beta+1)\int_\Omega u^{-2-\beta}|\nabla u|^2dx + c\end{aligned}$$

and

$$\beta\mu\int_\Omega u^{1-\beta}dx \le \beta\mu\|u\|_{L^\infty}^{1-\beta} \le c.$$

Therefore we get

$$\frac{d}{dt}\int_\Omega u^{-\beta}dx + \beta\mu\int_\Omega u^{-\beta}dx \le c, \tag{III.56}$$

and solving this differential inequality, we can obtain

$$\int_\Omega u^{-\beta}dx \le c_\beta. \tag{III.57}$$

Since

$$1 = |\Omega| = \int_\Omega \frac{u^{\frac{\beta}{2}}}{u^{\frac{\beta}{2}}}dx,$$

note by the Hölder inequality that

$$\int_\Omega \frac{u^{\frac{\beta}{2}}}{u^{\frac{\beta}{2}}} \le \left[\int_\Omega u^\beta dx\right]^{\frac{1}{2}}\left[\int_\Omega \frac{1}{u^\beta}dx\right]^{\frac{1}{2}},$$

After some computations, we get

$$c_\beta^{-1} \le \int_\Omega u^\beta dx \le \left|\int_\Omega u dx\right|^\beta.$$

Thus, there is a positive lower bound for $\int_\Omega u$ as in (IV.22), with

$$\varepsilon_1 = c_\beta^{-\frac{1}{\beta}}.$$

□

Lemma III.10. *Let u^* be the periodic solution to (III.3) of period T, then, there exists $\epsilon_2 > 0$ such that*

$$u^* > \epsilon_2.$$

Proof. We divide by u^* in (III.3) and integrate over $(0,T)$ to obtain the equation

$$\frac{1}{\mu}\frac{u_t^*}{u^*} + u^* = 1 + f^* \ge \epsilon_1.$$

We define $\epsilon_2 := \frac{1}{2}\min\{u_0^*, \epsilon_1\}$ and proceed by contradiction. Suppose that there exists t_0 such that $u^*(t_0) = \epsilon_2$ and $u^*(t) > \epsilon_2$ for all $t \in (0, t_0)$ and

$$u_t^*(t_0) \le 0. \tag{III.58}$$

We replace in the equation to obtain

$$\frac{1}{\mu}\frac{u_t^*(t_0)}{\epsilon_2} + \epsilon_2 \geq \epsilon_1,$$

equivalent to

$$\frac{1}{\mu}\frac{u_t^*(t_0)}{\epsilon_2} \geq \epsilon_1 - \epsilon_2 > 0,$$

which contradicts (III.58) and the proof ends. □

We now define

$$k_1(t) := \int_\Omega \left(u - \int_\Omega u dx\right)^2 dx \tag{III.59}$$

which is clearly a positive function.

Lemma III.11. *Under assumptions of Theorem III.2 we have*

$$\int_0^\infty k_1(t)dt \leq c < \infty, \tag{III.60}$$

with a positive constant c.

Proof. We integrate the first equation of (III.1) over Ω to obtain

$$\begin{aligned}\frac{1}{\mu}\frac{d}{dt}\int_\Omega u dx &= \int_\Omega u(1+f-u)dx = \\ &= \int_\Omega \left(u - \int_\Omega u dx\right)(1+f-u)dx + \int_\Omega u dx\left(1 + \int_\Omega f dx - \int_\Omega u dx\right) = \\ &= -\int_\Omega \left(u - \int_\Omega u dx\right)\left(\int_\Omega u dx - u\right)dx + \int_\Omega \left(u - \int_\Omega u dx\right)(f-f^*)dx \\ &+ \int_\Omega u\left(1 + f^* - \int_\Omega u dx\right)dx + \int_\Omega u\left(\int_\Omega (f-f^*)dx\right)dx.\end{aligned}$$

Since f and f^* are uniformly bounded, we have

$$\int_\Omega \left(u - \int_\Omega u dx\right)(f-f^*)dx \leq \delta k_1 + c(\delta)\|f-f^*\|_{L^1(\Omega)},$$

for $\delta > 0$ satisfying

$$(1-\delta)\mu > \frac{\chi^2}{16}\max\left\{\int_\Omega u_0 dx, \frac{1}{\mu}(1+\|f\|_{L^\infty(\Omega)})\right\}.$$

Therefore, it yields

$$\frac{1}{\mu}\frac{d}{dt}\int_\Omega u dx \leq -(1-\delta)k_1 + \int_\Omega u\left(1+f^* - \int_\Omega u dx\right)dx + c(\delta)\|f-f^*\|_{L^1(\Omega)}.$$

By dividing the last inequality by $\int_\Omega u dx$, we arrive to

$$\frac{d}{dt}\ln\left(\int_\Omega u dx\right) \leq \mu\left[-\frac{(1-\delta)k_1}{\int_\Omega u dx} + 1 + f^* - \int_\Omega u dx\right] + \frac{\mu c(\delta)}{\int_\Omega u dx}\|f - f^*\|_{L^1(\Omega)}. \quad \text{(III.61)}$$

In the same fashion, we divide equation (III.3) by u^* to obtain

$$\frac{d}{dt}(\ln u^*) = \mu(1 + f^* - u^*). \quad \text{(III.62)}$$

Now, we subtract (III.62) to (III.61)

$$\frac{d}{dt}\left(\ln\left(\int_\Omega u dx\right) - \ln u^*\right) \leq \mu\left[-\frac{(1-\delta)k_1}{\int_\Omega u dx} + u^* - \int_\Omega u dx\right] + \frac{\mu c(\delta)}{\int_\Omega u dx}\|f - f^*\|_{L^1(\Omega)}. \quad \text{(III.63)}$$

For the sake of simplicity, let us consider the following functions

$$F_1 := \int_\Omega \frac{u}{u^*} dx - 1 + \ln u^* - \int_\Omega \ln u dx; \quad F_2 := \ln\left(\int_\Omega u dx\right) - \ln u^*. \quad \text{(III.64)}$$

Such functionals present a similar form of that previously used in several works on related chemotaxis problems, e.g., in [6].
Notice that $h : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ defined by

$$h(s) := s - 1 - \ln s$$

satisfies $h(s) \geq 0$ for any $s > 0$, and $\lim_{s\to 0^+} h(s) = +\infty$. Since

$$F_1 = \int_\Omega h\left(\frac{u}{u^*}\right) dx$$

we have that $F_1 \geq 0$ and thanks to Lemma III.9,

$$F_2 \geq -c > -\infty, \qquad \text{for any } t > 0.$$

Then, the following inequality hold

$$\begin{aligned}
\frac{d}{dt}F_1 &= \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right) + \mu(1 + f^* - u^*) - \int_\Omega \frac{u_t}{u} dx \\
&= \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right) + \mu(1 + f^* - u^*) \\
&+ \int_\Omega \left[-\frac{|\nabla u|^2}{u^2} + \chi\frac{\nabla u \nabla v}{u} - \mu(1 + f - u)\right] dx \\
&\leq \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right) + \mu\|f^* - f\|_{L^1(\Omega)} + \mu\left(\int_\Omega u dx - u^*\right) + \frac{\chi^2}{4}\int_\Omega |\nabla v|^2 \\
&\leq \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right) + \mu\|f^* - f\|_{L^1(\Omega)} + \mu\left(\int_\Omega u dx - u^*\right) \\
&+ \frac{\chi^2}{4}\left[-\frac{d}{dt}\left(\frac{\int_\Omega (v - \int_\Omega v dx)^2 dx}{2}\right) + \frac{\int_\Omega (u - \int_\Omega u dx)^2 dx}{4}\right]
\end{aligned}$$

and

$$\frac{d}{dt}F_2 \leq \mu\left[-\frac{(1-\delta)k_1}{\int_\Omega udx} - \int_\Omega udx\right] + \frac{\mu c(\delta)}{\int_\Omega udx} + \mu u^* \|f - f^*\|_{L^1(\Omega)}.$$

Therefore, we have,

$$\begin{aligned}\frac{d}{dt}(F_1+F_2) \leq & -\frac{\mu(1-\delta)k_1}{\int_\Omega udx} + c\|f-f^*\|_{L^1(\Omega)} \\ & + \frac{\chi^2}{16}k_1 + \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) - \frac{\chi^2}{8}\frac{d}{dt}\left(\frac{\int_\Omega (v-\int_\Omega vdx)^2 dx}{2}\right).\end{aligned} \tag{III.65}$$

Now, since $F_1 \geq 0$, and $F_2 \geq -c$, after integration over $(0,\rho)$ we obtain

$$\left[\frac{\mu(1-\delta)}{sup_{t\in(0,\rho)}\{\int_\Omega udx\}} - \frac{\chi^2}{16}\right]\int_0^\rho k_1 dt \leq \left[\frac{\int_\Omega udx}{u^*}\right]_0^\rho + c\mu\int_0^\rho \|f-f^*\|_{L^1(\Omega)}dt + c_0 \leq c.$$

Thanks to assumptions (III.14), election of δ and Lemma III.2, taking limits as $\rho \to \infty$, we arrive to

$$\int_0^\infty k_1 dt \leq c < \infty \tag{III.66}$$

and the proof ends. □

Lemma III.12. *Under assumptions of Theorem III.2, there exists a positive constant $c < \infty$ such that*

$$\int_0^\infty \int_\Omega |\Delta v|^2 + |\nabla v|^2 dxdt \leq c.$$

Proof. First we notice that, after integration in the second equation in (III.1), it yields

$$\int_\Omega udx = \int_\Omega vdx + \tau\int_\Omega v_t dx.$$

Then, we have that

$$\tau v_t - \Delta v + (v - \int_\Omega vdx) = u - \int_\Omega udx + \tau\int_\Omega v_t dx.$$

We multiply by $-\Delta v$ and integrate over Ω the previous equation, taking into account the following identities

$$-\int_\Omega \left(v_t - \int_\Omega v_t\right)\Delta v dx = \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla v|^2 dx;$$

$$-\int_\Omega \left[\Delta v(v - \int_\Omega vdx)\right]dx = \int_\Omega |\nabla v|^2 dx$$

and

$$-\int_\Omega \left[\Delta v(u - \int_\Omega udx)\right]dx \leq \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\int_\Omega \left|u - \int_\Omega u\right|^2 dx$$

we get the inequality

$$\frac{d}{dt}\frac{\tau}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla v|^2 dx \leq \int_\Omega \left|u - \int_\Omega u\right|^2 dx.$$

After integration over $(0,\infty)$ and using Lemma III.11 we obtain the wished result. □

Lemma III.13. *Under assumptions of Theorem III.2, there exists a positive constant $c < \infty$, such that the following inequality holds*

$$\int_0^\infty \int_\Omega |\nabla u|^2 dxdt \leq c.$$

Proof. As in Lemma III.11, we derivate $F_1 + F_2$ to obtain

$$\begin{aligned}\frac{d}{dt}(F_1 + F_2) &\leq \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \int_\Omega \left[-\frac{|\nabla u|^2}{u^2} + \chi\frac{\nabla u \nabla v}{u}\right] dx \\ &\leq \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} dx + \frac{\chi^2}{2}\int_\Omega |\nabla v|^2 dx.\end{aligned}$$

After integration, in view of Lemma III.12, we arrive to

$$\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} dxdt \leq c < \infty.$$

In view of the boundedness of u we have

$$\int_0^\infty \int_\Omega |\nabla u|^2 dxdt \leq \|u\|^2_{L^\infty(\Omega)} \int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} dxdt \leq c < \infty$$

and the proof ends. □

Lemma III.14. *We assume that the hypotheses of Theorem III.2 are fulfilled. There exists a positive constant $c < \infty$, independent of t such that the following holds*

$$\int_\Omega |\nabla u|^2 dx \leq c.$$

Proof. We multiply the first equation (III.1) by $-\Delta u$ and integrate by parts to obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\Delta u|^2 dx = \chi \int_\Omega \Delta u[\nabla u \nabla v + u\Delta v]dx - \mu \int_\Omega \Delta u u(1 + f - u)dx.$$

Since ∇v is uniformly bounded by (I.3), we have

$$\chi \int_\Omega \Delta u[\nabla u \nabla v + u\Delta v]dx \leq \frac{1}{2}\int_\Omega |\Delta u|^2 dx + c_9 \int_\Omega |\nabla u|^2 dx + c_9 \int_\Omega |\Delta v|^2 dx$$

and

$$-\mu \int_\Omega u(1+f-u)\Delta u dx \leq \frac{c_{10}}{2}\int_\Omega |\nabla u|^2 dx + c_{10}\int_\Omega \nabla u \nabla f dx \leq c_{10}\int_\Omega |\nabla u|^2 dx + \frac{c_{10}}{2}\int_\Omega |\nabla f|^2 dx,$$

with c_9 and c_{10} positive constants independent of t. Then,

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\Delta u|^2 dx \leq (c_9 + c_{10})\int_\Omega |\nabla u|^2 dx + \frac{c_{10}}{2}\int_\Omega |\nabla f|^2 dx + c_9 \int_\Omega |\Delta v|^2 dx$$

After integration and having in mind assumption (III.12) and the previous lemmas, we get the result. □

Lemma III.15. *Let k_1 be defined in (III.59), then, under assumptions (III.5)-(III.14), there exists a positive constant $\tilde{c}_2 < \infty$ such that*

$$|k_1'| \leq \tilde{c}_2, \qquad \textit{for } t > 0.$$

Proof. In view of Lemma III.1 we have that $k_1 \in C^1(0,\infty)$. Then, we get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \left(u - \int_\Omega udx\right)^2 dx = \int_\Omega u_t \left(u - \int_\Omega udx\right) dx,$$

$$\int_\Omega u_t \left(u - \int_\Omega udx\right) dx = -\int_\Omega |\nabla u|^2 dx + \chi \int_\Omega u \nabla u \nabla v dx$$
$$+ \int_\Omega \left(u - \int_\Omega udx\right) u(1 + f - u)dx.$$

Then, by applying the Young inequality we have

$$\left| -\int_\Omega |\nabla u|^2 dx + \chi \int_\Omega u\nabla u \nabla v dx \right| \leq \int_\Omega |\nabla u|^2 dx + c\|u\|^2_{L^\infty(\Omega_\infty)} \int_\Omega |\nabla v|^2 dx.$$

Boundedness of u, assumption (III.11), Lemma III.1 and Lemma III.14 imply the result.

□

As a consequence of Lemma III.11 and Lemma III.15 we obtain the asymptotic behavior of the solutions by applying Lemma I.7.

Proof of Theorem III.2. We consider k_1 defined in (III.59), then, thanks to Lemma III.15 we have that the function $k_1 \in C^{1+\alpha}$, for some $\alpha \in (0,1)$. Due to Lemma III.11 and Lemma III.15 we obtain that

$$\|u - \int_\Omega udx\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty. \tag{III.67}$$

Now, we define k_2 as follows

$$k_2(t) := \left(\int_\Omega udx - u^*\right)^2$$

and consider $F_2(t)$ defined in (III.64) by

$$F_2(t) = \ln \int_\Omega udx - \ln u^*.$$

So, by derivation we get

$$\frac{d}{dt}F_2 = \frac{\int_\Omega u_t dx}{\int_\Omega udx} - \frac{u_t^*}{u^*},$$

which implies

$$\frac{d}{dt}F_2 = \mu\left(\frac{\int (uf)dx}{\int udx} - \frac{\int_\Omega u^2}{\int_\Omega udx} + u^* - f^*\right),$$
$$= \mu\left(\frac{\int (u(f - f^*)dx}{\int udx} - \frac{\int_\Omega u(u - u^*)}{\int_\Omega udx}\right)$$

and

$$\frac{d}{dt}F_2 + \frac{\mu u^*}{\int_\Omega udx}\left(\int_\Omega udx - u^*\right) = \mu\frac{\int (u(f-f^*)dx}{\int udx} - \frac{\mu k_1}{\int_\Omega udx}.$$

We multiply by F_2 the previous inequality to obtain

$$\frac{d}{dt}\frac{1}{2}F_2^2 + \frac{\mu u^*}{\int_\Omega udx}\left(\int_\Omega udx - u^*\right)F_2 \leq c|F_2|k_1(t) + c\|f-f^*\|_{L^1(\Omega)}. \tag{III.68}$$

Applying the Mean Value Theorem we claim

$$\left(\int_\Omega udx - u^*\right)F_2 = \xi F_2^2,$$

for some $\xi \in \left[u^*, \int_\Omega udx\right]$ if $u^* < \int_\Omega udx$, or $\xi \in \left[\int_\Omega udx, u^*\right]$, otherwise. As a consequence of Lemma III.9 and Lemma III.10, we get

$$\frac{\mu u^*}{\int_\Omega udx}\left(\int_\Omega udx - u^*\right)F_2 \geq cF_2^2$$

for some positive constant c. After integration in (III.68), it results

$$\int_0^\infty F_2^2 dt \leq c < \infty.$$

Notice that, due to Lemma III.2, we have

$$k_2 \leq cF_2^2,$$

for some positive constant c and it implies with the previous bound of F_2^2 that

$$\int_0^\infty k_2 dt \leq c < \infty. \tag{III.69}$$

In view of Lemma III.2, (III.11) and (III.2), we have that

$$\left|\int_\Omega udx\right| < c, \quad |u^*| < |u_0^*|+1+\|f\|_{L^\infty(\Omega_\infty)}, \quad |u_t^*| \leq \mu(|u_0^*|+1+\|f\|_{L^\infty(\Omega_\infty)})(1+\|f\|_{L^\infty(\Omega_\infty)})$$

and

$$\left|\int_\Omega u_t dx\right| \leq \mu\|u\|_{L^\infty(\Omega_\infty)}((1+\|f\|_{L^\infty(\Omega_\infty)} + \|u\|_{L^\infty(\Omega_\infty)}).$$

Since

$$k_2' = 2\left(\int_\Omega udx - u^*\right)\left(\int_\Omega u_t dx - u_t^*\right),$$

where

$$\int_\Omega u_t dx = \mu\int_\Omega u(1-u+f) < c$$

and

$$u_t^* = \mu u^*(1-u^*+f^*) < c,$$

we have that

$$|k_2'| \leq c < \infty. \tag{III.70}$$

As before, we have, due to (III.69) and (III.70), that

$$k_2 \to 0 \quad \text{as} \quad t \to \infty. \tag{III.71}$$

Since

$$\int_\Omega |u - u^*|^2 dx \le k_1 + k_2,$$

by relations (III.67) and (III.71), we get

$$\|u - u^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty.$$

To obtain

$$\|v - v^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty,$$

we proceed as before and we define

$$k_3 := \int_\Omega |v - v^*|^2 dx.$$

We take squares in both sides of the equation

$$\tau \frac{d}{dt}(v - v^*) - \Delta v + (v - v^*) = u - u^*$$

and integrate over Ω,

$$\int_\Omega \left| \tau \frac{d}{dt}(v - v^*) \right|^2 dx + \frac{d}{dt}\tau \left[\frac{1}{2} \int_\Omega |v - v^*|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx \right]$$
$$+ \int_\Omega |\Delta v|^2 \, dx + \int_\Omega |\nabla (v - v^*)|^2 \, dx + \int_\Omega |(v - v^*)|^2 \, dx \le \int_\Omega |u - u^*|^2 dx.$$

After integration in time we claim

$$\int_0^\infty \int_\Omega \left| \tau \frac{d}{dt}(v - v^*) \right|^2 dx dt + \frac{\tau}{2} k_3 + \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \int_0^\infty k_3 dt \le c,$$

i.e.,

$$\int_0^\infty k_3 dt \le c. \tag{III.72}$$

We multiply by $-\Delta v_t$ and integrate over $\Omega \times (0, \infty)$ the second equation of (III.1) to obtain

$$\frac{\tau}{2} \int_0^\infty \int_\Omega |\nabla v_t| dx dt + \int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla v|^2 dx \le 2 \int_0^\infty \int_\Omega |\nabla u|^2 dx dt + c(u_0) < c.$$

In the same fashion it yields

$$\int_\Omega \left| \tau \frac{d}{dt}(v - v^*) \right|^2 dx \le c \int_\Omega |\Delta v|^2 \, dx + c \int_\Omega |v - v^*|^2 \, dx + c \int_\Omega |u - u^*|^2 dx \le c$$

and therefore

$$|k_3'| \le \frac{1}{2} \int_\Omega \left| \tau \frac{d}{dt}(v - v^*) \right|^2 dx + \frac{1}{2} \int_\Omega |(v - v^*)|^2 \, dx \le c. \tag{III.73}$$

Relations (III.72), (III.73) and Lemma I.7 end the proof. □

Chapter IV

Nonlocal model

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain connected open set whose boundary is regular enough. In this chapter we will be concerned with the following Keller-Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u^m \nabla v) + u(a_0 - a_1 u^\alpha + a_2 \displaystyle\int_\Omega u^\alpha dx), & (x,t) \in \Omega_T, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u^\gamma, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{IV.1}$$

where α, m, $\chi, \gamma, a_i \in \mathbb{R}$ and $\Omega_T = \Omega \times (0,T)$. We consider a coupled system of two PDEs with nonlinear chemotaxis coefficient "χu^m" and logistic growth in the biological population including a nonlocal interaction in the form

$$u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx), \quad \text{for } \alpha \geq 1$$

and $a_i \in \mathbb{R}$. This logistic growth describes the local competition of the individuals of the species for the resources of the environment and the global cooperation to survive. In particular, the coefficient "a_0", sometimes also called *Malthusian parameter*, induces an exponential growth for low density populations if $a_0 > 0$. If $a_1 > 0$, the mechanism that limits the growth of biological species u is given by the term $-a_1 u^\alpha$ and it generalizes the most frequent case $\alpha = 1$. At the time that the population grows, the competitive effect of the local term $a_1 u^\alpha$ becomes more influential. We also consider a non-local term in the logistic source as "$a_2 \int_\Omega u^\alpha dx$" describing the effects of the total mass of the species in the growth of the population. If $a_2 < 0$ there is a non-local competition among the individuals of the species and if $a_2 > 0$, individuals cooperate globally to survive although they compete locally. In the last case, the effects of $a_1 u^\alpha$ and $a_2 \int_\Omega u^\alpha$ balance the system.

The non-linear nature of the chemotaxis term has been studied in the literature by different authors, as we can found in Horstmann [52] and references therein. The exponent m indicates nonlinearities with respect to u in the tactic sensitivity functions; intuitively, there is a reinforcement of movement in direction of ∇v where the population u is greater than 1 and weaker if $u < 1$. The growth term with $\alpha \geq 1$ induces a negative feedback that slows growth as populations approach their maximum size and a stronger intra-specific

concurrence (via exponents of the involved density).
In the present study, we work under the following assumptions:

- Parameter $\alpha \in \mathbb{R}$ is such that
$$\alpha \geq 1. \tag{IV.2}$$

- The exponent of the tactic sensitivity function fulfills
$$m > 1. \tag{IV.3}$$

- The following relation among the parameters of the problem holds
$$\alpha + 1 > m + \gamma. \tag{IV.4}$$

- The uniform boundedness of the the term $\int_\Omega u^\alpha dx$ is proved provided
$$a_1 > a_2|\Omega|. \tag{IV.5}$$

- The initial data u_0, v_0 satisfy
$$u_0, v_0 \in W^{2,q}(\Omega), \quad \text{for any } q < \infty, \tag{IV.6}$$
$$\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{IV.7}$$

In this chapter we study, under these assumptions, the properties of (IV.1), that is, the existence of global classical bounded solutions and their asymptotic behavior. The main result of this chapter concerns the asymptotic behavior of the solutions of the system. By a straightforward computation, one observes that system (IV.1) possesses the constant steady state given by (u^*, v^*), where
$$u^* = \frac{a_0^{\frac{1}{\alpha}}}{(a_1 - a_2)^{\frac{1}{\alpha}}}, \qquad v^* = (u^*)^\gamma. \tag{IV.8}$$

Our asymptotic stability result is enclosed in the following theorem.

Theorem IV.1. *Let assumptions (IV.2)-(IV.7) hold. Then, the solution of (IV.1) has the following asymptotic behavior*
$$\|u(\cdot,t) - u^*\|_{L^2(\Omega)} + \|v(\cdot,t) - (u^*)^\gamma\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

The chapter is organized as follows: in Section 2 we obtain existence and global boundedness of solutions by proving some necessary estimates. Section 3 is devoted to the proof of asymptotic behaviour of the solutions, the result is presented in Theorem IV.1.

1 Global Existence of Solutions

We start our global existence discussion with a local existence result.

1.1 Local existence

Lemma IV.1. *Let $\alpha, a_1 > 0$, $a_0, a_2, m, \gamma \in \mathbb{R}$. Then, for any nonnegative initial data (u_0, v_0) fulfilling $u_0 \in C^{\omega}(\bar{\Omega})$ $(0 < \omega < 1)$, $v_0 \in W^{1,\infty}(\bar{\Omega})$, system (IV.1) possesses a classical solution (u, v) from $C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap C(\bar{\Omega} \times [0, T_{max}))$ with the maximal existence time $T_{max} \in (0, \infty]$, and $u > 0$ on $\Omega \times (0, T_{max})$, i.e.,*

$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + t = \infty.$$

This basic result on local existence and uniqueness of classical solutions is obtained applying the well-known results of Amann [4] and it is guaranteed by the following lemma. The proof is similar to the proof in [Lemma 3.3, [78]] or the previous chapters, therefore we omit the details.

1.2 Estimates

Before we state our result on global existence and boundedness of solutions, we need the following estimates. First, a bound for the total mass of the solution is done.

Lemma IV.2. *Under assumptions (IV.2) and (IV.5), the total mass of the solution of (IV.1) is bounded, i.e.,*

$$\int_\Omega u dx \leq C, \tag{IV.9}$$

for some positive constant C independent of t.

Proof. By integrating over Ω the first equation of (IV.1), one gets

$$\frac{d}{dt}\int_\Omega u dx = a_0 \int_\Omega u dx - a_1 \int_\Omega u^{\alpha+1} dx + a_2 \int_\Omega u \int_\Omega u^\alpha dx dx. \tag{IV.10}$$

Note that

$$\frac{d}{dt}\int_\Omega u dx \leq a_0 \int_\Omega u dx - a_1 \int_\Omega u^{\alpha+1} dx + (a_2)_+ \int_\Omega u \int_\Omega u^\alpha dx dx, \tag{IV.11}$$

where

$$(a_2)_+ = \begin{cases} a_2 & \text{if } a_2 \geq 0, \\ 0 & \text{if } a_2 < 0. \end{cases} \tag{IV.12}$$

Then, thanks to the Hölder inequality we get

$$\int_\Omega u dx \int_\Omega u^\alpha dx \leq \left(\int_\Omega u^{\alpha+1} dx\right)^{1/\alpha+1} \left(\int_\Omega u^{\alpha+1} dx\right)^{\alpha/\alpha+1} |\Omega|^{\frac{\alpha+1}{\alpha+1}}, \tag{IV.13}$$

therefore

$$(a_2)_+ \int_\Omega u dx \int_\Omega u^\alpha dx \leq (a_2)_+ |\Omega| \int_\Omega u^{\alpha+1} dx.$$

Hence,

$$\begin{aligned} \frac{d}{dt}\int_\Omega u dx &\leq a_0 \int_\Omega u dx - [a_1 - (a_2)_+|\Omega|] \int_\Omega u^{\alpha+1} dx \\ &\leq a_0 \int_\Omega u dx - [a_1 - (a_2)_+|\Omega|] \left(\int_\Omega u dx\right)^{\alpha+1} \frac{1}{|\Omega|^\alpha}, \end{aligned} \tag{IV.14}$$

where the last inequality holds by the Hölder's inequality, assuming $a_1 > (a_2)_+|\Omega|$. The results follows by solving the differential inequality. □

Lemma IV.3. *Let $\alpha, a_1 > 0$, $a_0, a_2, m, \gamma \in \mathbb{R}$. Under the assumption $m + \gamma < \alpha + 1$, then for any initial data satisfying (IV.7) and any $p < \infty$, the solution to system (IV.1) satisfies*

$$\|u\|_{L^p(\Omega)} \leq c, \quad \textit{for any } t > 0$$

for some positive constant c independent of t.

Proof. We multiply the first equation of (IV.1) by u^{p-1}, for $p \geq 1$, and integrate over Ω:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx - \chi\int_\Omega u^{p-1}\nabla(u^m\nabla v)dx \\ &+ a_0\int_\Omega u^p dx - a_1\int_\Omega u^{p+\alpha}dx + a_2\int_\Omega u^p dx\int_\Omega u^\alpha dx \\ &\leq -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx - \chi\int_\Omega u^{p-1}\nabla(u^m\nabla v)dx \\ &+ (a_0 + a_2\int_\Omega u^\alpha dx)\int_\Omega u^p dx - a_1\int_\Omega u^{p+\alpha}dx.\end{aligned} \tag{IV.15}$$

We consider now the term

$$\begin{aligned}-\chi\int_\Omega u^{p-1}\nabla(u^m\nabla v)dx &= \frac{\chi m}{p+m-1}\int_\Omega u^{p+m-1}\Delta v dx \\ &\leq \epsilon\int_\Omega u^{p+\alpha}dx + A\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx,\end{aligned}$$

for

$$\epsilon = \frac{a_1 - a_2|\Omega|}{5}$$

and

$$A = \frac{\alpha-m+1}{p+m-1}\left(\frac{p+\alpha}{p+m-1}\right)^{-\frac{p+\alpha}{\alpha-m+1}}\left(\frac{a_1-a_2|\Omega|}{5}\right)^{-\frac{p+m-1}{\alpha-m+1}}\left(\frac{\chi m}{p+m-1}\right)^{\frac{p+\alpha}{\alpha-m+1}}, \tag{IV.16}$$

which is bounded for any p. The terms

$$I_1 := a_0\int_\Omega u^p dx, \quad \text{and} \quad I_2 := a_2\int_\Omega u^\alpha dx\int_\Omega u^p dx$$

are splitted as follows

$$I_1 \leq \epsilon\int_\Omega u^{p+\alpha}dx + c(\epsilon,p) - \frac{1}{p}\int_\Omega u^p dx,$$

where

$$c(\epsilon,p) := c = \frac{|\Omega|^{\frac{p+\alpha}{\alpha}}}{(\epsilon\frac{p+\alpha}{p})^{\frac{p}{\alpha}}\frac{p+\alpha}{\alpha}}, \tag{IV.17}$$

and

$$I_2 \leq a_2|\Omega|\int_\Omega u^{p+\alpha}dx$$

by applying the Young's inequality. Then, (IV.15) becomes

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{p}\int_\Omega u^p dx \le -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx \\ + A\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx - (a_1 - a_2|\Omega| - 2\epsilon)\int_\Omega u^{p+\alpha}dx + c,\end{aligned} \tag{IV.18}$$

which can be written equivalently as

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{p}\int_\Omega u^p dx + (a_1 - a_2|\Omega| - 2\epsilon)\int_\Omega u^{p+\alpha}dx \\ \le -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + A\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx + c.\end{aligned} \tag{IV.19}$$

Let us define

$$B := \min\{1, (a_1 - a_2|\Omega| - 2\epsilon)\},$$

and multiply by e^t to get

$$\frac{1}{p}\frac{d}{dt}\left[e^t\int_\Omega u^p dx\right] + Be^t\int_\Omega u^{p+\alpha}dx \le e^t A\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx + e^t c.$$

After integration in $(0,T)$ we arrive to

$$\begin{aligned}\frac{1}{p}\left[e^T\int_\Omega u^p dx - \int_\Omega u_0^p dx\right] + B\int_0^T e^t\int_\Omega u^{p+\alpha}dx \\ \le A\int_0^T e^t\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx + e^T c.\end{aligned} \tag{IV.20}$$

Thanks to Corollary I.3 it results

$$A\left[\int_0^T e^t\int_\Omega |\Delta v|^{\frac{p+\alpha}{\alpha-m+1}}dx \le A'\int_0^T e^t\int_\Omega u^{(p+\alpha)\frac{\gamma}{\alpha-m+1}}dx + \|v_0\|_{W^{2,p}(\Omega)}\right],$$

where we call $A' := c_{cor}A$, with c_{cor} being the constant of the statement of Corollary I.3 and A given by (IV.16). In view of assumption (IV.4) $\frac{\gamma}{\alpha-m+1} < 1$ and the Young's inequality give us

$$A'\int_0^T e^t\int_\Omega u^{(p+\alpha)\frac{\gamma}{\alpha-m+1}}dx \le Ke^T + B\int_0^T e^t\int_\Omega u^{p+\alpha}dx,$$

with

$$K = \frac{(A'|\Omega|)^{\frac{\alpha-m+1}{\alpha+1-m-\gamma}}}{(B\frac{\alpha-m+1}{\gamma})^{\frac{\gamma}{\alpha+1-m-\gamma}}\frac{\alpha-m+1}{\alpha+1-m-\gamma}}. \tag{IV.21}$$

Then, (IV.20) yields

$$\frac{1}{p}\left[e^T\int_\Omega u^p dx - \int_\Omega u_0^p dx\right] \le (c+K)e^T.$$

We multiply by e^{-T} to get

$$\int_\Omega u^p dx \le c(p, u_0),$$

where

$$c(p, u_0) = p(c + K) + e^{-T} \int_\Omega u_0^p dx,$$

for c defined in (IV.17) and K defined in (IV.21). □

Lemma IV.4. *Under assumptions (IV.2)-(IV.7), the solution to system (IV.1) satisfies*

$$\|u\|_{L^\infty(\Omega)} \le c, \quad \text{for any } t > 0$$

for some positive constant c independent of t.

Proof. Since

$$\lim_{p\to\infty} c^{\frac{1}{p}}(p, u_0) < c < \infty$$

we conclude the uniform bound of u in $L^\infty(\Omega)$ in $(0, T)$. In view of the independence of c and K respect to T we take limits when $T \to \infty$ to end the proof.

□

Theorem IV.2 (Global existence of solutions)**.** *Let $\alpha, a_1 > 0$, $a_0, a_2, m, \gamma \in \mathbb{R}$. Under the assumption $m+\gamma < \alpha+1$, then for any initial data satisfying assumptions (IV.2)-(IV.7), the solution to system (IV.1) satisfies*

$$\|u\|_{L^\infty(\Omega)} \le c, \quad \text{for any } t > 0$$

for some positive constant c independent of t.

Proof. As a consequence of Lemmas IV.1 and IV.4 we obtain the global existence of classical bounded solutions. □

2 Asymptotic behavior

In this section we present the proof of Theorem IV.1, i.e. the convergence of the solution of (IV.1) to the constant steady state

$$(u^*, v^*) = \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2|\Omega|}}, \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2|\Omega|}} \right)^\gamma \right).$$

The proof is divided into two steps: first of all we get the convergence of the solution (u, v) to its average,

$$\left(\frac{1}{|\Omega|} \int_\Omega u dx, \frac{1}{|\Omega|} \int_\Omega v dx \right),$$

to obtain later the convergence of the average to the constant (u^*, v^*) defined in (IV.8). One of the key points in the proof of the main result is to obtain a lower bound for the average of u. The result is embedded in the following lemma.

Lemma IV.5. *Let u be a solution to (IV.1). Assume $a_2 > 0$. Under the assumptions of Theorem IV.1, there exists a positive constant $\varepsilon_1 > 0$ independent of t, such that the following inequality holds:*

$$\int_\Omega u dx \geq \varepsilon_1, \quad \text{for any } t > 0. \tag{IV.22}$$

The proof is identical to the the proof of Lemma III.9 and presented in the previous chapter.
Let us now consider the nonnegative functions k_1 and k_2

$$k_1(t) := \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right)^2 dx, \tag{IV.23}$$

$$k_2(t) := \left(\frac{1}{|\Omega|}\int_\Omega u dx - u^* \right)^2. \tag{IV.24}$$

We first introduce the following estimate.

Lemma IV.6. *Under assumptions of Theorem IV.1 we have*

$$\int_0^\infty k_1(t)dt + \int_0^\infty k_2(t)dt \leq c < \infty, \tag{IV.25}$$

with a positive constant c.

Proof. We integrate the first equation of (IV.1) over Ω to obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u dx &= \int_\Omega u \left[a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx \right] dx \\ &= \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right) \left[a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx \right] dx \\ &+ \frac{1}{|\Omega|}\int_\Omega u dx \left(a_0|\Omega| - (a_1 - a_2|\Omega|) \int_\Omega u^\alpha dx \right). \end{aligned} \tag{IV.26}$$

Since

$$\int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right) dx = 0,$$

we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u dx &= -a_1 \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right) u^\alpha dx + \frac{1}{|\Omega|}\int_\Omega u dx \left(a_0|\Omega| - (a_1 - a_2|\Omega|) \int_\Omega u^\alpha dx \right) \\ &= -a_1 \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right) \left(u^\alpha - \left(\frac{1}{|\Omega|}\int_\Omega u dx \right)^\alpha \right) dx \\ &+ \frac{1}{|\Omega|}\int_\Omega u dx \left(a_0|\Omega| - (a_1 - a_2|\Omega|) \int_\Omega u^\alpha dx \right). \end{aligned}$$

We may thus invoke Lemma I.10 to conclude that

$$-a_1 \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right) \left(u^\alpha - \left(\frac{1}{|\Omega|}\int_\Omega u dx \right)^\alpha \right) dx \leq -\epsilon_2 \alpha a_1 \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx \right)^2 dx.$$

In view of $\alpha \geq 1$, assumption (IV.5) warrants that

$$-(a_1 - a_2|\Omega|) \int_\Omega u^\alpha dx \leq -(a_1 - a_2|\Omega|)|\Omega|^{1-\alpha} \left(\int_\Omega u dx \right)^\alpha .$$

Therefore, in equation (IV.26) we can estimate

$$\begin{aligned} \frac{d}{dt} \int_\Omega u dx \leq & - \epsilon_2 \alpha a_1 \int_\Omega \left(u - \frac{1}{|\Omega|} \int_\Omega u dx \right)^2 dx \\ & + \frac{1}{|\Omega|} \int_\Omega u dx \left(a_0 |\Omega| - (a_1 - a_2|\Omega|)|\Omega|^{1-\alpha} \left(\int_\Omega u dx \right)^\alpha \right) . \end{aligned}$$

We now only need to observe that

$$a_0|\Omega| - (a_1 - a_2|\Omega|)|\Omega|^{1-\alpha} \left(\int_\Omega u dx \right)^\alpha = |\Omega|(a_1 - a_2|\Omega|) \left((u^*)^\alpha - \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) .$$

We have, then,

$$\begin{aligned} & \int_\Omega \frac{u}{|\Omega|} dx |\Omega|(a_1 - a_2|\Omega|) \left((u^*)^\alpha - \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) \\ & = -|\Omega|(a_1 - a_2|\Omega|) \int_\Omega \left(u^* - \frac{u}{|\Omega|} \right) dx \left((u^*)^\alpha - \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) \\ & + |\Omega|(a_1 - a_2|\Omega|) \int_\Omega u^* dx \left((u^*)^\alpha - \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) \end{aligned} \tag{IV.27}$$

thus, we infer by Lemma I.10

$$\begin{aligned} & \int_\Omega u dx \left(a_0 - (a_1 - a_2|\Omega|) \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) \\ & \leq -(a_1 - a_2|\Omega|) \epsilon_2 \alpha \left(\frac{1}{|\Omega|} \int_\Omega u dx - u^* \right)^2 \\ & + |\Omega|^2 (a_1 - a_2|\Omega|) u^* \left((u^*)^\alpha - \left(\frac{1}{|\Omega|} \int_\Omega u dx \right)^\alpha \right) \end{aligned} \tag{IV.28}$$

which, by means of a straightforward computation, implies

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{|\Omega|} \int_\Omega u dx - u^* \right) + \phi(t) \left(\frac{1}{|\Omega|} \int_\Omega u dx - u^* \right) \\ & + \epsilon_2 a_1 \alpha \int_\Omega \left(u - \frac{1}{|\Omega|} \int_\Omega u dx \right)^2 dx + (a_1 - a_2|\Omega|) \epsilon_2 \alpha \left(\frac{1}{|\Omega|} \int_\Omega u dx - u^* \right)^2 \leq 0, \end{aligned} \tag{IV.29}$$

for

$$\phi(t) = \alpha \xi^{\alpha - 1}$$

where

$$\begin{cases} \xi \in \left(u^*, |\Omega|^{-1} \int_\Omega u dx \right), & \text{if } u^* \leq |\Omega|^{-1} \int_\Omega u dx, \\ \xi \in \left(|\Omega|^{-1} \int_\Omega u dx, u^* \right), & \text{otherwise.} \end{cases}$$

We denote by

$$\Phi(t) = \int_0^t \phi(s)ds,$$

and hence

$$\frac{d}{dt}e^{\Phi(t)}\left(\frac{1}{|\Omega|}\int_\Omega udx - u^*\right) + \epsilon_2 a_1 \alpha e^{\Phi(t)}\int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega udx\right)^2 dx \\ + (a_1 - a_2)\epsilon_2 \alpha e^{\Phi(t)}\left(\frac{1}{|\Omega|}\int_\Omega udx - u^*\right)^2 \leq 0. \tag{IV.30}$$

After an integration in time, since $e^{\Phi(t)} \geq 1$, we obtain

$$\int_0^\infty k_1(t)dt < c, \qquad \int_0^\infty k_2(t)dt < c, \tag{IV.31}$$

which ends the proof. □

To end the proof of Theorem IV.1, we need some boundary properties for the derivative of the solution (u, v). Similar results have been obtained in [87] for parabolic-ODE chemotactic systems.

Lemma IV.7. *Under assumptions of Theorem IV.1, there exists a positive constant* $c_{10} < \infty$ *such that*

$$\int_0^\infty \int_\Omega \left(|\Delta v|^2 + |\nabla v|^2\right) dxdt \leq c_{10}.$$

Proof. An integration over Ω of the second equation in (IV.1) shows that

$$\int_\Omega u^\gamma dx = \int_\Omega vdx - \int_\Omega v_t dx.$$

Moreover, we have

$$v_t - \Delta v + (v - \int_\Omega vdx) = u^\gamma - \int_\Omega u^\gamma dx + \int_\Omega v_t dx.$$

We multiply by $-\Delta v$ and integrate over Ω the previous equation, taking into account the following identities

$$-\int_\Omega \left(v_t - \int_\Omega v_t\right)\Delta vdx = \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla v|^2 dx;$$

$$-\int_\Omega \left[\Delta v(v - \int_\Omega vdx)\right] dx = \int_\Omega |\nabla v|^2 dx$$

and

$$-\int_\Omega \left[\Delta v(u^\gamma - \int_\Omega u^\gamma dx)\right] dx \leq \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\int_\Omega \left|u^\gamma - \int_\Omega u^\gamma\right|^2 dx \\ \leq \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\gamma^2 {\xi_1}^{2(\gamma-1)}\int_\Omega \left|u - \int_\Omega u\right|^2 dx,$$

with $\xi_1 \in (\int_\Omega u, \|u\|_{L^\infty(\Omega)})$. After substituting the above inequalities, we get

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla v|^2 dx \leq \frac{1}{2}\gamma^2 {\xi_1}^{2(\gamma-1)}\int_\Omega \left|u - \int_\Omega u\right|^2 dx.$$

We integrate over $(0, \infty)$ and thanks to Lemma IV.6 to conclude the proof. □

Proposition 3.4 in [29] gives the Schauder estimate of ∇u, the arguments for which are standard in order to obtain a time-dependent bound for ∇u in $L^\infty(\Omega)$. Since the non-local term $\int_\Omega u^\alpha$ in (IV.1) is uniformly bounded, we can use this result. We omit the demonstration for not bringing anything else to the point. The result as we need is as follows:

Lemma IV.8. *We assume that the hypotheses of Theorem IV.1 are fulfilled. There exists a positive constant $c_{11} < \infty$, independent of t, such that*

$$\int_\Omega |\nabla u|^2 dx \leq c_{11}.$$

Proof. We multiply the first equation (IV.1) by $-\Delta u$ and integrate by parts to obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\Delta u|^2 dx =&\chi \int_\Omega \Delta u[mu^{m-1}\nabla u \nabla v + u^m \Delta v]dx \\ &- \int_\Omega \Delta u u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha)dx.\end{aligned}$$

Since ∇v is uniformly bounded, we have

$$\begin{aligned}\chi \int_\Omega \Delta u[mu^{m-1}\nabla u\nabla v + u^m \Delta v]dxdx \leq& \varepsilon m\|u\|^{m-1}_{L^\infty(\Omega)} \int_\Omega |\Delta u|^2 dx \\ &+ c_\varepsilon \int_\Omega |\nabla u|^2 dx + c_\varepsilon \int_\Omega |\Delta v|^2 dx\end{aligned}$$

and

$$-\int_\Omega u\Delta u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha) \leq a_0 \int_\Omega |\nabla u|^2 dx - a_1(\alpha+1)\int_\Omega |\nabla u|^2 u^\alpha dx + a_2 \int_\Omega u^\alpha |\nabla u|^2 dx.$$

Then, for $\varepsilon < 1/(m\|u\|^{m-1}_{L^\infty(\Omega)})$, after integration and previous lemmas we get the result. □

Lemma IV.9. *Let k_i, $i = 1, 2$ be defined in (IV.23) and (IV.24), then, under assumptions (IV.2)-(IV.5), there exists two positive constants $\tilde{c}_1 < \infty$, $\tilde{c}_2 < \infty$ such that*

$$|k_1'| \leq \tilde{c}_1, \qquad \text{for } t > 0 \tag{IV.32}$$

and

$$|k_2'| \leq \tilde{c}_2, \qquad \text{for } t > 0. \tag{IV.33}$$

Proof. We now only need to observe that as a consequence of Lemmas IV.1 and IV.2 we have that $k_i \in C^1(0, \infty)$. An explicit integration of (IV.1) shows that we can find an estimate as follows

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx\right)^2 dx &= \int_\Omega u_t \left(u - \frac{1}{|\Omega|}\int_\Omega u dx\right) dx \\ = -\int_\Omega |\nabla u|^2 dx + \chi \int_\Omega u^m \nabla u \nabla v dx& \\ + \int_\Omega \left(u - \frac{1}{|\Omega|}\int_\Omega u dx\right) u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx)dx.&\end{aligned} \tag{IV.34}$$

Here, the Young's inequality warrants that

$$\begin{aligned}&\left|-\int_\Omega |\nabla u|^2 dx + \chi \int_\Omega u^m \nabla u \nabla v dx\right| \\ &\le \int_\Omega |\nabla u|^2 dx + c\|u\|^{2m}_{L^\infty(\Omega_\infty)} \int_\Omega |\nabla v|^2 dx.\end{aligned} \tag{IV.35}$$

Boundedness of u and the previous lemmas imply (IV.32). In the same fashion we prove (IV.33). □

Taking into account Lemma IV.6 and Lemma IV.9 we obtain the asymptotic behavior of the solutions by applying I.7.

Proof of Theorem IV.1 (Continuation). To obtain

$$\|v - v^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty,$$

we proceed as before (see also [87] for similar proof) and we define

$$k_3 := \int_\Omega |v - v^*|^2 dx.$$

We take squares in both sides of the equation

$$\frac{d}{dt}(v - v^*) - \Delta v + (v - v^*) = u^\gamma - (u^*)^\gamma$$

and integrate over Ω

$$\begin{aligned}&\int_\Omega \left|\frac{d}{dt}(v - v^*)\right|^2 dx + \frac{d}{dt}\left[\frac{1}{2}\int_\Omega |v - v^*|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx\right] \\ &+ \int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla(v - v^*)|^2 dx + \int_\Omega |(v - v^*)|^2 dx \\ &\le \int_\Omega |u^\gamma - (u^*)^\gamma|^2 dx.\end{aligned}$$

After integration in time we claim

$$\int_0^\infty \int_\Omega \left|\frac{d}{dt}(v - v^*)\right|^2 dxdt + \frac{\tau}{2}k_3 + \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_0^\infty k_3 dt \le c,$$

i.e.,

$$\int_0^\infty k_3 dt \le c. \tag{IV.36}$$

We multiply now the second equation in (IV.1) by $-\Delta v_t$ and integrate over $\Omega \times (0, \infty)$ to obtain

$$\frac{1}{2}\int_0^\infty \int_\Omega |\nabla v_t| dxdt + \int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla v|^2 dx \le 2\int_0^\infty \int_\Omega |\nabla u|^2 dxdt + c(u_0) < c.$$

In the same fashion it yields

$$\int_\Omega \left|\frac{d}{dt}(v - v^*)\right|^2 dx \le c\int_\Omega |\Delta v|^2 dx + c\int_\Omega |v - v^*|^2 dx + c\int_\Omega |u^\gamma - (u^*)^\gamma|^2 dx \le c$$

and therefore

$$|k_3'| \leq \frac{1}{2}\int_\Omega \left|\frac{d}{dt}(v - v^*)\right|^2 dx + \frac{1}{2}\int_\Omega |(v - v^*)|^2\, dx \leq c. \qquad \text{(IV.37)}$$

Relations (IV.36), (IV.37) and Lemma I.7 end the proof. □

Our last result concerns the exponential decay of the solutions to 0 in a non-local competitive scenario.

Theorem IV.3. *Consider $a_1 > 0$ and assume $a_0 < 0$. Then, if $a_2 < 0$ the solution to (IV.1) satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq c_1 e^{-c_2 t}, \qquad \text{(IV.38)}$$

where $c_1, c_2 > 0$.

Proof. Integrating in (IV.1) we have that

$$\frac{d}{dt}\int_\Omega u dx = a_0 \int_\Omega u dx - a_1 \int_\Omega u^{\alpha+1} dx + a_2 \int_\Omega u dx \int_\Omega u^\alpha dx. \qquad \text{(IV.39)}$$

Since $a_2 < 0$ one has

$$\frac{d}{dt}\int_\Omega u dx \leq a_0 \int_\Omega u dx - a_1 \int_\Omega u^{\alpha+1} dx,$$

then,

$$\frac{d}{dt}\int_\Omega u dx \leq a_0 \int_\Omega u dx - a_1 \int_\Omega u^{\alpha+1} dx \leq -c_4 \int_\Omega u dx. \qquad \text{(IV.40)}$$

By the Hölder inequality, we obtain

$$-a_1 \int_\Omega u^\alpha dx \leq -\frac{a_1}{|\Omega|}\left(\int_\Omega u dx\right)^\alpha,$$

and then, by calling $y(t) := \int_\Omega u dx$,

$$y'(t) \leq a_0 y(t) - a_1 |\Omega|^{-1} y^\alpha(t).$$

After integration, we get

$$\|u\|_{L^1} \leq e^{-c_4 t}\|u_0\|_{L^1(\Omega)}.$$

Finally, by the Gagliardo-Nirenberg inequality and the boundedness of ∇u in L^∞-norm,

$$\|u\|_{L^\infty} \leq C_{GN}\left(\|\nabla u\|_{L^\infty}^{\frac{d}{d+1}} \|u\|_{L^1(|\Omega)}^{\frac{1}{d+1}} + \|u\|_{L^1(\Omega)}\right) \leq c_1 e^{-c_2 t}.$$

□

Chapter V

Parabolic-ODE model

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain connected open set whose boundary is regular enough. In this chapter we consider the case when the chemical substance "v" is considered non-diffusive, i.e., once it is secreted by the biological species "u", it is maintained up to degradation. The evolution of "v" is given in terms of a general function "h" satisfying some technical assumptions presented in this chapter.
The problem is posed in a bounded domain $\Omega \subset \mathbb{R}^d$, with regular boundary $\partial\Omega$ and the system of equations is presented as follows

$$\begin{cases} u_t = \Delta u - div(\chi u \nabla v) + \mu u(1 - u + f(x,t)), & x \in \Omega, \ \ t > 0, \\ v_t = h(u,v), & x \in \Omega, \ \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} - u\chi \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0. \end{cases} \tag{V.1}$$

Throughout the chapter we assume, without loss of generality, that $|\Omega| = 1$ and we denote by g the function

$$g(v) := e^{\chi v}. \tag{V.2}$$

We work under the following hypotheses:

1. Function h fulfills

$$h \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap C^2(\mathbb{R}^2_+); \tag{V.3}$$

$$\frac{\partial h}{\partial u} > 0 \quad \text{and} \quad \frac{\partial h}{\partial v} \leq -\epsilon_v < 0; \tag{V.4}$$

$$\frac{\partial h}{\partial v} + u\chi \frac{\partial h}{\partial u} < 0. \tag{V.5}$$

Moreover, there exists a positive constant $c > 0$ such that

$$-h(0,v) \leq c e^{\chi v}, \tag{V.6}$$

$$h(0,0) \geq 0 \tag{V.7}$$

and

$$0 \leq h(0,0) < \varepsilon/\mu\chi, \quad \text{with} \quad \epsilon > 0. \tag{V.8}$$

2. Let us define

$$\epsilon := \frac{\mu}{2\mu + 2\mu\|f\|_{L^\infty(\Omega)} + 2},$$

$$c_4 := \max\left\{\frac{6\|u_0\|_{L^\infty(\Omega)}}{\min\{1,\mu\}}, \frac{3}{\epsilon^2}, \frac{6c\chi}{\mu}, c_3, 1\right\}$$

and

$$c_5 := \max\{3\|u_0\|_{L^\infty(\Omega)}, \frac{3\mu}{2\epsilon^2}, c\chi e c_4\},$$

for c_3 defined in Lemma V.5 Then, we have that

$$\limsup_{s\to\infty} h\left(c_5 e^{\chi s}, s\right) \le -\epsilon_0, \tag{V.9}$$

with $\epsilon_0 > 0$, small enough.

3. There exists a periodic function f^*, of period T, verifying

$$\|f(x,t) - f^*(t)\|_{L^\infty(\Omega)} \to 0, \quad \text{as} \quad t \to \infty, \tag{V.10}$$

$$\inf_{x\in\Omega} f(x,t) < f^*(t) < \sup_{x\in\Omega} f(x,t)$$

and

$$\int_0^\infty \|f(x,t) - f^*(t)\|_{L^1(\Omega)} \le c < \infty.$$

There exists $\varepsilon > 0$ such that

$$-1 + \varepsilon < f(x,t). \tag{V.11}$$

4. The positive initial data (u_0, v_0) of (V.1) satisfy, $(u_0, v_0) \not\equiv (0,0)$ and

$$0 \le u_0 \in L^\infty(\Omega) \cap W^{1,s}(\Omega), \quad \underline{v} \le v_0 \le \overline{v}, \tag{V.12}$$

for some $s > \max\{4, n\}$ and we assume

$$\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{V.13}$$

The above conditions cover the example

$$h(u,v) = ue^{-\chi v} - av,$$

with $a > 0$ and

$$h(u,v) = \frac{ue^{-\chi v} + v}{1+v} - v.$$

Our particular analysis will address the initial-boundary value problem (V.1) in a bounded open domain $\Omega \subset \mathbb{R}^d$, where the initial data are as in (V.12). We shall firstly address the basic issue of global solvability. The result is presented in the following theorem.

Theorem V.1. *Let Ω be a bounded open domain of $\mathbb{R}^d$ with regular boundary and suppose that assumptions (V.3)–(V.9) and (V.12) hold. Then, there exists a unique pair of nonnegative functions (u,v) which forms a global solution*

$$(u,v) \in C([0,\infty), (W^{1,s}(\Omega))^2) \cap C^1((0,\infty), (W^{1,s}(\Omega))' \times W^{1,s}(\Omega))$$

to the problem (V.1) for any initial data $(u_0, v_0) \in (W^{1,s}(\Omega))^2$, with $s > \max\{4, d\}$. Furthermore, the solution is uniformly bounded, i.e.,

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C < \infty.$$

Secondly, we study the asymptotic properties of the solutions. We introduce the function u^* defined in the Introduction, which for the reader's convenience we reproduce here:

$$u^*(t) = \frac{u_0^* e^{\int_0^t \mu(1+f^*(s))ds}}{1 + u_0^* \displaystyle\int_0^t \mu e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}, \tag{V.14}$$

for u_0^* defined by

$$u_0^* := \frac{e^{\int_0^T \mu(1+f^*(s))ds} - 1}{\mu \int_0^T e^{\int_0^\tau \mu(1+f^*(s))ds} d\tau}$$

and f^* in (V.10). Notice that u^* satisfies equation

$$u_t^* = \mu(1 + f^* - u^*), \tag{V.15}$$

and it is homogeneous in space and periodic in time function. We denote by $v^*(t)$, the solution of the ordinary differential equation

$$v_t^* = h(u^*, v^*). \tag{V.16}$$

The asymptotic result is enclosed in the following theorem:

Theorem V.2. *Assume (V.3)-(V.12) and let us denote by (u, v) the corresponding solution of (V.1) from Theorem V.1. Then, (u, v) has the following asymptotic behavior*

$$\|u(x,t) - u^*(t)\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \|v(x,t) - v^*(t)\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t \to \infty,$$

where u^* and v^* are given by (V.14) and (V.16), respectively.

This chapter is organized as follows: in Section 1 we prove the existence of a unique pair of classical solutions. A first key step consists in obtaining a maximal weak solution following [4], and then obtain boundedness of the solution. As a crucial ingredient in our derivation of a L^∞ bound for u we employ a Alikakos-Moser-type iterative procedure. By means of these and some further higher regularity properties will assert the statements on global existence and boundedness of u and v from Theorem V.1. Our collection of estimates of Section 1 will moreover turn out to be sufficient to derive the stabilization result from Theorem V.2 in Section 2 through an analysis into two steps. First, we prove that the solution converge to their respective averages, i.e.,

$$\int_\Omega u dx, \quad \int_\Omega v dx,$$

using energy estimates to conclude that these averages converge to the functions u^* and v^*, respectively.

1 Global existence of solutions

The present section is devoted to the proof of Theorem V.1. We firstly address the basic issue of local existence of the solution.

Lemma V.1. *If the assumptions of Theorem V.1 hold, there exist a maximal time $T_{max} > 0$ and a unique maximal positive solution to (V.1) such that*

$$(u,v) \in C([0,T_{max}),(W^{1,s}(\Omega))^2) \cap C^1((0,T_{max}),(W^{1,s}(\Omega))' \times W^{1,s}(\Omega)),$$

as well as

$$\limsup_{t\to T_{max}} \left(\|u\|_{W^{1,s}(\Omega)} + \|v\|_{W^{1,s}(\Omega)} + t\right) = \infty.$$

Proof. We consider the system (6.2) of [4] where

$$u_1 = u, \quad u_2 = v, \quad \mathcal{A}_1 u = -\Delta u, \quad \mathcal{A}_2(u,v)v = div(u\chi\nabla v),$$

$$f_1(\cdot,t,u,v) = \mu u(1-u+f), \quad f_2(\cdot,t,u,v) = h(u,v)$$

and

$$\mathcal{B}_1 u = \frac{\partial u}{\partial \nu}, \quad \mathcal{B}_2 v = -u\chi\frac{\partial v}{\partial \nu}.$$

We can rewrite then (V.1) as follows

$$\begin{cases} u_t + \mathcal{A}_1 u + \mathcal{A}_2 v = f_1(\cdot,t,u,v), & x\in\Omega, \quad (0,T_{max}),\\ v_t = f_2(\cdot,t,u,v), & x\in\Omega, \quad (0,T_{max}),\\ \mathcal{B}_1 u + \mathcal{B}_2 v = 0, & x\in\partial\Omega, \quad (0,T_{max}). \end{cases}$$

To end the proof we apply Amann [4] Theorem 6.4 and consider maximal interval of existence. Since the result is standard, we omit the details. □

Lemma V.2. *Under the assumptions of Theorem V.1, the solution to (V.1) is nonnegative, i.e.,*

$$u \geq 0, \qquad v \geq 0, \qquad \text{for} \quad t \in (0,T_{max}).$$

Proof. In order to see the non-negativity of u we introduce the following change of variables:

$$u = g(v)\tilde{u}, \qquad \text{for} \quad g(v) = e^{\chi v}. \tag{V.17}$$

Then we can rewrite the first equation in (V.1) as

$$u_t = \tilde{u}_t g(v) + \tilde{u}g'(v)v_t = \tilde{u}_t g(v) + \chi\tilde{u}g(v)h.$$

Now, deriving with respect to the spatial variable in the previous equation we get

$$\nabla u = g(v)\nabla\tilde{u} + \tilde{u}g'(v)\nabla v = g(v)\nabla\tilde{u} + \chi\tilde{u}g(v)\nabla v,$$

$$\Delta u = g(v)\Delta\tilde{u} + 2\chi g(v)\nabla\tilde{u}\nabla v + \chi^2\tilde{u}g(v)|\nabla v|^2 + \chi\tilde{u}g(v)\Delta v,$$

and

$$\nabla(\chi u\nabla v) = \chi g(v)\nabla\tilde{u}\nabla v + \chi^2\tilde{u}g(v)|\nabla v|^2 + \chi\tilde{u}g(v)\Delta v.$$

Then, the first equation of (V.1) becomes

$$g(v)\tilde{u}_t = g(v)\Delta\tilde{u} + \chi g(v)\nabla\tilde{u}\nabla v + \mu g(v)\tilde{u}(1-\tilde{u}g(v)+f) - \chi\tilde{u}g(v)h(\tilde{u}g(v),v).$$

We multiply by $e^{-\chi v}$ to get

$$\tilde{u}_t = \Delta \tilde{u} + \chi \nabla \tilde{u} \nabla v + \mu \tilde{u}(1 - \tilde{u} g(v) + f) - \chi \tilde{u} h(\tilde{u} g(v), v). \quad \text{(V.18)}$$

Notice that the equation for v remains as an ordinary differential equation

$$v_t = h(\tilde{u} g(v), v). \quad \text{(V.19)}$$

So, the original system (V.1) becomes (V.18)-(V.19) together with the initial data

$$\tilde{u}(x,0) = \tilde{u}_0(x) = \frac{u_0(x)}{g(v_0(x))}, \quad v(x,0) = v_0(x),$$

and the Neumann boundary condition

$$\frac{\partial \tilde{u}}{\partial \nu} = 0.$$

Finally, the Maximum Principle for parabolic equations and the regularity of h prove the non-negativity of u, taking into account that

$$[\mu \tilde{u}(1 - \tilde{u} g(v) + f) - \chi \tilde{u} h(\tilde{u} g(v), v)]|_{\tilde{u}=0} = 0.$$

Hypotheses (V.4) and (V.7) on h and the Maximum Principle applied to (V.19) prove

$$0 \leq v.$$

in view of $h(u,0) \geq 0$ for $u \geq 0$. □

Let us first collect some basic properties thereof which in our subsequent analysis will play important roles not only by providing some useful fundamental regularity features, but also by establishing the first quantitative information on large time behavior.

Lemma V.3. *The total mass of the component u of the solution to (V.1) is bounded:*

$$\int_\Omega u \leq \max\left\{(1 + \|f\|_{L^\infty(\Omega_\infty)}), \|u_0\|_{L^1(\Omega)}\right\} := c_1. \quad \text{(V.20)}$$

Proof. After integration over Ω in the first equation of (V.1), it shows that

$$\frac{d}{dt}\int_\Omega u dx \leq \mu(1 + \|f(x,t)\|_{L^\infty(\Omega_\infty)}) \int_\Omega u dx - \mu \int_\Omega u^2 dx. \quad \text{(V.21)}$$

Thanks to the Cauchy-Schwarz inequality, since $|\Omega| = 1$,

$$\left|\int_\Omega u dx\right|^2 \leq \int_\Omega u^2 dx,$$

we directly obtain

$$\frac{d}{dt}\int_\Omega u dx \leq \mu(1 + \|f(x,t)\|_{L^\infty(\Omega_\infty)}) \int_\Omega u dx - \mu \left|\int_\Omega u dx\right|^2. \quad \text{(V.22)}$$

Finally, (V.20) is a consequence of the Maximum Principle applied to (V.22), i.e.,

$$\int_\Omega u dx \leq \max\left\{(1 + \|f\|_{L^\infty(\Omega_\infty)}), \|u_0\|_{L^1(\Omega)}\right\}.$$

□

Lemma V.4. *The solution to (V.1) satisfies*

$$\int_t^{t+t_0}\int_\Omega u^2 dxds \le c_2, \tag{V.23}$$

for all $t \in (0, T_{max} - t_0)$*, where* $t_0 = \min\{1, \frac{1}{2}T_{max}\}$ *and*

$$c_2 := c_1\Bigg((1 + \|f\|_{L^\infty(\Omega_\infty)}) + \frac{1}{\mu}\Bigg).$$

Proof. We integrate (V.21) over the interval $(t, t+t_0)$ for $t_0 = \min\{1, \frac{1}{2}T_{max}\}$ to obtain

$$\int_\Omega u(\cdot, t+t_0)dx - \int_\Omega u(\cdot,t)dx \le \mu(1 + \|f\|_{L^\infty(\Omega_\infty)})\int_t^{t+t_0}\int_\Omega udxds - \mu\int_t^{t+t_0}\int_\Omega u^2dxds,$$

$\forall t \in (0, T_{max} - t_0)$, or, equivalently,

$$\int_t^{t+t_0}\int_\Omega u^2dxds \le \big(1 + \|f\|_{L^\infty(\Omega_\infty)}\big)\int_t^{t+t_0}\int_\Omega udxds + \frac{1}{\mu}\int_\Omega u(\cdot,t)dx.$$

By the previous lemma it follows

$$\begin{aligned}\int_t^{t+t_0}\int_\Omega u^2dxds &\le (1 + \|f\|_{L^\infty(\Omega_\infty)})\int_t^{t+t_0} c_1 ds + \frac{c_1}{\mu} \le (1 + \|f\|_{L^\infty(\Omega_\infty)})c_1 t_0 + \frac{c_1}{\mu}\\ &\le c_1\Bigg((1 + \|f\|_{L^\infty(\Omega_\infty)})t_0 + \frac{1}{\mu}\Bigg).\end{aligned}$$

Finally, since $t_0 \le 1$ we have

$$\int_t^{t+t_0}\int_\Omega u^2dxds \le c_1\Bigg((1 + \|f\|_{L^\infty(\Omega_\infty)}) + \frac{1}{\mu}\Bigg) := c_2$$

thereby completes the proof. □

Lemma V.5. *There exists a positive constant* c_3 *defined by*

$$c_3 := \frac{c_2}{e^{t_0} - 1},$$

for $t_0 := \min\{1, T_{\max}/2\}$ *such that*

$$\int_0^t e^{s-t}\int_\Omega u^2dxds \le c_3.$$

Proof. Notice that

$$\int_0^t e^{s-t}\int_\Omega u^2dxds \le \sum_{n=0}^{N-1}\int_{nt_0}^{(n+1)t_0} e^{s-Nt_0}\int_\Omega u^2dxds + \int_{Nt_0}^t\int_\Omega u^2dxds$$

for some $N \in \mathbb{N}$, such that $Nt_0 < t \le (N+1)t_0$. Then, we get

$$\begin{aligned}\int_0^t e^{s-t}\int_\Omega u^2dxds &\le \sum_{n=0}^{N-1} e^{(n-N+1)t_0}\int_{nt_0}^{(n+1)t_0}\int_\Omega u^2dxds + \int_{Nt_0}^t\int_\Omega u^2dxds\\ &\le c_2\sum_{n=0}^{N} e^{(n-N+1)t_0} \le \frac{c_2}{e^{t_0}-1}.\end{aligned}$$

□

We are now prepared to perform an iterative argument of Alikakos-Moser type in order to derive $L^\infty(\Omega)$ bounds for u and v.
The proof starts with the following lemma.

Lemma V.6. *Let $g(v)$ be defined by (V.17), then, for $p \geq 2$ the following estimate holds*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p g^{1-p}dx \quad &\leq -p(p-1)\int_\Omega \frac{u^{p-2}}{g^{p-3}}\left|\nabla \frac{u}{g}\right|^2 dx \\ &+p\mu\left(1+\|f\|_{L^\infty(\Omega_\infty)}\right)\int_\Omega u^p g^{1-p}dx \\ &-p\mu\int_\Omega u^{p+1}g^{1-p}dx + c(p-1)\chi\int_\Omega u^p g^{2-p}dx,\end{aligned} \tag{V.24}$$

where c is the constant given in assumption (V.6).

Proof. We proceed by induction in p, then, for $p \geq 2$ we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p g^{1-p}dx = p\int_\Omega u^{p-1}u_t g^{1-p}dx + (1-p)\chi\int_\Omega u^p g^{1-p}h dx \\ &= p\int_\Omega u^{p-1}(\Delta u - \chi\nabla(u\nabla v))g^{1-p}dx + p\mu\int_\Omega u^p(1-u+f)g^{1-p}dx \\ &+(1-p)\chi\int_\Omega u^p g^{1-p}h dx.\end{aligned} \tag{V.25}$$

For the first integral in (V.25) we infer that

$$\begin{aligned}&p\int_\Omega u^{p-1}g^{1-p}(\Delta u - \chi\nabla(u\nabla v))dx = -p\int_\Omega \nabla[u^{p-1}g^{1-p}](\nabla u - \chi u\nabla v)dx \\ &= -p\int_\Omega [(p-1)u^{p-2}g^{1-p}\nabla u + \chi(1-p)g^{1-p}u^{p-1}\nabla v](\nabla u - \chi u\nabla v)dx \\ &= -p(p-1)\int_\Omega u^{p-2}g^{1-p}(\nabla u - \chi u\nabla v)^2 dx \\ &= -p(p-1)\int_\Omega u^{p-2}g^{1-p}(e^{\chi v}\nabla(ue^{-\chi v}))^2 dx.\end{aligned}$$

From the expression of the above identity, we deduce

$$p\int_\Omega u^{p-1}(\Delta u - \chi\nabla(u\nabla v))g^{1-p}dx \leq 0. \tag{V.26}$$

We look now at the last term of (V.25). By the Mean Value Theorem and assumption (V.6) we have

$$h(u,v) = \left.\frac{\partial h}{\partial u}\right|_{(\xi,v)} u + h(0,v) \geq -cg(v),$$

then,

$$(1-p)\chi\int_\Omega u^p g^{1-p}h dx \leq (p-1)c\chi\int_\Omega u^p g^{2-p}dx. \tag{V.27}$$

Moreover, for the restant term of (V.25), we have

$$p\mu\int_\Omega u^p g^{1-p}(1-u+f)dx \leq p\mu(1+\|f\|_{L^\infty(\Omega_\infty)})\int_\Omega u^p g^{1-p}dx - p\mu\int_\Omega u^{p+1}g^{1-p}dx. \tag{V.28}$$

We now replace (V.26), (V.27) and (V.28) in (V.25) to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p g^{1-p}dx \quad &\leq -p(p-1)\int_\Omega \frac{u^{p-2}}{g^{p-3}}\left|\nabla \frac{u}{g}\right|^2 dx \\ &+p\mu\left(1+\|f\|_{L^\infty(\Omega_\infty)}\right)\int_\Omega u^p g^{1-p}dx \\ &-p\mu\int_\Omega u^{p+1}g^{1-p}dx + c\chi(p-1)\int_\Omega u^p g^{2-p}dx,\end{aligned}$$

which yields (V.24) and the proof ends. □

Lemma V.7. *Let us consider $p \geq 1$ and $g(v)$ as in (V.17). Let ϵ be a positive constant defined by*

$$\epsilon := \frac{\mu}{2\mu + 2\mu\|f\|_{L^\infty(\Omega)} + 2}, \tag{V.29}$$

then, there exist $c_4 > 0$ and $c_5 > 0$ given by

$$c_4 := \max\left\{\frac{6\|u_0\|_{L^\infty(\Omega)}}{\min\{1,\mu\}}, \frac{3}{\epsilon^2}, \frac{6c\chi}{\mu}, c_3, 1\right\}$$

and

$$c_5 := \max\{3\|u_0\|_{L^\infty(\Omega)}, \frac{3\mu}{2\epsilon^2}, c\chi e c_4\},$$

for c_3 as in Lemma V.5, such that

$$\int_0^t e^{s-t}\int_\Omega u^{p+1}g^{1-p}dxds \leq c_4^p, \tag{V.30}$$

and

$$\int_\Omega u^p g^{1-p}dx \leq c_5^p. \tag{V.31}$$

Proof. For $p = 1$ the result is a consequence of Lemma V.3 and Lemma V.4. For $p \geq 2$ we proceed by induction and assume the result for $p-1$, i.e.,

$$\int_\Omega u^{p-1}g^{2-p}dx \leq c_5^{p-1}, \qquad \int_0^t e^{s-t}\int_\Omega u^p g^{2-p}dx \leq c_4^{p-1}. \tag{V.32}$$

Taking $p \geq 2$, thanks to (V.24), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p g^{1-p}dx \quad &\leq p\mu\left(1+\|f\|_{L^\infty(\Omega_\infty)}\right)\int_\Omega u^p g^{1-p}dx \\ &-p\mu\int_\Omega u^{p+1}g^{1-p}dx + c\chi(p-1)\int_\Omega u^p g^{2-p}dx.\end{aligned} \tag{V.33}$$

We first recall the Young's inequality:

$$\frac{1}{\epsilon}u^p \leq \frac{p}{p+1}u^{p+1} + \frac{1}{(p+1)\epsilon^{p+1}};$$

multiplying it by g^{1-p} we get

$$\frac{1}{\epsilon}u^p g^{1-p} \leq u^{p+1}g^{1-p} + \frac{1}{(p+1)\epsilon^{p+1}}g^{1-p},$$

which is equivalent to

$$-u^{p+1}g^{1-p} \le -\frac{1}{\epsilon}u^p g^{1-p} + \frac{1}{(p+1)\epsilon^{p+1}}g^{1-p}.$$

We integrate in space over Ω, and in view of $g^{1-p} \le 1$, we obtain

$$-\frac{p\mu}{2}\int_\Omega u^{p+1}g^{1-p}dx \le -\frac{p\mu}{2\epsilon}\int_\Omega u^p g^{1-p} + \frac{p\mu}{2(p+1)\epsilon^{p+1}}. \tag{V.34}$$

Thanks to the definition (V.29) of ϵ, we have

$$p\mu\left(1 + \|f\|_{L^\infty(\Omega_\infty)}\right) - \frac{p\mu}{2\epsilon} \le -1.$$

We replace (V.34) into (V.33) to get

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p g^{1-p}dx \quad &\le -\int_\Omega u^p g^{1-p}dx + \frac{\mu}{2\epsilon^{p+1}} - \frac{p\mu}{2}\int_\Omega u^{p+1}g^{1-p}dx \\ &+c\chi(p-1)\int_\Omega u^p g^{2-p}dx. \end{aligned} \tag{V.35}$$

By solving the differential equation (V.35) after integration in time, we obtain

$$\begin{aligned} &\int_\Omega u^p g^{1-p}dx \le e^{-t}\int_\Omega u_0^p g^{1-p}(v_0)dx + \frac{\mu}{2\epsilon^{p+1}} \\ &-\frac{p\mu}{2}\int_0^t e^{s-t}\int_\Omega u^{p+1}g^{1-p}dxds + c\chi(p-1)\int_0^t e^{s-t}\int_\Omega u^p g^{2-p}dxds. \end{aligned} \tag{V.36}$$

Dropping the nonpositive term and making use of a favorable cancellation, it yields

$$\int_\Omega u^p g^{1-p}dx \le \|u_0\|^p_{L^\infty} + \frac{\mu}{2\epsilon^{p+1}} + c\chi(p-1)c_4^{p-1} \tag{V.37}$$

and

$$\int_0^t e^{s-t}\int_\Omega u^{p+1}g^{1-p}dx \le \frac{2}{p\mu}\|u_0\|^p_{L^\infty} + \frac{1}{p\epsilon^{p+1}} + \frac{2c\chi}{\mu}c_4^{p-1}.$$

Then, it results

$$\begin{aligned} \int_0^t e^{s-t}\int_\Omega u^{p+1}g^{1-p}dx \quad &\le 3\max\left\{\frac{2}{p\mu}\|u_0\|^p_{L^\infty}, \frac{1}{p\epsilon^{p+1}}, \frac{2c\chi}{\mu}c_4^{p-1}\right\} \\ &\le \max\left\{\frac{6}{\mu}\|u_0\|^p_{L^\infty}, \frac{3}{\epsilon^{p+1}}, \frac{6c\chi}{\mu}c_4^{p-1}\right\} \\ &\le \left[\max\left\{\tfrac{6\|u_0\|_{L^\infty}}{\min\{1,\mu\}}, \tfrac{3}{\epsilon^2}, c_4, 1\right\}\right]^p. \end{aligned}$$

By definition of c_4 we get

$$\int_0^t e^{s-t}\int_\Omega u^{p+1}g^{1-p}dx \le c_4^p$$

and due to (V.37), the following inequality holds

$$\begin{aligned} \int_\Omega u^p g^{1-p}dx \quad &\le 3\max\{\|u_0\|^p_{L^\infty}, \frac{\mu}{2\epsilon^{p+1}}, c\chi(p-1)c_4^{p-1}\} \\ &\le \max\{3\|u_0\|_{L^\infty}, \frac{3\mu}{2\epsilon^2}, c\chi e c_4\}^p = c_5^p. \end{aligned}$$

□

Lemma V.8. *Under the assumptions of Theorem V.1, we have*

$$\left\| \frac{u}{g} \right\|_{L^\infty(\Omega)} \le c_5, \tag{V.38}$$

where c_5 has been defined in Lemma V.7.

Proof. According to Lemma V.7 we have that

$$\int_\Omega u^p g^{1-p} dx \le c_5^p$$

and therefore

$$\left[\int_\Omega u^p g^{1-p} dx \right]^{\frac{1}{p}} \le c_5.$$

Since

$$\left[\int_\Omega u^p g^{-p} dx \right]^{\frac{1}{p}} \le \left[\int_\Omega u^p g^{1-p} dx \right]^{\frac{1}{p}},$$

we take limits when $p \to \infty$, to obtain (V.38). □

Lemma V.9. *Suppose that (V.4), (V.6) and (V.9) hold. Then there exists a positive constant $\overline{v} < \infty$ such that the solution v of (V.1) satisfies*

$$v(x,t) < \overline{v}(t).$$

Proof. By contradiction, we assume that for any $\overline{v} > \|v_0\|_{L^\infty(\Omega)}$ there exists $t_0 > 0$ such that $v(x, t_0) = \overline{v}$ which is the first t_0 fulfilling this condition. Since by assumption (V.12) $v_0 < \overline{v}$, v must be an increasing function in a neighborhood of t_0. Then, by applying (V.9), we obtain

$$v_t(t_0) = h(u, \overline{v}) = h\left(\frac{u}{g(\overline{v})} g(\overline{v}), \overline{v} \right) - h\left(c_5 g(\overline{v}), \overline{v}\right) + h\left(c_5 g(\overline{v}), \overline{v}\right)$$

then, as h is increasing in the first variable, we have

$$v_t(t_0) \le h\left(c_5 g(\overline{v}), \overline{v}\right).$$

Thanks to assumption (V.9) we have that for $\overline{v}$ large enough

$$v_t(t_0) < 0,$$

which is a contradiction and the proof ends. □

The above results entail the claimed qualitative properties of u:

Lemma V.10. *Under assumptions of Theorem V.1, the solution u is uniformly bounded by*

$$\|u\|_{L^\infty(\Omega)} \le e^{\chi \overline{v}} c_5.$$

Proof. The result is a consequence of Lemma V.9 and Lemma V.8. □

Proof of Theorem V.1.
The global existence of (u, v) over $\Omega \times (0, \infty)$ is a direct consequence of the local existence and the uniform boundedness of (u, v) in L^∞ established in the previous lemmas. □

2 Asymptotic behavior

The main propose of this section is to prove Theorem V.2, i.e., to obtain the convergence of the solution (u, v) to (u^*, v^*). The proof is divided into two steps: first of all we get the convergence of the solution u to its average $\int_\Omega u$, to obtain later the convergence of the average to the periodic function u^* given by (V.14). For it, we need to prove the boundedness of $|\nabla v|$ in $L^2(\Omega)$. The result is enclosed in the following lemma.

Lemma V.11. *Suppose that the assumptions of Theorem V.2 hold. Then, there exists $c_7 > 0$, independent of t, such that*

$$\int_\Omega |\nabla v|^2 dx \le c_7,$$

where v is the solution of (V.1).

Proof. We consider equation (V.24), for $p = 2$, and integrate over $(0, t)$ to obtain, after routinary computations and thanks to Lemma V.5, Lemma V.7 and Lemma V.8

$$\int_0^t e^{\epsilon(s-t)} \int_\Omega g \left|\nabla \frac{u}{g}\right|^2 dxds \le c_5, \tag{V.39}$$

for any $\epsilon > 0$. Recalling that v satisfies

$$v_t = h(u, v) = h\left(\frac{u}{g} g, v\right),$$

then taking gradients we get

$$\frac{d}{dt}\nabla v - [u\chi h_u + h_v]\nabla v = h_u g \nabla \frac{u}{g}. \tag{V.40}$$

Now, we multiply (V.40) by ∇v and integrate over Ω to obtain, in view of assumptions (V.4) and (V.5)

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla v|^2 + \epsilon_0 \int_\Omega |\nabla v|^2 \le \int_\Omega g\nabla\frac{u}{g}\nabla v,$$

and therefore, by the Young's inequality

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{\epsilon_0}{2}\int_\Omega |\nabla v|^2 \le \frac{c}{\epsilon_0}\int_\Omega g\left|\nabla\frac{u}{g}\right|^2 dx.$$

After integration we get

$$\int_\Omega |\nabla v|^2 \le e^{-\epsilon_0 t}\int_\Omega |\nabla v_0|^2 + \frac{2c}{\epsilon_0}\int_0^t e^{\epsilon_0(s-t)}\int_\Omega g\left|\nabla\frac{u}{g}\right|^2 dx.$$

and thanks to (V.39), we conclude the lemma. □

Lemma V.12. *For $u \in L^\infty(\Omega_\infty)$ and $|\nabla v| \in L^\infty(0, \infty : L^2(\Omega))$, there exists a positive constant $c_{\frac{2}{3}} > 0$ such that*

$$\int_\Omega u dx \ge c_{\frac{2}{3}}.$$

Proof. We proceed as in Mizukami-Yokota [74] Lemma 4.2. and multiply the equation of u by $u^{-\beta}e^{\chi\beta v}$ for some $\beta \in (1,2)$, after integration by parts we obtain

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx = \chi\beta\int_\Omega h(u,v)e^{\chi\beta v}u^{-\beta}dx + (1-\beta)\beta\int_\Omega e^{\chi\beta v}u^{1-\beta}|\nabla u - \chi u\nabla v|^2dx + \mu\beta\int_\Omega e^{\chi\beta v}u^{1-\beta}(1+f-u).$$

Since $\beta \in (1,2)$, we have that

$$(1-\beta)\beta\int_\Omega e^{\chi\beta v}u^{1-\beta}|\nabla u - \chi u\nabla v|^2dx \leq 0.$$

Notice that, thanks to the Mean Value Theorem it yields $h(u,v) = h(0,0) + h_u(\xi_1,0)u + h_v(u,\xi_2)v$, for some (ξ_1,ξ_2). Assumptions implies

$$\int_\Omega h(u,v)e^{\chi\beta v}u^{-\beta}dx \leq h(0,0)\int_\Omega e^{\chi\beta v}u^{-\beta}dx + c\int_\Omega e^{\chi\beta v}u^{2-\beta}dx.$$

Therefore we have that

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \leq \chi\beta h(0,0)\int_\Omega e^{\chi\beta v}u^{-\beta}dx - \mu\beta\varepsilon\int_\Omega e^{\chi\beta v}u^{1-\beta} + (\mu\beta + c)\int_\Omega e^{\chi\beta v}u^{2-\beta}.$$

Then

$$\frac{d}{dt}\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \leq \beta(\chi h(0,0) - \mu\varepsilon)\int_\Omega e^{\chi\beta v}u^{-\beta}dx + c.$$

In view of assumption over h, for β close enough to 2, we get, by the Maximum Principle that

$$\int_\Omega e^{\chi\beta v}u^{1-\beta}dx \leq c$$

and the non-negativity of v implies

$$\int_\Omega u^{1-\beta}dx \leq \tilde{c}.$$

Since

$$|\Omega| = \int_\Omega \frac{u^{\frac{\beta-1}{\beta}}}{u^{\frac{\beta-1}{\beta}}}dx,$$

the Hölder inequality implies

$$\int_\Omega \frac{u^{\frac{\beta-1}{\beta}}}{u^{\frac{\beta-1}{\beta}}}dx \leq \left[\int_\Omega udx\right]^{\frac{\beta-1}{\beta}}\left[\int_\Omega \frac{1}{u^{\beta-1}}dx\right]^{\frac{1}{\beta}}.$$

After some computations, the proof ends. □

The following lemma has been proved in Chapter III (Lemma III.10).

Lemma V.13. *Under assumption (V.10), the solution to (V.15) defined in (V.14) admits a lower bound*

$$u^* \geq \varepsilon_1,$$

for some $\varepsilon_1 > 0$.

We now define the positive function

$$k_1(t) := \int_\Omega \left(u - \int_\Omega u dx\right)^2 dx, \tag{V.41}$$

thus we achieve the following.

Lemma V.14. *Under the assumptions of Theorem V.2, there exists a positive constant c_8 independent of t such that the following estimate holds*

$$\int_0^\infty k_1(t) dt \le c_8 < \infty. \tag{V.42}$$

Proof. The proof follows the arguments of the previous chapter, where the problem is studied for a fully parabolic system. For readers convenience, we present the details. We integrate the first equation of (V.1) over Ω and in view of

$$\int_\Omega \left(u - \int_\Omega u dx\right) dx = \int_\Omega \left(u - \int_\Omega u dx\right) f^*(t) dx = \int_\Omega u dx \left(u - \int_\Omega u dx\right) dx = 0,$$

we obtain

$$\begin{aligned}
\frac{1}{\mu}\frac{d}{dt}\int_\Omega u dx &= \int_\Omega u(1 + f - u) dx \\
&= \int_\Omega \left(u - \int_\Omega u dx\right)(1 + f - u) dx + \int_\Omega u dx \left(1 + \int_\Omega f dx - \int_\Omega u dx\right) \\
&= \int_\Omega \left(u - \int_\Omega u dx\right)\left(\int_\Omega u dx - u\right) dx + \int_\Omega \left(u - \int_\Omega u dx\right)(f - f^*) dx \\
&\quad + \int_\Omega u \left(1 + f^* - \int_\Omega u dx\right) dx + \int_\Omega u \left(\int_\Omega (f - f^*) dx\right) dx.
\end{aligned}$$

Since f and f^* are uniformly bounded, we have

$$\int_\Omega u(f - f^*) dx = \int_\Omega \left(u - \int_\Omega u dx\right)(f - f^*) dx + \int_\Omega u dx \int_\Omega (f - f^*)\, dx,$$

and

$$\int_\Omega \left(u - \int_\Omega u dx\right)(f - f^*) dx \le \delta k_1 + c(\delta)\|f - f^*\|_{L^1(\Omega)},$$

for any $\delta > 0$. We take $\delta = \frac{\mu}{4}$ and then

$$\frac{d}{dt}\int_\Omega u dx \le -\frac{\mu}{2} k_1(t) + \mu \int_\Omega u \left(1 + f^* - \int_\Omega u dx\right) dx + c\|f - f^*\|_{L^1(\Omega)}.$$

We divide by $\int_\Omega u dx$ to get

$$\frac{d}{dt}\ln\left(\int_\Omega u dx\right) \le -\frac{\mu k_1}{2\int_\Omega u dx} + \mu\left(1 + f^* - \int_\Omega u dx\right) + \frac{c}{\int_\Omega u dx}\|f - f^*\|_{L^\infty(\Omega)}.$$

Since u^* satisfies

$$\frac{d}{dt}(\ln u^*) = \mu(1 + f^* - u^*),$$

we have

$$\frac{d}{dt}\left(\ln\left(\int_\Omega udx\right) - \ln u^*\right) \leq -\frac{\mu k_1(t)}{2\int_\Omega udx} + \mu\left(u^* - \int_\Omega udx\right) + \frac{c}{\int_\Omega udx}\|f - f^*\|_{L^1(\Omega)}. \quad \text{(V.43)}$$

Now, we consider the following functions, as in the previous chapter,

$$F_1 := \int_\Omega \frac{u}{u^*}dx - 1 + \ln u^* - \int_\Omega \ln udx; \quad F_2 := \ln\left(\int_\Omega udx\right) - \ln u^*. \quad \text{(V.44)}$$

Notice that $F_1 \geq 0$ and $F_2 \geq c_0$. Let c_1 be defined in (V.20), then

$$\frac{d}{dt}F_2 + \mu\left(\int_\Omega udx - u^*\right) \leq -\frac{\mu}{2c_1}k_1(t) + \frac{c}{c_{\frac{2}{3}}}\|f - f^*\|_{L^1(\Omega)}, \quad \text{(V.45)}$$

and also

$$\frac{d}{dt}F_1 = \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu(1 + f^* - u^*) - \int_\Omega \frac{u_t}{u}dx$$
$$= \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu(1 + f^* - u^*) + \int_\Omega\left[-\frac{|\nabla u|^2}{u^2} + \chi\frac{\nabla u\nabla v}{u} - \mu(1 + f - u)\right]dx.$$

We take gradients in the equation of v to have

$$\frac{d}{dt}\nabla v - h_v\nabla v = h_u\nabla u.$$

Multiplying by λ and integrate over Ω we get

$$\frac{d}{dt}\int_\Omega \frac{\lambda}{2}|\nabla v|^2 - \int_\Omega \lambda h_v|\nabla v|^2 = \int_\Omega \lambda h_u\nabla u\nabla vdx.$$

Now we add both expressions to obtain

$$\frac{d}{dt}\left(F_1 + \int_\Omega \frac{\lambda}{2}|\nabla v|^2\right) - \int_\Omega \lambda h_v|\nabla v|^2 = \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu(1 + f^* - u^*)$$
$$+ \int_\Omega\left[-\frac{|\nabla u|^2}{u^2} + \frac{\nabla v\nabla u}{u}\left[\chi + \lambda uh_u\right] - \mu(1 + f - u)\right]dx.$$

We apply the Cauchy-Schwarz inequality to the term $\frac{\nabla v\nabla u}{u}\left[\chi + \lambda uh_u\right]$,

$$\frac{\nabla v\nabla u}{u}\left[\chi + \lambda uh_u\right] \leq \frac{|\nabla u|^2}{u^2} + |\nabla v|^2\frac{1}{4}\left[\chi + \lambda uh_u\right]^2,$$

then, operating we achieve

$$\frac{d}{dt}\left(F_1 + \int_\Omega \frac{\lambda}{2}|\nabla v|^2\right) - \int_\Omega \lambda h_v|\nabla v|^2 \leq \frac{d}{dt}\left(\frac{\int_\Omega udx}{u^*}\right) + \mu(1 + f^* - u^*) +$$
$$+ \int_\Omega\left[|\nabla v|^2\frac{1}{4}\left[\chi + \lambda uh_u\right]^2 - \mu(1 + f - u)\right]dx,$$

which is reduced to

$$\begin{aligned} \frac{d}{dt}\left(F_1+\int_\Omega \frac{\lambda}{2}|\nabla v|^2\right) &\leq \int_\Omega \frac{|\nabla v|^2}{4}\left(\lambda^2u^2h_u^2+\lambda(4h_v+2\chi u h_u)+\chi^2\right) \\ &+\frac{d}{dt}\frac{\displaystyle\int_\Omega udx}{u^*}+\mu(1+f^*-u^*)-\int_\Omega \mu(1+f-u)dx. \end{aligned} \tag{V.46}$$

Due to the discriminant of the polynomial

$$p(\lambda)=\lambda^2u^2h_u^2+\lambda(4h_v+2\chi u h_u)+\chi^2 \tag{V.47}$$

is given by $16h_v(h_v+\chi u h_u)$, which is positive, we have two different roots λ_+ and λ_- which are both positive. Since

$$\lambda_\pm := \frac{-2h_v-\chi u h_u \pm 2\sqrt{h_v(h_v+\chi u h_u)}}{u^2h_u^2},$$

we have that

$$\lambda_- \geq 0.$$

Then, we take $\lambda=\lambda_-$ to obtain

$$\begin{aligned} \frac{d}{dt}\left(F_1+\int_\Omega \frac{\lambda_-}{2}|\nabla v|^2\right) &\leq \frac{d}{dt}\frac{\int_\Omega udx}{u^*}+\mu(1+f^*-u^*)-\int_\Omega \mu(1+f-u)dx \\ &\leq \frac{d}{dt}\frac{\int_\Omega udx}{u^*}+\mu\|f^*-f\|_{L^1(\Omega)}+\mu\left(\int_\Omega udx-u^*\right). \end{aligned}$$

Through the inequality (V.45) it results

$$\frac{d}{dt}\left(F_1+F_2+\int_\Omega \frac{\lambda_-}{2}|\nabla v|^2\right)+k_1 \leq \frac{d}{dt}\frac{\int_\Omega udx}{u^*}+\mu\|f^*-f\|_{L^1(\Omega)}.$$

After integration over $(0,T)$ and taking limits when $T\to\infty$ we conclude the lemma. □

Lemma V.15. *Under the assumptions of Theorem V.2 the following estimate holds*

$$\int_\Omega |\nabla v|^2+\int_0^\infty\int_\Omega |\nabla v|^2dxdt \leq c_9<\infty, \tag{V.48}$$

with c_9 a positive constant.

Proof. We first notice that $p(\lambda)$ defined in (V.47) achieves its minimum at

$$\lambda_0=\frac{-2h_v-\chi u h_u}{u^2h_u^2}$$

and

$$p(\lambda_0)=-4\frac{h_v^2+\chi u h_u h_v}{u^2h_u^2} \leq -\frac{4\epsilon_v^2}{c_5^2\|h_u\|^2_{L^\infty(A)}} := -p_0<0,$$

where $A=[0,\|u\|_{L^\infty}]\times[0,\|v\|_{L^\infty}]$ is a compact set of $\mathbb{R}^2$. Due to (V.46) we get

$$\begin{aligned} &\frac{d}{dt}\left(F_1+\int_\Omega \frac{\lambda_0}{2}|\nabla v|^2\right)+p_0\int_\Omega \frac{|\nabla v|^2}{4} \leq \\ &\leq \frac{d}{dt}\frac{\int_\Omega udx}{u^*}+\mu(1+f^*-u^*)-\int_\Omega \mu(1+f-u)dx. \end{aligned}$$

We now proceed as in Lemma V.14 and we obtain

$$\frac{d}{dt}\left(F_1+F_2+\int_\Omega \frac{\lambda_0}{2}|\nabla v|^2\right)+k_1+\frac{p_0}{4}\int_\Omega \frac{\lambda_0}{2}|\nabla v|^2 \le \frac{d}{dt}\frac{\int_\Omega u dx}{u^*}+\mu\|f^*-f\|_{L^1(\Omega)}.$$

After integration over $(0,\infty)$, in view of

$$\lambda_0 \ge \frac{\epsilon_v}{\|u\|_{L^\infty}\|h_u\|_{L^\infty(A)}} > 0$$

and we end the proof. □

We have the following boundedness property:

Lemma V.16. *Under the assumptions of Theorem V.2, there exists a positive constant* c_{10} *such that the following inequality holds*

$$\int_0^\infty \int_\Omega |\nabla u|^2 dx dt \le c_{10}.$$

Proof. The proof is similar to the one given in the previous chapter. For completeness we reproduce below the main ideas of the proof.

$$\begin{aligned}\frac{d}{dt}(F_1+F_2) &\le \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right)+\int_\Omega\left[-\frac{|\nabla u|^2}{u^2}+\chi\frac{\nabla u \nabla v}{u}\right]dx\\ &\le \frac{d}{dt}\left(\frac{\int_\Omega u dx}{u^*}\right)-\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}dx+\frac{\chi^2}{2}\int_\Omega |\nabla v|^2 dx.\end{aligned}$$

After integration, by Lemma V.15, we obtain

$$\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} dx dt \le c_{11} < \infty.$$

In view of the boundedness of u we have

$$\int_0^\infty \int_\Omega |\nabla u|^2 dx dt \le \|u\|^2_{L^\infty(\Omega)} \int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} dx dt \le c_{10} < \infty$$

and the proof ends. □

Lemma V.17. *Under assumptions (V.4)-(V.15), there exists a positive constant* $c_{12} < \infty$ *independent of* t *such that*

$$k_1' \le c_{12}, \qquad \textit{for } t > 0,$$

where k_1 *is defined in (V.41).*

Proof. The following relations hold:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \left(u-\int_\Omega u dx\right)^2 dx = \int_\Omega u_t\left(u-\int_\Omega u dx\right)dx$$

and

$$\begin{aligned}\int_\Omega u_t\left(u-\int_\Omega u dx\right)dx &= -\int_\Omega |\nabla u|^2 dx+\chi\int_\Omega u\nabla u \nabla v dx\\ &\quad +\mu\int_\Omega u(1+f-u)\left(u-\int_\Omega u dx\right)dx.\end{aligned}$$

By applying the Young's inequality we have

$$-\int_\Omega |\nabla u|^2 dx + \chi \int_\Omega u \nabla u \nabla v dx \leq c_{12} \|u\|^2_{L^\infty(\Omega_\infty)} \int_\Omega |\nabla v|^2 dx.$$

Boundedness of u, Lemma V.15 and assumption (V.5) imply the result. □

Lemma V.18. *Under assumptions of Theorem V.2 we have*

$$\|u - u^*\|_{L^2(\Omega)} \to 0, \quad \textit{as} \quad t \to \infty.$$

Proof. We consider k_1 defined in (V.41), then, thanks to Lemma V.14 and Lemma V.17 we have

$$\left\| u - \int_\Omega u dx \right\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{V.49}$$

Now, we define k_2 as follows

$$k_2(t) := \left(\int_\Omega u dx - u^* \right)^2.$$

Recalling $F_2(t)$ defined in (V.44) by

$$F_2(t) = \ln \int_\Omega u dx - \ln u^*,$$

due to (V.41), we get

$$\frac{d}{dt} F_2 + \mu \left(\int_\Omega u dx - u^* \right) \leq -k_1(t) + \mu \|f - f^*\|_{L^1(\Omega)}.$$

We multiply by F_2 and by the Mean Value Theorem we claim

$$\frac{d}{dt} F_2^2 + 2\mu\xi F_2^2 \leq |F_2| k_1(t) + \mu |F_2| \|f - f^*\|_{L^1(\Omega)} \leq 2c_5^2 \left(k_1(t) + \mu \|f - f^*\|_{L^1(\Omega)} \right),$$

for some $\xi \in \left[u^*, \int_\Omega u dx\right]$ if $u^* < \int_\Omega u dx$ or $\xi \in \left[\int_\Omega u dx, u^*\right]$ otherwise. After integration it results

$$\int_0^\infty F_2^2 dt \leq c_{13} < \infty.$$

Notice that Lemma V.3 implies

$$k_2 \leq c_{14} F_2^2$$

for some positive constant c_{14}. Therefore, there exists $c_{15} > 0$ such that

$$\int_0^\infty k_2 dt \leq c_{15} < \infty. \tag{V.50}$$

In view of Lemma V.3, (V.10) and Lemma V.10 it is easy to see that

$$|k_2'| \leq c_{16} < \infty. \tag{V.51}$$

Now, by Lemma I.8, (V.50) and (V.51) we obtain

$$k_2 \to 0 \quad \text{as} \quad t \to \infty. \tag{V.52}$$

Since

$$\int_\Omega |u-u^*|^2 dx \le k_1+k_2,$$

by taking into account (V.49) and (V.52), we get

$$\|u-u^*\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad t\to\infty$$

and the proof ends. □

In order to obtain

$$\|v-v^*\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t\to\infty,$$

we proceed as before in the following lemma.

Lemma V.19. *Under assumptions (V.3)-(V.12), the solution v fulfills*

$$\|v(x,t)-v^*(t)\|_{L^2(\Omega)} \to 0, \quad \text{as } t\to\infty.$$

Proof. By the Mean Value Theorem, it follows

$$\begin{aligned} v_t - v_t^* &= h(u,v)-h(u^*,v^*) = h(u,v)-h(u,v^*)+h(u,v^*)-h(u^*,v^*) \\ &= \left.\frac{\partial h}{\partial v}\right|_{(u,\eta)}(v-v^*) + \left.\frac{\partial h}{\partial u}\right|_{(\xi,v^*)}(u-u^*). \end{aligned}$$

We call $z=v-v^*$, by multiplying by z the above equation and after integrating over Ω, it yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx &= \int_\Omega h_v z^2 dx + \int_\Omega h_u z(u-u^*)dx \le \int_\Omega h_v z^2 dx + \\ &+ \|h_u z\|_{L^\infty(\Omega)} \int_\Omega (u-u^*)^2 dx, \end{aligned}$$

where we have applied the Hölder inequality to the last term. Now, by assumption (V.4) it results

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx \le -\epsilon_v \int_\Omega z^2 dx + \theta(t),$$

where $\theta(t)$ is uniformly bounded. We obtain the result by solving the differential inequality. □

Proof of Theorem V.2.
The proof is a consequence of Lemma V.18 and Lemma V.19. □

Chapter VI

Fundamentals of the Generalized Finite Difference Method

In this chapter we introduce the computational method which we use as tool for finding the numerical solution of all the models from the previous chapters. This tool is the *Generalized Finite Difference Method* (GFDM). The meshless (or meshfree) method is based on the Taylor series expansion together with a moving least squares procedure. The GDF method was first introduced by Jensen [61] and Liszka and Orkisz [72]. Benito, Gavete and Ureña [12] have studied the influence of several factors and developed the explicit formulae and h-adaptive method for the solution of the PDEs.
The GFDM presents serveral important aspects. Among others, the discretization of the problem (the numerical scheme) does not depend on the nodes of the domain in a rigid way. Since the distribution of the nodes (*cloud of nodes*) in the domain is not fixed, the convergence problems are often solved by changing the cloud or by refinement, which is not a difficult task. Also, the discretization of the spatial partial derivatives uses a very simple expression (depending only on the distribution of a few nodes, as we explain in the next section), so the treatment of nonlinearities is straightforward. Finally, one of the most powerful advantages of the method is the possibility of using a very irregular and complicated domain and clouds of nodes with high accuracy and efficiency. Hence, the GFDM is a rather appropiate computational method for solving problems arising in Applied Mathematics, Physics and Engineering.
The following pages of this document are devoted to the implementation of the meshless method called *Generalized Finite Difference Method* (GFDM) for solving numerically all the previous chemotactic models. For the sake of completeness, let us first introduce the basics of the method, though we refer the reader to [12], [113] and the references therein for a more detailed comprehension of the procedure. Later, in the following chapters, we derive the discretization of each one of the previous models by means of the GFDM and study the conditional convergence of the numerical solutions of the explicit scheme to the continuous solutions of the systems. We finally present several numerical examples in order to illustrate, first, the validity and neccesity of the assumptions of the continuous models and, second, the applicability, accuracy and efficiency of the GFDM for solving these problems.

1 Preliminaries

In this section we explain the procedure to obtain the discretization of the partial derivatives of a function in terms of this meshless method.

Remark VI.1. *In the following chapters we use the lower case letters for the solutions of the continuous model (maintaining the previous notation), i.e., u_i^n stands for the continuous function $u(x_i, n\Delta t)$, and the upper case for the approximate values given by the numerical schemes, i.e., U_i^n stands for the discrete value of the numerical solution at x_i and $n\Delta t$.*

Let Ω be a domian in $\mathbb{R}^2$ and $M = x_1, x_2, ...x_N$ a discretization of such domian. For the ease of notation and, without loss of generality, let us consider a set of s different points of M, say $V = \{x_1, x_2, ..., x_s\}$, and denote a generic interior point of $M - V$ as x_0. There are different criteria such as distance, quadrant or octant criteria can be used to select the nodes of the star, as illustrated in Figure VI.1.

In order to obtain the equation of the star for each of the these points, we look at the trun-

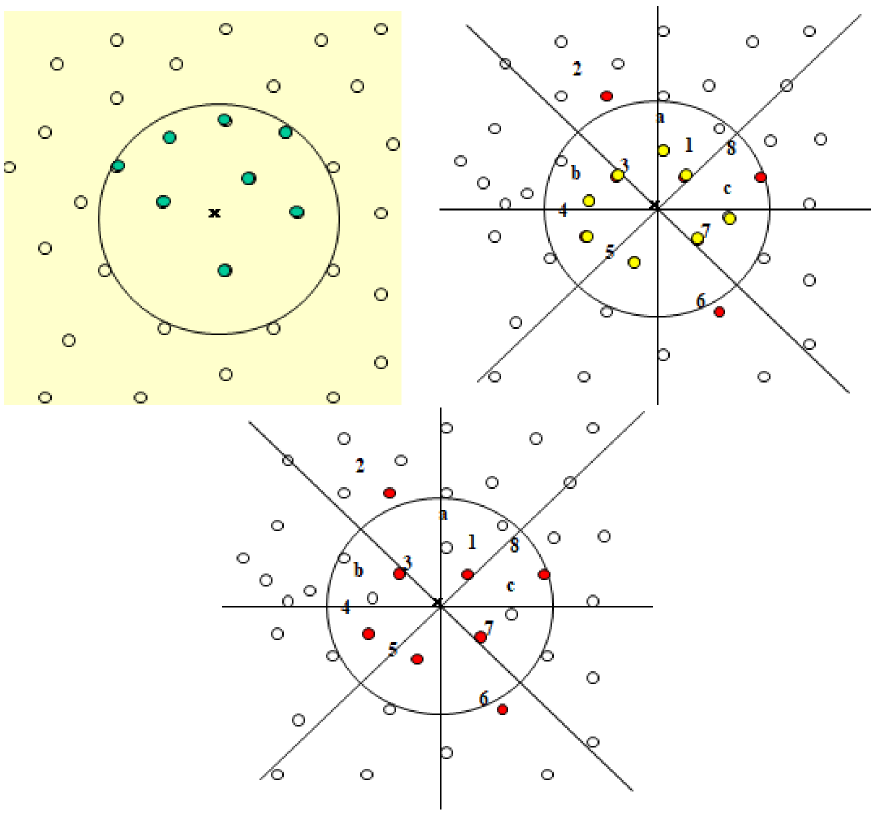

Figure VI.1: Distance, quadrant and octant criteria.

cated second order Taylor series expansion at $\mathbf{x}_0$ of the function f where $f(\mathbf{x}_i, n\Delta t) = f_i^n$ (although we omit the time dependence since it is not neccesary now) is the approximation of the continuous solution of the problem,

$$F_i = F_0 + (\mathbf{x}_i - \mathbf{x}_0)\nabla F_0 + \frac{1}{2}(\mathbf{x}_i - \mathbf{x}_0)^T H_{F_0}(\mathbf{x}_i - \mathbf{x}_0) + \mathcal{O}(3). \quad \text{(VI.1)}$$

We define now the vectors $\mathbf{d} = \left(\frac{\partial F_0}{\partial x}, \frac{\partial F_0}{\partial y}, \frac{\partial^2 F_0}{\partial x^2}, \frac{\partial^2 F_0}{\partial y^2}, \frac{\partial^2 F_0}{\partial x \partial y}\right)$ and $\mathbf{c}_i = (h_i, k_i, \frac{h_i^2}{2}, \frac{k_i^2}{2}, h_i k_i)$, where as usual, we take $h_i = x_i - x_0$ and $k_i = y_i - y_0$. Let us define the sum of the weighted quadratic errors as the operator B:

$$B(\mathbf{d}) = \sum_{i=1}^{s} (F_0 - F_i + \mathbf{c}_i^T \mathbf{d})^2 w_i^2 + \mathcal{O}(h_i^2, k_i^2),$$

where $w_i = w(h_i, k_i)$ are positive symmetrical weighting functions and decreasing in magnitude as the distance to the center increases, as defined in Lancaster and Salkauskas [66], [68]. Some weighting functions as potentials ($\frac{1}{\text{dist}^4}$ or $\frac{1}{\text{dist}^6}$) or exponential, $e^{-\text{dist}^2}$, can be used (more details can be found in [40]). By minimizing B with respect to $\mathbf{d}$, the following linear system is achieved

$$\sum_{i=1}^{s} w_i^2 \mathbf{c}_i \mathbf{c}_i^T \mathbf{d} = -\sum_{i=1}^{s} w_i^2 (F_0 - F_i) \mathbf{c}_i.$$

Let us define $A := \sum_{i=1}^{s} w_i^2 \mathbf{c}_i \mathbf{c}_i^T$, that is,

$$\boldsymbol{A} = \begin{pmatrix} h_1 & h_2 & \cdots & h_s \\ k_1 & k_2 & \cdots & k_s \\ \vdots & \vdots & \vdots & \vdots \\ h_1 k_1 & h_2 k_2 & \cdots & h_s k_s \end{pmatrix} \begin{pmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \cdots & \\ & & & \omega_s^2 \end{pmatrix} \begin{pmatrix} h_1 & k_1 & \cdots & h_1 k_1 \\ h_2 & k_2 & \cdots & h_2 k_2 \\ \vdots & \vdots & \vdots & \vdots \\ h_s & k_s & \cdots & h_s k_s \end{pmatrix} \tag{VI.2}$$

Properties of $\mathbf{A}$ are well-known, as can be consuted in [40]. In particular, it is positive definite and, therefore,

$$\mathbf{d} = -F_0 \sum_{i=1}^{s} w_i^2 A^{-1} \mathbf{c}_i + \sum_{i=1}^{s} F_i w_i^2 A^{-1} \mathbf{c}_i + \mathcal{O}(h_i^2, k_i^2).$$

For simplicity, we write

$$\boldsymbol{\lambda}_0 = \sum_{i=1}^{s} w_i^2 \mathbf{A}^{-1} \mathbf{c}_i, \quad \boldsymbol{\lambda}_i = w_i^2 \mathbf{A}^{-1} \mathbf{c}_i.$$

Then, introducing the notation $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02}, \lambda_{03}, \lambda_{04}, \lambda_{05})^T$ (and similarly for $\boldsymbol{\lambda}_i$), we express the partial derivatives of the function as a linear combination of the approximated values of f in the points of the star:

$$\begin{cases} \dfrac{\partial F(\mathbf{x}_0, n\Delta t)}{\partial x} = -\lambda_{01} F_0^n + \displaystyle\sum_{i=1}^{s} \lambda_{i1} F_i^n + \mathcal{O}(h_i^2, k_i^2), \\ \dfrac{\partial F(\mathbf{x}_0, n\Delta t)}{\partial y} = -\lambda_{02} F_0^n + \displaystyle\sum_{i=1}^{s} \lambda_{i2} F_i^n + \mathcal{O}(h_i^2, k_i^2), \\ \dfrac{\partial^2 F(\mathbf{x}_0, n\Delta t)}{\partial x^2} + \dfrac{\partial^2 F(\mathbf{x}_0, n\Delta t)}{\partial y^2} = -(\lambda_{03} + \lambda_{04}) F_0^n + \displaystyle\sum_{i=1}^{s} (\lambda_{i3} + \lambda_{i4}) F_i^n \\ \qquad + \mathcal{O}(h_i^2, k_i^2). \end{cases} \tag{VI.3}$$

Remark VI.2. *Since the only second derivatives appearing in all systems is the laplacian operator, we discretize it as*

$$\Delta^{\text{discrete}} F_0 = -\lambda_{00} F_0 + \sum_{i=1}^{s} \lambda_{i0} F_i,$$

where, clearly, we have $\lambda_{00} = \lambda_{03} + \lambda_{04}$ *and* $\lambda_{i0} = \lambda_{i3} + \lambda_{i4}$.

The time derivative approximation is computed with the classical advance difference formula, of first order,

$$\frac{\partial F(\mathbf{x}_0, n\Delta t)}{\partial t} = \frac{F_0^{n+1} - F_0^n}{\Delta t} + \mathcal{O}(\Delta t). \tag{VI.4}$$

2 Previous numerical results

In this section we state some known results that we use in the proof of the convergence of the GFD explicit scheme in the following chapters.

Lemma VI.1. *Let* $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$. *If there exists some matrix norm such that* $\|A\| < 1$, *then*

$$\lim_{k\to\infty} A^k = \boldsymbol{0}.$$

Lemma VI.2. *Assume* $A \in \mathfrak{M}_{n\times n}(\mathbb{R})$, *then the following are equivalent:*

(i) $\lim_{k\to\infty} A^k = \boldsymbol{0}$,

(ii) $\rho(A) < 1$,

where $\rho(\cdot)$ *stands for the spectral radius.*

The proof of the two previous results can be found in Isaacson and Keller [56].

Remark VI.3. *For all the following chapters, we compute the difference between the numerical solution and the exact values (continuous solutions or steady states) using the* l^∞ *norm, defined by*

$$\|v\|_{l^\infty(\Omega)} = \max_{i=0,\dots,N}\{|v_i|\},$$

where N *represents the number of nodes of our discretization.*

Remark VI.4. *Note that in the following chapters we compare the numerical solution of the problem with the asymptotic solution (not the exact one, since there is no explicit known solution). This explains the possible difference between our numerical values and the continuous ones at small times. Also notice that we may choose a very distant initial data (computed in* l^∞ *norm as in Remark VI.3) from the asymptotic limit, provided enough regularity, and this may result in a difference between the discrete and continuous values at small times.*

Chapter VII

Numerical solution of the parabolic-elliptic model

Our first numerical study is dedicated to the parabolic-elliptic system (II.1), which we recall here (under the assumptions made in Chapter II):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(\mathbf{x},t) - u), & \mathbf{x} \in \Omega,\ t > 0, \\ -\Delta v + v = u, & \mathbf{x} \in \Omega,\ t > 0, \\ u(0,\mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \mathbf{x} \in \partial\Omega,\ t > 0, \end{cases} \tag{VII.1}$$

with function f fulfilling

$$\lim_{t\to\infty} \sup_{\mathbf{x}\in\Omega} |f(\mathbf{x},t) - f^*(t)| = 0, \tag{VII.2}$$

for some periodic in time function f^*. Three particular cases are studied in this chapter. Firstly, we consider the case $\mu = 0$, that is to say, there is no logistic source. It is well-known that the absence of such term may end up in the blow-up of solutions (i.e., solutions become unbounded at finite time). More precisely, in [77], the author proved that the solution of system (IV.1) (when $\mu = 0$) blows up in finite time under the condition

$$\int_\Omega u_0(\mathbf{x}) d\mathbf{x} > \frac{8\pi}{\chi}.$$

Secondly, for $0 < \chi < \mu$, we consider $f = 0$, which is extensively studied in the literature. For instance, in [109] the authors proved that all solutions of the non-stationary system approach the steady state $(1,1)$ for large times. Finally, we consider a function $f(\mathbf{x},t)$ fulfilling the assumptions made in Chapter II with the periodic asymptotic behavior stated in (VII.2). We present numerical examples of all three cases in Section 2, for both regular and irregular domains.

The chapter is structured in the following way: first, we obtain the GFD explicit scheme which is the discrete version of the continuous model (VII.1). Then, we prove the conditional convergence of the discrete solution to the continuous one. The next section provides numerical examples where under certain assumptions we find blow-up solutions and periodic asymptotic solutions under the hypotheses of Chapter II. Finally, we reproduce the same study for the generalized model with two species of Chapter II.

1 GFD scheme and convergence

We derive the discretization of the first equation using the GFD explicit formulae given in (VI.3). The existence and uniqueness of solutions of system (VII.1) are proved for the case $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, for system (VII.1) though we consider the case $d = 2$. Then, we write the first equation as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \chi \nabla u \cdot \nabla v - \chi u(v-u) + \mu u(1+f(\mathbf{x},t)-u) = \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \chi \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \chi \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u^2(\chi-\mu) - \chi uv + \mu u + \mu u f(\mathbf{x},t).\end{aligned}$$

In order to deal with the homogeneous Neumann boundary conditions,

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,$$

we use the central difference, which is of second order, if the domain Ω has regular boundary (in the sense of the distribution of nodes) and GFD formulae in other case. The conditional convergence of the GFD explicit scheme in 2D is addressed in the following result.

Theorem VII.1. *Let $u, v \in C^{2,1}_{loc}(\bar{\Omega} \times [0,\infty))$ the solution to the system (VII.1). The GFD explicit scheme*

$$\begin{cases} U_0^{n+1} = U_0^n + \Delta t \left[-\lambda_{00} U_0^n + \sum_{i=1}^{s} \lambda_{i0} U_i^n \right] - \\ \quad - \chi \Delta t \left(-\lambda_{01} U_0^n + \sum_{i=1}^{s} +\lambda_{i1} U_i^n \right) \left(-\lambda_{01} V_0^n + \sum_{i=1}^{s} +\lambda_{i1} V_i^n \right) - \\ \quad - \chi \Delta t \left(-\lambda_{02} U_0^n + \sum_{i=1}^{s} +\lambda_{i2} U_i^n \right) \left(-\lambda_{02} V_0^n + \sum_{i=1}^{s} +\lambda_{i2} V_i^n \right) + \\ \quad + \Delta t \left[(U_0^n)^2 (\chi - \mu) - \chi U_0^n V_0^n + \mu U_0^n \right] + \mathcal{O}(\Delta t, h_i^2, k_i^2) \\ V_0^n - \left[-\lambda_{00} V_0^n + \sum_{i=1}^{s} \lambda_{i0} V_i^n \right] = U_0^n + \mathcal{O}(\Delta t, h_i^2, k_i^2) \end{cases} \tag{VII.3}$$

is convergent under the condition

$$0 < \Delta t < \frac{2 + |\lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}|}{(1 + |\lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}|)(A_2 + A_3) + B_1},$$

where

$$\begin{aligned} A_2 :=& \sum_{i=1}^{s} |\lambda_{i0}| + |\chi \lambda_{01} V_0^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \lambda_{02} V_0^n| \sum_{i=1}^{s} |\lambda_{i2}| \\ &+ |\chi \sum_{i=1}^{s} \lambda_{i1} v_i^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \sum_{i=1}^{s} \lambda_{i2} v_i^n| \sum_{i=1}^{s} |\lambda_{i2}|, \\ A_3 :=& \lambda_{00} + \chi(\lambda_{01})^2 v_0^n + \chi(\lambda_{02})^2 v_0^n - \chi \left(\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} v_i^n + \lambda_{02} \sum_{i=1}^{n} \lambda_{i2} v_i^n \right) \\ &- (\chi - \mu)(U_0^n + u_0^n) + \chi v_0^n - \mu(1 + f(x_0, y_0, n\Delta t)), \end{aligned} \tag{VII.4}$$

and

$$B_1 := \left|\chi[(\lambda_{01})^2 - (\lambda_{02})^2]U_0^n + \chi\lambda_{01}\sum_{i=1}^{s}\lambda_{i1}U_i^n + \chi\lambda_{02}\sum_{i=1}^{s}\lambda_{i2}U_0^n - \chi u_0^n\right|$$
$$+ |\chi\lambda_{01}u_0^n|\sum_{i=1}^{s}|\lambda_{i1}| + |\chi\lambda_{02}u_0^n|\sum_{i=1}^{s}|\lambda_{i2}| + \chi|\sum_{i=1}^{s}\lambda_{i1}u_i^n|\sum_{i=1}^{s}|\lambda_{i1}|+$$
$$+ \chi|\sum_{i=1}^{s}\lambda_{i2}u_i^n|\sum_{i=1}^{s}|\lambda_{i2}|.$$

Proof. We take the difference between GFD scheme (VII.3) and the expression for the exact solution, where we use the notation, also from now on, $\tilde{u}_j^n = u_j^n - U_j^n$; $\tilde{v}_j^n = v_j^n - V_j^n$ (U_j^n stands for the approximate solution at time $n\Delta t$ and point j and u_j^n for the exact solution). From the second equation of (VII.1) we obtain

$$u = v - \Delta v \Rightarrow \tilde{u}_0^n = \tilde{v}_0^n + \lambda_{00}\tilde{v}_0^n - \sum_{i=1}^{s}\lambda_{i0}\tilde{v}_i^n.$$

We arrive then to the following in the first equation

$$\begin{aligned}
\tilde{u}_0^{n+1} - \tilde{u}_0^n = \Delta t\left(-\lambda_{00}\tilde{u}_0^n + \sum_{i=1}^{s}\lambda_{i0}\tilde{u}_i^n\right)&- \\
- \chi\Delta t\Bigg[\left(-\lambda_{01}U_0^n + \sum_{i=1}^{s}\lambda_{i1}U_i^n\right)\left(-\lambda_{01}V_0^n + \sum_{i=1}^{s}\lambda_{i1}V_i^n\right)& \\
- \left(-\lambda_{01}u_0^n + \sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(-\lambda_{01}v_0^n + \sum_{i=1}^{s}\lambda_{i1}v_i^n\right)\Bigg]&- \\
- \chi\Delta t\Bigg[\left(-\lambda_{02}U_0^n + \sum_{i=1}^{s}\lambda_{i2}U_i^n\right)\left(-\lambda_{02}V_0^n + \sum_{i=1}^{s}\lambda_{i2}V_i^n\right)& \\
- \left(-\lambda_{02}u_0^n + \sum_{i=1}^{s}\lambda_{i2}u_i^n\right)\left(-\lambda_{02}v_0^n + \sum_{i=1}^{s}\lambda_{i2}v_i^n\right)\Bigg]&+ \\
+ \Delta t(\chi - \mu)[(u_0^n)^2 - (U_0^n)^2] - \Delta t\chi[U_0^nV_0^n - u_0^nv_0^n]&+ \\
+ \Delta t\mu\tilde{u}_0^n[1 + f(x_0, y_0, n\Delta t)] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).&
\end{aligned} \tag{VII.5}$$

Now, we put:

$$(U_0^n)^2 - (u_0^n)^2 = \tilde{u}_0^n(u_0^n + U_0^n), \tag{VII.6}$$

$$U_0^nV_0^n - u_0^nv_0^n = U_0^nV_0^n - U_0^nv_0^n + U_0^nv_0^n - u_0^nv_0^n = U_0^n\tilde{v}_0^n + \tilde{u}_0^nv_0^n, \tag{VII.7}$$

and

$$
\begin{aligned}
&(\lambda_{01})^2 U_0^n V_0^n - \lambda_{01} U_0^n \sum_{i=1}^{s} \lambda_{i1} V_i^n - \lambda_{01} V_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n + \\
&+ \left(\sum_{i=1}^{s} \lambda_{i1} V_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} U_i^n \right) - \left[(\lambda_{01})^2 u_0^n v_0^n - \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} v_i^n - \right. \\
&\left. - \lambda_{01} v_0^n \sum_{i=1}^{s} \lambda_{i1} u_i^n + \left(\sum_{i=1}^{s} \lambda_{i1} v_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} u_i^n \right) \right] \\
&= (\lambda_{01})^2 [U_0^n V_0^n - u_0^n v_0^n] - \lambda_{01} \left[U_0^n \sum_{i=1}^{s} \lambda_{i1} V_i^n - u_0^n \sum_{i=1}^{s} \lambda_{i1} v_i^n \right] - \\
&- \lambda_{01} \left[V_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n - v_0^n \sum_{i=1}^{s} \lambda_{i1} u_i^n \right] \\
&+ \left[\left(\sum_{i=1}^{s} \lambda_{i1} V_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} U_i^n \right) - \left(\sum_{i=1}^{s} \lambda_{i1} v_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} u_i^n \right) \right].
\end{aligned}
\tag{VII.8}
$$

We can rewrite identity (VII.8) in the equivalent form

$$
\begin{aligned}
(VII.8) = &(\lambda_{01})^2 [U_0^n \tilde{v}_0^n + \tilde{u}_0^n v_0^n] - \lambda_{01} \left(\tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i1} v_i^n + u_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n \right) \\
&- \lambda_{01} \left(\tilde{v}_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n + v_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n \right) + \\
&+ \left(\sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} v_i^n \right) + \left(\sum_{i=1}^{s} \lambda_{i1} u_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n \right).
\end{aligned}
\tag{VII.9}
$$

Now, substituting relations (VII.6), (VII.7) and (VII.9) all together in (VII.5), we get

$$
\begin{aligned}
\tilde{u}_0^{n+1} - \tilde{u}_0^n = \Delta t & \left(-\lambda_{00}\tilde{u}_0^n + \sum_{i=1}^{s} \lambda_{i0}\tilde{u}_i^n \right) - \\
& - \Delta t\chi(\lambda_{01})^2[U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n] - \Delta t\chi(\lambda_{02})^2[U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n] \\
& + \Delta t\chi\lambda_{01}\left(\tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i1} v_i^n + u_0^n \sum_{i=1}^{s} \lambda_{i1}\tilde{v}_i^n \right) + \\
& + \Delta t\chi\lambda_{02}\left(\tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i2} v_i^n + u_0^n \sum_{i=1}^{s} \lambda_{i2}\tilde{v}_i^n \right) \\
& + \Delta t\chi\lambda_{01}\left(\tilde{v}_0^n \sum_{i=1}^{s} \lambda_{i1} u_i^n + v_0^n \sum_{i=1}^{s} \lambda_{i1}\tilde{u}_i^n \right) + \\
& + \Delta t\chi\lambda_{02}\left(\tilde{v}_0^n \sum_{i=1}^{s} \lambda_{i2} u_i^n + v_0^n \sum_{i=1}^{s} \lambda_{i2}\tilde{u}_i^n \right) \\
& - \Delta t\chi\left(\sum_{i=1}^{s} \lambda_{i1}\tilde{u}_i^n \right)\left(\sum_{i=1}^{s} \lambda_{i1} v_i^n \right) - \Delta t\chi\left(\sum_{i=1}^{s} \lambda_{i2}\tilde{u}_i^n \right)\left(\sum_{i=1}^{s} \lambda_{i2} v_i^n \right) \\
& - \Delta t\chi\left(\sum_{i=1}^{s} \lambda_{i1} u_i^n \right)\left(\sum_{i=1}^{s} \lambda_{i1}\tilde{v}_i^n \right) - \Delta t\chi\left(\sum_{i=1}^{s} \lambda_{i2} u_i^n \right)\left(\sum_{i=1}^{s} \lambda_{i2}\tilde{v}_i^n \right) \\
& + \Delta t\tilde{u}_0^n(\chi - \mu)(u_0^n + U_0^n) - \Delta t\chi(U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n) + \\
& + \Delta t\mu\tilde{u}_0^n[1 + f(x_0, y_0, n\Delta t)] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).
\end{aligned}
$$

Now, it yields

$$
\begin{aligned}
\tilde{u}_0^{n+1} = \tilde{u}_0^n \Bigg[& 1 - \lambda_{00}\Delta t - \Delta t\chi(\lambda_{01})^2 v_0^n - \Delta t\chi(\lambda_{02})^2 v_0^n + \\
& + \Delta t\chi\left(\lambda_{01}\sum_{i=1}^{s}\lambda_{i1}v_i^n + \lambda_{02}\sum_{i=1}^{n}\lambda_{i2}v_i^n\right) \\
& + \Delta t(\chi-\mu)(u_0^n + U_0^n) - \Delta t\chi v_0^n + \Delta t\mu[1 + f(x_0, y_0, n\Delta t)]\Bigg] + \\
& + \Delta t\left[\sum_{i=1}^{s}\tilde{u}_i^n + \chi\lambda_{01}v_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n + \chi\lambda_{02}v_0^n\sum_{i=1}^{s}\lambda_{i2}\tilde{u}_i^n\right. \\
& \left. - \left(\chi\sum_{i=1}^{s}\lambda_{i1}v_i^n\right)\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n - \left(\chi\sum_{i=1}^{s}\lambda_{i2}v_i^n\right)\sum_{i=1}^{s}\lambda_{i2}\tilde{u}_i^n\right] + \\
& + \Delta t\tilde{v}_0^n\left[\left(-\chi(\lambda_{01})^2 - \chi(\lambda_{02})^2\right)U_0^n + \chi\lambda_{01}\sum_{i=1}^{s}\lambda_{i1}U_i^n + \right. \\
& \left. + \chi\lambda_{02}\sum_{i=1}^{s}\lambda_{i2}U_i^n - \chi U_0^n\right] + \Delta t\left[\chi\lambda_{01}u_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right. \\
& + \chi\lambda_{02}u_0^n\sum_{i=1}^{s}\lambda_{i2}\tilde{v}_i^n - \chi\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n \\
& \left. - \chi\left(\sum_{i=1}^{s}\lambda_{i2}u_i^n\right)\sum_{i=1}^{s}\lambda_{i2}\tilde{v}_i^n\right] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).
\end{aligned} \tag{VII.10}
$$

Let us define $\tilde{u}^n = \max_{i\in\{0,\ldots,s\}}|\tilde{u}_i^n|$ and $\tilde{v}^n = \max_{i\in\{0,\ldots,s\}}|\tilde{v}_i^n|$. We rewrite (VII.10) as follows

$$
\begin{aligned}
\tilde{u}^{n+1} \leq \tilde{u}^{n} \Bigg[\Bigg| 1 - \Delta t \Bigg(& \lambda_{00} + \chi(\lambda_{01})^2 v_0^n + \chi(\lambda_{02})^2 v_0^n - \\
& \chi \Bigg(\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} V_i^n + \lambda_{02} \sum_{i=1}^{n} \lambda_{i2} v_i^n \Bigg) - (\chi - \mu)(u_0^n + U_0^n) + \chi v_0^n \\
& - \mu(1 + f(x_0, y_0, n\Delta t)) \Bigg) \Bigg| + \Delta t \Bigg(\sum_{i=1}^{s} |\lambda_{i0}| + |\chi \lambda_{01} v_0^n| \sum_{i=1}^{s} |\lambda_{i1}| \\
& + |\chi \lambda_{02} v_0^n| \sum_{i=1}^{s} |\lambda_{i2}| + |\chi \sum_{i=1}^{s} \lambda_{i1} v_i^n| \sum_{i=1}^{s} |\lambda_{i1}| + \\
& + |\chi \sum_{i=1}^{s} \lambda_{i2} v_i^n| \sum_{i=1}^{s} |\lambda_{i2}| \Bigg) \Bigg] + \Delta t \tilde{v}^n \Bigg[\Bigg| \chi[-(\lambda_{01})^2 - (\lambda_{02})^2] U_0^n + \\
& + \chi \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} U_i^n + \chi \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_i^n - \chi U_0^n \Bigg| \\
& + |\chi \lambda_{01} u_0^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \lambda_{02} u_0^n| \sum_{i=1}^{s} |\lambda_{i2}| + \\
& + \chi |\sum_{i=1}^{s} \lambda_{i1} u_i^n| \sum_{i=1}^{s} |\lambda_{i1}| + \chi |\sum_{i=1}^{s} \lambda_{i2} u_i^n| \sum_{i=1}^{s} |\lambda_{i2}| \Bigg] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).
\end{aligned}
\tag{VII.11}
$$

For the sake of simplicity, let us put

$$\tilde{u}^{n+1} \leq A\tilde{u}^n + B\tilde{v}^n. \tag{VII.12}$$

Notice that since

$$\left| \tilde{v}_0^n + \lambda_{00} \tilde{v}_0^n - \sum_{i=1}^{s} \lambda_{i0} \tilde{v}_i^n \right| = |\tilde{u}_0^n|, \tag{VII.13}$$

we have

$$\tilde{v}^n \left[|1 + \lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}| \right] \geq \tilde{u}^n. \tag{VII.14}$$

Now, by combining (VII.12) with (VII.14) we get

$$\tilde{u}^{n+1} \leq \left(A \left[1 + |\lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}| \right] + B \right) \tilde{v}^n. \tag{VII.15}$$

The stabilibity follows from the condition

$$\alpha := A \left[1 + |\lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}| \right] + B < 1. \tag{VII.16}$$

Let us call $A = A_1 + \Delta t A_2 = |1 - \Delta t A_3| + \Delta t A_2$ and $B = B_1 \Delta t$. Hence, the following holds

$$|1 - \Delta t A_3| < \frac{1 - B_1 \Delta t}{1 + |\lambda_{00}| + \sum_{i=1}^{s} |\lambda_{i0}|} - \Delta t A_2, \tag{VII.17}$$

or equivalently,

$$-\frac{1-B_1\Delta t}{1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}+\Delta t A_2<1-\Delta t A_3<\frac{1-B_1\Delta t}{1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}-\Delta t A_2. \quad \text{(VII.18)}$$

Then, we have

$$1+\frac{1-B_1\Delta t}{1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}-\Delta t A_2>\Delta t A_3>1-\frac{1-B_1\Delta t}{1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}+\Delta t A_2. \quad \text{(VII.19)}$$

From the LHS inequality of (VII.19) we obtain

$$\Delta t<\frac{2+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}{(1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|)(A_2+A_3)+B_1}. \quad \text{(VII.20)}$$

From the RHS we get

$$\Delta t>\frac{|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|}{(1+|\lambda_{00}|+\sum_{i=1}^{s}|\lambda_{i0}|)(A_3-A_2)-B_1}. \quad \text{(VII.21)}$$

Now, since $A_2 > A_3$ and $B_1 > 0$, the denominator of the RHS of (VII.21) is negative, so we have

$$\Delta t > 0.$$

□

2 Numerical Results

In this section we illustrate the numerical results obtained by solving the system (VII.1), using both regular and irregular clouds of points as seen in Figure VII.1 (441 nodes in each one) in the domain $\Omega = [0,1]\times[0,1]$. We use an 8-node scheme, chosen by the distance criterion together with weight function $w = \frac{1}{dist^4}$. For all the numerical examples we put $\Delta t = 0.001$ and compute the error using Remark VI.3. We present examples of three cases: the first is dedicated to the model with $\mu = 0$, where we expect to find blow-up solutions for large enough initial data. In the second case we consider the common logistic source, $\mu u(1-u^2)$, so the numerical solution tends to $(1,1)$. Finally, we provided examples for the case when f tends to a periodic function f^*.

2.1 Case 1

In our first case we consider no source term in the u-equation, that is to say, we consider the minimal Keller-Segel parabolic-elliptic system:

$$\begin{cases} \partial_t u = \Delta u - div(\chi u \nabla v), & \mathbf{x}\in\Omega, \quad t>0, \\ -\Delta v + v = u, & \mathbf{x}\in\Omega, \quad t>0, \\ u(\mathbf{x},0)=u_0(\mathbf{x}), & \mathbf{x}\in\Omega, \\ \dfrac{\partial u}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0, & \mathbf{x}\in\partial\Omega. \end{cases}$$

As stated, we expect to find blow up of solutions in finite time for a large enough initial data.

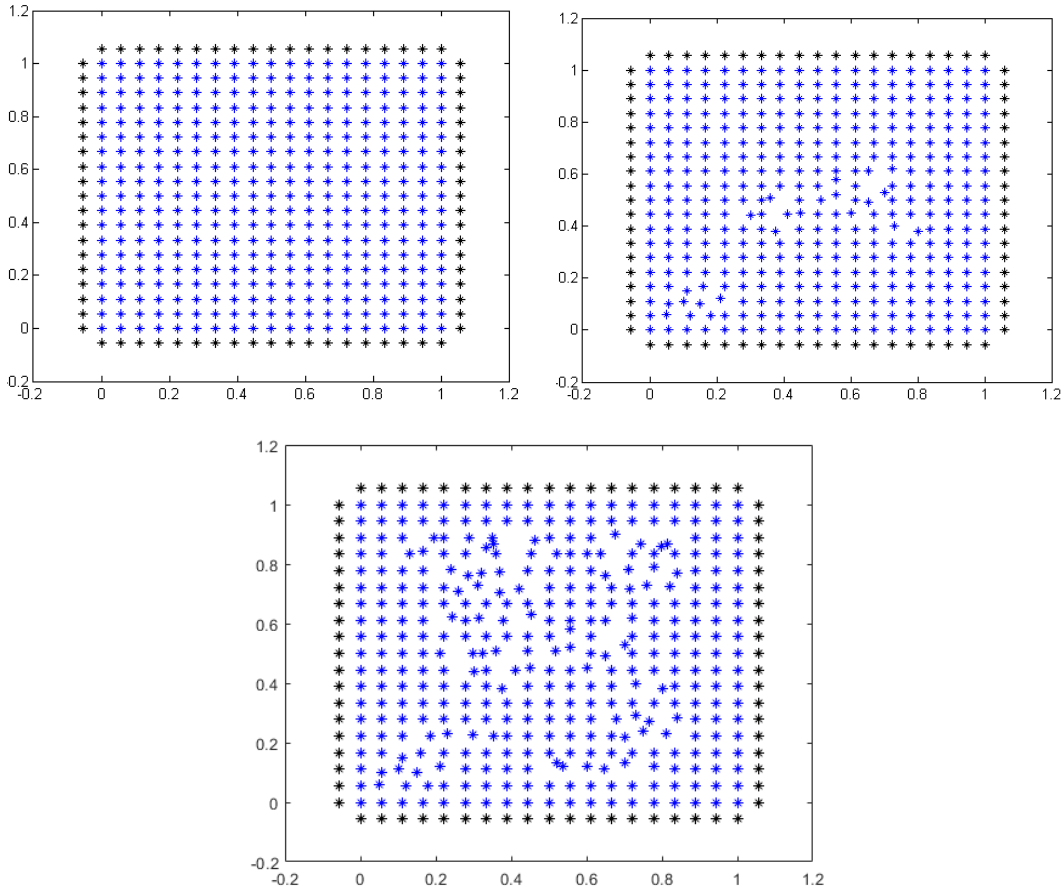

Figure VII.1: Regular and irregular clouds of points

Example 1: $\mu = 0$

For this example we choose $\chi = 0.2$ and $u_0(\mathbf{x}) = 150$, fulfilling $\int_\Omega u_0(\mathbf{x}) > 40\pi$. We use the irregular cloud of points of Figure VII.1. Table VII.1 shows the l^∞-norm of the solution. We obtain that the solution blows up before 1.65 seconds. We present the numerical solution (both u and v) to the system in Figure VII.2 and obtain that, after very slow growth, the solutions become unbounded at finite time.

T(s)	0.3	1	1.6	1.625	1.65
$\|U\|_{l^\infty(\Omega)}$	150.000	150.0041	449.9381	1.4231e+03	-
$\|V\|_{l^\infty(\Omega)}$	150.000	150.0004	165.8602	184.2503	-

Table VII.1: Values of $\|U\|_{l^\infty(\Omega)}$ and $\|V\|_{l^\infty(\Omega)}$ for different time values in the Example 1.

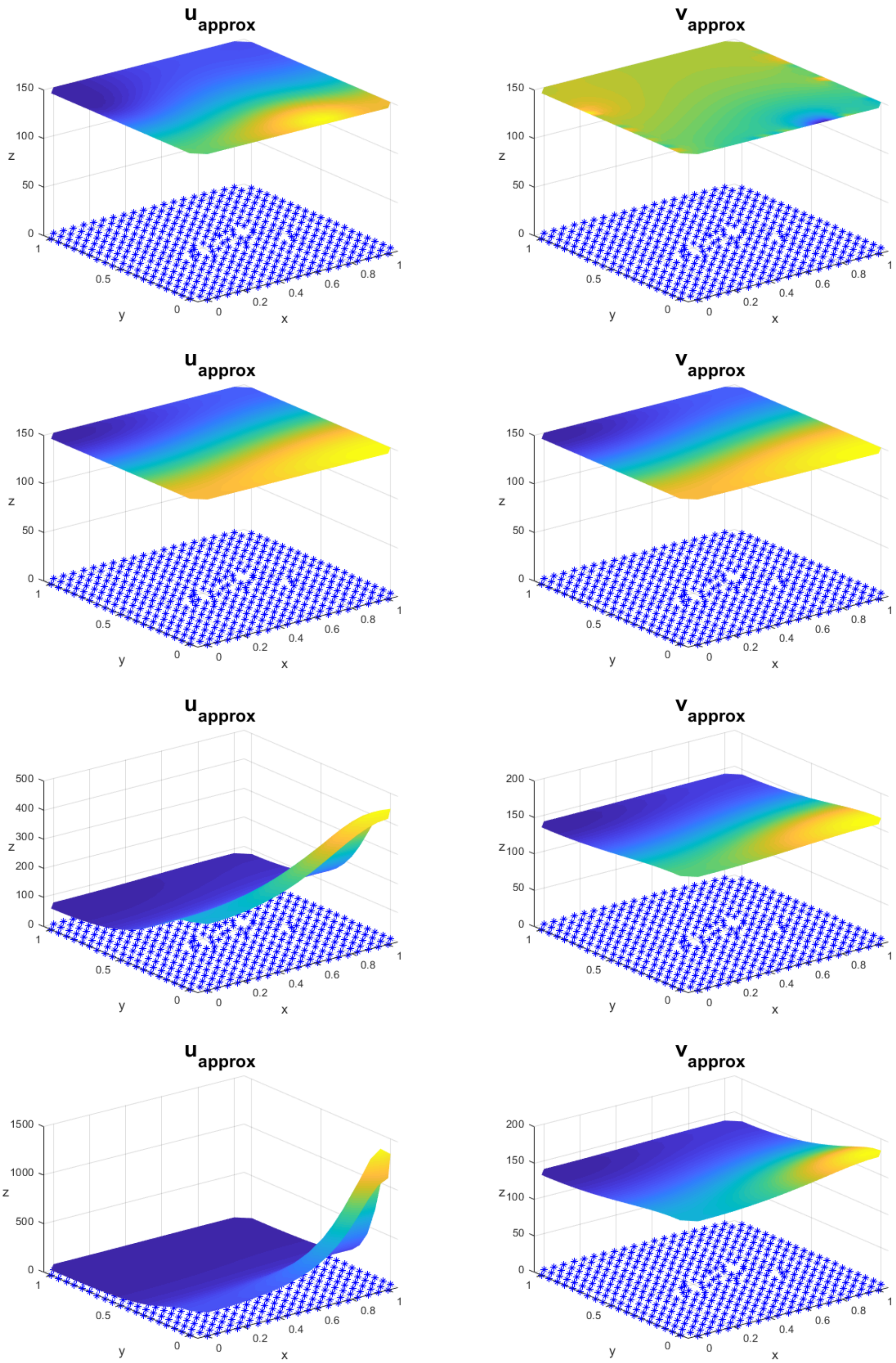

Figure VII.2: U, V-solution for 0.3, 1, 1.6 and 1.625 seconds in the Example 1.

2.2 Case 2

In this second case we use the GFDM to solve system (VII.1) numerically for $f(\mathbf{x}, t) = 0$. It is known that the solution to the parabolic-elliptic system has the following asymptotic behavior:

$$\lim_{t\to\infty} [\|u(\cdot,t) - 1\|_{l^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{l^\infty(\Omega)}] = 0, \tag{VII.22}$$

for any nonnegative continuous initial data $u_0(\mathbf{x})$ (as it was proved in Tello and Winkler [109]).

Example 2

In this second example we consider the following function $f(\mathbf{x}, t) = 0$ and initial data $u_0(\mathbf{x}) = 3e^{-10((x-0.5)^2+(y-0.5)^2)}$. We choose the parameters to be $\mu = 0.7$ and $\chi = 0.2$. In Table VII.2 we present the obtained values of the maximum difference between the approximate solution, U, and the steady state in a regular cloud of points. Figure VII.3 displays the solution to the u-equation.

T(s)	0	0.03	0.06	0.1	1	6
$\|U-1\|_{l^\infty(\Omega)}$	2	0.4953	0.2373	0.1354	0.0616	0.0020
$\|V-1\|_{l^\infty(\Omega)}$	-	0.1247	0.1158	0.1104	0.0616	0.0020

Table VII.2: Values of $\|U-1\|_{l^\infty(\Omega)}$ and $\|V-1\|_{l^\infty(\Omega)}$ for different time values in the Example 2.

Example 3

Now, we consider an irregular cloud of points for the initial data $u_0(\mathbf{x}) = 0.1e^{-10((x-0.3)^2+((y-0.3)^2)}$, the same function $f(\mathbf{x}, t) = 0$ and also $\mu = 0.7$, $\chi = 0.2$. Figure VII.4 provides the solution to the U-equation for different time values. Table VII.3 presents the values of the maximum difference between the numerical solution and the asymptotic value. Due to the small initial density of population, it takes greater time to

T(s)	0	0.03	0.06	0.3	1	6	10
$\|U-1\|_{l^\infty(\Omega)}$	1	0.9979	0.9931	0.9712	0.9497	0.3633	0.0335
$\|V-1\|_{l^\infty(\Omega)}$	-	0.9764	0.9753	0.9688	0.9498	0.3635	0.0335

Table VII.3: Values of $\|U-1\|_{l^\infty(\Omega)}$ and $\|V-1\|_{l^\infty(\Omega)}$ for different time values for the Example 3 in the second cloud of points of Figure VII.1.

become uniform for the solution.

Example 4

We consider for this example the initial data

$$u_0(\mathbf{x}) = 2e^{-10((x-0.8)^2+((y-0.8)^2)}$$

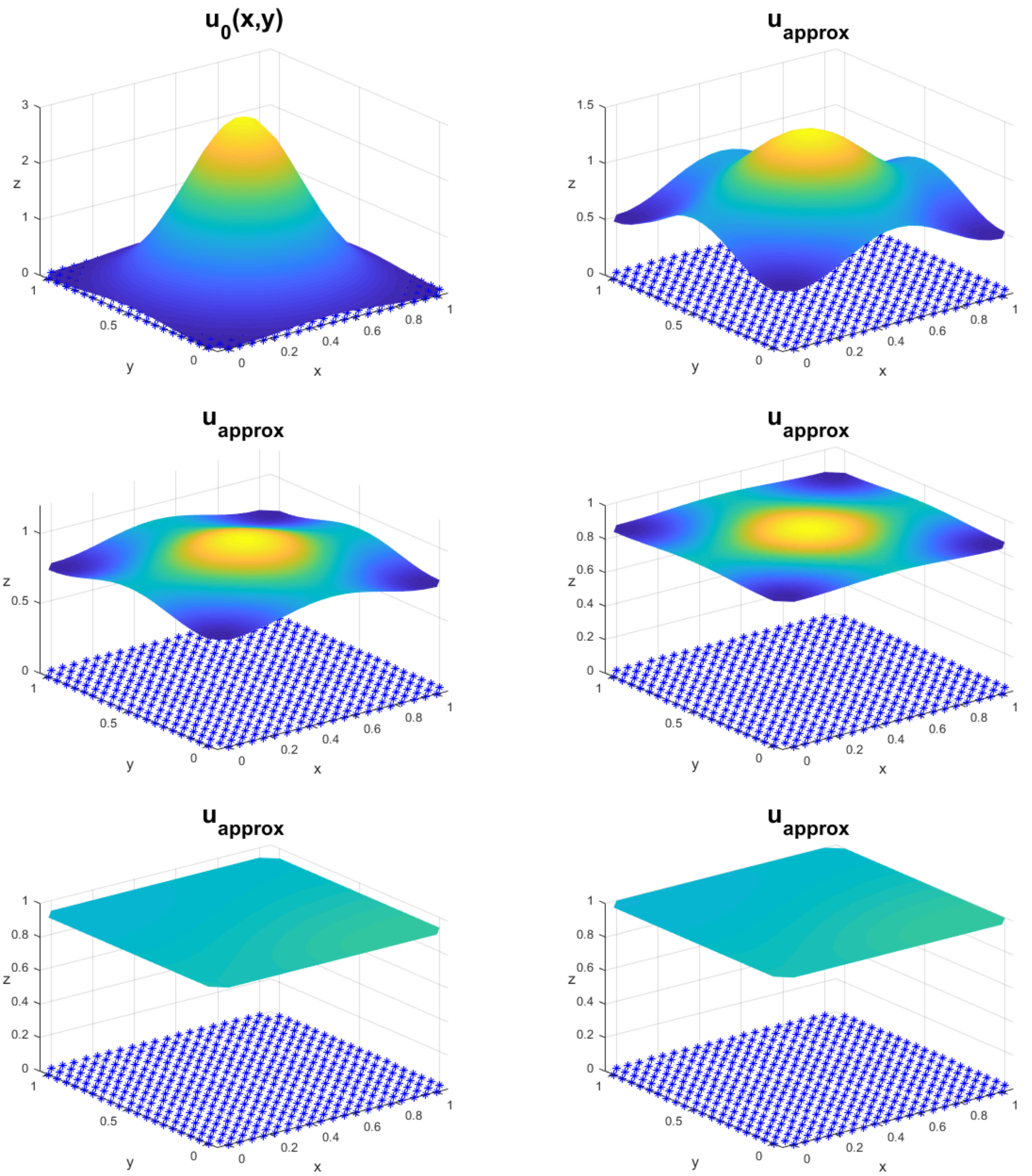

Figure VII.3: U-solution for 0, 0.03, 0.06, 0.1, 1 and 6 seconds in the Example 2.

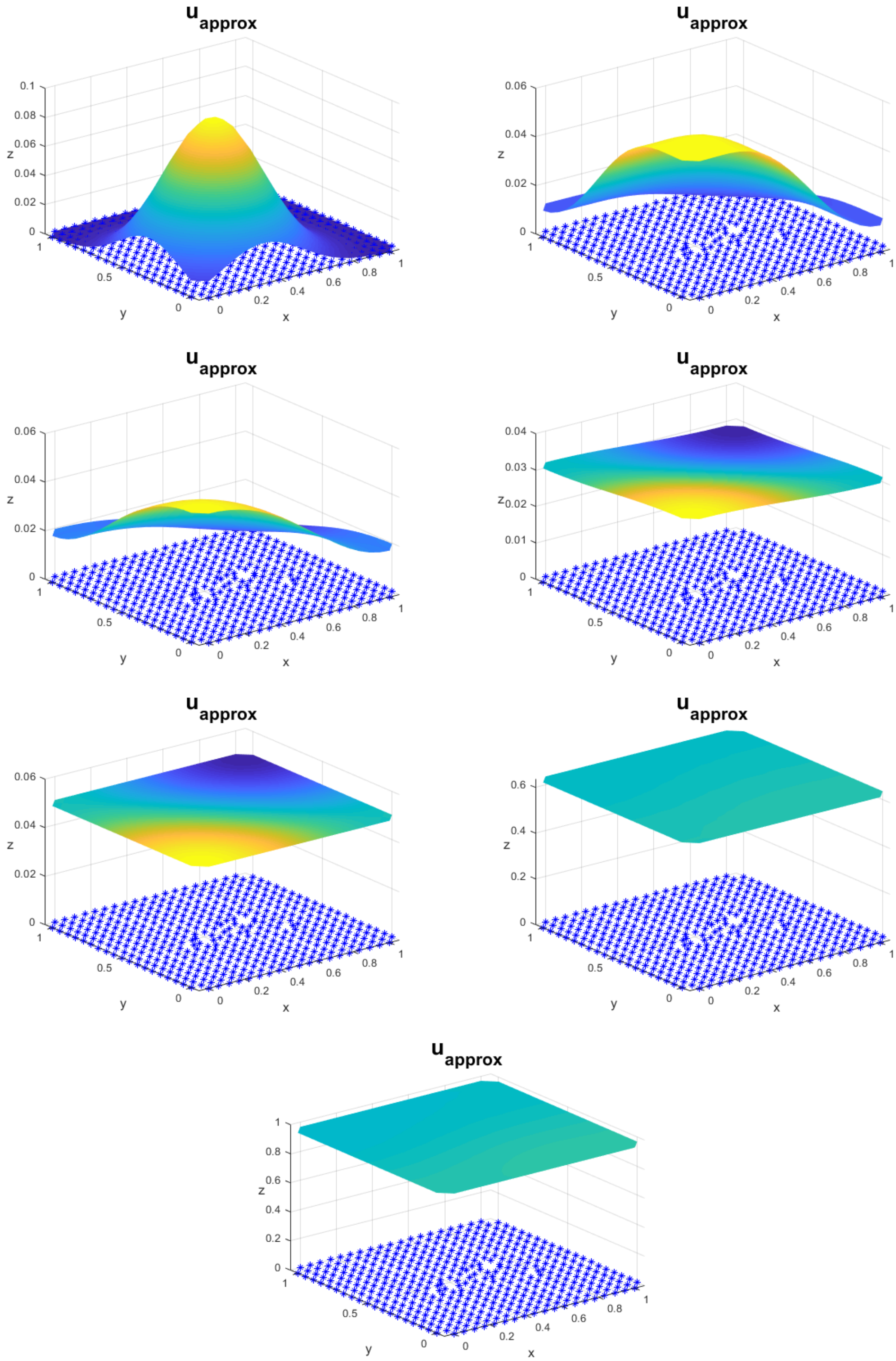

Figure VII.4: U-solution for 0, 0.03, 0.06, 0.3, 1, 6 and 10 seconds in the Example 3.

and the parameters $\mu = 1.5$ and $\chi = 0.5$ in the third cloud of points of Figure VII.1. Table VII.4 shows the l^∞ norm of the difference between the value of the numerical solution and the expected limit at different times. Figure VII.5 plots the $U-$solution at such times.

T(s)	0	0.5	1	3	6	10	15
$\|U-1\|_{l^\infty(\Omega)}$	1	0.4270	0.2545	0.0167	1.8865e-04	4.6750e-07	2.5868e-10
$\|V-1\|_{l^\infty(\Omega)}$	-	0.4202	0.2545	0.0167	1.8868e-04	4.6757e-07	2.5875e-10

Table VII.4: Values of $\|U-1\|_{l^\infty(\Omega)}$ and $\|V-1\|_{l^\infty(\Omega)}$ for different time values for the Example 4 in the third cloud of points of Figure VII.1.

2.3 Case 3

The aim of this third case is to present the asymptotic behavior of the solution (u, v) of the system (VII.1) when we consider a function $f(\mathbf{x}, t)$ fulfilling (VII.2). In order to solve system (VII.1) numerically, we choose the function $f(\mathbf{x}, t)$ to be

$$f(\mathbf{x}, t) = \frac{\cos t}{4 + \sin t} + \frac{x - y}{1 + t^2}, \quad (x, y) \in \Omega = [0, 1] \times [0, 1],\ t > 0.$$

It is easily checked that this function fulfils all the assumptions stated in Chapter II. Then, we find the $2\pi-$periodic function

$$f^*(t) = \frac{\cos t}{4 + \sin t},$$

which, gives us the $2\pi-$periodic limit of the solution (u, v),

$$u^*(t) = \frac{4 + \sin(t)}{4 - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}}.$$

As in the previous one, we divide this case into the regular and irregular cloud of points of Figure VII.1, to see that in both situations, the solution to (VII.1) inherits the periodic behavior of the function $f(\mathbf{x}, t)$:

$$\lim_{t\to\infty} [\|u(\cdot, t) - u^*(t)\|_{l^\infty(\Omega)} + \|v(\cdot, t) - u^*(t)\|_{l^\infty(\Omega)}] = 0. \qquad \text{(VII.23)}$$

Example 5

As stated, let us consider the function $f(\mathbf{x}, t) = \frac{\cos t}{4+\sin t} + \frac{x-y}{1+t^2}$. Assume $u_0(\mathbf{x}) = 5e^{-10((x-0.6)^2+(y-0.6)^2)}$ as initial data and the relation of parameters $\mu = 1$ and $\chi = 0.3$. Table VII.5 presents the values of the limit function $u^*(t)$ and maximum difference between this and the numerical solution.

Figure VII.6 shows the asymptotic solution u^* (solid line) and the most distant values of approximation at different times. As we see, the numerical solution is also periodic and the $l^\infty(\Omega)$ of the difference is small for large enough times.

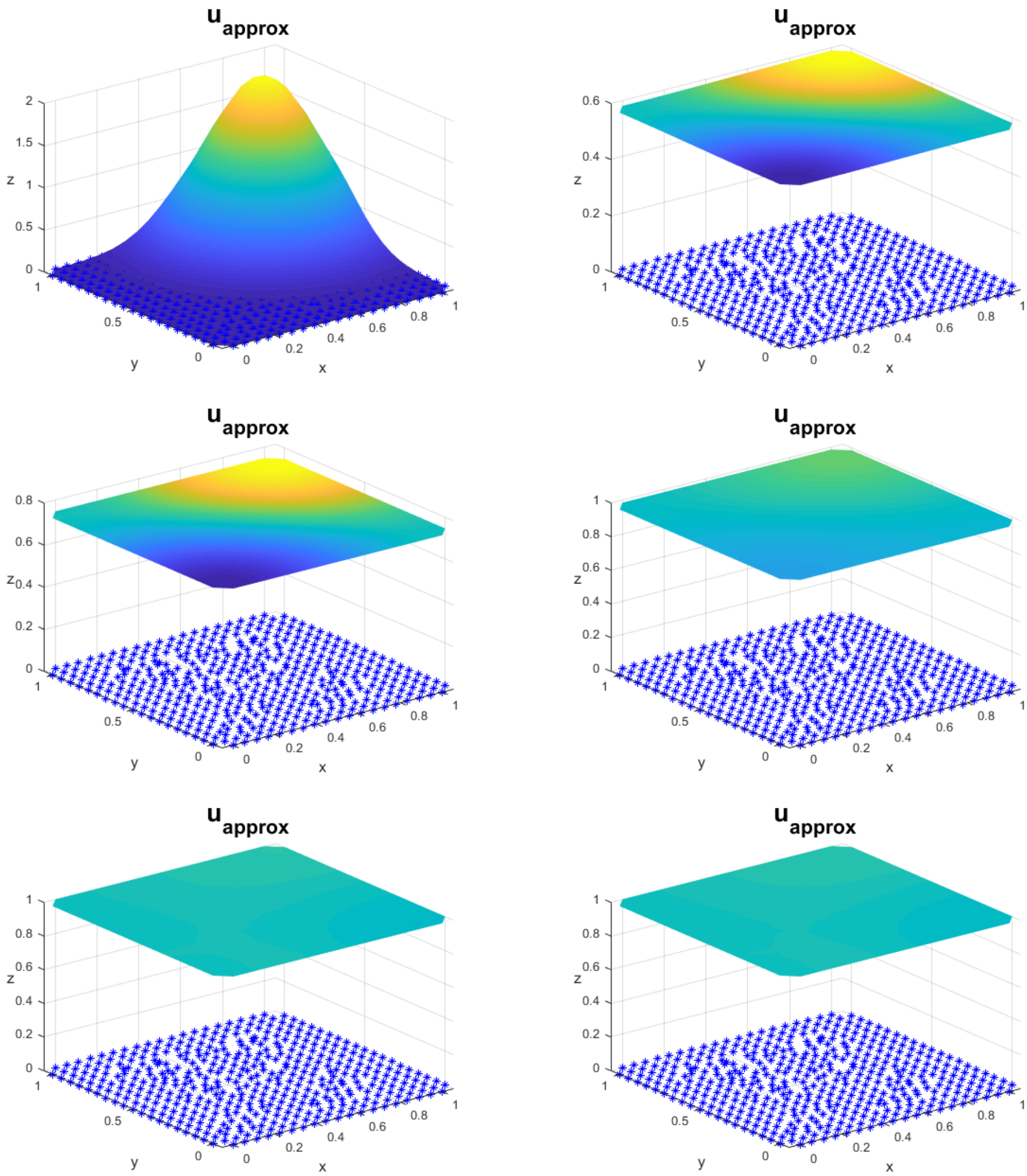

Figure VII.5: U-solution for 0, 0.5, 1, 3, 6 and 10 seconds in the Example 4.

T(s)	1.5	3	4.5	6	1.5+2π	3+2π
u^*	1.1197	0.9070	0.8357	1.1007	1.1200	0.9075
$\|U-u^*\|_{l^\infty(\Omega)}$	0.0707	1.4721e-02	5.1705e-03	3.1418e-03	1.4587e-03	9.4562e-04
$\|V-u^*\|_{l^\infty(\Omega)}$	0.0432	0.0079	0.0021	7.9842e-04	2.6320e-04	5.3863e-05

Table VII.5: Values of the function $u^*(t)$ and the differences $\|U-u^*\|_{l^\infty(\Omega)}$ and $\|V-u^*\|_{l^\infty(\Omega)}$ in the Example 5 in the regular cloud of points of Figure VII.1.

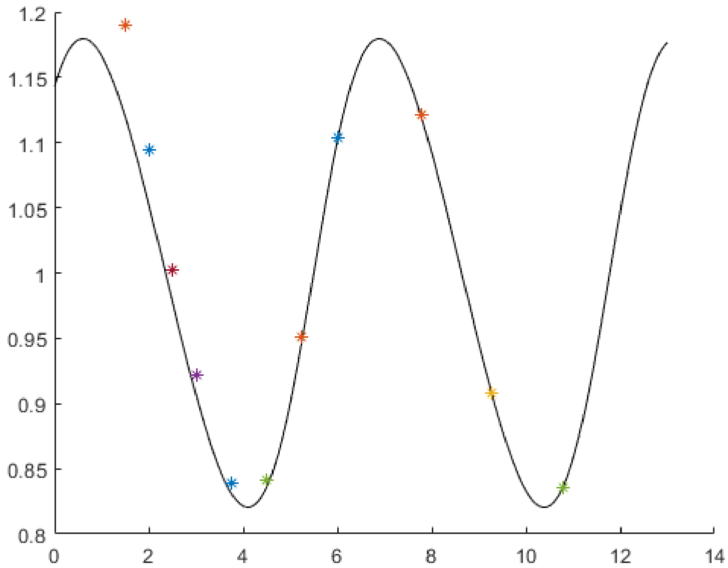

Figure VII.6: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the most distant value of the approximate solution U at such time in Example 5.

Example 6

For this last example we also consider the function $f(\mathbf{x},t)$ of the previous one as well as the same relation of parameters. Let us use now the initial data

$$u_0(\mathbf{x}) = e^{-10[(x-0.7)^2+(y-0.7)^2]},$$

and test the method in the second cloud of points of Figure VII.1. Table VII.6 presents the values of the limit function $u^*(t)$ and maximum difference between this and the numerical solution. Note that in Figure VII.7, it takes longer for the approximate solution to reach

T(s)	1.5	3	6	1.5+2π	3+2π	6+2π
u^*	1.1197	0.9070	1.1007	1.1200	0.9075	1.1001
$\|U-u^*\|_{l^\infty(\Omega)}$	0.4401	0.1120	0.0118	0.0028	0.0012	6.9721e-04
$\|V-u^*\|_{l^\infty(\Omega)}$	0.4234	0.1058	0.0098	0.0014	2.8953e-04	1.0059e-04

Table VII.6: Values of the function $u^*(t)$ and the differences $\|U-u^*\|_{l^\infty(\Omega)}$ and $\|V-u^*\|_{l^\infty(\Omega)}$ in the Example 6 in the second cloud of points of Figure VII.1.

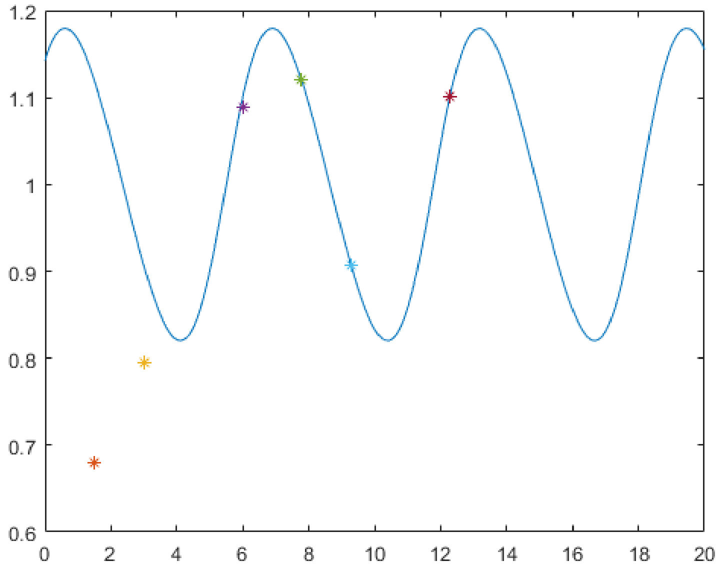

Figure VII.7: The solid line corresponds to the graphic of the function u^*, the stars to the most distant value of the approximate solution U at such time in Example 6.

the limit value $u^*(t)$ because the initial data is smaller than in Example 5. This means that the population's density grows slowly at initial times. Also notice that $u^*(t)$ is not a solution of the system but an asymptotic value for the exact solution.

3 Generalization

In this section we study the discrete version of (VII.24) and its numerical asymptotic behavior.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi_1 \nabla(u\nabla w) + \mu_1 u\left(1 - a_1 v - u + f_1(x,t)\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - \chi_2 \nabla(v\nabla w) + \mu_2 v\left(1 - v - a_2 u + f_2(x,t)\right), & x \in \Omega,\ t > 0, \\ -\Delta w + w = \alpha u + \beta v & x \in \Omega,\ t > 0. \end{cases} \tag{VII.24}$$

We proved in Chapter II that the solution to (VII.24) presents the following asymptotic behavior

$$\lim_{t\to\infty}\left(\|u-u^*\|_{L^\infty(\Omega)} + \|v-v^*\|_{L^\infty(\Omega)} + \|w-\alpha u^* - \beta v^*\|_{L^\infty(\Omega)}\right) = 0,$$

where (u^*, v^*) is the unique positive T-periodic solution of the system

$$\begin{cases} \widetilde{u}_t(t) = \mu_1 \widetilde{u}(t)\left(1 - a_1\widetilde{v}(t) - \widetilde{u}(t) + f_1^*(t)\right), \\ \widetilde{v}_t(t) = \mu_2 \widetilde{v}(t)\left(1 - a_2\widetilde{u}(t) - \widetilde{v}(t) + f_2^*(t)\right). \end{cases} \tag{VII.25}$$

Let us consider the explicit formulae (VI.3) of Chapter VI together with the first time derivative approximation (VI.4). The first equation of (VII.24) is discretized as

$$\begin{cases} U_0^{n+1} = U_0^n + \Delta t \left[-\lambda_{00} U_0^n + \sum_{i=1}^{s} \lambda_{i0} U_i^n \right] - \\ \quad - \chi_1 \Delta t \left(-\lambda_{01} U_0^n + \sum_{i=1}^{s} \lambda_{i1} U_i^n \right) \left(-\lambda_{01} W_0^n + \sum_{i=1}^{s} \lambda_{i1} W_i^n \right) - \\ \quad - \chi_1 \Delta t \left(-\lambda_{02} U_0^n + \sum_{i=1}^{s} \lambda_{i2} U_i^n \right) \left(-\lambda_{02} W_0^n + \sum_{i=1}^{s} \lambda_{i2} W_i^n \right) + \\ \quad + \Delta t \mu_1 U_0^n \left[1 - a_1 V_0^n - U_0^n - f_1(\mathbf{x}_0, n\Delta t) \right] + \\ \quad + \Delta t \chi_1 U_0^n (-W_0^n + \alpha U_0^n + \beta V_0^n) + \mathcal{O}(\Delta t, h_i^2, k_i^2), \end{cases} \tag{VII.26}$$

and the second one as

$$\begin{cases} V_0^{n+1} = V_0^n + \Delta t \left[-\lambda_{00} V_0^n + \sum_{i=1}^{s} \lambda_{i0} V_i^n \right] - \\ \quad - \chi_2 \Delta t \left(-\lambda_{01} V_0^n + \sum_{i=1}^{s} \lambda_{i1} V_i^n \right) \left(-\lambda_{01} W_0^n + \sum_{i=1}^{s} \lambda_{i1} W_i^n \right) - \\ \quad - \chi_2 \Delta t \left(-\lambda_{02} V_0^n + \sum_{i=1}^{s} \lambda_{i2} V_i^n \right) \left(-\lambda_{02} W_0^n + \sum_{i=1}^{s} \lambda_{i2} W_i^n \right) + \\ \quad + \Delta t \mu_2 V_0^n \left[1 - V_0^n - a_2 U_0^n - f_2(\mathbf{x}_0, n\Delta t) \right] + \\ \quad + \Delta t \chi_2 V_0^n (-W_0^n + \alpha U_0^n + \beta V_0^n) + \mathcal{O}(\Delta t, h_i^2, k_i^2). \end{cases} \tag{VII.27}$$

Third equation becomes, simply,

$$W_0^n - \left[-\lambda_{00} W_0^n + \sum_{i=1}^{s} \lambda_{i0} W_i^n \right] = \alpha U_0^n + \beta V_0^n. \tag{VII.28}$$

The following stability criterion is verified by the GFDM solutions of (VII.24), i.e., a conditional convergence result for the numerical model is given in the following theorem.

Theorem VII.2. *Let $u, v, w \in C^{2,1}(\bar{\Omega} \times [0, \infty))$ be the exact solution to system (VII.24). The GFD explicit scheme (VII.26)–(VII.28) is convergent under the condition*

$$\Delta t < \omega \tag{VII.29}$$

for some given positive ω defined by (VII.51) depending on the parameters of the problem and the distribution of the points.

Proof. Since the exact values u, v and w must fulfil system (VII.24), we take the difference between the exact expression and the GFD scheme. For simplicity, let us call $\tilde{u}_0^n := u_0^n - U_0^n$, the difference of the continuous and the discrete solution at the point $\mathbf{x}_0$ (and in the same manner we define $\tilde{u}_i^n, \tilde{v}_i^n$ and $\tilde{w}_i^n$). By the great symmetry of the scheme, we only perform the computations explicitly for the most significant terms, as the rest are analogously treated.

First, we compute the term containing the chemotaxis constant χ_1 of (VII.26):

$$
\begin{aligned}
-\chi_1 \left(-\lambda_{01} u_0^n + \sum_{i=1}^{s} \lambda_{i1} u_i^n\right) \left(-\lambda_{01} w_0^n + \sum_{i=1}^{s} \lambda_{i1} w_i^n\right) = \\
-\chi_1 \lambda_{01}^2 u_0^n w_0^n + +\chi_1 \lambda_{01} w_0^n \sum_{i=1}^{s} \lambda_{i1} u_i^n + \chi_1 \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} w_i^n - \\
-\chi_1 \sum_{i=1}^{s} \lambda_{i1} u_i^n \sum_{i=1}^{s} \lambda_{i1} w_i^n.
\end{aligned}
\tag{VII.30}
$$

For the first term on the right side in expression (VII.30) we have

$$
\begin{aligned}
-\chi_1 (\lambda_{01})^2 u_0^n w_0^n + \chi_1 (\lambda_{01})^2 U_0^n W_0^n \pm \chi_1 (\lambda_{01})^2 U_0^n w_0^n = \\
-\chi_1 (\lambda_{01})^2 \tilde{u}_0^n w_0^n - \chi_1 (\lambda_{01})^2 U_0^n \tilde{w}_0^n.
\end{aligned}
\tag{VII.31}
$$

Operating in a similar fashion, it yields

$$
\begin{aligned}
\chi_1 \lambda_{01} w_0^n \sum_{i=1}^{s} \lambda_{i1} u_i^n - \chi_1 \lambda_{01} W_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n \pm \chi_1 \lambda_{01} w_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n = \\
\chi_1 \lambda_{01} w_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n + \chi_1 \lambda_{01} \tilde{w}_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n,
\end{aligned}
\tag{VII.32}
$$

as well as

$$
\begin{aligned}
\chi_1 \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} w_i^n - \chi_1 \lambda_{01} U_0^n \sum_{i=1}^{s} \lambda_{i1} W_i^n \pm \chi_1 \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} W_i^n = \\
\chi_1 \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{w}_i^n + \chi_1 \lambda_{01} \tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i1} W_i^n.
\end{aligned}
\tag{VII.33}
$$

For the last term in (VII.30), by adding $\pm \chi_1 \sum_{i=1}^{s} \lambda_{i1} U_i^n \sum_{i=1}^{s} \lambda_{i1} w_i^n$, we get

$$
\begin{aligned}
-\chi_1 \sum_{i=1}^{s} \lambda_{i1} u_i^n \sum_{i=1}^{s} \lambda_{i1} w_i^n + \chi_1 \sum_{i=1}^{s} \lambda_{i1} U_i^n \sum_{i=1}^{s} \lambda_{i1} W_i^n = \\
-\chi_1 \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n \sum_{i=1}^{s} \lambda_{i1} w_i^n - \chi_1 \sum_{i=1}^{s} \lambda_{i1} U_i^n \sum_{i=1}^{s} \lambda_{i1} \tilde{w}_i^n.
\end{aligned}
\tag{VII.34}
$$

Also, we have that

$$
-(u_0^n)^2 + (U_0^n)^2 = -\tilde{u}_0^n (u_0^n + U_0^n), \tag{VII.35}
$$

and

$$
-a_1 u_0^n w_0^n + a_1 U_0^n W_0^n = -a_1 \tilde{u}_0^n w_0^n - a_1 U_0^n \tilde{w}_0^n. \tag{VII.36}
$$

Now, by considering (VII.30)–(VII.36) and after applying the same processing to the rest of the terms, we obtain

$$
\begin{aligned}
\tilde{u}_0^{n+1} = {} & \tilde{u}_0^n - \Delta t \lambda_{00} \tilde{u}_0^n + \Delta t \sum_{i=1}^{s} \lambda_{i0} \tilde{u}_i^n - \Delta t \chi_1 (\lambda_{01})^2 \tilde{u}_0^n w_0^n - \\
& - \Delta t \chi_1 (\lambda_{01})^2 U_0^n \tilde{w}_0^n + \Delta t \chi_1 \lambda_{01} w_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n + \Delta t \chi_1 \lambda_{01} \tilde{w}_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n + \\
& + \Delta t \chi_1 \lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{w}_i^n + \Delta t \chi_1 \lambda_{01} \tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i1} W_i^n - \\
& - \Delta t \chi_1 \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n \sum_{i=1}^{s} \lambda_{i1} w_i^n - \Delta t \chi_1 \sum_{i=1}^{s} \lambda_{i1} U_i^n \sum_{i=1}^{s} \lambda_{i1} \tilde{w}_i^n - \\
& - \Delta t \chi_1 (\lambda_{02})^2 \tilde{u}_0^n w_0^n - \Delta t \chi_1 (\lambda_{02})^2 U_0^n \tilde{w}_0^n + \Delta t \chi_1 \lambda_{02} w_0^n \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n + \\
& + \Delta t \chi_1 \lambda_{02} \tilde{w}_0^n \sum_{i=1}^{s} \lambda_{i2} U_i^n + \Delta t \chi_1 \lambda_{02} u_0^n \sum_{i=1}^{s} \lambda_{i2} \tilde{w}_i^n + \\
& + \Delta t \chi_1 \lambda_{02} \tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i2} W_i^n - \Delta t \chi_1 \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \sum_{i=1}^{s} \lambda_{i2} w_i^n - \\
& - \Delta t \chi_1 \sum_{i=1}^{s} \lambda_{i2} U_i^n \sum_{i=1}^{s} \lambda_{i2} \tilde{w}_i^n + \Delta t \alpha \chi_1 \tilde{u}_0^n (u_0^n + U_0^n) \\
& + \Delta t \mu_1 \tilde{u}_0^n - a_1 \mu_1 \Delta t \tilde{u}_0^n v_0^n - a_1 \mu_1 \Delta t U_0^n \tilde{v}_0^n - \Delta t \mu_1 \tilde{u}_0^n (u_0^n + U_0^n) + \\
& + \mu_1 \tilde{u}_0^n f_1(\mathbf{x}_0, n\Delta t) + \beta \chi_1 \Delta t \tilde{u}_0^n v_0^n + \beta \chi_1 \Delta t U_0^n \tilde{v}_0^n - \chi_1 \Delta t \tilde{u}_0^n w_0^n - \\
& - \chi_1 \Delta t U_0^n \tilde{w}_0^n.
\end{aligned}
\tag{VII.37}
$$

Let us take bounds in the last expression and call $\tilde{u}^n = \max_{i=0,\ldots,s} \{|\tilde{u}_i^n|\}$ (also we define $\tilde{v}^n$ and $\tilde{w}^n$ in the same way), so we arrive to

$$
\tilde{u}^{n+1} \leq A_1 \tilde{u}^n + B_1 \tilde{v}^n + C_1 \tilde{w}^n, \tag{VII.38}
$$

where the positive coefficients A_1, B_1 and C_1 are clearly given by

$$\begin{aligned}
A_1 :=& \Bigg|1 - \Delta t\Bigg(\lambda_{00} + \chi_1\lambda_{01}^2 w_0^n - \chi_1\lambda_{01}\sum_{i=1}^{s}\lambda_{i1}W_i^n + \chi_1\lambda_{02}^2 w_0^n - \\
&- \chi_1\lambda_{02}\sum_{i=1}^{s}\lambda_{i2}W_i^n - \alpha\chi_1(u_0^n + U_0^n) - \mu_1 + a_1\mu_1 v_0^n + \\
&+ \mu_1(u_0^n + U_0^n) - \mu_1 f_1(\mathbf{x}_0, n\Delta t) - \beta\chi_1 v_0^n + \chi_1 w_0^n\Bigg)\Bigg| + \\
&+ \Delta t\Bigg(\sum_{i=1}^{s}|\lambda_{i0}| + |\chi_1\lambda_{01}w_0^n|\sum_{i=1}^{s}|\lambda_{i1}| + \\
&+ |\chi_1\sum_{i=1}^{s}\lambda_{i1}w_i^n|\sum_{i=1}^{s}|\lambda_{i1}| + |\chi_1\lambda_{02}w_0^n|\sum_{i=1}^{s}|\lambda_{i2}| + \\
&+ |\chi_1\sum_{i=1}^{s}\lambda_{i2}w_i^n|\sum_{i=1}^{s}|\lambda_{i2}|\Bigg),
\end{aligned} \tag{VII.39}$$

$$B_1 := \Delta t|-a_1\mu_1 U_0^n \tilde{v}_0^n + \beta\chi_1 U_0^n \tilde{v}_0^n|, \tag{VII.40}$$

and

$$\begin{aligned}
C_1 :=& \Delta t\Bigg|-\chi_1\lambda_{01}^2 U_0^n + \chi_1\lambda_{01}\sum_{i=1}^{s}\lambda_{i1}U_i^n - \chi_1\lambda_{02}^2 U_0^n + \\
&+ \chi_1\lambda_{02}\sum_{i=1}^{s}\lambda_{i2}U_i^n - -\chi_1\alpha_1 U_0^n\Bigg| + \Delta t|\chi_1\lambda_{01}u_0^n|\sum_{i=1}^{s}|\lambda_{i1}| + \\
&+ \Delta t|\chi_1\sum_{i=1}^{s}\lambda_{i1}U_i^n|\sum_{i=1}^{s}|\lambda_{i1}| + \Delta t|\chi_1\lambda_{02}u_0^n|\sum_{i=1}^{s}|\lambda_{i2}| + \\
&+ \Delta t|\chi_1\sum_{i=1}^{s}\lambda_{i2}U_i^n|\sum_{i=1}^{s}|\lambda_{i2}|.
\end{aligned} \tag{VII.41}$$

We rewrite now A_1 as $A_1 = |1 - \Delta t A_1'| + \Delta t A_1''$, for an obvious choice of A_1', A_1''. By applying the previous arguments to (VII.27), we can write

$$\tilde{v}^{n+1} \leq A_2\tilde{u}^n + B_2\tilde{v}^n + C_2\tilde{w}^n. \tag{VII.42}$$

For the third equation of (VII.24),

$$\tilde{w}^n\left[|1 + \lambda_{00}| + \sum_{i=1}^{s}|\lambda_{i0}|\right] \geq \alpha\tilde{u}^n + \beta\tilde{v}^n. \tag{VII.43}$$

So, in particular,

$$\tilde{u}^n \leq \alpha^{-1}\tilde{w}^n\left[|1 + \lambda_{00}| + \sum_{i=1}^{s}|\lambda_{i0}|\right] := D_1\tilde{w}^n, \tag{VII.44}$$

$$\tilde{v}^n \leq \beta^{-1}\tilde{w}^n\left[|1 + \lambda_{00}| + \sum_{i=1}^{s}|\lambda_{i0}|\right] := E_1\tilde{w}^n.$$

In view of (VII.43) and (VII.44), we can formulate (VII.38) and (VII.42) as follows

$$\tilde{u}^{n+1} \leq (A_1 D_1 + B_1 E_1 + C_1)\tilde{w}^n := \rho_u \tilde{w}^n \tag{VII.45}$$

and

$$\tilde{v}^{n+1} \leq (A_2 D_1 + B_2 E_1 + C_2)\tilde{w}^n := \rho_v \tilde{w}^n. \tag{VII.46}$$

In order to guarantee the convergence of the numerical scheme, we impose that

$$\max\{\rho_u, \rho_v\} < 1. \tag{VII.47}$$

Consider the first entry of the maximum and notice that the condition is equivalent to (let us write $B_1 = \Delta t B_1^{'}, C_1 = \Delta t C_1^{'}$)

$$|1 - \Delta t A_1^{'}| < \frac{1 - \Delta t(B_1^{'} E_1 + C_1^{'})}{D_1} - \Delta t A_1^{''}. \tag{VII.48}$$

One inequality holds directly by assuming

$$\Delta t < \frac{D_1 + 1}{D_1(A_1^{'} + A_1^{''}) + \xi}, \tag{VII.49}$$

where we call $\xi := B_1^{'} E_1 + C_1^{'}$ (similarly, $\eta := A_2^{'} D_2 + C_2^{'}$). The other inequality is also directly fulfilled since it is equivalent to

$$\Delta t > \frac{D_1 - 1}{D_1(A_1^{'} - A_1^{''}) - \xi} \tag{VII.50}$$

and the denominator is negative. The same arguments are used for the second entry of (VII.47) and that finishes the proof. In Theorem VII.2, in (VII.29), we have

$$\omega := \min\left\{\frac{D_1 + 1}{D_1(A_1^{'} + A_1^{''}) + \xi}, \frac{E_1 + 1}{E_1(B_2^{'} + B_2^{''}) + \eta}\right\}. \tag{VII.51}$$

□

3.1 Numerical examples

In this section we illustrate the application of the GFDM for solving the parabolic-parabolic-elliptic system (VII.24) by testing the method using the irregular clouds of points of Figure VII.8, both of them containing 437 nodes. In all the following computations we use an 8-points star. We choose $\Delta t = 0.001$, fulfilling the assumption made in Theorem VII.2. We present examples of the asymptotic convergence of the solution to the periodic functions $u^*(t), v^*(t)$.

Remark VII.1. *Note that for this generalized version of the parabolic-elliptic model we do not possess the explicit expressions of the asymptotic values u^* and v^*. Therefore we perform standard numerical method in (VII.25). In particular, we use the ODE45 function of Matlab R2019b.*

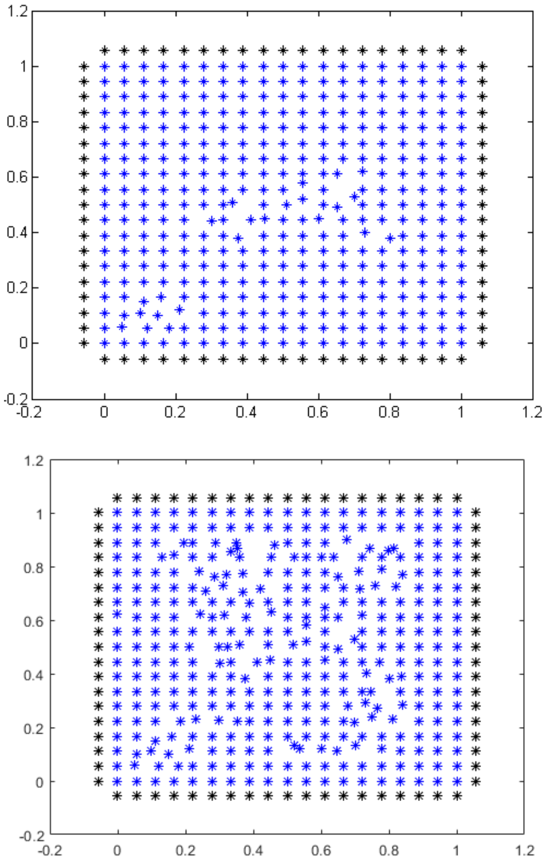

Figure VII.8: Clouds of points.

Example 1

Consider for this first example, in the first irregular cloud of points, the following initial data

$$u_0(x,y) = e^{-10[(x-0.2)^2+(y-0.2)^2]} + e^{-10[(x-0.8)^2+(y-0.8)^2]},$$

$$v_0(x,y) = e^{-10[(x-0.5)^2+(y-0.5)^2]}.$$

As parameters, we choose

$$\beta = \alpha = 1, \quad \chi_1 = \chi_2 = 0.5, \quad \mu_1 = \mu_2 = 1.5, \quad a_1 = a_2 = 0.5$$

and we take the periodic functions in the time variable

$$f_1(x,y,t) = \frac{\cos(t)}{4+\sin(t)} + \frac{x-y}{t^2+1},$$

$$f_2(x,y,t) = \frac{2\cos(t)}{2+\sin(t)} + \frac{\sin^2(t/2)+2\sin(t)}{1+\cos^2(t/2)}.$$

In Table VII.7 we show the l^∞-norm of the difference between the numerical solution and the asymptotic continuous one for times $t = 1, 2.5$ and 5 seconds. In Figures VII.9,

time(s)	1	2.5	5
$\|U-u^*\|_{l^\infty}$	0.0025	0.0007	$1.220 \cdot 10^{-3}$
$\|V-v^*\|_{l^\infty}$	0.0806	0.0005	$3.117 \cdot 10^{-3}$
$\|W-w^*\|_{l^\infty}$	0.1128	0.0073	$1.553 \cdot 10^{-3}$

Table VII.7: Values of l^∞ of the errors in the Example 1.

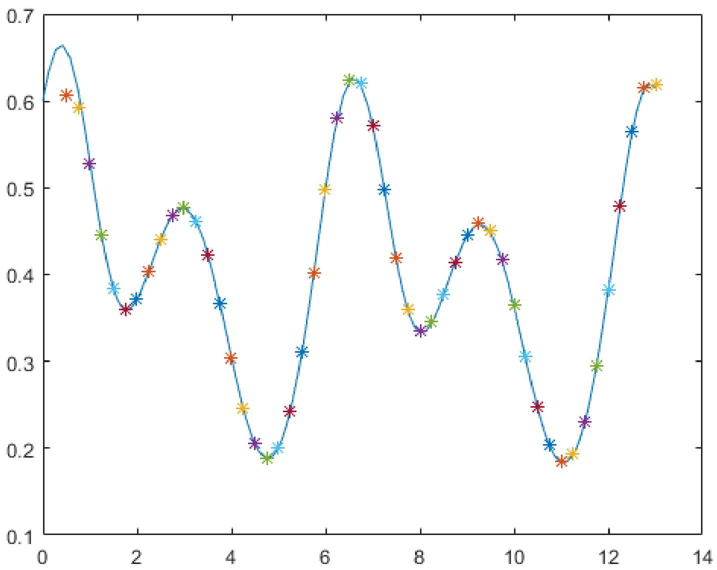

Figure VII.9: Asymptotic numerical $u-$solution (stars) and continuous solution (solid lines) for the Example 1.

VII.10 and VII.11 we plot the continuous asymptotic values (solid blue lines) and the most distant values (that is to say, where the greatest error is performed) of the discrete asymptotic limit (stars). As seen before, after one second the approximation given by the GFD scheme is accurate.

Example 2

In the second irregular cloud of points of Figure VII.8, we take the following initial data

$$u_0(x,y) = 2(x^2+y^2), \quad v_0(x,y) = (x-1)^2 + (y-1)^2,$$

and parameters

$$\chi_1 = 0.5, \chi_2 = -1, \quad \mu_1 = 7, \mu_2 = 9, \quad a_1 = 0.25, a_2 = 0.15,$$
$$\alpha = 0.5, \quad \beta = 1.5.$$

As source terms we pick

$$f_1(x,y,t) = \frac{\cos(t)}{4+\sin(t)} + \frac{x-y}{t^2+1},$$

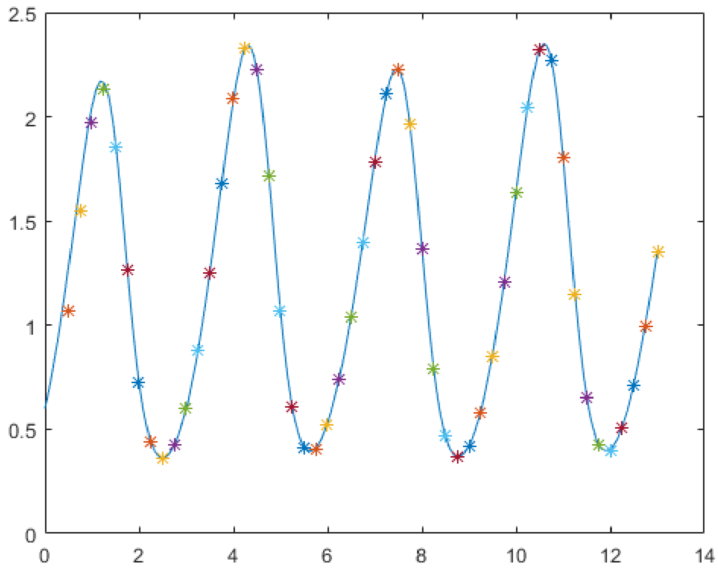

Figure VII.10: Asymptotic numerical $v-$solution (stars) and continuous solution (solid lines) for the Example 1.

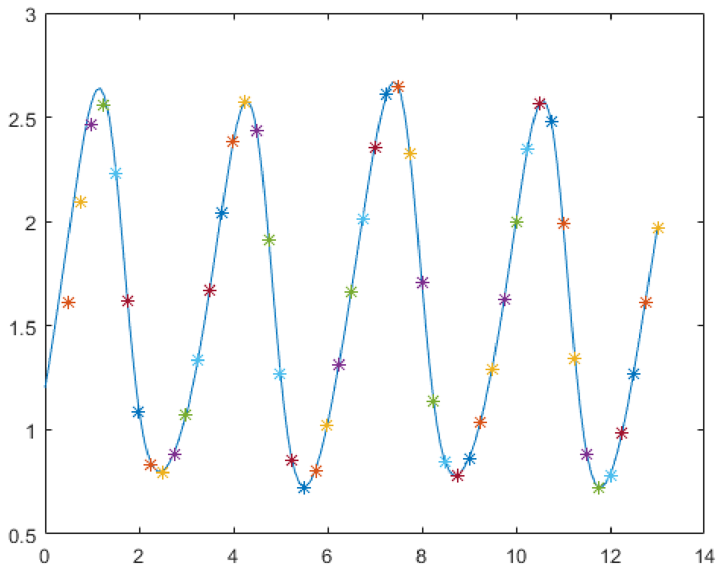

Figure VII.11: Asymptotic numerical $w-$solution (stars) and continuous solution (solid lines) for the Example 1

time(s)	1	2.5	5
$\|U-u^*\|_{l^\infty}$	0.1568	0.0347	$9.016 \cdot 10^{-3}$
$\|V-v^*\|_{l^\infty}$	0.0165	0.0033	$8.771 \cdot 10^{-3}$
$\|W-w^*\|_{l^\infty}$	0.0095	0.0059	$3.137 \cdot 10^{-3}$

Table VII.8: Values of l^∞ of the errors in the Example 2.

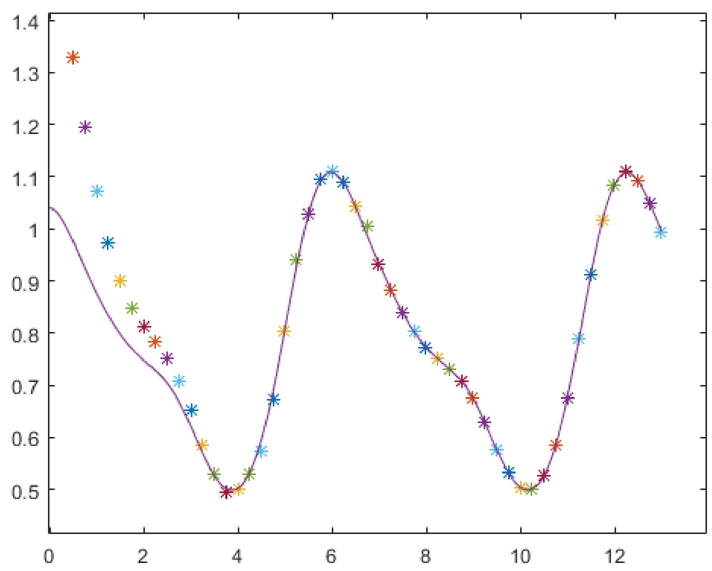

Figure VII.12: Asymptotic numerical $u-$solution (stars) and continuous solution (solid lines) for the Example 2.

$$f_2(x,y,t) = \frac{\sin(t)}{4+\sin(t)}.$$

Table VII.8 collects the $l^\infty-$norm of the difference between the numerical and the asymptotic exact solution for different times. Figures VII.12, VII.13 and VII.14 plots, respectively, the graph of the asymptotic solution (solid lines) and the value of the numerical solution where the greatest error is obtained.

Example 3: blow-up solutions

For this last example we choose the initial data:

$$u_0(x,y) = 3(\sin(x+y)+1), \quad v_0(x,y) = \cos(x+y)+1,$$

in the first irregular cloud of points. Let us take as parameters

$$\chi_1 = 8, \chi_2 = 3, \quad \mu_1 = 2, \mu_2 = 1, \quad a_1 = 0.25, a_2 = 0.5,$$

$$\alpha = \beta = 1,$$

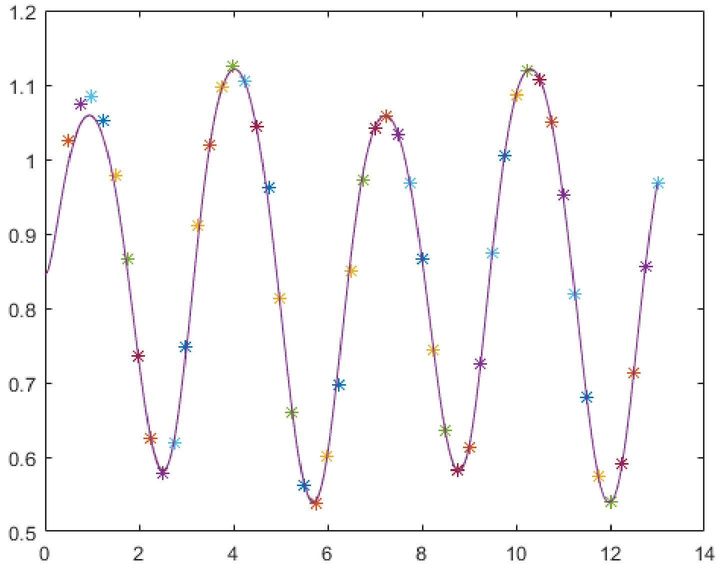

Figure VII.13: Asymptotic numerical $v-$solution (stars) and continuous solution (solid lines) for the Example 2.

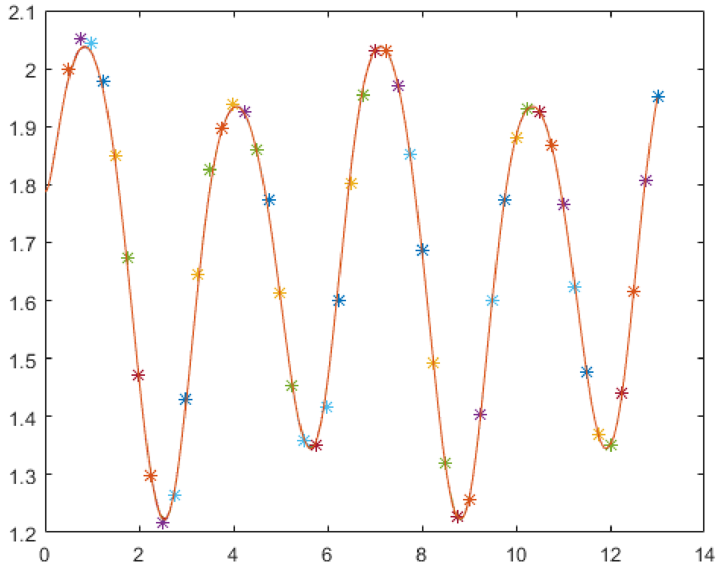

Figure VII.14: Asymptotic numerical $w-$solution (stars) and continuous solution (solid lines) for the Example 2.

time(s)	0.01	0.05	0.105
$\|U\|_{l^\infty}$	5.9212	9.7983	$1.3811 \cdot 10^{29}$
$\|V\|_{l^\infty}$	5.8007	6.5209	$8.5660 \cdot 10^{25}$
$\|W\|_{l^\infty}$	10.1008	8.1186	$1.0944 \cdot 10^{13}$

Table VII.9: Values of l^∞ of the solutions in the Example 3. At time 0.11 seconds, solutions becomes unbounded.

clearly not fulfulling (II.68). We put as functions f_1 and f_2:

$$f_1(x,y,t) = \frac{\cos(t)}{4+\sin(t)} + \frac{x-y}{t^2+1},$$

$$f_2(x,y,t) = \frac{2\cos(t)}{2+\sin(t)} + \frac{\sin^2(t/2)+2\sin(t)}{1+\cos^2(t/2)}$$

as in the first example. The maximum value of the discrete solution for different small times can be seen in Table VII.9 and the plots of the U and V solutions in Figure VII.15. This numerical example shows that blow-up solutions occur when this relation is not assumed and initial data is large enough. This makes us conjeture that some kind of generalization of the well-known condition $\int_\Omega u_0(\mathbf{x})d\mathbf{x} > \frac{8\pi}{\chi}$ must hold for this problem. This is an open problem.

4 Conclusions

We have derived the discretization of the modified Keller-Segel system (VII.1) and its generalization (VII.24) and found the conditions under which the GFD scheme is convergent. For the first model, for Case 1, where there is no source term, we obtain numerical solutions which blow up in finite time for large enough initial data in the discrete model, in accordance with the analytical studies. For Case 2, (case $f(\mathbf{x},t)=0$), we obtain asymptotic convergence to $(1,1)$ as stated for different initial data, in both regular and irregular clouds of points. No differences between these are found. Notice that the diffusion is faster when a large amount of the initial data is above the threshold value 1. When the initial population's density is small, it takes much longer to reach this value. For Case 3, we have obtained an approximation to the solution of the system which inherits the periodic behavior of the function $f^*(t)$. It is also remarkable that the elliptic equation for the chemical substance, v, models a fast diffusion process. Therefore the v-component of the solution becomes rather uniform at small times.

For the generalization to two biological species and one chemical substance, the discrete GFD scheme recovers the periodic asymptotic behavior of the continuous solution.

Furthermore, examples of functions f, f_1, f_2 and f^* are explicitly given in this chapter showing that the assumptions made on them in Chapter II are not too restrictive. The results of Sections 1 and 2 have been recently published in [7]. The results enclosed in Section 3 are submitted to a scientific journal and waiting for publication in [89].

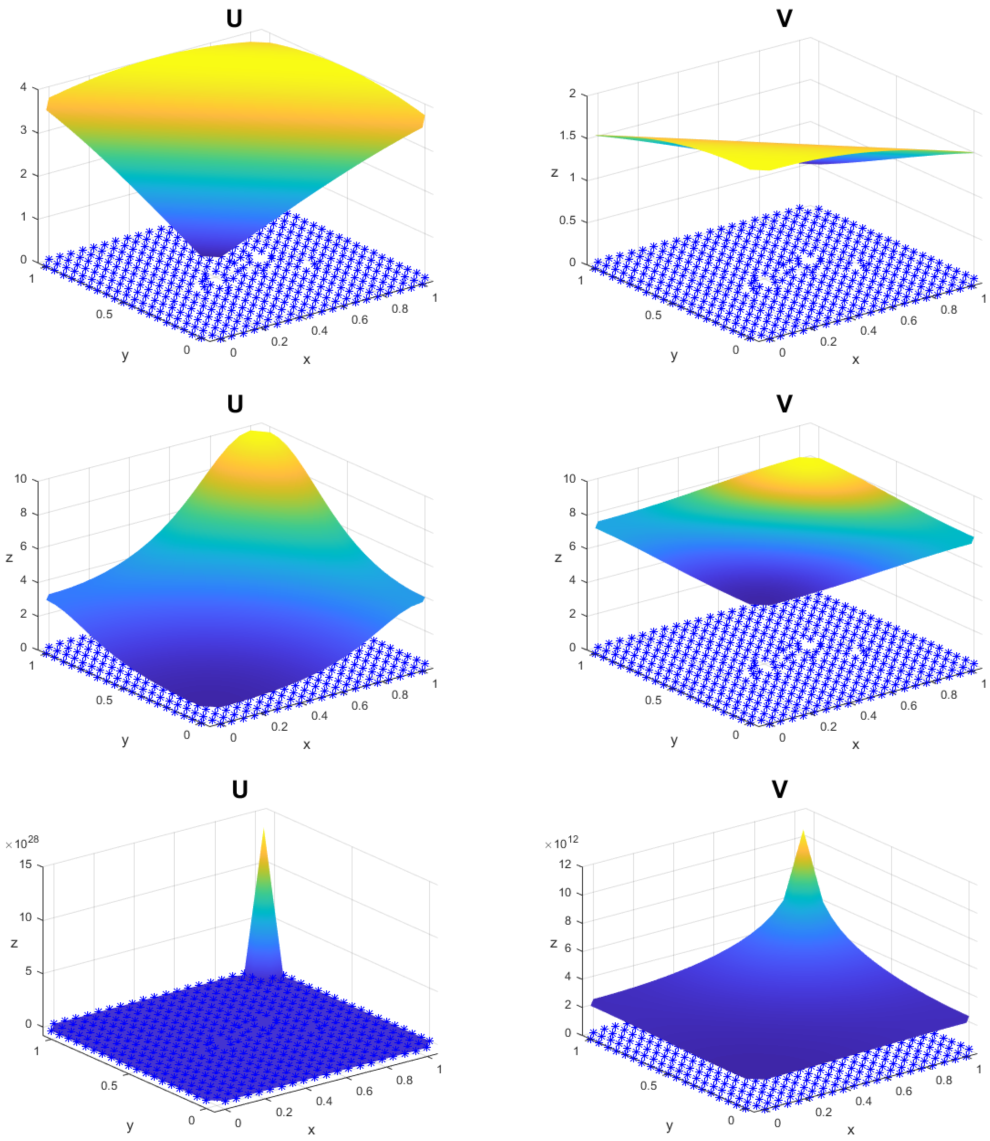

Figure VII.15: $U, V-$discrete solutions for times 0, 0.05 and 0.105 seconds in the Example 3.

Chapter VIII

Numerical solution of the parabolic-parabolic model

We continue our numerical study of the chemotaxis PDEs models with the fully parabolic case, where now the rate of production of the chemical substance is comparable to its diffusion. For completeness we reproduce system (III.1),

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(\mathbf{x}, t) - u), & \mathbf{x} \in \Omega, \quad t > 0, \\ \tau v_t - \Delta v + v = u, & \mathbf{x} \in \Omega, \quad t > 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \mathbf{x} \in \partial\Omega, \quad t > 0, \end{cases} \tag{VIII.1}$$

where $f(\mathbf{x}, t)$ is bounded and converges to a homogeneous in space and periodic in time function $f^*(t)$, i.e.,

$$\int_0^\infty \|f - f^*\|_{L^1(\Omega)} dt \leq c < \infty, \tag{VIII.2}$$

with $f^* = f^*(t)$ independent of $\mathbf{x}$ and periodic in time of period T. The global existence of the solutions and its asymptotic behavior for a range of parameters and certain initial data are obtained in Chapter III. Recall the asymptotic behavior of the solution is in the sense

$$\|u - u^*\|_{L^2(\Omega)} + \|v - v^*\|_{L^2(\Omega)} \to 0, \qquad \text{as } t \to \infty. \tag{VIII.3}$$

for the periodic in time functions u^* and v^* given in Chapter III, i.e., the solutions of

$$u_t^* = \mu u(1 + f(\mathbf{x}, t) - u),$$

and

$$\tau v_t^* = u^* - v^*.$$

As in the previous chapter we propound a numerical explicit scheme and prove its conditional convergence towards the continuous solution of (VIII.1) and, later, we present several examples where the assumptions of Chapter III are verified.

1 GFD scheme

Let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain. By using the explicit formulae (VI.3) for the approximation of the spatial derivatives and (VI.4) for the time derivative approximation, we obtain the following 2-dimensional GFD explicit scheme:

$$\begin{cases} U_0^{n+1} = U_0^n + \Delta t \left[-\lambda_{00} U_0^n + \sum_{i=1}^{s} \lambda_{0i} U_i^n - \chi U_0^n \left(-\lambda_{00} V_0^n + \sum_{i=1}^{s} \lambda_{0i} V_i^n \right) \right] - \\ \quad - \chi \Delta t \left(-\lambda_{01} U_0^n + \sum_{i=1}^{s} + \lambda_{i1} U_i^n \right) \left(-\lambda_{01} V_0^n + \sum_{i=1}^{s} + \lambda_{i1} V_i^n \right) - \\ \quad - \chi \Delta t \left(-\lambda_{02} U_0^n + \sum_{i=1}^{s} + \lambda_{i2} U_i^n \right) \left(-\lambda_{02} V_0^n + \sum_{i=1}^{s} + \lambda_{i2} V_i^n \right) - \\ \quad + \Delta t \mu U_0^n \left[1 - U_0^n + f(\mathbf{x}_0, n\Delta t) \right] + \mathcal{O}(\Delta t, h_i^2, k_i^2) \\ V_0^{n+1} = V_0^n \left[1 - \frac{\Delta t}{\tau}(1 + \lambda_{00}) \right] + \frac{\Delta t}{\tau} U_0^n + \frac{\Delta t}{\tau} \sum_{i=1}^{s} \lambda_{0i} V_i^n + \mathcal{O}(\Delta t, h_i^2, k_i^2). \end{cases} \tag{VIII.4}$$

Let U_j^n be the approximated U-solution at time $n\Delta t$ (similarly V_j^n) and u_j^n the value of the exact u-solution (similarly v_j^n). For the sake of simplicity, let us name some of the expressions which appear in the proof of the conditional convergence of the scheme:

$$\begin{aligned} A_1 := \Bigg[\Bigg| & \chi(\lambda_{01})^2 v_0^n + \chi(\lambda_{02})^2 v_0^n - \\ & - \chi \left(\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} v_i^n + \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} v_i^n \right) \\ & - (\chi - \mu)(U_0^n + u_0^n) + \chi v_0^n - \mu(1 + f(x_0, y_0, n\Delta t)) \Bigg| \\ & + |\chi \lambda_{01} V_0^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \lambda_{02} V_0^n| \sum_{i=1}^{s} |\lambda_{i2}| \\ & + |\chi \sum_{i=1}^{s} \lambda_{i1} v_i^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \sum_{i=1}^{s} \lambda_{i2} v_i^n| \sum_{i=1}^{s} |\lambda_{i2}| \Bigg], \end{aligned} \tag{VIII.5}$$

$$\begin{aligned} B_1 := \Bigg[\Bigg| & \chi[-(\lambda_{01})^2 - (\lambda_{02})^2] U_0^n + \chi \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} U_i^n \\ & + \chi \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_0^n - \chi U_0^n \Bigg| + |\chi \lambda_{01} u_0^n| \sum_{i=1}^{s} |\lambda_{i1}| \\ & + |\chi \lambda_{02} u_0^n| \sum_{i=1}^{s} |\lambda_{i2}| + \chi| \sum_{i=1}^{s} \lambda_{i1} u_i^n| \sum_{i=1}^{s} |\lambda_{i1}| \\ & + \chi| \sum_{i=1}^{s} \lambda_{i2} u_i^n| \sum_{i=1}^{s} |\lambda_{i2}| \Bigg], \end{aligned} \tag{VIII.6}$$

Theorem VIII.1. *Let $u, v \in \mathcal{C}^4(\Omega_\infty)$ be the exact solution of (VIII.1). Let $\tau > 0$ and χ and μ as in (III.14). Then, the GFD explicit scheme (VIII.4) is convergent if*

$$\Delta t < \frac{2}{\lambda_{00} + \sum_{i=1}^{s} |\lambda_{i0}| + A_1 + B_1}, \tag{VIII.7}$$

where A_1 and B_1 are given by (VIII.5) and (VIII.6).

Proof. As in the previous chapeter, we take the difference between GFD scheme (VIII.4) and the expression for the exact solution. Also, we call $\tilde{u}_j^n = u_j^n - U_j^n$, $\tilde{v}_j^n = v_j^n - V_j^n$. The following expression yields

$$\begin{aligned}
\tilde{u}_0^{n+1} =& \tilde{u}_0^n + \Delta t(\chi - \mu)[(U_0^n)^2 - (u_0^n)^2] - \Delta t\chi[U_0^n V_0^n - u_0^n v_0^n] \\
&+ \Delta t\mu\tilde{u}_0^n[1 + f(x_0, y_0, n\Delta t)] + \Delta t\left(-\lambda_{00}\tilde{u}_0^n + \sum_{i=1}^{s}\lambda_{i0}\tilde{u}_i^n\right) \\
&- \chi\Delta t\left[\left(-\lambda_{01}U_0^n + \sum_{i=1}^{s}\lambda_{i1}U u_i^n\right)\left(-\lambda_{01}V_0^n + \sum_{i=1}^{s}\lambda_{i1}V_i^n\right)\right. \\
&\left.- \left(-\lambda_{01}u_0^n + \sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(-\lambda_{01}v_0^n + \sum_{i=1}^{s}\lambda_{i1}v_i^n\right)\right] \\
&- \chi\Delta t\left[\left(-\lambda_{02}U_0^n + \sum_{i=1}^{s}\lambda_{i2}U_i^n\right)\left(-\lambda_{02}V_0^n + \sum_{i=1}^{s}\lambda_{i2}V_i^n\right)\right. \\
&\left.- \left(-\lambda_{02}u_0^n + \sum_{i=1}^{s}\lambda_{i2}u_i^n\right)\left(-\lambda_{02}v_0^n + \sum_{i=1}^{s}\lambda_{i2}v_i^n\right)\right] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).
\end{aligned} \tag{VIII.8}$$

Now, we use some known identities in order to rewrite the terms in (VIII.8):

$$(U_0^n)^2 - (u_0^n)^2 = \tilde{u}_0^n(U_0^n - u_0^n), \tag{VIII.9}$$

$$U_0^n V_0^n - u_0^n v_0^n = U_0^n V_0^n - U_0^n v_0^n + U_0^n v_0^n - u_0^n v_0^n = U_0^n \tilde{v}_0^n + \tilde{u}_0^n v_0^n, \tag{VIII.10}$$

and

$$\begin{aligned}
&(\lambda_{01})^2 U_0^n V_0^n - \lambda_{01}U_0^n\sum_{i=1}^{s}\lambda_{i1}V_i^n - \lambda_{01}V_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n \\
&+ \left(\sum_{i=1}^{s}\lambda_{i1}V_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}U_i^n\right) \\
&- \left[(\lambda_{01})^2 u_0^n v_0^n - \lambda_{01}u_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n - \lambda_{01}v_0^n\sum_{i=1}^{s}\lambda_{i1}u_i^n\right] \\
&- \left(\sum_{i=1}^{s}\lambda_{i1}v_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right) \\
&= (\lambda_{01})^2[U_0^n V_0^n - u_0^n v_0^n] - \lambda_{01}\left[U_0^n\sum_{i=1}^{s}\lambda_{i1}V_i^n - u_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n\right] \\
&- \lambda_{01}\left[V_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n - v_0^n\sum_{i=1}^{s}\lambda_{i1}u_i^n\right] \\
&+ \left[\left(\sum_{i=1}^{s}\lambda_{i1}V_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}U_i^n\right) - \left(\sum_{i=1}^{s}\lambda_{i1}v_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\right].
\end{aligned} \tag{VIII.11}$$

We can reformulate (VIII.11)

$$
\begin{aligned}
(VIII.11) = (\lambda_{01})^2[U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n] - \lambda_{01}\left(\tilde{u}_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n + u_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right) \\
- \lambda_{01}\left(\tilde{v}_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n + v_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n\right) + \left(\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}v_i^n\right) \\
+ \left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right).
\end{aligned}
\tag{VIII.12}
$$

Now, substituting the corresponding terms obtained in (VIII.9), (VIII.10) and (VIII.12) in (VIII.8), we have

$$
\begin{aligned}
&\tilde{u}_0^{n+1} - \tilde{u}_0^n = \Delta t\left(-\lambda_{00}\tilde{u}_0^n + \sum_{i=1}^{s}\lambda_{i0}\tilde{u}_i^n\right) - \Delta t\chi(\lambda_{01})^2[U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n] \\
&- \Delta t\chi(\lambda_{02})^2[U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n] + \tilde{u}_0^n v_0^n] \\
&+ \Delta t\chi\lambda_{01}\left(\tilde{u}_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n + u_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right) \\
&+ \Delta t\chi\lambda_{01}\left(\tilde{v}_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n + v_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n\right) \\
&+ \Delta t\chi\lambda_{02}\left(\tilde{u}_0^n\sum_{i=1}^{s}\lambda_{i2}v_i^n + u_0^n\sum_{i=1}^{s}\lambda_{i2}\tilde{v}_i^n\right) \\
&+ \Delta t\chi\lambda_{02}\left(\tilde{v}_0^n\sum_{i=1}^{s}\lambda_{i2}U_i^n + v_0^n\sum_{i=1}^{s}\lambda_{i2}\tilde{u}_i^n\right) \\
&- \Delta t\chi\left(\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}v_i^n\right) - \Delta t\chi\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right) \\
&- \Delta t\chi\left(\sum_{i=1}^{s}\lambda_{i2}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i2}v_i^n\right) - \Delta t\chi\left(\sum_{i=1}^{s}\lambda_{i2}u_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i2}\tilde{v}_i^n\right) \\
&+ \Delta t\tilde{u}_0^n(\chi - \mu)(U_0^n - u_0^n) - \Delta t\chi(U_0^n\tilde{v}_0^n + \tilde{u}_0^n v_0^n) + \Delta t\mu\tilde{u}_0^n[1 + f(x_0, y_0, n\Delta t)] \\
&+ \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2))
\end{aligned}
$$

Therefore, it yields

$$
\begin{aligned}
\tilde{u}_0^{n+1} = \tilde{u}_0^n \Bigg[& 1 - \lambda_{00}\Delta t - \Delta t \chi (\lambda_{01})^2 v_0^n - \Delta t \chi (\lambda_{02})^2 v_0^n \\
& + \Delta t (\chi - \mu)(u_0^n + U_0^n) \\
& + \Delta t \chi \left(\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} v_i^n + \lambda_{02} \sum_{i=1}^{n} \lambda_{i2} v_i^n \right) \\
& - \Delta t \chi v_0^n + \Delta t \mu [1 + f(x_0, y_0, n\Delta t)] \Bigg] \\
& + \Delta t \Bigg[\sum_{i=1}^{s} \lambda_{i0} \tilde{u}_i^n + \chi \lambda_{01} v_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n + \chi \lambda_{02} v_0^n \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \\
& - \left(\chi \sum_{i=1}^{s} \lambda_{i1} v_i^n \right) \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n - \left(\chi \sum_{i=1}^{s} \lambda_{i2} v_i^n \right) \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \Bigg] \\
& + \Delta t \tilde{v}_0^n \Bigg[\left(-\chi (\lambda_{01})^2 - \chi (\lambda_{02})^2 \right) U_0^n + \chi \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} U_i^n \\
& + \chi \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_i^n - \chi U_0^n \Bigg] \\
& + \Delta t \Bigg[\chi \lambda_{01} U_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n + \chi \lambda_{02} v_0^n \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \\
& - \chi \left(\sum_{i=1}^{s} \lambda_{i1} u_0^n \right) \sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n - \chi \left(\sum_{i=1}^{s} \lambda_{i2} u_0^n \right) \sum_{i=1}^{s} \lambda_{i2} \tilde{v}_i^n \Bigg] \\
& + \mathcal{O}(\Delta t (\Delta t, h_i^2, k_i^2)).
\end{aligned}
\tag{VIII.13}
$$

Let us define $\tilde{u}^n = \max_{i \in \{0,\ldots,s\}} |\tilde{u}_i^n|$ and $\tilde{v}^n = \max_{i \in \{0,\ldots,s\}} |\tilde{v}_i^n|$. We rewrite (VIII.13) as follows

$$
\begin{aligned}
\tilde{u}^{n+1} \leq \tilde{u}^n \Bigg[\Bigg| 1 - \Delta t \Bigg(& \lambda_{00} + \chi(\lambda_{01})^2 v_0^n + \chi(\lambda_{02})^2 v_0^n \\
& - \chi\Bigg(\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} v_i^n + \lambda_{02} \sum_{i=1}^{n} \lambda_{i2} v_i^n \Bigg) \\
& - (\chi - \mu)(u_0^n + U_0^n) + \chi v_0^n - \mu(1 + f(x_0, y_0, n\Delta t)) \Bigg) \Bigg| \\
& + \Delta t \Bigg(\sum_{i=1}^{s} |\lambda_{i0}| + |\chi \lambda_{01} V_0^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \lambda_{02} V_0^n| \sum_{i=1}^{s} |\lambda_{i2}| \\
& + |\chi \sum_{i=1}^{s} \lambda_{i1} v_i^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \sum_{i=1}^{s} \lambda_{i2} v_i^n| \sum_{i=1}^{s} |\lambda_{i2}| + \Bigg) \Bigg] \\
& + \Delta t \tilde{v}^n \Bigg[\Bigg| \chi[-(\lambda_{01})^2 - (\lambda_{02})^2] U_0^n + \chi \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} u_i^n \\
& + \chi \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_0^n + -\chi U_0^n \Bigg| \\
& + |\chi \lambda_{01} u_0^n| \sum_{i=1}^{s} |\lambda_{i1}| + |\chi \lambda_{02} u_0^n| \sum_{i=1}^{s} |\lambda_{i2}| + \\
& + \chi | \sum_{i=1}^{s} \lambda_{i1} u_i^n | \sum_{i=1}^{s} |\lambda_{i1}| + \chi | \sum_{i=1}^{s} \lambda_{i2} u_i^n | \sum_{i=1}^{s} |\lambda_{i2}|| \Bigg] + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)).
\end{aligned}
\tag{VIII.14}
$$

We consider now the second equation of (VIII.1) and subtract the expression for the exact solution, arriving to the following:

$$
\tilde{v}_0^{n+1} = \tilde{v}_0^n \left[1 - \frac{\Delta t}{\tau}(1 + \lambda_{00}) \right] + \frac{\Delta t}{\tau} \tilde{u}_0^n + \frac{\Delta t}{\tau} \sum_{i=1}^{s} \lambda_{0i} \tilde{v}_i^n + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)). \tag{VIII.15}
$$

Therefore, by the definition of $\tilde{u}^n$ and $\tilde{v}^n$, we get

$$
\tilde{v}^{n+1} \leq \frac{\Delta t}{\tau} \tilde{u}^n + \left[\left| 1 - \frac{\Delta t}{\tau}(1 + \lambda_{00}) \right| + \frac{\Delta t}{\tau} \sum_{i=1}^{s} |\lambda_{0i}| \right] \tilde{v}^n + \mathcal{O}(\Delta t(\Delta t, h_i^2, k_i^2)). \tag{VIII.16}
$$

For the sake of simplicity, we join the above equations (VIII.15) and (VIII.16) in the form of the matrix inequality

$$
\begin{pmatrix} \tilde{u}^{n+1} \\ \tilde{v}^{n+1} \end{pmatrix} \leq \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \tilde{u}^n \\ \tilde{v}^n \end{pmatrix}, \tag{VIII.17}
$$

where M_{21} and M_{22} are exlicitly defined by (VIII.16) and M_{11}, M_{12} are given by

$$
M_{11} = |1 - \Delta t \lambda_{00}| + \Delta t \sum_{i=1}^{s} |\lambda_{i0}| + A_1 \Delta t, \quad M_{12} = \Delta t B_1, \tag{VIII.18}
$$

for an obvious choice of A_1 and B_1. Then, square matrix in (VIII.17) is rewritten as

$$M = \begin{pmatrix} |1-\Delta t\lambda_{00}| + \Delta t \sum_{i=1}^{s} |\lambda_{i0}| + A_1 \Delta t & \Delta t B_1 \\ \frac{\Delta t}{\tau} & \left|1 - \frac{\Delta t(1+\lambda_{00})}{\tau}\right| + \frac{\Delta t}{\tau} \sum_{i=1}^{s} |\lambda_{i0}| \end{pmatrix}. \tag{VIII.19}$$

We consider the $\|\cdot\|_1$ norm as the maximum sum by rows of the elements of M and

$$\|M\|_1 = |1-\Delta t\lambda_{00}| + \Delta t \sum_{i=1}^{s} |\lambda_{i0}| + A_1\Delta t + \Delta t B_1$$

Hence, $\|M\|_1 < 1$ is equivalent to

$$|1-\Delta t\lambda_{00}| < 1 - \Delta t \sum_{i=1}^{s} |\lambda_{i0}| - A_1\Delta t - \Delta t B_1,$$

which holds since (VIII.7), i.e.,

$$\Delta t < \frac{2}{\lambda_{00} + \sum_{i=1}^{s} |\lambda_{i0}| + A_1 + B_1}.$$

We conclude the proof of Theorem VIII.1 by using lemmas VI.1 and VI.2. □

2 Numerical examples

For our numerical examples we test the method over the irregular cloud points given by Figure VIII.1, (which has 441 nodes) and we take $\Delta t = 0.001$, fulfilling the assumptions made in Theorem VIII.1. We use an 8-node scheme, chosen by the distance criterion together with the weight function $w = \frac{1}{dist^4}$. In order to solve system (VIII.1) numerically we consider two different functions f and their corresponding periodic functions f^* verifying (VIII.2).

For all the presented examples we make use of the error norm of Remark VI.3. In the following figures, the solid line corresponds to the asymptotic limit of the continuous solutions, $u^*(t), v^*(t)$ and the stars to the values of the discrete solutions, U, V where the greatest error is achieved.

2.1 Case 1

As a first example we consider function $f(\mathbf{x}, t)$ in (VIII.1) as

$$f(\mathbf{x}, t) = \frac{\cos t}{4+\sin t} + \frac{x-y}{1+t^2}, \quad (x,y) \in \Omega = [0,1]\times[0,1], \ t > 0.$$

It is easily checked that this function fulfils all assumptions stated in Chapter III. Then, we find the $2\pi-$periodic function

$$f^*(t) = \frac{\cos t}{4+\sin t},$$

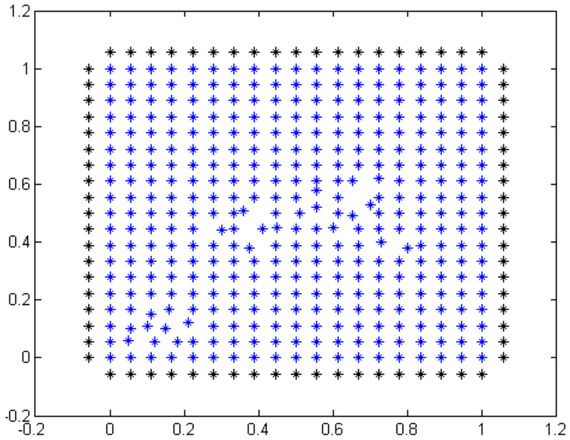

Figure VIII.1: Irregular clouds of points

which, by (III.2), provides us the $2\pi-$periodic limit of the solution u,

$$u^*(t) = \frac{4+\sin(t)}{4-\frac{\cos(t)}{2}+\frac{\sin(t)}{2}}, \tag{VIII.20}$$

and the limit of v, as in (III.4), i.e.,

$$v^*(t) = e^{-\frac{t}{\tau}}v^*(0) + \frac{1}{\tau}\int_0^t e^{\frac{s-t}{\tau}}u^*(s)ds. \tag{VIII.21}$$

We propound the following two numerical examples to show that the solution to (VIII.1) inherits the periodic behavior of the function $f(\mathbf{x}, t)$:

$$\lim_{t\to\infty}[\|u(\cdot,t)-u^*(t)\|_{l^\infty(\Omega)} + \|v(\cdot,t)-v^*(t)\|_{l^\infty(\Omega)}] = 0, \tag{VIII.22}$$

Example 1

For this first example we consider $\mu = 1, \tau = 1$ and $\chi = 0.3$. Also, we take the following initial data:

$$u_0(x,y) = 2e^{-10[(x-0.1)^2+(y-0.1)^2]}, \quad v_0(x,y) = \frac{1}{2}e^{-10[(x-1.2)^2+(y-1.2)^2]}.$$

As stated in the theory, we expect to find the convergence of the discrete solutions to the periodic functions given by (VIII.20) and (VIII.21) in the sense of (VIII.22).
In Table VIII.1 we show the $\|\cdot\|_{l^\infty}$ norm of the difference between the numerical solution and the uniform asymptotic limit at different times. Figures VIII.2 and VIII.3 illustrate the periodic functions $u^*(t), v^*(t)$ (solid lines) and the most distant value of the U, V-solution, respectively.

Example 2

Now, we consider $\mu = 1, \tau = 1$ and $\chi = 0.4$. As initial data we pick

$$u_0(x,y) = e^{-10[(x-0.2)^2+(y-0.2)^2]} + e^{-10[(x-0.8)^2+(y-0.8)^2]},$$

T(s)	3.72	6.86	10	13.14	16.28
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0465	0.0016	4.3710e-04	6.0287e-04	1.3454e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.1828	0.0175	0.0014	4.0000e-04	7.5000e-04

Table VIII.1: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 1 and Case 1

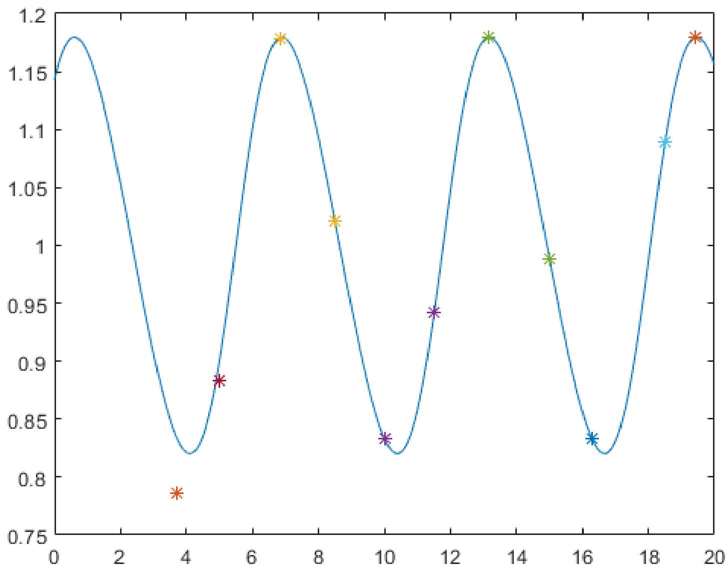

Figure VIII.2: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the most distant value of the approximate solution U at such time in Example 1 and Case 1.

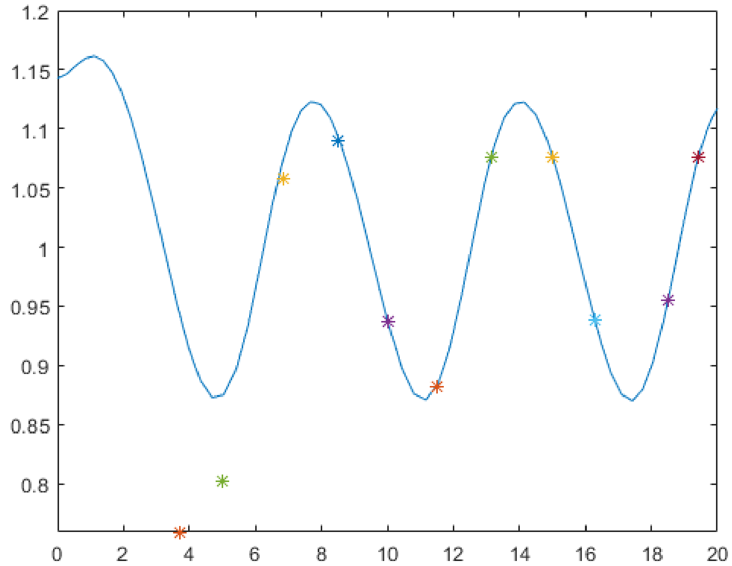

Figure VIII.3: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the most distant value of the approximate solution V at such time in Example 1 and Case 1.

T(s)	3.72	6.86	10	13.14	16.28
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0245	0.0038	5.3710e-04	6.0287e-04	1.3454e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.1251	0.0107	9.0000e-04	6.0000e-04	3.0600e-04

Table VIII.2: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 2 and Case 1.

$$v_0(x, y) = e^{-10[(x-1.1)^2+(y-0.5)^2]}.$$

In Table VIII.2 we present the l^∞ norm of the difference between the numerical solution and the value of the asymptotic limits at different times. Figures VIII.4 and VIII.5 display the functions $u^*(t), v^*(t)$ (solid lines) and the most distant values of the U, V-solution (stars) at different times.

2.2 Case 2

Now we pick up a second example for a function that checks the hypotheses of Chapter III and [86],

$$f(\mathbf{x}, t) = \frac{2\cos(2t)}{2 + \sin(2t)} + \frac{\sin^2(t) + 2\sin(2t)}{1 + \cos^2(t)}$$

$(x, y) \in \Omega = [0, 1] \times [0, 1],\ t > 0$. Then $f^* = f$ is the $\pi-$periodic function of the problem and, by (III.2), for

$$u^*(0) = 1$$

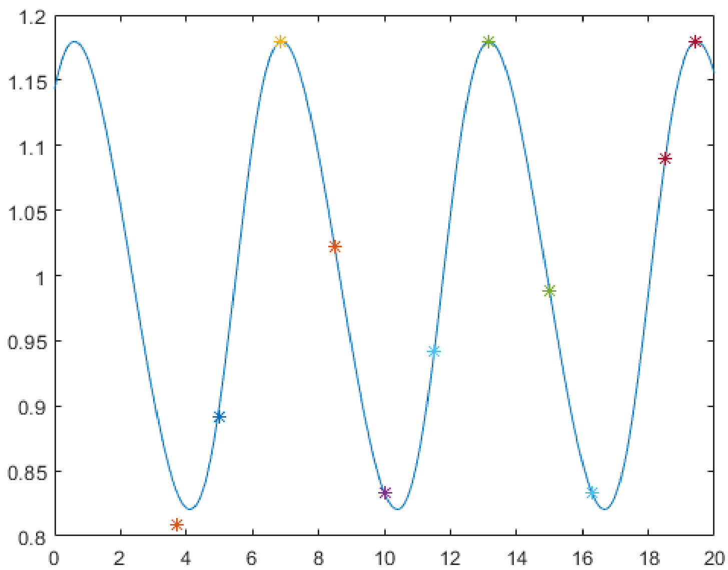

Figure VIII.4: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the most distant value of the approximate solution U at different times in Example 2 and Case 1.

we meet the expression of the $\pi-$periodic limit of the solution u,

$$u^*(t) = \frac{2 + \sin(2t)}{1 + \cos^2(2t)}, \tag{VIII.23}$$

and the limit v^* of v, as in (VIII.21).

Example 3

Now we consider case 2 and $\mu = 1, \tau = 1$ and $\chi = 0.4$. As initial data we put

$$u_0(x, y) = 2e^{-10[(x-0.2)^2+(y-0.2)^2]},$$

$$v_0(x, y) = e^{-10[(x-0.9)^2+(y-0.5)^2]}.$$

In Table VIII.3 we observe the l^∞ norm of the difference between the numerical solution and the value of the asymptotic limits at different times. Figures VIII.6 and VIII.7 show the functions $u^*(t), v^*(t)$ (solid lines) and the most distant values of the U, V-solution (stars) at different times.

Example 4

For this fourth example we also consider $\mu = 1, \tau = 1$ and $\chi = 0.4$, and initial data

$$u_0(x, y) = e^{-10[x^2+y^2]},$$

$$v_0(x, y) = 0.5e^{-10[(x-0.5)^2+(y-0.5)^2]}.$$

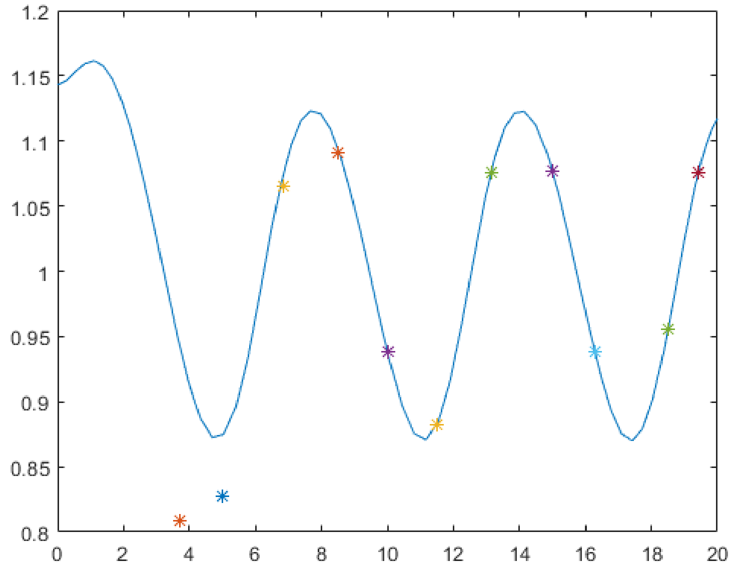

Figure VIII.5: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the most distant value of the approximate solution V at different times in Example 2 and Case 1.

T(s)	3.72	6.86	10	13.14	16.28
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0131	1.9008e-04	3.4604e-04	3.4627e-04	3.4467e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0804	0.0047	7.1677e-04	6.5885e-04	6.8582e-04

Table VIII.3: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 1 in Case 2.

Table VIII.4 displays the l^∞ norm of the difference between the numerical solution and the value of the asymptotic limits at different times. Figures VIII.8 and VIII.9 show the functions $u^*(t), v^*(t)$ (solid lines) and the most distant values of the U, V-solution (stars) at different times.

2.3 Influence of the number of nodes

For this fifth example we test the method, for the same data, over two different clouds of points. The first one has 117 nodes and the second one has 957. Both clouds can be seen in Figure VIII.10. Table VIII.5 provides the values for times 3.72, 6.86 and 10 seconds for the first one whereas Table VIII.6 shows the same for the second one.

As can be clearly seen, an increment of the number of nodes produces a more accurate approximation, though the time consumption increases as well.

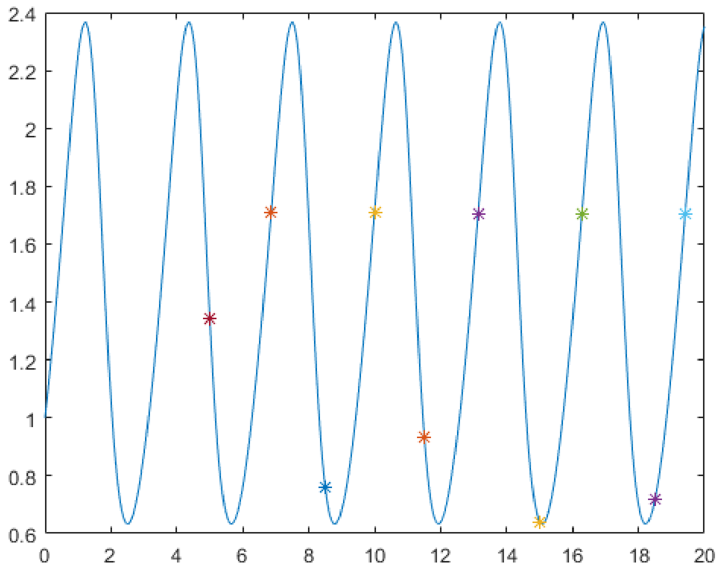

Figure VIII.6: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the most distant value of the approximate solution U at different times in Example 1 and Case 2.

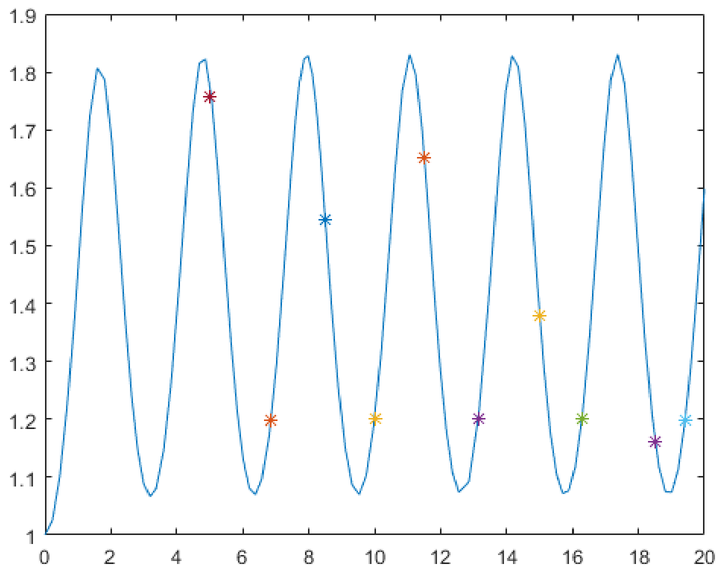

Figure VIII.7: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the most distant value of the approximate solution V at different times in Example 1 and Case 2.

T(s)	3.72	6.86	10	13.14	16.28	19.42
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.1011	9.3989e-04	3.3275e-04	3.4611e-04	3.4466e-04	3.4304e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.2671	0.0176	0.0013	6.8601e-04	6.8700e-04	6.4743e-04

Table VIII.4: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 2 in Case 2.

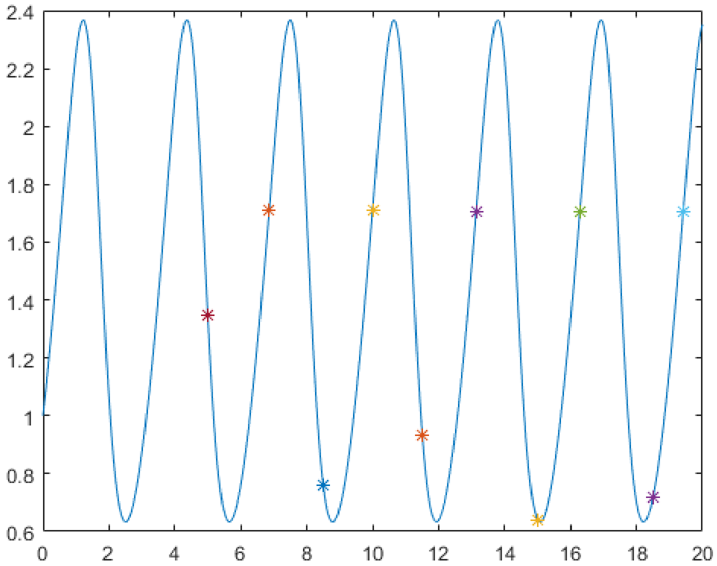

Figure VIII.8: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the most distant value of the approximate solution U at different times in Example 2 and Case 2.

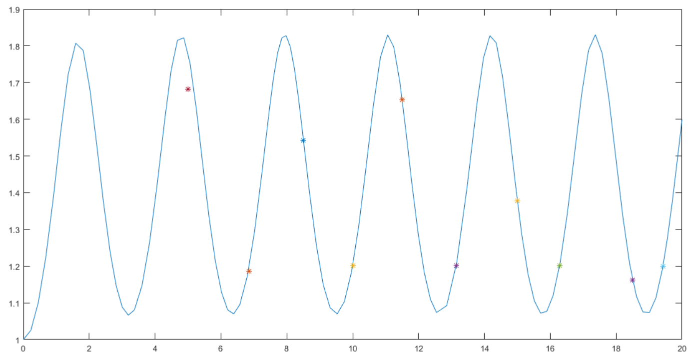

Figure VIII.9: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the most distant value of the approximate solution V at different times in Example 2 and Case 2.

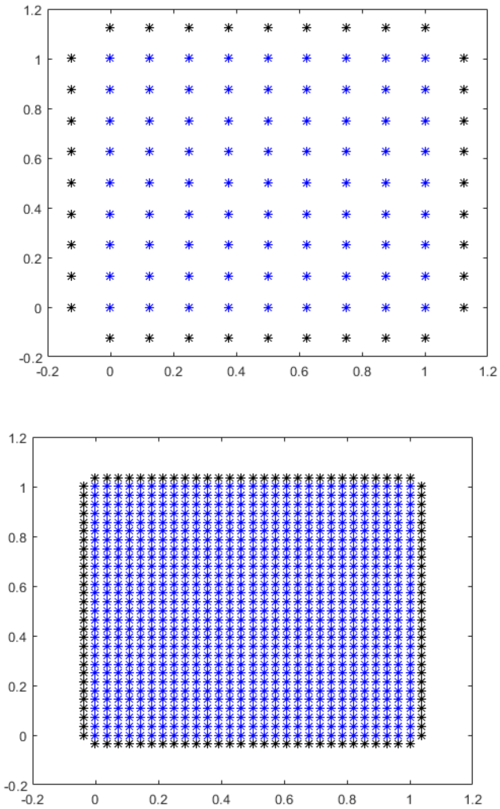

Figure VIII.10: Clouds of points with 117 and 957 nodes

2.4 Influence of the time increment

The aim of this last example is to test the explicit scheme for different time increments, Δt. We choose as initial data the following:

$$u_0(x,y) = 2e^{-10[(x-0.3)^2+(y-0.3)^2]},$$

$$v_0(x,y) = \frac{1}{2}e^{-10[(x-0.3)^2+(y-0.3)^2]}.$$

We take $\mu = 1$,$\chi = 0.3$ and the function of Case 1. Let us select 3 different values of the time increment, i.e., $\Delta t = 0.00125, 0.001$ and 0.0005. The results are displayed in Table VIII.7, VIII.8 and VIII.9, for times 5, 7.5 and 10 seconds. All Δt are in the range of

T(s)	3.72	6.86	10
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.1030	9.4331e-04	3.3273e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.2707	0.0176	0.0013

Table VIII.5: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ for the cloud of points of 117 nodes and the function f^* of Case 2.

T(s)	3.72	6.86	10
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0091	5.8513e-04	1.9470e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.2401	0.0171	8.1949e-04

Table VIII.6: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|U - v^*(t)\|_{l^\infty(\Omega)}$ for the cloud of points of 957 nodes and the function f^* of Case 2.

T(s)	5	7.5	10
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0106	0.0022	7.9729e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0360	0.0046	5.1769e-04

Table VIII.7: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ for $\Delta t = 0.00125$ and the function f^* of Case 1.

T(s)	5	7.5	10
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0106	0.0022	7.7545e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0358	0.0046	4.9424e-04

Table VIII.8: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ for $\Delta t = 0.001$ and the function f^* of Case 1.

T(s)	5	7.5	10
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0106	0.0022	7.3179e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0356	0.0043	4.4738e-04

Table VIII.9: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ for $\Delta t = 0.0005$ and the function f^* of Case 1.

(VIII.7), so the scheme is convergent. As expected and seen in the tables, a smaller time increment results in a smaller error.

3 Conclusions

We have derived the discretization of the fully parabolic PDEs system given by (VIII.1) by means of the GFDM. In Theorem VIII.1 we have obtained the conditional convergence of the method for solving this nonlinear system and, also, we have given the explicit condition for the convergence. Numerical tests are presented to validate the asymptotic behavior of the solution stated in the theory, and to demonstrate the accuracy and efficiency of the GFDM applied to this highly nonlinear system of coupled parabolic PDEs over irregular domains. These results are enclosed in the recently published paper [8].

Chapter IX

Numerical solution of the nonlocal model

In this chapter we consider the parabolic-parabolic system of quasilinear PDEs with a nonlocal term, describing the interactions between the population's density of a biological species, "u", and the chemical substance, "v", responsible for the chemotactic process. Although this model has been studied in Chapter IV, we reproduce the system for completeness:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \mathrm{div}(u^m \nabla v) + u\left(a_0 - a_1 u^\alpha + a_2 \displaystyle\int_\Omega u^\alpha d\mathbf{x}\right), & \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u^\gamma, & \Omega \times (0,\infty), \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \partial\Omega \times (0,\infty), \end{cases} \tag{IX.1}$$

The global existence and boundedness of solutions as well as their asymptotic behavior have already been proved in Chapter IV. In few words, provided

$$\alpha \geq 1, \quad m > 1, \quad \alpha + 1 > m + \gamma \text{ and } a_1 > a_2|\Omega|,$$

the solution converges to the constant steady state

$$(u^*, v^*)\left(\frac{a_0^{\frac{1}{\alpha}}}{(a_1 - a_2|\Omega|)^{\frac{1}{\alpha}}}, \left(\frac{a_0^{\frac{1}{\alpha}}}{(a_1 - a_2|\Omega|)^{\frac{1}{\alpha}}}\right)^\gamma\right)$$

if $a_0 > 0$ and $a_2 > 0$. In the case $a_0 < 0$ and $a_2 < 0$, then the constant steady state becomes $(u^*, v^*) = (0, 0)$.

In order to determine the numerical evidence for the cases that are not covered by assumptions (IV.2)-(IV.7), we give several examples using the GFD method over the computational domain $\Omega = [0,1] \times [0,1]$. The numerical results show that all conditions in the statement of the Theorem IV.1 (Chapter IX.2) play a relevant role in the behavior of the solution of (IV.1). The suppression of some of the above conditions, together with the election of large enough initial data, may end up in the existence of blow-up of the solutions.

We divide the chapter into three sections. In Section 1 we obtain the GFD scheme that we shall use as discrete version of (IX.1) and prove the conditional convergence of the numerical solution to the continuous one. In Section 2 we present several numerical examples to illustrate the asymptotic stability of the system. We finally give some conclusions.

1 GFD scheme

The nonlocal term can be expressed by means of the Taylor series expansion as

$$\begin{aligned}\int_\Omega F(x+x_0,y+y_0)d\mathbf{x} = \int_\Omega \Bigg(F(\mathbf{x}_0) + x\frac{\partial F}{\partial x}(\mathbf{x}_0) + y\frac{\partial F}{\partial y}(\mathbf{x}_0) \\ + \frac{x^2}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{x}_0) + xy\frac{\partial^2 F}{\partial x \partial y}(\mathbf{x}_0) + \frac{y^2}{2}\frac{\partial^2 F}{\partial y^2}(\mathbf{x}_0) + R_2 \Bigg) d\mathbf{x},\end{aligned} \tag{IX.2}$$

where R_2 verifies

$$\lim_{(x,y)\to(0,0)} \frac{R_2}{\|(x,y)\|^2} = 0.$$

Using the approximations given by (VI.3) in Chapter VI, we obtain the following 2-dimensional GFD explicit scheme:

$$\left\{\begin{aligned} U_0^{n+1} =& U_0^n + \Delta t \left[-\lambda_{00}U_0^n + \sum_{i=1}^s \lambda_{i0}U_i^n - \chi(U_0^n)^m \left(-\lambda_{00}V_0^n + \sum_{i=1}^s \lambda_{i0}V_i^n \right)\right] \\ &- \chi(U_0^n)^{m-1}\Delta t \left(-\lambda_{01}U_0^n + \sum_{i=1}^s +\lambda_{i1}U_i^n \right)\left(-\lambda_{01}V_0^n + \sum_{i=1}^s \lambda_{i1}V_i^n \right) \\ &- \chi(U_0^n)^{m-1}\Delta t \left(-\lambda_{02}U_0^n + \sum_{i=1}^s +\lambda_{i2}U_i^n \right)\left(-\lambda_{02}V_0^n + \sum_{i=1}^s \lambda_{i2}V_i^n \right) \\ &+ \Delta t U_0^n \Bigg[a_0 - a_1(U_0^n)^\alpha + a_2 \Bigg((U_0^n)^\alpha + \frac{\alpha}{2}(U_0^n)^{\alpha-1}\Bigg(-(\lambda_{01}+\lambda_{02})U_0^n \\ &+ \sum_{i=1}^s (\lambda_{i1}+\lambda_{i2})U_i^n \Bigg) + \frac{\alpha(\alpha-1)}{6}(U_0^n)^{\alpha-2}\left(-\lambda_{00}U_0^n + \sum_{i=1}^s \lambda_{i0}U_i^n \right) \\ &+ \frac{\alpha(\alpha-1)}{4}(U_0^n)^{\alpha-2}\left(-\lambda_{05}U_0^n + \sum_{i=1}^s \lambda_{i5}U_i^n \right)\Bigg)\Bigg] + \mathcal{O}(h_i^2,k_i^2) \\ V_0^{n+1} =& -\lambda_{00}V_0^n + \sum_{i=1}^s \lambda_{i0}V_0^n - V_0^n + (U_0^n)^\gamma + \mathcal{O}(\Delta t(h_i^2,k_i^2)). \end{aligned}\right. \tag{IX.3}$$

Remark IX.1. *Note that the identities of (IX.3) hold since $\Omega = [0,1]\times[0,1]$. In other case the coefficients resulting from (IX.2) may vary although the following theorem remains true.*

The following result proves that the explicit scheme given by (IX.3) is conditionally convergent for this fully parabolic case.

Theorem IX.1. *Let (u, v) be the exact solution of (IX.1). Let $a_1 > 0$, $a_0, a_2 \in \mathbb{R}$ and $m, \alpha, \gamma \geq 1$. Then, the GFD explicit scheme (IX.3) is convergent if the time increment, Δt, satifies*

$$\Delta t < \frac{2}{\lambda_{00} + \sum_{i=1}^{s} |\lambda_{i0}| + |\Phi| + |\Psi|}, \tag{IX.4}$$

where Φ and Ψ are given by

$$\begin{aligned}
\Phi :&= -\chi m(m-1)((\lambda_{01})^2 + (\lambda_{02})^2)\xi_1^{m-2} \\
&+ \chi m(m-1)\xi_2^{m-2}\lambda_{01}V_0^n \sum_{i=1}^{s} \lambda_{i1}U_i^n + \chi m^2 \xi_3^{m-1}\lambda_{01} \sum_{i=1}^{s} \lambda_{i1}V_i^n \\
&- \chi m(m-1)\xi_4^{m-2}\left(\sum_{i=1}^{s} \lambda_{i1}U_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i1}V_i^n\right) + \chi m(m-1)\xi_5^{m-2}\lambda_{05}V_0^n \sum_{i=1}^{s} \lambda_{i2}U_i^n \\
&+ \chi m^2 \xi_6^{m-1}\lambda_{02} \sum_{i=1}^{s} \lambda_{i2}V_i^n - \chi m(m-1)\xi_7^{m-2}\left(\sum_{i=1}^{s} \lambda_{i2}U_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i2}V_i^n\right) \\
&\chi m \xi_8^{m-1}\lambda_{00}V_0^n - \chi m \xi_9^{m-1} \sum_{i=1}^{s} \lambda_{i0}V_i^n + a_0 - (a_1 - a_2)(\alpha+1)\xi_{10}^{\alpha} \\
&- \frac{a_2\alpha(\alpha+1)\xi_{11}^{\alpha}}{2}(\lambda_{01} + \lambda_{02}) + \frac{a_2\alpha^2\xi_{12}^{\alpha-1}}{2} \sum_{i=1}^{s}(\lambda_{i1} + \lambda_{i2})U_i^n \\
&- \frac{a_2\alpha^2(\alpha-1)\xi_{13}^{\alpha-1}}{6}\lambda_{00} + \frac{a_2\alpha(\alpha-1)^2\xi_{14}^{\alpha-2}}{6} \sum_{i=1}^{s} \lambda_{i0}U_i^n \\
&- \frac{a_2\alpha^2(\alpha-1)\xi_{15}^{\alpha-2}}{4}\lambda_{05} + \frac{a_2\alpha(\alpha-1)^2\xi_{16}^{\alpha-1}}{4} \sum_{i=1}^{s} \lambda_{i5}U_i^n \\
&+ |\chi m(u_0^n)^{m-1}V_0^n\lambda_{01} \sum_{i=1}^{s} \lambda_{i1}| + |\chi m(u_0^n)^{m-1} \sum_{i=1}^{s} \lambda_{i1}V_i^n| \sum_{i=1}^{s} |\lambda_{i1}| \\
&+ |\chi m(u_0^n)^{m-1}V_0^n\lambda_{02} \sum_{i=1}^{s} \lambda_{i2}| + |\chi m(u_0^n)^{m-1} \sum_{i=1}^{s} \lambda_{i2}V_i^n| \sum_{i=1}^{s} |\lambda_{i2}| \\
&\frac{|a_2|\alpha}{2}(u_0^n)^{\alpha} \sum_{i=1}^{s} |\lambda_{i1} + \lambda_{i2}| + \left|\frac{a_2\alpha(\alpha-1)(u_0^n)^{\alpha-1}}{6}\right| \sum_{i=1}^{s} |\lambda_{i0}| \\
&+ \left|\frac{a_2\alpha(\alpha-1)(u_0^n)^{\alpha-1}}{4}\right| \sum_{i=1}^{s} |\lambda_{i5}|,
\end{aligned} \tag{IX.5}$$

and

$$
\begin{aligned}
|\Psi| := \Big| & -\chi m(u_0^n)^m[(\lambda_{01})^2+(\lambda_{02})^2] + \chi m(u_0^n)^{m-1}\lambda_{01}\sum_{i=1}^s \lambda_{i1}u_i^n \\
& + \chi m(u_0^n)^{m-1}\lambda_{02}\sum_{i=1}^s \lambda_{i2}u_i^n - \chi(u_0^n)^m\lambda_{00}\Big| \\
& + |\chi m(u_0^n)^m\lambda_{01}|\sum_{i=1}^s|\lambda_{i1}| + |\chi m(u_0^n)^{m-1}\sum_{i=1}^s\lambda_{i1}u_i^n|\sum_{i=1}^s|\lambda_{i1}| \\
& + |\chi m(u_0^n)^m\lambda_{02}|\sum_{i=1}^s|\lambda_{i2}| + |\chi m(u_0^n)^{m-1}\sum_{i=1}^s\lambda_{i2}u_i^n|\sum_{i=1}^s|\lambda_{i2}| \\
& + |\chi(u_0^n)^m|\sum_{i=1}^s|\lambda_{i0}|,
\end{aligned}
\tag{IX.6}
$$

for some $\xi_j \in (u_i^n, U_i^n)\cup(U_i^n, u_i^n)$, $\forall j\in\{1,...,16\}, \forall i\in\{0,1,...,s\}$.

Proof. Consider the first equation of (IX.3) (approximate solution) and subtract the same expression for the exact solution, i.e., in terms of u_i^n and v_i^n (which stands for the exact solution at time $n\Delta t$). Let us call $\tilde{u}_i^n := U_i^n - u_i^n$ (similarly for $\tilde{v}_i^n$) and take the maximum of $\tilde{u}_i^n$ among all nodes of the star, that is, $\tilde{u}^n := \max_{i=0,...,s}|\tilde{u}_i^n|$. Then, after some straightforward computations (in the previous chapters further details are done, in particular Chapter VIII) together with the Mean Value Theorem applied to the functions

$$f(t) = t^\delta, \quad \delta = m, \alpha,$$

we arrive to the following

$$\tilde{u}^{n+1} \le \tilde{u}^n\left[|1-\Delta t\lambda_{00}| + \Delta t\sum_{i=1}|\lambda_{i0}| + \Delta t|\Phi|\right] + \Delta t|\Psi|\tilde{v}^n, \tag{IX.7}$$

where Φ and Ψ are defined in (IX.5) and (IX.6), respectively. If we perform the same computations with the second equation of (IX.3) and its expression for the exact solution, it yields

$$\frac{\tilde{v}_0^n - \tilde{v}_0^n}{\Delta t} = -(\lambda_{00}+1)\tilde{v}_0^n + \sum_{i=1}^s\lambda_{i0}\tilde{v}_i^n + \gamma\xi_{17}^{\gamma-1}\tilde{u}_0^n. \tag{IX.8}$$

Again, by taking bounds and rewritting (IX.8) in terms of $\tilde{u}^n, \tilde{v}^n$, we get

$$\tilde{v}^{n+1} \le \tilde{u}^n\Delta t\gamma\xi_{17}^{\gamma-1} + \tilde{v}^n[|1-\Delta t(\lambda_{00}+1)| + \Delta t\sum_{i=1}^s|\lambda_{i0}|]. \tag{IX.9}$$

Therefore, inequalities (IX.7) and (IX.9) can be expressed as

$$\begin{pmatrix}\tilde{u}^{n+1}\\ \tilde{v}^{n+1}\end{pmatrix} \le \begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\tilde{u}^n\\ \tilde{v}^n\end{pmatrix} \tag{IX.10}$$

for an obvious choice of M_{lr}, with $r, l \in \{1,2\}$. Convergence of the explicit scheme is assured if the eigenvalues of (M_{rl}) are all smaller than 1. Consider the $\|\cdot\|_1$ norm as the maximum sum by rows. Clearly, since $\Delta t << 1$, $\|M\|_1 = M_{11} + M_{12}$ and, by assuming,

$$\Delta t < \frac{2}{\lambda_{00} + \sum_{i=1}^s|\lambda_{i0}| + |\Phi| + |\Psi|}, \tag{IX.11}$$

we have that $\|M\|_1 < 1$. Therefore, Lemma VI.1 implies $\lim_{k\to\infty} M^k = \mathbf{0}$. Finally, by Lemma VI.2 we obtain that the spectral radius is less that 1, which proves the result. □

Remark IX.2. *The above result covers only the parabolic-parabolic case. The conditional convergence of the GFD explicit scheme for the parabolic-elliptic (without nonlocal terms) case has been obtained in Chapter VII.*

2 Numerical examples

In this section we present numerical examples of the applicability of the GFD scheme given by (IX.3) for solving the non-linear non-local system (IX.1). For our simulations we use as time step $\Delta t = 0.001$ and consider as discretization of the domain $\Omega = [0,1]\times[0,1]$ both regular and irregular clouds of points of Figure IX.1, each one with 437 nodes.

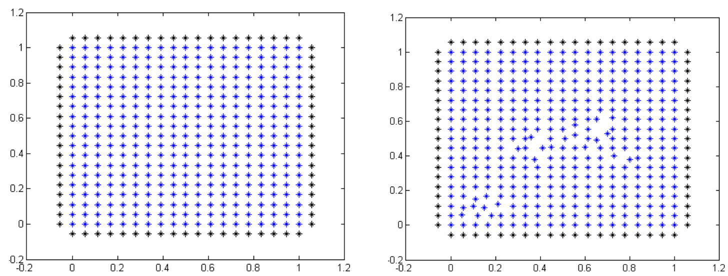

Figure IX.1: Regular and irregular clouds of points

For all examples we compute the difference between the numerical solution and the constant steady states using the l^∞ norm

$$\|v\|_{l^\infty(\Omega)} = \max_{i=0,\dots,N}\{|v_i|\},$$

as stated in Chapter VI. In order to illustrate the accuracy of this meshless method, let us first find the steady states of the problem.

2.1 Steady states

In order to study the asymptotic behavior of system (IX.1) we look at the constant steady states (u^*, v^*). We assume, without loss of generality, that $|\Omega| = 1$.

Lemma IX.1. *Assume $a_0, a_1, \alpha, \gamma > 0$, $a_1 > |a_2|$ then the constant steady state of (IX.1) is*

$$(u^*, v^*) = \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2}}, \left(\sqrt[\alpha]{\frac{a_0}{a_1 - a_2}}\right)^{\gamma}\right). \tag{IX.12}$$

Proof. The steady states of system (IX.1) are the solutions to the homogeneous one

$$\begin{cases} 0 = a_0 - a_1(u^*)^\alpha + a_2\displaystyle\int_\Omega (u^*)^\alpha d\mathbf{x}, \\ 0 = -v^* + (u^*)^\gamma. \end{cases} \tag{IX.13}$$

T(s)	0	2.5	5	7.5	10
$\|\frac{\sqrt{2}}{2} - U\|_{l^\infty(\Omega)}$	1.2929	0.0024	0.0001	0e-03	0e-04
$\|0.5 - V\|_{l^\infty(\Omega)}$	0.5000	0.0436	0.0036	0.2929e-03	0.2401e-4

Table IX.1: Values of $\|\frac{\sqrt{2}}{2} - U\|_{l^\infty(\Omega)}$ and $\|0.5 - V\|_{l^\infty(\Omega)}$ in the Example 1.

Since $|\Omega| = 1$, the first equation leads us to

$$(u^*)^\alpha = \frac{a_0}{a_1 - a_2},$$

and by the second equation we arrive to the result for v^* as in (IX.12). □

Remark IX.3. *The above result covers the following two particular cases:*

1. *Clearly, if we make $\alpha = m = \gamma = 1$ and $\partial_t v = 0$, the result is the one obtained by Negreanu and Tello [81].*
2. *Also, by making $a_0 = r$, $a_1 = \mu$ and $a_2 = 0$, Ding et al [29] obtained the convergence of the solution to the constant steady state (u^*, v^*).*

Asymptotic stability of the steady state has been obtained in both papers.

Remark IX.4. *If $a_0, a_2 \leq 0$, for any $a_1 > 0$, the solution converges asymptotically to 0, again in accordance with [29].*

2.2 Case 1

Example 1

In this first example we solve the case which appears in [29], that is to say, the case of Remark IX.3.2. Therefore we fix $a_2 = 0$. Consider $u_0(x, y) = 2x^2$ and $v_0(x, y) = \exp(-10[(x - 1.2)^2 + (y - 1)^2])$. Assume the following relation of parameters: $m = 1, \alpha = 2, \gamma = 2, \chi = 0.5$ and $a_0 = 2, a_1 = 4$. Then, by theory, we expect to find convergence towards $(\frac{\sqrt{2}}{2}, \frac{1}{2})$. We consider the regular cloud of points of Figure IX.1.

Table IX.1 shows the values of the l^∞ norm of the difference of the solution and the asymptotic value at different times, and Figure IX.2 the discrete solution at 0, 2.5 and 10 seconds.

Example 2

We now consider $a_0 = a_2 = -1$ and $a_1 = 1$ for initial data $u_0(x, y) = \exp(-10[(x - 0.5)^2 + (y - 0.5)^2])$ and $v_0(x, y) = 0.5 \exp(-10[(x - 0.2)^2 + (y - 1)^2])$ in the irregular cloud of points. Also we put $m = 1, \alpha = 2, \gamma = 1$ and $\chi = 0.5$. As stated in Remark IX.4 we expect to find asymptotic decay of the solution to 0.

Table IX.2 presents the l^∞ norm of the numerical solution at times 0, 2.5, 5, 7.5 and 10 seconds and Figure IX.3 the plot of the solution at 0, 2.5 and 10 seconds. As stated, solution tends to zero very rapidly.

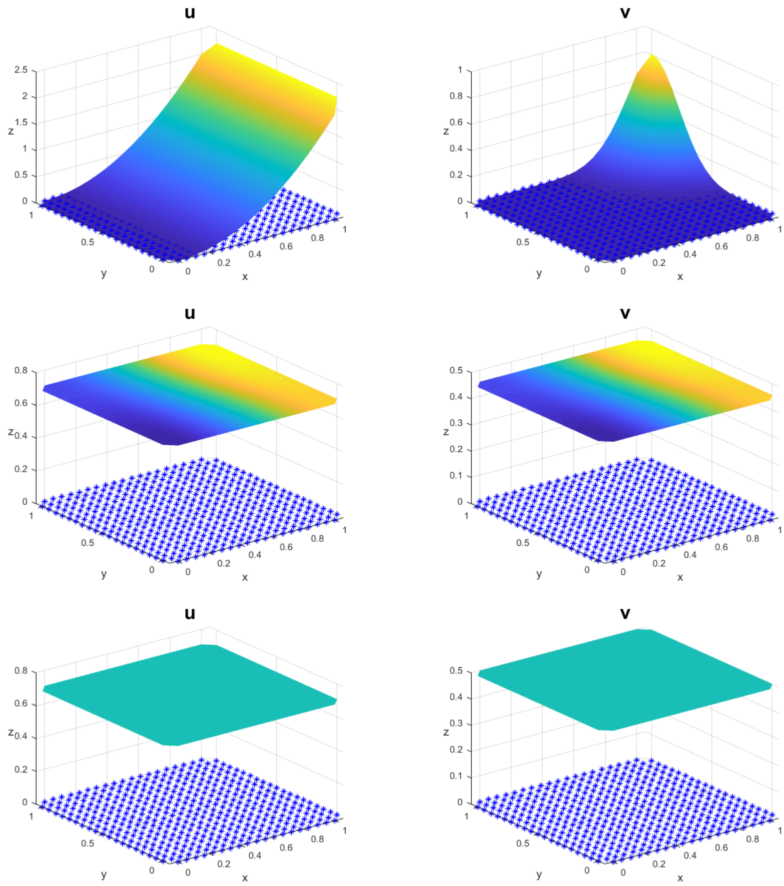

Figure IX.2: U, V-solution for 0, 2.5 and 10 seconds in the Example 1.

T(s)	0	2.5	5	7.5	10
$\|U\|_{l^\infty(\Omega)}$	0.9990	0.0234	0.0019	0.0002	0.0129e-03
$\|V\|_{l^\infty(\Omega)}$	0.4975	0.0639	0.0100	0.0012	0.1321e-03

Table IX.2: Values of $\|U\|_{l^\infty(\Omega)}$ and $\|V\|_{l^\infty(\Omega)}$ in the Example 2.

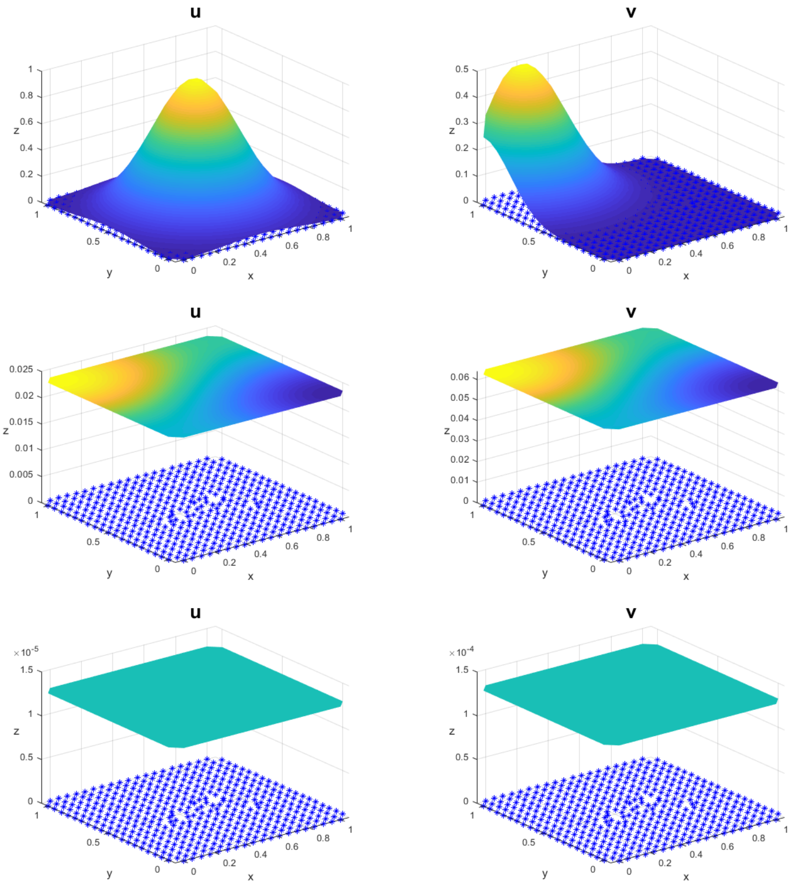

Figure IX.3: U, V-solution for 0, 2.5 and 10 seconds in the Example 2.

T(s)	0	2.5	5	7.5	10
$\|0.5 - U\|_{l^\infty(\Omega)}$	1.4951	0.0366	0.0032	0.0003	0.0218e-3
$\|0.5 - V\|_{l^\infty(\Omega)}$	0.5000	0.1143	0.0172	0.0021	0.2247e-3

Table IX.3: Values of $\|0.5 - U\|_{l^\infty(\Omega)}$ and $\|0.5 - V\|_{l^\infty(\Omega)}$ in the Example 3.

T(s)	0	2.5	5	7.5	10
$\|\frac{\sqrt{3}}{3} - U\|_{l^\infty(\Omega)}$	0.5774	0.0003	0.0000	0e-03	0e-04
$\|\frac{1}{3} - V\|_{l^\infty(\Omega)}$	0.6667	0.0419	0.0034	0.2815e-03	0.2308e-4

Table IX.4: Values of $\|\frac{\sqrt{3}}{3} - U\|_{l^\infty(\Omega)}$ and $\|\frac{1}{3} - V\|_{l^\infty(\Omega)}$ in the Example 4.

Example 3

Consider as initial data $u_0(x,y) = \exp(-10[(x-0.1)^2 + (y-0.1)^2])$ and $v_0(x,y) = 0.7\exp(-10[(x-1.2)^2 + (y-1)^2])$. We choose the relation of parameters: $m = 1, \alpha = 1, \gamma = 1$. As coefficients of the logistic term we take $a_0 = 1, a_1 = 3$ and $a_2 = 1$. For the chemotactic constant we put $\chi = 0.5$. Hence, in accordance with (IX.12) we expect to find convergence to $(1/2, 1/2)$. Table IX.3 shows the maximum distance between the numerical solution and the limit value (0.5 for both U and V). Figure IX.4 illustrates the numerical solution at the initial time and 2.5 and 10 seconds.

Example 4

For this fourth example we take the following initial data over the irregular cloud of points

$$u_0(x,y) = e^{-10[(x-0.5)^2+(y-0.5)^2]}, \quad v_0(x,y) = 1 - 0.5e^{-10[(x-0.2)^2+(y-1)^2]}.$$

We put $m = 2, \alpha = 2, \gamma = 2, \chi = 0.5$ and $a_0 = 3, a_1 = 8$ and $a_2 = -1$. Then, $(u^*, v^*) = (\frac{\sqrt{3}}{3}, \frac{1}{3})$.

Table IX.4 shows the differences, in l^∞ norm, of the asymptotic limit and the discrete solution. In Figure IX.5 we plot the numerical solution at 0, 2.5 and 10 seconds.

2.3 Case 2

Example 5

In this example we consider the parabolic-elliptic system (0.1) of [81] with $f = 0$ and $\lambda = 1$ (also $m = \alpha = \gamma = 1$). We choose the irregular clouds of points of Fig.1 and initial data

$$u_0(x,y) = 1 - e^{-10[(x-0.4)^2+(y-0.7)^2]}$$

and parameters $a_0 = a_2 = 1$ and $a_1 = 5$. Therefore, convergence towards $(\frac{1}{4}, \frac{1}{4})$ is expected.

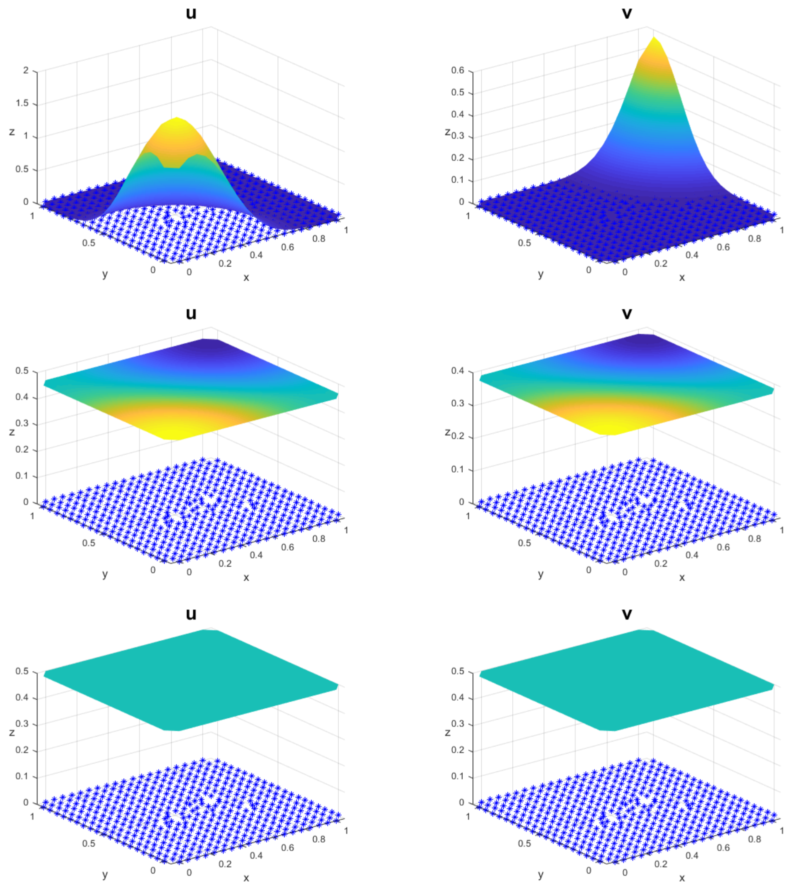

Figure IX.4: U, V-solution for 0, 2.5 and 10 seconds in the Example 3.

T(s)	0	2.5	5	7.5	10
$\|\frac{1}{4} - U\|_{l^\infty(\Omega)}$	0.75	0.0141	0.0011	8.9589e-05	7.3423e-06
$\|\frac{1}{4} - V\|_{l^\infty(\Omega)}$	-	0.0141	0.0011	8.9679e-05	7.3497e-06

Table IX.5: Values of $\|\frac{1}{4} - U\|_{l^\infty(\Omega)}$ and $\|\frac{1}{4} - V\|_{l^\infty(\Omega)}$ in the Example 5.

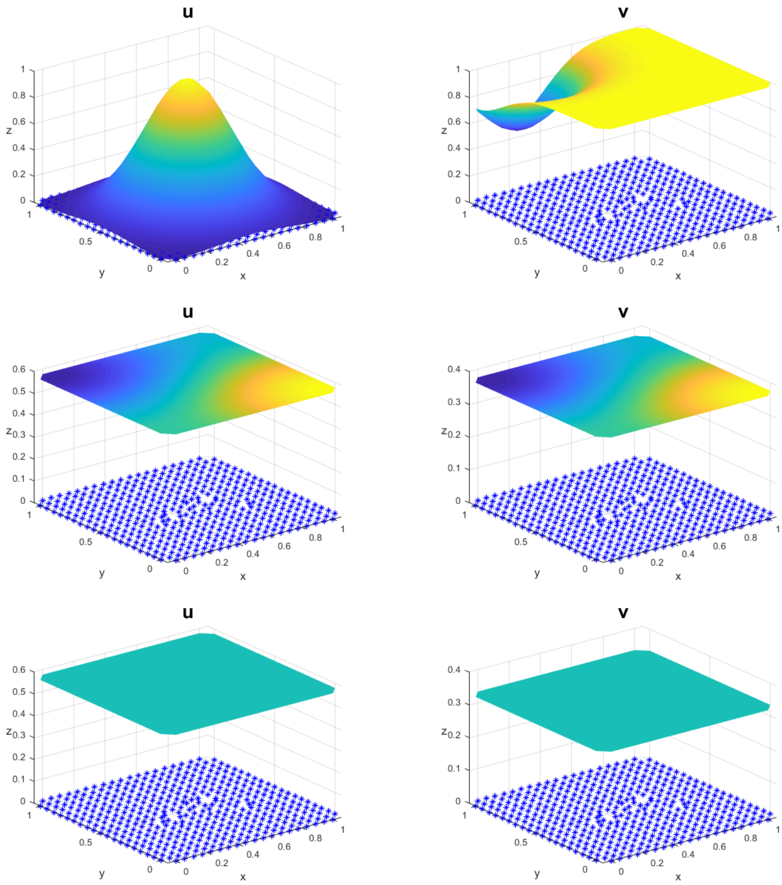

Figure IX.5: U, V-solution for 0, 2.5 and 10 seconds in the Example 4.

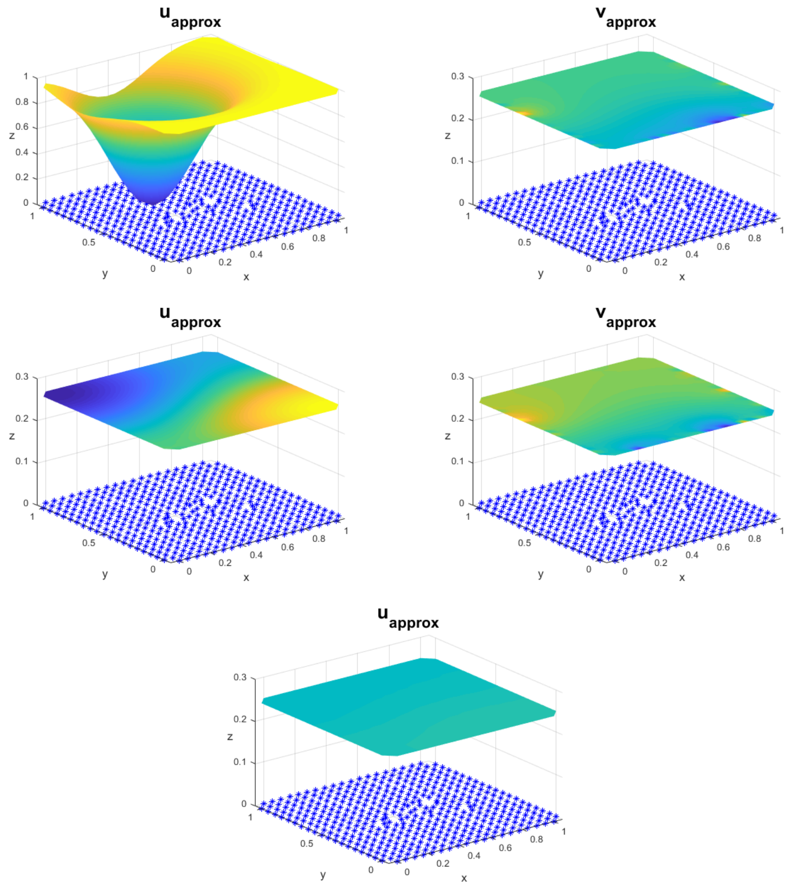

Figure IX.6: U, V-solution for 0, 2.5 and 10 seconds in the Example 5.

T(s)	0	2.5	5	7.5	10
$\|1-U\|_{l^\infty(\Omega)}$	1	0.0097	6.5381e-05	4.3836e-07	2.9387e-09
$\|1-V\|_{l^\infty(\Omega)}$	-	0.0097	6.5512e-05	4.3925e-07	2.9487e-09

Table IX.6: Values of $\|1-U\|_{l^\infty(\Omega)}$ and $\|1-V\|_{l^\infty(\Omega)}$ in the Example 6.

Example 6

In this final example we also solve numerically the parabolic-elliptic system, now with local competition among the individuals of the biological species. We put $a_0 = 2, a_1 = 1$ and $a_2 = -1$, so the constant steady state in this case is $(1,1)$. We consider

$$u_0(x,y) = e^{-10[(x-0.2)^2+(y-0.2)^2]} + e^{-10[(x-0.8)^2+(y-0.8)^2]}$$

Table IX.6 shows the l^∞ norm of the difference between the numerical solution and the steady state. Figure IX.7 displays the U, V-solutions at different times.

3 Conclusions

We have derived the discretization of the non-linear and non-local quasilinear system of parabolic PDEs in 2D using the GFDM. Also we have proved under which conditions the convergence can be expected. This meshless method does not reliance on the geometry of the domain or node distribution. Therefore it can be easily applied for solving highly non-linear PDEs over complicated and realistic domains.

All the numerical results are in accordance with the theoretical asymptotic stability results. We have provided different simulations considering the most significant cases: first, the case where non-local interactions don't occur, and we have compared our numerical results with the asymptiotic behavior obtained by Ding et al in [29]. Second, the case in which all the coefficients of the logistic source are negative, we reached that numerical solution decays to zero, also as stated in [29]. Finally we have extended our numerical study to the case in which individuals cooperate ($a_2 > 0$) and compete ($a_2 < 0$) in both parabolic-parabolic and parabolic-elliptic cases, where the numerical solution indicates the validity of the results of Chapter IV and also the extension of the parabolic-elliptic to the fully parabolic problem considered by Negreanu and Tello in [81].

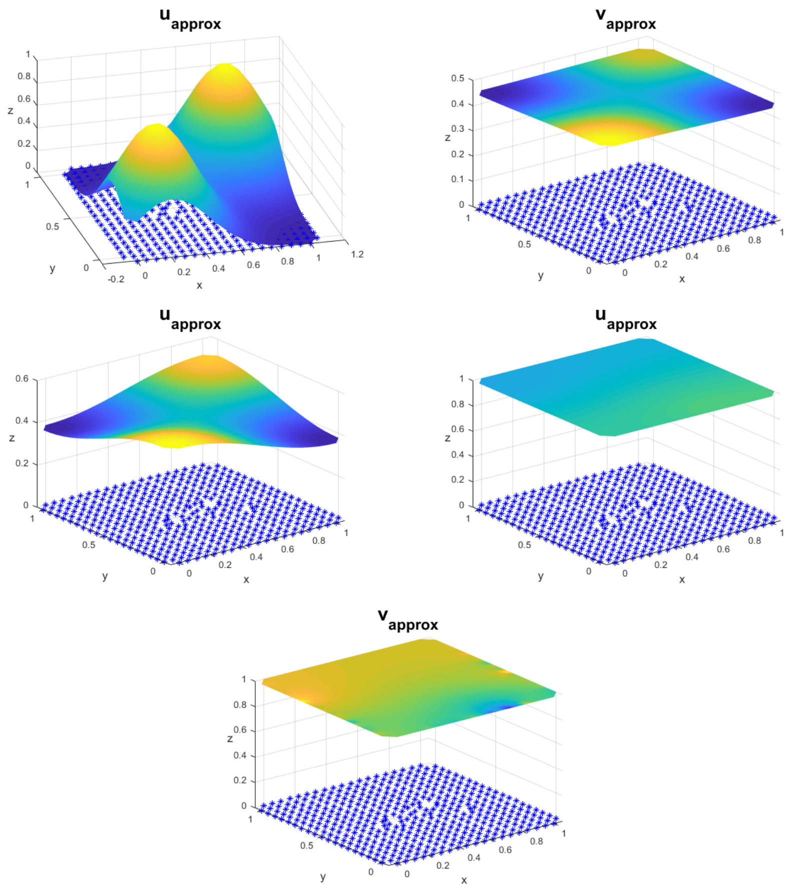

Figure IX.7: U, V-solution for 0, 0.1 and 2.5 seconds in the Example 6.

Chapter X

Numerical solution of the parabolic-ODE model

In this chapter we obtain the conditional convergence of the GDF scheme for the discretization of system (X.1) and we give the explicit conditions that the time increment, Δt, must fulfil in order to have it. The discrete numerical solution converges to the asymptotic periodic functions $u^*(t)$, $v^*(t)$. This means that some environmental periodicity conditions affect the behavior of the populations' density of a biological species, "u" and a chemical substance, "v", related by a chemotactic process. In other words, we prove that the discrete solution obtained by applying the GFD method to (X.1) preserves the same behavior of the continuous one. We also illustrate with our experiments that the Generalized Finite Difference Method solves this strongly coupled highly nonlinear parabolic-ODE system efficiently and with high accuracy over regular and irregular domains.
In Chapter V, we have proved that for all sufficiently smooth initial data $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, $v(\mathbf{x}, 0) = v_0(\mathbf{x})$, $\mathbf{x} \in \Omega$, the parabolic-ODE (where we consider the chemical substance to be non-diffusive) possesses a unique global-in-time classical solution that is bounded in $\Omega \times (0, \infty)$, with $\Omega \subset \mathbb{R}^d$, for $d \geq 1$. Recall the model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) + \mu u(1 + f(\mathbf{x}, t) - u), & \mathbf{x} \in \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial t} = h(u, v), & \mathbf{x} \in \Omega, \quad t > 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), v(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \mathbf{x} \in \partial\Omega, \quad t > 0 \end{cases} \tag{X.1}$$

In this chapter we prove that the convergence in space and in time of the classical solution is maintained for the discrete model. Also, let us remember the assumption

$$\frac{\partial h}{\partial v} < 0 \tag{X.2}$$

since we use it in the proof of the main result of this chapter. As in the two previous chapters, the present one is organized as follows: in the first section we introduce the numerical scheme which discretizes (X.1), we study the convergence of the GFD explicit scheme and we prove the main result of the paper, Theorem X.1. In the second section, extensive numerical experiments (convergence studies in space and in time, long-time simulations, etc.) are presented to illustrate the efficiency and robustness of the developed numerical algorithms. We finally present some conclusions.

1 GFD scheme

By using the explicit formulae of the GFD method introduced in Chapter VI, we obtain the following 2-dimensional GFD explicit scheme:

$$\begin{cases} U_0^{n+1} = U_0^n + \Delta t \left[-\lambda_{00} U_0^n + \sum_{i=1}^{s} \lambda_{i0} U_i^n - \chi U_0^n \left(-\lambda_{00} V_0^n + \sum_{i=1}^{s} \lambda_{i0} V_i^n \right) \right] \\ \qquad - \chi \Delta t \left(-\lambda_{01} U_0^n + \sum_{i=1}^{s} \lambda_{i1} U_i^n \right) \left(-\lambda_{01} V_0^n + \sum_{i=1}^{s} \lambda_{i1} V_i^n \right) \\ \qquad - \chi \Delta t \left(-\lambda_{02} U_0^n + \sum_{i=1}^{s} \lambda_{i2} U_i^n \right) \left(-\lambda_{02} V_0^n + \sum_{i=1}^{s} \lambda_{i2} V_i^n \right) \\ \qquad + \Delta t \mu U_0^n \left[1 - U_0^n + f(\mathbf{x}_0, n\Delta t) \right] \\ V_0^{n+1} = V_0^n + h(U_0^n, V_0^n) \Delta t. \end{cases} \tag{X.3}$$

The main result regarding the convergence of the proposed numerical scheme (X.3) is as follows:

Theorem X.1. *Let u, v be the exact solution of (X.1). Let be $\frac{\partial h}{\partial v}(u,v) < 0$, then the GFDM explicit scheme (X.3) is convergent if the time increment Δt verifies*

$$\frac{1}{\alpha - \frac{\partial h}{\partial v}} < \Delta t < \frac{3}{\alpha - \frac{\partial h}{\partial v}} \tag{X.4}$$

where

$$\begin{aligned} \alpha := \Bigg| & -\lambda_{00}(1 + \chi v_0^n) - \chi[(\lambda_{01})^2 + (\lambda_{02})^2] V_0^n \\ & + \chi \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} V_0^n + \chi \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} V_0^n - \chi \sum_{i=1}^{s} \lambda_{i0} V_i^n \\ & + \mu[1 - (u_0^n + U_0^n) + f(\boldsymbol{x}_0, n\Delta t)] \Bigg| + \Delta t \left(\sum_{i=1}^{s} |\lambda_{i0}| \right. \\ & + \chi |\lambda_{01}| v_0^n \sum_{i=1}^{s} |\lambda_{i1}| + \chi \left| \sum_{i=1}^{s} \lambda_{i1} V_i^n \right| \sum_{i=1}^{s} |\lambda_{i1}| \\ & \left. + \chi \left| \sum_{i=1}^{s} \lambda_{i2} V_i^n \right| \sum_{i=1}^{s} |\lambda_{i2}| + \chi |\lambda_{02}| v_0^n \sum_{i=1}^{s} |\lambda_{i2}| \right). \end{aligned} \tag{X.5}$$

Proof. As before, let U_j^n be the approximated U–solution at time $n\Delta t$ (similarly V_j^n) and v_j^n the value of the exact u– solution (similarly v_j^n). Also, we call $\tilde{u}_j^n = U_j^n - u_j^n$ and $\tilde{v}_j^n = v_j^n - V_j^n$. Let us take the difference between the GFD scheme (X.3) and the expression for the exact solution. We obtain the following:

$$
\begin{aligned}
\tilde{u}_0^{n+1} = \tilde{u}_0^n + \Delta t \Bigg[& - \lambda_{00}\tilde{u}_0^n + \sum_{i=1}^{s} \lambda_{i0}\tilde{u}_i^n - \chi[(\lambda_{01})^2 + \\
& + (\lambda_{02}^2)][\tilde{u}_0^n V_0^n - u_0^n \tilde{v}_0^n] + \chi\lambda_{01}\tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i1} V_i^n + \\
& + \chi\lambda_{01} u_0^n \sum_{i=1}^{s} \lambda_{i1}\tilde{v}_i^n + \chi\lambda_{01}\tilde{v}_0^n \sum_{i=1}^{s} \lambda_{i1} U_i^n \\
& + \chi\lambda_{01} v_0^n \sum_{i=1}^{s} \lambda_{i1}\tilde{u}_i^n - \left(\sum_{i=1}^{s} \lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i1} V_i^n\right) \\
& - \left(\sum_{i=1}^{s} \lambda_{i1} u_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i1}\tilde{v}_i^n\right) + \chi m_{02}\tilde{u}_0^n \sum_{i=1}^{s} \lambda_{i2} V_i^n \\
& + \chi\lambda_{02} u_0^n \sum_{i=1}^{s} \lambda_{i2}\tilde{v}_i^n + \chi\lambda_{02}\tilde{v}_0^n \sum_{i=1}^{s} \lambda_{i2} U_i^n + \chi\lambda_{02} v_0^n \sum_{i=1}^{s} \lambda_{i2}\tilde{u}_i^n \\
& - \left(\sum_{i=1}^{s} \lambda_{i2}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i2} V_i^n\right) - \left(\sum_{i=1}^{s} \lambda_{i2} u_i^n\right)\left(\sum_{i=1}^{s} \lambda_{i2}\tilde{v}_i^n\right) \\
& + \mu\tilde{u}_0^n[1 - (u_0^n + U_0^n) + f(\mathbf{x}_0, n\Delta t)] \Bigg].
\end{aligned}
\tag{X.6}
$$

After rearranging the terms, it yields:

$$
\begin{aligned}
\tilde{u}_0^{n+1} = \tilde{u}_0^n \Bigg[1 - \Delta t \Bigg(& -\lambda_{00}(1+\chi v_0^n) - \chi[(\lambda_{01})^2 + (\lambda_{02})^2] V_0^n \\
& + \chi\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} V_0^n + \chi\lambda_{02} \sum_{i=1}^{s} \lambda_{i2} V_0^n - \chi \sum_{i=1}^{s} \lambda_{i0} V_i^n \\
& + \mu[1 - (u_0^n + U_0^n) + f(\mathbf{x}_0, n\Delta t)] \Bigg) \Bigg] + \\
& + \tilde{v}_0^n \chi \Delta t \Bigg[- [(\lambda_{01})^2 + (\lambda_{02})^2] u_0^n + \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} U_i^n + \\
& + \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_i^n + \lambda_{00} u_0^n \Bigg] + \Delta t \Bigg[\sum_{i=1}^{s} \lambda_{i0} \tilde{u}_i^n + \\
& + \chi\lambda_{01} v_0^n \sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n - \chi \left(\sum_{i=1}^{s} \lambda_{i1} \tilde{u}_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} V_i^n \right) \\
& - \chi \left(\sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i2} V_i^n \right) + \chi\lambda_{02} v_0^n \sum_{i=1}^{s} \lambda_{i2} \tilde{u}_i^n \Bigg] \\
& + \chi \Delta t \Bigg[\lambda_{01} u_i^n \sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n - \left(\sum_{i=1}^{s} \lambda_{i1} u_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i1} \tilde{v}_i^n \right) \\
& - \left(\sum_{i=1}^{s} \lambda_{i2} u_i^n \right) \left(\sum_{i=1}^{s} \lambda_{i2} \tilde{v}_i^n \right) + \lambda_{02} u_i^n \sum_{i=1}^{s} \lambda_{i2} \tilde{v}_i^n \Bigg].
\end{aligned}
\tag{X.7}
$$

Now, we take bounds and call $\tilde{u}^n = \max_{i=0,\dots,s}\{|\tilde{u}_i^n|\}$ (the same applies for $\tilde{v}^n$). Then, we get

$$\begin{aligned}
\tilde{u}^{n+1} \leq \tilde{u}^n \Bigg[\Bigg| 1 - \Delta t \Bigg(& -\lambda_{00}(1+\chi v_0^n) - \chi[(\lambda_{01})^2 + (\lambda_{02})^2] V_0^n \\
& + \chi\lambda_{01} \sum_{i=1}^{s} \lambda_{i1} V_0^n + \chi\lambda_{02} \sum_{i=1}^{s} \lambda_{i2} V_0^n - \chi \sum_{i=1}^{s} \lambda_{i0} V_i^n \\
& + \mu[1 - (u_0^n + U_0^n) + f(\mathbf{x}_0, n\Delta t)] \Bigg) \Bigg| + \Delta t \Bigg(\sum_{i=1}^{s} |\lambda_{i0}| \\
& + \chi|\lambda_{01}| v_0^n \sum_{i=1}^{s} |\lambda_{i1}| + \chi \Bigg| \sum_{i=1}^{s} \lambda_{i1} V_i^n \Bigg| \sum_{i=1}^{s} |\lambda_{i1}| \\
& + \chi \Bigg| \sum_{i=1}^{s} \lambda_{i2} V_i^n \Bigg| \sum_{i=1}^{s} |\lambda_{i2}| + \chi|\lambda_{02}| v_0^n \sum_{i=1}^{s} |\lambda_{i2}| \Bigg) \Bigg] + \\
& + \tilde{v}^n \chi \Delta t \Bigg[\Bigg| -[(\lambda_{01})^2 + (\lambda_{02})^2] u_0^n + \lambda_{01} \sum_{i=1}^{s} \lambda_{i1} U_i^n \\
& + \lambda_{02} \sum_{i=1}^{s} \lambda_{i2} U_i^n + \lambda_{00} u_0^n \Bigg| + |\lambda_{01}| u_i^n \sum_{i=1}^{s} |\lambda_{i1}| \\
& + \Bigg| \sum_{i=1}^{s} \lambda_{i1} u_i^n \Bigg| \sum_{i=1}^{s} |\lambda_{i1}| + \Bigg| \sum_{i=1}^{s} \lambda_{i2} u_i^n \Bigg| \sum_{i=1}^{s} |\lambda_{i2}| \\
& + |\lambda_{02}| u_i^n \sum_{i=1}^{s} |\lambda_{i2}| \Bigg].
\end{aligned} \tag{X.8}$$

For the second equation of (X.1) we have

$$\tilde{v}_0^{n+1} = \tilde{v}_0^n + \Delta t \frac{\partial h}{\partial u}\bigg|_{(\xi, v_0^n)} \tilde{u}_0^n + \Delta t \frac{\partial h}{\partial v}\bigg|_{(u_0^n, \eta)} \tilde{v}_0^n, \tag{X.9}$$

where we have applied the Mean Value Theorem twice for some $\xi \in (u_0^n, U_0^n) \cup (U_0^n, u_0^n)$, $\eta \in (v_0^n, V_0^n) \cup (V_0^n, v_0^n)$. Hence, by taking again the maximum for all indices $i = 0, ..., s$, we reach the expression

$$\tilde{v}^{n+1} \leq \tilde{u}^n \left| \frac{\partial h}{\partial u} \right| + \tilde{v}^n \left| 1 + \Delta t \frac{\partial h}{\partial v} \right|. \tag{X.10}$$

We express the expressions (IX.8) and (IX.10) in matrix form, in the following sense

$$\begin{pmatrix} \tilde{u}^{n+1} \\ \tilde{v}^{n+1} \end{pmatrix} \leq \begin{pmatrix} |1 - \Delta t \cdot \alpha| & B \\ C & |1 + \Delta t \partial_v h| \end{pmatrix} \begin{pmatrix} \tilde{u}^n \\ \tilde{v}^n \end{pmatrix}. \tag{X.11}$$

The characteristic polynomial of the square matrix has, at most, two roots fulfilling

$$\Big| |1 - \Delta t \cdot \alpha| + |1 + \Delta t \partial_V h| \Big| \leq |\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2| < 1. \tag{X.12}$$

If we impose

$$\left| 2 - \Delta t \left(\alpha - \frac{\partial h}{\partial v} \right) \right| < 1, \tag{X.13}$$

then (X.12) is verified. Therefore, the LHS inequality leads us to

$$\Delta t < \frac{3}{\alpha - \frac{\partial h}{\partial v}}, \tag{X.14}$$

whereas the second one

$$\Delta t > \frac{1}{\alpha - \frac{\partial h}{\partial v}}. \tag{X.15}$$

Notice that the denominator in (X.14)–(X.15) is positive due to assumption (X.2), i.e., $\partial_v h < 0$. So Δt can be always chosen such that the GFD explicit scheme (X.3) is convergent. □

Remark X.1. *Observe that (X.2) is enough to guarantee the convergence without adding extra assumptions on the problem.*

2 Numerical examples

In this section we present the numerical results obtained by solving the system (X.1), using two irregular clouds of points as seen in Figure X.1 (441 nodes in each one) in the domain $\Omega = [0,1] \times [0,1]$. We use an 8-node scheme, chosen by the distance criterion together with weight function $w = \frac{1}{dist^4}$. For all numerical examples we put $\Delta t = 0.001$.

We divide the section into two different cases, depending on the choice of function h. For each case, we provide two examples where we consider two different functions $f(\mathbf{x}, t)$:

1. Firstly, we take

$$f(\mathbf{x}, t) = \frac{\cos t}{4 + \sin t} + \frac{x - y}{1 + t^2}, \tag{X.16}$$

   in $(x, y) \in \Omega = [0,1] \times [0,1]$. Therefore, we can find the $2\pi-$periodic function

$$f^*(t) = \frac{\cos t}{4 + \sin t},$$

   and then the asymptotic limit is

$$u^*(t) = \frac{4 + \sin(t)}{4 - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}}.$$

2. Secondly, we consider

$$f(\mathbf{x}, t) = f^*(t) = \frac{2\cos(2t)}{2 + \sin(2t)} + \frac{\sin^2(t) + 2\sin(2t)}{1 + \cos^2(t)} \tag{X.17}$$

   in $(x, y) \in \Omega = [0,1] \times [0,1]$. Hence, this time the $\pi-$periodic non-constant steady state is

$$u^*(t) = \frac{2 + \sin(2t)}{1 + \cos^2(2t)}.$$

In order to find the asymptotic value v^* we perform an standard numerical method, as in the previous chapter we use the function ODE45 of Matlab2019b, in the equation

$$v_t^* = h(u^*, v^*),$$

for each choice of the function h.

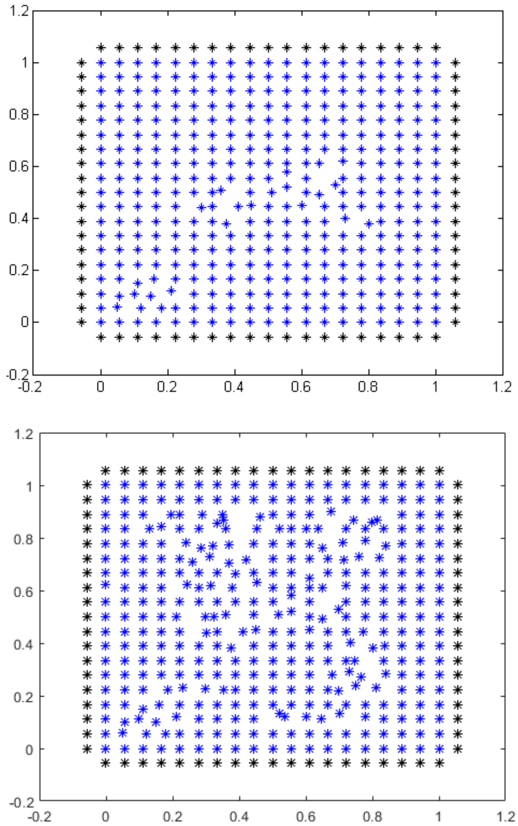

Figure X.1: Irregular clouds of points

2.1 Case 1

In this first case, we choose function $h(u, v)$ to be

$$h(u, v) = ue^{-\chi v} - v, \tag{X.18}$$

which fulfils all assumptions made in Chapter V (published in [87]).

Example 1

We select the following initial data

$$u_0(x, y) = e^{-10[(x-0.1)^2+(y-0.1)^2]} + e^{-10[(x-0.9)^2+(y-0.9)^2]},$$

$$v_0(x, y) = 0.7e^{-10[(x-0.5)^2+(y-0.5)^2]},$$

and parameters $\mu = 1, \chi = 0.3$. Table X.1 illustrates the l^∞ norm of the difference between the numerical solution and the asymptotic values u^*, v^* at several times. In Figure X.2 and X.3 we sketch the graphs of the periodic functions $u^*(t), v^*(t)$ (solid lines) and the most distant values of the discrete solution (that is to say, the value of the approximate solution where the greatest error in l^∞ norm is performed) at different times. As expected, the numerical results shown in Tables X.1 and Figures X.2 and X.3, respectively, confirm the theoretical result of Chapter V (and also the ones in [87]). The numerical solution given

T(s)	3.72	6.86	13.14	16.28
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0436	0.0014	5.5900e-04	1.5783e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.1023	0.0088	3.9862e-04	2.1900e-04

Table X.1: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 1 and Case 1

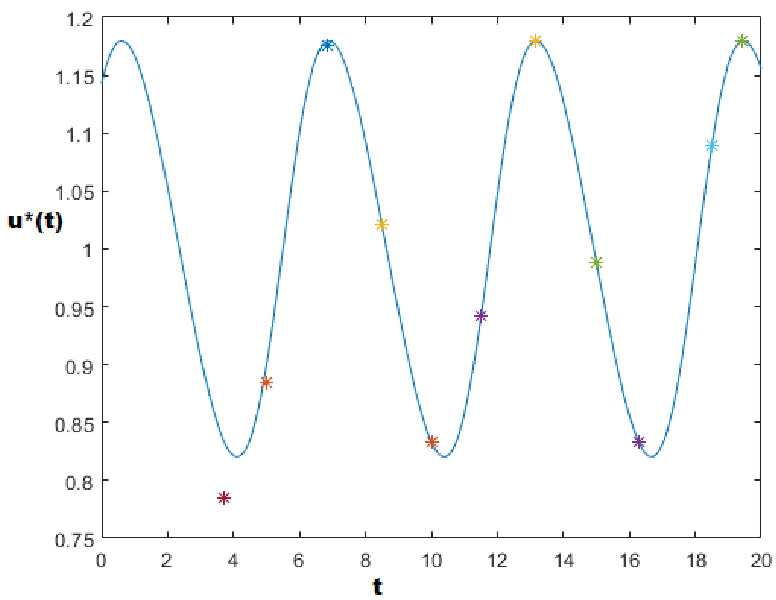

Figure X.2: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the value of the approximate solution U where we obtain the greatest error at 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 1 and Case 1.

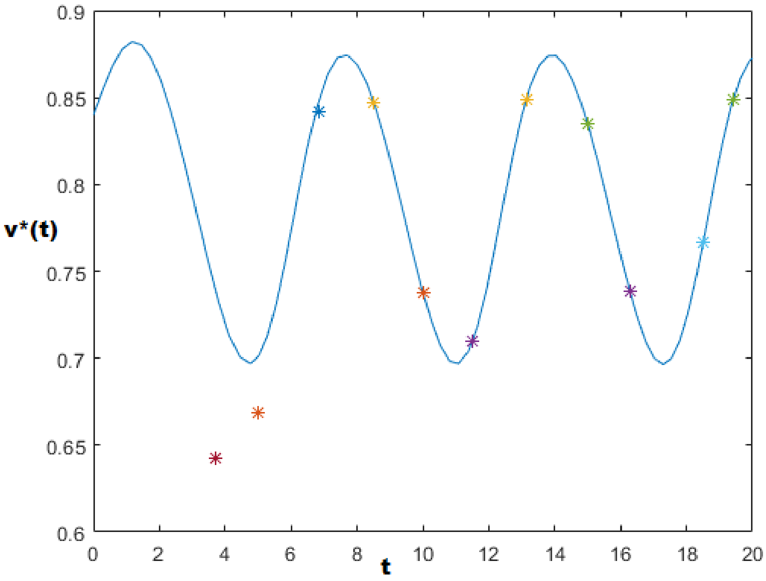

Figure X.3: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the most distant value of the approximate solution V at 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 1 and Case 1.

by GFD scheme converges to the periodic asymptotic limit of system (X.1). In accordance with the theory, the approximate solution inherits the periodic behavior of the function f^* at large times.

T(s)	3.72	13.14	19.42
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0339	3.4624e-04	3.4304e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0634	9.0783e-04	3.9845e-04

Table X.2: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 2 and Case 1

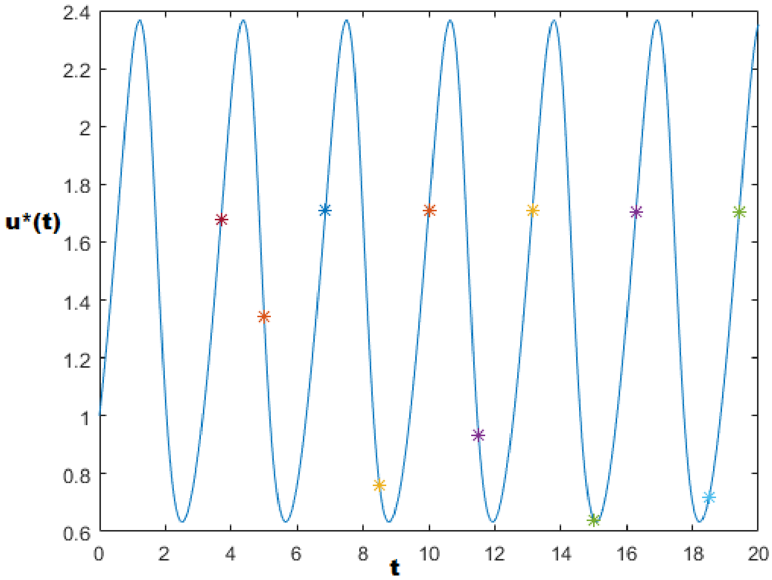

Figure X.4: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the value of the approximate solution, U, with the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 2 and Case 1.

Example 2

We consider now the function given by (X.15) and the initial data

$$u_0(x,y) = e^{-10[(x-0.2)^2+(y-0.2)^2]}, \quad v_0(x,y) = e^{-10[(x-0.8)^2+(y-0.8)^2]}$$

together with the parameters $\mu = 1, \chi = 0.3$.
As before, we illustrate in Table X.2 the maximum difference between the limit value and the numerical solutions. Figure X.4 and X.5 reflect the periodic functions $u^*(t), v^*(t)$ (solid lines) and the values of the discrete solution at the node where the greatest error is achieved (stars) at different times.
As stated before, the convergence of the GFD scheme is clearly featured in the above figures. The numerical solution given by the explicit scheme behaves as the periodic functions (u^*, v^*) as t increases.

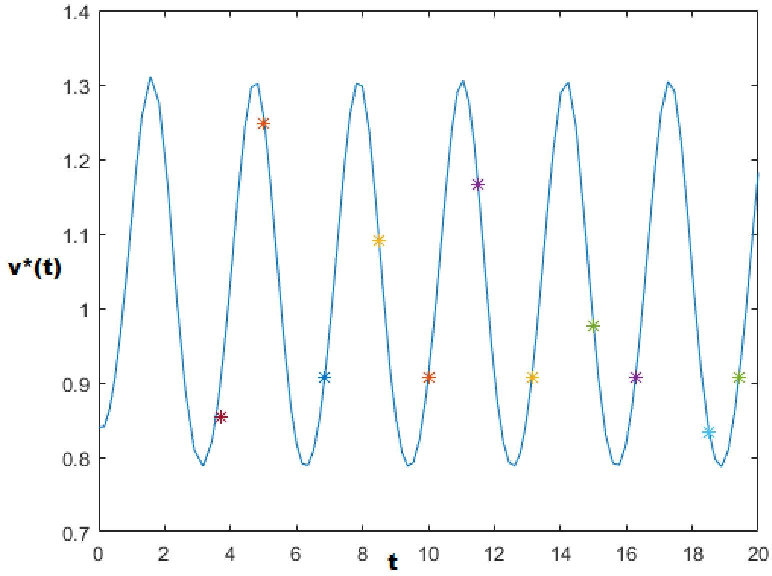

Figure X.5: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the value of the approximate solution, V, with the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 2 and Case 1.

2.2 Case 2

We consider now the function to describe the growth rate of the chemical substance

$$h(u,v) = \frac{ue^{-\chi v} + v}{1+v} - v. \tag{X.19}$$

It is easily checked that h fulfils the assumptions made in Chapter V (see also [87]). We provide two more examples with different functions $f(\mathbf{x}, t)$.

Example 3

Assume now that the initial data of this example are of the form

$$u_0(x,y) = 2e^{-10[(x-0.5)^2+(y-0.5)^2]}, \quad v_0(x,y) = e^{(x-0.5)^2+(y-0.5)^2},$$

and take the parameters as $\mu = 1, \chi = 0.5$.
As previously mentioned, Table X.3 shows the l^∞ norm of the difference between the solution given by the GFD scheme and asymptotic solution. Figure X.6 and X.7 present the behavior of the periodic functions $u^*(t), v^*(t)$ (solid lines) and the most distant values of the discrete solution at different times.
Tables X.3 and Figure X.6 and X.7 display that, for any initial data, the discrete solution of (X.1) presents the same asymptotic periodic behavior of the continuous model, proved in Chapter V. For small times, the numerical solution may differ from the functions (u^*, v^*) since these represent the limit of the continuous solution and not the solution itself.

T(s)	3.72	6.86	10	16.28
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0093	0.0010	5.4652e-04	1.6059e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0357	0.0026	4.1827e-04	6.6184e-05

Table X.3: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 3 and Case 2.

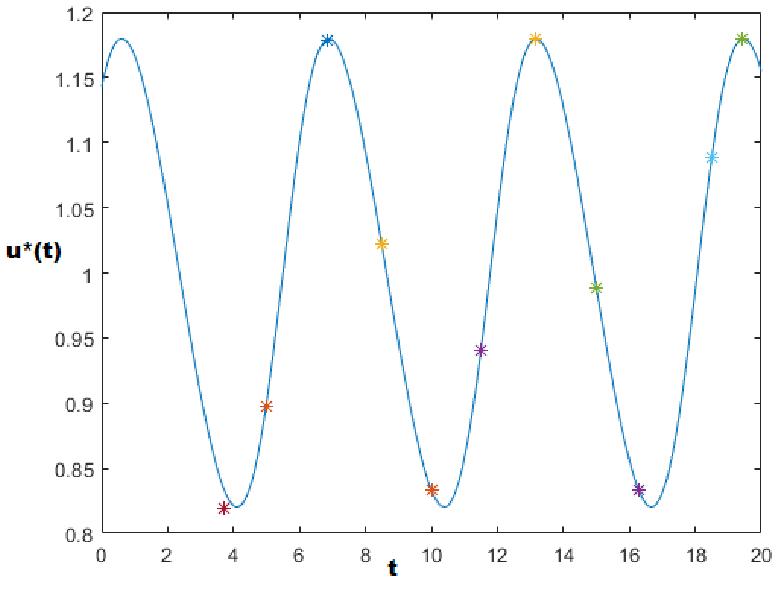

Figure X.6: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the value of the approximate solution, U, performing the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 3 and Case 2.

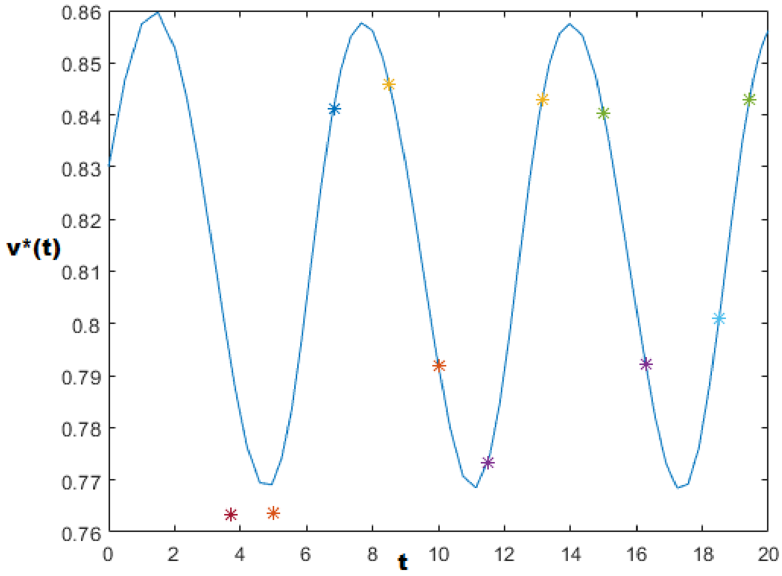

Figure X.7: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the value of the approximate solution, V, where we obtain the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 3 and Case 2.

T(s)	3.72	10	19.42
$\|U - u^*(t)\|_{l^\infty(\Omega)}$	0.0326	3.4328e-04	3.4302e-04
$\|V - v^*(t)\|_{l^\infty(\Omega)}$	0.0464	1.9974e-04	7.0348e-05

Table X.4: Values of $\|U - u^*(t)\|_{l^\infty(\Omega)}$ and $\|V - v^*(t)\|_{l^\infty(\Omega)}$ in the Example 4 and Case 2.

Example 4

Now we consider again

$$u_0(x, y) = e^{-10[(x-0.2)^2+(y-0.2)^2]}, \quad v_0(x, y) = e^{-10[(x-0.8)^2+(y-0.8)^2]},$$

and $\mu = 1, \chi = 0.5$ over the second cloud of points of Figure X.1. In Table X.4 we resume the maximum difference between the theoretical values u^*, v^* and the numerical solution. Figure X.8 and X.9 show the asymptotic limits of the problem (solid lines) and the values of the numerical solution (stars).

3 Conclusions

We have derived the discretization of the modified Keller-Segel system (X.1) by means of the GFD explicit scheme (X.3). Also, we have proved the conditional convergence of this scheme to the continuous model of the system and we have given the explicit conditions that the time increment, Δt must fulfil in order to obtain convergence of the

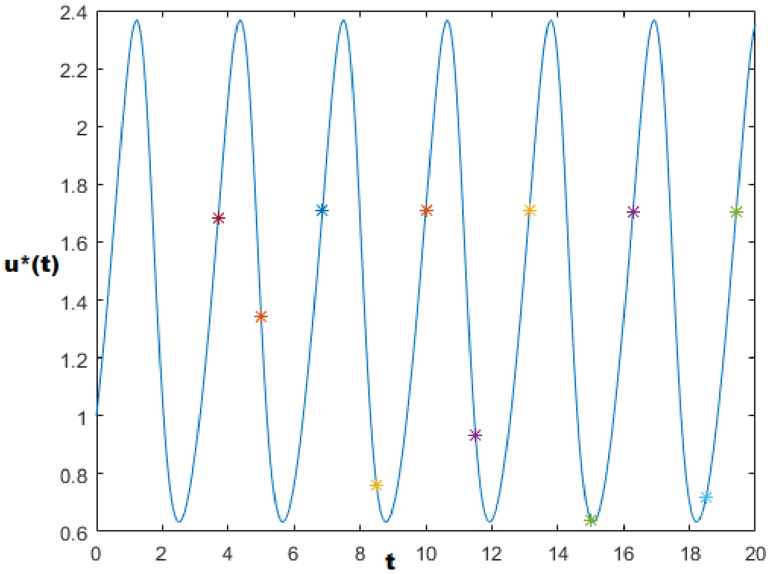

Figure X.8: The solid line corresponds to the graphic of the function $u^*(t)$, the stars to the value of the approximate solution, U, with the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 4 and Case 2.

method. An interesting remark from this proof is the fact that the condition for the convergence relies strongly in the assumption of the continuous model, $\partial_v h < 0$. In order to illustrate the convergence of the Generalized Finite Difference Method for solving this PDE-ODE problem, and also the validity of the results stated in Chapter V, we have provided four examples with different functions f and h, in the conditions of [87], and tested the GFD method over two irregular cloud of points. As stated for the continuous model, and expected once we have proved the conditional convergence of the scheme, the discrete numerical solution converges to the asymptotic periodic functions $u^*(t), v^*(t)$. This means that some environmental periodicity conditions affect the behavior of the populations' density of a biological species, "u" and a chemical substance, "v", related by a chemotactic process. The results of this chapter are published in [10].

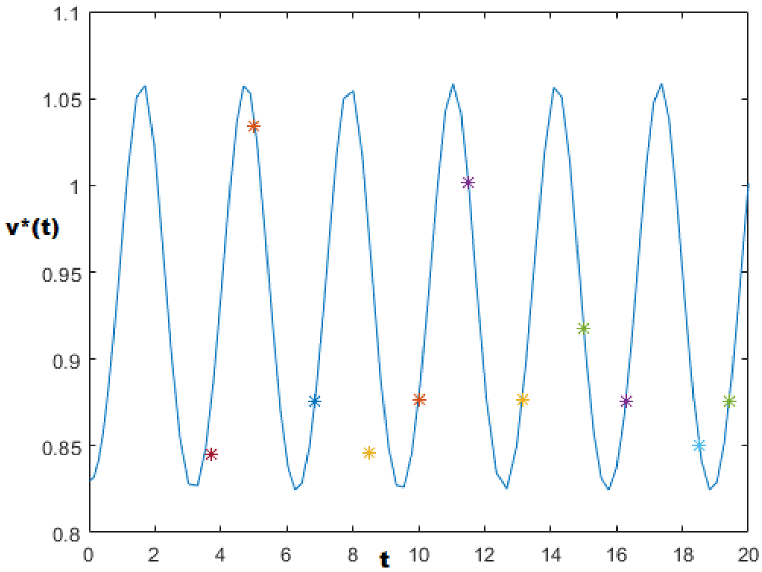

Figure X.9: The solid line corresponds to the graphic of the function $v^*(t)$, the stars to the value of the approximate solution, V, performing the greatest error at times 3.72, 5, 6.86, 8.5, 10, 11.5, 13.14, 15, 16.28, 18.5 and 19.42 in Example 4 and Case 2.

Chapter XI

Chemotaxis-haptotaxis model

In this chapter we focus on a mathematical model describing the process of cells invasion in the surrounding extracellular matrix. We present a mathematical model introduced in [21] which is a simplification of that presented in [20] focusing on key events of the cells invasion process of tissue by cancerous cells, on the role of matrix degrading enzymes and on the potential competition between chemotaxis and haptotaxis.
Tumor cells find a variety of substratum-bound factors that can influence their migration directed to different stages in the process of tumor invasion and metastasis. Such factors can promote the targeted movement of tumor cells by at least two mechanisms, called chemotaxis and haptotaxis.
Initially, Chaplain and Lolas in [20]–[21] developed a mathematical model consisting of three partial differential equations describing the evolution in time and space of the system variables. It is assumed that the key physical variables are tumor cell density (denoted by u), protein density of the extracellular matrix (denoted by w) and the concentration of urokinase plasminogen activator (denoted by v) each of them considered at $x \in \Omega$ and time $t > 0$. Through this paper $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ is a bounded domain with a regular boundary. The model is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - div(\chi u \nabla v) - div(\xi u \nabla w) + \mu u(1 - u - w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial t} = -vw, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{XI.1}$$

It is natural to assume that there is no-flux of tumor cells or protease across the boundary of the domain, $\partial\Omega$ due to the vitro experimental protocol in which invasion takes place within an isolated system (see [21] for more details)

$$-\frac{\partial u}{\partial \nu} + \chi u \frac{\partial v}{\partial \nu} + \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{XI.2}$$

The three-component chemotaxis-haptotaxis model (XI.1) is an extension of the two-component chemotaxis model proposed in 1970 by Keller and Segel [63] by taking $w = 0$.

This final chapter has the following structure: in Section 1 we analyze the local stability of the constant steady states. Section 2 is devoted to the analytical study of the GFD explicit scheme where we prove the main result of this section, enclosed in Theorem XI.1. In Section 3 we present several numerical examples over regular and irregular domains which show the applicability of the method. Finally, some conclusions are obtained in Section 4.

1 Steady states and linearization

In this section we consider the local stability of the constant equilibrium solutions (u^*, v^*, w^*), the steady states of the nonlinear system (XI.1) which verify

$$\begin{cases} 0 = \mu u(1-u-w) \\ 0 = u - v \\ 0 = -vw, \end{cases} \tag{XI.3}$$

i.e., $(u^*, v^*, w^*) = (1, 1, 0)$ and $(u^*, v^*, w^*) = (0, 0, w^*)$, with $w^* \geq 0$.

From the principle of linearized stability for quasilinear parabolic problems, if all eigenvalues of the linearized system (XI.1) at an equilibrium are of negative real parts, then the equilibrium is locally asymptotically stable in $W^{1,p}(\Omega)$, with $p > d$, as can be seen, for example, in [30].

The linearized problem of (XI.1) at any constant equilibrium is expressed by the system

$$\begin{pmatrix} \phi_t \\ \psi_t \\ \eta_t \end{pmatrix} = L(\xi, \chi) \begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix} = D \begin{pmatrix} \Delta\phi \\ \Delta\psi \\ \Delta\eta \end{pmatrix} + J_{(u,v,w)} \begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix}, \tag{XI.4}$$

with

$$D = \begin{pmatrix} 1 & -\chi u & -\xi u \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_{(u,v,w)} = \begin{pmatrix} \mu(1-u-w) - \mu u & 0 & -\mu u \\ 1 & -1 & 0 \\ 0 & -w & v \end{pmatrix}.$$

The stability of (u^*, v^*, w^*) is obtained by the eigenvalue problem

$$L(\xi, \chi) \begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix} = \beta \begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix},$$

i.e.,

$$\begin{cases} \Delta\phi - \chi u^* \Delta\psi - \xi u^* \Delta\eta + \mu(1 - 2u^* - w^*)\phi - \mu u^* \eta = \beta\phi \\ \Delta\psi + \phi - \psi = \beta\psi \\ -w^*\psi - v^*\eta = \beta\eta \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = \dfrac{\partial\eta}{\partial\nu} = 0, \end{cases} \tag{XI.5}$$

where β is an eigenvalue of (XI.5) with corresponding eigenfunction (ϕ, ψ, η). Let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of eigenvalues

$$0 = \alpha_0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq ..., \qquad \lim_{n\to\infty} \alpha_n = \infty$$

of operator $-\Delta$ with Neumann boundary conditions defined on the space

$$S = \left\{ \varphi \in W^{2,r}(\Omega) : \frac{\partial \varphi}{\partial \nu} = 0 \right\},$$

and let $\zeta_n(x)$ be its normalized eigenfunction, $n \in \mathbb{N}$. The linear part of the operator in (XI.1) is normally elliptic and it has only point spectrum. By using Fourier expansion, there exist $\{\phi_n\}, \{\psi_n\}$ and $\{\eta_n\}$ such that

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n \zeta_n(x), \quad \psi(x) = \sum_{n=0}^{\infty} \psi_n \zeta_n(x), \quad \eta(x) = \sum_{n=0}^{\infty} \eta_n \zeta_n(x)$$

and due to $(\phi, \psi, \eta) \neq (0,0,0)$, it implies $(\phi_n, \psi_n, \eta_n) \neq (0,0,0)$.
Multiplying (XI.5) by ζ_n and integrating over Ω, with $\|\zeta_n\|_{L^2(\Omega)} = 1$ we get

$$\begin{cases} -\alpha_n \phi_n - \chi u^* \alpha_n \psi_n - \xi u^* \alpha_n \eta_n + \mu(1 - 2u^* - w^*)\phi_n - \mu u^* \eta_n = \beta \phi_n \\ -\alpha \psi_n + \phi_n - \psi_n = \beta \psi_n \\ -w^* \psi_n - v^* \eta_n = \beta \eta_n \end{cases} \tag{XI.6}$$

equivalent to

$$A_n \begin{pmatrix} \phi_n \\ \psi_n \\ \eta_n \end{pmatrix} = \beta \begin{pmatrix} \phi_n \\ \psi_n \\ \eta_n \end{pmatrix},$$

with

$$A_n = \begin{pmatrix} -\alpha_n + \mu(1 - 2u^* - w^*) & \chi u^* \alpha_n & \xi u^* \alpha_n - \mu u^* \\ 1 & -1 - \alpha_n & 0 \\ 0 & -w^* & -v^* \end{pmatrix}.$$

If β is an eigenvalue of (XI.5), then there exists $n \in \mathbb{N}$, such that β is an eigenvalue of A_n and if (ϕ_n, ψ_n, η_n) is an eigenvector associated with β for A_n, then it is an eigenvector associated with β for (XI.5). Thus, the eigenvalue problem (XI.5) can be reduced to a sequence of matrix eigenvalue problems.
We know that a constant equilibrium (u^*, v^*, w^*) is locally asymptotically stable with respect to the original PDE system if and only if for every $n \in \mathbb{N}$, all the eigenvalues of A_n have negative real part and it is instable if and only if there exists $n \in \mathbb{N}$ such that A_n has at least one eigenvalue with nonnegative real part.
With a direct calculation, one finds that the eigenvalues of A_n for

$$(u^*, v^*, w^*) = (1, 1, 0)$$

are solutions of the characteristic polynomial

$$(\beta + 1)[\beta^2 - \beta(-2\alpha_n - \mu - 1) + (\alpha_n + \mu)(\alpha_n + 1) - \chi \alpha_n] = 0.$$

Thus $\beta_1 = -1$ and the other eigenvalues β_2, β_3 verify

$$\beta_2 + \beta_3 = -2\alpha_n - \mu - 1 < 0$$

and

$$\begin{aligned} \beta_2 \beta_3 &= [(\alpha_n + \mu) + (\alpha_n + 1)]^2 - 4(\alpha_n + \mu)(\alpha_n + 1) + 4\chi \alpha_n = \\ &= (\alpha_n + \mu)^2 + (\alpha_n + 1)^2 - 2(\alpha_n + \mu)(\alpha_n + 1) + 4\chi \alpha_n = \\ &= (\alpha_n + \mu - \alpha_n - 1)^2 + 4\chi \alpha_n > 0, \end{aligned} \tag{XI.7}$$

so $(1,1,0)$ is locally asymptotically stable.
At $(u^*, v^*, w^*) = (0,0,w^*)$, with $w^* \geq 0$, we get

$$A_n = \begin{pmatrix} -\alpha_n - \mu w^* & 0 & 0 \\ 1 & -1-\alpha_n & 0 \\ 0 & -w^* & 0 \end{pmatrix}.$$

As $\beta = 0$ is an eigenvalue, we cannot conclude about the local stability of $(0,0,w^*)$ using its corresponding linealized system. In the next sections we will see numerically the behavior of the solution around this point.

Note that

$$w(x,t) = w_0(x) e^{-\int_0^t V(x,s)ds}.$$

Assuming $w_0 \geq 1$, since $w_t = -vw$, there exists $\delta > 0$ such that

$$w(x,t) \leq w_0(x) - \delta.$$

2 GFD scheme

Let us consider a bounded domain $\Omega \subset \mathbb{R}^2$. The first equation of (XI.1) reads as:

$$\begin{aligned} u_t = \Delta u - \chi \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \chi \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - \xi \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} - \xi \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} \\ - u(\chi \Delta v + \xi \Delta w) + \mu u(1-u-w). \end{aligned} \tag{XI.8}$$

Then, the GFD scheme for system (XI.1) is:

$$\begin{aligned} \frac{U_0^{n+1} - U_0^n}{\Delta t} &= -\lambda_{00} U_0^n + \sum_{i=1}^s \lambda_{i0} U_i^n + \mu U_0^n (1 - U_0^n - W_0^n) \\ &- \chi \left(-\lambda_{01} U_0^n + \sum_{i=1}^s \lambda_{i1} U_i^n \right) \left(-\lambda_{01} V_0^n + \sum_{i=1}^s \lambda_{i1} V_i^n \right) \\ &- \chi \left(-\lambda_{02} U_0^n + \sum_{i=1}^s \lambda_{i2} U_i^n \right) \left(-\lambda_{02} V_0^n + \sum_{i=1}^s \lambda_{i2} V_i^n \right) \\ &- \xi \left(-\lambda_{01} U_0^n + \sum_{i=1}^s \lambda_{i1} U_i^n \right) \left(-\lambda_{01} W_0^n + \sum_{i=1}^s \lambda_{i1} W_i^n \right) \\ &- \xi \left(-\lambda_{02} U_0^n + \sum_{i=1}^s \lambda_{i2} U_i^n \right) \left(-\lambda_{02} W_0^n + \sum_{i=1}^s \lambda_{i2} W_i^n \right) \\ &- U_0^n \Bigg[\chi \left(-\lambda_{00} V_0^n + \sum_{i=1}^s \lambda_{i0} V_i^n \right) + \\ &+ \xi \left(-\lambda_{00} W_0^n + \sum_{i=1}^s \lambda_{i0} W_i^n \right) \Bigg] + \mathcal{O}(\Delta t, h_i^2, k_i^2), \end{aligned} \tag{XI.9}$$

$$\frac{V_0^{n+1} - V_0^n}{\Delta t} = -\lambda_{00} V_0^n + \sum_{i=1}^s \lambda_{i0} V_i^n - V_0^n + U_0^n + \mathcal{O}(\Delta t, h_i^2, k_i^2), \tag{XI.10}$$

and

$$\frac{W_0^{n+1} - W_0^n}{\Delta t} = -V_0^n W_0^n + \mathcal{O}(\Delta t). \tag{XI.11}$$

Our main result with respect to the proposed numerical scheme is as stated below.

Theorem XI.1. *Let (u, v, w) be the exact solution to system (XI.1). Then, the GFD explicit scheme given by (XI.9), (XI.10) and (XI.11) is convergent if the following holds:*

$$\Delta t \leq \frac{2}{\sum_{i=1}^{s} |\lambda_{i0}| + \lambda_{00} + \alpha + \beta(\chi + \xi) + \xi\mu|u_0^n|}, \tag{XI.12}$$

where

$$\begin{aligned}\alpha = & |\lambda_{00}(\chi v_0^n + \xi w_0^n) - \sum_{i=1}^{s} \lambda_{i0}(\chi v_i^n + \xi w_i^n)| \\ & + \left(|\lambda_{01}| + \sum_{i=1}^{s} |\lambda_{i1}|\right)\left|\sum_{i=1}^{s} \lambda_{i1}(\chi v_i^n + \xi w_i^n) - \lambda_{01}(\chi v_i^n + \xi w_i^n)\right| \\ & + \left(|\lambda_{02}| + \sum_{i=1}^{s} |\lambda_{i2}|\right)\left|\sum_{i=1}^{s} \lambda_{i2}(\chi v_i^n + \xi w_i^n) - \lambda_{02}(\chi v_i^n + \xi w_i^n)\right| \\ & + \mu|1 - U_0^n - u_0^n - w_0^n|,\end{aligned}$$

and

$$\begin{aligned}\beta := & |U_0^n|\left(\lambda_{00} + \sum_{i=1}^{s} |\lambda_{i0}|\right) + \left(\lambda_{01} + \sum_{i=1}^{s} |\lambda_{i1}|\right)\left|\sum_{i=1}^{s} \lambda_{i1} u_i^n - \lambda_{01} u_0^n\right| \\ & + \left(\lambda_{02} + \sum_{i=1}^{s} |\lambda_{i2}|\right)\left|\sum_{i=1}^{s} \lambda_{i2} u_i^n - \lambda_{02} u_0^n\right|.\end{aligned}$$

Proof. Let us denote by U_i^n the approximate U-solution at time n and node i (respetively V_i^n, W_i^n) and u_i^n the exact u-solution at time n and node i (respetively v_i^n, w_i^n). First, some computations in (XI.9) lead us to

$$
\begin{aligned}
\frac{U^{n+1}-U_0^n}{\Delta t} &= \lambda_{00}U_0^n(-1+\chi V_0^n+\xi W_0^n)+\sum_{i=1}^{s}\lambda_{i0}U_i^n-\chi U_0^n\sum_{i=1}^{s}\lambda_{i0}V_i^n \\
&-\xi U_0^n\sum_{i=1}^{s}\lambda_{i0}W_i^n-\chi[(\lambda_{01})^2+(\lambda_{02})^2]W_0^nV_0^n \\
&-\xi[(\lambda_{01})^2+(\lambda_{02})^2]U_0^nW_0^n \\
&+\chi\lambda_{01}\left(U_0^n\sum_{i=1}^{s}\lambda_{i1}V_i^n+V_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n\right) \\
&+\chi\lambda_{02}\left(U_0^n\sum_{i=1}^{s}\lambda_{i2}V_i^n+V_0^n\sum_{i=1}^{s}\lambda_{i2}U_i^n\right) \\
&+\xi\lambda_{01}\left(U_0^n\sum_{i=1}^{s}\lambda_{i1}W_i^n+W_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n\right) \\
&+\xi\lambda_{02}\left(U_0^n\sum_{i=1}^{s}\lambda_{i2}W_i^n+W_0^n\sum_{i=1}^{s}\lambda_{i2}U_i^n\right) \\
&-\chi\left(\sum_{i=1}^{s}\lambda_{i1}U_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}V_i^n\right)-\chi\left(\sum_{i=1}^{s}\lambda_{i2}U_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i2}V_i^n\right) \\
&-\xi\left(\sum_{i=1}^{s}\lambda_{i1}U_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}W_i^n\right)-\xi\left(\sum_{i=1}^{s}\lambda_{i2}U_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i2}W_i^n\right) \\
&+\mu U_0^n(1-U_0^n-W_0^n)+\mathcal{O}(\Delta t,h_i^2,k_i^2).
\end{aligned}
\tag{XI.13}
$$

Notice that, since u, v and w are the exact solution of the system, they also solve the discrete equation (XI.13). We take the difference between (XI.13) and the same expression for the exact solution. We call $\tilde{u}_i^n = U_i^n - u_i^n$ (similarly for $\tilde{v}_i^n$ and $\tilde{w}_i^n$) and notice the following relations:

$$
\begin{aligned}
&\lambda_{00}U_0^n(-1+\chi V_0^n+\xi W_0^n)-\lambda_{00}u_0^n(-1+\chi v_0^n+\xi w_0^n)= \\
&=\lambda_{00}\tilde{u}_0^n(-1+\chi v_0^n+\xi w_0^n)+\lambda_{00}U_0^n(\chi\tilde{v}_0^n+\xi\tilde{w}_0^n).
\end{aligned}
\tag{XI.14}
$$

We also have

$$
-\chi U_0^n\sum_{i=1}^{s}\lambda_{i0}V_i^n+\chi u_0^n\sum_{i=1}^{s}\lambda_{i0}v_i^n=-\chi\tilde{u}_0^n\sum_{i=1}^{s}\lambda_{i0}V_i^n+\chi u_0^n\sum_{i=1}^{s}\lambda_{i0}\tilde{v}_i^n,
\tag{XI.15}
$$

as well as

$$
-U_0^nV_0^n+u_0^nv_0^n=-\tilde{u}_0^nV_0^n-u_0^n\tilde{v}_0^n.
\tag{XI.16}
$$

On the other hand we get

$$
\begin{aligned}
&U_0^n\sum_{i=1}^{s}\lambda_{i1}V_i^n+V_0^n\sum_{i=1}^{s}\lambda_{i1}U_i^n-u_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n-v_0^n\sum_{i=1}^{s}\lambda_{i1}u_i^n= \\
&=U_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n+V_0^n\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n+\tilde{u}_0^n\sum_{i=1}^{s}\lambda_{i1}v_i^n+\tilde{v}_0^n\sum_{i=1}^{s}\lambda_{i1}u_i^n,
\end{aligned}
\tag{XI.17}
$$

as well as

$$
\begin{aligned}
&\left(\sum_{i=1}^{s}\lambda_{i1}U_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}V_i^n\right)-\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}v_i^n\right)=\\
&=-\left(\sum_{i=1}^{s}\lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}V_i^n\right)-\left(\sum_{i=1}^{s}\lambda_{i1}u_i^n\right)\left(\sum_{i=1}^{s}\lambda_{i1}\tilde{v}_i^n\right).
\end{aligned}
\tag{XI.18}
$$

Finally, it yields

$$
\begin{aligned}
&\mu U_0^n(1-U_0^n-W_0^n)-\mu u_0^n(1-u_0^n-w_0^n)=\\
&=\mu\tilde{u}_0^n(1-(U_0^n+u_0^n)-W_0^n)-\mu u_0^n\tilde{w}_0^n.
\end{aligned}
\tag{XI.19}
$$

We arrive then to the identity:

$$
\begin{aligned}
&\frac{\tilde{u}_0^{n+1}-\tilde{u}_0^n}{\Delta t} = \tilde{u}_0^n \Bigg[\lambda_{00}(-1+\chi v_0^n+\xi w_0^n) - \chi\sum_{i=1}^s \lambda_{i0}V_i^n - \xi\sum_{i=1}^s \lambda_{i0}W_i^n \\
&- \chi V_0^n[(\lambda_{01})^2+(\lambda_{02})^2] - \xi W_0^n[(\lambda_{01})^2+(\lambda_{02})^2] \\
&+ \chi\lambda_{01}\sum_{i=1}^s \lambda_{i1}v_i^n + \chi\lambda_{02}\sum_{i=1}^s \lambda_{i2}v_i^n + \xi\lambda_{01}\sum_{i=1}^s \lambda_{i1}w_i^n \\
&+ \xi\lambda_{02}\sum_{i=1}^s \lambda_{i2}w_i^n + \mu\tilde{u}_0^n(1-(U_0^n+u_0^n)-W_0^n)\Bigg] \\
&+ \tilde{v}_0^n\Bigg[\chi\lambda_{00}U_0^n - \chi u_0^n[(\lambda_{01})^2+(\lambda_{02})^2] + \chi\lambda_{01}\sum_{i=1}^s \lambda_{i1}u_i^n \\
&+ \chi\lambda_{02}\sum_{i=1}^s \lambda_{i2}u_i^n\Bigg] + \tilde{w}_0^n\Bigg[\xi\lambda_{00}u_0^n - \xi u_0^n[(\lambda_{01})^2+(\lambda_{02})^2] \\
&+ \xi\lambda_{01}\sum_{i=1}^s \lambda_{i1}u_i^n + \xi\lambda_{02}\sum_{i=1}^s \lambda_{i2}u_i^n - \mu u_0^n\Bigg] \\
&+ \sum_{i=1}^s \lambda_{i0}\tilde{u}_i^n - \chi u_0^n\sum_{i=1}^s \lambda_{i0}\tilde{v}_i^n - \xi u_0^n\sum_{i=1}^s \lambda_{i0}\tilde{w}_i^n \\
&+ \chi\lambda_{01}U_0^n\sum_{i=1}^s \lambda_{i1}\tilde{v}_i^n + \chi\lambda_{01}V_0^n\sum_{i=1}^s \lambda_{i1}\tilde{u}_i^n + \chi\lambda_{02}U_0^n\sum_{i=1}^s \lambda_{i2}\tilde{v}_i^n + \\
&+ \chi\lambda_{02}V_0^n\sum_{i=1}^s \lambda_{i2}\tilde{u}_i^n + \xi\lambda_{01}U_0^n\sum_{i=1}^s \lambda_{i1}\tilde{w}_i^n + \xi\lambda_{01}W_0^n\sum_{i=1}^s \lambda_{i1}\tilde{u}_i^n \\
&+ \xi\lambda_{02}U_0^n\sum_{i=1}^s \lambda_{i2}\tilde{w}_i^n + \xi\lambda_{02}W_0^n\sum_{i=1}^s \lambda_{i2}\tilde{u}_i^n \\
&- \chi\left(\sum_{i=1}^s \lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^s \lambda_{i1}V_i^n\right) - \chi\left(\sum_{i=1}^s \lambda_{i1}u_i^n\right)\left(\sum_{i=1}^s \lambda_{i1}\tilde{v}_i^n\right) \\
&- \chi\left(\sum_{i=1}^s \lambda_{i2}\tilde{u}_i^n\right)\left(\sum_{i=1}^s \lambda_{i2}V_i^n\right) - \chi\left(\sum_{i=1}^s \lambda_{i2}u_i^n\right)\left(\sum_{i=1}^s \lambda_{i2}\tilde{v}_i^n\right) \\
&- \xi\left(\sum_{i=1}^s \lambda_{i1}\tilde{u}_i^n\right)\left(\sum_{i=1}^s \lambda_{i1}W_i^n\right) - \xi\left(\sum_{i=1}^s \lambda_{i1}u_i^n\right)\left(\sum_{i=1}^s \lambda_{i1}\tilde{w}_i^n\right) \\
&- \xi\left(\sum_{i=1}^s \lambda_{i2}\tilde{u}_i^n\right)\left(\sum_{i=1}^s \lambda_{i2}W_i^n\right) - \xi\left(\sum_{i=1}^s \lambda_{i2}u_i^n\right)\left(\sum_{i=1}^s \lambda_{i2}\tilde{w}_i^n\right) + \mathcal{O}(\Delta t, h_i^2, k_i^2).
\end{aligned}
\tag{XI.20}
$$

Let us call $\tilde{u} = \max_i\{|\tilde{u}_i^n|\}$ (similarly for $\tilde{v}$ and $\tilde{w}$). Then, we write (XI.20) as

$$\tilde{u}^{n+1} \leq A_1\tilde{u}^n + B_1\tilde{v}^n + C_1\tilde{w}^n, \tag{XI.21}$$

where

$$\begin{aligned}
A_1 = & |1-\lambda_{00}\Delta t| + \Delta t\sum_{i=1}^{s}|\lambda_{i0}| + \Delta t|\lambda_{00}(\chi v_0^n+\xi w_0^n) - \sum_{i=1}^{s}\lambda_{i0}(\chi v_i^n+\xi w_i^n)| \\
& + \Delta t\left(|\lambda_{01}| + \sum_{i=1}^{s}|\lambda_{i1}|\right)\left|\sum_{i=1}^{s}\lambda_{i1}(\chi v_i^n+\xi w_i^n) - \lambda_{01}(\chi v_i^n+\xi w_i^n)\right| \\
& + \Delta t\left(|\lambda_{02}| + \sum_{i=1}^{s}|\lambda_{i2}|\right)\left|\sum_{i=1}^{s}\lambda_{i2}(\chi v_i^n+\xi w_i^n) - \lambda_{02}(\chi v_i^n+\xi w_i^n)\right| \\
& + \Delta t\mu|1-U_0^n-u_0^n-w_0^n|, \\
B_1 = & \Delta t\Bigg[\chi|U_0^n|\left(\lambda_{00}+\sum_{i=1}^{s}|\lambda_{i0}|\right) + \chi\left(\lambda_{01}+\sum_{i=1}^{s}|\lambda_{i1}|\right)\left|\sum_{i=1}^{s}\lambda_{i1}u_i^n - \lambda_{01}u_0^n\right| \\
& + \chi\left(\lambda_{02}+\sum_{i=1}^{s}|\lambda_{i2}|\right)\left|\sum_{i=1}^{s}\lambda_{i2}u_i^n - \lambda_{02}u_0^n\right|\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
C_1 = & \Delta t\Bigg[\xi|U_0^n|\left(\lambda_{00}+\sum_{i=1}^{s}|\lambda_{i0}|\right) + \xi\left(\lambda_{01}+\sum_{i=1}^{s}|\lambda_{i1}|\right)\left|\sum_{i=1}^{s}\lambda_{i1}u_i^n - \lambda_{0i}u_0^n\right| \\
& + \xi\left(\lambda_{02}+\sum_{i=1}^{s}|\lambda_{i2}|\right)\left|\sum_{i=1}^{s}\lambda_{i2}u_i^n - \lambda_{02}u_0^n\right| + \mu|u_0^n|\Bigg].
\end{aligned}$$

Also, for the $v, w-$equations, we obtain

$$\begin{aligned}
\tilde{v}^{n+1} & \leq \left(|1-(\lambda_{00}+1)\Delta t| + \Delta t\sum_{i=1}^{s}|\lambda_{i0}|\right)\tilde{v}^n + \Delta t\tilde{u}^n \\
& := \Delta t\tilde{u}^n + B_2\tilde{v}^n
\end{aligned} \tag{XI.22}$$

and

$$\tilde{w}^{n+1} \leq |1-\Delta tV_0^n|\tilde{w}^n + \Delta t w_0^n\tilde{v}^n := \Delta t w_0^n\tilde{v}^n + C_2\tilde{w}^n \tag{XI.23}$$

Let us rewrite (XI.21), (XI.22) and (XI.23) as

$$\begin{pmatrix} \tilde{u}^{n+1} \\ \tilde{v}^{n+1} \\ \tilde{w}^{n+1} \end{pmatrix} \leq \begin{pmatrix} A_1 & B_1 & C_1 \\ \Delta t & B_2 & 0 \\ 0 & \Delta t W_0^n & C_2 \end{pmatrix} \begin{pmatrix} \tilde{u}^n \\ \tilde{v}^n \\ \tilde{w}^n \end{pmatrix}. \tag{XI.24}$$

we consider the matrix

$$\mathfrak{A} = \begin{pmatrix} A_1 & B_1 & C_1 \\ \Delta t & B_2 & 0 \\ 0 & \Delta t w_0^n & C_2 \end{pmatrix}, \tag{XI.25}$$

and we use the matrix norm $N_1(\mathfrak{A}) = \max_{i=1,2,3}\{\sum_{j=1}^{3}|a_{ij}|\}$. From their definition it is clear that $N_1(\mathfrak{A}) = A_1 + B_1 + C_1$. Notice that by assumption

$$\Delta t \leq \frac{2}{\sum_{i=1}^{s}|\lambda_{i0}| + \lambda_{00} + \alpha + \beta(\chi+\xi) + \xi\mu|u_0^n|}.$$

Applying Lemma VI.1 we have that $\lim_{k\to\infty}\mathfrak{A}^k = \mathbf{0}$. Now, by Lemma VI.2, this is equivalent to $\rho(\mathfrak{A}) < 1$, that is, the greatest absolute value of all eigenvalues of matrix $\mathfrak{A}$ is less than 1, which implies the convergence of the explicit scheme under the condition (XI.12). □

T(s)	0.5	1	3	6	10
$\|U\|_{l^\infty(\Omega)}$	0.3657	0.4672	0.8004	0.9763	0.9994
$\|V\|_{l^\infty(\Omega)}$	0.1139	0.2013	0.5523	0.9039	0.9951
$\|W\|_{l^\infty(\Omega)}$	0.9686	0.8952	0.4207	0.0435	0.0009

Table XI.1: Values of $\|U\|_{l^\infty(\Omega)}$, $\|V\|_{l^\infty(\Omega)}$ and $\|W\|_{l^\infty(\Omega)}$ for different time values in Example 1.

3 Numerical examples

In this section we illustrate the application of the GFDM for solving the chemotaxis-haptotaxis system given by (XI.1). We test the method using the regular and irregular clouds of points of Figure XI.1, both of them containing 437 nodes. We choose $\Delta t = 0.001$, fulfilling the assumption made in Theorem XI.1. We present examples of the asymptotic convergence of the solution to the constant steady-state $(u^*, v^*, w^*) = (1, 1, 0)$, and under some assumptions on the initial data to the other steady state $(u^*, v^*, w^*) = (0, 0, w^*)$. For the all examples we compute the numerical error according to (VI.3).

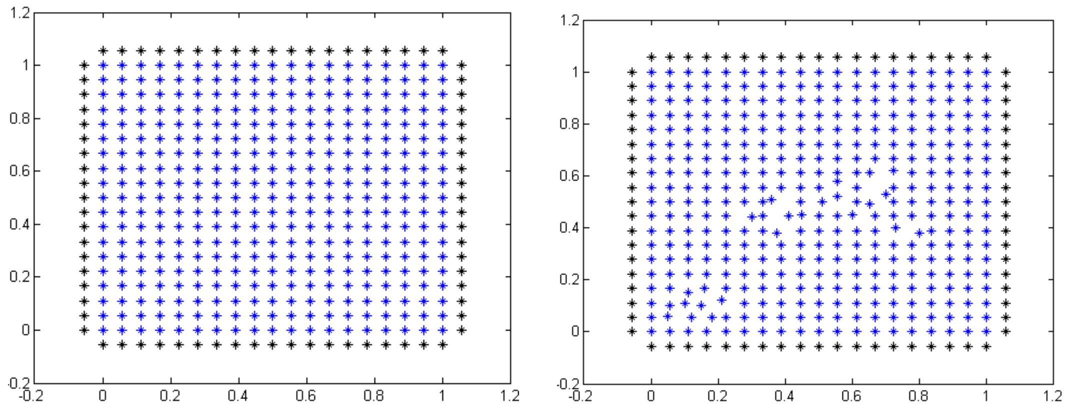

Figure XI.1: Regular and irregular clouds of points

Example 1

We consider for this first case the regular clouds of points of figure XI.1 and the following initial data:

$$u_0(x,y) = e^{-10[(x-0.2)^2+(y-0.2)^2]}, \quad v_0(x,y) = 0.7e^{-10[(x-1.2)^2+(y-1)^2]},$$

$$w_0(x,y) = x^2.$$

As parameters, we choose $\chi = \xi = 0.5$ and $\mu = 1.5$. Table XI.1 shows the maximum value of the discrete solution at times $t = 0.5, 1, 3, 6$ and 10. Figure XI.2 plots the solution at 0.5 and 10 seconds. In accordance with the theory, the numerical solution reaches the constant steady state $(1, 1, 0)$.

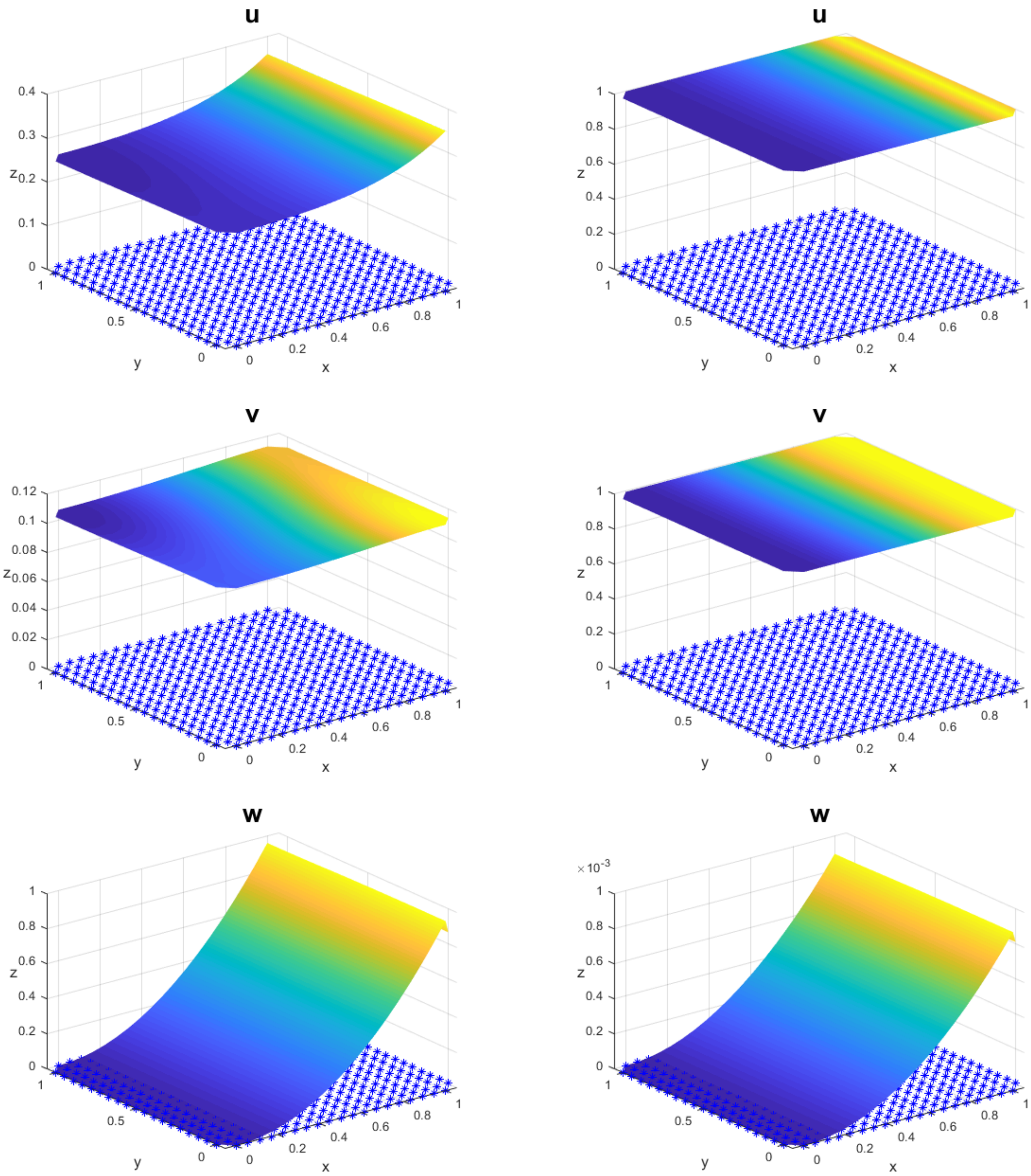

Figure XI.2: U, V, W-solution for 0.5 and 10 seconds in the Example 1.

T(s)	0.5	1	3	6	10	15
$\|U\|_{l^\infty(\Omega)}$	0.1641	0.1759	0.3039	0.7189	0.9845	0.9999
$\|V\|_{l^\infty(\Omega)}$	0.1859	0.1759	0.2304	0.5667	0.9479	0.9990
$\|W\|_{l^\infty(\Omega)}$	0.9129	0.8345	0.5691	0.1838	0.0076	0.0001

Table XI.2: Values of $\|U\|_{l^\infty(\Omega)}$, $\|V\|_{l^\infty(\Omega)}$ and $\|W\|_{l^\infty(\Omega)}$ for different time values in Example 2.

Example 2

Now, let us consider the irregular cloud of points of Figure XI.1. As initial data we put

$$u_0(x,y) = 2e^{-10[x^2+y^2]}, \quad v_0(x,y) = 0.7e^{-10[(x-0.5)^2+(y-0.5)^2]},$$

$$w_0(x,y) = 1 - \frac{1}{2}e^{-10[(x-0.5)^2+(y-0.5)^2]}$$

and the same parameters as in Example 1. Again, since the initial data fulfil all assumptions we obtain convergence to the steady state $(1,1,0)$. Table XI.2 shows the values of the maximum value of the numerical solution at 0.5, 1, 3, 6, 10 and 15 seconds and Figure XI.3 displays the solution at times 0.5 and 15 over the irregular cloud of points.

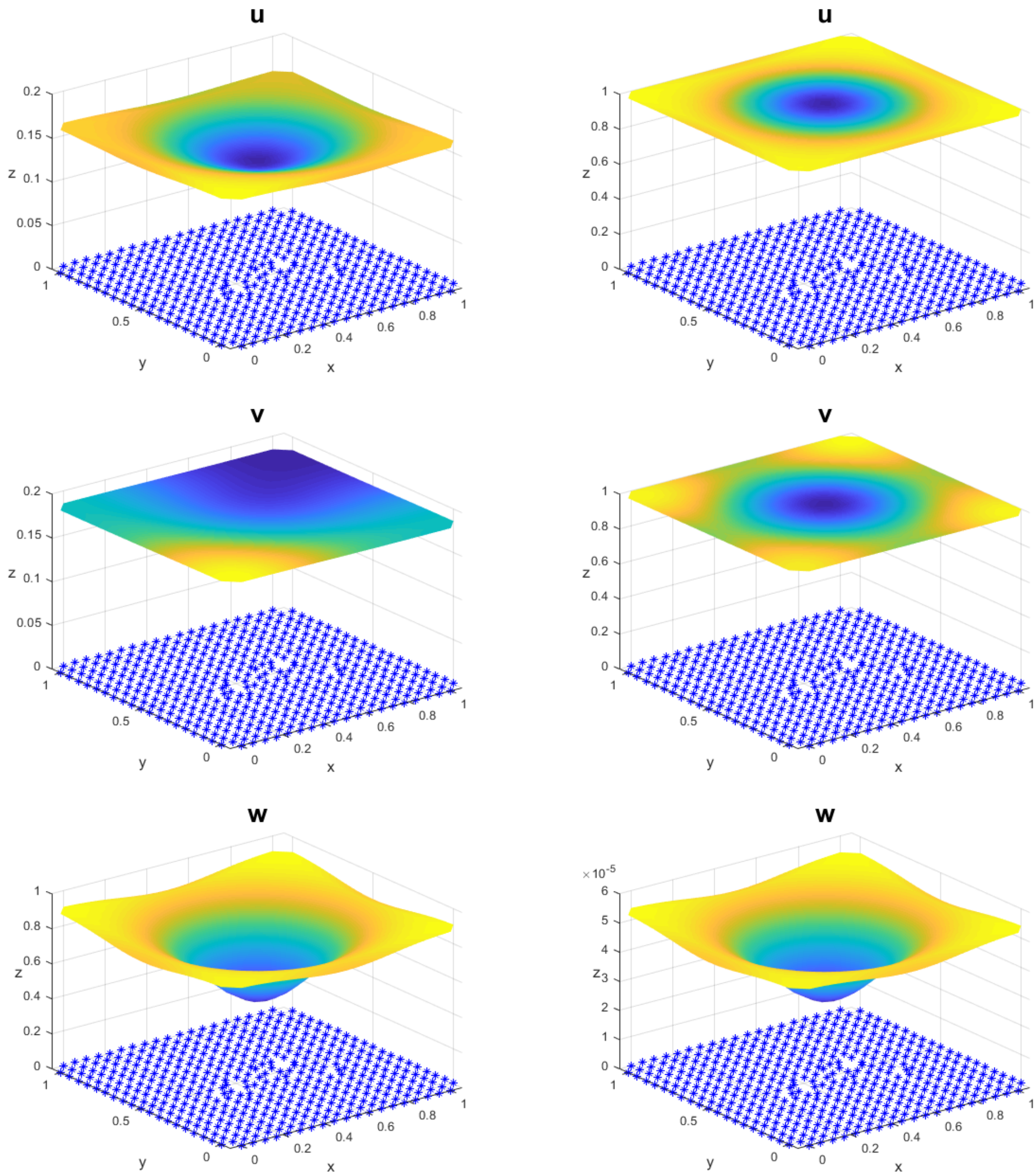

Figure XI.3: U, V, W-solution for 0.5 and 15 seconds in the Example 2.

T(s)	0.5	1	3	6	10
$\|U\|_{l^\infty(\Omega)}$	0.1897	0.0781	0.0033	0.0000	0.0000
$\|V\|_{l^\infty(\Omega)}$	0.0942	0.0731	0.0141	0.0008	0.0000
$\|W\|_{l^\infty(\Omega)}$	4.8559	4.6561	4.3207	4.2599	4.2565

Table XI.3: Values of $\|U\|_{l^\infty(\Omega)}$, $\|V\|_{l^\infty(\Omega)}$ and $\|W\|_{l^\infty(\Omega)}$ for different time values in Example 3.

Example 3

Let us now consider the same parameters as in Example 1 and the following initial data:

$$u_0(x,y) = 2e^{-10[x^2+y^2]}, \quad v_0(x,y) = e^{-10[x^2+y^2]},$$

$$w_0(x,y) = 1 + 4x^2.$$

Clearly, the condition $w_0(x,y) \leq 1$ does not hold and therefore we cannot expect convergence to $(1,1,0)$. Table XI.3 shows the l^∞ norm of the discrete solution at different times. Figure XI.3 illustrates the numerical solution at different times. As stated, for w_0 large enough we find convergence towards the non-constant steady state $(0,0,w_0(\mathbf{x})-\delta)$, for some $\delta > 0$ depending on w_0, v_0, χ, ξ and μ.

Example 4: finite-time blow up

In this example we look for unbounded solutions. As stated in the introduction of this chapter, it was proved in [45] that blow-up occurs in absence of logistic term whenever

$$\int_\Omega u_0(\mathbf{x})d\mathbf{x} > \frac{8\pi}{\chi}.$$

Therefore we take as parameters $\chi = 4, \xi = 5, \mu = 0$ and initial data

$$u_0(x,y) = 3\pi, \quad v_0(x,y) = \frac{1}{2}e^{-10[x^2+y^2]},$$

$$w_0(x,y) = 1 - \frac{1}{2}e^{-10[(x-0.5)^2+(y-0.5)^2]}.$$

Table XI.4 shows the l^∞ norm of the numerical solution at times 0.01, 0.05, 0.075 and 0.1 seconds. Figure XI.5 exhibits the discrete solution at 0.01, 0.05 and 0.1 seconds. The three components of the solution become unbounded for any time greater than 0.1 seconds, in accordance with the theory.

4 Conclusions

We have studied the local stability of the constant steady states and obtained the discretization of system (XI.1) using a Generalized Finite Difference Method explicit scheme for the two dimensional case. Conditional convergence has been obtained for the numerical scheme. We have presented several numerical example over regular and irregular domains. Applicability of the method does not depend on the distribution of nodes, as it is clear

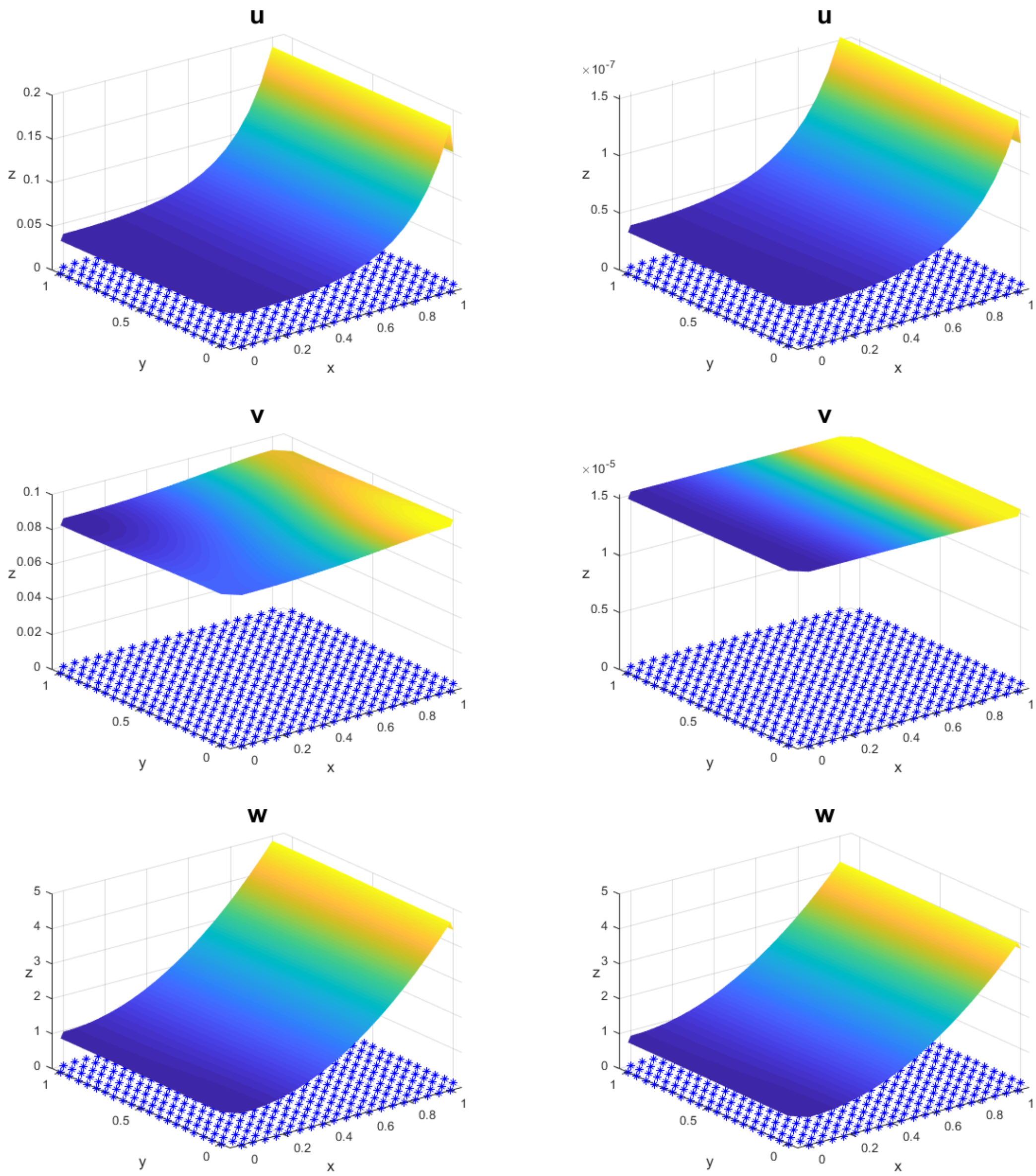

Figure XI.4: U, V, W-solution for 0.5 and 10 seconds in the Example 3.

T(s)	0.01	0.05	0.075	0.1
$\|U\|_{l^\infty(\Omega)}$	16.2053	38.0371	82.7087	8.8751e+19
$\|V\|_{l^\infty(\Omega)}$	0.4742	0.9767	1.5146	1.7496e+10
$\|W\|_{l^\infty(\Omega)}$	0.9962	0.9830	0.9647	2.4761e+03

Table XI.4: Values of $\|U\|_{l^\infty(\Omega)}$, $\|V\|_{l^\infty(\Omega)}$ and $\|W\|_{l^\infty(\Omega)}$ for 0.5 seconds in Example 4.

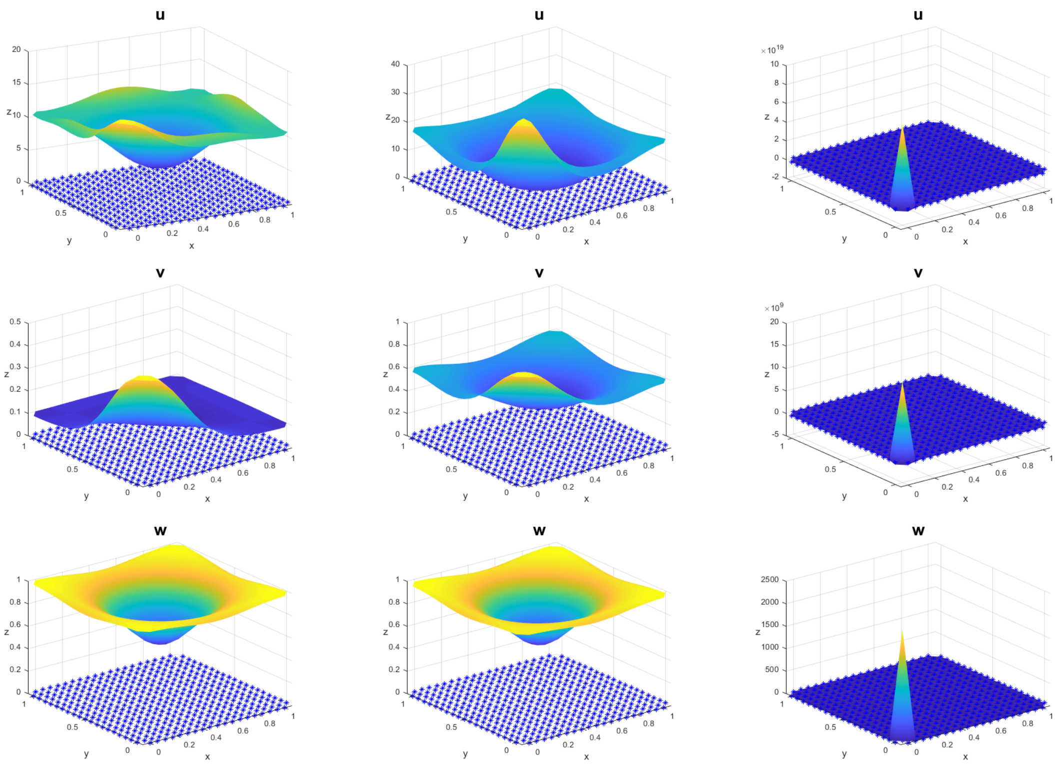

Figure XI.5: U, V, W-solution for 0.01, 0.05 and 0.1 seconds in the Example 4.

from the examples. All the numerical examples, even the cases in which blow-up occurs, are in accordance with the theoretical results concerning asymptotic behavior of the solutions of this highly non-linear chemotaxis-haptotaxis system. The results of this chapter have been published in [9].

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