Superfluid–Mott insulator transition and Bose–Einstein Condensation of phonons in ion traps

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Abstract. We show that transverse phonons in a set of trapped ions under the action of lasers are described by a Bose–Hubbard model whose parameters can be externally adjusted. The hopping of the phonons between different phonons is provided by the Coulomb interaction. On the other hand, the nonlinear terms in the interaction of the ions with a standing–wave provide us with the phonon–phonon interaction. We investigate the possibility of observing several quantum many–body phenomena, including (quasi) Bose–Einstein Condensation as well as a superfluid–Mott insulator quantum phase transition.

Keywords: spin systems, trapped ions, quantum theory
PACS: 03.67.Pp, 42.50.vk, 75.10.Jm

INTRODUCTION

Ultracold bosons present a variety of fascinating phenomena, including Bose–Einstein Condensation (BEC) [1], and the superfluid–Mott insulator (SI) quantum phase transition in the Bose–Hubbard model (BHM) [2]. At present, atomic gases at ultra low temperatures constitute a unique system, where these effects can be observed [3, 4, 5]. In this work we show that phonons in trapped ions provide us with a system of ultracold bosons where the physical parameters describing the hopping of phonons between ions, and the phonon–phonon interaction can be adjusted by means of the choice of trapping conditions, and the use of lasers, respectively. It would be possible, for example, to change adiabatically the phonon-phonon interaction in such a way that the superfluid–Mott insulator quantum phase transition could be observed. Furthermore, the theoretical and experimental progress experienced by the field of trapped ion quantum information [6, 7, 8, 9, 10] can be applied here to study novel possibilities that, so far, have not been possible with atoms in optical lattices, for example by means of the individual measure and manipulation of ions.

Phonons in trapped ions are associated with the motion of ions in the trap. The motion of single ions can be described by a quantum harmonic oscillator whose energy is the trapping frequency. In the absence of any other effect, phonons would, thus, behave like bosons that are localized at each ion. However, the Coulomb interaction can couple the motion of distant ions, and can provide a mechanism for hopping of phonons between ions. Furthermore, if there are non–linearities in the potential that is experienced by the ions, these would correspond to phonon–phonon interactions. The situation is analogous

CP770, *Atomic Physics 19,* edited by L. G. Marcassa, K. Helmerson, and V. S. Bagnato

to an optical lattice, with ions playing the role of lattice sites. If the trapping frequency is much larger than any other energy scale, then phonon non–conserving processes are strongly suppressed. Phonon number conservation in our setup is in contrast with the case of usual solid-state systems, where the fact that phonons are not conserved prevent the system to reach the regime of BEC and superfluidity.

Let us consider a set of N trapped ions confined by external electric potentials, and which move around their equilibrium positions. The motion of the ions is described by the Hamiltonian $H = K + V_0 + V_{\mathrm{Coul}}$, where K is the kinetic energy, V_0 the trapping potential, and V_{Coul} the Coulomb interaction between ions. We will assume that the motion of the ions along one particular direction, say $\mathbf{x}$, is decoupled from the motion along the other directions. Under the condition that the displacements of the ions are small enough, one can approximate V_0, V_{Coul} by terms that are quadratic in the displacements. The trapping term in the $\mathbf{x}$ direction is given by $12m\omega^2 x_i^2$, where ω is the trapping frequency. On the other hand the second order terms from the Coulomb energy are of the form $(e^2/d_{i,j}^3)x_i x_j$, where $d_{i,j}$ is the distance between ions. We can define the ratio between the confinement potential, and the Coulomb energy, $\beta := e^2/(d_0^3 m\omega^2) \ll 1$. Under condition $\beta \ll 1$, a reasonable approximate description of our system is to consider that phonons are localized at each ion, that is $H = K + V_0 \approx \hbar\omega \sum_i a_i^\dagger a_i$, with $a_i^\dagger$ the creation operator of one phonon located at ion i. On the other hand, the Coulomb energy includes hopping terms between different ions, of the form $(e^2/d_{i,j}^3)(a_i^\dagger + a_i)(a_j^\dagger + a_j)$. In principle, V_{Coul} includes terms $a_i a_j$ that do not conserve the total number of phonons. However, under condition $\beta \ll 1$, these processes can be neglected in a rotating wave approximation, and the Coulomb interaction can be approximated by $(e^2/d_{i,j}^3)(a_i^\dagger a_j + h.c.)$. That is, if ω is large enough, the energetic cost to destroy or create a phonon is too large for the Coulomb energy.

Nonlinearities in the motion of trapped ions can be induced by placing the ions near the maximum of a standing wave, something which will induce an AC–Stark shift proportional to $\cos(kx_i)^2 \approx 1 - (kx_i)^2 + (1/3)(kx_i)^4$, where k is the wave-vector of the laser. Note that $(kx_i)^4$ includes a repulsive Hubbard interaction $a_i^{\dagger 2} a_i^2$. We that terms from the standing–wave that do not conserve phonon number can be neglected by the very same reasons exposed above for the case of the Coulomb interaction. By placing ions at the mininum, one obtains an attractive phonon–phonon interaction.

INTERACTING RADIAL PHONONS IN A COULOMB CHAIN

There are several physical set–ups which realize a BHM as explained above. In the following we will concentrate in the simplest one, which consists of ions in a linear trap and which gives rise to a 1D system. Let us emphasize, however, that with ions in microtraps [11] or in Penning traps [12] one can realize higher dimensional situations.

In a linear trap, ions are arranged in a Coulomb chain. Phonons moving along the chain cannot be used in the way we described above since for them $\beta 1$ [15, 16]. However, transverse phonons corresponding to the radial modes fulfill $\beta \ll 1$ and thus are perfectly suited for our purposes. The radial phonon dispersion relation has a large gap of the order of ω, giving rise to the phonon–number conservation, and has a bandwidth of the order of $\beta\omega$. Therefore, we take **x** as one of the transverse directions and **z** the trap axis. The Hamiltonian that describes the motion in a chain with N ions is given by:

$$
\begin{aligned}
V_0 &= \frac{1}{2}m\sum_{i=1}^{N}\left(\omega_x^2 x_i^2+\omega_y^2 y_i^2+\omega_z^2 z_i^2\right) \\
V_{\mathrm{Coul}} &= \sum_{i>j}^{N}\frac{e^2}{\sqrt{(z_i-z_j)^2+(x_i-x_j)^2+(y_i-y_j)^2}}.
\end{aligned} \tag{1}
$$

where ω_α, $\alpha = x,y,z$, are the trapping frequencies in each direction, and we define β_α as the corresponding ratios between Coulomb and trapping energy. If $\omega_{x,y} \gg \omega_z$ the ions form a chain along the **z** axis and occupy equilibrium positions z_i^0. Phonons in the **x** direction can be described approximately by:

$$
\begin{aligned}
H_{x0} &= \sum_{i=1}^{N}(\omega_x+\omega_{x,i})a_i^\dagger a_i+\sum_{j>i}^{N}t_{i,j}(a_i^\dagger a_j+a_i a_j^\dagger) \\
\omega_{x,i} &= -\frac{1}{2}\sum_{j'\neq i}^{N}\frac{e^2/(m\omega_x^2)}{|z_i^0-z_{j'}^0|^3}\ \omega_x,\ \ t_{i,j}=\frac{1}{2}\frac{e^2/(m\omega_x^2)}{|z_i^0-z_{j'}^0|^3}\ \omega_x.
\end{aligned} \tag{2}
$$

$\omega_{x,i}$, $t_{i,j}$ are spatial dependent shifts of the trapping frequency, and effective hopping energies, respectively. Note that both of them are of the order of $\beta_x\omega_x$. The approximations that lead to (3) are: (i) In V_{Coul}, we keep only second order terms in the displacements of the ions around the equilibrium positions. Higher order terms are of the form x^4, x^2y^2, zx^2 [13]. They can be neglected assuming that $x_0/d_0, z_0/d_0 \ll 1$, with x_0, z_0 the size of an individual ion's wave-packet. x_0 can be estimated by the size of the ground state in the radial trapping frequency and for typical parameters (d_0 = 5 μm, ω_x = 10 MHz, also used below) we have $x_0/d_0 \approx 10^{-3}$. In the case of z_0, one has to consider the collective nature of the axial modes, because $\beta_z \gg 1$ if $N \gg 1$. If we consider axial modes at a finite temperature, z_0 is given by the thermal fluctuations of the position of the ion. In the limit $N \gg 1$, we can estimate $z_0^2 \approx \frac{\hbar}{2m\omega_z\sqrt{\beta_z\log\beta_z}}\frac{k_BT}{\hbar\omega_z}$, which means that $z_0/d_0 \approx 10^{-3}k_BT/\hbar\omega_z$, with ω_z = 100 kHz, $N = 100$ (see [14]). (ii) We consider $\beta_x \ll 1$, and neglect the phonon number non–conserving terms in the couplings of the form $x_ix_j \propto (a_i^\dagger+a_i)(a_j^\dagger+a_j)$.

We include the effect of a standing wave in H_x, such that a repulsive phonon–phonon interaction is induced [7]:

$$
H_{sw} = F\sum_{i=1}^{N}|0\rangle_i\langle 0|\cos^2(kx_i). \tag{3}
$$

$|0\rangle_i$ is the internal ground state of the ions. In the following we will assume that ions stay always in $|0\rangle_i$, and expand the standing-wave in the Lamb-Dicke parameter,

$\eta = kx_0/\hbar\omega_x$:

$$H_{sw} = F\sum_{i=1}^{N}(1+\eta^2(a_i+a_i^\dagger)^2+\frac{1}{3}\eta^4(a_i+a_i^\dagger)^4+\mathscr{O}(\eta^6)). \tag{4}$$

The fourth order contribution contains a Hubbard interaction, $U{a_i^\dagger}^2 {a_i}^2$, with $U = 2F\eta^4$ (note that $U < 0$ if the ions are placed at the minimum). The other terms in Eq. (4) are: (i) phonon conserving terms that just give corrections to the trapping frequency; (ii) phonon non–conserving terms, that rotate with frequency ω_x. The non–conserving contributions can be adiabatically eliminated if $F\eta^2/\omega_x \ll 1$. For example, in the case of the second order terms, $F\eta^2(a_i^2 + {a_i^\dagger}^2)$, a perturbative calculation shows that they only give harmonic corrections of the form $((F\eta^2)^2/\omega_x)(-2a_i^\dagger a_i - 1) + \mathscr{O}(F\eta^2)^3/\omega_x^2$. Thus the contributions from non–conserving term, either give corrections to the trapping frequency, or can be neglected when compared to U.

The final Hamiltonian takes the form of a BHM:

$$H_x = H_{x0} + \sum_{i=1}^{N} U{a_i^\dagger}^2 a_i^2, \tag{5}$$

where we include in H_{x0} the corrections from the standing wave. Note that as long as the number of phonons is conserved, ω_x in H_{x0} is a global chemical potential that does not play any role in the description of the system.

We discuss now the properties of the solutions of the non-interacting Hamiltonian, H_{x0}. A quite unexpected result is that the Coulomb interaction induces the confinement of the radial phonons. This is due to the fact that the distance between ions is larger at the sides than at the center of the chain, and is well described by a quadratic dependence on the position of the ions. Thus, the corrections $\omega_{x,i}$ in Eq. (3) are smaller for the ions placed at the center of the chain, in such a way that the radial phonon field is confined (Fig. 1). The harmonic phonon confinement can be estimated by means of Eq. (3) in the limit $N \gg 1$. In this case, the distance between ions at site i satisfies [15]:

$$\frac{1}{(d(i')/d_0)^3} \approx \alpha - \gamma\left(\frac{i'}{N}\right)^2, \;\; \alpha = 3.4, \; \gamma = 18, \tag{6}$$

where $i' = i - N/2$. One can use Eq. (6) to describe qualitatively the dependence of $\omega_{x,i}$ with the position. We include only Coulomb interaction between nearest–neighbors in order to get analytical results. The spatial dependent part of the non–interacting boson Hamiltonian is given, in this approximation, by:

$$\begin{aligned} &H_{x0}/(\beta_x\omega_x) = \\ &\sum_{i=1}^{N}\frac{1}{2}\gamma(\frac{i'}{N})^2 a_i^\dagger a_i + \frac{1}{2}\left(\alpha - \gamma(\frac{i'}{N})^2\right)(a_i^\dagger a_{i+1} + h.c.). \end{aligned} \tag{7}$$

In the limit of many ions and low energies, the continuum limit in this expression describes a one dimensional system of bosons trapped by the frequency $\omega_c \approx (8/N)\beta_x\omega_x$.

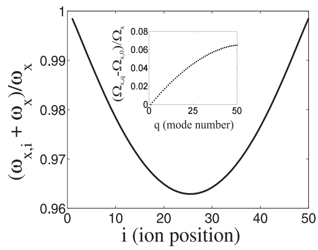

FIGURE 1. Phonon trapping potential (3) for the radial modes of a Coulomb chain with $N = 50$ ions, and $\beta_x = 10^{-2}$, as a function of the ion position along the chain. Inset: Spectrum of the radial collective modes, $\Omega_{x,q}$, that diagonalize H_{x0}.

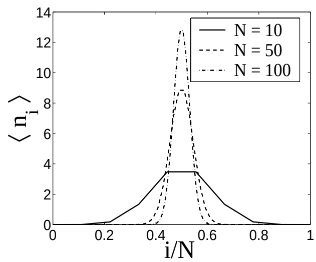

FIGURE 2. Mean phonon number $\langle n_i \rangle = \langle a_i^\dagger a_i \rangle$ along Coulomb chains with different number of ions, N, in the state with $N_{ph} = N$ phonons in the radial lowest mode. The width of the wavefunction in units of N, is $\propto 1/\sqrt{N}$, in accord with the scaling for the phonon trapping frequency, $\omega_c \propto 1/N$.

The lowest collective modes in the exact spectrum show a linear dispersion that is well described by our estimation for ω_c (see inset of Fig. 1). Thus, we get the conclusion that radial phonons in ions in linear Paul traps are naturally confined by an approximate harmonic potential (see Fig. 2).

SUPERFLUID-MOTT INSULATOR TRANSITION AND BEC IN COULOMB CHAINS

Our ideas lead to the following two proposals of experiments with linear Paul traps:

(i) Superfluid-Mott insulator transition and creation of a superfluid phonon state by adiabatic evolution. Hamiltonian (5) describes a BHM with the peculiarity that hopping terms are positive, with a range that is longer than the usual nearest-neighbor hopping in optical lattices. However we can understand the properties of our system by means of the better known model with nearest-neighbor hopping only [2, 17]. Let us consider $t = (1/2)\beta_x \hbar\omega_x$, the characteristic hopping energy. If the total number of phonons N_{ph} is commensurate with the number of ions N, then, for values $U \gg t$, the ground state of Hamiltonian (5) is a Mott insulator, well described by a product of Fock states of N_{ph}/N in each ion (note that phonon confinement, $\hbar\omega_c$ is also of order t, so that condition $U \gg t$

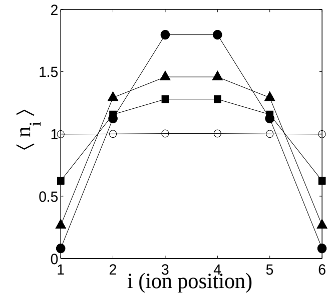

FIGURE 3. Mean phonon number at each ion in the ground state of a Coulomb chain with $N = 6$ ions, and $N_{ph} = 6$ phonons. $\beta_x = 0.01$, so that the nearest neighbor hopping terms are $t \approx 5\ 10^{-3}\omega_x$. Black circles represent the phonon density without standing wave ($U = 0$), and show the confinement due to the phonon trapping potential. Empty circles: Mott phase when a standing wave is applied with $F\eta^2 = 0.1\omega_x$, $\eta^2 = 0.1$, and $U \approx 0.02\omega_x > t$. Squares: $U = 0.01\omega_x$, Triangles: $U = 5\ 10^{-3}\omega_x$.

ensures a uniform phonon density). On the other hand, the ground state for $U \ll t$ is a superfluid with all the phonons in the lowest energy level. In Fig. 3, we present the results of an exact numerical diagonalization of the complete phonon Hamiltonian (that is, including also the phonon number non-conserving terms) for the case $N_{ph}/N = 1$. The transition from the superfluid to the Mott insulator, with one phonon per site, is evident in the evolution of the phonon density as a function of the interaction U.

The properties of the BHM, allows us to propose an experimental sequence that would lead to the observation of the SI quantum phase transition: *(1)* The ion chain is cooled to the state with zero radial phonons by laser cooling. *(2)* Starting with a value $U \gg t$, the eigenstates of the system are well described by Fock states localized at each ion. The ground state of the phonon system can be created by means of sequencies of blue/red-side band transitions, in a method that has been successfully implemented with single trapped ions (see [8]). *(3)* The value of U is varied adiabatically down to a given value U_f, in such a way that the system remains in the ground state. At a given critical value $U_f \approx t$, the system undergoes a transition to a phonon superfluid. *(4)* The measurement of the ground state can be accomplished by the coupling of the transverse phonons to a given internal transition. One could apply, for example, a red sideband pulse with intensity g, for a short time t. Under such conditions, the probability of inducing a transition to the excited internal state is $\propto \sum_n (\sin(\sqrt{n}gt))^2 P(n) \approx \sum_n ng^2t^2P(n)$, where $P(n)$ is the probability of having n phonons. This method would allow us to measure the mean phonon number. By resolving individually the photoluminescence from each ion, one could observe features of the BEC, or SI transition in the variations of the phonon density along the chain. One could also apply well known methods to determine $P(n)$, or even the whole quantum tomography of the phonon quantum states [8, 10].

(ii) (Quasi) Bose–Einstein Condensation by evaporative laser cooling. We propose an experiment that is akin to the usual BEC of cold atoms in harmonic traps. First, we note that techniques for cooling of trapped ions, like laser cooling [10, 18] can only be used to destroy phonons. The existence of the trapping phonon potential in ion traps allows us to propose the combination of laser cooling with the idea of evaporative cooling. A

possible experimental sequence would be as follows: *(1)* Start with a Coulomb chain after usual Doppler cooling, that is, a chain with a given number of phonons per site, and induce a small phonon–phonon interaction $U \ll t$, so that the system remains in the weak interacting regime. *(2)* Apply laser cooling at the sides of the Coulomb chain, in such a way that the higher energy phonons on the top of the confinement potential are destroyed (evaporated). *(3)* The interaction U induces collisions that thermalize the phonons to a lower temperature. Several cycles of laser cooling / thermalization could be applied until the system is cooled below the critical temperature. Detection of the BEC could be accomplished along the same lines exposed above for the case of the BHM. Note that in the case of Coulomb chains (1D) considered here, (quasi) BEC is possible in finite size systems only.

We have shown that phonons in a system of trapped ions can be manipulated in such a way that they undergo BEC, or a SI transition. The main ingredients of our proposal are: *(1)* The fact that phonons can have a large energy gap that suppresses processes that do no conserve the number of phonons. *(2)* Phonon–phonon interactions (anharmonicities) can be induced by placing the ions in a standing–wave. *(3)* In the particular case of ions in a linear trap, radial phonons would be suitable for this proposal, with the advantage that the Coulomb interaction provides us with an approximately harmonic phonon confinement.

CONCLUSIONS

In this work we have exposed only a few applications of this idea, but phonons in trapped ions could be used to study quantum phases with a degree of controllability that is not possible with cold neutral atoms. Individual addressing would allow us to design Hubbard Hamiltonians with local interactions that change at will from site to site. Different directions of the radial modes, or different internal states of the ions could play the role of effective spins [19] for the phonons. On the other hand, one could also reach the regime dominated by the repulsive interaction and create, thus, a Tonks-Girardeau gas of phonons [20] in a Coulomb chain (this idea was implemented recently with optical lattices [21]). In a very promising approach, 2D systems of arrays of microtraps [11], or ions in Penning traps [12], could be considered, because phonons transverse to the crystal plane satisfy the conditions required by our proposal.

ACKNOWLEDGMENTS

Work supported by the European projects CONQUEST, RESQ and TOPQIP, and the Kompetenznetzwerk Quanteninformationsverarbeitung der Bayerischen Staatsregierung.

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