WEIGHTED ESTIMATES FOR BOCHNER-RIESZ OPERATORS ON LORENTZ SPACES

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Abstract. We present new estimates in the setting of weighted Lorentz spaces of operators satisfying a limited Rubio de Francia condition; namely T is bounded on $L^p(v)$ for every v in an strictly smaller class of weights than the Muckenhoupt class A_p. Important examples will be the Bochner-Riesz operators B_λ with $0 < \lambda < \frac{n-1}{2}$, sparse operators, Hörmander multipliers with limited regularity and rough operators with $\Omega \in L^r(\Sigma)$.

1. Introduction

Let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n,$$

be the Fourier transform of $f \in L^1(\mathbb{R}^n)$ and let $a_+ = \max\{a, 0\}$ denote the positive part of $a \in \mathbb{R}$. Given $\lambda > 0$, the Bochner-Riesz operator B_λ is defined by

$$\widehat{B_\lambda f}(\xi) = \left(1 - |\xi|^2\right)_+^\lambda \hat{f}(\xi).$$

These operators were first introduced by S. Bochner in [6] and, since then, they have been widely studied (see [7, 8, 9, 14, 22, 26, 40, 42, ...]). The case $\lambda = 0$ corresponds to the so-called disc multiplier, which is unbounded on $L^p(\mathbb{R}^n)$ if $n \geq 2$ and $p \neq 2$ (see [21]). When $\lambda > \frac{n-1}{2}$, it is known that $B_\lambda f$ is controlled by the Hardy-Littlewood maximal function Mf. As a consequence, all weighted inequalities for M are also satisfied by B_λ. The value $\lambda = \frac{n-1}{2}$ is called the critical index. In this case, X. Shi and Q. Sun [38] proved that $B_{\frac{n-1}{2}}$ is bounded on $L^p(v)$ for every $1 < p < \infty$ and $v \in A_p$. The weak-type inequality for $p = 1$ was first settled by M. Christ [16], who showed that $B_{\frac{n-1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and the corresponding weighted weak-type inequality was obtained by A. Vargas in [42], where she proved that $B_{\frac{n-1}{2}}$ is bounded from $L^1(v)$ to $L^{1,\infty}(v)$ for every $v \in A_1$.

Below the critical index, $0 < \lambda < \frac{n-1}{2}$, B_λ is not bounded on $L^p(\mathbb{R}^n)$ for the whole range $1 < p < \infty$. For instance, in dimension $n = 2$, L. Carleson and P. Sjölin [8] proved that B_λ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > 1$ and

$$\lambda > \max\left(2\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right),$$

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or equivalently, $0 < \lambda < \frac{1}{2}$ and

$$\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}.$$

Moreover, A. Seeger [37] showed that the corresponding weak-type inequality at the endpoint $B_\lambda : L^{\frac{4}{3+2\lambda}}(\mathbb{R}^n) \to L^{\frac{4}{3+2\lambda},\infty}(\mathbb{R}^n)$ also holds.

For higher dimensions, it is already well known that B_λ is not bounded on $L^p(\mathbb{R}^n)$ for $p \le \frac{2n}{n+1+2\lambda}$ or $p \ge \frac{2n}{n-1-2\lambda}$ (see for instance [19, Theorem 8.15] or [22, Proposition 5.2.3]). Furthermore, it was conjectured the following:

Conjecture 1.1 (Bochner-Riesz Conjecture)**.** *B_λ is bounded on $L^p(\mathbb{R}^n)$ if $p > 1$ and*

$$\lambda > \lambda(p) = \max\left(n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right),$$

or equivalently, $0 < \lambda < \frac{n-1}{2}$ and

$$\frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda}. \tag{1.1}$$

This has an equivalent formulation (see for instance [28]): let $\mathbb{1}_{[-1/4,1/4]} \le \chi \le \mathbb{1}_{[-1/2,1/2]}$ be a Schwartz function, and set S_τ to be the Fourier multiplier with symbol $\chi((|\xi| - 1)/\tau)$, $\tau > 0$.

Conjecture 1.2. *For $p > 1$ such that $n\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2}$,*

$$S_\tau : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \qquad 0 < \tau < 1,$$

with constant controlled by $C_\varepsilon \tau^{-\varepsilon}$ for all $0 < \varepsilon < 1$.

However, the Bochner-Riesz conjecture only has been partially answered and the best results known are currently due to Bourgain and Guth [7] (see also Lee [30]). We summarize it here: if $q = \max(p, p')$ satisfies

$$q > \begin{cases} \dfrac{2(4n+3)}{4n-3} & n \equiv 0 \,(\mathrm{mod}\ 3), \\ \dfrac{2n+1}{n-1} & n \equiv 1 \,(\mathrm{mod}\ 3), \\ \dfrac{4(n+1)}{2n-1} & n \equiv 2 \,(\mathrm{mod}\ 3), \end{cases}$$

then the Bochner-Riesz Conjecture holds for $0 < \lambda < \frac{n-1}{2}$ and p in the range of (1.1). Moreover, under the condition $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$, the corresponding weak-type inequality at the endpoint $B_\lambda : L^{\frac{2n}{n+1+2\lambda}} \to L^{\frac{2n}{n+1+2\lambda},\infty}$ was settled by M. Christ [15] and extended to $\lambda = \frac{n-1}{2(n+1)}$ by T. Tao [40], while it remains unknown for the range $0 < \lambda < \frac{n-1}{2(n+1)}$. This is the so called endpoint Bochner-Riesz conjecture (see for instance [41]) and we observe that if it holds for some λ, then by duality and interpolation the Bochner-Riesz conjecture holds for such λ as well.

Now, let us consider for a positive and locally integrable function w in $\mathbb{R}^+$ the primitive $W(t) = \int_0^t w(s)ds$. Given an operator T, let us define the class

$$\mathcal{W}(T) = \left\{0 \le w \in L^1_{\mathrm{loc}}(\mathbb{R}^+) : (Tf)^*(t)W(t) \le C_w \int_0^\infty f^*(s)w(s)ds, \, \forall t > 0, \, \forall f \in \mathcal{M}(\mathbb{R}^n)\right\}$$

and its corresponding norm $\|w\|_{\mathcal{W}(T)}$ to be the infimum of all constants C_w. Then, one can immediately see that

$$w(s) = s^{\frac{1-n+2\lambda}{2n}} \in \mathcal{W}(B_\lambda) \qquad \Longrightarrow \qquad \text{Bochner-Riesz conjecture holds for } B_\lambda.$$

The original motivation of this paper was to study the class $\mathcal{W}(T)$ not only for the Bochner-Riesz operators but also for other interesting operators in harmonic analysis having in common some properties connected with weighted estimates; namely,

$$T : L^p(v) \longrightarrow L^{p,\infty}(v),$$

for every v in a certain subclass of the Muckenhoupt weights A_p.

In particular, and as a consequence of our main results, we will obtain boundedness on weighted Lorentz spaces $\Lambda^p(w)$ defined by

$$\Lambda^p(w) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) : ||f||_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t)\, dt\right)^{1/p} < \infty \right\}.$$

Continuing with the boundedness of the operators B_λ, let us mention that, above the critical index it is reduced to the study of the Hardy-Littlewood maximal operator over these spaces (see [3]), while for the critical index $\lambda = \frac{n-1}{2}$ this was done in [4]. So we will concentrate in values for λ ranging below the critical index.

In two recent papers [26, 28], new weighted estimates for B_λ have been proved using the fact that the Bochner-Riesz operators can be dominated by sparse type operators. As far as we know, these are the best weighted estimates known together with the results for the (2,2)-strong type inequality in [9, 14]. We summarize the results here.

Definition 1.3. *Given $0 \le \alpha, \beta \le 1$ and $1 \le p < \infty$, let us define the class of weights*

$$A_{p;(\alpha,\beta)} = \{0 < v \in L^1_{loc}(\mathbb{R}^n) : v = v_0^\alpha v_1^{\beta(1-p)}, v_j \in A_1\}.$$

Remark 1.4. *We observe that*

$$v \in A_{p;(\alpha,\beta)} \qquad \Longrightarrow \qquad v \in A_{1+\beta(p-1)}$$

Let us start stating the result for dimension $n = 2$ where the conjecture has completely been solved.

Proposition 1.5. *Let $n = 2$ and $0 < \lambda < 1/2$.*

(1) [9, 28] *For every $\frac{4}{3} \le q_0 \le 4$,*

$$B_\lambda : L^{q_0}(v) \longrightarrow L^{q_0}(v), \qquad v \in A_{q_0;(2\lambda,2\lambda)}.$$

(2) [26, Corollary 1.3]

$$B_\lambda : L^{\frac{4}{3+2\lambda}}(v) \to L^{\frac{4}{3+2\lambda},\infty}(v), \qquad v \in A_{\frac{4}{3+2\lambda};\left(\frac{2\lambda}{3+2\lambda},0\right)} \ (i.e.\ v^{\frac{3+2\lambda}{2\lambda}} \in A_1).$$

In particular, in this case, the operator norm is bounded by $c(n,\lambda)\,\|v\|_{A_1}^{\frac{7+4\lambda}{4}}$.

For $n > 2$ we have the following result.

Proposition 1.6. *Let* $0 < \lambda < \frac{n-1}{2}$.

(1) [9, 28] *If* $q_0 = 2$ *or* q_0 *is such that* $\frac{2n}{n+1} < q_0 < 2$ *and Conjecture 1.2 holds with* $p = q_0$,

$$B_\lambda : L^{q_0}(v) \longrightarrow L^{q_0}(v), \qquad v \in A_{q_0;\left(\frac{2\lambda}{n-1},\frac{2\lambda}{n-1}\right)}.$$

And, by duality,

$$B_\lambda : L^{q_0'}(v) \longrightarrow L^{q_0'}(v), \qquad v \in A_{q_0';\left(\frac{2\lambda}{n-1},\frac{2\lambda}{n-1}\right)}.$$

(2) [14] *If* $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$, *then*

$$B_\lambda : L^2(v) \longrightarrow L^2(v), \qquad \forall v \in A_{2;\left(\frac{1+2\lambda}{n},0\right)} \cup A_{2;\left(0,\frac{1+2\lambda}{n}\right)}.$$

This type of weighted inequalities are also satisfied by other operators such as the Hörmander Fourier multiplier $m \in M(s,l)$ with $l < n$ (see [27]), among many others. And this is the starting point of this paper. We want to use the information about weighted estimates on Lebesgue spaces to study the class $\mathcal{W}(T)$ and, as a consequence, to obtain boundedness on weighted Lorentz spaces. To this end, we need to adapt the ideas in [4] where the case $\lambda = \frac{n-1}{2}$ (for the Bochner-Riesz operator) and the case $l = n$ (for the Hörmander multiplier) were considered and the following result was proved (see definitions in Section 2): if T satisfies that

$$T : L^1(v) \to L^{1,\infty}(v), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi(||v||_{A_1})$, with φ an increasing function on $(0,\infty)$, then,

$$B_1^{\mathcal{R}} \cap B_\infty^* \subset \mathcal{W}(T)$$

with $||w||_{\mathcal{W}(T)} \le C_1 ||w||_{B_1^{\mathcal{R}}} \varphi\left(C_2 ||w||_{B_\infty^*}\right)$ for some universal constants $C_1, C_2 > 0$.

We write $A \lesssim B$ if there exists a positive constant $C > 0$, independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

Definition 1.7. *We say that* f *is quasi-decreasing, and we denote by* $f \approx\downarrow$, *if*

$$f(t) \lesssim f(s), \qquad 0 < s \le t.$$

If $-f$ *is quasi-decreasing we say that* f *is quasi-increasing and we denote it by* $f \approx\uparrow$.

The first main result is the following:

Theorem 1.8. *Let* $1 < p_0 < \infty$, $0 \le \alpha \le 1$, $0 < \beta \le 1$ *and let* T *be a sublinear operator such that*

$$T : L^{p_0}(v) \longrightarrow L^{p_0,\infty}(v), \qquad \forall v \in A_{p_0;(\alpha,\beta)},$$

with constant less than or equal to $\varphi(||v||_{A_{1+\beta(p_0-1)}})$*, with* φ *an increasing function on* $(0,\infty)$*. Set*

$$p_+ = \frac{p_0}{1-\alpha}, \qquad p_-' = \frac{p_0'}{1-\beta}.$$

Then,

$$B_{\frac{1}{p_-}} \cap B^*_{p_+} \subset \mathcal{W}(T),$$

or equivalently, if there exists some $\varepsilon > 0$ *so that*

$$\frac{W(t)}{t^{\frac{1}{p_-}-\varepsilon}} \approx\downarrow \quad \text{and} \quad \frac{W(t)}{t^{\frac{1}{p_+}+\varepsilon}} \approx\uparrow,$$

then $w \in \mathcal{W}(T)$*. Moreover, it holds that*

$$\|w\|_{\mathcal{W}(T)} \leq \Phi(\|w\|_{B_{\frac{1}{p_-}}}, \|w\|_{B^*_{p_+}}),$$

where Φ *is an increasing function in each variable depending on* φ*.*

By taking $T = M\circ M$ with M the Hardy littlewood maximal operator, it is clear that the above result is false with $\varepsilon = 0$ since T satisfies the hypothesis of our theorem with $\alpha = 1$ and $\beta = 1$ but it is not of weak type $(1,1)$.

Remark 1.9. *Recall that a weight* v *belongs to the reverse Hölder class* RH_s*,* $1 < s < \infty$*, if for every measurable cube* $Q \subset \mathbb{R}^n$*,*

$$\left(\frac{1}{|Q|}\int_Q v^s\right)^{1/s} \lesssim \frac{1}{|Q|}\int_Q v.$$

It is known that $v \in A_p \cap RH_s$ *if and only if* $v^s \in A_r$ *with* $r = s(p-1)+1$ *(see for instance* [24, (P6)]*) so that when* $\alpha > 0$*,*

$$A_{p_0;(\alpha,\beta)} = A_{\frac{p_0}{p_-}} \cap RH_{\left(\frac{p_+}{p_0}\right)'},$$

which keeps some symmetry with the class of weights $B_{\frac{1}{p_-}} \cap B^*_{p_+}$*.*

Corollary 1.10. *Let* $0 < p < \infty$*. Under the hypotheses of Theorem 1.8, if there exists some* $\varepsilon > 0$ *so that*

$$\frac{W(t)}{t^{\frac{p}{p_-}-\varepsilon}} \approx\downarrow \quad \text{and} \quad \frac{W(t)}{t^{\frac{p}{p_+}+\varepsilon}} \approx\uparrow, \tag{1.2}$$

we have that

$$T : \Lambda^p(w) \longrightarrow \Lambda^p(w).$$

Our next main result reads as follows:

Theorem 1.11. *Let* $1 \leq p_0 < \infty$*,* $0 < \alpha \leq 1$*, and let* T *be a sublinear operator such that*

$$T : L^{p_0,1}(v^\alpha) \longrightarrow L^{p_0,\infty}(v^\alpha), \qquad \forall v \in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$*, where* φ *is an increasing function. Then,*

$$B^{\mathcal{R}}_{\frac{1}{p_0}} \cap B^*_{\frac{p_0}{1-\alpha}} \subset \mathcal{W}(T),$$

or equivalently, if there exists some $\varepsilon > 0$ *so that*

$$\frac{W(t)}{t^{\frac{1}{p_0}}} \approx\downarrow \quad \textit{and} \quad \frac{W(t)}{t^{\frac{1-\alpha}{p_0}+\varepsilon}} \approx\uparrow,$$

then $w \in \mathcal{W}(T)$. *Moreover,*

$$\|w\|_{\mathcal{W}(T)} \lesssim \|w\|_{B^{\mathcal{R}}_{\frac{1}{p_0}}}^{\frac{1}{p_0}} \begin{cases} \varphi\left(C\,\|w\|_{B^*_{\frac{p_0}{1-\alpha}}}^{\frac{p_0}{\alpha}}\right), & 0<\alpha<1, \\ \varphi\left(C\,\|w\|_{B^*_\infty}\right), & \alpha = 1. \end{cases}$$

Corollary 1.12. *Let* $0 < p < \infty$ *and let us assume the hypotheses of Theorem 1.11.*
a) If

$$\frac{W(t)}{t^{\frac{p}{p_0}}} \approx\downarrow \quad \textit{and} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{p(1-\alpha)}{p_0}+\varepsilon}} \approx\uparrow,$$

then

$$T : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w).$$

b) If

$$\exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{p}{p_0}-\varepsilon}} \approx\downarrow \quad \textit{and} \quad \frac{W(t)}{t^{\frac{p(1-\alpha)}{p_0}+\varepsilon}} \approx\uparrow,$$

then

$$T : \Lambda^{p}(w) \longrightarrow \Lambda^{p}(w).$$

Remark 1.13. *Although Theorems 1.8 and 1.11 are stated with* T *being a sublinear operator, which is enough for the applications that we present in Section 4, they can be proved without this hypothesis. In fact, we only need this condition when applying interpolation in Corollaries 1.10 and 1.12.*

The paper is organized as follows. In Section 2 we present some technical lemmas and previous results which shall be used later on, and Section 3 contains our main results. The last section will be devoted to obtain estimates in the setting of weighted Lorentz spaces for the Bochner-Riesz operators in the critical range, among others, such as sparse operators, Hörmander multipliers with limited regularity and rough operators with $\Omega \in L^r(\Sigma)$.

2. Definitions, previous results and some technical lemmas

Given $0 < p, q < \infty$, w a positive locally integrable function defined on $\mathbb{R}^+$ and $W(t) = \int_0^t w(r)\,dr$, the weighted Lorentz space $\Lambda^{p,q}(w)$ is defined by the condition $||f||_{\Lambda^{p,q}(w)} < \infty$ where

$$||f||_{\Lambda^{p,q}(w)} = \begin{cases} \left(\displaystyle\int_0^\infty f^*(t)^q W(t)^{\frac{q}{p}-1} w(t)dt\right)^{\frac{1}{q}}, & q < \infty, \\ \displaystyle\sup_{t>0} f^*(t) W(t)^{\frac{1}{p}}, & q = \infty, \end{cases}$$

where λ_f and f^* are, respectively, the distribution function and the decreasing rearrangement of f defined by

$$\lambda_f(t) := |\{|f| > t\}|, \qquad f^*(t) := \inf\{y > 0 : \lambda_f(y) \leq t\}.$$

Observe that if $w = 1$, $\Lambda^{p,q}(w)$ coincides with the more usual case $L^{p,q}$. We should also emphasize here that for $0 < q < \infty$,

$$\Lambda^{p,q}(w) = \Lambda^{q}(\widetilde{w}), \qquad \widetilde{W}(t) \approx W(t)^{\frac{q}{p}}.$$

and, similar, $\Lambda^{p,\infty}(w) = \Lambda^{1,\infty}(\tilde{w})$, with $\tilde{W}(t) \approx W(t)^{\frac{1}{p}}$. Besides, these spaces satisfy the embeddings

$$\Lambda^{p,q_0}(w) \hookrightarrow \Lambda^{p,q_1}(w)$$

continuously for $0 < q_0 \le q_1 \le \infty$. For further information about these notions and related topics see [5, 12]. Therefore, with this notation,

$$w \in \mathcal{W}(T) \qquad \Longleftrightarrow \qquad T : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w),$$

with

$$||w||_{\mathcal{W}(T)} = ||T||_{\Lambda^1(w)\to\Lambda^{1,\infty}(w)} = \sup_{\|f\|_{\Lambda^1(w)}\le 1} \|Tf\|_{\Lambda^{1,\infty}(w)}.$$

Let us consider the Hardy-Littlewood maximal operator M, defined for locally integrable functions on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x. It is known [34] that for every $1 < p < \infty$,

$$M : L^p(v) \longrightarrow L^p(v) \qquad \Longleftrightarrow \qquad v \in A_p,$$

where $v \in A_p$ if v is a positive and locally integrable function in $\mathbb{R}^n$ (i.e., v is a weight in $\mathbb{R}^n$) such that

$$||v||_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q v(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{\frac{1}{1-p}}dx\right)^{p-1} < \infty.$$

Moreover, if $1 \le p < \infty$,

$$M : L^p(v) \longrightarrow L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_p,$$

where $v \in A_1$ if v is a weight in $\mathbb{R}^n$ such that

$$||v||_{A_1} = \left\|\frac{Mv}{v}\right\|_\infty < \infty.$$

Also, in the context of restricted weak type inequalities, the following result was proved in [17, 25]:

$$M : L^{p,1}(v) \longrightarrow L^{p,\infty}(v) \qquad \Longleftrightarrow \qquad v \in A_p^{\mathcal{R}},$$

where a weight $v \in A_p^{\mathcal{R}}$ if

$$\|v\|_{A_p^{\mathcal{R}}} = \sup_Q \frac{1}{|Q|}\|\chi_Q\|_{L^p(v)}\|\chi_Q v^{-1}\|_{L^{p',\infty}(v)} < \infty.$$

Moreover,

$$\|M\|_{L^{p,1}(v)\to L^{p,\infty}(v)} \lesssim \|v\|_{A_p^{\mathcal{R}}}.$$

Concerning the boundedness of M on weighted Lorentz spaces we have (see [3]) that for every $p > 0$,

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff w \in B_p,$$

where

$$||w||_{B_p} = \sup_{t>0} \frac{\int_0^\infty \min(1, \frac{t}{r})^p w(r)dr}{W(t)}$$

and

$$\|M\|_{\Lambda^p(w)} := \|M\|_{\Lambda^p(w)\to\Lambda^p(w)} \lesssim \|w\|_{B_p}.$$

Also, since for $\tilde{W}(t) \approx W(t)^{\frac{1}{p}}$,

$$M : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w) \quad \iff \quad M : \Lambda^1(\tilde{w}) \to \Lambda^{1,\infty}(\tilde{w}),$$

and we have that ([10, 13])

$$M : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w) \quad \iff \quad w \in B_p^{\mathcal{R}},$$

where $w \in B_p^{\mathcal{R}}$ is defined by

$$||w||_{B_p^{\mathcal{R}}} = \sup_{0<s\le t<\infty} \frac{sW(t)^{1/p}}{tW(s)^{1/p}} < \infty.$$

Further,

$$||M||_{\Lambda^{p,1}(w)\longrightarrow\Lambda^{p,\infty}(w)} \lesssim ||w||_{B_p^{\mathcal{R}}}.$$

For $0 < q < p < \infty$, it is easy to see that $B_q^{\mathcal{R}} \subsetneq B_p \subsetneq B_p^{\mathcal{R}}$ with $\|w\|_{B_p^{\mathcal{R}}} \lesssim \|w\|_{B_p}^{1/p} \le \frac{p}{p-q} \|w\|_{B_q^{\mathcal{R}}}^{q/p}$. Indeed, it is known that ([3, 35]) for $0 < p < \infty$,

$$w \in B_p \iff \exists \varepsilon > 0 : \frac{W(t)}{t^{p-\varepsilon}} \approx\downarrow. \tag{2.1}$$

Also, we say that $w \in \Delta_2$ if there exists $C > 0$ such that $W(2r) \le CW(r)$, for all $r > 0$ and it holds that, for every $0 < p < \infty$, $B_p^{\mathcal{R}} \subsetneq \Delta_2$, and if $w \in \Delta_2$, $\Lambda^p(w)$ is a quasi-Banach r.i. function space (see [12]).

Let us recall now, that given a r.i. space $\mathbb{X}$, the associate space $\mathbb{X}'$ is defined by the condition

$$||f||_{\mathbb{X}'} = \sup_{||g||_{\mathbb{X}}\le 1} \int_0^\infty f^*(t)g^*(t)dt < \infty.$$

To prove our main results we need several estimates on the boundedness of the operators $M_\alpha f = M\left(f^{1/\alpha}\right)^\alpha$, $0 < \alpha \le 1$, on the corresponding associate spaces $(\Lambda^p(w))'$. To this end, the following classes of weights are going to be fundamental.

Definition 2.1 ([36]). *A weight $w \in B_\infty^*$ if and only if*

$$||w||_{B_\infty^*} = \sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(r)}{r}\, dr < \infty.$$

Definition 2.2 ([29, 36]). *Let $0 < p < \infty$. A weight $w \in B_p^*$ if and only if*

$$||w||_{B_p^*} = \sup_{t>0} \frac{1}{W(t)} \int_0^t \left(\frac{t}{r}\right)^{1/p} w(r)\, dr < \infty.$$

It is known that ([29, 33]) for every $0 < p < \infty$,

$$w \in B_p^* \iff \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{1}{p}+\varepsilon}} \approx\uparrow . \tag{2.2}$$

Besides, direct from the definition, for $0 < q \le p < \infty$,

$$B_q^* \subseteq B_p^* \subseteq B_\infty^*,$$

and, for every $0 < p < \infty$,

$$B_p^* = \bigcup_{0<q<p} B_q^*. \tag{2.3}$$

Indeed, (2.3) also holds for $p = \infty$:

Lemma 2.3.

$$w \in B_\infty^* \iff \exists \varepsilon > 0 : \frac{W(t)}{t^{\varepsilon}} \approx\uparrow .$$

Proof. Since $B_p^* \subseteq B_\infty^*$, by (2.2), it is enough to see that if $w \in B_\infty^*$, then there exists some $p > 0$ such that $w \in B_p^*$. Let $\overline{W} : (0,\infty) \to (0,\infty)$ be defined as

$$\overline{W}(\lambda) := \sup\left\{ \frac{W(t)}{W(s)} : 0 < t \le \lambda s \right\} = \sup_{x\in[0,\infty)} \frac{W(\lambda x)}{W(x)}.$$

It was seen ([1, Lemma 2.6]) that $w \in B_\infty^*$ is equivalent to the existence of some $\mu \in (0,1)$ such that $\overline{W}(\mu) < 1$. What's more, arguing similar as in [1, Lemma 2.7], it can be seen that $w \in B_p^*$ is equivalent to the existence of some $\mu \in (0,1)$ such that $\overline{W}(\mu) < \mu^{1/p}$. Therefore, given $w \in B_\infty^*$ and taking p great enough, it holds that $\overline{W}(\mu) < 1$ implies $\overline{W}(\mu) < \mu^{1/p}$. □

Hence, from Lemma 2.3, (2.1), (2.2) and the definition of $B_p^{\mathcal{R}}$, it follows immediately that for every $0 < p < \infty$, $0 < q \le \infty$,

$$w \in B_{\frac{1}{p}} \cap B_q^* \iff \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{1}{p}-\varepsilon}} \approx\downarrow \text{ and } \frac{W(t)}{t^{\frac{1}{q}+\varepsilon}} \approx\uparrow,$$

and

$$w \in B_{\frac{1}{p}}^{\mathcal{R}} \cap B_q^* \iff \frac{W(t)}{t^{\frac{1}{p}}} \approx\downarrow \text{ and } \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{1}{q}+\varepsilon}} \approx\uparrow .$$

Example 2.4. *a) Let $0 < p < \infty$ and take $w(r) = r^{\gamma-1}$. Then,*

$$w \in B_p \iff \gamma < p \quad \textit{and} \quad w \in B_p^{\mathcal{R}} \iff \gamma \le p$$

with $\|w\|_{B_p} = \frac{p}{p-\gamma}$ and $\|w\|_{B_p^{\mathcal{R}}} = 1$. Moreover,

$$w \in B_p^* \iff \gamma > \frac{1}{p} \quad \textit{and} \quad w \in B_\infty^* \iff \gamma > 0$$

with $\|w\|_{B_p^*} = \frac{\gamma}{\gamma-\frac{1}{p}}$ and $\|w\|_{B_\infty^*} = \frac{1}{\gamma}$.

b) Let $1 \le p < q < \infty$ and set, for every $m \in \mathbb{N}$,

$$w_m(r) = \left(1 + \log_+ \frac{1}{r}\right)^m r^{\frac{1}{p}-1}.$$

By induction on m, it is easy to see that $w_m \in B_p^{\mathcal{R}} \cap B_q^*$ with

$$\|w_m\|_{B^{\mathcal{R}}_{\frac{1}{p}}} = 1 \quad \text{and} \quad \|w_m\|_{B_q^*} \lesssim \left[\left(\frac{pq}{q-p}\right)\left(\frac{m+1}{m}\right)\right]^m (m+1)!.$$

c) Similarly, let $1 \le p < q < \infty$ and set, for every $m \in \mathbb{N}$,

$$\tilde{w}_m(r) = \big(1 + \log_+ r\big)^{-m} r^{\frac{1}{p}-1}.$$

Then, $\tilde{w}_m \in B_p^{\mathcal{R}} \cap B_q^*$ with

$$\|\tilde{w}_m\|_{B^{\mathcal{R}}_{\frac{1}{p}}} \lesssim 1 \quad \text{and} \quad \|\tilde{w}_m\|_{B_q^*} \lesssim \left[\frac{pq}{q-p}\right]^m (m+1)^{m+1}.$$

Lemma 2.5. *Let $0 < p \le 1$. If $w \in B_p^{\mathcal{R}}$, then for every measurable set $E \subseteq \mathbb{R}^n$,*

$$||\chi_E||_{(\Lambda^p(w))'} \le \|w\|_{B_p^{\mathcal{R}}} \frac{|E|}{W(E)^{1/p}}.$$

Proof. It is known [12, Theorem 2.4.7] that

$$||h||_{(\Lambda^p(w))'} = \sup_{t>0} \frac{\int_0^t h^*(s)ds}{W(t)^{\frac{1}{p}}},$$

and hence

$$||\chi_E||_{(\Lambda^p(w))'} = \sup_{t>0} \frac{\min(t,|E|)}{W(t)^{\frac{1}{p}}} \le ||w||_{B_p^{\mathcal{R}}} \frac{|E|}{W^{1/p}(|E|)}.$$

□

Proposition 2.6 ([2, 4]). *For every $0 < p < \infty$,*

$$||M||_{(\Lambda^p(w))'} \lesssim ||w||_{B_\infty^*} \qquad \text{and} \qquad ||M||_{(\Lambda^{p,1}(w))'} \lesssim ||w||_{B_\infty^*}.$$

From the above results we are now ready to estimate the boundedness of M_α, $0 < \alpha < 1$, on the associate spaces $(\Lambda^p(w))'$ for $0 < p \le 1$. To do so, we need this technical lemma:

Lemma 2.7. *Let $0 < p \le 1$, $0 < \alpha < 1$ and*

$$V(t) = \inf_{r>0} \left\{ \max\left(\frac{t}{r}, 1\right) W(r)^{\frac{1}{p}} \right\}.$$

If

$$A_V := \sup_{t>0} \frac{1}{V(t)} \int_0^t \left(\frac{1}{s}\int_0^s \left[\frac{V(r)}{r}\right]^{1/\alpha} dr\right)^\alpha ds < \infty,$$

then

$$||M_\alpha f||_{(\Lambda^p(w))'} \lesssim A_V ||f||_{(\Lambda^p(w))'}.$$

Moreover, if $M_\alpha : (\Lambda^p(w))' \to (\Lambda^p(w))'$, and $w \in \Delta_2$, then $A_V < \infty$.

Proof. Since

$$||h||_{(\Lambda^p(w))'} = \sup_{t>0} \frac{1}{W(t)^{\frac{1}{p}}} \int_0^t h^*(s)\, ds$$

and $\int_0^t h^*(s)\, ds$ is a concave function, we have that

$$\int_0^t h^*(s)\, ds \le ||h||_{(\Lambda^p(w))'} W(t)^{\frac{1}{p}} \iff \int_0^t h^*(s)\, ds \le ||h||_{(\Lambda^p(w))'} V(t),$$

and hence

$$||h||_{(\Lambda^p(w))'} = \sup_{t>0} \frac{1}{V(t)} \int_0^t h^*(s)\, ds = ||h||_{(\Lambda^p(dV))'}.$$

First assume that $A_V < \infty$. Then,

$$\begin{aligned} ||M_\alpha f||_{(\Lambda^p(w))'} &= \sup_{t>0} \frac{1}{V(t)} \int_0^t \Big((M(f^{1/\alpha}))^*(s) \Big)^\alpha\, ds \\ \approx\ & \sup_{t>0} \frac{1}{V(t)} \int_0^t \left(\frac{1}{s} \int_0^s f^*(r)^{1/\alpha}\, dr \right)^\alpha ds \le \sup_{t>0} \frac{1}{V(t)} \int_0^t \left(\frac{1}{s} \int_0^s f^{**}(r)^{1/\alpha}\, dr \right)^\alpha ds \\ \le\ & A_V ||f||_{(\Lambda^p(dV))'} = A_V ||f||_{(\Lambda^p(w))'}. \end{aligned}$$

Conversely, the first observation is that since $Mf \le M(f^{1/\alpha}))^\alpha$, the boundedness of M_α implies the boundedness of M and hence, since $w \in \Delta_2$ (so that $(\Lambda^p(w))'$ is a Banach space [12]) we already know that the derivative of V must satisfy $dV \in B^*_\infty$ ([4]).

Now, since V is concave, the function $f(r) = \dfrac{V(r)}{r}$ is decreasing and hence, if we take $g(x) = f(C_n|x|^n)$, $x \in \mathbb{R}^n$, where C_n is the volume of the unit sphere in $\mathbb{R}^n$, we have that $g^* = f$ and thus

$$\begin{aligned} A_V &= \sup_{t>0} \frac{1}{V(t)} \int_0^t \left(\frac{1}{s} \int_0^s \left(\frac{V(r)}{r} \right)^{1/\alpha} dr \right)^\alpha ds \approx \left\| \Big(M(g^{1/\alpha}) \Big)^\alpha \right\|_{(\Lambda^p(w))'} \\ &\lesssim ||g||_{(\Lambda^p(w))'} = \sup_{t>0} \frac{1}{V(t)} \int_0^t \frac{V(r)}{r}\, dr \le ||dV||_{B^*_\infty} < \infty. \end{aligned}$$

□

Lemma 2.8. *Let* $0 < p \le 1$ *and* $0 < \alpha < 1$. *If* $w \in B^*_{\frac{1}{(1-\alpha)p}}$, *then*

$$||M_\alpha f||_{(\Lambda^p(w))'} \lesssim \frac{\alpha}{(1-\alpha)^{\frac{p+1}{p}}} \|w\|^{1/p}_{B^*_{\frac{1}{(1-\alpha)p}}} ||f||_{(\Lambda^p(w))'}.$$

Proof. Observe that, by Lemma 2.7, it is enough to see that

$$A_V \lesssim \frac{\alpha}{(1-\alpha)^{\frac{p+1}{p}}} \|w\|^{1/p}_{B^*_{\frac{1}{(1-\alpha)p}}},$$

and we have seen in the proof of Lemma 2.7, that

$$A_V \approx \sup_{t>0} \frac{1}{W(t)^{\frac{1}{p}}} \int_0^t \left(\frac{1}{s} \int_0^s \left[\frac{V(r)}{r} \right]^{1/\alpha} dr \right)^\alpha ds.$$

Now take $s > 0$. By the Minkowski's inequality applied with the exponent $\dfrac{1}{\alpha} > 1$,

$$\begin{aligned}\left(\frac{1}{s}\int_0^s \left[\frac{V(r)}{r}\right]^{1/\alpha} dr\right)^{\alpha} &= \left(\frac{1}{s}\int_0^s \left(\int_0^{\frac{V(r)}{r}} dx\right)^{1/\alpha} dr\right)^{\alpha}\\ &\le \frac{1}{s^{\alpha}}\int_0^\infty \left(\int_0^\infty \chi_{(0,s)}(r)\chi_{\left(0,\frac{V(r)}{r}\right)}(x)\,dr\right)^{\alpha} dx \qquad (2.4)\\ &= \frac{1}{s^{\alpha}}\int_0^\infty \lambda^{\alpha}_{\chi_{(0,s)}\frac{V}{\cdot}}(x)\,dx = \frac{\alpha}{s^{\alpha}}\int_0^s \frac{V(r)}{r}\,\frac{dr}{r^{1-\alpha}}.\end{aligned}$$

Therefore, using that $V(t) \le W(t)^{\frac{1}{p}}$ and the Minkowski's inequality again with exponent $\dfrac{1}{p} \ge 1$, we obtain that

$$\begin{aligned}&\frac{1}{W(t)^{\frac{1}{p}}}\int_0^t \left(\frac{1}{s}\int_0^s \left[\frac{V(r)}{r}\right]^{1/\alpha} dr\right)^{\alpha} ds \le \frac{1}{W(t)^{\frac{1}{p}}}\int_0^t \frac{\alpha}{s^{\alpha}}\int_0^s \frac{W(r)^{\frac{1}{p}}}{r}\,\frac{dr}{r^{1-\alpha}}\,ds\\ \le\;& \frac{\alpha}{1-\alpha}\left(\frac{t^{1-\alpha}}{W(t)^{\frac{1}{p}}}\int_0^t \frac{W(r)^{\frac{1}{p}}}{r}\,\frac{dr}{r^{1-\alpha}}\right) \le \frac{1}{(1-\alpha)^{1/p}}\left(\frac{1}{W(t)}\int_0^t \left(\frac{t}{r}\right)^{(1-\alpha)p} w(r)\,dr\right)^{1/p}\\ \le\;& \frac{1}{(1-\alpha)^{1/p}}\,\|w\|^{1/p}_{B^*_{\frac{1}{(1-\alpha)p}}},\end{aligned}$$

and the result follows. □

Lemma 2.9. *Let $0 < p \le 1$ and $0 < \alpha < 1$. If $w \in B^*_{\frac{1}{(1-\alpha)p}}$, then*

$$\|M_\alpha f\|_{(\Lambda^{p,1}(w))'} \lesssim \frac{\alpha}{(1-\alpha)^{\frac{1+3p}{p}}}\,\|w\|^{\frac{1}{p}}_{B^*_{\frac{1}{(1-\alpha)p}}}\,\|f\|_{(\Lambda^{p,1}(w))'}\,.$$

Proof. Recall that $\Lambda^{p,1}(w) = \Lambda^1(\tilde{w})$ with $\tilde{w} = W^{1/p-1}w$. Therefore, by Lemma 2.8

$$\|M_\alpha\|_{(\Lambda^{p,1}(w))'} = \|M_\alpha\|_{(\Lambda^1(\tilde{w}))'} \le \frac{\alpha}{(1-\alpha)^2}\,\|\tilde{w}\|_{B^*_{\frac{1}{1-\alpha}}}\,.$$

Now, integrating by parts and using the Minkowski's inequality we obtain that

$$\begin{aligned}&\int_0^t \left(\frac{t}{r}\right)^{1-\alpha} \tilde{w}(r)\,dr = \int_0^t \left(\frac{t}{r}\right)^{1-\alpha} W(r)^{1/p-1}w(r)\,dr\\ \le\;& pW(t)^{\frac{1}{p}} + p\frac{t^{1-\alpha}}{1-\alpha}\int_0^t \frac{W(r)^{\frac{1}{p}}}{r^{1-\alpha}}\,\frac{dr}{r}\\ \le\;& pW(t)^{\frac{1}{p}} + \frac{p}{(1-\alpha)^{1+\frac{1}{p}}}\left(\int_0^t \left(\frac{t}{r}\right)^{(1-\alpha)p} w(r)\,dr\right)^{1/p}\\ \le\;& pW(t)^{\frac{1}{p}} + \frac{p}{(1-\alpha)^{1+\frac{1}{p}}}\,\|w\|^{1/p}_{B^*_{\frac{1}{(1-\alpha)p}}}\,W(t)^{\frac{1}{p}},\end{aligned}$$

and the result follows. ☐

Definition 2.10. *Given* $1 \leq p_0 < \infty$ *and* $0 \leq \alpha, \beta \leq 1$*, set*

$$p_+ = \frac{p_0}{1-\alpha}, \quad p'_- = \frac{p'_0}{1-\beta},$$

and, for every $p \in (p_-, p_+)$*, take* $0 \leq \alpha(p), \beta(p) \leq 1$ *such that*

$$p_+ = \frac{p}{1-\alpha(p)}, \quad p'_- = \frac{p'}{1-\beta(p)}.$$

To end this section, we present an extrapolation theorem that we shall need for our purposes (see [9, Theorem 2.7] and [20, Theorem 7.1]).

Theorem 2.11. *Assume that for some family of pairs of functions* (f, g)*, for some* $1 \leq p_0 < \infty$ *and* $0 \leq \alpha, \beta \leq 1$*,* α *and* β *not both identically zero, and for all* $v \in A_{p_0;(\alpha,\beta)}$

$$||g||_{L^{p_0,\infty}(v)} \leq \varphi(||v||_{A_{p_0}})||f||_{L^{p_0}(v)},$$

where φ *is an increasing function in* $(0,\infty)$*. Then, for every* $p_- < q < p_0$ *and for every* $v \in A_{q;(\alpha(q),\beta(q))}$

$$||g||_{L^{q,\infty}(v)} \lesssim \varphi\left(C\left\|v^{\frac{1}{\alpha(q)}}\right\|_{A_{r_q}}^{\frac{\alpha(p)(p_0-p_-)}{(q-p_-)}}\right)||f||_{L^q(v)}$$

with $r_q = 1 + \frac{\beta(q)}{\alpha(q)}(q-1)$*.*

3. Proof of our main results

Proof of Theorem 1.8. Assume first that $\alpha < 1$ (i.e., $p_+ < \infty$) and let $w \in B_{\frac{1}{p_-}} \cap B^*_{p_+}$. Using (2.3), we have that

$$B^*_{p_+} = B^*_{\frac{p_-}{1-\alpha(p_-)}} = \bigcup_{p_- < q < p_0} B^*_{\frac{p_-}{1-\alpha(q)}},$$

with

$$\alpha(q) = 1 - \frac{q}{p_+} \qquad \text{and} \qquad \alpha(p_-) = 1 - \frac{p_-}{p_+},$$

and hence, we have that there exists some $p_- < q < p_0 \leq p_+$ such that $w \in B^*_{\frac{p_-}{1-\alpha(q)}}$ and $0 < \alpha(q) < 1$, $0 < \beta(q) \leq 1$. Besides, by Theorem 2.11, for every $v \in A_{q;(\alpha(q),\beta(q))}$,

$$||Tf||_{L^{q,\infty}(v)} \lesssim \varphi\left(C\left\|v^{\frac{1}{\alpha(q)}}\right\|_{A_{r_q}}^{\alpha(q)\left(\frac{p_0-p_-}{q-p_-}\right)}\right)||f||_{L^q(v)},$$

with $r_q = 1 + \frac{\beta(q)}{\alpha(q)}(q-1)$. So, let us see that $w \in \mathcal{W}(T)$, or equivalently,

$$||Tf||_{\Lambda^{1,\infty}(w)} \lesssim ||f||_{\Lambda^1(w)}.$$

Let $\delta = \frac{1}{p_-} \leq 1$. Since we have that $w \in B_\delta$, we can define

$$R_\delta f(x) := \left[R\left(f^{1/\delta}\right)(x)\right]^\delta = \left[\sum_{k=0}^{\infty} \frac{M^k\left(f^{1/\delta}\right)(x)}{\left(2||M||_{\Lambda^\delta(w)}\right)^k}\right]^\delta.$$

Then,

i) $f(x) \leq R_\delta f(x)$ a.e. $x \in \mathbb{R}^n$,
ii) $R(f^{1/\delta}) \in A_1$ with $\left\| R(f^{1/\delta}) \right\|_{A_1} \leq 2 \left\| M \right\|_{\Lambda^\delta(w)}$,
iii) and, since $\Lambda^\delta(w) \subset \Lambda^{\delta,\infty}(w)$ and $\Lambda^{\delta,\infty}(w)$ is a Banach space for $w \in B_\delta$ (see [39]),

$$||R_\delta f||_{\Lambda^{1,\infty}(w)} \quad \leq \quad \left(\sum_{k=0}^{\infty} \frac{||M^k(f^{1/\delta})||_{\Lambda^{\delta,\infty}(w)}}{\Big(2||M||_{\Lambda^\delta(w)}\Big)^k} \right)^\delta \leq 2^\delta ||f||_{\Lambda^1(w)}.$$

Moreover, by Lemma 2.8, since $w \in B^*_{\frac{1}{(1-\alpha(q))\delta}}$, we can define, for an arbitrary nonnegative function $h \in (\Lambda^\delta(w))'$,

$$S_{\alpha(q)} h(x) := \left[S\left(h^{1/\alpha(q)} \right)(x) \right]^{\alpha(q)} = \left[\sum_{k=0}^{\infty} \frac{M^k(h^{1/\alpha(q)})(x)}{\Big(2||M_{\alpha(q)}||_{(\Lambda^\delta(w))'}\Big)^{k/\alpha(q)}} \right]^{\alpha(q)}, \tag{3.1}$$

so that

i) $h(x) \leq S_{\alpha(q)} h(x)$ a.e. $x \in \mathbb{R}^n$,
ii) $S(h^{1/\alpha(q)}) \in A_1$ with $\left\| S(h^{1/\alpha(q)}) \right\|_{A_1} \leq \Big(2 \left\| M_{\alpha(q)} \right\|_{(\Lambda^\delta(w))'} \Big)^{1/\alpha(q)}$,
iii) and, since $(\Lambda^\delta(w))'$ is a Banach space (see [12]),

$$\|S_{\alpha(q)} h\|_{(\Lambda^\delta(w))'} \quad \leq \quad \sum_{k=0}^{\infty} \frac{\left\| M^k_{\alpha(q)} h \right\|_{\left(\Lambda^\delta(w)\right)'}}{\Big(2||M||_{\left(\Lambda^\delta(w)\right)'}\Big)^k} \leq 2||h||_{(\Lambda^\delta(w))'}.$$

Let $y > 0$ and observe that since $\delta(q - p_-) = \beta(q)(q-1)$,

$$v = R_\delta f(x)^{p_- - q} S_{\alpha(q)}\left(\chi_{\{g>y\}}\right) = \left[R\left(f^{1/\delta} \right) \right]^{\beta(q)(1-q)} \left[S\left(\chi_{\{g>y\}}\right) \right]^{\alpha(q)} \in A_{q;(\alpha(q),\beta(q))}.$$

Hence, for every $\gamma > 0$,

$$\begin{aligned} & \lambda_{Tf}(y) \leq \lambda_{R_\delta f}(\gamma y) + \int_{\{|Tf|>y, R_\delta f \leq \gamma y\}} dx \\ \leq \quad & \lambda_{R_\delta f}(\gamma y) + \gamma^{q-p_-} \frac{y^q}{y^{p_-}} \int_{\{|Tf|>y\}} R_\delta f(x)^{p_- - q} S_{\alpha(q)}\left(\chi_{\{|Tf|>y\}}\right)(x)\, dx \\ \lesssim \quad & \lambda_{R_\delta f}(\gamma y) + \gamma^{q-p_-} \frac{\varphi\left(C \left\| v^{\frac{1}{\alpha(q)}} \right\|_{A_{r_q}}^{\alpha(q)\left(\frac{p_0 - p_-}{q - p_-}\right)} \right)^q}{y^{p_-}} \int_{\mathbb{R}^n} f^{p_-}(x) S_{\alpha(q)}\left(\chi_{\{Tf>y\}}\right)(x)\, dx. \end{aligned}$$

Now, by the Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} f^{p_-}(x) S_{\alpha(q)}\left(\chi_{\{|Tf|>y\}}\right)(x)\,dx &\leq ||f||_{\Lambda^1(w)}^{1/\delta} ||S_{\alpha(q)}\left(\chi_{\{|Tf|>y\}}\right)||_{(\Lambda^\delta(w))'} \\ &\lesssim ||f||_{\Lambda^1(w)}^{1/\delta} ||\chi_{\{|Tf|>y\}}||_{(\Lambda^\delta(w))'} \\ &\lesssim ||f||_{\Lambda^1(w)}^{1/\delta} ||w||_{B_\delta}^{1/\delta} \frac{\lambda_{Tf}(y)}{W^{1/(\delta)}(\lambda_{Tf}(y))},\end{aligned}$$

where in the last estimate we have used Lemma 2.5. Therefore, since $\delta = \frac{1}{p_-}$,

$$\begin{aligned} & yW\lambda_{Tf}(y) \\ \lesssim\ & \max\left(yW\left(\lambda_{R_{\frac{1}{p_-}}f}(\gamma y)\right), \gamma^{\frac{q}{p_-}-1} ||w||_{B_{\frac{1}{p_-}}} \varphi\left(C\left\| v^{\frac{1}{\alpha(q)}} \right\|_{A_{r_q}}^{\alpha(q)\left(\frac{p_0-p_-}{q-p_-}\right)}\right)^{q/p_-} ||f||_{\Lambda^1(w)} \right) \\ \lesssim\ & \max\left(\frac{1}{\gamma}, \gamma^{\frac{q}{p_-}-1} ||w||_{B_{\frac{1}{p_-}}} \varphi\left(C\left\| v^{\frac{1}{\alpha(q)}} \right\|_{A_{r_q}}^{\alpha(q)\left(\frac{p_0-p_-}{q-p_-}\right)}\right)^{q/p_-} \right) ||f||_{\Lambda^1(w)}. \end{aligned}$$

Thus, taking the supremum in $y > 0$ and the infimum in $\gamma > 0$,

$$||Tf||_{\Lambda^{1,\infty}(w)} \lesssim ||w||_{B_{\frac{1}{p_-}}}^{\frac{p_-}{q}} \varphi\left(C\left\| v^{\frac{1}{\alpha(q)}} \right\|_{A_{r_q}}^{\alpha(q)\left(\frac{p_0-p_-}{q-p_-}\right)}\right) ||f||_{\Lambda^1(w)}.$$

Finally, since

$$v = \left(S_{\alpha(q)}h\right)\left(R_\delta f\right)^{p_- - q} = \left[S(h^{1/\alpha(q)}) R(f^{p_-})^{1-r_q}\right]^{\alpha(q)},$$

then (see [20, Lemma 2.1])

$$\left\| v^{\frac{1}{\alpha(q)}} \right\|_{A_{r_q}} \leq \left\| S(h^{1/\alpha(q)}) \right\|_{A_1} \|R(f^{p_-})\|_{A_1}^{r_q-1} \leq 2^{1/\alpha(q)+r_q-1} \left\| M_{\alpha(q)} \right\|_{\left(\Lambda^{\frac{1}{p_-}}(w)\right)'}^{1/\alpha(q)} \|M\|_{\Lambda^{\frac{1}{p_-}}(w)}^{r_q-1},$$

so that

$$||Tf||_{\Lambda^{1,\infty}(w)} \leq C_w ||f||_{\Lambda^1(w)},$$

with

$$C_w \lesssim ||w||_{B_{\frac{1}{p_-}}}^{\frac{p_-}{q}} \varphi\left(C\left[\frac{\alpha(q)}{(1-\alpha(q))^{1+p_-}} \|w\|_{B^*_{\frac{p_-}{(1-\alpha(q))}}}^{p_-} \right]^{\frac{p_0-p_-}{q-p_-}} \|w\|_{B_{\frac{1}{p_-}}}^{\frac{p_0-p_-}{p_-}} \right).$$

If $\alpha = 1$, then $p_+ = \infty$ and $w \in B_{\frac{1}{p_-}} \cap B^*_\infty$. So, arguing identically as before but with $q = p_0$ and $\alpha(q) = 1$, in addition to use Proposition 2.6 instead of Lemma 2.8 to define (3.1), we get that

$$||Tf||_{\Lambda^{1,\infty}(w)} \leq C_w ||f||_{\Lambda^1(w)},$$

with

$$C_w \lesssim ||w||_{B_{\frac{1}{p_-}}}^{\frac{p_-}{p_0}} \varphi\left(\|w\|_{B^*_\infty} \|w\|_{B_{\frac{1}{p_-}}}^{\frac{p_0-p_-}{p_-}} \right).$$

□

Proof of Corollary 1.10. Let $\tilde{w} := W^{1/p-1}w$. Hence, observe that

$$\begin{aligned} w \in B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}} \qquad &\Longleftrightarrow \qquad \exists \varepsilon > 0 : \ \frac{W(t)}{t^{\frac{p}{p_-}-\varepsilon}} \approx\downarrow \quad \text{and} \quad \frac{W(t)}{t^{\frac{p}{p_+}+\varepsilon}} \approx\uparrow \\ &\Longleftrightarrow \qquad \tilde{w} \in B_{\frac{1}{p_-}} \cap B^*_{p_+} \end{aligned}$$

Therefore, if $w \in B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$, by Theorem 1.8, we have that

$$T : \Lambda^1(\tilde{w}) \longrightarrow \Lambda^{1,\infty}(\tilde{w}),$$

and hence

$$T : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w).$$

Now if $w \in B_{\frac{p}{p_-}} \cap B^*_{\frac{p_+}{p}}$, there exists some $\delta > 0$ such that $w \in B_{\frac{p-\delta}{p_-}} \cap B^*_{\frac{p_+}{p-\delta}}$ and $w \in B_{\frac{p+\delta}{p_-}} \cap B^*_{\frac{p_+}{p+\delta}}$. Hence,

$$T : \Lambda^{p-\delta,1}(w) \to \Lambda^{p-\delta,\infty}(w) \qquad \text{and} \qquad T : \Lambda^{p+\delta,1}(w) \to \Lambda^{p+\delta,\infty}(w),$$

and the result follows by interpolation on Lorentz spaces [12, Theorem 2.6.5].

□

Proof of Theorem 1.11. The proof essentially follows the same steps as the proof of Theorem 1.8 with few modifications. For the sake of completeness, we have added it here.

Let $w \in B^{\mathcal{R}}_{\frac{1}{p_0}} \cap B^*_{\frac{p_0}{1-\alpha}}$. First, by Lemma 2.9 (when $0 < \alpha < 1$) and Proposition 2.6 (when $\alpha = 1$) we can define, for an arbitrary nonnegative function $h \in \left(\Lambda^{\frac{1}{p_0},1}(w)\right)'$,

$$R_\alpha h(x) := \left[R\left(h^{1/\alpha}\right)(x)\right]^\alpha = \left[\sum_{k=0}^{\infty} \frac{M^k(h^{1/\alpha})(x)}{\left(2\left\|M_\alpha\right\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'}\right)^{k/\alpha}}\right]^\alpha .$$

Then,

i) $h(x) \le R_\alpha h(x)$ a.e. $x \in \mathbb{R}^n$,

ii) $R_\alpha h \in A_1$ with $\left\|R(h^{1/\alpha})\right\|_{A_1} \le \left(2\left\|M_\alpha\right\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'}\right)^{1/\alpha}$,

iii) and, since $\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'$ is a Banach space (see [12]),

$$\left\|R_\alpha h\right\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'} \le 2\left\|h\right\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'}. \tag{3.2}$$

Let $y > 0$ and observe that $v = R\left(\chi_{\{Tf>y\}}\right) \in A_1$. Then, we have

$$\begin{aligned}\lambda_{Tf}(y) &\leq \int_{\{|Tf|>y\}} R_\alpha\left(\chi_{\{|Tf|>y\}}\right)(x)\,dx \leq \frac{\varphi\left(\|v\|_{A_1}\right)^{p_0}}{y^{p_0}}\,\|f\|^{p_0}_{L^{p_0,1}(v^\alpha)}\\ &\approx \frac{\varphi\left(\|v\|_{A_1}\right)^{p_0}}{y^{p_0}}\left(\int_0^\infty\left[\int_{\{f>z\}} R_\alpha\left(\chi_{\{Tf>y\}}\right)(x)\,dx\right]^{\frac{1}{p_0}} dz\right)^{p_0}\\ &\lesssim \frac{\varphi\left(\|v\|_{A_1}\right)^{p_0}}{y^{p_0}}\,\|f\|^{p_0}_{\Lambda^1(w)}\left\|\chi_{\{Tf>y\}}\right\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'},\end{aligned}$$

where in the last estimate we have used the Hölder's inequality and the property (3.2) of R_α. Now, since $\Lambda^{\frac{1}{p_0},1}(w)=\Lambda^1(\tilde{w})$ with $\tilde{W}(t)\approx W(t)^{p_0}$, by Lemma 2.5

$$\lambda_{Tf}(y)\lesssim \frac{\varphi\left(\|v\|_{A_1}\right)^{p_0}}{y^{p_0}}\,\|f\|^{p_0}_{\Lambda^1(w)}\,\|w\|_{B^{\mathcal{R}}_{\frac{1}{p_0}}}\,\frac{\lambda_{Tf}(y)}{W^{p_0}(\lambda_{Tf}(y))}.$$

Thus,

$$\|Tf\|_{\Lambda^{1,\infty}(w)}\lesssim \|w\|^{1/p_0}_{B^{\mathcal{R}}_{\frac{1}{p_0}}}\,\varphi\left(\|v\|_{A_1}\right)\|f\|_{\Lambda^1(w)},$$

and since

$$\|v\|_{A_1}\leq\left(2\,\|M_\alpha\|_{\left(\Lambda^{\frac{1}{p_0},1}(w)\right)'}\right)^{\frac{1}{\alpha}}\lesssim\begin{cases}\|w\|^{\frac{p_0}{\alpha}}_{B^*_{\frac{p_0}{1-\alpha}}}, & 0<\alpha<1,\\ \|w\|_{B^*_\infty}, & \alpha=1,\end{cases}$$

the result follows. ☐

Proof of Corollary 1.12. The proof follows the same pattern as in Corollary 1.10. ☐

4. Applications to the boundedness of operators

4.1. Bochner-Riesz B_λ.

Corollary 4.1. *Let $n=2$, $0<\lambda<\frac{1}{2}$ and $\frac{4}{3}\leq q_0\leq 4$. If there exists some $\varepsilon>0$ so that*

$$\frac{W(t)}{t^{\frac{1+2\lambda(q_0-1)}{q_0}-\varepsilon}}\approx\downarrow \quad\text{and}\quad \frac{W(t)}{t^{\frac{1-2\lambda}{q_0}+\varepsilon}}\approx\uparrow$$

then

$$w\in\mathcal{W}(B_\lambda).$$

Proof. By Proposition 1.5 (1), we have that

$$p_+=\frac{q_0}{1-2\lambda}\quad\text{and}\quad p_-=\frac{q_0}{1+2\lambda(q_0-1)}$$

and the result follows by Theorem 1.8. ☐

Corollary 4.2. *Let $n = 2$ and $0 < \lambda < \frac{1}{2}$. If*

$$\frac{W(t)}{t^{\frac{3+2\lambda}{4}}} \approx\downarrow \quad \textit{and} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{3}{4}+\varepsilon}} \approx\uparrow \tag{4.1}$$

then

$$w \in \mathcal{W}(B_\lambda) \qquad \textit{with} \qquad \|w\|_{\mathcal{W}(B_\lambda)} \lesssim \|w\|_{B^{\mathcal{R}}_{\frac{3+2\lambda}{4}}}^{\frac{3+2\lambda}{4}} \|w\|_{B^*_{\frac{4}{3}}}^{\frac{7+4\lambda}{2\lambda}}.$$

Proof. By Proposition 1.5 (2), we have that

$$p_0 = \frac{4}{3+2\lambda} \quad \text{and} \quad \alpha = \frac{2\lambda}{3+2\lambda}$$

and the result follows by Theorem 1.11. □

Remark 4.3. *As a first example of Corollary 4.2, if we let $w(s) = s^{\frac{2\lambda-1}{4}}$ then we conclude that $w \in \mathcal{W}(B_\lambda)$; that is*

$$B_\lambda : L^{\frac{4}{3+2\lambda},1} \longrightarrow L^{\frac{4}{3+2\lambda},\infty}$$

and, as expected, we obtain that the Bochner-Riesz conjecture holds for B_λ when $n = 2$.

We also obtain spaces for which the boundedness of B_λ was not previously known. For instance, taking w_m and $\tilde{w}_m$ as in Examples 2.4, we have that

$$\Lambda^{\frac{4}{3+2\lambda},1}(w_m) \hookrightarrow L^{\frac{4}{3+2\lambda},1} \hookrightarrow \Lambda^{\frac{4}{3+2\lambda},1}(\tilde{w}_m) \quad \textit{and} \quad \Lambda^{\frac{4}{3+2\lambda},\infty}(w_m) \hookrightarrow L^{\frac{4}{3+2\lambda},\infty} \hookrightarrow \Lambda^{\frac{4}{3+2\lambda},\infty}(\tilde{w}_m),$$

and

$$B_\lambda : \Lambda^{\frac{4}{3+2\lambda},1}(w_m) \longrightarrow \Lambda^{\frac{4}{3+2\lambda},\infty}(w_m), \qquad B_\lambda : \Lambda^{\frac{4}{3+2\lambda},1}(\tilde{w}_m) \longrightarrow \Lambda^{\frac{4}{3+2\lambda},\infty}(\tilde{w}_m).$$

Corollary 4.4. *Let $n > 2$, $0 < \lambda < \frac{n-1}{2}$ and $q_0 = 2$ or let q_0 be such that $\frac{2n}{n+1} < q_0 < 2$ and Conjecture 1.2 holds with $p = q_0$. If $q \in \{q_0, q_0'\}$ and there exists some $\varepsilon > 0$ so that*

$$\frac{W(t)}{t^{\frac{n-1+2\lambda(q-1)}{q(n-1)}-\varepsilon}} \approx\downarrow \quad \textit{and} \quad \frac{W(t)}{t^{\frac{n-1-2\lambda}{q(n-1)}+\varepsilon}} \approx\uparrow$$

then

$$w \in \mathcal{W}(B_\lambda).$$

Proof. By Proposition 1.6 (1), we have that for $q \in \{q_0, q_0'\}$,

$$p_+ = \frac{q(n-1)}{n-1-2\lambda} \quad \text{and} \quad p_- = \frac{q(n-1)}{n-1+2\lambda(q-1)}$$

and the result follows by Theorem 1.8. □

Corollary 4.5. *Let $n > 2$ and $\frac{n-1}{2(n+1)} < \lambda < \frac{n-1}{2}$. If*

$$\frac{W(t)}{t^{\frac{1}{2}}} \approx\downarrow \quad \textit{and} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{n-1-2\lambda}{2n}+\varepsilon}} \approx\uparrow \quad \textit{or} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\frac{n+1+2\lambda}{2n}-\varepsilon}} \approx\downarrow \quad \textit{and} \quad \frac{W(t)}{t^{\frac{1}{2}+\varepsilon}} \approx\uparrow$$

then

$$w \in \mathcal{W}(B_\lambda).$$

Proof. By Proposition 1.6 (2), we have that either

$$p_0 = 2 \quad \text{and} \quad \alpha = \frac{1+2\lambda}{n}$$

and the result follows by Theorem 1.11, or

$$p_+ = 2 \quad \text{and} \quad p_- = \frac{2n}{n+1+2\lambda}$$

and the result follows by Theorem 1.8. $\square$

4.2. **Sparse operators.** These operators have become very important due to its role in the so called A_2 conjecture consisting in proving that if T is a Calderón-Zygmund operator then

$$||Tf||_{L^2(v)} \lesssim ||v||_{A_2} ||f||_{L^2(v)}.$$

This result was first obtained by T.P. Hytönen [23] (see also A.K. Lerner [31, 32]). Let us give the precise definition. A general dyadic grid $\mathcal{D}$ is a collection of cubes in $\mathbb{R}^n$ satisfying the following properties:

(i) For any cube $Q \in \mathcal{D}$, its side length is 2^k for some $k \in \mathbb{Z}$.
(ii) Every two cubes in $\mathcal{D}$ are either disjoint or one is wholly contained in the other.
(iii) For every $k \in \mathbb{Z}$ and given $x \in \mathbb{R}^n$, there is only one cube in $\mathcal{D}$ of side length 2^k containing it.

Let $0 < \eta < 1$, a collection of cubes $\mathcal{S} \subset \mathcal{D}$ is called η-sparse if, for every $Q \in \mathcal{S}$, there exist pairwise disjoint measurable sets $E_Q \subset Q$ with $|E_Q| \geq \eta |Q|$.

Definition 4.6. *Let $1 \leq r < \infty$. Given a sparse family of cubes $\mathcal{S} \subset \mathcal{D}$, the sparse operator is defined by*

$$\mathcal{A}_{r,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f^r(y)\, dy \right)^{1/r} \chi_Q(x), \qquad x \in \mathbb{R}^n.$$

The boundedness of $A_{1,\mathcal{S}} f$ over the Lorentz spaces $\Lambda^p(w)$ was settled in [4]. Here we deal with the case $1 < r < \infty$. To do so, first we present the following restricted weak-type inequality for the sparse operator $\mathcal{A}_{r,\mathcal{S}}$. The proof follows by duality using the same ideas as in [11, Theorem 4.1] with the obvious modifications.

Proposition 4.7. *Given $1 < r \leq p < \infty$ and a weight $v \in A^{\mathcal{R}}_{\frac{p}{r}}$. The sparse operator $\mathcal{A}_{r,\mathcal{S}}$ satisfies the weak-type estimate*

$$\|\mathcal{A}_{r,\mathcal{S}} f\|_{L^{p,\infty}(v)} \lesssim \|v\|_{A^{\mathcal{R}}_{\frac{p}{r}}}^{\frac{r+p}{r}} \|f\|_{L^{p,1}(v)}.$$

Observe that, in particular, for $p = r$ we have $v \in A_1$ with $p_0 = r$ and $\alpha = 1$. Consequently, by Theorem 1.11 and Corollary 1.12:

Corollary 4.8. *If*

$$\frac{W(t)}{t^{\frac{1}{r}}} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \frac{W(t)}{t^{\varepsilon}} \approx\uparrow,$$

then

$$w \in \mathcal{W}(\mathcal{A}_{r,\mathcal{S}}) \qquad \text{with} \qquad \|w\|_{\mathcal{W}(A_{r,\mathcal{S}})} \lesssim \|w\|_{B^{\mathcal{R}}_{\frac{1}{r}}}^{\frac{1}{r}} \|w\|_{B^*_\infty}^{2}.$$

Moreover, if for $0<p<\infty$,

$$\frac{W(t)}{t^{\frac{p}{r}}}\approx\downarrow \quad \textit{and} \quad \exists\varepsilon>0:\ \frac{W(t)}{t^{\varepsilon}}\approx\uparrow,$$

then

$$\mathcal{A}_{r,\mathcal{S}}:\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w).$$

4.3. **Hörmander multipliers.** Let m be a bounded function on $\mathbb{R}^n$ and consider the Fourier multiplier $T_mf(x)=(m\hat{f})^{\vee}(x)$. Set $s>1$ and let $\gamma=(\gamma_1,\dots,\gamma_n)$ a multi-index of nonnegative integers with $|\gamma|=\gamma_1+\cdots+\gamma_n$. It is said that $m\in M(s,l)$ if

$$\sup_{R>0}\left(R^{s|\gamma|-n}\int_{R<|x|<2R}|D^{\gamma}m(x)|^{s}dx\right)^{1/s}<\infty, \quad \forall|\gamma|\le l.$$

Proposition 4.9. *[*[27]*] Let* $1<s\le 2$, $\frac{n}{s}<l<n$ *and* $m\in M(s,l)$*. Then:*

(i)

$$T_m:L^1(v^{\frac{l}{n}})\to L^{1,\infty}(v^{\frac{l}{n}}), \qquad \forall v\in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$ *with* φ *an increasing function,*

(ii)

$$T_m:L^{\frac{n}{l}}(v)\to L^{\frac{n}{l},\infty}(v), \qquad \forall v\in A_1,$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$*, with* φ *an increasing function.*

Consequently, by Theorem 1.11 and Corollary 1.12:

Corollary 4.10. *Let* $1<s\le 2$, $\frac{n}{s}<l<n$ *and* $m\in M(s,l)$*. If*

$$\frac{W(t)}{t}\approx\downarrow \quad \textit{and} \quad \exists\varepsilon>0:\ \frac{W(t)}{t^{\frac{n-l}{n}+\varepsilon}}\approx\uparrow \qquad \textit{or} \qquad \frac{W(t)}{t^{\frac{l}{n}}}\approx\downarrow \quad \textit{and} \quad \exists\varepsilon>0:\ \frac{W(t)}{t^{\varepsilon}}\approx\uparrow,$$

then

$$w\in\mathcal{W}(T_m).$$

Moreover, if for $0<p<\infty$,

$$\frac{W(t)}{t^{p}}\approx\downarrow \quad \textit{and} \quad \exists\varepsilon>0:\ \frac{W(t)}{t^{\frac{p(n-l)}{n}+\varepsilon}}\approx\uparrow \qquad \textit{or} \qquad \frac{W(t)}{t^{\frac{pl}{n}}}\approx\downarrow \quad \textit{and} \quad \exists\varepsilon>0:\ \frac{W(t)}{t^{\varepsilon}}\approx\uparrow,$$

then

$$T_m:\Lambda^{p,1}(w)\to\Lambda^{p,\infty}(w).$$

4.4. **Rough operators.** Let $\Sigma=\Sigma_{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ and, for $1<r<\infty$, take $\Omega\in L^r(\Sigma)$ to be a positive function homogeneous of degree zero such that $\int_{\Sigma}\Omega=0$. Let us consider the rough operator

$$T_{\Omega}f(x)=p.v.\int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^n}f(x-y)dy=\lim_{\varepsilon>0}\int_{|y|>\varepsilon}\frac{\Omega(y')}{|y|^n}f(x-y)dy,$$

where $y'=\frac{y}{|y|}$, $y\neq 0$.

Proposition 4.11 ([18]). *If* $v \in A_1$,

$$T_\Omega : L^{r'}(v) \to L^{r',\infty}(v),$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$ *with* φ *an increasing function.*

Consequently, by Theorems 1.8 and 1.11 and Corollaries 1.10 and 1.12:

Corollary 4.12. *If*

$$\frac{W(t)}{t^{\frac{1}{r'}}} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \ \frac{W(t)}{t^\varepsilon} \approx\uparrow,$$

then

$$w \in \mathcal{W}(T_\Omega).$$

Moreover, for $0 < p < \infty$, *if*

$$\frac{W(t)}{t^{\frac{p}{r'}}} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \ \frac{W(t)}{t^\varepsilon} \approx\uparrow,$$

then

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w).$$

Also, if we assume that Ω satisfies the L^r-Dini condition, that is

$$\int_0^1 \omega_r(\delta) \frac{d\delta}{\delta} < +\infty,$$

where

$$\omega_r(\delta) = \sup_{|\rho|<\delta} \left(\int_\Sigma |\Omega(\rho x) - \Omega(x)|^r d\sigma \right)^{1/r},$$

with ρ any rotation of Σ and $|\rho| = \sup_{x\in\Sigma} |\rho x - x|$, Proposition 4.11 (ii) can be improved as follows:

Proposition 4.13 ([27]). *If* $v \in A_1$,

$$T_\Omega : L^1(v^{\frac{1}{r'}}) \to L^{1,\infty}(v^{\frac{1}{r'}}),$$

with constant less than or equal to $\varphi\left(\|v\|_{A_1}\right)$ *with* φ *an increasing function.*

Consequently, by Theorem 1.11 and Corollary 1.12:

Corollary 4.14. *For* $0 < p < \infty$, *if*

$$\frac{W(t)}{t^p} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \ \frac{W(t)}{t^{\frac{p}{r}+\varepsilon}} \approx\uparrow \qquad \text{or} \qquad \frac{W(t)}{t^{\frac{p}{r'}}} \approx\downarrow \quad \text{and} \quad \exists \varepsilon > 0 : \ \frac{W(t)}{t^\varepsilon} \approx\uparrow,$$

then

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w).$$

Remark 4.15. *We should finally mention that although the property of being increasing of* φ *in Propositions 4.9, 4.11 and 4.13 is known, the sharp expression for such function is unknown.*

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