

Nonlinear Michelson interferometer for improved quantum metrology

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We examine quantum detection via a Michelson interferometer embedded in a gas with Kerr nonlinearity. This nonlinear interferometer is illuminated by pulses of classical light. This strategy combines the robustness against practical imperfections of classical light with the improvement provided by nonlinear processes. Regarding ultimate quantum limits, we stress that, as a difference with linear schemes, the nonlinearity introduces pulse duration as a new variable into play along with the energy resources.

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I. INTRODUCTION

Precise measurements are crucial in physics since they constitute the link between theory and nature. Accurate measurements can promote or reject a theory. Besides, precise detection and monitoring are fundamental for technology and other applications of science.

A critical contribution of the quantum theory to metrology is that quantum fluctuations would limit the resolution to some ultimate limits depending on the energy resources employed in the process [1], usually counted as the number of particles.

Since standard metrology is based on linear processes, previously known quantum limits heavily depend on an implicit assumption of linearity. Thus, a new frontier arises if we consider that the signal may be detected via nonlinear processes. The key point is that nonlinear schemes allow us to reach larger resolution than linear ones for the same resources. Moreover, the improvement holds even when using probes in classical states. This is of much relevance concerning robustness against practical imperfections, which can be deadly for schemes based on nonclassical probe states [2–4]. Quantum nonlinear metrology has been studied and proven experimentally in very different physical contexts [5–13]. In particular, this is the case for light propagation in Kerr-type nonlinear media, that has already demonstrated its usefulness in the context of precise detection [14–16].

In this paper we present a further feature of quantum detection involving nonlinear processes. This is that resolution depends not only on the number of photons, but also on the duration of the pulse (this is both on the number of particles and on the rate at which they are employed). This is in sharp contrast to linear schemes where the duration of the pulse plays no role. In this way a new variable appears which may be advantageously used to improve detection performance beyond previously accepted limits.

II. SCHEME

To develop this point let us consider signals encoded as length variations that can be typically detected via a Michelson interferometer schematized in Fig. 1. In the absence of a signal the two arms are equal $\ell_2 = \ell_1 = \ell_0$. For simplicity we

will consider the signal manifests as an anticorrelated length change as $\ell_1 = \ell_0 - x/2$, $\ell_2 = \ell_0 + x/2$ as is expected to be the case by the pass of a gravitational wave [17].

In order to involve nonlinear effects, we assume that the interferometer is embedded in a gas displaying a Kerr nonlinearity. The light is made of classical-light pulses of mean frequency ω , duration τ , and cross section A . The propagation in the nonlinear medium can be conveniently expressed in terms of an intensity-dependent index of refraction,

$$n = n_0 + \tilde{n}I = n_0(1 + \chi N), \quad (1)$$

here, \tilde{n} is the nonlinear coefficient, n_0 is the linear index, I is the light intensity, N is the number of photons of each pulse, and χ expresses the nonlinear phase shift per photon,

$$I \simeq \frac{\hbar\omega}{A\tau}N, \quad \chi = \frac{\tilde{n}}{n_0} \frac{\hbar\omega}{A\tau}, \quad (2)$$

where this is a definition of χ whereas the I versus N relation is an estimation good enough for our purposes as far as the exact equivalence would require the specific spectral distribution of the pulse.

After Eq. (1), the light propagation within the interferometer is described in the quantum domain by the unitary operator $U = U_1 U_2$ with

$$U_j = e^{i\varphi_j G_j}, \quad G_j = \hat{N}_j + \frac{\chi}{2} \hat{N}_j^2, \quad (3)$$

with $\varphi_j = k\ell_j$ where $k = n_0\omega/c$ is the wave number and $\hat{N}_j = a_j^\dagger a_j$ represents the photon-number operator in each arm $j = 1, 2$. We further assume that the signal induces an arm length difference x small compared with the length of the pulse $c\tau$. Any other changes produced in the optical constants of the media are assumed to lead to negligible effects.

We illuminate the interferometer just by one of the input ports (as usual the vacuum is at the other input) by a classical-like pure coherent state $|\alpha\rangle$ with a mean number of photons $|\alpha|^2 = N \gg 1$ [exactly the same results are obtained if the probe state before the input beam splitter is in the phase-averaged mixed state $\int_0^{2\pi} d\theta |\alpha e^{i\theta}\rangle \langle \alpha e^{i\theta}| / (2\pi)$]. The light state in the internal modes $a_{1,2}$ of the interferometer leaving the 50% beam splitter can be expressed as the product of coherent states $|\alpha/\sqrt{2}\rangle_1 |\alpha/\sqrt{2}\rangle_2$, whereas the light state reaching the beam splitter after propagation within the interferometer is $U_1 U_2 |\alpha/\sqrt{2}\rangle_1 |\alpha/\sqrt{2}\rangle_2$.

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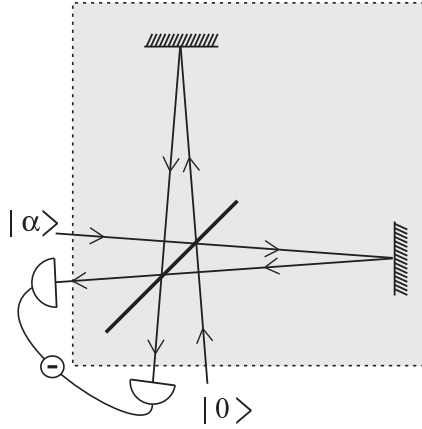


FIG. 1. Scheme of a Michelson interferometer embedded in a nonlinear medium.

The measurement is carried out by registering the difference in the number of photons recorded by two detectors at the output ports of the interferometer. The corresponding operator can be expressed in terms of the internal modes of the interferometer $a_{1,2}$ as $M = i(a_2^\dagger a_1 - a_1^\dagger a_2)$.

The sources of technical noise will be taken into account by their most typical consequences, such as phase randomization (e.g., caused by fluctuations in the optical properties of the medium), thermalization (e.g., thermal photons coming from undesired residual sources), as well as the usual finite quantum efficiency of the detectors.

III. SIGNAL DETECTION AND UNCERTAINTY

The signal x produces a phase shift that alters the statistics of the observed M , shifting its mean value (see the Appendix for details),

$$\langle M \rangle = \eta N e^{-N\chi^2 k^2 x^2/8} e^{-\sigma^2/2} \sin \left[kx \left(1 + \chi \frac{N}{2} \right) \right], \quad (4)$$

where η is the quantum efficiency of the detectors and σ is the variance of the random relative phase.

In order to be detected, the signal-induced shift $\langle M \rangle$ must be larger than the background quantum noise ΔM at $x = 0$. Taking into account the noise sources commented on above we get

$$(\Delta M)^2 = \eta N + \eta^2 N^2 \sigma^2 + \eta N N_t, \quad (5)$$

where N_t is the mean number of the undesired thermal photons. Since the nonlinearity complicates the noise analysis we postpone the details of the calculation to an Appendix to not interrupt the analysis.

We can estimate the resolution of the detection of x via the noise-to-signal ratio as

$$(\Delta x)^2 = \frac{(\Delta M)^2}{\left| \frac{d\langle M \rangle}{dx} \right|^2}, \quad (6)$$

leading to

$$(\Delta x)^2 = \frac{1 + \eta N \sigma^2 + N_t}{\eta k^2 N (1 + \chi N/2)^2}. \quad (7)$$

This holds provided that $\chi N k x \ll 1$, $N_t \ll N$, and $\sigma \ll 1$. Moreover, we have assumed $\chi k \ell_0 \simeq 2\pi m$ for integer m . Otherwise, without this last condition the nonlinearity would deeply disturb the probe state by producing coherent superpositions of distinguishable states [18], and the standard interferometric measurement M would become useless requiring more advanced detection strategies beyond the scope of the present analysis (see the Appendix for further details). The nonlinearity will have a noticeable effect for $\chi N \gg 1$, which is compatible with the above assumption $\chi N k x \ll 1$ provided that kx is small enough $kx \ll 1$.

This might be compared with the case when the nonlinear medium is absent $\chi = 0$,

$$(\Delta x|_{\text{lin}})^2 = \frac{1 + N_t}{\eta k^2 N}, \quad (8)$$

and we have further assumed that in such a case propagation occurs in vacuum and the phase randomization can be safely neglected $\sigma = 0$.

In the ideal case that the phase randomization and thermal effects might be ignored $\sigma = N_t = 0$ and $\chi N \gg 1$ we get the following improvement of the nonlinear versus the linear scheme,

$$(\Delta x)^2 = \frac{4(\Delta x|_{\text{lin}})^2}{\chi^2 N^2} \rightarrow \frac{4}{\eta k^2 \chi^2 N^3}. \quad (9)$$

We recall that the duration of the pulses is embedded in the nonlinear phase shift per photon χ in Eq. (2), and so, the lesser τ , the larger χ , and the larger the resolution.

IV. DISCUSSION

We can roughly estimate the amount of noise reduction with parameters within the reach of current technology. For the sake of simplicity and to fix the main ideas let us first consider the ideal case where the effect of technical noise is negligible $\eta N \sigma^2 + N_t \ll 1$ and $\eta \simeq 1$. Regarding numerical values we can address two extreme situations: standard natural nonlinearities and giant nonlinearities achieved via atomic coherence. Throughout we will assume that the index in darkness is on the order of unity $n_0 \simeq 1$.

The typical natural nonlinearities in gases can be on the order of $\tilde{n} \simeq 10^{-17} \text{ cm}^2/\text{W}$ [19]. As to the pulse parameters let us assume a pulse duration of $\tau \simeq 1 \text{ ps}$, light power of $P \simeq 1 \text{ PW}$, and beam cross section of $A \simeq 10^{-9} \text{ m}^2$, which leads in the visible spectrum to $N \simeq 10^{21}$ photons per pulse and a nonlinear shift per photon of $\chi \simeq 10^{-18}$ so that

$$\Delta x \simeq 10^{-3} \Delta x|_{\text{lin}} \simeq 10^{-21} \text{ m}. \quad (10)$$

The condition $\chi N k x \ll 1$ means the following condition on the signal $x \ll 10^{-10} \text{ m}$. On the other hand, the condition $\chi k \ell_0 \simeq 2\pi m$ leads to an extremely large interferometer even for $m = 1$ since in such a case $\ell_0 \simeq 10^{12} \text{ m}$. Thus the $m = 0$ situation should be addressed as suggested in the Appendix.

Things are completely different if we consider the giant nonlinearities achieved via electromagnetically induced transparency, leading to Kerr coefficients on the order of $\tilde{n} \simeq 10^{-2} \text{ cm}^2/\text{W}$ as reported in Ref. [20] and similarly large values in other configurations, such as $\tilde{n} \simeq 10^{-5} \text{ cm}^2/\text{W}$ in Ref. [21]. Such large values allow for alleviating the

requirements on the light probe state. For example we may have $\tau \simeq 100$ ps, $P \simeq 1$ MW, and $A \simeq 10^{-6}$ m², which leads in the visible spectrum to $N \simeq 10^{14}$ and $\chi \simeq 10^{-8}$ so that

$$\Delta x \simeq 10^{-6} \Delta x|_{\text{lin}} \simeq 10^{-20} \text{ m}. \quad (11)$$

The condition $\chi k \ell_0 \simeq 2\pi m$ leads to a more practical interferometer with $\ell_0 \simeq 100$ m, whereas the condition $\chi N k x \ll 1$ implies $x \ll 10^{-13}$ m. Thus a detectable signal should be composed in the range of 10^{-13} m $\gg x \gg 10^{-20}$ m. This fits perfectly well with the expected signals due to the pass of a gravitational wave in a 100-m-long interferometer, which are 10^{-15} m $\gg x \gg 10^{-20}$ m [22]. Notably, smaller τ and/or A , such as the beam-size values reached in Ref. [23] with current technology, may lead even to room-size interferometers with similar performance.

Finally, we may estimate the maximum effect of imperfections so that the good effects of nonlinearity are not spoiled. The condition we are looking for is derived from $\eta N \sigma^2 + N_t \ll \chi^2 N^2$ that is satisfied if, roughly speaking, $\sigma \ll \chi \sqrt{N/\eta}$ and $N_t \ll \chi^2 N^2$. For the natural nonlinearity and $\eta \simeq 1$ we get $\sigma \ll 10^{-8}$, $N_t \ll 10^6$, whereas for giant nonlinearities we get much less limiting bounds $\sigma \ll 10^{-1}$, $N_t \ll 10^{12}$.

V. CONCLUSIONS

Summarizing, nonlinearity not only can improve resolution beyond linear limits, but also can introduce a new variable into play. The signal uncertainty depends not only on the number of probe photons N , but also on the duration of the pulse τ through the nonlinear effect per photon χ in Eq. (2). This is because optical nonlinearity is sensible to light intensity rather than just energy or photon number. In particular, after Eq. (7) we may conjecture an optimum ultimate quantum limit (that would require nonclassical probes to be reached) scaling as

$$\Delta x \propto \frac{\tau A \lambda^2}{N^2}, \quad (12)$$

in terms of the probe-free parameters, where λ is the wavelength. This result may be particularly useful for example in situations of frequent monitoring where small pulse durations and large repetition rate of the interrogating pulse may be of interest. In this regard, the availability to obtain large beam intensities by shortening pulses seems a more feasible condition than increasing energy resources as required in usual linear quantum metrology.

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APPENDIX: CALCULUS

After the relation $aF(a^\dagger a) = F(a^\dagger a + 1)a$ valid for any F we get that

$$U^\dagger a_j U = e^{i\varphi_j} e^{iz_j} e^{i2z_j \hat{N}_j} a_j, \quad (A1)$$

where $\varphi_j = k\ell_j$, $z_j = \varphi_j \chi/2$, and $U = U_1 U_2$ is the global transformation. When evaluating the mean values of $a_1^\dagger a_2$ and its Hermitian conjugate on coherent states $|\beta\rangle$ with $\beta = \alpha/\sqrt{2}$, we will get expressions of the form

$$\langle \beta | U^\dagger a_j U | \beta \rangle = e^{i\varphi_j} e^{iz_j} \langle \beta | e^{i2z_j \hat{N}_j} a_j | \beta \rangle, \quad (A2)$$

that can be easily evaluated taking into account that

$$\langle \beta | e^{i2z_j \hat{N}_j} a_j | \beta \rangle = \beta \langle \beta | \beta e^{i2z_j} \rangle = \beta e^{|\beta|^2 (e^{i2z_j} - 1)}. \quad (A3)$$

Besides the finite quantum efficiency of the detectors we will consider some further typical forms of practical noise, such as thermalization and phase randomization. These common noise forms can have different physical origins, such as fluctuations of the optical properties of the medium, random variations of the complex amplitude from pulse to pulse, and so on. They can be addressed at once by performing the replacements,

$$a_1^\dagger a_2 \rightarrow e^{i\phi} (\sqrt{\eta} a_1^\dagger + \sqrt{1-\eta} b_1^\dagger) (\sqrt{\eta} a_2 + \sqrt{1-\eta} b_2), \quad (A4)$$

where ϕ is a random phase that we will assume to be Gaussian distributed with zero mean and variance σ^2 , η is the quantum efficiency in the detection, and b_j 's are uncorrelated field modes in thermal states with $(1-\eta)\langle b_1^\dagger b_1 \rangle = (1-\eta)\langle b_2^\dagger b_2 \rangle = N_t/2$ with $N_t \ll N$ and $\langle b_1 \rangle = \langle b_2 \rangle = \langle b_1^\dagger b_2 \rangle = 0$.

1. Mean value

Taking all this into account the mean value of M can be obtained after a long but straightforward calculation as

$$\begin{aligned} \langle M \rangle = \eta N \exp \left\{ \frac{1}{2} N [\cos(2z_1) + \cos(2z_2) - 2] \right\} \\ \times \sin \left\{ \phi + \varphi_2 - \varphi_1 + z_2 - z_1 \right. \\ \left. + \frac{N}{2} [\sin(2z_2) - \sin(2z_1)] \right\}. \end{aligned} \quad (A5)$$

Before the ϕ integration several natural considerations seem in order to get simpler and meaningful expressions. A required condition is that the factor in the real exponential should be close to zero, otherwise the final uncertainty Δx would increase exponentially with N . This is because the uncertainty ΔM will contain always a photon-counting noise term independent of the arm lengths. Thus we have to consider that in the absence of signal $z_0 = m\pi$, where m is any integer. This may be achieved by properly adjusting the fixed arm length ℓ_0 depending on χ . Alternatively we may consider that the Kerr transformation induced by the fixed length ℓ_0 may be compensated by another Kerr transformation with a nonlinear susceptibility of the opposite sign in propagation conditions insensitive to the signal value x . An alternative approach may follow the strategy in Ref. [24] by comparing outputs for two consecutive pulses experiencing alternatively linear and nonlinear transformations.

Thus, considering that the signal induces a very small variation in z_j around $z_0 \simeq m\pi$ we have

$$\langle M \rangle \simeq \eta N e^{-\chi^2 N k^2 x^2 / 8} \sin \left[\phi + kx \left(1 + \chi \frac{N}{2} \right) \right], \quad (A6)$$

which can be obtained after Eq. (A5) by a series expansion of the harmonic functions within the exponential and the sine function where we have also neglected the $z_2 - z_1$ term not multiplied by N . Carrying out the ϕ integration over a Gaussian distribution with zero mean and variance σ^2 we get

$$\langle M \rangle \simeq \eta N e^{-\chi^2 N k^2 x^2 / 8} e^{-\sigma^2 / 2} \sin \left[kx \left(1 + \chi \frac{N}{2} \right) \right]. \quad (\text{A7})$$

Finally, considering typical values for the variables involved the real exponentials can be safely approximated by unity. Moreover, the expected signals are small enough $\chi N k x \ll 1$ so that there is a linear relationship between M and x , which is usually an implicit assumption leading to Eq. (6),

$$\langle M \rangle \simeq \eta N \left[kx \left(1 + \chi \frac{N}{2} \right) \right]. \quad (\text{A8})$$

This is compatible with the fact that χN can be very large having observable effects. This is because it determines the value of the x derivative in the denominator of Eq. (6) and such a derivative need not be small. On the contrary, the best situation holds when χN is large enough to imply a noticeable reduction in signal uncertainty as analyzed in detail in Secs. III and IV.

2. Uncertainty

Next we address the evaluation of $(\Delta M)^2$ at $\ell_2 = \ell_1 = \ell_0$, this is $x = 0$ so that $\langle M \rangle = 0$ and $(\Delta M)^2 = \langle M^2 \rangle$. After

Eq. (A4) the effect of thermalization and finite efficiency means that M^2 should be replaced by

$$\eta^2 M_0^2 + \eta(1 - \eta)[(2b_2^\dagger b_2 + 1)\hat{N}_1 + (2b_1^\dagger b_1 + 1)\hat{N}_2], \quad (\text{A9})$$

where to avoid confusion we denote by M_0 when evaluating M in the noiseless case. Other terms lead to null or negligible contributions. It is worth noting that the last term is not affected by the U_j transformation nor by the random phase.

Then we can compute $\langle M_0^2 \rangle$, where

$$M_0^2 = 2\hat{N}_1\hat{N}_2 + \hat{N}_1 + \hat{N}_2 - a_1^{\dagger 2}a_2^2 - a_2^{\dagger 2}a_1^2. \quad (\text{A10})$$

The first terms depending just on \hat{N}_j are invariant under the transformations U_j whereas for the remaining two terms we can use the fact that $a^2 F(a^\dagger a) = F(a^\dagger a + 2)a^2$ and then proceed as above, to get, before phase randomization,

$$\langle M_0^2 \rangle = \frac{N^2}{2} + N - \frac{N^2}{2} \cos(2\phi), \quad (\text{A11})$$

and after the random-phase average,

$$\langle M_0^2 \rangle = \frac{N^2}{2} + N - \frac{N^2}{2} e^{-2\sigma^2} \simeq N + \sigma^2 N^2, \quad (\text{A12})$$

where the approximation holds for $\sigma \ll 1$. Finally, collecting the contributions in Eqs. (A9) and (A12) we finally get

$$(\Delta M)^2 \simeq \eta N + \eta^2 \sigma^2 N^2 + \eta N N_t, \quad (\text{A13})$$

leading to Eq. (7).

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