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Results in Mathematics

Isoperimetric Inequality, *p*-parabolicity and Doubling Graphs

Álvaro Martínez-Pérez and José M. Rodríguez

Abstract. Herein we study the relationship on graphs between being (metric) doubling and these two properties: being p-parabolic and satisfying the Cheeger isoperimetric inequality. We prove that if a uniform graph G satisfies the (Cheeger) isoperimetric inequality, then G is not (metric) doubling and see that the converse is not true. We also prove that if G is a doubling graph with doubling constant C, then it is p-parabolic for every $p \geq \log_2(C)$ and see that the converse is not true. Furthermore, we see that being doubling does not imply being p-parabolic for every $1 < p < \infty$. Finally, we see that a quasi-isometry between manifolds whose Ricci curvature is bounded below preserves being doubling and also, that an manifold with bounded Ricci curvature below is doubling if and only any uniform graph quasi-isometric to it is doubling.

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1. Introduction

Given a graph $G = (V(G), E(G))$ with the usual length metric d_G for which every edge has length 1, let us denote, as usual, by $B(v, k)$ the open ball of radius k centered at v and $S(v, k) := \{w \in V(G) \,|\, d_G(v, w) = k\}$.

A metric space is *(metric) doubling* if there exists a constant C such that every ball in the space can be covered by at most C balls of half the radius.

Álvaro Martínez-Pérez and José M. Rodríguez have contributed equally to this work.

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Recall that a nontrivial measure μ on a metric space X is said to be *doubling* if the measure of any ball is finite and there is a constant $c \geq 1$ such that $0 < \mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$.

Although these two concepts are different, there are relationships between them:

If a doubling measure exists on a metric space X, then X itself is a metric doubling space. Conversely, every complete metric doubling space admits a doubling measure. Also, doubling spaces are exactly the spaces with a finite Assouad dimension.

Doubling metrics are a crucial concept in metric analysis. For a comprehensive introduction to this topic, we refer to [3] and [12].

Definition 1. The *combinatorial Cheeger isoperimetric constant* of a graph G is defined to be

$$h(G) = \inf_U \frac{|\partial U|}{|U|},$$

where U ranges over all non-empty finite subsets of vertices in G, $\partial U = \{v \in G \,|\, d_G(v, U) = 1\}$ and $|U|$ denotes the cardinality of the set U.

A graph G satisfies the (Cheeger) *isoperimetric inequality* if $h(G) > 0$, since this means that

$$|U| \leq h(G)^{-1} |\partial U|$$

for every finite set of vertices U.

A graph G is said to be μ-*uniform* if each vertex p of V has at most μ neighbors, i.e.,

$$\sup \big\{ |N(p)| \,\big|\, p \in V(G) \big\} \leq \mu.$$

If a graph G is μ-uniform for some constant μ we say that G is *uniform*.

A graph G is defined as p-parabolic if all positive p-superharmonic functions on G are constant. This is equivalent to the absence of a p-Green's function, which is a positive fundamental solution of the p-Laplace-Beltrami operator. In [19], several results regarding the p-parabolicity of graphs and trees were obtained, particularly characterizing p-parabolicity for a substantial class of trees. In another study, [20], further characterizations of p-parabolicity were achieved using graph decompositions. This work also provided necessary and sufficient conditions for a uniform hyperbolic graph to be p-parabolic based on its boundary at infinity. Additionally, in [21] it was proved that a uniform hyperbolic graph satisfying the (Cheeger) isoperimetric inequality is non-p-parabolic for every $1 < p < \infty$.

Herein we want to study the relationship on graphs between being (metric) doubling and these two properties: being p-parabolic and satisfying the Cheeger isoperimetric inequality.

In sect. 2 we prove that if a uniform graph G satisfies the (Cheeger) isoperimetric inequality, then G is not (metric) doubling and see that the converse is not true.

In sect.3 we prove that if G is a doubling graph with doubling constant C, then it is p-parabolic for every $p \geq \log_2(C)$ and see that the converse is not true.

In sect. 4 we see that being doubling does not imply being p-parabolic for every $1 < p < \infty$.

The use of discrete structures to approximate Riemannian manifolds has proven to be a valuable tool in studying large-scale properties. M. Kanai approached this by defining a graph, known as the ε-net of the manifold, ensuring that the manifold and its ε-net are quasi-isometric. Numerous studies have followed Kanai's methodology or explored the relationship between the large-scale behavior of a manifold and its associated graph. Typically, the large-scale structure of the manifold or corresponding graph is preserved by quasi-isometries, which allow for significant distortion of local geometry. One key large-scale property maintained by quasi-isometries is Gromov hyperbolicity (see, e.g., [6,11]). In [16,17], and [18], M. Kanai investigated several geometric properties, including isoperimetric inequalities, Poincaré-Sobolev inequalities, parabolicity, growth rate of the volume of balls, and Liouville-type theorems, for a broad class of Riemannian manifolds under certain local geometric conditions. Kanai proved that these properties are preserved under quasi-isometries (see also [1,4,8,9]). Additionally, quasi-isometries preserve the parabolic Harnack inequality (see [4,5]) and various estimates on the transition probabilities of random walks, such as heat kernel estimates.

Furthermore, Holopainen and Soardi, among others (see [7,13,14,22]), showed that the existence of non-trivial solutions to a wide class of partial differential equations is also preserved under quasi-isometries.

Section 5 proves a technical result necessary for sect. 6.

In sect. 6 we prove that a quasi-isometry between n-manifolds whose Ricci curvature is bounded below preserves being doubling. The same result is obtained for uniform graphs. Also, an n-manifold with Ricci curvature bounded below is doubling if and only if its ε-net is doubling. Therefore, given an n-manifold with Ricci curvature bounded below and any uniform graph quasi-isometric to it, then the manifold is doubling if and only if the graph is doubling.

2. Cheeger Isoperimetric Inequality and (metric) Doubling Graphs

If G is a μ-uniform graph and $v \in V(G)$, then number of vertices in $B(v, m)$ is at most

$$1 + \mu + \mu^2 + \cdots + \mu^{n-1} = \frac{\mu^n}{\mu - 1}. \tag{1}$$

Theorem 1. *If a uniform graph G satisfies the (Cheeger) isoperimetric inequality, then G is not (metric) doubling.*

Proof. Suppose that G is μ-uniform and satisfies the (Cheeger) isoperimetric inequality with constant $K = h(G)^{-1}$. Then, for any vertex $v_0 \in V(G)$:

$$|B(v_0, n)| \le K|S(v_0, n)| \quad \forall n \in \mathbb{N}.$$

In particular,

$$|B(v_0, n)| \le |B(v_0, m)| \le K|S(v_0, m)| \quad \forall n \le m < 2n.$$

Therefore,

$$\frac{n}{K}|B(v_0, n)| \le |B(v_0, 2n)|.$$

Suppose that G is (metric) doubling with constant C and assume that $n > K \cdot C$. Then, there are at most C vertices $y_1, \dots, y_m$ such that $B(v_0, 2n) \subset \cup_{i=1}^{m} B(y_i, n)$. In particular, $|B(v_0, 2n)| \le \sum_{i=1}^{m} |B(y_i, n)|$. Thus, there is some vertex $v_1 = y_j$ such that

$$|B(v_1, n)| \ge \frac{1}{m}|B(v_0, 2n)| \ge \frac{1}{C}|B(v_0, 2n)| \ge \frac{n}{K \cdot C}|B(v_0, n)|.$$

Repeating the argument with v_i for each $i \in \mathbb{N}$, we can find a vertex v_{i+1} such that

$$|B(v_{i+1}, n)| \ge \frac{n}{K \cdot C}|B(v_i, n)| \ge \left(\frac{n}{K \cdot C}\right)^i |B(v_0, n)|.$$

Thus, since $\frac{n}{K \cdot C} > 1$, at some point there is some i_0 such that for $i > i_0$ we have $|B(v_i, n)| > \frac{\mu^n - 1}{\mu - 1}$, leading to contradiction with (1). Therefore, G is not (metric) doubling. □

We are going to see that the converse of Theorem 1 does not hold. We need some definitions.

Given two sequences of positive integers $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, with $2 \le r_n \le N$ for every $n \ge 1$ and some constant N, the *Cantor tree* $(T_{L,R}, v_0)$ is a rooted tree such that the root, v_0, has degree r_1, the vertices at distance $\ell_1 + \cdots + \ell_{n-1}$ have degree $r_n + 1$, and any other vertex has degree two. Note that $(T_{L,R}, v_0)$ is uniform since $R = \{r_n\}_{n=1}^{\infty}$ is a bounded sequence.

The *Cantor tree* (T_C, v_0) is a rooted tree such that the root, v_0, has degree two and any other vertex has degree three, i.e., $(T_C, v_0) = (T_{L,R}, v_0)$ with $\ell_n = 1$ and $r_n = 2$ for every $n \ge 1$.

Proposition 2. *Given sequences* $L = \{\ell_n\}_{n=1}^{\infty}$ *and* $R = \{r_n\}_{n=1}^{\infty}$*, the Cantor tree* $(T_{L,R}, v_0)$ *satisfies the (Cheeger) isoperimetric inequality if and only if the sequence*

$$\sum_{k=1}^{n-1} \frac{\ell_k}{r_{k+1} \cdots r_n} + \ell_n$$

is bounded.

Proof. One can check that the Cantor tree $(T_{L,R}, v_0)$ satisfies the (Cheeger) isoperimetric inequality if and only there exists a constant K such that

$$|B(v_0, \ell_1 + \cdots + \ell_n)| \le K\ |\partial B(v_0, \ell_1 + \cdots + \ell_n)|$$

for every n.

Since

$$|B(v_0, \ell_1 + \cdots + \ell_n)| = 1 + \sum_{k=1}^{n-1} \ell_k r_1 \cdots r_k + (\ell_n - 1) r_1 \cdots r_n$$
$$|\partial B(v_0, \ell_1 + \cdots + \ell_n)| = r_1 \cdots r_n,$$

the Cantor tree $(T_{L,R}, v_0)$ satisfies the (Cheeger) isoperimetric inequality if and only if the sequence

$$\sum_{k=1}^{n-1} \frac{\ell_k}{r_{k+1} \cdots r_n} + \ell_n$$

is bounded. □

Remark 1. The converse of Theorem 1 does not hold since no Cantor tree is doubling and Proposition 2 gives that there are Cantor trees without (Cheeger) isoperimetric inequality.

3. Parabolicity and Doubling Graphs

Given a function u on a graph G, define the *p-modulus of its discrete gradient* $|\nabla_G u|_p$ and its *discrete p-Dirichlet integral* $D_{p,G}(u)$, respectively, by

$$|\nabla_G u|_p(x) := \left(\sum_{y \in N(x)} \big| u(y) - u(x) \big|^p \right)^{1/p},$$
$$D_{p,G}(u) := \sum_{x \in G} |\nabla_G u|_p^p(x) = 2 \sum_{vw \in E(G)} \big| u(v) - u(w) \big|^p,$$

where the edges are considered unoriented.

For a finite subset S of G, the *p-capacity* of S is defined by
$$\operatorname{cap}_p S = \operatorname{cap}_p(S, G) = \inf \Big\{ D_{p,G}(u) : u \text{ function on } G \text{ with finite support, } u|_S = 1 \Big\}.$$

The following well-known result relates the p-capacity and the p-parabolicity.

Theorem 3. *Given $1 < p < \infty$, a uniform graph G is p-parabolic if and only if $\operatorname{cap}_p S = 0$ for some (and then for every) non-empty finite subset of $S \subset G$.*

For a proof of Theorem 3, see [18, Proposition 6] and [13, Final remark 5.16]. Note that the definition of discrete p-Dirichlet integral in [13] is slightly different, but both are equivalent.

We recently proved the following result:

Theorem 4 *[21, Theorem 11]. If G is a uniform hyperbolic graph with $h(G) > 0$, then G is non-p-parabolic for every $1 < p < \infty$.*

Also, parabolicity is somehow connected with the idea of a moderate growth of the space as we can see in the following result for Cantor trees.

Theorem 5 *[19, Theorem 7]. Given $1 < p < \infty$ and sequences $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, the Cantor tree $(T_{L,R}, v_0)$ is p-parabolic if and only if*

$$\sum_{k=1}^{\infty} \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} = \infty.$$

Thus, it makes sense to wonder if there is some relation between being doubling and being p-parabolic.

In [21, Theorem 16] we proved the following result (for $p = 2$ this result is classical, see e.g., [10, Theorem 6.10]).

Theorem 6. *If G is a uniform graph such that*

$$\sum_{k=1}^{\infty} \Big(\frac{k}{|B_G(u,k)|} \Big)^{1/(p-1)} = \infty \tag{2}$$

for some $1 < p < \infty$ and $u \in V(G)$, then G is p-parabolic.

Using this, we can prove that a doubling uniform graph is p-parabolic if p is big enough.

Theorem 7. *If G is a doubling graph with doubling constant C, then it is p-parabolic for every $p \geq \log_2(C)$.*

Proof. Suppose that G is a doubling graph with constant C. Then, trivially, G is C-uniform. Also, for every vertex $u \in G$,

$$|B(u, 2^i)| \leq C^i |B(u,1)| = C^i.$$

Also, for every k such that $2^{i-1} < k \leq 2^i$,

$$|B(u,k)| \leq |B(u, 2^i)| \leq C^i.$$

Thus,

$$\begin{aligned}\sum_{k=1}^{\infty} \Big(\frac{k}{|B_G(u,k)|} \Big)^{1/(p-1)} &\geq 1^{\frac{1}{p-1}} + \sum_{i=1}^{\infty} 2^{i-1} \Big(\frac{2^{i-1}+1}{C^i} \Big)^{\frac{1}{p-1}} \\ &> \sum_{i=1}^{\infty} 2^{i-1} \Big(\frac{2^{i-1}}{C^i} \Big)^{\frac{1}{p-1}} \\ &= \frac{1}{C^{\frac{1}{p-1}}} \sum_{i=1}^{\infty} \Big(\frac{2^{\frac{p}{p-1}}}{C^{\frac{1}{p-1}}} \Big)^{i-1},\end{aligned}$$

and this is a geometric series with ratio $\frac{2^{\frac{p}{p-1}}}{C^{\frac{1}{p-1}}}$. So, this series is divergent if

$$\frac{2^{\frac{p}{p-1}}}{C^{\frac{1}{p-1}}} \geq 1 \quad \Leftrightarrow \quad 2^p \geq C \quad \Leftrightarrow \quad p \geq \log_2(C).$$

Therefore, the result follows from Theorem 6. □

Question 1. Is a doubling uniform graph p-parabolic for every $1 < p < \infty$? Or at least, how small can we assume p to be?

The converse of Theorem 7 is not true. To build a space that is p-parabolic for every $1 < p < \infty$ but fails to be doubling, let us recall the following from [19].

A *cut set* C for a rooted, geodesically complete tree (T, v) is a subset C of T such that $v \notin C$ and for every isometric embedding $f : [0, \infty) \to T$ with $f(0) = v$ there exists a unique $t_0 > 0$ such that $f(t_0) \in C$ (see [15]).

If c is any vertex of the rooted tree (T, v), then the subtree of (T, v) determined by c is

$$T_c = \{x \in T \,|\, c \in [v, x]\},$$

where $[v, x]$ denotes the unique geodesic in T joining v and x.

Given a rooted tree (T, v_0), a set of vertices in T, $\mathcal{C} = \{c_i\}_{i\in I}$, is called a *subcut set* in T if $T_{c_i} \cap T_{c_j} = \emptyset$ for every $i \neq j$ and $\mathcal{C}$ is not a cut set.

Given a rooted tree (T, v_0) and a cut set or a subcut set $\mathcal{C} = \{c_i\}_{i\in I}$, let us denote $T_{\mathcal{C}} = T \setminus \cup_{i\in I} T_{c_i}$.

Proposition 8 *[19, Proposition 7] . Let (T, v_0) be a uniform geodesically complete rooted tree, $\mathcal{C}$ a subcut set in T and $1 < p < \infty$. Then, T is p-parabolic if and only if $T_{\mathcal{C}}$ and T_c are p-parabolic for every $c \in \mathcal{C}$.*

Example 1. Consider a geodesic ray T_0 starting on a root vertex v_0 and denote v_k the vertex in T_0 such that $d_{T_0}(v_0, v_k) = k$ for each $k \in \mathbb{N}$. Now, let (T, v_0) be the rooted tree obtained by attaching to each vertex v_k, $k \geq 1$, the root of another geodesic ray, T_k. Notice that for every $n \in \mathbb{N}$, there are n points $y_k = S(v_0, 2n-1) \cap T_k$, with $1 \leq k \leq n$, in the ball $B(v_0, 2n)$ such that $d_T(y_i, y_j) > n$. Therefore, T is not doubling. However, considering the subcut set in T defined by the vertices in T_k adjacent to v_k it is trivial to check, by Proposition 8, that T is p-parabolic for every $1 < p < \infty$.

We need the following result.

Proposition 9 *[19, Proposition 6]. If a uniform graph G contains a non-p-parabolic subgraph G' for some $1 < p < \infty$, then G is non-p-parabolic.*

The following is a natural complement of Proposition 8.

Proposition 10. *Let (T, v_0) be a uniform geodesically complete rooted tree, $\mathcal{C}$ a cut set in T and $1 < p < \infty$. Then, T is p-parabolic if and only if T_c is p-parabolic for every $c \in \mathcal{C}$.*

Proof. By Proposition 9, if T is p-parabolic, then T_c is p-parabolic for every $c \in \mathcal{C}$.

Assume now that T_c is p-parabolic for every $c \in \mathcal{C}$. First, let us see that $T_{\mathcal{C}}$ is bounded and, therefore, compact. Suppose, otherwise, that $T_{\mathcal{C}}$ is unbounded. Since T is uniform, $S(v_0, i)$ is finite for every $i \in \mathbb{N}$. Then, there is some vertex $x_1 \in S(v_0, 1) \cap T_{\mathcal{C}}$ such that $(T_{\mathcal{C}})_{x_1}$ is unbounded. Again, for every $i \in \mathbb{N}$, there is some vertex $x_i \in S(v_0, i) \cap (T_{\mathcal{C}})_{x_{i-1}}$ such that $(T_{\mathcal{C}})_{x_i}$ is unbounded. Therefore, there is an isometric embedding $f : [0, \infty) \to T$ with $f(0) = v_0$ and $f(i) = x_i$ for every $i \in \mathbb{N}$, such that $f([0, \infty)) \subset T_{\mathcal{C}}$ and $f([0, \infty)) \cap \mathcal{C} = \emptyset$, leading to contradiction. Thus, $T_{\mathcal{C}}$ is bounded and, since T is uniform, compact.

Then, let $u : V(T_{\mathcal{C}}) \to \mathbb{R}$ be such that $u(v) = 1$ for every vertex $v \in T_{\mathcal{C}}$. Since $T_{\mathcal{C}}$ is finite, there is a finite number of vertices $c_1, \ldots, c_k \in \mathcal{C}$. Since T_{c_i} is p-parabolic, there is a function $u_i^n : V(T_{c_i}) \to \mathbb{R}$ such that u_i^n has finite support, $u_i^n(c_i) = 1$ and $D_{p,T_{c_i}}(u_i^n) < \frac{1}{2^i n}$. Let us define $\bar{u}^n : V(T) \to \mathbb{R}$ such that $\bar{u}^n(v) = u(v) = 1$ for every $v \in T_{\mathcal{C}}$ and $\bar{u}^n(v) = u_i^n(v)$ for every $v \in T_{c_i}$ and every $1 \le i \le k$.

Thus,

$$D_{p,T}(\bar{u}^n) = \sum_{i=1}^{k} D_{p,T_{c_i}}(u_i^n) < \sum_{i=1}^{k} \frac{1}{2^i n} < \frac{1}{n}$$

and it is readily seen that $\bar{u}^n$ has finite support. Therefore, by Theorem 3, T is p-parabolic. □

4. A Partial Answer to Question 1

First, let us introduce some background.

A function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \ge 1$, $b \ge 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \le d_Y(f(x_1), f(x_2)) \le a\, d_X(x_1, x_2) + b\,, \text{ for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \ge 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \le c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [16]).

A complete manifold with polynomial growth of degree d is p-parabolic for all $p \ge d$. For instance a complete n-dimensional manifold with non negative Ricci curvature is p-parabolic for all $p \ge n$. Conversely, a manifold of isoperimetric dimension d is not p-parabolic for any $p < d$. Consequently, $\mathbb{R}^n$ is p-parabolic if and only if $p \ge n$. For any uniform graph or manifold with bounded geometry X, we can introduce an invariant $d_{par} \in [1, \infty]$, called its parabolic dimension, such that the manifold X is p-parabolic if $p > d_{par}(X)$ and p-hyperbolic if $p < d_{par}(X)$. This is a quasi-isometric invariant. Hence, $d_{par}(\mathbb{R}^n) = n$. See [23].

Theorem 7 implies that if G is a doubling graph with doubling constant C, then $d_{par}(G) \leq \log_2(C)$.

The following results are trivial extensions of [17, Theorem 2 and Corollary 7] for the general case of p-parabolicity.

Theorem 11. *Given $1 < p < \infty$ and $\varepsilon > 0$, let X be a complete Riemannian manifold with bounded geometry, and let P be an ε-net in X. Then, X is p-parabolic if and only if P is p-parabolic.*

Theorem 12. *If P and Q are quasi-isometric uniform graphs, then P is p-parabolic if and only if Q is p-parabolic for any fixed $1 < p < \infty$.*

Corollary 1. *Suppose that X (respectively, Y) is a complete Riemannian manifold with bounded geometry or a uniform graph. If X and Y are quasi-isometric, then X is p-parabolic if and only if Y is p-parabolic for any fixed $1 < p < \infty$.*

Consider the graph $\mathbb{Z}^n$, i.e., the Cayley graph of the finitely generated group $(\mathbb{Z}^n, +)$.

Proposition 13. *$\mathbb{Z}^n$ is a doubling graph which is not p-parabolic for $1 < p < n$.*

Proof. $\mathbb{Z}^n$ is a doubling graph for each positive integer n. Since the inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ is a quasi-isometry, Corollary 1 implies that $\mathbb{Z}^n$ is p-parabolic if and only if $p \geq n$. □

5. Hyperbolic Balls

In this section we prove Proposition 14, which is a technical result needed to prove later Proposition 16.

If $B(r)$ is any ball of radius r in $\mathbb{H}^n$ $(n \geq 2)$, then

$$\mathrm{Vol}(B(r)) = \mathrm{Vol}(S^{n-1}) \int_0^r \sinh^{n-1}(t)\, dt,$$

where $\mathrm{Vol}(S^{n-1})$ is the total volume of the Euclidean $(n-1)$-sphere of radius 1.

We are going to prove that for each $a > 1$ and $n \geq 2$, the function

$$\frac{\mathrm{Vol}(B(ar))}{\mathrm{Vol}(B(r))} = \frac{\int_0^{ar} \sinh^{n-1}(t)\, dt}{\int_0^{r} \sinh^{n-1}(t)\, dt} \tag{3}$$

is increasing on r, see Proposition 14.

We need some preliminary facts.

Fix $a > 1$ and let $I_a(0, \infty)$ be the set of increasing functions $f : (0, \infty) \to (0, \infty)$ such that $f(ar)/f(r)$ is an increasing function on $r \in (0, \infty)$.

The following fact is elementary.

Lemma 1. *If* $a_1/b_1 \le a_2/b_2$*, then*

$$\frac{a_1}{b_1} \le \frac{a_1+a_2}{b_1+b_2}\,.$$

Lemma 2. *If* $f \in I_a(0,\infty)$ *and* $\lambda > 0$*, then* $f^\lambda \in I_a(0,\infty)$ *and* $\int_0^r f(t)\,dt \in I_a(0,\infty)$.

Proof. The first statement is direct, let us prove the second one. If $0 < s < r < t < R$, then

$$f(as)f(r) \le f(ar)f(s), \quad f(ar)f(t) \le f(at)f(r),$$

$$f(r)\int_0^r f(as)\,ds \le f(ar)\int_0^r f(s)\,ds, \quad f(ar)\int_r^R f(t)\,dt \le f(r)\int_r^R f(at)\,dt,$$

$$f(r)\int_0^{ar} f(s)\,ds \le af(ar)\int_0^r f(s)\,ds, \quad af(ar)\int_r^R f(t)\,dt \le f(r)\int_{ar}^{aR} f(t)\,dt,$$

$$\frac{\int_0^{ar} f(s)\,ds}{\int_0^r f(s)\,ds} \le a\,\frac{f(ar)}{f(r)}\,, \quad a\,\frac{f(ar)}{f(r)} \le \frac{\int_{ar}^{aR} f(t)\,dt}{\int_r^R f(t)\,dt}\,.$$

Consequently, Lemma 1 implies

$$\frac{\int_0^{ar} f(t)\,dt}{\int_0^r f(t)\,dt} \le \frac{\int_0^{ar} f(t)\,dt + \int_{ar}^{aR} f(t)\,dt}{\int_0^r f(t)\,dt + \int_r^R f(t)\,dt} = \frac{\int_0^{aR} f(t)\,dt}{\int_0^R f(t)\,dt}\,.$$

□

Corollary 2. *If* $f \in I_a(0,\infty)$ *and* $\lambda > 0$*, then* $\int_0^r f^\lambda \in I_a(0,\infty)$.

Proposition 14. *The function* $\int_0^r \sinh^{n-1}(t)\,dt$ *belongs to* $I_a(0,\infty)$ *for every* $a > 1$ *and* $n \ge 2$.

Proof. By Corollary 2, it suffices to check that the hyperbolic sine belongs to $I_a(0,\infty)$ for each $a > 1$.

Let us check first that the function $u(t) = t \operatorname{cotanh} t$ is increasing on $(0,\infty)$:

$$u'(t) = \frac{(\cosh t + t\sinh t)\sinh t - t\cosh t\cosh t}{\sinh^2 t} = \frac{\cosh t \sinh t - t}{\sinh^2 t} > 0.$$

Hence, for each $t > 0$ and $a > 1$,

$$\frac{at\cosh at}{\sinh at} > \frac{t\cosh t}{\sinh t}\,,$$
$$a\cosh at \sinh t > \sinh at \cosh t.$$

Finally, for each fixed $a > 1$, consider the function

$$v(t) = \frac{\sinh at}{\sinh t}\,.$$

We have

$$v'(t) = \frac{a \cosh at \sinh t - \sinh at \cosh t}{\sinh^2 t} > 0$$

and the result follows. □

6. Boundedly Doubling Spaces and Quasi-isometries

Definition 2. We say that a metric space (X, d) is *boundedly doubling* if for every $R > 0$ there is a constant C_R such that every ball of radius $2k \leq 2R$ can be covered by at most C_R balls with radius k.

Proposition 15. *Uniform graphs are boundedly doubling.*

Proof. Assume that G is μ-uniform.

Consider any ball $B(w, 2R)$ in G. Then

$$|B(w, 2R)| \leq 1 + \mu + \cdots + \mu^{\lceil 2R \rceil - 1} = \frac{\mu^{\lceil 2R \rceil} - 1}{\mu - 1}.$$

In particular, $B(w, 2R)$ is covered by at most $\frac{\mu^{\lceil 2R \rceil}-1}{\mu-1} := C_R$ balls of any radius $k > 0$, where C_R depends only on μ and R. □

Let $V(r)$ denote the volume of a geodesic ball in the simply connected complete Riemannian n-manifold of constant curvature $-\kappa^2$, with $\kappa > 0$.

The following result is a restatement of Lemma 2.3 in [16]:

Lemma 3. *Let X be a complete Riemannian n-manifold whose Ricci curvature is bounded from below by $-(n-1)\kappa^2$ for some $\kappa > 0$, and let P be an ε-separated subset of X. Then,*

$$\#\{p \in P : x \in B(p, r)\} \leq \frac{V(2r + \varepsilon/2)}{V(\varepsilon/2)}$$

for all $r > 0$ and for all $x \in X$.

Lemma 4. *For each $a > 1$, the function $\frac{V(ar)}{V(r)}$ is increasing on r.*

Proof. It is well known (see [2, III.4.1]) that

$$V(r) = \text{Vol}(S^{n-1}) \frac{1}{\kappa^n} \int_0^{\kappa r} \sinh^{n-1}(t)\, dt.$$

Then, Proposition 14 implies that $\frac{V(ar)}{V(r)}$ is increasing on r. □

Proposition 16. *If X is a complete Riemannian n-manifold whose Ricci curvature is bounded below by $-(n-1)\kappa^2$ for some $\kappa > 0$, then X is boundedly doubling.*

Proof. Consider any $R > 0$ and any ball $B(x, 2k)$ in X with $k \leq R$. By Lemma 3 with $r = 2k$ and $\varepsilon = k$, $B(x, 2k)$ contains at most a finite number $\nu(k)$ of k-separated points, where $\nu(k) = \frac{V(5k/2)}{V(k/2)}$. Therefore, $B(x, 2k)$ is covered by $\nu(k)$ balls of radius k.

By Lemma 4, for every $k \leq R$, $\nu(k) \leq \nu(R)$ and $\nu(R)$ only depends on R, the curvature bound and the dimension of the manifold. Hence, X is boundedly doubling. □

Theorem 17. *Given two quasi-isometric metric spaces X and Y, if X is doubling and Y is boundedly doubling, then Y is doubling.*

Proof. Consider $f : G_1 \to G_2$ and $g : G_2 \to G_1$ two ε-full (a, b)-quasi-isometries such that for every vertex $y \in Y$ $d_Y(y, f \circ g(y)) < \varepsilon$. Consider any fixed point $y_0 \in Y$.

If $0 < k \leq \max\{4b, 2\varepsilon\}$, then, since Y is boundedly doubling, $B(y_0, 2k)$ is covered by at most C_1 balls of radius k, where C_1 depends only on b and ε.

Now, suppose that $k > \max\{4b, 2\varepsilon\}$.

Since g is an (a, b)-quasi-isometry, $g\big(B(y_0, 2k)\big) \subset B(g(y_0), a2k + b)$ and if X is C-doubling, the ball $B(g(y_0), a2k + b)$ is covered by C balls of radius $\frac{a2k+b}{2}$ and, inductively, C^N balls of radius $\frac{a2k+b}{2^N}$ and centers $\{x_i\}_{i=1}^{C^N}$. Let N be such that $2^N > 8a^2 + a$.

Notice that

$$\frac{a2k + b}{2^N} < \frac{\frac{k}{2} - b}{a} \Leftrightarrow \frac{4a^2k + 2ab}{k - 2b} < 2^N$$

and since $k > 4b$, that is, $k - 2b > \frac{k}{2}$, we have that

$$\frac{4a^2k + 2ab}{k - 2b} < \frac{4a^2k + 2ab}{k/2} = 8a^2 + \frac{4ab}{k} < 8a^2 + a < 2^N.$$

Thus, $f\big(B(x_i, \frac{a2k+b}{2^N})\big)$ is contained in the ball $B(f(x_i), a(\frac{a2k+b}{2^N}) + b) \subset B(f(x_i), \frac{k}{2})$. Now, since

$$g(B(y_0, 2k)) \subset B(g(y_0), a2k + b) \subset \cup_{i=1}^{C^N} B\Big(x_i, \frac{a2k + b}{2^N}\Big)$$

and $d_{G_2}(y, f \circ g(y)) < \varepsilon < \frac{k}{2}$ for every $y \in Y$, it follows that

$$B(y_0, 2k) \subset \cup_{i=1}^{C^N} B(f(x_i), k)$$

where $C^N := C_2$ depends only on the constants C and a.

Hence, if $C' := \max\{C_1, C_2\}$, Y is C'-doubling. □

Corollary 3. *Given two boundedly doubling, quasi-isometric metric spaces X and Y, then X is doubling if and only if Y is doubling.*

Corollary 4. *If G_1 and G_2 are quasi-isometric uniform graphs, then G_1 is doubling if and only if G_2 is doubling.*

Corollary 5. *If X and Y are quasi-isometric n-manifolds whose Ricci curvature is bounded below , then X is doubling if and only if Y is doubling.*

Corollary 6. *If X is an n-manifold whose Ricci curvature is bounded below and P is an ε-net in X, then X is doubling if and only if P is doubling.*

Corollary 7. *If X is an n-manifold whose Ricci curvature is bounded below and G is a uniform graph such that X and G are quasi-isometric, then X is doubling if and only if G is doubling.*

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Declarations

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References

[1] Cantón, A., Fernández, J.L., Pestana, D., Rodríguez, J.M.: On harmonic functions on trees. Poten. Anal. **15**, 199–244 (2001)

[2] Chavel, I.: Riemannian Geometry: a modern introduction. Cambridge University Press, New York (2006)

[3] Coifman, R. R., Weiss, G., Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math., **242**, Springer-Verlag, Berlin, (1971). https://doi.org/10.1007/BFb0058946.

[4] Coulhon, T., Saloff-Coste, L.: Variétés riemanniennes isométriques à l'infini. Rev. Matem. Iberoamericana **11**, 687–726 (1995)

[5] Delmotte, T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Mat. Iberoam. **15**, 181–232 (1999)

[6] Ghys, E., de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics, Volume 83, Birkhäuser, 1990

[7] Granados, A., Pestana, D., Portilla, A., Rodríguez, J.M.: Stability of p-parabolicity under quasi-isometries. Math. Nachr. **295**, 536–559 (2022)

[8] Granados, A., Pestana, D., Portilla, A., Rodríguez, J.M., Tourís, E.: Stability of the injectivity radius under quasi-isometries and applications to isoperimetric inequalities, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM **112**, 1225–1247 (2018)

[9] Granados, A., Pestana, D., Portilla, A., Rodríguez, J.M., Tourís, E.: Stability of the volume growth rate under quasi-isometries. Rev. Mat. Complut. **33**(1), 231–270 (2020)

[10] Grigor'yan, A., Introduction to Analysis on Graphs. University Lecture Series, vol. 71. American Mathematical Society (AMS), Providence (2018)

[11] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75–263

[12] Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer-Verlag, Universitext. New York (2001)0-387-95104-0

[13] Holopainen, I.: Rough isometries and p-harmonic functions with finite Dirichlet integral. Rev. Mat. Iberoam. **10**, 143–176 (1994)

[14] Holopainen, I., Soardi, P.M.: p-harmonic functions on graphs and manifolds. Manuscr. Math. **94**, 95–110 (1997)

[15] Hughes, B.: Trees and ultrametric spaces: a categorical equivalence. Adv. Math. **189**, 148–191 (2004)

[16] Kanai, M.: Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds. J. Math. Soc. Japan **37**, 391–413 (1985)

[17] Kanai, M.: Rough isometries and the parabolicity of Riemannian manifolds. J. Math. Soc. Japan **38**, 227–238 (1986)

[18] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata,: Lecture Notes in Math. 1201. Springer **1986**, 122–137 (1985)

[19] Martínez-Pérez, A., Rodríguez, J.M.: On p-parabolicity of Riemannian manifolds and graphs. Rev. Mat. Complut. **35**, 179–198 (2022). https://doi.org/10.1007/s13163-021-00387-x

[20] Martínez-Pérez, A., Rodríguez, J.M.: Parabolicity on graphs. Results Math. **79**, 70 (2024). https://doi.org/10.1007/s00025-023-02095-y

[21] Martínez-Pérez, A., Rodríguez, J.M.: Parabolicity and Cheeger's constant on graphs. Rev. Real Acad. Cien. Exac. Fís. Natur. Serie A. Mat. **118**, 139 (2024). https://doi.org/10.1007/s13398-024-01639-3

[22] Soardi, P. M., Potential Theory in infinite networks. Lecture Notes in Math., Volume 1590, Springer-Verlag, 1994

[23] Troyanov, M.: Parabolicity of manifolds. Siberian Adv. Math. **9**(4), 125–150 (1999)

Álvaro Martínez-Pérez
Universidad Complutense de Madrid
Departamento de Álgebra, Geometría y Topología
Plaza de Ciencias, 3
Madrid 28040 Madrid
Spain
e-mail: alvaro36@ucm.es

José M. Rodríguez
Universidad Carlos III de Madrid, ROR: https://ror.org/03ths8210
Departamento de Matemáticas
Avenida de la Universidad, 30 (edificio Sabatini)
Leganés 28911 Madrid
Spain
e-mail: jomaro@math.uc3m.es

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