Documento de trabajo
9420

ASYMPTOTIC PROPERTIES OF MEAN LENGTH ESTIMATORS FOR THE M/G/1 QUEUE

ANTONIO RODRIGO

FACULTAD DE CIENCIAS ECONOMICAS Y EMPRESARIALES
UNIVERSIDAD COMPLUTENSE
Campus de Somosaguas 28223 MADRID

ASYMPTOTIC PROPERTIES OF MEAN LENGTH ESTIMATORS FOR THE M/G/1 QUEUE

Antonio Rodrigo

Department of Economic Analysis,
Facultad de C.C. E.E. y Empresariales. U. Complutense de Madrid.
Campus de Somosaguas, 28223 Madrid, Spain.
E-mail: ececo03@sis.ucm.es

Keywords: *M/G/1 queues*; *direct estimator*; *Chapman-Kolmogorov equations*; *accuracy.*

ABSTRACT

In this article we estimate the mean number of customers in an *M/G/1* queue in a steady state. We use a direct method allowing for any possible service distribution. We give an approximate expression for the variance of the estimator.We also give numerical examples illustrating these results, based on data generated by simulating known models.

INTRODUCTION

Inference about the mean number of customers, q, in the system in the steady state has been dealt with by many authors in a Markovian context. Two methods have been used: a direct method (see for example Reynolds(1972) and Gafarian(1966)) and an indirect method using an estimator for the traffic intensity, ρ, (see Lilliefors(1966)).

One difficulty with the estimation of q is the dependence among the variables involved in the sample. Schruben and Kulkarni

(1982) showed that the estimator given by Lilliefors(1966), which does not consider this dependence, is not adequate for estimating q. This problem can be solved using occupation cycles in the sample as in Rodrigo(1994)). In general non Markovian queues it is still possible to use occupation cycles, at least theoretically, but the distribution functions involved are usually unknown. Hence it is preferable to use a direct (or integral) estimator.

The classical generalization of an $M/M/1$ queue considers general service time distributions, i.e., an $M/G/1$ queue. As usual let λ be the arrival rate and let μ be the service rate in the exponential case. Let X be the duration of a service, $B(x)$ its (unknown) corresponding distribution, $B^*(s)$ the Laplace transform of $B(x)$ and β_i, $i\geq 1$, the i-th central moment. Suppose we observe an $M/G/1$ ergodic queue (i.e. we observe Poisson arrivals at rate λ and a service channel with a general distribution, $B(x)$, $x\geq 0$, and traffic intensity $\rho=\lambda E[X]$ smaller than one) over the time interval $(0,T]$ where T is a fixed time period. The mean number of customers in the system in steady state can be estimated (see Reynolds (1972)) by

$$\hat{q}=\frac{1}{T}\int_0^T N(t)dt$$

where $N(t)$, $t\in(0,\infty)$, is an ergodic stochastic process representing the number of customers in the system at time t. Under the assumption that at time $t=0$ the system is in steady state, $\hat{q}$ is an unbiased estimator of q and its mean square error is given by $Var[\hat{q}]$. Our purpose in this article is to obtain some approximate expression for this variance. In order to do this we need some previous results. Let $R(t)=E[N(t)N(0)]-E[N(t)]E[N(0)]$ be the auto-covariance function, which can be written as

$$R(t)=\sum_{n=1}^{\infty} n\sum_{m=1}^{\infty} mp_m\left(p_n(t|m)-p_n\right)$$

where $p_n(t|m)=\Pr[N(t)=n|N(0)=m]$, $n\geq 0$, $m\geq 0$, are the transition probabilities and $p_n=\lim_{t\to\infty}p_n(t|m)$, $n\geq 1$, the stationary probabilities. It is well known that $Var[\hat{q}]$ can be expressed as

$$\frac{2}{T}\int_0^T R(t)dt - \frac{2}{T^2}\int_0^T tR(t)dt. \tag{1}$$

When $R(t)\in L^1(0,\infty)$ (which will be proved below) and $N(t)$ is ergodic we can write

$$Var[\hat{q}]=\frac{2}{T}V+o(T^{-1}), \qquad V=\int_0^\infty R(t)dt. \tag{2}$$

It can be proved using the results of Zeifman(1991) that in $M/M/1$ queues the asymptotic error is $O(T^{-2})$. This rate of convergence can also be obtained (see Györfi et al. (1990) and Zeifman (1991)) albeit under rather strong conditions for other queues. But generally only an $o(1/T)$ rate of convergence can be guaranteed.

Reynolds(1972) finds this variance for the Markovian case when $\mu=1$ although there is an error in his formula. In the next section we derive this variance for the $M/M/1$ case with general service rate. In section three we do the same allowing for any service time distribution. Finally in section-4 we give some numerical examples.

2. THE VARIANCE IN AN M/M/1 QUEUE

Given (2) we only need to calculate the constant V. From (1) this constant can be written as

$$V=\sum_{n=1}^{\infty} n\sum_{m=1}^{\infty} mp_m\int_0^\infty\Big(p_n(t|m)-p_n\Big)dt. \tag{3}$$

Now, if $p_n(t|m)$ and p_n were known, we could get V from (3). However it is also possible to calculate V using the Chapman-Kolmogorov equations which in transient state (see Kleinrock (1974)) are given by

$$\frac{dp_n(t|m)}{dt}=-(\lambda+\mu)p_n(t|m)+\lambda p_{n-1}(t|m)+\mu p_{n+1}(t|m), \quad n\geq 1$$
$$\frac{dp_0(t|m)}{dt}=-\lambda p_0(t|m)+\mu p_1(t|m). \tag{4}$$

In steady state these equations become

$$0=-(\lambda+\mu)p_n+\lambda p_{n-1}+\mu p_{n+1}, \quad n\geq 1$$
$$0=-\lambda p_0+\mu p_1. \tag{5}$$

Subtracting equation (5) from (4) we have

$$p_n-\delta_{nm}=-(\lambda+\mu)V_n(m)+\lambda V_{n-1}(m)+\mu V_{n+1}(n), \quad n\geq 1$$
$$p_0-\delta_{0m}=-(\lambda+\mu)V_0(m)+\mu V_1(m) \tag{6}$$

where

$$V_i(j)=\int_0^\infty (p_i(t|j)-p_i)dt \quad \text{and} \quad \delta_{ij}=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases}; \quad i,j\in\mathbb{N}.$$

Multiplying by mp_m in (6) and adding for all $m\in\mathbb{N}$ we get

$$qp_n-np_n=-(\lambda+\mu)V_n+\lambda V_{n-1}+\mu V_{n+1}, \quad n\geq 1$$
$$qp_0=-\lambda V_0+\mu V_1 \tag{7}$$

where

$$V_i=\sum_{m=1}^{\infty} mp_m V_i(m).$$

Using the appropiate generating functions, (7) becomes

$$qP(z)-zP'(z)=(\lambda z+\mu z^{-1}-\lambda-\mu)V(z)+\mu(1-z^{-1})V_0 \tag{8}$$

where

$$P(z)=\sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad V(z)=\sum_{n=0}^{\infty} V_n z^n.$$

From (3) and (8) we have that $V=V'(1)$. Taking derivatives in (8) at $z=1$ and using the normalizing condition (i.e. the sum of V_i, for all $i\in\mathbb{N}$, is zero since $N(t)$ is a birth-death process) we obtain

$$(q-1)P'(1)-P''(1)=\mu V_0$$
$$(q-2)P''(1)-P'''(1)=2(\lambda-\mu)V'(1)-2\mu V_0$$
$$V(1)=1 \tag{9}$$

where

$$P(z)=(1-\rho)(1-\rho z)^{-1}$$

(see Kleinrock(1974)). Therefore $V=\rho(1+\rho)\mu^{-1}(1-\rho)^{-4}$. Thus we have proved the following theorem:

Theorem-1. In $M/M/1$ ergodic queues the variance of the direct estimator $\hat{q}$ is given by

$$Var[\hat{q}]=\frac{2\rho(1+\rho)}{\mu(1-\rho)^4 T} + O(T^{-2}) \tag{10}$$

■

3. THE VARIANCE IN AN M/G/1 QUEUE

We now use the supplementary variable method to reduce the problem of finding the variance in an *M/G/1* queue to the Markovian case. Proceeding as we did in the previous section we obtain the analogous constant V. Let $\xi(t)$ be the time elapsed while the customer is being served at time t. Let $p_n(x)dx$ be the stationary probability and $p_n(t;x|m)dx=\Pr[N(t)=n, x\le\xi(t)<x+dx|N(0)=m]$, $n\ge1$, be the transition probabilities when $x\le\xi(t)<x+dx$. The autocovariance function can be written as

$$R(t)=\int_0^\infty R(t;x)dx, \qquad R(t;x)=\sum_{n=1}^\infty n\sum_{m=1}^\infty mp_m\Big(p_n(t;x|m)-p_n(x)\Big)$$

and the constant V in (2) as

$$V=\int_0^\infty R(t)dt=\int_0^\infty V(x)dx, \qquad V(x)=\int_0^\infty R(t;x)dt.$$

To obtain V we first calculate the function $V(x)$. For *M/G/1* queues the Chapman-Kolmogorov equations in transient state are given by

$$\frac{\partial p_0(t|m)}{\partial t}=-\lambda p_0(t|m)+\int_0^\infty b(x)p_1(t;x|m)dx$$

$$\frac{\partial p_n(t;x|m)}{\partial t}+\frac{\partial p_n(t;x|m)}{\partial x}=-(\lambda+b(x))p_n(t;x|m) + \lambda(1-\delta_{1n})p_{n-1}(t;x|m), \qquad n\ge1$$

$$p_n(t;0|m)=\lambda\delta_{1n}p_0(t|m)+\int_0^\infty b(x)p_{n+1}(t;x|m)dx$$

where $b(x)=dB(x)/(1-B(x))$ is the instantaneous failure rate of $B(x)$. The corresponding stationary equations are

$$\lambda p_0=\int_0^\infty b(x)p_1(x)dx$$

$$\frac{dp_n(x)}{dx}=-(\lambda+b(x))p_n(x)+\lambda(1-\delta_{1n})p_{n-1}(x), \qquad n\ge1$$

$$p_n(0)=\lambda\delta_{1n}p_0+\int_0^\infty b(x)p_{n+1}(x)dx.$$

Proceeding as we did in the previous section we get $V(x)=V'(1;x)$ as the solution of the system of linear differential equations

$$qp_0=-\lambda V_0+\int_0^\infty b(x)V_1(x)dx$$

$$qP(z;x)-zP'_z(z;x)+\frac{\partial V(z;x)}{dx}=(\lambda z-\lambda-b(x))V(z;x) \tag{11}$$

$$V(z;0)=-\lambda(1-z)V_0-qp_0+\frac{1}{z}\int_0^\infty b(x)V(z;x)dx \tag{12}$$

together with the normalizing condition

$$V_0+\int_0^\infty V(1;x)dx=0 \tag{13}$$

where

$$V(z;x)=\sum_{n=1}^\infty V_n(x)z^n\ ,\qquad V_n(x)=\sum_{m=1}^\infty mp_m\int_0^\infty\Big(p_n(t;x|m)-p_n(x)\Big)dt$$

$$V_0=\sum_{m=1}^\infty mp_m\int_0^\infty\Big(p_0(t|m)-p_0\Big)dt \quad\text{and}\quad P(z;x)=\sum_{n=1}^\infty p_n(x)z^n.$$

Solving for V we have the following theorem:

Theorem-2. If an *M/G/1* ergodic queue with service distribution function $B(x)$, $x\in\mathbb{R}^+$, has the first four moments, the term V in the expression for the variance of the integral estimator $\hat{q}$ in (2) is

$$V=\frac{1}{2(1-\rho)}\Big(A_0+A_1k_1+A_2k_2+\beta_1^2k_3\Big)$$

where

$$A_0=\lambda\rho q(2+\rho)\beta_2+(\lambda^2\beta_3/3)(1-2q+\rho-\rho q-2\rho^2+\lambda^2\beta_2)+ + (\lambda^3/4)[(1+\rho)\beta_4-2q\beta_2^2]-2\rho^2q\beta_1$$

$$A_1=2\rho q\beta_1^2+(1-q+\rho-2\rho q-2\rho^2)\beta_2+\lambda(1+\rho)\beta_3+\lambda^2\beta_2^2$$

$$A_2=(2-2\rho-q)\beta_1^2+(1+5\rho/2)\beta_2$$

$$k_1=\lambda+\lambda^3\beta_2(2(1-\rho))^{-1},\quad k_2=\lambda^3(6(1-\rho)^2)^{-1}\Big(2(1-\rho)(3\beta_2+\lambda\beta_3)+3\lambda^2\beta_2^2\Big)$$

$$k_3=\lambda^4(4(1-\rho)^3)^{-1}\Big((1-\rho)^2(\lambda\beta_4+4\beta_3)+2\lambda\beta_2(1-\rho)(2\lambda\beta_3+3\beta_2)+3\lambda^3\beta_2^3\Big).$$

Proof. (see appendix). ∎

4. NUMERICAL EXAMPLES

We calculate now the value of the constant V for some particular cases and give some estimations of the parameter q by simulating some known models (note that in an $M/G/1$ queue $q=\rho+\rho^2\beta_2(2(1-\rho)\beta_1^2)^{-1}$; see Kleinrock(1974)). Note that V is strongly dependent on the coefficient of variation (cv) of service time and that it increases with cv (see table-I). Obviously, V tends to infinity if $\rho\uparrow 1$.

TABLE-I

We find values for the constant V for the deterministic, Erlang, exponential and hiperexponential cases for different values of the traffic intensity and a mean service time equal to one.

	Deterministic service time=1	Erlang, N=2 μ=2	exponential μ=1	hiperexp. N=2 μ_1=2 p_1=.4 μ_2=.75 p_2=.6
$\lambda=\rho=.1$	.0648929	.1111497	.1676574	.2141342
$\lambda=\rho=.2$	.1796224	.3498780	.5859375	.8008464
$\lambda=\rho=.3$	.4045346	.8886962	.6243230	2.336225
$\lambda=\rho=.4$	.8987654	2.200000	4.320988	6.455602
$\lambda=\rho=.5$	2.145833	5.765625	12.00000	18.43056
$\lambda=\rho=.6$	5.934375	17.21602	37.50001	58.70627
$\lambda=\rho=.7$	21.15231	65.16077	146.9136	232.7539
$\lambda=\rho=.8$	120.9333	389.4001	900.0001	1434.178
$\lambda=\rho=.9$	2193.860	7278.770	17099.98	27264.05

Estimations of q are drawn in the figures below by simulating the above four queues for $\lambda=.5$ (fifth row of table-I) during the time interval (0,500]. We suppose that at time t=0 a customer starts the service. Assuming that the service time distribution is known, except for a parameter $\theta\in\Theta\subset\mathbb{R}^n$, the direct estimator $\hat{q}$ is compared with an "indirect estimator", $q^*=q(\hat{\theta})$, obtained using the expression for the mean number of customers in the steady state. In the first three examples we only need an estimator for the traffic intensity ρ to get an expression for q^*. The estimator of ρ can be obtained using the maximum likelihood estimators of λ and of the mean service time, β_1. Note that we have assumed that the only unknown parameters are λ and μ. In the hiperexponential case the maximum likelihood estimators are more complex. We assume β_2 to be known in order to obtain easy indirect estimators.

In fig-1 we have assumed that the initial number n_0 of customers in the system is equal to the mean number in the steady state. Note that there are not too many differences between both estimators and that we obtain the worst estimations in the hiper-exponential case. More differences appear when ρ is close to one (fig-3) or n_0 is away from q (fig-2).

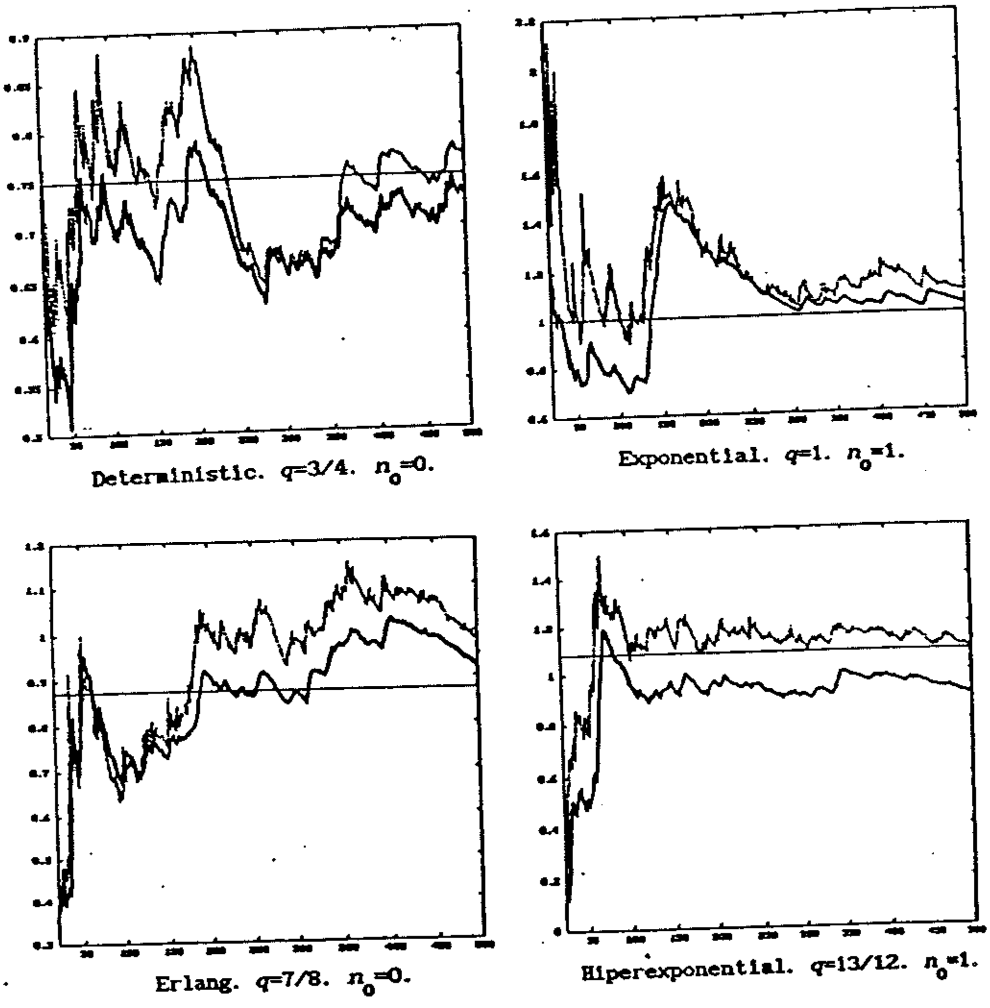


FIG-1. The initial number of customers in the queue has been assumed to be equal to the integer part of the mean number in steady state (n_0=[q]). ρ=λ=.5. We denote the direct estimator by (=) and the indirect estimator by (-).

We observe that the estimations are strongly dependent on the initial number of customers in the queue (fig-1 and 2). Note that if n_0 is away from q (for instance when n_0=20) the indirect estimator is less influenced by that number than the direct one (fig-2).

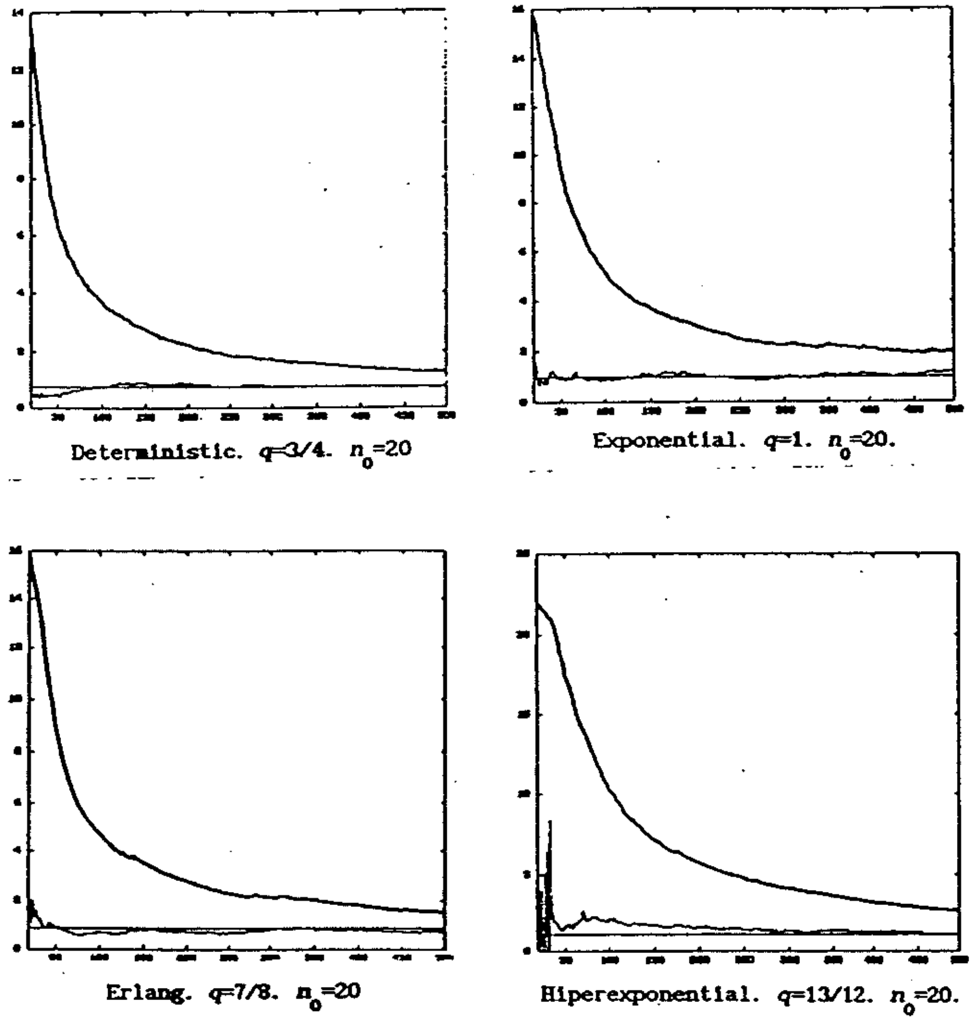
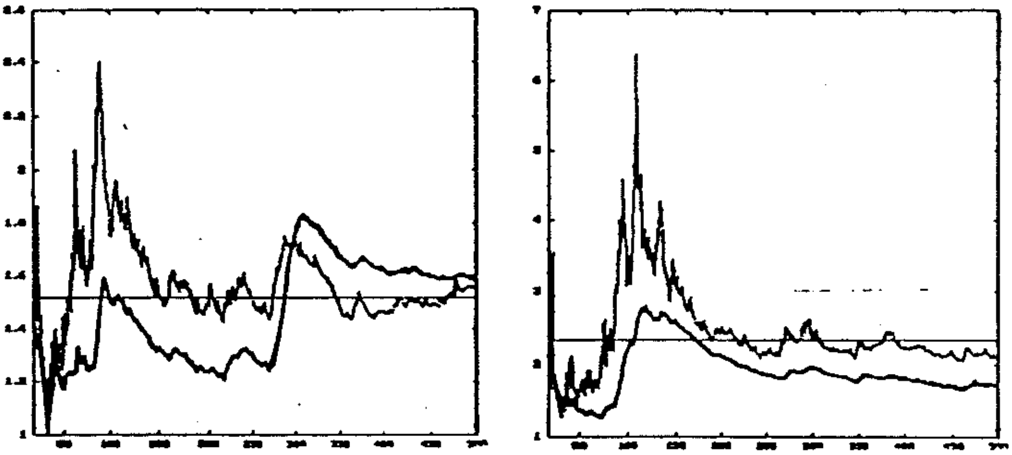


FIG-2. n_0=20, $\rho=\lambda=.5$, direct estimator(=), indirect estimator(-).

In fig-3 we simulate the determitistic and exponential cases when λ=.7. We observe that the indirect estimator usually gives the best estimations. In the exponential case, when $\rho\geq.5$, it can be proved (see Rodrigo(1994)) that the indirect estimator has smaller asymptotic variance than the direct one. No results have been reported in the literature using a general service time distribution. However when ρ is away from zero I conjecture that we would obtain an analogous result, at least for distributions in the exponential family. (Note that in the deterministic case this variance only depends on the variance of the maximum likelihood estimator of λ).



Deterministic. $q=7/3$. $n_0=2$. Exponential. $q=91/60$. $n_0=1$.

FIG-3. $n_0=[q]$, $\rho=\lambda=.7$, direct estimator(•), indirect estimator(-).

Note that when ρ is close to one, we need a long observation time to obtain "good" estimations of q. The observation time could be approximated using the asymptotic variance given in theorem-2 when $\hat{q}$ is asypmtotically Gaussian, (e.g., if $\lambda=.9$ and the service time is identically one, then we would need about 204,442 units of time to get an error smaller than .1 with a 95% confidence level).

Finally note that the service time distribution is usually either unknown or dependent on a large number of parameters and hence a direct estimator will always be preferable.

APPENDIX

To prove theorem-2 we introduce the following notation. Let $\theta(z;x)=-qP(z;x)+zP'_z(z;x)$ where $P(z;x)$ is the function defined in section-4. At $z=1$ this function can be written as $\vartheta(1;x)=\bar{\vartheta}(x)(1-B(x))$ where $\bar{\vartheta}(x)$ is a polynomial that we calculate in the lemmas below. Let

$$I(\vartheta(x))=I^1(\vartheta(x))=\int\bar{\vartheta}(x)dx, \quad I^i(\vartheta(x))=\int I^{i-1}(\vartheta), \quad i=2,3,\dots$$

be the integral operator and let $I^i(\vartheta(\beta))$, $i=2,3,\dots$, be the values this operator takes after replacing x^j by the central moment β_j, $1\le j\le n$, where n is the degree of the polynomial $I^i(\vartheta(x))$ $i\ge1$. Let

$$k_0=P(1;0),\ k_1=P'_z(1;0),\ k_2=P''_{zz}(1;0),\ k_3=P'''_{zzz}(1;0)$$

$$a_0=V(1;0),\ a_1=V'_z(1;0),\ a_2=V''_{zz}(1;0)$$

be the initial conditions of the system of differential equations we now proceed to solve. To get an explicit expression for the constant V we need to use the three following lemmas:

Lemma-1. If β_i, $1\le i\le4$, exist and $N(t)$ is an ergodic process we then have

$$P(1;x)=\lambda(1-B(x))$$

$$P'_z(1;x)=(\lambda^2x+k_1)(1-B(x))$$

$$P''_{zz}(1;x)=(\lambda^3x^2+2\lambda k_1x+k_2)(1-B(x))$$

$$P'''_{zzz}(1;x)=(\lambda^4x^3+3\lambda^2k_1x^2+3\lambda k_2x+k_3)(1-B(x))$$

where

$$k_1=\lambda+\lambda^3\beta_2(2(1-\rho))^{-1}, \quad k_2=\lambda^3(6(1-\rho)^2)^{-1}\left(2(1-\rho)(3\beta_2+\lambda\beta_3)+3\lambda^2\beta_2^2\right)$$

$$k_3=\lambda^4(4(1-\rho)^3)^{-1}\left((1-\rho)^2(\lambda\beta_4+4\beta_3)+2\lambda\beta_2(1-\rho)(2\lambda\beta_3+3\beta_2)+3\lambda^3\beta_2^3\right).$$

Proof. Note that

$$Q(z)=p_0+\int_0^\infty P(z;x)dx \tag{14}$$

where $Q(z)$ is the generating function corresponding to the number of customers in the system in steady state (Pollaczek-Khinchin formula; see Kleinrock(1974)). Taking derivatives in equation (14) the lemma is proved.■

Remark. This lemma can also be proved using the stationary Chapman

-Kolmogorov equations. That, together with (14), allows us to obtain the Pollaczek-Khinchin formula.

Lemma-2. Under the conditions imposed in lemma-1 we have

$$V(1;x)=(I(\vartheta(x))+a_0)(1-B(x))$$

$$V'_z(1;x)=(I(\vartheta'_z(x))+\lambda I^2(\vartheta(x))+\lambda a_0 x+a_1)(1-B(x))$$

$$V''_{zz}(1;x)=(I(\vartheta''_{zz}(x))+2\lambda I^2(\vartheta'_z(x))+2\lambda^2 I^3(\vartheta(x)) + $$
$$+ \lambda^2 a_0 x^2+2\lambda a_1 x+a_2)(1-B(x))$$

where

$$a_0=I(\vartheta'_z(\beta))-I(\vartheta(\beta))$$

$$a_1=\frac{1}{2(1-\rho)}\Big((\lambda^2\beta_2-2\rho)a_0+I(\vartheta''_{zz}(\beta))+2\lambda I^2(\vartheta'_z(\beta)) + $$
$$+ 2\lambda^2 I^3(\vartheta(\beta))-2\lambda I^2(\vartheta(\beta))\Big).$$

(Note that it is not necessary to calculate the constant a_2).

Proof. Taking derivatives with respect to z at $z=1$ in equation (11) we obtain the expressions for $V_z^{i)}(1;x)$, $i=0,1,2$. Taking derivatives in equation (12), also with respect to z at $z=1$, we get

$$(1-\rho)a_0=\lambda V_0+I(\vartheta'_z(\beta))+\lambda I^2(\vartheta(\beta))-I(\vartheta(\beta)). \tag{15}$$

From (13) we have $V_0=-I^2(\vartheta(\beta))-a_0\beta_1$. This, together with (15), permits us to obtain the constant a_0. Finally, differentiating equation (12) twice with respect to z at $z=1$ we have

$$2(1-\rho)a_1= (\lambda^2\beta_2-2\rho+2)[I(\vartheta'_z(\beta))-I(\vartheta(\beta))]+I(\vartheta''_{zz}(\beta))+2\lambda I^2(\vartheta'_z(\beta)) + $$
$$+ 2\lambda^2 I^3(\vartheta(\beta))-2\lambda I^2(\vartheta(\beta))-2I(\vartheta'_z(\beta))+2I(\vartheta(\beta))$$

and from this we obtain the constant a_1. ∎

Lemma-3. Under the assumptions of lemmas 1 and 2 we have

$$I^i(\vartheta(\beta))=(k_1-\lambda q)\frac{1}{i!}\beta_i+\frac{\lambda^2}{(i+1)!}\beta_{i+1}, \quad i=1,2,3$$

$$I^i(\vartheta'_z(\beta)) = [(1-q)k_1+k_2]\frac{1}{i!}\beta_i+[\lambda^2(1-q)+2\lambda k_1]\frac{1}{(i+1)!}\beta_{i+1} + $$
$$+ \frac{2\lambda^3}{(i+2)!}\beta_{i+2}, \quad i=1,2$$

$$I(\vartheta''_{zz}(\beta)) = [(2-q)k_2+k_3]\beta_1+\frac{1}{2}[2\lambda(2-q)k_1+3\lambda k_2]\beta_2 +$$

$$+ \frac{1}{3}[\lambda^3(2-q)+3\lambda^2 k_1]\beta_3 + \frac{\lambda^4}{4}\beta_4.$$

Proof. This can be easily proved integrating the formulas appearing in lemma-1.■

Proof of theorem-2. In section three we found that $V'_z(1;x)=V(x)$. Integrating the expression for $V'_z(1;x)$ in lemma-2, where $x\in(0,\infty)$, we get

$$V=I^2(\vartheta'_z(\beta))+\lambda I^3(\vartheta(\beta))+\lambda a_0\beta_2/2+a_1\beta_1. \qquad (16)$$

Using (16) and the lemmas 2 and 3 the theorem is proved.■

BIBLIOGRAPHY

Gafarian, A. V. and Ancker, C. J. (1966). Mean value estimation from digital computer simulations, Oper. Res. 14, 25-44.

Györfi, L., Härdle, W., Sarda, P. and Vieu, P. (1990). Notes in Nonparametric Statistics, Springer-Verlag, Berlin.

Kleinrock, L. (1974). Queueing Systems, Vol.I, Wiley, New York.

Lilliefors, H. W. (1966). Some confidence intervals for queues, Oper. Res. 14, 723-727.

Reynolds, J. F. (1972). Asymptotic properties of mean length estimators for finite Markov queue, Oper. Res. 20, 1, 52-57.

Rodrigo, A. (1994). Inference in ergodic queues using occupation cycles, Dept. of Economic Analysis. Technical Report #9407, Universidad Complutense de Madrid.

Schruben, L. and Kulkarni, R. (1982). Some consequences of estimating parameters for the *M/M/*1 queue, Oper. Res. Letters, 1, 2, 75-78.

Zeifman, A. I. (1991). Some estimates of the rate of convergence for birth and death processes, J. Appl. Prob. 28, 268-277.