Risk Neutral Valuation and Linear Programming

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Content

- Risk neutral valuation
- Methodological framework.
- Utility and Relative Entropy.
- Calibration
- Kullback-Leibler relative entropy minimization
- Absolute deviation minimization
- Comparison of Methods
- Pricing and Hedging.
- Conclusions

Risk neutral valuation

- Parametric modelization (ex: Bl-Sch world):

hypothesis → Model → Estimation → Results →

→ Decisions → Consequences

- Are the models correct?
- Inverse engineering as an alternative style of modelization: let the real world deliver the most general tool for valuing (RNP).

Risk neutral valuation

- The formulation of unrealistic hypothesis can drive to ill conclusions.
- Main idea: not to formulate any hypothesis but rather tracking some theoretical milestones against the real world (inverse methods – Jackwerth, 2004).

Risk Neutral Valuation: retrieving Risk Neutral Probabilities

- Let us assume APT.
- Then we would like to know the Risk Neutral Probabilities (RNP)…
- …for we could then price any asset, particularly **illiquid ones,** we could even try to replicate them to hedge risk.
- **Inverse engineering**: we try to retrieve RNP from the observed assets prices.

Risk Neutral Valuation: two main approaches

- Parametric methods to recover the RNP:
  - Recovering the RNP through the second derivatives of option values (Chang, 2002), by means of non-linear least squares (Breeden, 1978):

$$\frac{\partial^2 C}{\partial K^2} = \pi_{S=K}$$

  - Using mixtures of log-normal (Melick, 2007 and Cheng, 2010) or normal distributions (Ritchey, 1990), assuming a student-t (Jong and Huisman, 2000) or a generalised lambda distribution (Corrado, 2001).

Risk Neutral Valuation: two main approaches

- Non-parametric methods for recovering the RNP:
  - Kernel estimation (Bondarenko, 2000), least squares (Yatchev and Härdle, 2006), curve fitting (Monteiro, 2008) or maximum entropy (Rubinstein, 1997; Branger, 2003; Rockinger and Jondeau 2002).
    - **Maximum entropy by means of Kullback-Leibler relative entropy minimisation (Samperi, 1997; Avellaneda, Buff, Friedman, Grandechamp, Kruck and Newman, 2011; Arrieta, 2012, 2013).**

A common methodological framework. Step 1

- All the methods for RNP Recovery share the following framework.
- **STEP 1:** Choosing some frequency distribution for the underlying asset price to be used as a prior distribution to seed the method: $(q_i)_{i=1}^{M}$
  - The prior distribution can be either some parametric model or a uniform distribution (for instance the one obtained by means of a Monte Carlo simulation of the asset prices).

A common methodological framework
Step 2

- **STEP 2:** Choosing a set of liquid assets j (benchmarks) with observed market prices C_j then writing them as discounted expectations with respect to the RNP $\hat{\pi}$.
  – Here is where we ask our output to verify ATP

A common methodological framework
Step 3

- **STEP 3:** Targeting a posterior distribution $\left(p_i \right)_{i=1}^{M}$ that will be the RNP. Two main streams have been followed here:
  - Determine the "best" member among a parametric family: **parametric methods**. Think for instance in the log-normal RNP of the Black-Scholes world. The "best" is calculated by means of some optimization criteria determining the parameters optimal values
  - Not restricting the calculation to any parametric family, giving way to the data for a maximum influence into the result, and a minimum influence to the theoretical model. This last will be restricted to the only design of the underlying price trajectories.

A common methodological framework . Step 4

- **STEP 4:** Selecting some optimization process such that the optimum is the posterior distribution or RNP. This optimization process consists in a set of constraints and an objective to optimize among the feasible set
  - The feasible set must always take into account the facts that both the result must be a probability distribution and the prices of the benchmarks written as discounted expectation with respect to the RNP must be equal, or at least be near to the market prices. Equality or closeness depends on the kind of applied methodology.
  - The objective expresses some condition to be fulfilled by the posterior with respect to the prior distribution. Usually this condition is expressed as a deviation or pseudo-distance between them, able to be financially and/or mathematically interpreted in some way.
- Mathematical program reached= Method's Complexity & stability

Utility and relative entropy

- The pseudo-*distance* between two distributions p and q is defined by means of

$$\tilde{D}_{\psi}\left(p|q\right) = \sum_i \psi\left(\frac{p_i}{q_i}\right) q_i$$

  - $\psi\left(x\right)$ is convex function.
- We want to minimize this pseudo-*distance* calibrated to the observed prices

Utility and relative entropy

- For each penalization function $\psi(x)$ there exists a corresponding concave utility $u(x)$
- Avellaneda proposes to use relative entropy (Kullback-Leibler pseudo-distance) coming from exponential utility

$$u(x) = -\exp(-\alpha x) \longleftrightarrow \psi(x) = x \log x$$

- We propose to use the total variation distance

$$u(x) = x \longleftrightarrow \psi(x) = |x-1|$$

coming from risk neutral utility function

Calibration

- The calibration problem consists in finding the RNP vector *p* such that the following equation is satisfied:

$$\sum_{i=1}^{\nu} p_i g_{ij} = C_j,\ j = 1, \ldots, N$$

  - C_j are the observed prices
  - g_{ij} if the payoff of each option *j* and path *i*.

Kullback-Leibler relative entropy minimization

- Utility function $u(x) = -\exp(-\alpha x)$
- Penalization function $\psi(x) = x \log x$
- Distance to the prior $D_\psi(p|q) = \log \nu \sum_i p_i \log p_i$
- Formulation of the optimization problem

$$\min_p \left\{ \log \nu \sum_i p_i \log p_i \right\}$$

$$\text{s.t.} \sum_{i=1}^{\nu} p_i g_{ij} = C_j,\ j = 1, \ldots, N$$

Absolute deviation minimization

- Utility function $u(x) = x, x \in \mathbb{R}$
- Penalization function $\psi(x) = |x-1|, x \in \mathbb{R}$
- Pseudo-Distance to the prior

$$D_\psi(p|q) = \sum_i \left| \frac{p_i}{q_i} - 1 \right| q_i = \sum_i |p_i - q_i|$$

Absolute deviation minimization

- Formulation of the optimization problem by means of Linear Goal Programming (LGP)

$$\min \sum_i \left(y_i^- + y_i^+ \right)$$

$$\text{s.t. } p_i - q_i + y_i^- - y_i^+ = 0, i = 1, \ldots, M$$

$$\sum_{i=1}^{M} p_i g_{ij} - C_j = 0, \; j = 1, \ldots, N$$

$$\sum_{i=1}^{M} p_i = 1$$

$$p_i, y_i^+, y_i^- \geq 0, i = 1, \ldots, M$$

Comparison of Methods

- Prices of 33 benchmarks given by Black-Scholes with 2000 paths

Benchmark	s0	r	sigma	t	k	Type
instr2	100	0	0.25	30	0	forward
instr4	100	0	0.25	60	0	forward
instr6	100	0	0.25	90	0	forward
instr23	100	0	0.25	30	90	call
instr25	100	0	0.25	60	90	call
instr27	100	0	0.25	90	90	call
instr37	100	0	0.25	30	100	call
instr39	100	0	0.25	60	100	call
instr41	100	0	0.25	90	100	call
instr51	100	0	0.25	30	110	call
instr53	100	0	0.25	60	110	call
instr55	100	0	0.25	90	110	call
instr86	100	0	0.25	30	90	put
instr88	100	0	0.25	60	90	put
instr90	100	0	0.25	90	90	put
instr100	100	0	0.25	30	100	put
instr102	100	0	0.25	60	100	put
instr104	100	0	0.25	90	100	put
instr114	100	0	0.25	30	110	put
instr116	100	0	0.25	60	110	put
instr118	100	0	0.25	90	110	put

Comparison of Methods

- Assets to be valued (among others):

	s0	r	sigma	t	k	option
Asset 5	100	100	0.25	75	0	forward
Asset 19	100	100	0.25	75	85	call
Asset 47	100	0	0.25	75	105	call

Comparison of Methods

- Weighted Monte Carlo calculations: We use Matlab to replicate the relative entropy minimization wrt the a priori distribution, using the iteration algorithm proposed in Elices and Giménez, 2006.
  - Prior probabilities q have been chosen equipobable.
  - Objective function:

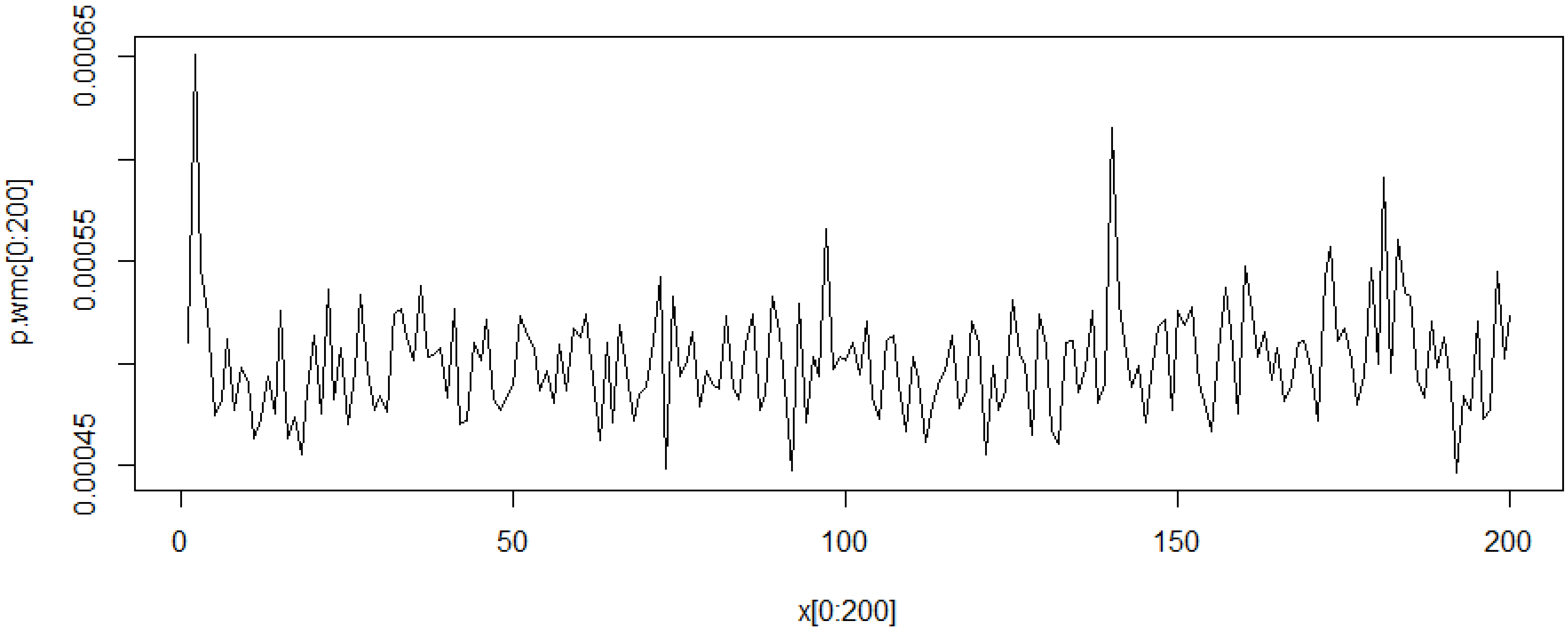
Minimize:
$$W(\lambda) = \log\left(\frac{1}{\nu}\sum_{i=1}^{\nu}\exp\left(\sum_{j=1}^{N} g_{ij}\lambda_i\right)\right) - \sum_{j=1}^{N}\lambda_j C_j$$

Comparison of Methods

- The goal programming optimization problem has been solved using GAMS to introduce the math. program and CPLEX being the solver.
- In order to obtain comparable results, the simulated cash-flows used were those obtained in Matlab for the KL distance minimization method.
- Objectives considered: q
  - Equiprobable (case LP0)
  - Parametric (normal) (case LP1)

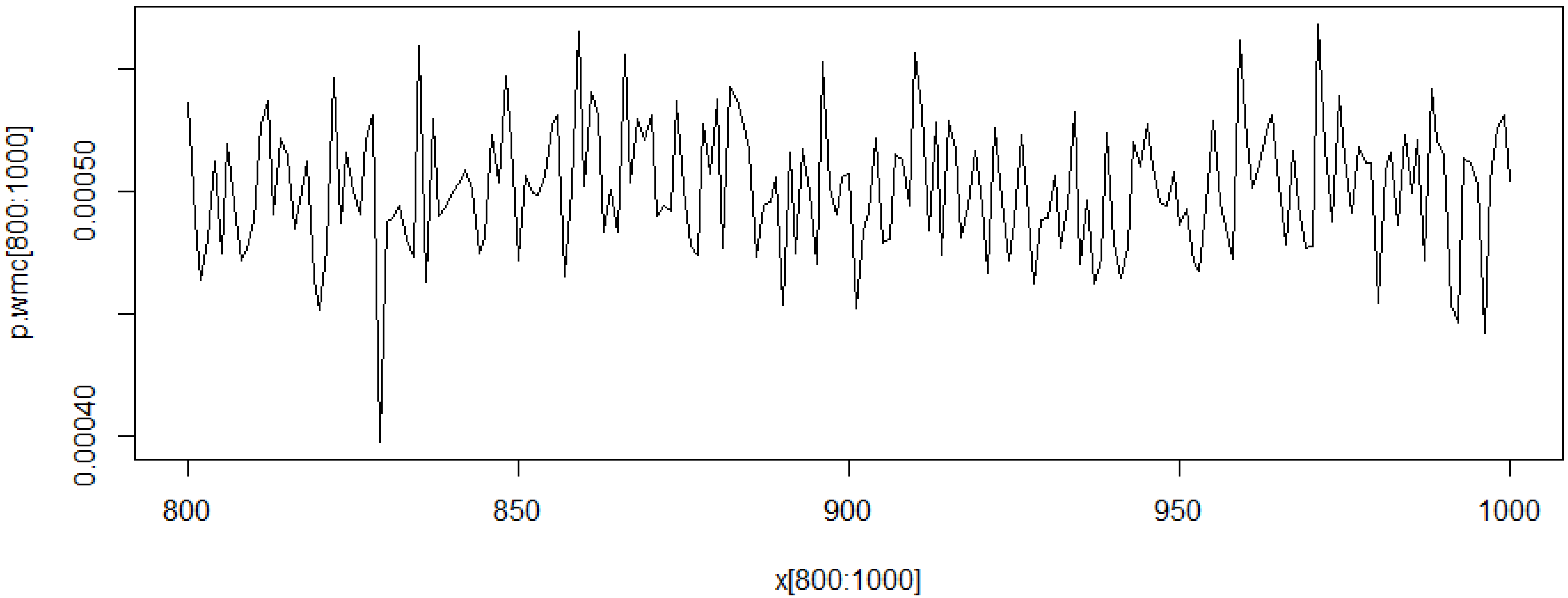
Comparison of Methods: WMC's RNPs

- Its shape over the first 200 scenarios:



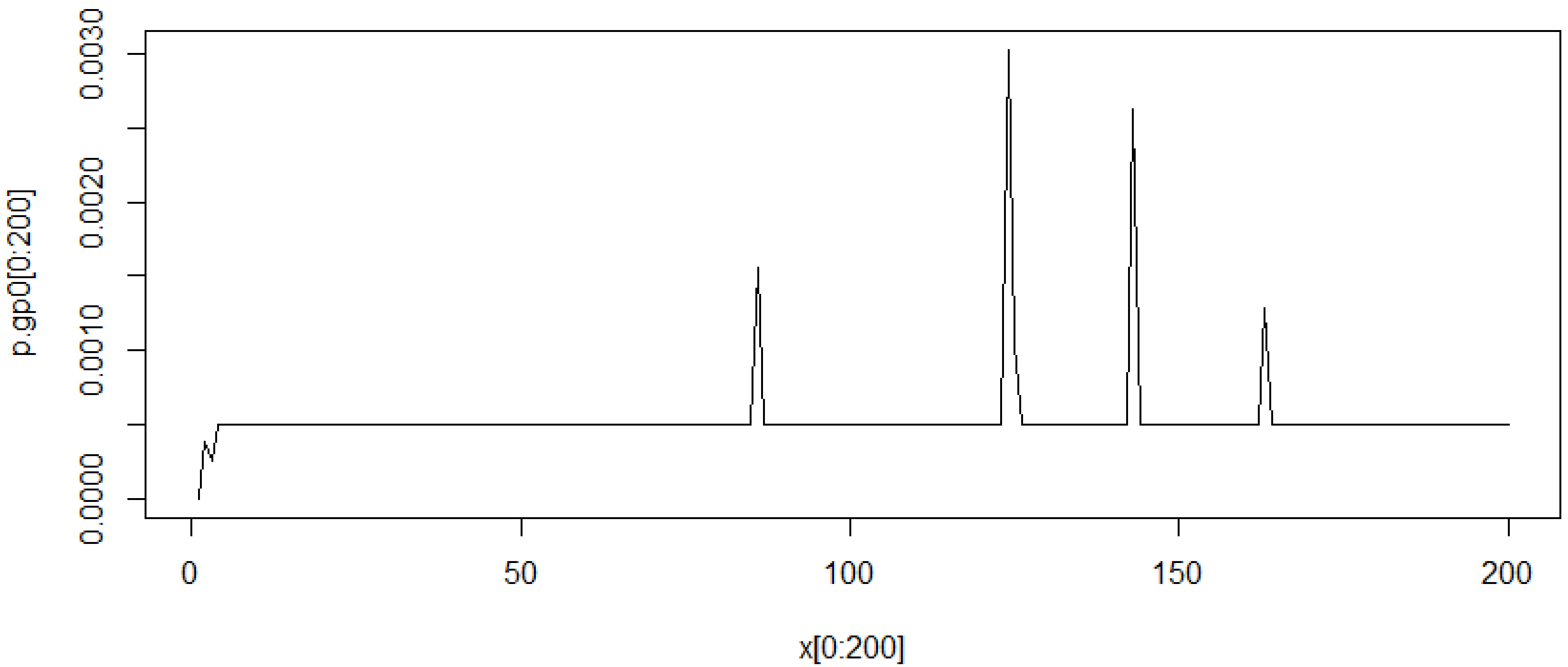
Comparison of Methods: WMC's RNPs

- Its shape over the following 200 scenarios:



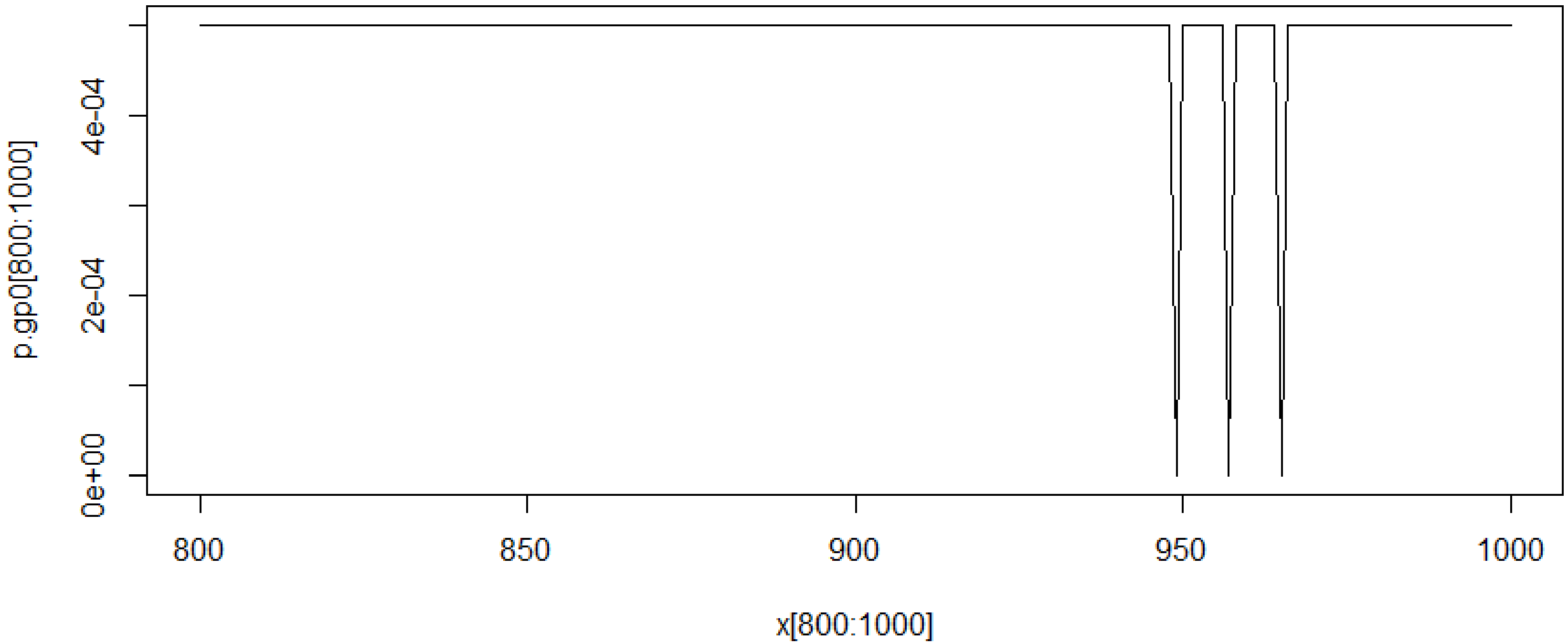
Comparison of Methods: LP0 RNP

- This is the RNP obtained in the case of equiprobable prior q., first 200 scenarios:



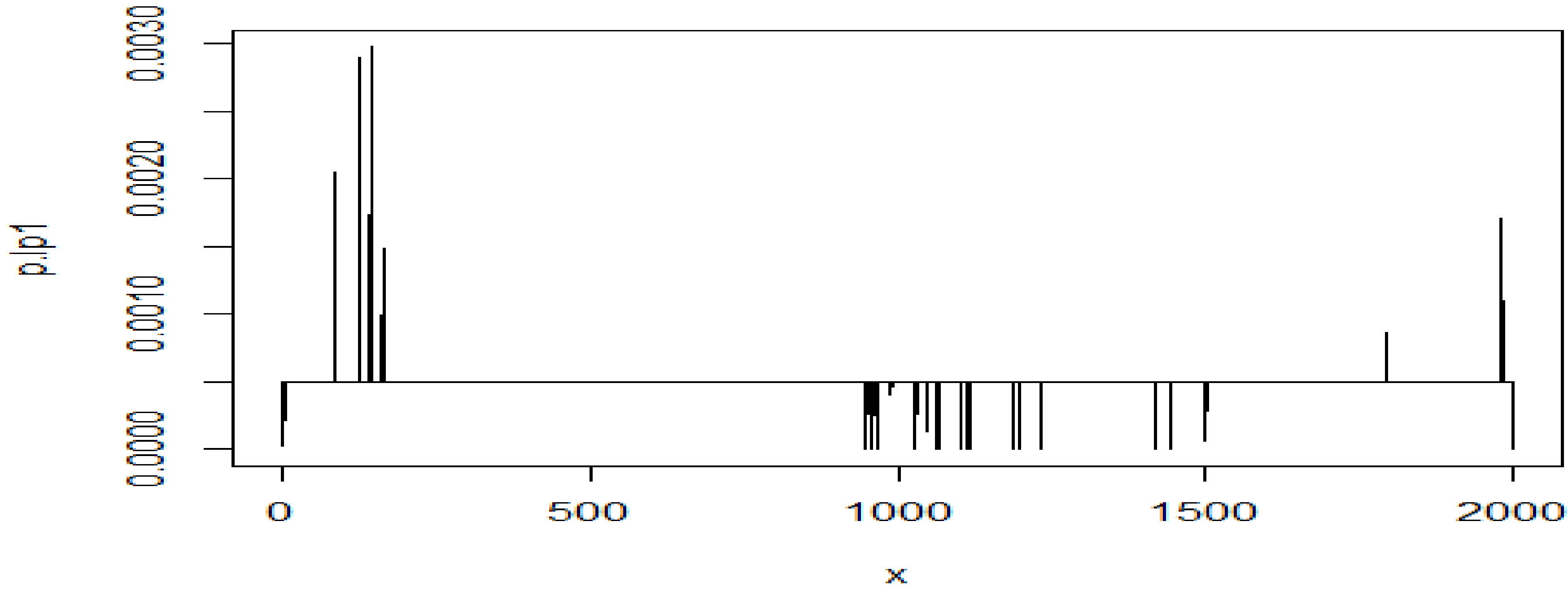
Comparison of Methods: LP0 RNP

- LP1 from scenario 200 to 400th



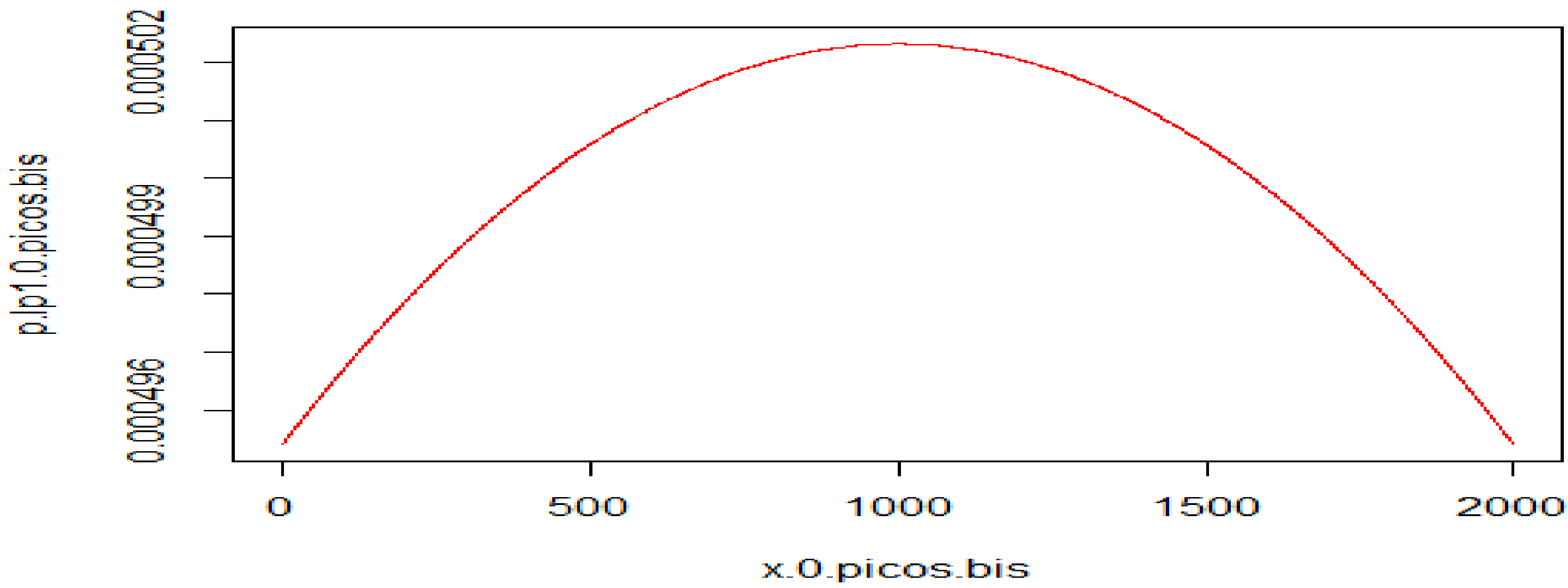
Comparison of Methods: LP1 RNP

- This is the RNP obtained in the case of normal prior q, general shape:



Comparison of Methods: LP1 RNP

- This is the RNP obtained in the normal prior q once it has been washed of 27 mass points



Comparison of Methods: Valuation

	Asset 5	Asset 17	Asset 49
Price_WMC	99.96132814	15.33394297	2.569520331
Price_LP0	99.96406714	15.34133482	2.574297005
Price_LP1	99.99897563	15.37663554	2.597898259
“True” Price	**100**	**15.36519723**	**2.582923106**
dev_WMC	-0.039%	-0.203%	-0.519%
dev_LP0	-0.036%	-0.155%	-0.334%
dev_LP1	-0.001%	0.074%	0.580%

Comparison of Methods: Maximum Entropies

- The entropies of the two RNPs have also been compared. It is supposed that the higher would be the entropy corresponding to the RNP the more efficient would be the "market". The entropy is given by:

$$H\left(\overline{x}\right) = \mathrm{E}\left\{I\left(\overline{x}\right)\right\} = E\left\{-\ln\left(p\left(\overline{x}\right)\right)\right\}$$

	LP1	LP0	WMC
Max. Entropy:	3.293645608	3.294127401	3.300511995

Following steps:

- Improving RNP calculations
- Pricing and hedging

Let us have a look here.

Pricing (PRIMAL):

$$\min_{\bar{x}=\left(\bar{p},\bar{y}^{+},\bar{y}^{-}\right)} \sum_{j=1}^{M} p_j e_j + \sum_{j=1}^{M}\left(y_j^{-} + y_j^{+}\right)$$

$$st:\begin{cases} p_j - \omega_j q_j + y_j^{-} - y_j^{+} = \bar{0},\ j=1,\ldots,M \\ G\bar{p} = \bar{c} \\ \bar{1}'\bar{p} = 1 \\ \bar{p} \geq 0 \end{cases}$$

$$\min_{\bar{x}=\left(\bar{p},\bar{y}^{\pm}\right)} z\left(\bar{p},\bar{y}^{+},\bar{y}^{-}\right) = \bar{k}\bar{x}$$

$$s.a:\begin{cases} \left(I_M, I_M, -I_M\right)\begin{pmatrix}\bar{p}\\ \bar{y}^{+}\\ \bar{y}^{-}\end{pmatrix} = \bar{q} \\ G\bar{p} = \bar{c} \\ \bar{1}'\bar{p} = 1 \\ \left(\bar{p},\bar{y}^{+},\bar{y}^{-}\right) \geq 0 \end{cases} \Leftrightarrow \begin{pmatrix} I_M & I_M & -I_M \\ D & 0 & 0 \\ \bar{1}'_M & 0 & 0 \end{pmatrix}\begin{pmatrix}\bar{p}\\ \bar{y}^{+}\\ \bar{y}^{-}\end{pmatrix} = \begin{pmatrix}\overline{\omega q}\\ \bar{c}\\ 1\end{pmatrix}$$

$$\bar{x} = \begin{pmatrix}\bar{p}\\ \bar{y}^{-}\\ \bar{y}^{+}\end{pmatrix} \in \mathbb{R}^{3M}$$

$$\bar{k} = \left(\bar{e}', \bar{1}'_M, \bar{1}'_M\right) \in M_{1\times 3M}$$

$$\bar{b} = \left(\left(\overline{\omega q}\right)', \bar{c}', 1\right) \in M_{1\times(M+N+1)}$$

$$A = \begin{pmatrix} 1 & \overset{(M)}{\cdots} & 0 & 1 & \overset{(M)}{\cdots} & 0 & -1 & \overset{(M)}{\cdots} & 0 \\ {}^{(M)}\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & -1 \\ G_{11} & \cdots & G_{1M} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ {}^{(N)}\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ G_{N1} & \cdots & G_{NM} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{(M+N+1)\times(3M)}$$

Hedging (DUAL)

$$\max_{\overline{w}} d\left(\overline{w}\right) = \overline{w}\overline{b}$$

$$st : \overline{w}A \leq \overline{k}$$

One dual variable for each primal constraint:

$$\overline{w} = \left(w_1, \ldots, w_M ; w_{M+1}, \ldots, w_{M+N} ; w_{M+N+1} \right) = \left(\overline{w}_p, \overline{w}_B, w_{M+N+1} \right) \in M_{1\times(M+N+1)}$$

$$\max_{\overline{w}} d\left(\overline{w}\right) = \overline{w} \begin{pmatrix} \left(\overline{\omega}\overline{q}\right)' \\ \overline{c}' \\ 1 \end{pmatrix} = q\sum_{j=1}^{M} w_j \omega_j + \sum_{i=1}^{N} c_i w_{M+i} + w_{M+N+1}$$

$$s.a : \left(\underset{1\times M}{\overline{w}_p}, \underset{1\times N}{\overline{w}_B}, \underset{1\times 1}{w_{M+N+1}} \right) \begin{pmatrix} \underset{M\times M}{I_M} & \underset{M\times M}{I_M} & \underset{M\times M}{-I_M} \\ \underset{N\times M}{G} & 0 & 0 \\ \underset{1\times M}{\overline{1}'}\ M & 0 & 0 \end{pmatrix} \leq \left(\overline{e}', \overline{1}'_M, \overline{1}'_M \right) \Leftrightarrow \left(\overline{w}_p I_M + \overline{w}_B G + w_{M+N+1}, \overline{w}_p I_M, \overline{w}_p \left(-I_M\right) \right) \leq \left(\overline{e}', \overline{1}'_M, \overline{1}'_M \right) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} w_1 + \overline{w}_B \overline{G}_{i1} + w_{M+N+1} \leq e_1 \\ \vdots \\ w_M + \overline{w}_B \overline{G}_{iM} + w_{M+N+1} \leq e_M \\ w_1 \leq 1 \\ \vdots \\ w_M \leq 1 \\ w_1 \geq -1 \\ \vdots \\ w_M \geq -1 \end{cases}$$

HEDGING (DUAL)

It is known that primal and dual optimal values * are equal. Therefore we can equal both objectives at their respectives optimae $\bar{x}^*, \bar{w}^*$:

$$z^* = z\left(\bar{p}^*, \bar{y}^{\pm *}\right) = d\left(\bar{w}^*\right) = d^* \Leftrightarrow \sum_{j=1}^{M} e_j p_j^* + \sum_{j=1}^{M}\left(y_j^{-*} + y_j^{+*}\right) = q\sum_{j=1}^{M} \omega_J w_j^* + \sum_{i=1}^{N} w_{M+i}^* c_i + w_{M+N+1}^*$$

$$\Leftrightarrow \sum_{j=1}^{M} e_j p_j^* \simeq \sum_{i=1}^{N} w_{M+i}^* c_i + q\sum_{j=1}^{M} \omega_J w_j^* + w_{M+N+1}^* - \sum_{j=1}^{M}\left(y_j^{-*} + y_j^{+*}\right) \qquad (1)$$

Illiquid Asset Value — Benchmarks Pofrtfolio — Difference

Let us suppose a change in one of the benchmark prices i_0=1,…,N.

Then the variation of the primal optimal value is given by its Kuhn-Tucker multiplier or dual variable w_{M+i_0}:

$$\frac{\partial z^*}{\partial c_{i_0}} = w_{M+i_0}^*$$

Therefore, to a unitary variation in one of the benchmark prices it would correspond a new replicated portfolio with a variation equal to w_{M+i_0} monetary units invested in the benchmark i_0.

Thank you very much for your attention