



Regular versus singular solutions in quasilinear indefinite problems with sublinear potentials [☆]

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Abstract

The aim of this paper is analyzing existence, multiplicity, and regularity issues for the positive solutions of the quasilinear Neumann problem

$$\begin{cases} -(u'/\sqrt{1+(u')^2})' = \lambda a(x)f(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Here, $(u'/\sqrt{1+(u')^2})'$ is the one-dimensional curvature operator, $\lambda \in \mathbb{R}$ is a parameter, which is generally taken positive, the weight $a(x)$ changes sign, and, in most occasions, the function $f(u)$ has a sublinear potential $F(u)$ at ∞ . Our discussion displays the manifold patterns occurring for these solutions, depending on the behavior of the potential $F(u)$ at $u = 0$, and, possibly, at infinity, and of the weight function $a(x)$ at its nodal points.

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1. Introduction

The main aim of this paper is analyzing the set of positive solutions, regular or singular, of the quasilinear Neumann problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)f(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.1}$$

Here, $(u'/\sqrt{1+(u')^2})'$ is the one-dimensional curvature operator, $\lambda \in \mathbb{R}$ is viewed as a parameter, which in most cases is positive, and the functions $a(x)$ and $f(u)$ are generally supposed to satisfy

$$(a_1) \quad a \in L^1(0, 1), \int_0^1 a(x) dx < \infty, \text{ and } a(x) > 0 \text{ on a set of positive measure,}$$

and, respectively,

$$(f_1) \quad f \in C(\mathbb{R}), f(0) = 0, f(u) > 0 \text{ if } u > 0, \text{ and, for some constants } h > 0, p > 0 \text{ and } q \in (0, 1),$$

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^p} = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{f(u)}{u^{-q}} = h. \tag{1.2}$$

Obviously, if $\lim_{u \rightarrow 0^+} \frac{f(u)}{u^p} = h_0 > 0$, then, by replacing $f(u)$ with $g(u) = f(u)/h_0$ and λ with $\mu = h_0\lambda$, the first condition of (1.2) is always met; in particular, $f'(0^+) = 1$ if $p = 1$.

By a *regular* solution of (1.1), we mean a function $u \in W^{2,1}(0, 1)$, which fulfills, for some $\lambda \in \mathbb{R}$, the equation a.e. in $(0, 1)$, as well as the boundary conditions. It is evident that u is a regular solution of (1.1) if and only if it satisfies

$$\begin{cases} -u'' = \lambda a(x)f(u)g(u'), & 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where

$$g(v) = (1 + v^2)^{\frac{3}{2}}. \tag{1.4}$$

In this paper we will also use, as we previously did in [39–43], the notion of bounded variation solution of (1.1). A function $u \in BV(0, 1)$ is said to be a *bounded variation* solution of (1.1) if the next identity holds

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}} dx + \int_0^1 \frac{Du^s}{|Du^s|} D\phi^s = \int_0^1 \lambda a f(u) \phi dx$$

for every $\phi \in BV(0, 1)$ such that $|D\phi^s|$ is absolutely continuous with respect to $|Du^s|$. Here, for any $v \in BV(0, 1)$, $Dv = Dv^a dx + Dv^s$ is the Lebesgue–Nikodym decomposition, with respect to the Lebesgue measure dx in \mathbb{R} , of the Radon measure Dv in its absolutely continuous part $Dv^a dx$, with density function Dv^a , and its singular part Dv^s . Further, $\frac{Dv^s}{|Dv^s|}$ denotes the density function of Dv^s with respect to its total variation $|Dv^s|$. We refer, e.g., to [3,4,42] for additional details on these concepts.

It is apparent that any regular solution is a bounded variation solution. When a bounded variation solution is not regular, it is called *singular*. Such solutions may exhibit jumps and, in principle, even more complex behaviors. Throughout this paper, all solutions will be bounded variation solutions, even if not emphasized explicitly.

A solution u of (1.1) is said to be *positive* if $\text{ess inf } u \geq 0$ and $\text{ess sup } u > 0$, whereas it is said *strictly positive* if $\text{ess inf } u > 0$. It is also said that a pair (λ, u) is a positive, or strictly positive, solution of (1.1) if u is a positive, or strictly positive, solution of (1.1), respectively, for some $\lambda \geq 0$. As this paper focuses attention on positive solutions, all solutions through it will be understood to be positive.

Throughout this paper, $\lambda_0 > 0$ stands for the positive principal eigenvalue of the linear weighted eigenvalue problem

$$\begin{cases} -\varphi'' = \lambda a(x)\varphi, & 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0. \end{cases} \tag{1.5}$$

According to a classical result of Brown and Lin [8] (see [38, Ch. 9] for the general theory), besides $\lambda = 0$, the problem (1.5) admits under (a_1) a unique positive eigenvalue $\lambda_0 > 0$, with a strictly positive eigenfunction φ , unique up to a positive multiplicative constant.

Hereafter, we denote by \mathcal{S}_r^+ the set of couples $(\lambda, u) \in [0, \infty) \times C^1[0, 1]$ such that (λ, u) is a positive regular solution of (1.1), together with $(0, 0)$ and $(\lambda_0, 0)$, its two possible bifurcation points from the trivial line $(\lambda, 0)$, $\lambda \in \mathbb{R}$. Similarly, we denote by \mathcal{S}_{bv}^+ the set of couples $(\lambda, u) \in [0, \infty) \times BV(0, 1)$ such that (λ, u) is a positive (bounded variation) solution of (1.1), together with $(0, 0)$ and $(\lambda_0, 0)$. By definition, the set of singular positive solutions of (1.1), denoted by \mathcal{S}_s^+ , is given by $\mathcal{S}_s^+ = \mathcal{S}_{bv}^+ \setminus \mathcal{S}_r^+$.

Let us observe that, according to (1.2), the potential of $f(u)$, defined by $F(u) = \int_0^u f(s) ds$ for all $u \in \mathbb{R}$, satisfies

$$\lim_{u \rightarrow 0^+} \frac{F(u)}{u^{p+1}} = \lim_{u \rightarrow 0^+} \frac{f(u)}{(p+1)u^p} = \frac{1}{p+1} > 0 \tag{1.6}$$

and, hence, it is *quadratic* at zero if $p = 1$, *subquadratic* if $0 < p < 1$, and *superquadratic* if $p > 1$. Similarly, we have that

$$\lim_{u \rightarrow \infty} \frac{F(u)}{u^{1-q}} = \lim_{u \rightarrow \infty} \frac{f(u)}{(1-q)u^{-q}} = \frac{h}{1-q} > 0 \tag{1.7}$$

and, therefore, $F(u)$ is *sublinear* at infinity, because $0 < 1 - q < 1$.

As the main goal of this paper is analyzing the existence and the interplay between the regular and the singular solutions of (1.1) under (f_1) , this work can be viewed as a natural continuation of [39–43] to cover the case where $F(u)$ is sublinear at infinity. There are strong motivations for studying this problem: a rather thorough discussion is presented in [39–43], together with a

wide list of relevant references, also witnessing the relevance of this topic for the applications. These references include some very classical works dating back to the three decades from the sixties up to the eighties and dealing with the prescribed mean curvature equation [5,10,34,55,57], or capillarity-type problems [13,19,24–28,31,32], as well as other more recent papers, often dedicated to finding positive solutions, either in higher dimensions [12,14,15,33,35,44,45,47,51] and possibly with radial symmetry [11,46,49,56], or in one dimension [6,7,9,16,48,50].

Note that the condition (1.2) implies that

$$\lim_{u \rightarrow \infty} f(u) = 0 \tag{1.8}$$

and hence $\|f\|_\infty = \max_{u \geq 0} f(u) < \infty$. For the validity of some of the results found in this paper we shall however impose a stronger condition than (f_1) . Since $f \in \mathcal{C}(\mathbb{R})$ and $f(0) = 0$ it follows from (1.8) that there exists a maximal $M > 0$ such that $f(M) = \|f\|_\infty$. The next assumption incorporates into (f_1) the monotonicity of $f(u)$ on each of the intervals $(0, M)$ and (M, ∞) :

(f_2) $f(u)$ satisfies (f_1) , $f \in \mathcal{C}^1[M, \infty)$, and it is increasing in $(0, M)$ and decreasing in (M, ∞) .

For any given $p > 0$, $q \in (0, 1)$ and $M > 0$, the function

$$f(u) = \begin{cases} u^p & \text{if } 0 \leq u \leq M, \\ \frac{M^{p+q}}{u^q} & \text{if } u > M \end{cases} \tag{1.9}$$

provides us with a paradigmatic example of function satisfying (f_2) . By regularizing it around M it is very easy to construct a family of functions satisfying (f_2) , with the same shape as (1.9) at $u = 0$ and $u = \infty$, and such that $f \in \mathcal{C}^1(0, \infty)$. In some cases, when using bifurcation methods, more regularity will be necessary, as to require that

(f_3) $f(u)$ satisfies $f \in \mathcal{C}^1(\mathbb{R})$, $f(0) = 0$, $f'(0) = 1$, $f(u) > 0$ if $u > 0$, and, for some constants $h > 0$ and $q \in (0, 1)$, $\lim_{u \rightarrow \infty} \frac{f'(u)}{-qu^{-q-1}} = h > 0$ holds.

Moreover, in some circumstances we will replace (a_1) with the stronger condition

(a_2) $a \in L^\infty(0, 1)$, $\int_0^1 a(x) dx < 0$, and there is $z \in (0, 1)$ such that $a(x) > 0$ for a.e. $x \in (0, z)$ and $a(x) < 0$ for a.e. $x \in (z, 1)$.

When (a_2) holds, by [39, Cor. 3.7], any positive singular solution (λ, u) of (1.1) satisfies

$$u|_{[0,z)} \in W_{\text{loc}}^{2,\infty}[0, z) \cap W^{1,1}(0, z) \text{ and is concave,}$$

$$u|_{(z,1]} \in W_{\text{loc}}^{2,\infty}(z, 1] \cap W^{1,1}(z, 1) \text{ and is convex.}$$

Moreover, $u'(x) < 0$ for every $x \in (0, z)$, $u'(x) \leq 0$ for every $x \in (z, 1)$, $u'(0) = u'(1) = 0$ and $u'(z^-) = u'(z^+) = -\infty$, where $u'(z^-)$ and $u'(z^+)$ stand for the left and right Dini derivatives of

u at z . Therefore, in this case, singular solutions can only develop jumps at z , the node of the function $a(x)$.

Throughout this paper, for any given $r < s$ and $V \in L^\infty(r, s)$, we denote by $\sigma[-D^2 + V(x); \mathcal{B}, (r, s)]$, with $D^2 = \frac{d^2}{dx^2}$, the lowest eigenvalue of the boundary value problem

$$\begin{cases} -w'' + V(x)w = \tau w, & r < x < s, \\ \mathcal{B}w(r) = \mathcal{B}w(s) = 0, \end{cases}$$

where \mathcal{B} stands either for the Neumann boundary operator, \mathcal{N} , or the Dirichlet boundary operator, \mathcal{D} . If (f_1) holds with $p \geq 1$, and u is a strictly positive regular solution of (1.1), i.e., (1.3) holds, then u must be a principal eigenfunction associated with the eigenvalue

$$\sigma[-D^2 - \lambda a(x) \frac{f(u)}{u} g(u'); \mathcal{N}, (0, 1)] = 0$$

and hence, e.g., by [38, Thm. 7.10], $\min u > 0$, i.e., u is strictly positive. Moreover, if $f(u)$ satisfies $f \in C^1(0, \infty)$ and (f_1) with $p \geq 1$, and (a_1) holds, then, from [42, Prop. 1.1] (see also [43, Lem. 2.1]), we find that $\lambda \geq 0$ if (1.1) possesses a strictly positive solution. Actually, the solution is constant in $[0, 1]$ if $\lambda = 0$. Thus, non-constant positive solutions of (1.1) can only arise for $\lambda > 0$. However, the situation is quite different if (f_1) holds with $0 < p < 1$. Indeed, as pointed out in [42, Rem. 1.8], in this case there may occur *dead core* solutions, i.e., solutions vanishing on some open interval, thus provoking the possible existence of positive regular solutions even for $\lambda < 0$. This can be shown by slightly modifying [42, Ex. 2] as follows. In this work we anyhow restrict our analysis to the case $\lambda \geq 0$.

Example. Let $f \in C(\mathbb{R})$ be such that $f(u) = \sqrt{u}$ if $0 \leq u \leq 1$. Then, the function defined by

$$u(x) = \begin{cases} \frac{1}{12^2} (\frac{2}{3^4} - x^4) & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{12^2} (x - \frac{2}{3})^4 & \text{if } \frac{1}{3} < x \leq \frac{2}{3}, \\ 0 & \text{if } \frac{2}{3} < x \leq 1, \end{cases}$$

satisfies $u \in W^{2,\infty}(0, 1)$ and it is a positive regular solution of (1.1) with $\lambda = -1$ and $a \in L^\infty(0, 1)$ defined by

$$a(x) = \begin{cases} \left(\frac{u'}{\sqrt{1 + (u')^2}} \right)' \frac{1}{\sqrt{u(x)}} & \text{if } 0 \leq x < \frac{2}{3}, x \neq \frac{1}{3}, \\ A & \text{if } \frac{2}{3} \leq x \leq 1, \end{cases}$$

for every constant $A \in \mathbb{R}$. In particular, so that (a_1) holds, the constant A can be chosen so that $\int_0^1 a(x) dx < 0$. It is worthy observing that, for this choice of the functions $f(u)$ and $a(x)$, by [43, Thm. 9.1], the problem (1.1) also admits positive regular solutions for sufficiently small $\lambda > 0$.

We describe below the main findings of this paper which concern, in the following order, with non-existence, existence, multiplicity and regularity properties of the positive solutions of (1.1).

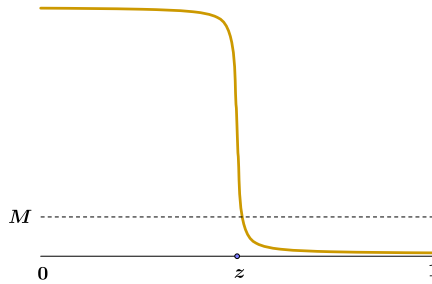


Fig. 1. The profile of the solutions for large $\lambda > 0$.

In Section 2 we establish that (1.1) cannot admit a singular solution for sufficiently small $\lambda > 0$ under conditions (f_1) and (a_1) , regardless the size of $p > 0$. Actually, (1.1) cannot admit any solution, neither singular nor regular, for sufficiently small $\lambda > 0$ if $p \geq 1$. These conclusions are optimal, because, due to [43, Thm. 9.1], the problem (1.1) has at least one positive regular solution (λ, u_λ) , for sufficiently small $\lambda > 0$, if $p \in (0, 1)$.

In Section 3 we show, for quadratic potentials at the origin ($p = 1$), the existence of two components of regular solutions of (1.1) bifurcating from the trivial line at $\lambda = 0$ and at $\lambda = \lambda_0$.

By the bifurcation theorems in [39], these components are subcomponents of some components of the set \mathcal{S}_{bv}^+ of the positive bounded variation solutions of (1.1). For superlinear and linear potentials at infinity it is already known from [40,41] that the regular solutions can develop singularities along these components. The existence of positive bounded variation solutions of (1.1) for all $\lambda > \lambda_0$ is guaranteed by Theorem 3.1.

Section 4 deals with subquadratic potentials at the origin ($0 < p < 1$). In this case the existence of positive bounded variation solutions of (1.1) for all $\lambda > 0$ follows from [42, Thm. 1.2]. The main goal of Section 4 is establishing the existence of a component of the set \mathcal{S}_r^+ of positive regular solutions bifurcating from $(0, 0)$, which is done in Theorem 4.2. This result complements [43, Thm. 9.1]. As the proof of [43, Thm. 9.1] is based on the direct method of calculus of variations it does not guarantee such structure information, provided instead by Theorem 4.2 which relies on the construction of sub- and supersolutions and the use of topological degree methods.

Section 5 focuses on superquadratic potentials at the origin ($p > 1$). From [42, Thm. 1.5] and [43, Thm. 10.1] the existence of two positive solutions, one of them regular and small, can be inferred for sufficiently large $\lambda > 0$. Theorem 5.2 establishes the existence of a component of the set \mathcal{S}_r^+ of positive regular solutions of (1.1) bifurcating from 0 as $\lambda \rightarrow \infty$. The proof of Theorem 5.2 relies on some elementary topological techniques based on the theory of superlinear indefinite problems developed in [2].

Section 6 ascertains, for every $p > 0$, the limiting profile of the regular solutions of (1.1) that are separated away from zero as $\lambda \rightarrow \infty$, should they exist, when $f(u)$ and $a(x)$ satisfy (f_2) and (a_2) . The assumption (a_2) entails that any regular solution u of (1.1) is decreasing in $[0, 1]$, and thus $u(0) = \|u\|_{L^\infty(0,1)}$. These solutions satisfy $u(0) > M$ (see (f_2)) for sufficiently large $\lambda > 0$ and grow up to infinity on $[0, z)$ at least at the rate $C_1\lambda^{1/q}$, while they decay to zero on $(z, 1]$ at least as $C_2\lambda^{-1/p}$, for some constants $C_1 > 0, C_2 > 0$, as sketched in Fig. 1.

According to Theorem 5.2, (1.1) possesses a subcontinuum of the set of regular solutions of (1.1) consisting of solutions (λ, u_λ) which satisfy

$$\lim_{\lambda \rightarrow \infty} \|u_\lambda\|_{C^1[0,1]} = 0.$$

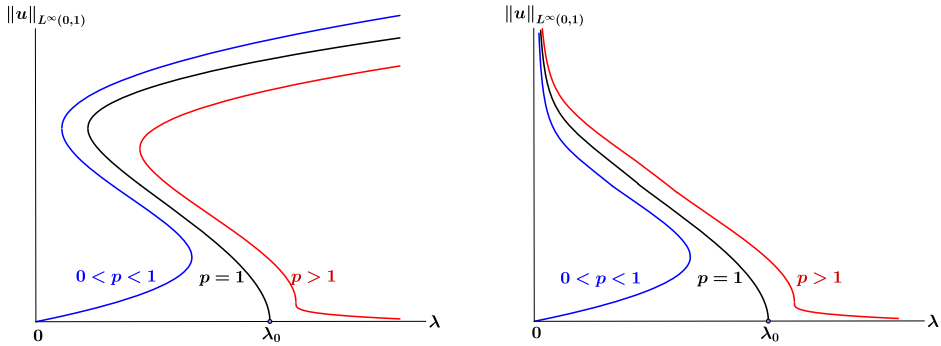


Fig. 2. Global bifurcation diagrams when $F(u)$ is sublinear at infinity (left), or $F(u)$ is superlinear at infinity (right), and $a(x)$ satisfies (a_2) and (1.10), according to the nature of $F(u)$ at the origin: subquadratic (blue, $0 < p < 1$), quadratic (black, $p = 1$), or superquadratic (red, $p > 1$). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

As these solutions are not separated away from zero for sufficiently large $\lambda > 0$, they must be left outside the analysis of Section 6 to ensure the validity of all our results therein.

Section 7 carries out a detailed discussion of the existence, and non-existence, of singular solutions. Precisely, Theorem 7.1 establishes a general criterion that allows to ascertain the local regularity of the bounded variation solutions of the equation

$$-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = h(x),$$

where $h \in L^1(0, 1)$ satisfies $h(x) \geq 0$ a.e. in $(z - \delta_1, z)$ and $h(x) \leq 0$ a.e. in $(z, z + \delta_2)$ for some $z \in (0, 1)$ and $\delta_1, \delta_2 > 0$, according to the behavior of $h(x)$ near its nodal point z . Based on this result, Theorem 7.2 shows that, under condition (a_2) , the problem (1.1) cannot admit singular solutions if

$$\text{either } \int_0^z \left(\int_x^z a(t) dt\right)^{-\frac{1}{2}} dx = \infty, \text{ or } \int_z^1 \left(\int_x^z a(t) dt\right)^{-\frac{1}{2}} dx = \infty. \quad (1.10)$$

This condition measures how smooth is the function $a(x)$ on the left, or on the right, of z ; it is easily seen that it holds whenever $a(x)$ is differentiable at z . Surprisingly, Theorem 7.2 holds true regardless the particular behavior of $f(u)$ at zero and at infinity, just requiring $f(u)$ to be continuous and positive. Thus, Theorem 7.1 is a quite general and versatile result, applying to a large variety of situations. In particular, it completes and sharpens, very substantially, some of our previous findings in [39–43]. As a direct consequence of this regularity result, the global bifurcation diagrams of the set of positive solutions of (1.1) look like those superimposed in Fig. 2 according to the decay rate of the potential at the origin, measured by $p > 0$, and at infinity. In the global bifurcation diagrams plotted in Fig. 2, as well as in the remaining figures, we are plotting the value of parameter λ versus the L^∞ -norm of u . Thus, each point on the plotted curves stands for a particular solution (λ, u) of (1.1). In these bifurcation diagrams, continuous lines are filled in by regular solutions, while dashed lines consist of singular solutions. Thanks to Theorem 7.2,

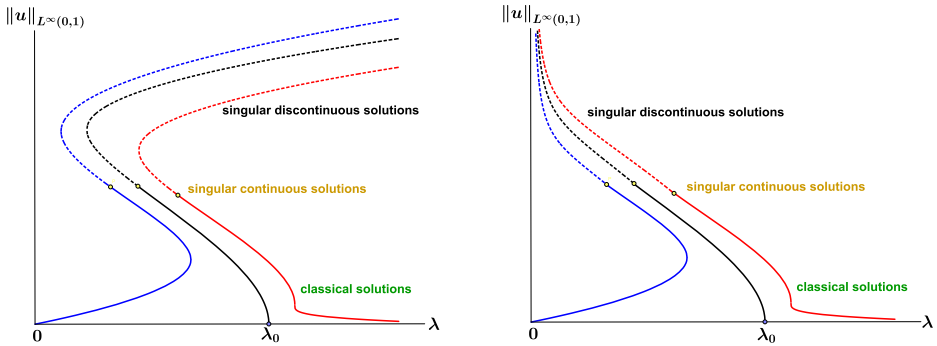


Fig. 3. Global bifurcation diagrams when $F(u)$ is sublinear at infinity (left), or superlinear at infinity (right), and $a(x)$ satisfies (a_2) and (1.11), according to the nature of $F(u)$ at the origin: subquadratic (blue, $0 < p < 1$), quadratic (black, $p = 1$), or superquadratic (red, $p > 1$).

the problem (1.1) cannot admit singular solutions not only for sublinear potentials at infinity but also for superlinear, or asymptotically linear, potentials at infinity. Thus, these findings complete, when $0 < p \neq 1$, the regularity result of [41] as well as the main theorem of [40], where the non-existence of singular solutions was only established for sufficiently small $\lambda > 0$.

The global bifurcation diagrams of Fig. 2 are in full agreement with the existence and non-existence results of [42] and [43], as well as with the new findings of this paper.

According to Theorem 7.2, the condition

$$\int_0^z \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx < \infty \quad \text{and} \quad \int_z^1 \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx < \infty \tag{1.11}$$

is necessary for the existence of a singular solution. This condition holds, for example, if

$$\text{ess lim}_{x \rightarrow z^-} a(x) > 0 > \text{ess lim}_{x \rightarrow z^+} a(x). \tag{1.12}$$

According to the results of [43], the small solutions of (1.1) must be regular. So, a further goal of Section 7 is analyzing the formation of singularities from these small regular solutions as λ varies. By Theorem 7.3, under conditions (a_2) and (1.11), there are examples of functions $f(u)$ satisfying (f_1) for which (1.1) possesses singular solutions. Moreover, regardless $f(u)$, when $a(x)$ satisfies (a_2) and (1.12), then, any sufficiently large solution of (1.1) for sufficiently large λ must be singular. Therefore, the solutions of (1.1) whose existence is guaranteed by [42, Thm. 1.4, Rem. 1.9] for sufficiently large $\lambda > 0$ must be singular. Fig. 3 provides us with six admissible bifurcation diagrams when the function $a(x)$ satisfies (a_2) and (1.12), according to the nature of $F(u)$ at infinity.

In strong contrast with the situations described in Fig. 2, under condition (1.12), the small solutions of (1.1) are regular, whereas the solutions far away from zero, for sufficiently large $\lambda > 0$, may become singular. Therefore, the small regular solutions on each of the components plotted in Fig. 3 develop singularities as they become sufficiently large at the points of the bifurcation diagrams separating continuous and dashed lines. Although the results of Section 7 guarantee the existence of singular solutions for sufficiently large $\lambda > 0$ for sublinear potential at infinity, we

were unable to guarantee the formation of singularities from the small regular solutions along the solution components of (1.1) just assuming the weaker condition (1.11). So, this problem remains open here.

2. Non-existence of solutions for small $\lambda > 0$ when $p \geq 1$

This section establishes the non-existence of positive solutions (regular or singular) for sufficiently small $\lambda > 0$ when $p \geq 1$. Recall that in this context the positive solutions of (1.1) are actually strictly positive. Our first result establishes the non-existence of singular solutions of (1.1) when $f(u)$ is globally bounded in $[0, \infty)$ and $\lambda \geq 0$ is sufficiently small.

Lemma 2.1. *Assume (f_1) and (a_1) . Then, the problem (1.1) has no positive singular solution for sufficiently small $\lambda \geq 0$.*

Proof. Let u be a positive bounded variation solution of (1.1) for some $\lambda \geq 0$. Set $h(x) = a(x)f(u(x))$, for a.e. $x \in [0, 1]$. Hence, u is a solution of the problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda h(x), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Since $\|h\|_{L^1(0,1)} \leq \|f\|_\infty \|a\|_{L^1(0,1)}$, there exists $\bar{\lambda} > 0$ such that $\lambda \|h\|_{L^1(0,1)} < 1$, for all $\lambda \in [0, \bar{\lambda})$. Thus, by the regularity result [39, Cor. 3.5], $u \in W^{2,1}(0, 1)$ and therefore u is a regular solution of (1.1). \square

The next result provides information on the asymptotic behavior of the positive, necessarily regular, solutions as $\lambda \rightarrow 0$.

Lemma 2.2. *Assume (f_1) and (a_1) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of positive regular solutions of (1.1) such that $\lambda_n > 0$ for all $n \geq 1$ and*

$$\lim_{n \rightarrow \infty} \lambda_n = 0. \tag{2.1}$$

Then, one has that

$$\lim_{n \rightarrow \infty} u_n = 0 \quad \text{in } W^{2,1}(0, 1). \tag{2.2}$$

Proof. Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be any sequence of positive regular solutions of (1.1) such that $\lambda_n > 0$, for all $n \geq 1$, and (2.1) holds. Let us set, for every n ,

$$\psi_n = \frac{-u'_n}{\sqrt{1+(u'_n)^2}} \in W^{1,\infty}(0, 1).$$

Pick any $x \in (0, 1]$. Integrating the equation of (1.1) in $[0, x]$ yields

$$\psi_n(x) = \lambda_n \int_0^x a(t) f(u_n(t)) dt$$

and hence

$$\|\psi_n\|_{L^\infty(0,1)} \leq \lambda_n \|f\|_\infty \|a\|_{L^1(0,1)}.$$

Consequently, by (2.1), we find that $\lim_{n \rightarrow \infty} \|\psi_n\|_{L^\infty(0,1)} = 0$ and therefore

$$\lim_{n \rightarrow \infty} \|u'_n\|_{L^\infty(0,1)} = 0. \tag{2.3}$$

For each n , let $x_n \in [0, 1]$ be such that

$$u_n(x_n) = \|u_n\|_{L^\infty(0,1)}. \tag{2.4}$$

Let us write, for all $n \geq 1$ and $x \in [0, 1]$,

$$u_n(x) = u_n(x_n) + \int_{x_n}^x u'_n(t) dt. \tag{2.5}$$

For a subsequence, still labeled by n , we have that either

$$\lim_{n \rightarrow \infty} u_n(x_n) = \infty, \tag{2.6}$$

or

$$\lim_{n \rightarrow \infty} u_n(x_n) = u_\omega \in [0, \infty). \tag{2.7}$$

In the former case, thanks to (2.3), (2.4) and (2.6), we infer from (2.5) that

$$\lim_{n \rightarrow \infty} \frac{u_n(x)}{\|u_n\|_{L^\infty(0,1)}} = 1 \quad \text{uniformly in } [0, 1].$$

By (f_1) , this implies that

$$\lim_{n \rightarrow \infty} \frac{f(u_n(x))}{\|u_n\|_{L^\infty(0,1)}^{-q}} = \lim_{n \rightarrow \infty} \frac{f(u_n(x))}{u_n^{-q}(x)} \lim_{n \rightarrow \infty} \frac{u_n^{-q}(x)}{\|u_n\|_{L^\infty(0,1)}^{-q}} = h \quad \text{uniformly in } [0, 1].$$

On the other hand, integrating the equation of (1.1) in $[0, 1]$ yields, for all $n \geq 1$,

$$\int_0^1 a(x) f(u_n(x)) dx = 0 \tag{2.8}$$

and, hence,

$$\int_0^1 a(x) \frac{f(u_n(x))}{\|u_n\|_{L^\infty(0,1)}^{-q}} dx = 0.$$

Thus, letting $n \rightarrow \infty$ and using (f_1) , we obtain that $h \int_0^1 a(x) dx = 0$. As $h > 0$, this contradicts (a_1) , which requires $\int_0^1 a(x) dx < 0$. So, (2.6) cannot occur. Consequently, the condition (2.7) holds. In this case, we infer from (2.5) and (2.3) that $\{u_n\}_{n \geq 1}$ converges to u_ω in $C^1[0, 1]$. Hence, letting $n \rightarrow \infty$ in (2.8) yields

$$f(u_\omega) \int_0^1 a(x) dx = 0.$$

Consequently, since $\int_0^1 a(x) dx < 0$, we get $f(u_\omega) = 0$. By (f_1) , we necessarily have that $u_\omega = 0$. Therefore, we can conclude from (2.5) that $\{u_n\}_{n \geq 1}$ converges to 0 in $C^1[0, 1]$, and actually, by (1.3), in $W^{2,1}(0, 1)$. This ends the proof. \square

The next result establishes the non-existence of positive solutions of (1.1) if $p \geq 1$ when $\lambda > 0$ is small. Whereas in case $0 < p < 1$, by [43, Thm. 9.1] and Lemma 2.1, (1.1) possesses only regular solutions for sufficiently small $\lambda > 0$.

Theorem 2.1. *Assume (f_1) , with $p \geq 1$, and (a_1) . Then, the problem (1.1) has no positive solutions for sufficiently small $\lambda > 0$.*

Proof. Suppose by contradiction that there exists a sequence $\{(\lambda_n, u_n)\}_{n \geq 1}$ of (strictly) positive solutions of (1.1) with $\lambda_n > 0$ for all n and such that $\lim_{n \rightarrow \infty} \lambda_n = 0$. By Lemmas 2.1 and 2.2, we can suppose that all these solutions are regular and (2.2) holds. Let us set, for every $n \geq 1$ and a.e. $x \in [0, 1]$,

$$a_n(x) = a(x) \left[1 + (u'_n(x))^2 \right]^{\frac{3}{2}} \frac{f(u_n(x))}{u_n^p(x)}.$$

Then, due to (1.3), each u_n satisfies

$$\begin{cases} -u''_n = \lambda_n a_n(x) u_n^p, & 0 < x < 1, \\ u'_n(0) = u'_n(1) = 0. \end{cases} \tag{2.9}$$

By (2.2), we have that $\lim_{n \rightarrow \infty} \left[1 + (u'_n(x))^2 \right]^{\frac{3}{2}} = 1$ uniformly in $[0, 1]$, and, by (1.2),

$$\lim_{n \rightarrow \infty} \frac{f(u_n(x))}{u_n^p(x)} = 1 \quad \text{uniformly in } [0, 1]. \tag{2.10}$$

Thus, from these facts, we infer that

$$\lim_{n \rightarrow \infty} a_n = a \quad \text{in } L^1(0, 1). \tag{2.11}$$

Subsequently, we define, for every $n \geq 1$, $v_n = \frac{u_n}{\|u_n\|_{L^\infty(0,1)}}$. By (2.9), each v_n satisfies

$$\begin{cases} -v_n'' = \lambda_n a_n(x) u_n^{p-1} v_n, & 0 < x < 1, \\ v_n'(0) = v_n'(1) = 0, \end{cases}$$

and thus

$$\|v_n''\|_{L^1(0,1)} \leq \lambda_n \|a_n\|_{L^1(0,1)} \|u_n\|_{L^\infty(0,1)}^{p-1}.$$

Hence, from (2.1), (2.11), (2.2) and the assumption $p \geq 1$, we find that $\lim_{n \rightarrow \infty} v_n'' = 0$ in $L^1(0, 1)$. Writing down, for every $n \geq 1$ and $x \in [0, 1]$,

$$v_n'(x) = v_n'(0) + \int_0^x v_n''(t) dt \quad \text{and} \quad v_n(x) = v_n(x_n) + \int_{x_n}^x v_n'(t) dt,$$

where $x_n \in [0, 1]$ is taken so that $v_n(x_n) = \|v_n\|_{L^\infty(0,1)} = 1$, it is easily seen that

$$\lim_{n \rightarrow \infty} v_n = 1 \quad \text{in } W^{2,1}(0, 1). \tag{2.12}$$

As $\int_0^1 a(x) f(u_n(x)) dx = 0$ holds for every $n \geq 1$, by (2.10) and (2.12), we get

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_0^1 a(x) f(u_n(x)) dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{\|u_n\|_{L^\infty(0,1)}^p} \int_0^1 a(x) f(u_n(x)) dx \\ &= \lim_{n \rightarrow \infty} \int_0^1 a(x) \frac{f(u_n(x))}{u_n^p(x)} \frac{u_n^p(x)}{\|u_n\|_{L^\infty(0,1)}^p} dx \\ &= \int_0^1 a(x) \left(\lim_{n \rightarrow \infty} \frac{f(u_n(x))}{u_n^p(x)} \right) \left(\lim_{n \rightarrow \infty} v_n^p(x) \right) dx = \int_0^1 a(x) dx, \end{aligned}$$

which is impossible, because we are assuming, by (a_1) , that $\int_0^1 a(x) dx < 0$. This contradiction ends the proof. \square

3. Global bifurcation from $(\lambda, u) = (0, 0)$ and $(\lambda, u) = (\lambda_0, 0)$ when $p = 1$

Our main goal in this section is to prove, under assumptions (a_1) and (f_1) , with $p = 1$, the existence of connected components of the set of the positive solutions of (1.1), which are indeed strictly positive if they are regular, or if condition (a_2) holds. Thus, we generally suppose that the

functions $a(x)$ and $f(u)$ satisfy (a_1) and (f_1) , with $p = 1$, except in the last theorem, where (a_2) and (f_3) , with $p = 1$, are assumed. In the subsequent analysis the weighted eigenvalue problem (1.5) plays a pivotal role.

We start recalling that, thanks to [42, Thm. 1.4, Rem. 1.9], the problem (1.1) admits positive solutions for sufficiently large $\lambda > 0$. Some changes in the proof yield the following sharper result, which seems optimal in the sense that (1.1) might not admit any positive solution for $\lambda \leq \lambda_0$.

Theorem 3.1. *Assume (f_1) , with $p = 1$, and (a_1) . Then, for every $\lambda > \lambda_0$, the problem (1.1) has at least one positive solution.*

Proof. Fix any $\lambda > \lambda_0$. We will find a positive bounded variation solution u of (1.1) as a global minimizer of the functional $\mathcal{J} : BV(0, 1) \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) = \int_0^1 (\sqrt{1 + (Du^a(x))^2} - 1) dx + \int_0^1 |Du^s| - \lambda \int_0^1 a(x) F(u(x)) dx.$$

It is plain that, without loss of generality, we can suppose that $F(u)$ is an even function.

We first prove that, under (f_1) and (a_1) , \mathcal{J} is coercive and bounded from below in $BV(0, 1)$. Indeed, setting $\kappa = \frac{h}{1-q}$, the condition (1.7) entails that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(u) - \kappa|u|^q| \leq \varepsilon|u|^q + c_\varepsilon \quad \text{for all } u \in \mathbb{R}. \tag{3.1}$$

Hence, setting

$$r = \int_0^1 u(x) dx \quad \text{and} \quad w = u - r \quad \text{for every } u \in BV(0, 1),$$

it follows from the Jensen inequality that

$$\begin{aligned} \mathcal{J}(u) &= \int_0^1 (\sqrt{1 + (Dw^a(x))^2} - 1) dx + \int_0^1 |Dw^s| - \lambda \int_0^1 a(x) F(u(x)) dx \\ &\geq \sqrt{1 + \|Dw^a\|_{L^1(0,1)}^2} - 1 + \int_0^1 |Dw^s| - \lambda \int_0^1 a(x) F(u(x)) dx. \end{aligned} \tag{3.2}$$

On the other hand, since $q \in (0, 1)$, by the Poincaré-Wirtinger inequality, we find that, for a.e. $x \in [0, 1]$,

$$|u(x)|^q - |r|^q \leq |w(x) + r|^q - |r|^q \leq |w(x)|^q \leq \|w\|_\infty^q \leq \|Dw\|^q,$$

where

$$\|Dw\| = \int_0^1 |Dw^a(x)| dx + \int_0^1 |D^s w|$$

is the variation of w . Thus, thanks to (3.1), we find that

$$\begin{aligned} \int_0^1 a(x)F(u(x)) dx &= \int_0^1 a(x) (F(u(x)) - \kappa|u(x)|^q) dx \\ &\quad + \kappa \int_0^1 a(x) (|u(x)|^q - |r|^q) dx + \kappa|r|^q \int_0^1 a(x) dx \\ &\leq \int_0^1 |a(x)| (\varepsilon|u(x)|^q + c_\varepsilon) dx + \kappa\|a\|_{L^1(0,1)}\|Dw\|^q + \kappa|r|^q \int_0^1 a(x) dx \\ &\leq \|a\|_{L^1(0,1)} ((\varepsilon + \kappa)\|Dw\|^q + \varepsilon|r|^q + c_\varepsilon) + \kappa|r|^q \int_0^1 a(x) dx. \end{aligned}$$

Consequently, applying this estimate to (3.2) easily yields

$$\begin{aligned} \mathcal{J}(u) &\geq \|Dw\| - \lambda\|a\|_{L^1(0,1)}(\varepsilon + \kappa)\|Dw\|^q \\ &\quad - \lambda \left(\kappa \int_0^1 a(x) dx + \varepsilon\|a\|_{L^1(0,1)} \right) |r|^q - \lambda c_\varepsilon\|a\|_{L^1(0,1)} - 1. \end{aligned}$$

Thus, as we are assuming that $\int_0^1 a(x) dx < 0$, we can take $\varepsilon > 0$ so small that

$$\kappa \int_0^1 a(x) dx + \varepsilon\|a\|_{L^1(0,1)} < 0.$$

Hence, it is plain that we can find two constants $A > 0, B > 0$ such that

$$\mathcal{J}(u) \geq A(\|Dw\| + |r|^q) - B. \tag{3.3}$$

Condition (3.3) implies that

$$\lim_{\|u\|_{BV(0,1)} \rightarrow +\infty} \mathcal{J}(u) = +\infty \quad \text{and} \quad \inf_{u \in BV(0,1)} \mathcal{J}(u) > -\infty.$$

Since the functional \mathcal{J} is lower semicontinuous with respect to the L^1 -convergence in $BV(0, 1)$, it is a classical fact (see, e.g., [19]) that \mathcal{J} admits a global minimizer $u \in BV(0, 1)$. Moreover, by [4], any minimizer of \mathcal{J} is a bounded variation solution of the problem (1.1).

Next, we will prove that, thanks to the choice $\lambda > \lambda_0$, u is non-trivial. To this end, it suffices to show that $\mathcal{J}(u) < 0$. Condition (1.6), with $p = 1$, implies that, for every sequence $\{s_n\}_{n \geq 1}$, with $s_n > 0$ for all $n \geq 1$, such that

$$\lim_{n \rightarrow \infty} s_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{F(s_n)}{s_n^2} = \frac{1}{2},$$

one has that

$$\lim_{n \rightarrow \infty} \frac{F(s_n \varphi(x))}{s_n^2 \varphi^2(x)} = \frac{1}{2} \quad \text{uniformly in } x \in [0, 1].$$

Thus, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 \left(\frac{(D^a \varphi(x))^2}{1 + \sqrt{1 + s_n^2 (D^a \varphi(x))^2}} - \lambda a(x) \frac{F(s_n \varphi(x))}{s_n^2 \varphi^2(x)} \varphi^2(x) \right) dx \\ = \frac{1}{2} \int_0^1 \left((D^a \varphi(x))^2 - \lambda a(x) \varphi^2(x) \right) dx = \frac{1}{2} \int_0^1 \left(1 - \frac{\lambda}{\lambda_0} \right) (D^a \varphi(x))^2(x) dx < 0. \end{aligned}$$

We therefore can conclude that

$$\mathcal{J}(s_n \varphi) = s_n^2 \int_0^1 \left(\frac{(D^a \varphi(x))^2}{1 + \sqrt{1 + s_n^2 (D^a \varphi(x))^2}} - \lambda a(x) \frac{F(s_n \varphi(x))}{s_n^2 \varphi^2(x)} \varphi^2(x) \right) dx < 0,$$

for large n . This clearly implies that $\mathcal{J}(u) < 0$.

Finally, we show that u can be chosen to be positive. Indeed, since

$$\mathcal{J}(|u|) = \mathcal{J}(u) \quad \text{for all } u \in BV(0, 1),$$

we see that if u is a global minimizer of \mathcal{J} , then $|u|$ is a global minimizer too. \square

For the reader’s convenience, we recall from Section 1 that S_{bv}^+ denotes the set of couples $(\lambda, u) \in [0, \infty) \times BV(0, 1)$ such that (λ, u) is a positive (bounded variation) solution of (1.1). Similarly, S_r^+ stands for the set of couples $(\lambda, u) \in [0, \infty) \times C^1[0, 1]$ such that (λ, u) is a positive regular solution of (1.1). Finally, $S_s^+ = S_{bv}^+ \setminus S_r^+$ is the set of the singular positive solutions of (1.1).

The following result, going back to [43, Thm. 3.1 and 3.2], establishes the existence of two components of S_r^+ bifurcating from $(\lambda, 0)$ at $\lambda = 0$ and at $\lambda = \lambda_0$. By a *component* of S_r^+ it is meant a closed connected subset of S_r^+ , equipped with the topology of $\mathbb{R} \times C^1[0, 1]$, which is maximal for the inclusion. Note that the regularity requirements on $f(u)$ in the next result have been slightly relaxed with respect to those imposed in [43]; they anyhow allow to apply the results in [37], in particular [37, Thm. 6.4.3], to achieve the conclusions. Subsequently, we denote by \mathcal{P}_λ the λ -projection operator, $\mathcal{P}_\lambda(\lambda, u) = \lambda$.

Theorem 3.2. Assume that $f \in C(\mathbb{R}) \cap C^1(-\eta, \eta)$, for some $\eta > 0$, $f'(0) = 1$, and (a_1) . Then, the following assertions hold:

- (a) there exists an unbounded component $\mathfrak{C}_{r,\lambda_0}^+$ of \mathcal{S}_r^+ such that
 - $(\lambda_0, 0) \in \mathfrak{C}_{r,\lambda_0}^+$;
 - $\mathcal{P}_\lambda(\mathfrak{C}_{r,\lambda_0}^+) \subseteq [0, \infty)$;
 - $\lambda = \lambda_0$ if $(\lambda, 0) \in \mathfrak{C}_{r,\lambda_0}^+$ with $\lambda \neq 0$;
 - $\min u > 0$ if $(\lambda, u) \in \mathfrak{C}_{r,\lambda_0}^+ \setminus \{(0, 0), (\lambda_0, 0)\}$.
- (b) there exists an unbounded component $\mathfrak{C}_{r,0}^+$ of \mathcal{S}_r^+ such that
 - $\{0\} \times [0, \infty) \subseteq \mathfrak{C}_{r,0}^+$;
 - $\mathcal{P}_\lambda(\mathfrak{C}_{r,0}^+) \subset [0, \infty)$;
 - $\lambda = \lambda_0$ if $(\lambda, 0) \in \mathfrak{C}_{r,0}^+$ with $\lambda \neq 0$;
 - $\min u > 0$ if $(\lambda, u) \in \mathfrak{C}_{r,0}^+ \setminus \{(0, 0), (\lambda_0, 0)\}$.

Moreover, when (f_1) holds, and hence $F(u)$ is sublinear at infinity, we have that

$$\mathfrak{C}_{r,0}^+ \cap \mathfrak{C}_{r,\lambda_0}^+ = \emptyset,$$

and, in particular, $(0, 0) \notin \mathfrak{C}_{r,\lambda_0}^+$ and $(\lambda_0, 0) \notin \mathfrak{C}_{r,0}^+$.

The last assertion of Theorem 3.2 is a direct consequence of Theorem 2.1 and shows that, much like in the cases when $F(u)$ is superlinear at infinity, or asymptotically linear at infinity, which have been previously treated in [43], [39] and [41], also when $F(u)$ is sublinear at infinity the two components $\mathfrak{C}_{r,0}^+$ and $\mathfrak{C}_{r,\lambda_0}^+$ are disjoint.

When, in addition, $f \in C^2(-\eta, \eta)$, then one can invoke [17, Thm. 1.7] in order to complement Theorem 3.2 with the next result, of a local nature, which basically goes back to [43, Thms. 4.1 and 4.2]. Theorem 3.3 also corrects a wrong assertion made in [43, Thm 4.2] concerning the bifurcation directions.

Theorem 3.3. Assume that $f \in C(\mathbb{R}) \cap C^v(-\eta, \eta)$, for some $\eta > 0$ and $v \geq 2$, $f'(0) = 1$, and (a_1) . Then, in a neighborhood of $(\lambda, u) = (0, 0)$, the component $\mathfrak{C}_{r,0}^+$ consists of the curve $\{(0, \kappa) : \kappa \in [0, \kappa_0)\}$ for some $\kappa_0 > 0$. Similarly, setting

$$V = \left\{ v \in C^1[0, 1] : \int_0^1 v(x)\varphi(x) dx = 0 \right\},$$

where φ is any positive eigenfunction associated with (1.5), there exist $\varepsilon > 0$ and two maps of class C^{v-1} , $\lambda : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ and $v : (-\varepsilon, \varepsilon) \rightarrow V$, such that

- (i) $\lambda(0) = \lambda_0$ and $v(0) = 0$;
- (ii) $(\lambda(s), s(\varphi + v(s)))$ solves (1.1) for all $s \in (-\varepsilon, \varepsilon)$;
- (iii) in a neighborhood of $(\lambda, u) = (\lambda_0, 0)$, $\mathfrak{C}_{r,\lambda_0}^+$ consists of the smooth arc of curve $(\lambda(s), s(\varphi + v(s)))$, with $s \in [0, \varepsilon)$.

Moreover, the following holds:

$$\lambda'(0) = -\lambda_0 f''(0) \frac{\int_0^1 \varphi(x)(\varphi'(x))^2 dx}{\int_0^1 (\varphi'(x))^2 dx} \tag{3.4}$$

and, if $v \geq 3$ and $f''(0) = 0$,

$$\lambda''(0) = -\frac{f'''(0) \int_0^1 \varphi^2(x)(\varphi'(x))^2 dx + \int_0^1 (\varphi'(x))^4 dx}{\int_0^1 (\varphi'(x))^2 dx}. \tag{3.5}$$

Thus, the component $\mathfrak{C}_{r,\lambda_0}^+$ bifurcates subcritically at $\lambda = \lambda_0$ if $f''(0) > 0$, or if $f''(0) = 0$ and

$$f'''(0) > -\frac{\int_0^1 (\varphi(x)')^4 dx}{\int_0^1 \varphi^2(x)(\varphi'(x))^2 dx},$$

while it does it supercritically if $f''(0) < 0$, or if $f''(0) = 0$ and

$$f'''(0) < -\frac{\int_0^1 (\varphi(x)')^4 dx}{\int_0^1 \varphi^2(x)(\varphi'(x))^2 dx}.$$

Proof. Since assertions (i)–(iii) follow from [43, Thms. 4.1 and 4.2], we only provide the proof of formulas (3.4) and (3.5). In the course of this proof, in order to simplify the notation, the dependence on x is not indicated. Set

$$u(s) = s(\varphi + v(s)) \quad \text{for all } s \in (-\varepsilon, \varepsilon).$$

Substituting $(\lambda, u) = (\lambda(s), u(s))$ in (1.3) and dividing by s , we find that

$$-(\varphi + sv_1 + o(s))'' = (\lambda_0 + s\lambda_1 + o(s))a(x)(\varphi + sv_1 + o(s)) \cdot \left[1 + \frac{f''(0)}{2}s(\varphi + sv_1 + o(s)) + o(s) \right] \left[1 + \frac{3}{2}(\varphi')^2s^2 + o(s^2) \right]$$

for sufficiently small s , where

$$\lambda_1 = \lambda'(0), \quad v_1 = \frac{dv}{ds}(0).$$

Particularizing at $s = 0$, we get

$$-\varphi'' = \lambda_0 a \varphi, \tag{3.6}$$

which is true by the definition of λ_0 and φ . Identifying terms of order s yields

$$-v_1'' = \lambda_0 a v_1 + \lambda_0 \frac{f''(0)}{2} a \varphi^2 + \lambda_1 a \varphi.$$

Multiplying this equation by φ and integrating by parts in $(0, 1)$, we find from (3.6) that

$$\frac{1}{2}\lambda_0 f''(0) \int_0^1 a\varphi^3 dx + \lambda_1 \int_0^1 a\varphi^2 dx = 0. \tag{3.7}$$

On the other hand, multiplying (3.6) by φ and φ^2 , respectively, and integrating by parts in $(0, 1)$, we get

$$\lambda_0 \int_0^1 a\varphi^2 dx = - \int_0^1 \varphi'' \varphi dx = \int_0^1 (\varphi')^2 dx > 0 \tag{3.8}$$

and

$$\lambda_0 \int_0^1 a\varphi^3 dx = - \int_0^1 \varphi'' \varphi^2 dx = \int_0^1 \varphi'(\varphi^2)' dx = 2 \int_0^1 \varphi(\varphi')^2 dx,$$

as φ is positive and not constant. Hence, by eliminating λ_1 in (3.7), thanks to (3.8), we find that

$$\lambda'(0) = \lambda_1 = -\frac{1}{2}\lambda_0 f''(0) \frac{\int_0^1 a\varphi^3 dx}{\int_0^1 a\varphi^2 dx} = -\lambda_0 f''(0) \frac{\int_0^1 \varphi(\varphi')^2 dx}{\int_0^1 (\varphi')^2 dx},$$

thus proving (3.4).

Subsequently, we suppose $\nu \geq 3$ and $f''(0) = 0$. Then, by (3.4) we have that $\lambda_1 = 0$ and hence $-v_1'' = \lambda_0 a v_1$. Thus, there exists $\alpha \in \mathbb{R}$ such that $v_1 = \alpha\varphi$. Therefore, since $v_1 \in V$, we find that $\alpha = 0$, which implies $v_1 = 0$. Consequently, substituting $(\lambda(s), u(s))$ in (1.3) and dividing by s yields

$$-(\varphi + s^2 v_2 + o(s^2))'' = (\lambda_0 + s^2 \lambda_2 + o(s^2))a(x)(\varphi + s^2 v_2 + o(s^2)) \cdot \left[1 + \frac{f'''(0)}{6} s^2 (\varphi + s^2 v_2 + o(s^2))^2 + o(s^2) \right] \left[1 + \frac{3}{2} s^2 (\varphi')^2 + o(s^2) \right],$$

where

$$\lambda_2 = \frac{1}{2} \frac{d^2 \lambda}{ds^2}(0), \quad v_2 = \frac{1}{2} \frac{d^2 v}{ds^2}(0).$$

Consequently, identifying terms of order s^2 , we obtain that

$$-v_2'' = \lambda_0 a v_2 + \frac{3}{2} \lambda_0 a \varphi (\varphi')^2 + \lambda_2 a \varphi + \frac{f'''(0)}{6} \lambda_0 a \varphi^3. \tag{3.9}$$

Thus, multiplying (3.9) by φ and integrating by parts in $(0, 1)$ gives

$$\frac{3}{2}\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx + \lambda_2 \int_0^1 a\varphi^2 dx + \frac{f'''(0)}{6}\lambda_0 \int_0^1 a\varphi^4 = 0$$

and hence, as $\int_0^1 a\varphi^2 dx > 0$ by (3.8),

$$\frac{1}{2}\lambda''(0) = \lambda_2 = -\frac{\frac{1}{6}f'''(0)\lambda_0 \int_0^1 a\varphi^4 dx + \frac{3}{2}\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx}{\int_0^1 a\varphi^2 dx}.$$

On the other hand, multiplying (3.6) by φ^3 and $\varphi(\varphi')^2$, respectively, and integrating by parts in $(0, 1)$, we get

$$\lambda_0 \int_0^1 a\varphi^4 dx = -\int_0^1 \varphi''\varphi^3 dx = 3 \int_0^1 \varphi^2(\varphi')^2 dx$$

and

$$\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx = -\int_0^1 \varphi(\varphi')^2\varphi'' dx = -\int_0^1 \varphi \frac{d}{dx} \left(\frac{1}{3}(\varphi')^3\right) dx = \frac{1}{3} \int_0^1 (\varphi')^4 dx > 0.$$

Thus, by using (3.8), we can conclude that

$$\lambda''(0) = -\frac{f'''(0) \int_0^1 \varphi^2(\varphi')^2 dx + \int_0^1 (\varphi')^4 dx}{\int_0^1 (\varphi')^2 dx}$$

and, therefore, (3.5) is proven. The statements concerning the bifurcation directions are obvious consequences of (3.4) and (3.5). \square

The next global bifurcation result holds true for bounded variation solutions of (1.1).

Theorem 3.4. *Assume (a_2) and (f_3) with $p = 1$. Then, there exist two subsets of \mathcal{S}_{bv}^+ , $\mathfrak{C}_{bv,0}^+$ and $\mathfrak{C}_{bv,\lambda_0}^+$, such that, for every $\rho > 2$,*

- $\mathfrak{C}_{bv,0}^+ = \{0\} \times [0, \infty)$;
- $\mathfrak{C}_{bv,0}^+ \cap \mathfrak{C}_{bv,\lambda_0}^+ = \emptyset$;
- $\mathfrak{C}_{bv,\lambda_0}^+$ is maximal in \mathcal{S}_{bv}^+ with respect to the inclusion, is connected in $\mathbb{R} \times BV(0, 1)$, having endowed $BV(0, 1)$ with the topology of the strict convergence (cf. [3, Def. 3.14]), and is unbounded in $\mathbb{R} \times L^\rho(0, 1)$;
- $(\lambda, 0) \in \mathfrak{C}_{bv,\lambda_0}^+$ if and only if $\lambda = \lambda_0$;
- $\text{ess inf } u > 0$ if $(\lambda, u) \in \mathfrak{C}_{bv,\lambda_0}^+$ with $u \neq 0$;

- there exists a neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times L^\rho(0, 1)$ such that $\mathfrak{C}_{bv, \lambda_0}^+ \cap U$ consists of regular solutions of (1.1), i.e.,

$$\mathfrak{C}_{bv, \lambda_0}^+ \cap U = \mathfrak{C}_{r, \lambda_0}^+ \cap U. \tag{3.10}$$

Proof. Condition (f_3) implies that, for every $\rho > 2$, there exists a constant $\kappa > 0$ such that

$$|f'(u)| \leq \kappa (|u|^{e-2} + 1) \quad \text{for all } u \in \mathbb{R}.$$

Therefore, Theorem 3.4 is a direct consequence of [39, Thm.1.1] and of Theorem 2.1. \square

Remark 3.1. According to (3.10), the small bounded variation solutions of (1.1) must be regular solutions, and thus $\mathfrak{C}_{r, \lambda_0}^+ \subseteq \mathfrak{C}_{bv, \lambda_0}^+$. One of the main goals of this paper is ascertaining, whether, or not, $\mathfrak{C}_{r, \lambda_0}^+$ is a proper subcomponent of $\mathfrak{C}_{bv, \lambda_0}^+$. Note that, whenever $\mathfrak{C}_{r, \lambda_0}^+ \subsetneq \mathfrak{C}_{bv, \lambda_0}^+$, regular solutions develop singularities along the same component.

4. Bifurcation from $(\lambda, u) = (0, 0)$ when $0 < p < 1$

Throughout this section, we assume that the functions $a(x)$ and $f(u)$ satisfy (a_2) and (f_1) with $0 < p < 1$, respectively. The main goal of this section is establishing the existence of a component of the set \mathcal{S}_r^+ of positive regular solutions bifurcating from $(0, 0)$. Our starting point is the next result which is a consequence of [43, Thm. 9.1].

Theorem 4.1. Assume (f_1) , with $p \in (0, 1)$, and (a_2) . Then, there exists $\eta > 0$ such that, for every $\lambda \in (0, \eta)$, the problem (1.1) has at least one positive regular solution, u_λ . Moreover, one has that

$$\lim_{\lambda \rightarrow 0} \|u_\lambda\|_{C^1[0,1]} = 0, \tag{4.1}$$

regardless each particular choice of u_λ .

As the proof of [43, Thm. 9.1] is based on the direct method of calculus of variations, Theorem 4.1 does not guarantee the existence of a component of \mathcal{S}_r^+ containing these solution pairs (λ, u_λ) . By relying instead on the construction of sub- and supersolutions and on the use of the topological degree, we can complement Theorem 4.1 as follows.

Theorem 4.2. Assume (f_1) , with $p \in (0, 1)$, and (a_2) . Then, there is a component $\mathfrak{C}_{r,0}^+$ of \mathcal{S}_r^+ such that $[0, \lambda_*) \subseteq \mathcal{P}_\lambda(\mathfrak{C}_{r,0}^+)$, for some $\lambda_* > 0$, and (4.1) holds, for every $(\lambda, u_\lambda) \in \mathfrak{C}_{r,0}^+$.

Proof. Without loss of generality, we can suppose in the course of this proof that $f \in C(\mathbb{R})$ is an odd function. By performing the change of variable

$$u = \varepsilon v, \quad \varepsilon = \lambda^{\frac{-1}{p-1}}, \tag{4.2}$$

the problem (1.1), or (1.3), can be equivalently written in the form

$$\begin{cases} -v'' = a(x)|v|^p \operatorname{sgn}(v) g(\varepsilon v') h(\varepsilon v), & 0 < x < 1, \\ v'(0) = v'(1) = 0, \end{cases} \tag{4.3}$$

where g is defined in (1.4) and

$$h(u) = \begin{cases} \frac{f(u)}{|u|^p \operatorname{sgn}(u)} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases} \tag{4.4}$$

According to (1.2), the problem (4.3) perturbs, as $\varepsilon > 0$ separates away from 0, from the semi-linear problem

$$\begin{cases} -v'' = a(x)|v|^p \operatorname{sgn}(v), & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{4.5}$$

We claim that the problem (4.5) admits a subsolution α and a supersolution β , with $\alpha(x) < \beta(x)$ for all $x \in [0, 1]$, such that every possible solution v of (4.5), with $v \geq \alpha$ in $[0, 1]$, satisfies $v(x) > \alpha(x)$ for all $x \in [0, 1]$ and, similarly, every possible solution v of (4.5), with $v \leq \beta$ in $[0, 1]$, satisfies $v(x) < \beta(x)$ for all $x \in [0, 1]$. This means that α and β are strict sub- and supersolutions according to, e.g., [18, Ch. III].

Construction of a subsolution. Let μ_1 be the unique positive eigenvalue, with an associated positive eigenfunction φ_1 , of the weighted problem

$$\begin{cases} -\varphi'' = \mu a(x) \varphi, & 0 < x < z, \\ \varphi(0) = \varphi(z) = 0. \end{cases}$$

Then, pick $c > 0$ so small that

$$\mu_1 [c\varphi_1(x)]^{1-p} \leq 1 \quad \text{for all } x \in [0, z] \tag{4.6}$$

and define

$$\alpha(x) = \begin{cases} c \varphi_1(x) & \text{if } 0 \leq x \leq z, \\ c \varphi_1'(z) (x - z) & \text{if } z < x \leq 1. \end{cases} \tag{4.7}$$

It is clear that $\alpha \in W^{2,\infty}(0, 1)$ and, since $p \in (0, 1)$, by (4.6) and (4.7), it satisfies

$$\begin{aligned} -\alpha''(x) &= -c \varphi_1''(x) = a(x) \mu_1 c \varphi_1(x) = a(x) \mu_1 [c \varphi_1(x)]^{1-p} [c\varphi_1(x)]^p \\ &\leq a(x) [c \varphi_1(x)]^p = a(x) \alpha^p(x) \quad \text{for a.e. } x \in (0, z) \end{aligned} \tag{4.8}$$

and

$$-\alpha''(x) = 0 \leq a(x) |\alpha(x)|^p \operatorname{sgn}(\alpha(x)) \quad \text{for a.e. } x \in (z, 1).$$

Further, we have that

$$\alpha'(0) = c\varphi_1'(0) > 0, \quad \alpha'(1) = c\varphi_1'(z) < 0.$$

Now, we will show that any solution v of (4.5) such that $v \geq \alpha$ in $[0, 1]$ also satisfies $v(x) > \alpha(x)$ for all $x \in [0, 1]$. Indeed, set $w = v - \alpha$ and suppose, by contradiction, that $\min w = 0$. Let $x_0 \in [0, 1]$ be such that $w(x_0) = 0$. Since $w \geq 0$ in $[0, 1]$ and

$$w'(0) = -\alpha'(0) < 0 < -\alpha'(1) = w'(1),$$

it follows that $x_0 \in (0, 1)$. Hence, $w'(x_0) = 0$.

Suppose that $x_0 \in (0, z]$. Then, as $v \geq \alpha \geq 0$ in $[0, z]$ and $p > 0$, we can infer from (4.8) that

$$-w''(x) = -v''(x) + \alpha''(x) \geq a(x)(v^p(x) - \alpha^p(x)) \geq 0 \quad \text{for a.e. } x \in (0, z).$$

Thus, w is concave in $[0, z]$ and hence

$$w(x) \leq w(x_0) + w'(x_0)(x - x_0) = 0 \quad \text{for all } x \in [0, z].$$

This implies that $w = 0$ in $[0, z]$, contradicting $w'(0) < 0$. Therefore, $x_0 \in (z, 1)$. Since $w(x_0) = 0$, we have that

$$v(x_0) = \alpha(x_0) = c\varphi_1'(z)(x_0 - z) < 0.$$

Thus, there exists an interval $J \subseteq (z, 1)$, with $x_0 \in J$, such that $v(x) < 0$ if $x \in J$ and

$$-w''(x) = -v''(x) = a(x)|v(x)|^p \operatorname{sgn}(v(x)) > 0 \quad \text{for a.e. } x \in J. \tag{4.9}$$

Hence, w is concave in J . Arguing as above, we find that $w = 0$ in J , thus contradicting the strict inequality in (4.9). Thus, we have proved that $v(x) > \alpha(x)$ for all $x \in [0, 1]$.

Construction of a supersolution. For every $k > 0$, let ζ_k denote the unique solution of the linear problem

$$\begin{cases} -\zeta'' = \left(a(x) - \int_0^1 a(t) dt \right) k^p, & 0 < x < 1, \\ \zeta'(0) = \zeta'(1) = 0, \quad \int_0^1 \zeta(t) dt = 0. \end{cases}$$

The Poincaré–Wirtinger inequality yields

$$\|\zeta_k\|_{L^\infty(0,1)} \leq \|\zeta_k'\|_{L^1(0,1)} \leq \|\zeta_k'\|_{L^\infty(0,1)} \leq \|\zeta_k''\|_{L^1(0,1)} \leq 2\|a\|_{L^1(0,1)} k^p. \tag{4.10}$$

Consequently, since $p \in (0, 1)$, the function β defined by $\beta = \zeta_k + k$ satisfies, for sufficiently large $k > 0$, $\min \beta > \max \alpha > 0$. Moreover, for a.e. $x \in [0, 1]$, we have that

$$\begin{aligned}
 -\beta''(x) &= -\zeta_k'' = a(x)k^p - k^p \int_0^1 a(t) dt \\
 &= a(x)\beta^p(x) + a(x)[k^p - \beta^p(x)] - k^p \int_0^1 a(t) dt
 \end{aligned}$$

and hence

$$-\beta''(x) = a(x)\beta^p(x) + k^p \left[a(x) \left(1 - \left(1 + \frac{\zeta_k(x)}{k} \right)^p \right) - \int_0^1 a(t) dt \right]. \tag{4.11}$$

Using (4.10) and the assumption $p \in (0, 1)$, it is easily seen that

$$\lim_{k \rightarrow \infty} \left[a(x) \left(1 - \left(1 + \frac{\zeta_k(x)}{k} \right)^p \right) \right] = 0 \quad \text{uniformly a.e. in } [0, 1].$$

Thus, since $\int_0^1 a(t) dt < 0$, we can conclude from (4.11) that, for sufficiently large $k > 0$,

$$-\beta''(x) \geq a(x)\beta^p(x) - \frac{1}{2}k^p \int_0^1 a(t) dt \quad \text{for a.e. } x \in [0, 1], \tag{4.12}$$

and hence $-\beta''(x) > a(x)\beta^p(x)$ for a.e. $x \in [0, 1]$. Thus, the function β is a supersolution of (4.5) satisfying the boundary conditions.

Now, we will show that any solution v of (4.5) such that $v \leq \beta$ in $[0, 1]$ satisfies $v(x) < \beta(x)$ for all $x \in [0, 1]$. Indeed, consider the function $w = \beta - v$ and suppose, by contradiction, that $\min w = 0$. Let $x_0 \in [0, 1]$ be such that $w(x_0) = 0$. Then, there exists an interval $J \subseteq [0, 1]$, with $x_0 \in J$, such that for a.e. $x \in J$

$$|a(x)(\beta^p(x) - |v(x)|^p \operatorname{sgn}(v(x)))| < -\frac{1}{2}k^p \int_0^1 a(t) dt \tag{4.13}$$

and hence, by (4.12), (4.5) and (4.13),

$$\begin{aligned}
 -w''(x) &= -\beta''(x) + v''(x) \\
 &\geq a(x)\beta^p(x) - a(x)|v(x)|^p \operatorname{sgn}(v(x)) - \frac{1}{2}k^p \int_0^1 a(t) dt > 0.
 \end{aligned} \tag{4.14}$$

Thus, w is concave in J . Arguing similarly, we find that $w = 0$ in J . So, contradicting the strict inequality in (4.14). Therefore, we have shown that $v(x) < \beta(x)$ for all $x \in [0, 1]$.

Degree computation. Note that any solution v of (4.5) such that $\alpha \leq v \leq \beta$ also satisfies

$$\|v'\|_\infty \leq \|a\|_{L^1(0,1)} \max\{|\min \alpha|^p, (\max \beta)^p\}.$$

Pick a constant

$$C > \|a\|_{L^1(0,1)} \max\{|\min \alpha|^p, (\max \beta)^p\}$$

and consider the open bounded subset of $C^1[0, 1]$ defined by

$$\Omega = \{v \in C^1[0, 1] : \alpha(x) < v(x) < \beta(x) \text{ for all } x \in [0, 1], \|v'\|_{L^\infty(0,1)} < C\}.$$

Since $\alpha(x) < \beta(x)$ for all $x \in [0, 1]$, Ω is non-empty. Let $\mathcal{T} : [0, \infty) \times C^1[0, 1] \rightarrow C^1[0, 1]$ denote the operator which sends any $(\varepsilon, v) \in [0, \infty) \times C^1[0, 1]$ to the unique solution $w \in W^{2,\infty}(0, 1)$ of the linear problem

$$\begin{cases} -w'' + w = a(x) |v|^p \operatorname{sgn}(v) g(\varepsilon v') h(\varepsilon v) + v, & 0 < x < 1, \\ w'(0) = w'(1) = 0. \end{cases}$$

It is plain that \mathcal{T} is completely continuous and its fixed points are precisely the solutions of the problem (4.3). As α and β are, respectively, a strict subsolution and a strict supersolution of (4.5), by our choice of the constant C , it follows that $\mathcal{T}(0, \cdot)$ has no fixed points on $\partial\Omega$. A standard argument (see [18, Ch. III]) also shows that

$$\operatorname{deg}_{LS}(\mathcal{I} - \mathcal{T}(0, \cdot), \Omega, 0) = 1.$$

Existence of continua. The boundedness of $\partial\Omega$ and the complete continuity of the operator \mathcal{T} guarantee the existence of some $\varepsilon^* > 0$ such that $\mathcal{T}(\varepsilon, \cdot)$ has no fixed point on $\partial\Omega$ for all $\varepsilon \in [0, \varepsilon^*]$. Consequently, the Leray-Schauder continuation theorem [36, p. 63] yields the existence of a continuum of solutions (ε, v) of the problem (4.3), where $\varepsilon \in [0, \varepsilon^*]$ and $v \in \Omega$, and hence of solutions (λ, u) of the problem (1.1), where $\lambda = \varepsilon^{1-p} \in [0, \lambda^*]$, with $\lambda^* = (\varepsilon^*)^{1-p}$, and $u = \lambda^{\frac{1}{1-p}} v$.

Let us verify that, for each $\varepsilon \in [0, \varepsilon^*]$, v is positive and, therefore, for every $\lambda \in (0, \lambda^*]$, u is positive. Indeed, otherwise, owing to the definition of α , there should exist $x_0 \in (z, 1]$ such that $v(x_0) = \min v < 0$ and $v'(x_0) = 0$. Then, one would infer from (4.3) the existence of an interval $J \subseteq (z, 1]$, with $x_0 \in J$, such that $v''(x) < 0$ for a.e. $x \in J$. This is clearly impossible at a minimum point which is also a critical point.

As in [53,54], by the Zorn lemma, this continuum of positive solutions can be eventually continued to a component $\mathcal{C}_{r,0}^+$ of the set of positive regular solutions of (1.1). Finally, by Lemma 2.2, (4.1) holds for sufficiently small $\lambda > 0$. \square

5. Bifurcation from $(\lambda, u) = (\infty, 0)$ when $p > 1$

Throughout this section, we assume that the functions $a(x)$ and $f(u)$ satisfy (a_2) and (f_1) with $p > 1$, respectively. In this case, by Theorem 2.1, the problem (1.1) cannot have any solution for sufficiently small $\lambda > 0$. The main goal of this section is establishing the existence of a

component of positive regular solutions of (1.1) bifurcating from 0 as $\lambda \rightarrow \infty$. From [42, Thm. 1.5] and [43, Thm. 10.1] the following result can be deduced.

Theorem 5.1. *Assume (f_1) , with $p \in (1, \infty)$, and (a_2) . Then, the problem (1.1) has at least two positive solution u_λ, v_λ for sufficiently large $\lambda > 0$. Moreover, u_λ is regular and can be chosen so that $\lim_{\lambda \rightarrow \infty} \|u_\lambda\|_{C^1[0,1]} = 0$.*

The next result complements Theorem 5.1 by establishing the existence of a component of the set \mathcal{S}_r^+ containing small solutions for sufficiently large $\lambda > 0$.

Theorem 5.2. *Assume (f_1) , with $p \in (1, \infty)$, and (a_2) . Then, there is a component $\mathfrak{C}_{r,\infty}^+$ of \mathcal{S}_r^+ such that $(\lambda^*, \infty) \subseteq \mathcal{P}_\lambda(\mathfrak{C}_{r,\infty}^+)$, for some $\lambda^* > 0$, and*

$$\lim_{\lambda \rightarrow \infty} \min\{\|u_\lambda\|_{C^1[0,1]} : (\lambda, u_\lambda) \in \mathfrak{C}_{r,\infty}^+\} = 0. \tag{5.1}$$

Theorems 5.1 and 5.2 confirm that the global bifurcation diagram of (1.1) looks like the right (red) plots of Figs. 2 and 3 show, according to the regularity properties of the function $a(x)$ at z .

Theorem 5.2 can also be deduced from some conclusions established in the proof of Theorem 10.1 in [43], which relies on Mawhin’s coincidence degree theory and on some estimates developed in [20] for an auxiliary problem. Indeed, from [43] we know that there exist a completely continuous operator $\mathcal{T} : [0, \infty) \times C^1[0, 1] \rightarrow C^1[0, 1]$ (canonically defined in the frame of the coincidence degree theory), whose fixed points are the positive regular solutions of (4.3), and an open bounded set Ω in $C^1[0, 1]$ (basically a spherical shell) such that $\text{deg}_{LS}(\mathcal{I} - \mathcal{T}(0, \cdot), \Omega, 0) = -1$. Consequently, like in the proof of Theorem 4.2, the Leray-Schauder continuation theorem [36, p. 63] yields the existence of a continuum of solutions (ε, v) of (4.3), where $\varepsilon \in (0, \varepsilon^*)$ for some $\varepsilon^* > 0$ and $v \in \Omega$, and hence of solutions (λ, u) of (1.1), where $\lambda = \varepsilon^{1-p} \in (\lambda^*, \infty)$, with $\lambda^* = (\varepsilon^*)^{1-p}$, and $u = \lambda^{\frac{1}{1-p}} v$.

An alternative argument is provided in the proof given below which instead combines bifurcation methods and topological techniques based on the theory of superlinear indefinite problems developed by Amann and López-Gómez in [2]. Since the arguments that we will exploit are by now quite standard, the most straightforward details will be omitted, however making, when necessary, precise references to the literature.

Proof. Like in the proof of Theorem 4.2 we suppose that $f \in \mathcal{C}(\mathbb{R})$ is an odd function and we make the change of variable (4.2). Then, the problem (1.1), or (1.3), can be equivalently written as (4.3), where g and h are defined by (1.4) and (4.4), respectively. By (1.2), this problem perturbs, as $\varepsilon \rightarrow 0$, from the semilinear boundary value problem (4.5). In the following proof we confine ourselves to treat the case where $p \geq 2$.

A related problem. Let us embed (4.5) into the following problem, depending on an auxiliary parameter $\mu \in \mathbb{R}$,

$$\begin{cases} -v'' = \mu v + a(x)|v|^p \text{sgn}(v), & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{5.2}$$

It is apparent that (4.5) can be obtained from (5.2) by freezing the value of the parameter μ at $\mu = 0$. Note that (5.2) is a simple one-dimensional prototype of the multidimensional model introduced in [2].

Bifurcation analysis. Since $\mu = 0$ is a simple algebraic eigenvalue of $-D^2$ under Neumann boundary conditions with associated eigenfunction 1, the local index of zero changes as μ crosses zero (see, e.g., [37, Thm. 5.6.2]). In other words, the associated (local) topological degree of the solution $(\mu, 0)$ changes as μ crosses the critical value $\mu = 0$. Thus, thanks to [37, Thm. 7.1.3], there is a component $\mathfrak{C}_{\mu,0}^+$ of the set of positive solutions of (5.2) in $\mathbb{R} \times C^1[0, 1]$ such that

$$(\mu, v) = (0, 0) \in \overline{\mathfrak{C}_{\mu,0}^+}$$

Actually, the fact that a global bifurcation phenomenon should occur at any critical value of the parameter where the degree changes is folklore since, at least, the pioneering work by P. H. Rabinowitz in [53]. Note that, thanks to the local bifurcation theorem of Crandall and Rabinowitz, [17], which applies as $p \geq 2$, $\mathfrak{C}_{\mu,0}^+$ is a curve of (at least) class C^1 in a neighborhood of $(0, 0)$.

Let $\{(\mu_n, v_n)\}_{n \geq 1}$ be a sequence of solutions of $\mathfrak{C}_{\mu,0}^+$, with $v_n \neq 0$, such that

$$\lim_{n \rightarrow \infty} (\mu_n, v_n) = (0, 0) \quad \text{in } \mathbb{R} \times C^1[0, 1]. \tag{5.3}$$

This sequence exists because, by construction, the component $\mathfrak{C}_{\mu,0}^+$ emanates from $(\mu, v) = (0, 0)$. Then, as it will become apparent below, we have that

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{\|v_n\|_\infty^{p-1}} = - \int_0^1 a(x) dx > 0 \tag{5.4}$$

and hence $\mu_n > 0$ for sufficiently large n . In particular, $\mathfrak{C}_{\mu,0}^+$ bifurcates supercritically from $(\mu, u) = (0, 0)$, i.e., there is neighborhood U of $(0, 0)$ such that $\mu > 0$ if $(\mu, v) \in \mathfrak{C}_{\mu,0}^+ \cap U$. To prove (5.4) one can argue as follows. Since

$$-v_n'' = \mu_n v_n + a(x)v_n^p, \quad n \geq 1, \tag{5.5}$$

we have that

$$\frac{v_n}{\|v_n\|_\infty} = (-D^2 + 1)^{-1} \left[\frac{v_n}{\|v_n\|_\infty} + \mu_n \frac{v_n}{\|v_n\|_\infty} + a(x) \frac{v_n}{\|v_n\|_\infty} v_n^{p-1} \right],$$

where $(-D^2 + 1)^{-1}$ stands for the resolvent operator of $-D^2 + 1$ under homogeneous Neumann boundary conditions. As $(-D^2 + 1)^{-1}$ is compact, there exists a subsequence of $\varphi_n := \frac{v_n}{\|v_n\|_\infty}$, $n \geq 1$, relabeled by n , such that

$$\lim_{n \rightarrow \infty} \varphi_n = \varphi \in C^2[0, 1] \quad \text{in } C^1[0, 1].$$

By (5.3), letting $n \rightarrow \infty$ it is easily seen that necessarily $\varphi = 1$ and, since this argument can be repeated along any subsequence, it becomes apparent that

$$\lim_{n \rightarrow \infty} \varphi_n = 1 \quad \text{in } \mathcal{C}^1[0, 1]. \tag{5.6}$$

On the other hand, integrating (5.5) in $[0, 1]$ and dividing by $\|v_n\|_\infty^p$ yields

$$\frac{\mu_n}{\|v_n\|_\infty^{p-1}} \int_0^1 \frac{v_n(x)}{\|v_n\|_\infty} dx = - \int_0^1 a(x) \left(\frac{v_n(x)}{\|v_n\|_\infty} \right)^p dx.$$

Consequently, letting $n \rightarrow \infty$ in this identity, (5.4) follows readily from (5.6). This shows that $\mathfrak{C}_{\mu,0}^+$ bifurcates towards the interval $\mu > 0$.

Suppose that (5.2) admits a positive solution, (μ, v) . Then, since $p > 1$, v is strictly positive and hence $(-D^2 - \mu)v = a(x)v^p(x) > 0$ for all $x \in [0, z]$. Moreover, $v(0) > 0$ and $v(z) > 0$. Thus, v provides us with a positive strict supersolution of $-D^2 - \mu$ in $(0, z)$ under homogeneous Dirichlet boundary conditions and, due to [38, Thm. 7.10], this is equivalent to the positivity of the associated principal eigenvalue, i.e.,

$$\sigma[-D^2 - \mu; \mathcal{D}, (0, z)] = \left(\frac{\pi}{z}\right)^2 - \mu > 0.$$

So, $\mu < \mu_* \equiv \left(\frac{\pi}{z}\right)^2$. Therefore, (5.2) cannot admit any positive solution if $\mu \geq \mu_*$. In particular, $\mathcal{P}_\mu(\mathfrak{C}_{\mu,0}^+) \subset (-\infty, \mu_*)$. Moreover, by the generalized a priori bounds of Amann and López-Gómez [2, Sect. 4], as we are working with a one-dimensional problem, for every compact interval $K \subset \mathbb{R}$, there is a constant $C = C(K) > 0$ such that $\|v\|_{\mathcal{C}^1[0,1]} \leq C$ for any positive solution (μ, v) of (5.2) with $\mu \in K$. Therefore, setting

$$\mu_c := \max_{(\mu, v) \in \mathfrak{C}_{\mu,0}^+} \mu,$$

we have that $\mu_c \in (0, \mu_*]$ and that $\mathcal{P}_\mu(\mathfrak{C}_{\mu,0}^+) = (-\infty, \mu_c]$, as illustrated in Fig. 4, where we are plotting μ , in abscissas, versus $\|v\|_{\mathcal{C}^1[0,1]}$ in ordinates. Thus, each solution (μ, v) of (5.1) is represented by a single point on some of the components plotted in the figure. Naturally, (5.1) might have other components of positive solutions, like \mathfrak{D}^+ .

Stability analysis. As the main technical device to get the uniqueness of the stable solution in [29, 30] is the Picone identity [52], which due to [22, Lem. 9.3] remains true for Neumann boundary conditions, the theory of Gómez-Reñasco and López-Gómez in [29,30] can be adapted to our present setting to show that the unique stable positive solutions of (5.2) are the minimal solutions of (5.2) for $\mu > 0$ (see [23]), i.e., those on the branch of $\mathfrak{C}_{\mu,0}^+$ plotted with a continuous line in Fig. 4. In particular, they are linearly asymptotically stable for all $\mu \in (0, \mu_c)$ and neutrally stable for $\mu = \mu_c$. Whereas, by [29,30], the remaining solutions, plotted with a dashed line in Fig. 4, must be linearly unstable.

Fixed point index calculations. Subsequently, the positive solutions of (5.2) are regarded as positive fixed points of the compact operator $\mathcal{K} : \mathbb{R} \times \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$ defined by

$$\mathcal{K}(\mu, v) = (-D^2 + 1)^{-1}[(\mu + 1)v + a(x)|v|^p \text{sgn}(v)]. \tag{5.7}$$

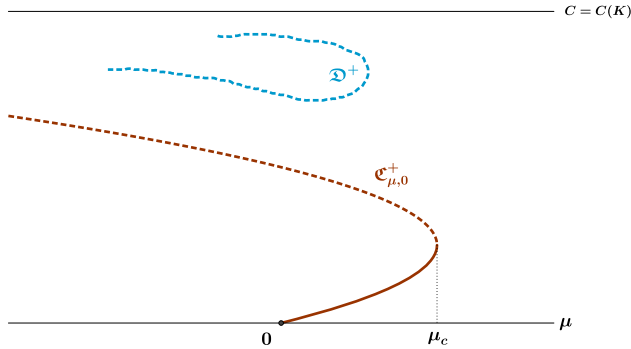


Fig. 4. The components $\mathfrak{C}_{\mu,0}^+$ and \mathfrak{D}^+ of S_r^+ ; $\mathfrak{C}_{\mu,0}^+$ bifurcates supercritically from $(\mu, 0)$ at $\mu = 0$ and goes backwards at some turning point $\mu_c > 0$.

Let B denote any bounded open subset of $C^1[0, 1]$ containing all non-negative fixed points (μ, v) of (5.7), with $\mu \in [-1, \mu_*]$. It exists by the uniform a priori bounds on compact subintervals of μ and the non-existence for $\mu \geq \mu_*$ already discussed before. Next, we will invoke the theory of fixed point index in cones as developed by H. Amann in his celebrated review [1], which we refer to for all properties of the fixed point index that will be used in the rest of this proof. Since B contains all non-negative fixed points of $\mathcal{K}(\mu, \cdot)$ for all $\mu \in [-1, \mu_*]$, the fixed point index of $\mathcal{K}(\mu, \cdot)$ on B with respect to the cone P of nonnegative functions in $C^1[0, 1]$ is well defined. Moreover,

$$i_P(\mathcal{K}(\mu, \cdot), B) = 0 \quad \text{for all } \mu \in [0, \mu_*]. \tag{5.8}$$

Indeed, by the invariance by homotopy of the index, for every $\mu \in [0, \mu_*]$, we have that

$$i_P(\mathcal{K}(\mu, \cdot), B) = i_P(\mathcal{K}(\mu_*, \cdot), B) = i_P(\mathcal{K}(\mu_*, \cdot), 0),$$

because 0 is the unique fixed point of $\mathcal{K}(\mu_*, \cdot)$ in P . For ascertaining the spectral radius of the linearized operator $D\mathcal{K}(\mu_*, 0)$, suppose $\varrho \in \mathbb{R}$ is an eigenvalue of $D\mathcal{K}(\mu_*, 0)$ associated with a positive eigenfunction φ . Then,

$$(-D^2 + 1)^{-1}[(\mu_* + 1)\varphi] = \varrho\varphi,$$

which can be equivalently expressed as

$$-\varphi'' = \left(\frac{\mu_* + 1}{\varrho} - 1\right)\varphi, \quad \varphi'(0) = \varphi'(1) = 0.$$

Integrating in $[0, 1]$ yields

$$0 = \left(\frac{\mu_* + 1}{\varrho} - 1\right) \int_0^1 \varphi(x) dx$$

and hence,

$$\varrho = \mu_* + 1 > 1,$$

because $\varphi(x) > 0$ in $(0, 1)$. Consequently, Lemma 13.1 in [1], which provides us the local indices through the spectral radius of the underlying linearized operators, guarantees that

$$i_P(\mathcal{K}(\mu_*, \cdot), 0) = 0.$$

Therefore, (5.8) holds.

Now, let v_μ denote the minimal positive solution of (5.2) for $\mu \in (0, \mu_c)$. Since it is linearly asymptotically stable for all $\mu \in (0, \mu_c)$ and neutrally stable for $\mu = \mu_c$, combining the Schauder formula [1, Thm. 11.4] with the analysis of Amann and López-Gómez [2, Sect. 7], it is easily seen that

$$i_P(\mathcal{K}(\mu, \cdot), v_\mu) = 1.$$

Therefore, by (5.8) and the additivity-excision property, taking into account that v_μ is non-degenerate, we find that there exists $\eta > 0$ such that

$$i_P(\mathcal{K}(\mu, \cdot), B \setminus \overline{B_\eta(v_\mu)}) = -1 \quad \text{for all } \mu \in (0, \mu_c/2),$$

where $B_\eta(v_\mu)$ denotes the ball of radius η centered at v_μ in $C^1[0, 1]$. The fact that $\eta > 0$ can be chosen uniform in $\mu \in (0, \mu_c/2]$ follows from the local uniqueness of v_μ at $\mu = 0$ obtained as a consequence of the local bifurcation theorem of Crandall and Rabinowitz [17]. Consequently, since $\lim_{\mu \rightarrow 0} v_\mu = 0$, by homotopy invariance, it is apparent that, setting $\Omega = B \setminus \overline{B_\eta(0)}$, for sufficiently small $\eta > 0$,

$$i_P(\mathcal{K}(0, \cdot), \Omega) = -1. \tag{5.9}$$

Next, note that the positive solutions of (4.3) are the positive fixed points of the compact operator $\mathcal{M} : \mathbb{R} \times C^1[0, 1] \rightarrow C^1[0, 1]$ defined by

$$\mathcal{M}(\varepsilon, v) = (-D^2 + 1)^{-1} [a(x)|v|^p \operatorname{sgn}(v) g(\varepsilon v') h(\varepsilon v)].$$

Since $\mathcal{K}(0, \cdot) = \mathcal{M}(0, \cdot)$, it follows from (5.9) that

$$i_P(\mathcal{M}(0, \cdot), \Omega) = -1.$$

Moreover, for sufficiently small $\varepsilon > 0$, (4.3) cannot admit a solution on $\partial\Omega$. On the contrary, suppose that there exists a sequence $\{(\varepsilon_n, v_n)\}_{n \geq 1}$ of solutions of (4.3) such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and $v_n \in \partial\Omega$ for all $n \geq 1$. Then, $v_n = \mathcal{M}(\varepsilon_n, v_n)$ for all $n \geq 1$ and, by compactness, there exists a subsequence of v_n , relabeled by n , such that $\lim_{n \rightarrow \infty} v_n = v_0 \in \partial\Omega$. Since $(\varepsilon, v) = (0, v_0)$ must be a positive solution of (5.2), this contradicts the fact that (5.2) cannot admit positive solutions on $\partial\Omega$. Therefore, by the homotopy invariance of the fixed point index, $i_P(\mathcal{M}(\varepsilon, \cdot), \Omega) = -1$ for sufficiently small $\varepsilon > 0$.

Conclusion of the proof. As in the last part of the proof Theorem 4.2, the Leray-Schauder continuation theorem [36, p. 63] guarantees the existence of a component $\mathcal{C}_{r,\infty}^+$ of S_r^+ such that $(\lambda^*, \infty) \subseteq \mathcal{P}_\lambda(\mathcal{C}_{r,\infty}^+)$, for some $\lambda^* > 0$, and for which (5.1) holds. \square

Remark 5.1. In this section we confined ourselves to considering the case where the function $a(x)$ satisfies condition (a_2) . However, similar conclusions could be established even when the function $a(x)$ changes sign finitely many times. Actually, the existence of multiple continua of solutions could be shown in this case. Indeed, as observed in [29] and then rigorously proven in [20], the problem

$$\begin{cases} -v'' = a(x)v^p, & 0 < x < 1, \\ v'(0) = v'(1) = 0 \end{cases}$$

might possess a high number of positive solutions according to the number of changes of sign of the weight function $a(x)$. The conjecture of [29] has been recently proven in [21] for symmetric weight functions $a(x)$.

6. Pointwise behavior of the regular solutions as $\lambda \rightarrow \infty$

In this section we ascertain the limiting profile of the regular positive solutions of (1.1) as $\lambda \rightarrow \infty$, should they exist, when $f(u)$ satisfies (f_2) with $p \in (0, 1]$ and $a(x)$ satisfies (a_2) . This analysis also provides us with the pointwise behavior of the regular positive solutions separated away from zero when $p > 1$. So, through this section we suppose that (1.1) possesses a sequence of regular positive solutions, $\{(\lambda_n, u_n)\}_{n \geq 1}$, such that

$$\lim_{n \rightarrow \infty} \lambda_n = \infty. \tag{6.1}$$

We recall that the assumption (a_2) on the function $a(x)$ entails that any regular positive solution u of (1.1) is decreasing in $[0, 1]$, as a result of its concavity in $[0, z)$ and its convexity in $(z, 1]$, and, in particular,

$$\|u\|_{L^\infty(0,1)} = u(0) \quad \text{and} \quad \|u'\|_{L^\infty(0,1)} = -u'(z). \tag{6.2}$$

We stress that u might not be strictly positive if $p \in (0, 1)$, as already pointed out in Section 1. However, should u vanish, this would happen in the interval $(z, 1]$, i.e., necessarily $u(x) > 0$ for each $x \in [0, z]$.

The next result characterizes the pointwise limit of $\{u_n(x)\}_{n \geq 1}$, as $n \rightarrow \infty$, for every $x \in [0, 1]$.

Lemma 6.1. *Assume (f_1) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) such that (6.1) holds. Then, for a.e. $x \in [0, 1]$ there exists*

$$\lim_{n \rightarrow \infty} u_n(x) \in \{0, \infty\}. \tag{6.3}$$

Proof. Integrating the equation of (1.1) on $[0, z]$ yields

$$\int_0^z a(x) f(u_n(x)) dx = \frac{1}{\lambda_n} \frac{-u'_n(z)}{\sqrt{1 + (u'_n(z))^2}} < \frac{1}{\lambda_n} \quad \text{for all } n,$$

and thus, letting $n \rightarrow \infty$, we find that $\lim_{n \rightarrow \infty} \int_0^z a(x)f(u_n(x)) dx = 0$. The convergence in $L^1(0, z)$ entails that there is a subsequence $\{u_{n_h}\}_{h \geq 1}$ of $\{u_n\}_{n \geq 1}$ such that

$$\lim_{h \rightarrow \infty} a(x)f(u_{n_h}(x)) = 0 \quad \text{a.e. in } [0, z].$$

Consequently, as $a(x) > 0$ a.e. in $[0, z]$, we find that $\lim_{h \rightarrow \infty} f(u_{n_h}(x)) = 0$ a.e. in $[0, z]$. Similarly, as $\int_0^1 a(x)f(u_n(x)) dx = 0$ for all n , we also have that

$$\lim_{n \rightarrow \infty} \int_z^1 a(x)f(u_n(x)) dx = 0$$

and therefore there is a subsequence $\{u_{n_k}\}_{k \geq 1}$ of $\{u_n\}_{n \geq 1}$ such that $\lim_{k \rightarrow \infty} f(u_{n_k}(x)) = 0$ a.e. in $[1, z]$. Since this argument can be repeated for any possible subsequence of $\{u_n\}_{n \geq 1}$, we infer that $\lim_{n \rightarrow \infty} f(u_n(x)) = 0$ a.e. in $[0, 1]$. As $f(u) > 0$ for all $u > 0$, (6.3) holds. \square

Note that Lemma 6.1 holds regardless the nature of the growth of $f(u)$ at $u = 0$, i.e., without any restriction on the size of $p > 0$.

Lemma 6.2. *Assume (f_1) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) such that (6.1) holds. Then, the cluster points of the sequence $\{u_n(0)\}_{n \geq 1}$ are either 0 or ∞ . Moreover, using (6.2), as soon as $p \in (0, 1]$, one has that*

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^\infty(0,1)} = \lim_{n \rightarrow \infty} u_n(0) = \infty. \tag{6.4}$$

Proof. Suppose that there are a constant $K > 0$ and a subsequence, relabeled by n , of $\{(\lambda_n, u_n)\}_{n \geq 1}$ such that

$$u_n(0) = \|u_n\|_{L^\infty(0,1)} \leq K \quad \text{for all } n. \tag{6.5}$$

We will show that this is impossible if $p \in (0, 1]$, while it implies

$$\lim_{n \rightarrow \infty} u_n(0) = 0 \tag{6.6}$$

if $p > 1$.

We first consider the case where $p \in (0, 1]$. Let us define the auxiliary function

$$\tilde{f}(u) = \begin{cases} f(u) & \text{if } u \leq K, \\ \frac{f(K)}{K}u & \text{if } u > K. \end{cases}$$

By construction, $\tilde{f} \in C(\mathbb{R})$ and, since $f(u)$ satisfies (f_1) with $p \in (0, 1]$, there exists $c > 0$ such that $\tilde{f}(u) \geq cu$ for all $u \geq 0$. Then, for every n , (λ_n, u_n) solves the auxiliary boundary value problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)\tilde{f}(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Thus, each u_n satisfies

$$-u_n''(x) = \lambda_n a(x)\tilde{f}(u_n(x))\left[1 + (u_n'(x))^2\right]^{\frac{3}{2}} \geq \lambda_n a(x) c u_n(x) \quad \text{a.e. in } (0, z)$$

and hence, u_n is a strictly positive supersolution of the problem

$$\begin{cases} -w'' = c \lambda_n a(x) w, & 0 < x < z, \\ w(0) = w(z) = 0. \end{cases} \tag{6.7}$$

Let μ_1 denote the unique positive eigenvalue, with a corresponding positive eigenfunction φ_1 , of the weighted eigenvalue problem

$$\begin{cases} -\varphi'' = \mu a(x) \varphi, & 0 < x < z, \\ \varphi(0) = \varphi(z) = 0. \end{cases}$$

If we pick a sufficiently large n so that $c \lambda_n > \mu_1$, then a suitable multiple of φ_1 provides us with a positive subsolution of (6.7) smaller than u_n , thus yielding the existence of a positive solution of (6.7). This solution would be a principal eigenfunction associated to $c \lambda_n > \mu_1$, contradicting the uniqueness of μ_1 . So, (6.4) holds in case $p \in (0, 1]$.

Now, consider the case where $p > 1$. Then, setting

$$\hat{f}(u) = \begin{cases} f(u) & \text{if } u \leq K, \\ f(K) & \text{if } u > K, \end{cases}$$

(6.5) entails that $f(u_n) = \hat{f}(u_n)$, for all n , and thus (λ_n, u_n) solves

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)\hat{f}(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

Integrating the equation above in $(0, z)$ yields

$$\int_0^z a(x)\hat{f}(u_n(x)) dx = \frac{1}{\lambda_n} \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} < \frac{1}{\lambda_n}$$

and consequently, by (6.1),

$$\lim_{n \rightarrow \infty} \int_0^z a(x)\hat{f}(u_n(x)) dx = 0.$$

This implies that, for a subsequence, still labeled by n , $\lim_{n \rightarrow \infty} (a(x)\hat{f}(u_n(x))) = 0$ a.e. in $[0, z]$ and hence $\lim_{n \rightarrow \infty} \hat{f}(u_n(x)) = 0$ a.e. in $[0, z]$. The definition of $\hat{f}(u)$ yields $\lim_{n \rightarrow \infty} u_n(x) = 0$ a.e. in $[0, z]$. Therefore, since u_n is decreasing, it becomes apparent that

$$\lim_{n \rightarrow \infty} u_n(x) = 0 \quad \text{for all } x \in (0, 1]. \tag{6.8}$$

Suppose, by contradiction, that (6.6) does not hold. Then, due to (6.5), there exists a subsequence, again labeled by n , such that

$$\lim_{n \rightarrow \infty} u_n(0) = L \in (0, K]. \tag{6.9}$$

By the concavity of u_n in $[0, z)$, for any given $y \in (0, z)$, we have that

$$u'_n(y) \leq \frac{u_n(y) - u_n(0)}{y}.$$

Hence, by (6.8) and (6.9), we find that, for a further subsequence, still labeled by n ,

$$\lim_{n \rightarrow \infty} u'_n(y) \leq \lim_{n \rightarrow \infty} \frac{u_n(y) - u_n(0)}{y} = -\frac{L}{y}. \tag{6.10}$$

By the concavity, we also have that

$$u_n(x) \leq u_n(y) + u'_n(y)(x - y) \quad \text{for all } x \in (y, z). \tag{6.11}$$

Note that the right hand side of (6.11) vanishes at $x_n = y - \frac{u_n(y)}{u'_n(y)}$. Thanks to (6.8) and (6.10), we get $\lim_{n \rightarrow \infty} x_n = y < z$ and hence, for sufficiently large n , $x_n < z$. This forces u_n to vanish in $(0, z)$. As this is impossible, we conclude that $L = 0$, i.e., (6.6) holds. \square

Under assumption (f_1) there exists $M > 0$, not necessarily unique, such that $f(M) = \|f\|_\infty$. Throughout the rest of this section, M is chosen so that $f(u) < f(M)$ for all $u > M$. Under assumption (f_2) , M is uniquely determined.

Lemma 6.3. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) such that (6.1) holds. Then,*

$$u_n(1) < M \quad \text{for sufficiently large } n. \tag{6.12}$$

Proof. On the contrary, assume that there is a subsequence, labeled again by n , such that $u_n(1) \geq M$ for all n . Then, since each u_n is decreasing, we have that $u_n(x) \geq M$ for all $x \in [0, 1]$. Accordingly, since $f \in C^1[M, \infty)$, the next identity holds for every $x \in [0, 1]$

$$\lambda_n a(x) = \left(\frac{1}{f(u_n(x))} \frac{-u'_n(x)}{\sqrt{1 + (u'_n(x))^2}} \right)' + \left(\frac{1}{f(u_n(x))} \right)' \frac{u'_n(x)}{\sqrt{1 + (u'_n(x))^2}}. \tag{6.13}$$

Integrating (6.13) in $[0, 1]$ yields

$$\begin{aligned} \lambda_n \int_0^1 a(x) dx &= \int_0^1 \left(\frac{1}{f(u_n(x))} \right)' \frac{u_n'(x)}{\sqrt{1 + (u_n'(x))^2}} dx \\ &= - \int_0^1 \frac{f'(u_n(x))}{f^2(u_n(x))} \frac{(u_n'(x))^2}{\sqrt{1 + (u_n'(x))^2}} dx. \end{aligned}$$

As $u_n(x) \geq M$ and $f(u)$ is decreasing in $[M, \infty)$, we have that $f'(u_n(x)) \leq 0$ for all $x \in [0, 1]$ and n . Hence, we obtain a contradiction with (a_2) , which requires $\int_0^1 a(x) dx < 0$. Therefore, (6.12) holds. \square

Throughout the remainder of this section we will assume that, in the case where $p > 1$, the sequence $\{(\lambda_n, u_n)\}_{n \geq 1}$, in addition to (6.1), satisfies

$$\liminf_{n \rightarrow \infty} u_n(0) > 0. \tag{6.14}$$

Under this assumption, according to Lemma 6.2, the condition (6.4) must hold, regardless the size of $p > 0$. Naturally, this property fails to be true in case $p > 1$ for the small regular positive solutions of (1.1) whose existence is guaranteed by Theorem 5.1.

Lemma 6.4. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Let x_ω be the unique point in the interval $(z, 1)$ where $\int_0^{x_\omega} a(x) dx = 0$. Then, for sufficiently large n , there exists a unique $x_n \in (0, x_\omega)$ such that*

$$u_n(x_n) = M. \tag{6.15}$$

Proof. Under the condition (6.14), Lemma 6.2 and Lemma 6.3 imply that $u_n(0) > M$ and $u_n(1) < M$ for sufficiently large n . Thus, since u_n is decreasing in $[0, 1]$, there exists a unique $x_n \in (0, 1)$ for which (6.15) holds. Necessarily, we have $u_n(x) \geq M$ for all $x \in [0, x_n]$. Thus, integrating (6.13) in $[0, x_n]$ we find that

$$\lambda_n \int_0^{x_n} a(x) dx = \frac{1}{f(M)} \frac{-u_n'(x_n)}{\sqrt{1 + (u_n'(x_n))^2}} - \int_0^{x_n} \frac{f'(u_n(x))}{f^2(u_n(x))} \frac{(u_n'(x))^2}{\sqrt{1 + (u_n'(x))^2}} dx > 0,$$

because $f'(u) < 0$ if $u > M$. Hence, we conclude that $\int_0^{x_n} a(x) dx > 0$ for sufficiently large n , and therefore $x_n < x_\omega$. This ends the proof. \square

Lemma 6.5. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta \in (0, z)$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in (z - \eta, x_\omega)$ for all $n \geq n_0$.*

Proof. On the contrary, suppose that there exists $\eta \in (0, z)$ such that $[0, z - \eta]$ contains a subsequence of $\{x_n\}_{n \geq 1}$, still labeled by n . Then, without loss of generality, we can further assume that $\lim_{n \rightarrow \infty} x_n = x_* \in [0, z - \eta]$. Since u_n is decreasing, it follows from (6.15) and Lemma 6.1 that

$$\lim_{n \rightarrow \infty} u_n(x) = 0 \quad \text{uniformly in } [z - \frac{\eta}{2}, 1]. \tag{6.16}$$

Hence, by the concavity of u_n in $[0, z)$, we see that, for sufficiently large n ,

$$u'_n(z - \frac{\eta}{2}) < \frac{u_n(z - \frac{\eta}{2}) - u_n(x_n)}{z - \frac{\eta}{2} - x_n} = \frac{u_n(z - \frac{\eta}{2}) - M}{z - \frac{\eta}{2} - x_n} < -\frac{1}{2} \frac{M}{z - \frac{\eta}{2} - x_*}.$$

In view of (6.16), this forces u_n to vanish, by its concavity, somewhere in $[0, z]$, for large n . This contradiction ends the proof. \square

Lemma 6.6. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Suppose, in addition, that, for some $y \in [0, 1)$, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, one has that $y + \varepsilon \leq x_n$ for all $n \geq n_0$. Then, there holds*

$$\lim_{n \rightarrow \infty} u_n(x) = \infty \quad \text{uniformly in } [0, y].$$

Proof. Since u_n is decreasing for all $n \geq 1$, it suffices to show that $\lim_{n \rightarrow \infty} u_n(y) = \infty$. This is an easy consequence of Lemma 6.1, because $u_n(y) > M$ for all $n \geq n_0$. \square

As a byproduct of Lemmas 6.5 and 6.6, the next result holds.

Corollary 6.1. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta \in (0, z)$,*

$$\lim_{n \rightarrow \infty} u_n(x) = \infty \quad \text{uniformly in } [0, z - \eta].$$

The next result does estimate the grow-up rate of u_n to infinity in $[0, z)$ as $n \rightarrow \infty$.

Theorem 6.1. *Assume (f_2) and (a_2) . Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta \in (0, z)$, there exists a constant $C > 0$ and an integer $n_0 \geq 1$ such that*

$$u_n(x) \geq C \lambda_n^{\frac{1}{q}} \quad \text{for all } x \in [0, z - \eta] \text{ and } n \geq n_0. \tag{6.17}$$

Proof. Indeed, by Lemma 6.5, there exists $n_0 \in \mathbb{N}$ such that $x_n \geq z - \frac{\eta}{2}$ for all $n \geq n_0$. Thus, fixing $n \geq n_0$ and $x \in [0, z - \eta]$, and integrating in $[x, x_n]$ the identity (6.13) we find that

$$\begin{aligned} \lambda_n \int_x^{x_n} a(s) ds &= \frac{1}{f(u_n(x))} \frac{u'_n(x)}{\sqrt{1 + (u'_n(x))^2}} + \frac{1}{f(M)} \frac{-u'_n(x_n)}{\sqrt{1 + (u'_n(x_n))^2}} \\ &\quad + \int_x^{x_n} \left(\frac{1}{f(u_n(s))} \right)' \frac{u'_n(s)}{\sqrt{1 + (u'_n(s))^2}} ds \end{aligned}$$

$$< \frac{1}{f(M)} - \int_x^{x_n} \left(\frac{1}{f(u_n(s))} \right)' \frac{-u'_n(s)}{\sqrt{1 + (u'_n(s))^2}} ds.$$

Since $u_n(s) > M$ for each $s \in (x, x_n)$, u_n is decreasing and f is decreasing in (M, ∞) , we find that

$$- \left(\frac{1}{f(u_n(s))} \right)' = \frac{f'(u_n(s)) u'_n(s)}{(f(u_n(s)))^2} > 0 \quad \text{in } (x, x_n).$$

Thus, the previous estimate implies that

$$\lambda_n \int_x^{x_n} a(s) ds < \frac{1}{f(M)} - \int_x^{x_n} \left(\frac{1}{f(u_n(s))} \right)' ds = \frac{1}{f(u_n(x))}$$

for all $x < x_n$ and $n \geq n_0$. Consequently, since for every $n \geq n_0$, one has that $x \leq z - \eta < z - \frac{\eta}{2} \leq x_n$, we find that, for all $x \in [0, z - \eta]$,

$$\frac{1}{f(u_n(x))} > \lambda_n \int_x^{x_n} a(s) ds > \lambda_n \int_{z-\eta}^{z-\frac{\eta}{2}} a(s) ds$$

and then

$$\frac{f(u_n(x))}{u_n^{-q}(x)} \int_{z-\eta}^{z-\frac{\eta}{2}} a(s) ds < \frac{u_n^q(x)}{\lambda_n}.$$

By Corollary 6.1, it follows from (1.2) that $\lim_{n \rightarrow \infty} \frac{f(u_n(x))}{u_n^{-q}(x)} = h$ uniformly in $[0, z - \eta]$. Therefore, we conclude that

$$\liminf_{n \rightarrow \infty} \frac{u_n^q(x)}{\lambda_n} \geq h \int_{z-\eta}^{z-\frac{\eta}{2}} a(s) ds \quad \text{uniformly in } [0, z - \eta].$$

Hence the estimate (6.17) follows. \square

Lemma 6.7. Assume (f_2) and (a_2) . Suppose further that

$$\text{ess sup}_{[z+\varepsilon, x_\omega]} a < 0 \quad \text{for all small } \varepsilon > 0. \tag{6.18}$$

Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Then, there exists

$$\lim_{n \rightarrow \infty} x_n = z. \tag{6.19}$$

Proof. On the contrary, assume that (6.19) is not true. Then, owing to Lemma 6.5, there exist $\eta > 0$ and a subsequence, still labeled by n , such that $z + \eta \leq x_n$ for all $n \geq 1$. Then, since for every $n \geq 1$ and $x \in [z, 1]$,

$$\left(\frac{1}{\sqrt{1 + (u'_n(x))^2}} \right)' = \lambda_n a(x) f(u_n(x)) u'_n(x) \geq 0,$$

integrating in $[z, x_n]$, we obtain

$$\begin{aligned} 1 &\geq \frac{1}{\sqrt{1 + (u'_n(x_n))^2}} - \frac{1}{\sqrt{1 + (u'_n(z))^2}} = \lambda_n \int_z^{x_n} a(x) f(u_n(x)) u'_n(x) dx \\ &\geq \lambda_n \int_{z+\frac{\eta}{2}}^{x_n} a(x) f(u_n(x)) u'_n(x) dx \geq -\lambda_n \operatorname{ess\,sup}_{[z+\frac{\eta}{2}, x_\omega]} a \int_{z+\frac{\eta}{2}}^{x_n} f(u_n(x)) u'_n(x) dx \\ &\geq -\lambda_n \operatorname{ess\,sup}_{[z+\frac{\eta}{2}, x_\omega]} a \int_M^{u_n(z+\frac{\eta}{2})} f(s) ds. \end{aligned}$$

Therefore, by (6.1), we obtain $\lim_{n \rightarrow \infty} \int_M^{u_n(z+\frac{\eta}{2})} f(s) ds = 0$, which implies $\lim_{n \rightarrow \infty} u_n(z + \frac{\eta}{2}) = M$, while, according to Lemma 6.6, $\lim_{n \rightarrow \infty} u_n(z + \frac{\eta}{2}) = \infty$. This contradiction ends the proof. \square

Remark 6.1. Under (a_2) , the condition (6.18) holds if, for instance, the function $a(x)$ is continuous in $(z, x_\omega]$.

As a direct consequence of Lemma 6.7, the next result holds.

Corollary 6.2. Assume (f_2) , (a_2) and (6.18). Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta \in (0, 1 - z)$,

$$\lim_{n \rightarrow \infty} u_n(x) = 0 \quad \text{uniformly in } [z + \eta, 1].$$

Proof. According to Lemma 6.7, for every $\eta \in (0, 1 - z)$, there exists n_0 such that, for all $n \geq n_0$, $x_n < z + \frac{\eta}{2}$ and hence $u_n(z + \eta) < u_n(x_n) = M$. Then, Lemma 6.1 yields $\lim_{n \rightarrow \infty} u_n(z + \eta) = 0$. Since u_n is decreasing, the conclusion follows. \square

Finally, the next result estimates the decay rate of u_n in the interval $(z, 1]$.

Theorem 6.2. Assume (f_2) , (a_2) and (6.18). Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular strictly positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta \in (0, 1 - z)$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$u_n(x) \leq C \lambda_n^{-\frac{1}{p}} \quad \text{for all } x \in [z + \eta, 1] \text{ and } n \geq n_0. \tag{6.20}$$

Proof. Pick $x \in [z + \eta, 1]$. By (6.19), there exists $n_0 \in \mathbb{N}$ such that

$$z - \frac{\eta}{2} < x_n < z + \frac{\eta}{2} < z + \eta \leq x \quad \text{for all } n \geq n_0. \tag{6.21}$$

Note that $0 < u_n(t) \leq M$ for all $n \geq n_0$ and $t \in [x_n, x]$. Thus, for any $n \geq n_0$, since $u_n(t)$ is decreasing, the composition $f(u_n(t))$ is decreasing in $[x_n, x]$ and hence

$$0 < \frac{1}{f(u_n(x_n))} \leq \frac{1}{f(u_n(t))} \leq \frac{1}{f(u_n(x))} \quad \text{for all } t \in [x_n, x].$$

Suppose $z \leq x_n$. Then, from the differential equation in (1.1), we get, for all $t \in [x_n, x]$,

$$0 < -\lambda_n a(t) = \left(\frac{u'_n(t)}{\sqrt{1 + (u'_n(t))^2}} \right)' \frac{1}{f(u_n(t))} \leq \left(\frac{u'_n(t)}{\sqrt{1 + (u'_n(t))^2}} \right)' \frac{1}{f(u_n(x))}$$

and hence integrating in $[x_n, x]$

$$\begin{aligned} -\lambda_n \int_{x_n}^x a(t) dt &\leq \frac{1}{f(u_n(x))} \int_{x_n}^x \left(\frac{u'_n(t)}{\sqrt{1 + (u'_n(t))^2}} \right)' dt \\ &= \frac{1}{f(u_n(x))} \left(\frac{u'_n(x)}{\sqrt{1 + (u'_n(x))^2}} - \frac{u'_n(x_n)}{\sqrt{1 + (u'_n(x_n))^2}} \right) \leq \frac{1}{f(u_n(x))}. \end{aligned}$$

By (6.21), we have that, for all $n \geq n_0$,

$$\int_{x_n}^x (-a(t)) dt \geq \int_{z + \frac{\eta}{2}}^{z + \eta} (-a(t)) dt > 0,$$

and hence

$$\lambda_n \int_{z + \frac{\eta}{2}}^{z + \eta} (-a(s)) ds \leq \frac{1}{f(u_n(x))}$$

for all $x \in [z + \eta, 1]$. Then, the estimate (6.20) follows readily from the fact that $\lim_{n \rightarrow \infty} \frac{f(u_n(x))}{u'_n(x)} = 1$, which is a consequence of (1.2) by Corollary 6.2.

Now, assume that $x_n < z$. Then, as above, from the differential equation in (1.1) we get, for all $t \in [z, x]$,

$$0 < -\lambda_n a(t) \leq \left(\frac{u'_n(t)}{\sqrt{1 + (u'_n(t))^2}} \right)' \frac{1}{f(u_n(x))}$$

and hence integrating in $[z, x]$

$$-\lambda_n \int_z^x a(t) dt \leq \frac{1}{f(u_n(x))} \left(\frac{u'_n(x)}{\sqrt{1+(u'_n(x))^2}} - \frac{u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \right) \leq \frac{1}{f(u_n(x))}.$$

Thus, thanks again to (6.21), we find, for all $n \geq n_0$,

$$\lambda_n \int_z^{z+\eta} (-a(t)) dt \leq \lambda_n \int_z^x (-a(t)) dt \leq \frac{1}{f(u_n(x))}$$

and the argument of the previous case allows to complete the proof. \square

The next result establishes that, in addition, the solutions u_n are rather flat on

$$I_\eta = [0, z - \eta] \cup [z + \eta, 1]$$

for sufficiently small $\eta > 0$ and large n .

Lemma 6.8. *Assume (f_2) , (a_2) and (6.18). Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular strictly positive solutions of (1.1) satisfying (6.1) and (6.14). Then, for every $\eta > 0$ small enough, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|u'_n(x)| \leq C \quad \text{for all } x \in I_\eta \text{ and } n \geq n_0 \tag{6.22}$$

and, actually,

$$\lim_{n \rightarrow \infty} u'_n(x) = 0 \quad \text{uniformly in } [z + \eta, 1].$$

Proof. Fix $\eta \in (0, z)$ and $x \in [0, z - \frac{\eta}{2}]$. By (6.19), there exists $n_0 \in \mathbb{N}$ such that $z - \frac{\eta}{2} \leq x_n$, for all $n \geq n_0$. Hence, it follows that $u_n(t) \geq M$ for all $n \geq n_0$ and all $t \in [0, z - \frac{\eta}{2}]$. Thus, since u_n is decreasing, the composition $f(u_n(t))$ is increasing in $[0, z - \frac{\eta}{2}]$. Consequently, integrating the differential equation in (1.1) in $[0, x]$, we find that

$$\lambda_n f(u_n(x)) \int_0^x a(t) dt \geq \int_0^x \lambda_n f(u_n(t)) a(t) dt = \frac{-u'_n(x)}{\sqrt{1+(u'_n(x))^2}}. \tag{6.23}$$

Suppose that (6.22) is false. Then, for a subsequence, still labeled by n , we have that

$$\lim_{n \rightarrow \infty} u'_n(z - \eta) = -\infty,$$

which implies $\lim_{n \rightarrow \infty} u'_n(x) = -\infty$ uniformly in $[z - \eta, z]$ and hence

$$\lim_{n \rightarrow \infty} \frac{-u'_n(x)}{\sqrt{1+(u'_n(x))^2}} = 1 \quad \text{uniformly in } [z - \eta, z]. \tag{6.24}$$

On the other hand, integrating the differential equation in $[z - \eta, z - \frac{\eta}{2}]$ yields

$$\frac{u'_n(z - \eta)}{\sqrt{1 + (u'_n(z - \eta))^2}} - \frac{u'_n(z - \frac{\eta}{2})}{\sqrt{1 + (u'_n(z - \frac{\eta}{2}))^2}} = \int_{z-\eta}^{z-\frac{\eta}{2}} \lambda_n f(u_n(x)) a(x) dx$$

and then, owing to (6.23) and (6.24),

$$\frac{u'_n(z - \eta)}{\sqrt{1 + (u'_n(z - \eta))^2}} - \frac{u'_n(z - \frac{\eta}{2})}{\sqrt{1 + (u'_n(z - \frac{\eta}{2}))^2}} \geq \int_{z-\eta}^{z-\frac{\eta}{2}} \frac{-u'_n(x)}{\sqrt{1 + (u'_n(x))^2}} \frac{a(x)}{\int_0^x a(t) dt} dx.$$

Therefore, letting $n \rightarrow \infty$ in this inequality, we find that

$$0 \geq \int_{z-\eta}^{z-\frac{\eta}{2}} \frac{a(x)}{\int_0^x a(t) dt} dx > 0,$$

which is impossible. This contradiction provides us with the uniform bound for u'_n on $[0, z - \eta]$.

Next, we prove (6.22), which obviously yields a uniform bound for u'_n on $[z + \eta, 1]$. According to Lemma 6.1 and Lemma 6.7, we have that $\lim_{n \rightarrow \infty} u_n(x) = 0$ uniformly in $[z + \eta, z]$. Since, for every $x \in [z + \eta, 1]$ and $n \geq 1$, $u_n(1) = u_n(x) + \int_x^1 u'_n(t) dt$, we infer that $\lim_{n \rightarrow \infty} \int_x^1 u'_n(t) dt = 0$. Consequently, as u_n is convex on $(z, 1]$, also (6.22) is proven. \square

As a byproduct of Lemma 6.8 the next result holds.

Corollary 6.3. Assume (f_2) , (a_2) and (6.18). Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of regular strictly positive solutions of (1.1) satisfying (6.1) and (6.14). Then, the following holds:

- (i) for every $\eta \in (0, z)$, $\lim_{n \rightarrow \infty} \frac{u_n(x)}{u_n(0)} = 1$ uniformly in $[0, z - \eta]$;
- (ii) for every $\eta \in (0, 1 - z)$, $\lim_{n \rightarrow \infty} u_n(x) = \lim_{n \rightarrow \infty} u'_n(x) = 0$ uniformly in $[z + \eta, 1]$.

Proof. For every $x \in [0, z - \eta]$ and $n \geq 1$, we have that $u_n(x) = u_n(0) + \int_0^x u'_n(t) dt$ and hence

$$\frac{u_n(x)}{u_n(0)} = 1 + \frac{\int_0^x u'_n(t) dt}{u_n(0)}. \tag{6.25}$$

Since, by Lemma 6.8,

$$\left| \frac{\int_0^x u'_n(t) dt}{u_n(0)} \right| \leq \frac{C(z - \eta)}{u_n(0)},$$

conclusion (i) follows from Corollary 6.1 by letting $n \rightarrow \infty$ in (6.25). As for the proof of (ii), the conclusion follows from Lemma 6.1, Lemma 6.7 and Lemma 6.8. \square

At the light of these results, for sufficiently large λ , the regular positive solutions of (1.1) bounded away from zero have the profile already shown in Fig. 1.

Although, due to Lemma 6.7, $\lim_{n \rightarrow \infty} x_n = z$, in general it is unknown whether or not $x_n = z$. According to Corollaries 6.1 and 6.2, the solutions grow-up to infinity in the interval $(0, z)$, whereas decay to zero on $(z, 1)$. Theorems 6.1 and 6.2 provide us with some sharp estimates for the growth and decay rates of u_n in $(0, z)$ and $(z, 1)$, respectively. According to Corollary 6.3, the larger is λ the flatter are the solutions on $(0, z)$ and $(z, 1)$.

7. Regularity versus singularity

Our aim in this section is to discuss the existence and the non-existence of singular solutions of problem (1.1).

7.1. A general regularity criterion

Based on some ideas from our previous papers [39–41], we establish here a criterion for ascertaining the local regularity of the bounded variation solutions of the equation

$$-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = h(x), \quad 0 < x < 1, \tag{7.1}$$

under the assumption

(h_1) $h \in L^1(0, 1)$ and there exist $z \in (0, 1)$, $\delta_1 > 0$ and $\delta_2 > 0$ such that $h(x) \geq 0$ a.e. in $(z - \delta_1, z)$ and $h(x) \leq 0$ a.e. in $(z, z + \delta_2)$.

A bounded variation solution of (7.1) is a function $u \in BV(0, 1)$ such that

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}} dx + \int_0^1 \frac{Du^s}{|Du^s|} D\phi^s = \int_0^1 h\phi dx$$

for every $\phi \in BV(0, 1)$ such that $\phi(0^+) = 0 = \phi(1^-)$ and $|D\phi^s|$ is absolutely continuous with respect to $|Du^s|$.

From the proof of [39, Prop. 3.6] we infer that, under (h_1), every bounded variation solution u of (7.1) satisfies the following conditions:

- u is concave in $(z - \delta_1, z)$ and convex in $(z, z + \delta_2)$,

$$u|_{(z-\delta_1, z)} \in W_{loc}^{2,1}(z - \delta_1, z) \cap W^{1,1}(z - \delta_1, z),$$

$$u|_{(z, z+\delta_2)} \in W_{loc}^{2,1}(z, z + \delta_2) \cap W^{1,1}(z, z + \delta_2),$$

and

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = h(x) \text{ a.e. in } (z - \delta_1, z + \delta_2); \tag{7.2}$$

- either $u \in W_{\text{loc}}^{2,1}(z - \delta_1, z + \delta_2)$, or else

$$u(z^-) \geq u(z^+) \quad \text{and} \quad u'(z^-) = -\infty = u'(z^+),$$

where $u'(z^-)$ and $u'(z^+)$ stand for the left and right Dini derivatives of u at z .

The following theorem determines whether u is regular or not, depending on the behavior of h near its nodal point z ; more precisely, on the integrability properties of the function $(\int_x^z h(t) dt)^{-\frac{1}{2}}$, as expressed by conditions (7.5) and (7.6) below. Note that, under assumption (h_1) , the continuous function $\int_x^z h(t) dt$ is non-increasing in $(z - \delta_1, z)$ and non-decreasing in $(z, z + \delta_2)$; thus, either

$$\int_x^z h(t) dt > 0 \quad \text{for all } x \in (z - \delta_1, z + \delta_2) \setminus \{z\}, \tag{7.3}$$

or there is $x_0 \in (z - \delta_1, z)$ such that

$$\int_x^z h(t) dt = 0 \quad \text{for all } x \in [x_0, z],$$

or there is $x_0 \in (z, z + \delta_2)$ such that

$$\int_x^z h(t) dt = 0 \quad \text{for all } x \in [z, x_0].$$

Hence, (7.3) is complementary of

$$\int_{x_0}^z h(t) dt = 0 \quad \text{for some } x_0 \in (z - \delta_1, z + \delta_2) \setminus \{z\}. \tag{7.4}$$

Theorem 7.1. *Assume (h_1) and let u be a bounded variation solution of (7.1). Then, the following assertions are true:*

- (a) $u \in W_{\text{loc}}^{2,1}(z - \delta_1, z + \delta_2)$ if

$$\text{either } \int_{z-\delta_1}^z \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx = \infty, \text{ or } \int_z^{z+\delta_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx = \infty; \tag{7.5}$$

(b) $u(z^-) > u(z^+)$ if (7.3) holds and there are $x_1 \in (z - \delta_1, z)$, $x_2 \in (z, z + \delta_2)$ such that

$$\int_{x_1}^z \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx < \infty, \quad \int_z^{x_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx < \infty, \tag{7.6}$$

$$\int_{x_1}^{x_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx \leq u(x_1) - u(x_2).$$

It is understood that condition (7.5) is satisfied whenever (7.4) holds.

Proof. By (h_1) , either (7.3), or (7.4), holds. Let us prove Part (a). Assume (7.4) with $x_0 \in (z - \delta_1, z)$, the argument being similar in case $x_0 \in (z, z + \delta_2)$. Then, integrating the equation (7.2) on (x_0, z) yields

$$\frac{-u'(x_0)}{\sqrt{1 + (u'(x_0))^2}} = \frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} - \int_{x_0}^z h(t) dt = \frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}}. \tag{7.7}$$

As $u' \in W_{loc}^{1,1}(z - \delta_1, z)$, and hence $\frac{|u'(x_0)|}{\sqrt{1 + (u'(x_0))^2}} < 1$, it follows from (7.7) that $u'(z^-)$ is finite.

Therefore, $u \in W_{loc}^{2,1}(z - \delta_1, z + \delta_2)$. Now suppose (7.3). Then, for every $t \in (z - \delta_1, z)$, integrating (7.2) in (t, z) , we obtain that

$$\frac{-u'(t)}{\sqrt{1 + (u'(t))^2}} = \frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} - \int_t^z h(s) ds \tag{7.8}$$

and thus

$$-u'(t) = \frac{\frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} - \int_t^z h(s) ds}{\sqrt{1 - \frac{u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} - \int_t^z h(s) ds}} \frac{1}{\sqrt{1 + \frac{u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} + \int_t^z h(s) ds}}. \tag{7.9}$$

Assume (7.5). Without loss of generality we can suppose that

$$\int_{z-\delta_1}^z \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx = \infty, \tag{7.10}$$

because the proof is similar if $\int_z^{z+\delta_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx = \infty$. As the function $\int_t^z h(s) ds$ is continuous and positive for $t \in (z - \delta_1, z)$, the condition (7.10) can be expressed as

$$\int_x^z \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} dt = \infty \quad \text{for all } x \in (z - \delta_1, z). \tag{7.11}$$

To prove that $u \in W_{loc}^{2,1}(z - \delta_1, z + \delta_2)$, suppose, on the contrary, that $u'(z^-) = -\infty$, that is, $\frac{-u'(z^-)}{\sqrt{1+(u'(z^-))^2}} = 1$. Hence, (7.9) implies that, for every $x \in (z - \delta_1, z)$,

$$-u'(x) = \frac{1 - \int_x^z h(t) dt}{\sqrt{2 - \int_x^z h(t) dt}} \frac{1}{\sqrt{\int_x^z h(t) dt}}. \tag{7.12}$$

As there exists $\eta \in (0, \delta_1)$ such that $\int_x^z h(t) dt \leq \frac{1}{2}$ for all $x \in (z - \eta, z)$, (7.12) implies

$$-u'(x) \geq \frac{1}{2\sqrt{2}} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}}.$$

Therefore, by (7.11), integrating this inequality in $(z - \eta, z)$ yields

$$u(z - \eta) - u(z^-) \geq \frac{1}{2\sqrt{2}} \int_{z-\eta}^z \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx = \infty,$$

which is a contradiction, because $u \in L^\infty(z - \delta_1, z + \delta_2)$. This ends the proof of Part (a).

In order to prove Part (b), observe that the first two inequalities in (7.6) are equivalent to

$$\int_{z-\delta_1}^{z+\delta_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx < \infty.$$

Moreover, without loss of generality we can suppose that $u'(x_1) \leq 0$ and $u'(x_2) \leq 0$. Indeed, otherwise there is $\hat{x}_1 \in (x_1, z)$ such that $u(\hat{x}_1) \geq u(x_1)$ and $u'(\hat{x}_1) \leq 0$, or $\hat{x}_2 \in (z, x_2)$ with $u(\hat{x}_2) \leq u(x_2)$ and $u'(\hat{x}_2) \leq 0$. Replacing x_1 by \hat{x}_1 , or x_2 by \hat{x}_2 , we are done.

We claim that, under condition (7.3),

$$0 \leq -u'(t) < \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} \quad \text{for all } t \in [x_1, x_2] \setminus \{z\}. \tag{7.13}$$

Pick $t \in [x_1, z)$. Since $u'(x_1) \leq 0$ and $u(x)$ is concave in $[x_1, z)$, we have that

$$0 \leq \frac{-u'(t)}{\sqrt{1+(u'(t))^2}} < 1 \quad \text{for all } t \in [x_1, z)$$

and

$$0 \leq \frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} \leq 1.$$

Hence, by (7.3), it follows from (7.8) that

$$0 \leq \frac{-u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} - \int_t^z h(s) ds < 1 \quad \text{for all } t \in [x_1, z).$$

Consequently, since $1 + \frac{u'(z^-)}{\sqrt{1 + (u'(z^-))^2}} \geq 0$, the validity of (7.13) for all $t \in [x_1, z)$ follows easily from (7.9). As the proof of (7.13) for $t \in (z, x_2]$ proceeds similarly, the technical details are omitted here.

Next, pick $x \in (z - \delta_1, z + \delta_2) \setminus \{z\}$. Integrating (7.13) we obtain the estimates

$$u(x) - u(z^-) < \int_x^z \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} dt \quad \text{for all } x \in (z - \delta_1, z), \tag{7.14}$$

$$u(z^+) - u(x) < \int_z^x \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} dt \quad \text{for all } x \in (z, z + \delta_2). \tag{7.15}$$

Taking $x = x_1$ in (7.14) and $x = x_2$ in (7.15) and adding up the two inequalities yields

$$u(x_1) - u(z^-) + u(z^+) - u(x_2) < \int_{x_1}^{x_2} \left(\int_x^z h(t) dt \right)^{-\frac{1}{2}} dx.$$

Thus, thanks to (7.6), we find

$$\int_{x_1}^{x_2} \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} dt > -u(z^-) + u(z^+) + \int_{x_1}^{x_2} \left(\int_t^z h(s) ds \right)^{-\frac{1}{2}} dt$$

and therefore, $u(z^-) > u(z^+)$, which ends the proof. \square

Remark 7.1. It is straightforward to see that similar conclusions hold by imposing in (h_1) , alternatively, that $h(x) \leq 0$ a.e. in $(z - \delta_1, z)$ and $h(x) \geq 0$ a.e. in $(z, z + \delta_2)$.

Remark 7.2. The conditions (7.5) and (7.6) measure the smoothness of the function $h(x)$ at the nodal point z . Indeed, (7.5) holds true, in particular, when $h(z^-) = \text{ess lim}_{x \rightarrow z^-} h(x) = 0$ and $h(x)$ has a bounded slope on the left of z , or when $h(z^+) = \text{ess lim}_{x \rightarrow z^+} h(x) = 0$ and $h(x)$ has a bounded slope on the right of z , while (7.5) fails if, for instance, both $h(z^-)$ and $h(z^+)$ exist, are finite and $h(z^-) > 0 > h(z^+)$. This way a classical regularity result, requiring the function $h(x)$ to be globally Lipschitz on $(0, 1)$ (see, e.g., [28]), is significantly improved in the frame of equation (7.1).

7.2. Non-existence of singular solutions

A direct consequence of Theorem 7.1 is the following result, which guarantees the regularity of all possible solutions of the problem (1.1) if the function $a(x)$ satisfies (a_2) and

$$(a_4) \text{ either } \int_0^z \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx = \infty, \text{ or } \int_z^1 \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx = \infty.$$

Theorem 7.2. Assume (a_2) , (a_4) , $\lambda > 0$, and

$$(f_4) \ f \in C(\mathbb{R}) \text{ satisfies } f(u) \geq 0 \text{ if } u \geq 0.$$

Then, any positive solution (λ, u) of (1.1) is regular.

Proof. Let (λ, u) , with $\lambda > 0$, be a positive solution of (1.1), and set

$$h(x) = \lambda a(x) f(u(x)) \quad \text{for a.e. } x \in (0, 1).$$

The composite function $f(u(x))$ lies in $L^\infty(0, 1)$ and, by (f_4) , satisfies $f(u(x)) \geq 0$ for a.e. $x \in (0, 1)$. Thus, by (a_2) , h satisfies (h_1) , with $\delta_1 = z$ and $\delta_2 = 1 - z$, and either (7.4) holds for some $x_0 \in [0, 1] \setminus \{z\}$, or (7.3) and (7.5) hold. By Theorem 7.1 (a), u is regular. \square

Theorem 7.2 holds true regardless the particular behavior of $f(u)$ at zero and at infinity: it only requires the continuity and positivity of $f(u)$. Thus, it is a quite general and versatile result that applies to a large variety of situations and allows to complete and sharpen several previous statements, such as the ones obtained in [39,40,42,43]. Indeed, assuming further (a_2) and (a_4) , the results in [42, Thms. 1.1–1.6], in [39, Thms. 5.13 and 5.14] and in [40, Thms. 1.1 and 6.1], combined with Theorem 7.2, provide the existence and the multiplicity of regular solutions. In particular, thanks to [40, Thms. 1.1 and 6.1] and Theorem 7.2 a wrong assertion in [43, Thm. 7.1] can be corrected and the situation completely clarified. Theorem 7.2 also guarantees that, in the frame of Theorem 3.4, one has, under (a_2) and (a_4) , $\mathcal{S}_{bv}^+ = \mathcal{S}_r^+$ and $\mathcal{C}_{bv,\lambda_0}^+ = \mathcal{C}_{r,\lambda_0}^+$, as illustrated in Fig. 2.

7.3. Existence of singular solutions

In this section we assume that (a_4) fails, i.e.,

$$(a_5) \ \int_0^z \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx < \infty \text{ and } \int_z^1 \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx < \infty.$$

The next result shows that (1.1) can admit singular solutions under (a_5) .

Theorem 7.3. Assume (a_2) and (a_5) . Then, the following assertions are true:

- (i) for every $p > 1$ and $q \in (0, 1)$, there exists a function $f(u)$ satisfying (f_1) for which (1.1) admits a singular solution (λ_s, u_s) for some $\lambda_s > 0$;
- (ii) for every $q \in (0, 1)$ and $\bar{\lambda} > 0$, there exist $\varepsilon > 0$ and a function $f(u)$ satisfying (f_1) with $p = 1$ such that (1.1) admits a singular solution (λ_s, u_s) for some $\lambda_s > 0$ with $|\lambda_s - \bar{\lambda}| < \varepsilon$.

Proof. For any given $p > 1$ and $q \in (0, 1)$, let $\tilde{f} \in C^1(\mathbb{R})$ be such that $\tilde{f}(u) > 0$ and $\tilde{f}'(u) \geq 0$ for all $u > 0$ and

$$\lim_{u \rightarrow 0^+} \frac{\tilde{f}(u)}{u^p} = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{\tilde{f}(u)}{u^q} = 1.$$

Then, thanks to [40, Thm. 1.1], the auxiliary problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)\tilde{f}(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{7.16}$$

possesses a singular solution (λ_s, u_s) for some $\lambda_s > 0$. Let $M > 0$ be such that

$$M > u_s(0) = \|u_s\|_\infty \tag{7.17}$$

and consider any function $f \in C^1(\mathbb{R})$ such that

$$f(u) = \begin{cases} \tilde{f}(u) & \text{if } u \leq M, \\ \tilde{g}(u) & \text{if } u > M, \end{cases} \tag{7.18}$$

where $\tilde{g} \in C[M, \infty)$ is any function such that

$$\lim_{u \rightarrow \infty} \frac{\tilde{g}(u)}{u^{-q}} = h, \tag{7.19}$$

for some constant $h > 0$. Then, by construction, $f(u)$ satisfies (f_1) and (λ_s, u_s) is a singular solution of (1.1), thus proving Part (i).

To prove Part (ii) one can proceed as follows. For any given $\bar{\lambda} > 0$, let $\tilde{f} \in C^1(\mathbb{R})$ be any function satisfying

$$\lim_{u \rightarrow 0^+} \frac{\tilde{f}(u)}{u} = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \tilde{f}(u) = \frac{1}{\bar{\lambda} \int_0^z a(x) dx}.$$

As due to [41, Thm. 1.1] the singular solutions of (7.16) bifurcate from infinity at $\lambda_\infty = \bar{\lambda}$, there exist $\varepsilon > 0$ and a singular solution (λ_s, u_s) of (7.16) for some $\lambda = \lambda_s > 0$ with $|\lambda_s - \bar{\lambda}| < \varepsilon$.

Let $M > 0$ be satisfying (7.17) and consider any function $f(u)$ of the form (7.18) with $\tilde{g} \in C[M, \infty)$ satisfying (7.19). Then, $f(u)$ satisfies (f_1) and (λ_s, u_s) is a singular solution of (1.1). \square

When $a(x)$ has a jump discontinuity at z , that is,

$$\operatorname{ess\,lim}_{x \rightarrow z^-} a(x) > 0 > \operatorname{ess\,lim}_{x \rightarrow z^+} a(x),$$

our next result shows that the problem (1.1) cannot admit regular solutions separated away from zero for sufficiently large $\lambda > 0$.

Theorem 7.4. Assume (f_2) , (a_2) ,

(a_6) there exist constants $A > 0$, $B > 0$ and $\eta > 0$ such that

$$a(x) \geq A \text{ for a.e. } x \in (z - \eta, z) \quad \text{and} \quad a(x) \leq -B \text{ for a.e. } x \in (z, z + \eta),$$

and (6.18). Then, the problem (1.1) cannot admit regular solutions separated away from zero for sufficiently large $\lambda > 0$.

By Remark 7.2, (a_6) implies (a_5) . We conjecture that, more generally, Theorem 7.4 remains true if $a(x)$ satisfies (a_5) instead of (a_6) .

Proof. Assume, by contradiction, that (1.1) possesses a sequence of positive regular solutions, $\{(\lambda_n, u_n)\}_{n \geq 1}$, such that

$$\lim_{n \rightarrow \infty} \lambda_n = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} u_n(0) > 0. \tag{7.20}$$

By Lemma 6.4, for sufficiently large n there exists a unique $x_n \in (0, 1)$ such that $u_n(x_n) = M$. From Lemma 6.7, we know that $\lim_{n \rightarrow \infty} x_n = z$. Without loss of generality, we can suppose that, for every $n \geq 1$, $x_n \in (z - \eta, z + \eta)$. We claim that, in addition,

$$\lim_{n \rightarrow \infty} u_n(z) = M. \tag{7.21}$$

To prove this, we will distinguish, for each n , between two different cases: either $x_n > z$, or $x_n \leq z$. Suppose that $x_n > z$. Then, integrating in $[z, x_n]$ the identity

$$\left(\frac{1}{\sqrt{1 + (u'_n(x))^2}} \right)' = \lambda_n a(x) f(u_n(x)) u'_n(x) \tag{7.22}$$

yields

$$\int_z^{x_n} a(x) f(u_n(x)) u'_n(x) dx = \frac{1}{\lambda_n} \left(\frac{1}{\sqrt{1 + (u'_n(x_n))^2}} - \frac{1}{\sqrt{1 + (u'_n(z))^2}} \right) \leq \frac{1}{\lambda_n}.$$

Thus, we infer from the first limit in (7.20) that

$$\lim_{n \rightarrow \infty} \int_z^{x_n} a(x) f(u_n(x)) u'_n(x) dx = 0. \tag{7.23}$$

Moreover, we have that

$$\begin{aligned} \int_z^{x_n} a(x) f(u_n(x)) u'_n(x) dx &= \int_z^{x_n} (-a(x)) f(u_n(x)) (-u'_n(x)) dx \\ &\geq B \int_z^{x_n} f(u_n(x)) (-u'_n(x)) dx = B \int_M^{u_n(z)} f(s) ds. \end{aligned}$$

Consequently, the following inequalities hold

$$0 \leq B \int_M^{u_n(z)} f(s) ds \leq \int_z^{x_n} a(x) f(u_n(x)) u'_n(x) dx.$$

Similarly, if $x_n \leq z$, we can obtain that

$$0 \leq A \int_{u_n(z)}^M f(s) ds \leq - \int_{x_n}^z a(x) f(u_n(x)) u'_n(x) dx.$$

Since $A > 0$ and $B > 0$, from (7.23) we can conclude that $\lim_{n \rightarrow \infty} \int_M^{u_n(z)} f(s) ds = 0$. Therefore, (7.21) holds. Next, integrating (7.22) in $[z - \eta, z + \eta]$ yields

$$\int_{z-\eta}^{z+\eta} a(x) f(u_n(x)) u'_n(x) dx = O(\lambda_n^{-1}) \quad \text{as } n \rightarrow \infty,$$

or, equivalently,

$$\int_{z-\eta}^z a(x) f(u_n(x)) (-u'_n(x)) dx = - \int_z^{z+\eta} a(x) f(u_n(x)) (-u'_n(x)) dx + O(\lambda_n^{-1}).$$

Thus, arguing as above, we get

$$A \int_{u_n(z)}^{u_n(z-\eta)} f(s) ds \leq \|a\|_{L^\infty(0,1)} \int_{u_n(z+\eta)}^{u_n(z)} f(s) ds + O(\lambda_n^{-1}).$$

Therefore, letting $n \rightarrow \infty$ in this estimate, (7.21) and Corollaries 6.1 and 6.2 imply

$$A \int_M^\infty f(s) ds \leq \|a\|_{L^\infty(0,1)} \int_0^M f(s) ds < \infty. \tag{7.24}$$

Since $A > 0$, (7.24) entails $\int_M^\infty f(s) ds < \infty$, which is impossible, as the assumption $q \in (0, 1)$, made in (f_1) , implies that $\int_M^\infty f(s) ds = \infty$. This contradiction shows that (1.1) cannot admit, for large $\lambda > 0$, positive regular solutions separated away from 0. \square

Under conditions (f_1) and (a_2) , the existence of solutions separated away from zero for sufficiently large $\lambda > 0$ is guaranteed, thanks to Lemma 6.2, when $p \in (0, 1)$ by [42, Thm. 1.2], or when $p = 1$ by [42, Thm. 1.4, Rem. 1.9]. Whereas in case $p > 1$, thanks to [42, Thm. 1.5], it is known that (1.1) admits, at least, two positive solutions for sufficiently large $\lambda > 0$. According to Theorem 7.4, under conditions (f_2) , (a_2) and (a_6) , all the solutions of (1.1) for sufficiently large $\lambda > 0$ are singular, except the ones perturbing from zero, whose existence was discussed in Section 5.

Remark 7.3. The proof of Theorem 7.4 actually provides us with singular solutions as $\lambda \rightarrow \infty$ for a much wider family of functions $f(u)$ than those satisfying (f_1) . Indeed, to fix ideas suppose that

$$a(x) = \begin{cases} A & \text{in } [0, z), \\ -B & \text{in } (z, 1], \end{cases} \tag{7.25}$$

for two positive constants, $A, B > 0$, such that

$$\int_0^1 a(x) dx = Az - B(1 - z) < 0 = (A + B)z - B < 0.$$

Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of positive regular solutions of (1.1) satisfying (7.20). Then, integrating in $[0, 1]$ the identity (7.22) yields $\int_0^1 a(x) f(u_n(x)) u_n'(x) dx = 0$ for all $n \geq 1$. Thus,

$$\int_0^z a(x) f(u_n(x)) u_n'(x) dx = - \int_z^1 a(x) f(u_n(x)) u_n'(x) dx$$

and hence, by (7.25), we find that, for every $n \geq 1$,

$$A \int_{u_n(z)}^{u_n(0)} f(s) ds = B \int_{u_n(1)}^{u_n(z)} f(s) ds. \tag{7.26}$$

Consequently, letting $n \rightarrow \infty$, from the analysis done in Section 6, we infer that

$$A \int_M^\infty f(s) ds = B \int_0^M f(s) ds. \tag{7.27}$$

Therefore, (7.27) is necessary in order that (1.1) can admit a regular solution for sufficiently large $\lambda > 0$. Obviously, it fails to be true when $f(u)$ satisfies (f_1) with $q \in (0, 1)$. Yet, even when $f(u)$ has a sufficiently fast decay at infinity so that $\int_M^\infty f(s) ds < \infty$, the identity (7.27) will never be satisfied, unless

$$B = \frac{\int_M^\infty f(s) ds}{\int_0^M f(s) ds} A.$$

In all these cases, it is possible to show that (1.1) cannot admit a regular solution for sufficiently large $\lambda > 0$, regardless the decay rate of $f(u)$ at infinity. This result sharpens Theorem 7.4 in the special case when $a(x)$ satisfies (7.25).

Note that if $f(u)$ and $a(x)$ satisfy (f_1) and (a_2) , then the identity (7.26) restricts the size of $u_n(0) = \|u_n\|_{L^\infty(0,1)}$ so that (λ_n, u_n) can be a regular solution of (1.1). Thus, under these assumptions, any sufficiently large solution must be singular.

More generally, when $a(x)$ satisfies (a_5) , instead of (a_6) , the next result holds.

Proposition 7.1. *Assume (f_1) , (a_2) , (a_5) and (6.18). Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of positive solutions of (1.1) satisfying (7.20). Suppose, in addition, that there exist constants $\eta > 0$ and $C > 0$ such that*

$$\lambda_n f(u_n(x)) \geq C \quad \text{if } 0 < |x - z| < \eta. \tag{7.28}$$

Then, for sufficiently large n , (λ_n, u_n) is a singular solution of (1.1) with $u_n(z^-) > u_n(z^+)$.

Proof. Let us set, for every $n \geq 1$,

$$h_n(x) = \lambda_n a(x) f(u_n(x)) \quad \text{for a.e. } x \in [0, 1].$$

For each n , the function $h_n(x)$ satisfies assumption (h_1) and

$$\int_x^z h_n(t) dt \geq C \int_x^z a(t) dt > 0 \quad \text{if } 0 < |x - z| < \eta.$$

In addition, $\int_x^z h_n(t) dt > 0$ for all $x \in [0, 1] \setminus \{z\}$, and hence

$$\int_{z-\eta}^{z+\eta} \left(\int_x^z h_n(t) dt \right)^{-\frac{1}{2}} dx \leq \frac{1}{\sqrt{C}} \int_{z-\eta}^{z+\eta} \left(\int_x^z a(t) dt \right)^{-\frac{1}{2}} dx < \infty.$$

Moreover, by Corollaries 6.1 and 6.2, we already know that $\lim_{n \rightarrow \infty} u_n(z - \eta) = \infty$ and $\lim_{n \rightarrow \infty} u_n(z + \eta) = 0$, which implies that, for sufficiently large n ,

$$u_n(z - \eta) - u_n(z + \eta) \geq \int_{z-\eta}^{z+\eta} \left(\int_x^z h_n(t) dt \right)^{-\frac{1}{2}} dx.$$

Therefore, the conclusion can be inferred from Theorem 7.1 (b). \square

Remark 7.4. By the definition of x_n , we have that

$$\lim_{n \rightarrow \infty} (\lambda_n f(u_n(x_n))) = \lim_{n \rightarrow \infty} (\lambda_n f(M)) = \infty$$

and, due to Lemma 6.7, $\lim_{n \rightarrow \infty} x_n = z$. Thus, the condition (7.28) seems rather natural to hold. Unfortunately, we were not able to exclude the existence of some sequence $\{y_{n_k}\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} y_{n_k} = z$ and $\lim_{k \rightarrow \infty} (\lambda_{n_k} f(u_{n_k}(y_{n_k}))) = 0$ in the general case when $a(x)$ satisfies (a_5) . So, it remains an open problem to characterize the existence of positive singular solutions of (1.1) when $F(u)$ is sublinear at infinity, unlike what we were able to do in [40,41] for potentials which are linear or superlinear at infinity.

Data availability

No data was used for the research described in the article.

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