

# HIGHER TYPE ADJUNCTION INEQUALITIES FOR DONALDSON INVARIANTS

VICENTE MUÑOZ

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**ABSTRACT.** We prove new adjunction inequalities for embedded surfaces in four-manifolds with non-negative self-intersection number using the Donaldson invariants. These formulas are completely analogous to the ones obtained by Ozsváth and Szabó [11] using the Seiberg-Witten invariants. To prove these relations, we give a fairly explicit description of the structure of the Fukaya-Floer homology of a surface times a circle. As an aside, we also relate the Floer homology of a surface times a circle with the cohomology of some symmetric products of the surface.

## 1. INTRODUCTION

In this paper, we prove new adjunction inequalities for embedded surfaces of non-negative self-intersection in a four-manifold  $X$  by making use of the Donaldson invariants of  $X$ . Our results are parallel to those of Ozsváth and Szabó [11] on adjunction inequalities from Seiberg-Witten theory, and once more, they provide a confirmation of the (yet conjectural) equivalence between the Donaldson invariants and the Seiberg-Witten invariants [12] and give a guideline on how this correspondence should be established. The scope of application of these new adjunction inequalities is somewhat narrow at present, as they provide new information only for 4-manifolds not of simple type or with  $b_1 > 0$ . So far no example of 4-manifold not of simple type with  $b^+ > 1$  has been given. This same situation applies to the results in [11]. In turn the description and use of the Fukaya-Floer homology is very enlightening and clarifies the meaning of the basic classes [9] and of the finite type condition.

The input for the analysis carried out here is the Floer homology [7] and Fukaya-Floer homology [8] of the three-manifold  $Y = \Sigma \times \mathbb{S}^1$ , where  $\Sigma$  is a surface of genus  $g \geq 1$ . Using the descriptions, in terms of generators and relations, of (the ring structure of)  $HF^*(Y)$  and  $HFF^*(Y, \mathbb{S}^1)$  given in [7] and [8], we are able to analyse the pieces of their artinian

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decompositions and check that they look like (a deformation of) the cohomology of some symmetric products of the surface  $\Sigma$ . This turns out to provide enough information to get new relations in the effective (Fukaya-)Floer homology, which are recast in the shape of adjunction inequalities for 4-manifolds with  $b^+ > 1$ .

In order to state our main results, let us set up some notation. Donaldson invariants for a (smooth, compact, oriented) 4-manifold  $X$  with  $b^+ > 1$  (and with a homology orientation) are defined as linear functionals [4]

$$D_X^w : \mathbb{A}(X) = \text{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \Lambda^* H_1(X) \rightarrow \mathbb{C},$$

where  $w \in H^2(X; \mathbb{Z})$ . As for the grading of  $\mathbb{A}(X)$ , we give degree  $4 - i$  to the elements in  $H_i(X)$  (complex coefficients are understood for the homology  $H_*(X)$  and cohomology  $H^*(X)$ ). We shall denote by  $x \in H_0(X; \mathbb{Z})$  the class of a point. Recall [8] that we say that a 4-manifold  $X$  with  $b^+ > 1$  is of finite type if there exists  $n \geq 0$  such that  $D_X^w((x^2 - 4)^n z) = 0$ , for all  $z \in \mathbb{A}(X)$ , and that we call the minimum such  $n$  the order of finite type of  $X$ . All 4-manifolds  $X$  with  $b^+ > 1$  are of finite type [8] and the order of finite type does not depend on  $w \in H^2(X; \mathbb{Z})$  (see [9]).

Our first main result is a bound on the order of finite type for 4-manifolds, which greatly improves the one given in [8, proposition 7.3] in the case where  $X$  has  $b_1 > 0$ .

**Theorem 1.1.** *Let  $X$  be a 4-manifold with  $b^+ > 1$  and suppose that there is an embedded surface  $\Sigma \subset X$  of genus  $g$  and either with self-intersection  $\Sigma^2 > 0$  or with  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd homology class. Then the order of finite type of  $X$  is less than or equal to  $g$ .*

In [9] we saw that the Donaldson invariants of a 4-manifold  $X$  with  $b^+ > 1$  always produce a distinguished set of cohomology classes  $K_i$  (called basic classes), regardless of  $X$  being of simple type or having  $b_1 = 0$ , extending thus the notion of basic classes given in [4]. These were obtained by looking at the asymptotics of  $D_X^w$  for large degrees. Also the set of basic classes is independent of  $w \in H^2(X; \mathbb{Z})$ . Recall the following criterium extracted from [9, proposition 15].

**Proposition 1.2.** *Let  $X$  be a 4-manifold with  $b^+ > 1$  and  $w \in H^2(X; \mathbb{Z})$ . Then  $K \in H^2(X; \mathbb{Z})$  is a basic class for  $X$  if and only if there exists  $z \in \mathbb{A}(X)$  such that  $D_X^w(ze^{tD+\lambda x}) = e^{Q(tD)/2+2\lambda+K \cdot tD}$ , for all  $D \in H_2(X)$ .  $\square$*

We want to define the order of finite type for a single basic class  $K$  for  $X$ . This should be the analogue of the expected dimension  $d(\mathfrak{s})$  of the Seiberg-Witten moduli space associated to a  $\text{spin}^c$  structure  $\mathfrak{s}$  with  $c_1(\mathfrak{s}) = K$  and non-zero Seiberg-Witten invariant  $SW_{X,\mathfrak{s}}$  (see [11]). In that case there exists  $SW_{X,\mathfrak{s}}(x^a \gamma_1 \cdots \gamma_r) \neq 0$  with  $2a + r = d(\mathfrak{s})$ , and  $\gamma_i \in H_1(X)$ . The heuristic comparison of the Donaldson and Seiberg-Witten theories makes

correspond  $x$  in Seiberg-Witten theory to  $\wp = (x^2 - 4) \in \mathbb{A}(X)$  in Donaldson theory. So we define

$$\tilde{\mathbb{A}}(X) = \mathbb{C}[\wp] \otimes \Lambda^* H_1(X),$$

where we give the following grading:  $d(\wp) = 2$ , and for any  $\gamma \in H_1(X)$ ,  $d(\gamma) = 1$ . Note that this grading has nothing to do with the grading of  $\mathbb{A}(X)$  under the inclusion  $\tilde{\mathbb{A}}(X) \subset \mathbb{A}(X)$ . The fact that any 4-manifold  $X$  with  $b^+ > 1$  is of finite type is transcribed as  $D_X^w(\wp^n z) = 0$ , for any  $z \in \mathbb{A}(X)$  and sufficiently large  $n$ . Now we can give the following two definitions.

**Definition 1.3.** Let  $X$  be a 4-manifold with  $b^+ > 1$ , and let  $b \in \tilde{\mathbb{A}}(X)$ . If  $K \in H^2(X; \mathbb{Z})$  then we say that  $K$  is a basic class for  $D_X^w(b \bullet)$  if there exists  $z \in \mathbb{A}(X)$  such that  $D_X^w(bz e^{tD+\lambda x}) = e^{Q(tD)/2+2\lambda+K \cdot tD}$ , for all  $D \in H_2(X)$ . This condition is independent of  $w \in H^2(X; \mathbb{Z})$ .

Using [9, theorem 6] we have that for  $b \in \tilde{\mathbb{A}}(X)$  and any homogeneous  $z \in \Lambda^* H_1(X)$ , there exist polynomials  $p_{i,z}, q_{i,z} \in \text{Sym}^* H^2(X) \otimes \mathbb{C}[\lambda]$ , for every basic class  $K_i$ , such that

$$(1) \quad D_X^w(bz e^{tD+\lambda x}) = \sum p_{i,z}(tD, \lambda) e^{Q(tD)/2+2\lambda+K_i \cdot tD} + \sum q_{i,z}(tD, \lambda) e^{-Q(tD)/2-2\lambda+iK_i \cdot tD},$$

for all  $D \in H_2(X)$ . Then the condition that  $K$  be a basic class for  $D_X^w(b \bullet)$  is equivalent to the existence of some homogeneous  $z \in \Lambda^* H_1(X)$  such that  $p_{i,z} \neq 0$  in (1) for  $K = K_i$ .

**Definition 1.4.** Let  $X$  be a 4-manifold with  $b^+ > 1$ , and  $K \in H^2(X; \mathbb{Z})$  a basic class for  $X$ . Then we define the order of finite type of  $K$  to be

$$d(K) = \max\{d(b) \mid K \text{ is a basic class for } D_X^w(b \bullet), b \in \tilde{\mathbb{A}}(X)\}.$$

*Remark 1.5.* We leave the proof of the following characterization of  $d(K)$  to the reader. Collecting together (as  $z$  runs through an homogeneous basis of  $\Lambda^* H_1(X)$ ) all the polynomials from [9, theorem 6], we have  $P_i, Q_i \in \text{Sym}^* H^2(X) \otimes \mathbb{C}[\lambda] \otimes \Lambda^* H^1(X)$ , for every basic class  $K_i$ , such that

$$D_X^w(z e^{tD+\lambda x}) = \sum P_i(tD, \lambda, z) e^{Q(tD)/2+2\lambda+K_i \cdot tD} + \sum Q_i(tD, \lambda, z) e^{-Q(tD)/2-2\lambda+iK_i \cdot tD},$$

for all  $D \in H_2(X)$  and  $z \in \Lambda^* H_1(X)$ . Then  $d(K_i) = \deg P_i = \deg Q_i$ , where the elements in  $H^2(X)$  have degree 2,  $\lambda$  has degree 2 and the elements in  $H^1(X)$  have degree 1.

The main results that we prove are refinements, for four-manifolds not of simple type or with  $b_1 > 0$ , of the adjunction inequality proved by Kronheimer and Mrowka [4]. They are analogues to theorem 1.1, theorem 1.3 and theorem 1.4 of [11], which are established in the context of Seiberg-Witten invariants.

**Theorem 1.6.** Let  $X$  be a 4-manifold with  $b^+ > 1$  and let  $\Sigma \subset X$  be an embedded surface of genus  $g \geq 1$  either with self-intersection  $\Sigma^2 > 0$  or with  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd homology class. Let  $b \in \tilde{\mathbb{A}}(\Sigma)$ . If  $K$  is a basic class for  $D_X^w(b \bullet)$  then

$$|K \cdot \Sigma| + \Sigma^2 + d(b) \leq 2g - 2.$$

Note in particular that theorem 1.1 follows from theorem 1.6. With addition of the condition  $b_1 = 0$  on  $X$  we get the stronger result

**Theorem 1.7.** *Let  $X$  be a 4-manifold with  $b^+ > 1$  and  $b_1 = 0$ , and let  $\Sigma \subset X$  be an embedded surface of genus  $g \geq 1$  either with self-intersection  $\Sigma^2 > 0$  or with  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd homology class. If  $K$  is a basic class for  $X$  then we have the following adjunction inequality*

$$|K \cdot \Sigma| + \Sigma^2 + 2d(K) \leq 2g - 2.$$

As in [11], theorem 1.7 may be generalised as

**Theorem 1.8.** *Let  $X$  be a 4-manifold with  $b^+ > 1$  and let  $\iota : \Sigma \hookrightarrow X$  be an embedded surface of genus  $g \geq 1$  either with self-intersection  $\Sigma^2 > 0$  or with  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd homology class. Let  $l$  be an integer so that there is a symplectic basis  $\{\gamma_i\}_{i=1}^{2g}$  of  $H_1(\Sigma)$  with  $\gamma_i \cdot \gamma_{g+i} = 1$ ,  $1 \leq i \leq g$ , satisfying that  $\iota_*(\gamma_j) = 0$  in  $H_1(X)$  for  $j = 1, \dots, l$ . Let  $b \in \hat{A}(\Sigma)$  be an element of degree  $d(b) \leq l + 1$ . If  $K$  is a basic class for  $D_X^w(b \bullet)$  then we have*

$$|K \cdot \Sigma| + \Sigma^2 + 2d(b) \leq 2g - 2.$$

A simple application is the following. Take  $X$  and  $\Sigma$  as in the statement of theorem 1.6 such that there is a basic class  $K$  of  $X$  with  $|K \cdot \Sigma| + \Sigma^2 = 2g - 2$  (for instance the manifolds  $B_g$  or  $C_g$  of [6, definition 25] will do, for any  $g \geq 1$ ). Let  $l \geq 1$  and consider the connected sum  $X' = X \# l(\mathbb{S}^1 \times \mathbb{S}^3)$ . Put  $\gamma_i$  for the loop corresponding to the  $\mathbb{S}^1$  factor in the  $i$ -th copy of  $\mathbb{S}^1 \times \mathbb{S}^3$  and let  $T_i \subset \mathbb{S}^1 \times \mathbb{S}^3$  be the (homologically trivial) torus of the natural elliptic fibration, for  $1 \leq i \leq l$ . Let  $\Sigma' = \Sigma \# T_1 \# \dots \# T_l \subset X'$  be the genus  $g + l$  surface obtained by performing internal connected sums. As  $D_{X'}^w(\gamma_1 \dots \gamma_l z) = D_X^w(z)$ , for any  $z \in A(X)$ , theorem 1.8 is sharp for  $X'$  and  $\Sigma'$  with  $b = \gamma_1 \dots \gamma_l$ . So the homology class of  $\Sigma'$  cannot be represented by an embedded surface  $S \subset X'$  with  $H_1(T_1) \oplus \dots \oplus H_1(T_l) \subset H_1(S)$  and genus strictly less than  $g + l$ .

In section 2 we describe the Fukaya-Floer homology  $HFF_g^*$  of the three manifold  $Y = \Sigma \times \mathbb{S}^1$ , where  $\Sigma$  is a surface of genus  $g \geq 1$ . This is a finite graded commutative  $\mathbb{C}[[t]]$ -algebra generated by elements  $\alpha$ ,  $\beta$  and  $\psi_i$ ,  $1 \leq i \leq 2g$ , canonically associated to generators of the homology  $H_*(\Sigma)$ . The eigenvalues of the  $\mathbb{C}[[t]]$ -linear (commuting) endomorphisms given by multiplication by  $\alpha$ ,  $\beta$  and  $\psi_i$ ,  $1 \leq i \leq 2g$ , form a discrete set. Actually there is a Jordan decomposition of  $HFF_g^*$  with respect to all of these endomorphisms, which corresponds with the artinian decomposition of  $HFF_g^*$  interpreted as a module over the free algebra  $\mathbb{C}[[t]][\alpha, \beta] \otimes \Lambda^*(\psi_1, \dots, \psi_{2g})$ . Each of the pieces  $\mathcal{H}_r$  of the decomposition  $HFF_g^* = \oplus \mathcal{H}_r$  is labelled by an integer  $r$  running between  $-(g - 1)$  and  $g - 1$ , and is the localization of  $HFF_g^*$  at a corresponding prime ideal. The piece  $\mathcal{H}_r$  controls the basic classes  $K$  such that  $K \cdot \Sigma = 2r$ , for any 4-manifold  $X$  with  $b^+ > 1$  and an embedded surface  $\Sigma \subset X$  with  $\Sigma^2 = 0$  (and representing an odd element in homology).

In section 4 we get the new relations coming out of the description of the Fukaya-Floer homology of  $Y$  and also how imposing “extra” conditions produces more relations. This is used in section 5 in a fairly straightforward way to prove our main results: theorem 1.1 and theorems 1.6–1.8.

For completeness, in section 3 we show how the artinian decomposition of the Floer homology  $HF^*(Y)$  is related to the cohomology of some symmetric products of  $\Sigma$ , looking like a deformation of their natural ring structures. This is what one would expect it to be, as the Seiberg-Witten-Floer homologies of  $Y$  (labelled by the  $\text{spin}^C$  structures on  $Y$ ) are a deformation of the cohomology rings of symmetric products of  $\Sigma$ , and probably isomorphic to the quantum cohomology of such spaces (see [10]).

## 2. STRUCTURE OF THE FUKAYA-FLOER HOMOLOGY OF $\Sigma \times \mathbb{S}^1$

Fix a surface  $\Sigma$  of genus  $g \geq 1$  and consider the 3-manifold  $Y = \Sigma \times \mathbb{S}^1$  with the  $SO(3)$ -bundle with  $w_2 = \text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$  and the loop  $\mathbb{S}^1 \subset Y = \Sigma \times \mathbb{S}^1$ . Associated to this triple we have defined Fukaya-Floer (co)homology groups  $HFF_g^* = HFF^*(Y, \mathbb{S}^1)$  which were (partially) determined in [8, section 5]. In this section it is our purpose to describe the artinian decomposition of the ring structure of  $HFF_g^*$ .

There are two basic properties of the Fukaya-Floer homology. The first one is the existence of relative Donaldson invariants. For every 4-manifold  $X_1$  with boundary  $\partial X_1 = Y$ ,  $w_1 \in H^2(X_1; \mathbb{Z})$  such that  $w_1|_Y = w \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$ ,  $z \in \mathbb{A}(X_1)$  and  $D_1 \subset X_1$  a 2-cycle with  $\partial D_1 = \mathbb{S}^1$ , one has a relative invariant

$$\phi^w(X_1, ze^{tD_1}) \in HFF_g^*.$$

The second basic property of the Fukaya-Floer homology is the existence of a pairing

$$\langle, \rangle : HFF_g^* \otimes HFF_g^* \rightarrow \mathbb{C}[[t]],$$

satisfying a gluing property for the Donaldson invariants [8, theorem 3.1]. Let  $X = X_1 \cup_Y X_2$  be a 4-manifold, split in two 4-manifolds  $X_1$  and  $X_2$  with boundary  $\partial X_1 = -\partial X_2 = Y$ , and  $w \in H^2(X; \mathbb{Z})$  satisfying  $w \cdot \Sigma \equiv 1 \pmod{2}$ . Put  $w_i = w|_{X_i} \in H^2(X_i; \mathbb{Z})$ . Let  $D \in H_2(X)$  be decomposed as  $D = D_1 + D_2$  with  $D_i \subset X_i$ ,  $i = 1, 2$ , 2-cycles with  $\partial D_1 = -\partial D_2 = \mathbb{S}^1$ . For  $z_i \in \mathbb{A}(X_i)$ ,  $i = 1, 2$ , it is

$$D_X^{(w, \Sigma)}(z_1 z_2 e^{tD}) = \langle \phi^{w_1}(X_1, z_1 e^{tD_1}), \phi^{w_2}(X_2, z_2 e^{tD_2}) \rangle,$$

where  $D_X^{(w, \Sigma)} = D_X^w + D_X^{w+\Sigma}$ . If  $X$  has  $b^+ = 1$  then the invariants are calculated for metrics on  $X$  giving a long neck to the splitting  $X = X_1 \cup_Y X_2$ .

Let  $A = \Sigma \times D^2$  be the product of  $\Sigma$  times a 2-dimensional disc and consider the horizontal section  $\Delta = \text{pt} \times D^2 \subset A$ . Put  $w = \text{P.D.}[\Delta] \in H^2(A; \mathbb{Z})$ . Let  $\{\gamma_i\}$  be a symplectic basis for

$H_1(\Sigma)$  with  $\gamma_i \cdot \gamma_{g+i} = 1$ ,  $1 \leq i \leq g$ . We have the following elements

$$(2) \quad \begin{cases} \alpha = 2\phi^w(A, \Sigma e^{t\Delta}) \in HFF_g^2 \\ \psi_i = \phi^w(A, \gamma_i e^{t\Delta}) \in HFF_g^3, \\ \beta = -4\phi^w(A, x e^{t\Delta}) \in HFF_g^4 \end{cases} \quad 1 \leq i \leq 2g$$

The Fukaya-Floer homology  $HFF_g^*$  is a  $\mathbb{C}[[t]]$ -algebra [8, section 5] generated by the elements (2) and the product for  $HFF_g^*$  is actually determined by the property

$$\phi^w(A, z_1 e^{t\Delta}) \phi^w(A, z_2 e^{t\Delta}) = \phi^w(A, z_1 z_2 e^{t\Delta}),$$

for  $z_1, z_2 \in \mathbb{A}(\Sigma)$ . The mapping class group of  $\Sigma$  acts on  $HFF_g^*$  factoring through an action of the symplectic group  $\mathrm{Sp}(2g, \mathbb{Z})$  on  $\{\psi_i\}$ . The invariant part,  $(HFF_g^*)_I$ , is generated by  $\alpha$ ,  $\beta$  and  $\gamma = -2 \sum \psi_i \psi_{g+i}$ . For  $0 \leq k \leq g$ , we define the primitive component of  $\Lambda^k = \Lambda^k(\psi_1, \dots, \psi_{2g})$  as

$$\Lambda_0^k = \Lambda_0^k(\psi_1, \dots, \psi_{2g}) = \ker(\gamma^{g-k+1} : \Lambda^k \rightarrow \Lambda^{2g-k+2}).$$

The spaces  $\Lambda_0^k$  are irreducible  $\mathrm{Sp}(2g, \mathbb{Z})$ -representations. We have the following structural result

**Theorem 2.1.** ([8, theorem 5.3]) *Let  $\Sigma$  be a surface of genus  $g \geq 1$ . Then  $HFF_g^*$  is, as  $\mathrm{Sp}(2g, \mathbb{Z})$  representation,*

$$(3) \quad HFF_g^* = \bigoplus_{k=0}^{g-1} \Lambda_0^k \otimes \frac{\mathbb{C}[[t]][\alpha, \beta, \gamma]}{J_{g-k}},$$

where  $J_r = (R_r^1, R_r^2, R_r^3)$  and  $R_r^i$  are defined recursively by setting  $R_0^1 = 1$ ,  $R_0^2 = 0$ ,  $R_0^3 = 0$  and putting, for all  $0 \leq r \leq g-1$ ,

$$\begin{cases} R_{r+1}^1 = (\alpha + f_{11})R_r^1 + r^2(1 + f_{12})R_r^2 + f_{13}R_r^3 \\ R_{r+1}^2 = (\beta + (-1)^{r+1}8 + f_{21})R_r^1 + f_{22}R_r^2 + (\frac{2r}{r+1} + f_{23})R_r^3 \\ R_{r+1}^3 = \gamma R_r^1 \end{cases}$$

for some (unknown) functions  $f_{ij} \in t\mathbb{C}[[t]][\alpha, \beta, \gamma]$ , dependent on  $r$  and  $g$ .  $\square$

The actual shape of the relations is not of much importance for our purposes. We intend now to understand the artinian decomposition of  $HFF_g^*$  in a fairly clean way to turn the algebra information of  $HFF_g^*$  into properties for the Donaldson invariants. To shorten the notation, we shall write

$$(4) \quad T_{g,k} = \frac{\mathbb{C}[[t]][\alpha, \beta, \gamma]}{J_{g-k}},$$

so that  $HFF_g^* = \bigoplus \Lambda_0^k \otimes T_{g,k}$ . Note that the ideals  $J_{g-k}$  depend (in principle) on  $g$  and  $k$ , and not only on the difference  $g-k$ . The discussion following proposition 5.1 in [8] shows that  $T_{g,k}$  is generated, as a free  $\mathbb{C}[[t]]$ -module, by  $\alpha^a \beta^b \gamma^c$ ,  $a+b+c < g-k$ .

**Proposition 2.2.** *There is a direct sum decomposition*

$$(5) \quad T_{g,k} = \bigoplus_{r=-(g-k-1)}^{g-k-1} R_{g,k,r},$$

where  $R_{g,k,r}$  are free  $\mathbb{C}[[t]]$ -modules. For  $r$  even, the eigenvalues of  $(\alpha, \beta, \gamma)$  in  $R_{g,k,r}$  are of the form  $(4ri + O(t), 8 + O(t), 0)$ . For  $r$  odd, the eigenvalues of  $(\alpha, \beta, \gamma)$  in  $R_{g,k,r}$  are of the form  $(4r + O(t), -8 + O(t), 0)$ . Here  $O(t)$  means any series  $f(t) \in t\mathbb{C}[[t]]$ .

*Proof.* From [8, lemma 5.2] we have that  $\gamma J_r \subset J_{r+1}$ , for all  $0 \leq r \leq g-1$ . It follows that  $\gamma^{g-k} \in J_{g-k}$ , i.e.  $\gamma^{g-k} = 0$  in  $T_{g,k}$ . So the only eigenvalue of  $\gamma$  is zero. To find the eigenvalues of  $\alpha$  and  $\beta$  we restrict to study the quotient  $\bar{T}_{g,k} = T_{g,k}/\gamma T_{g,k}$ , which is the space  $\bar{\mathcal{F}}_{g-k}$  of [8, proposition 5.11]. By [8, lemma 5.12] there is an exact sequence

$$(6) \quad \bigoplus_{\substack{-(g-k) \leq r \leq g-k \\ r \equiv g-k \pmod{2}}} R_{g-k+1,r} \hookrightarrow \bar{T}_{g,k-1} \twoheadrightarrow \bar{T}_{g,k}$$

where  $R_{g-k+1,r}$  is a free  $\mathbb{C}[[t]]$ -module of rank 1, such that for  $r$  even,  $\alpha = 4ri + O(t)$  and  $\beta = 8 + O(t)$ , and for  $r$  odd,  $\alpha = 4r + O(t)$  and  $\beta = -8 + O(t)$ . Starting with  $\bar{T}_{g,g} = 0$ , we find by descending induction that the eigenvalues of  $\bar{T}_{g,k}$  are as in the statement.

Now let  $R_{g,k,r} \subset T_{g,k}$  be the  $\mathbb{C}[[t]]$ -submodule generated by all vectors  $v \in T_{g,k}$  annihilated by some power of either  $\alpha - (4r + f(t))$  if  $r$  is odd, or  $\alpha - (4ri + f(t))$  if  $r$  is even, where  $f(t) \in t\mathbb{C}[[t]]$  is a series (depending in principle on  $v$ ). Then an application of the Chinese Remainder Theorem yields that  $T_{g,k} = \bigoplus_{r=-(g-k-1)}^{g-k-1} R_{g,k,r}$  over  $\mathbb{C}[[t]]$  (we only need to use that two polynomials  $p_{1,t}(X), p_{2,t}(X) \in \mathbb{C}[[t]][X]$  are coprime over  $\mathbb{C}[[t]]$  if  $p_{1,0}(X), p_{2,0}(X) \in \mathbb{C}[X]$  have no common roots). There is an alternative way of looking at this. Take the maximal ideal  $\mathfrak{m}_r \subset \mathbb{C}[[t]][\alpha, \beta, \gamma]$  given as  $\mathfrak{m}_r = (t, \alpha - 4r, \beta - 8, \gamma)$  if  $r$  is odd,  $\mathfrak{m}_r = (t, \alpha - 4ri, \beta + 8, \gamma)$  if  $r$  is even. Then  $R_{g,k,r} = (T_{g,k})_{\mathfrak{m}_r}$  is the localisation of  $T_{g,k}$  at  $\mathfrak{m}_r$ .  $\square$

Now we shall filter  $T_{g,k}$  (and also each of the  $R_{g,k,r}$ ) by the ideals generated by the powers of  $\gamma$  and consider the associated graded rings

$$(7) \quad \text{Gr}_{\gamma} T_{g,k} = \bigoplus_{i \geq 0} \frac{\gamma^i T_{g,k}}{\gamma^{i+1} T_{g,k}} = \bigoplus_{r=-(g-k-1)}^{g-k-1} \text{Gr}_{\gamma} R_{g,k,r}.$$

The reason for doing this is that the associated graded rings are always easier to describe than the rings themselves.

**Lemma 2.3.** *For any  $i = 0, 1, \dots, g-k-1$ , there is a well-defined map*

$$T_{g,k+i}/\gamma T_{g,k+i} \xrightarrow{\gamma^i} \gamma^i T_{g,k}/\gamma^{i+1} T_{g,k},$$

which moreover is an isomorphism.

*Proof.* From [8, lemma 5.2] we have  $\gamma J_r \subset J_{r+1} \subset J_r$ , for all  $1 \leq r \leq g-1$ , so that  $\gamma^i J_{g-k-i} \subset J_{g-k}$  and the map of the statement is well-defined. Surjectivity is obvious. To prove injectivity we work as follows. Note that  $\alpha^a \beta^b$ ,  $a+b < g-k-i$ , are a basis for  $T_{g,k+i}/\gamma T_{g,k+i}$ . Then  $\alpha^a \beta^b \gamma^i$ ,  $a+b < g-k-i$ , generate the  $\mathbb{C}[[t]]$ -module  $\gamma^i T_{g,k}/\gamma^{i+1} T_{g,k}$ . So  $\text{rk}_{\mathbb{C}[[t]]}(\gamma^i T_{g,k}/\gamma^{i+1} T_{g,k}) \leq \binom{g-k-i+1}{2}$  and

$$\text{rk}_{\mathbb{C}[[t]]} T_{g,k} = \text{rk}_{\mathbb{C}[[t]]} (\text{Gr}_\gamma T_{g,k}) = \sum \text{rk}_{\mathbb{C}[[t]]} \left( \frac{\gamma^i T_{g,k}}{\gamma^{i+1} T_{g,k}} \right) \leq \sum \binom{g-k-i+1}{2} = \text{rk}_{\mathbb{C}[[t]]} T_{g,k}.$$

Therefore equality must hold,  $\text{rk}_{\mathbb{C}[[t]]}(\gamma^i T_{g,k}/\gamma^{i+1} T_{g,k}) = \binom{g-k-i+1}{2}$ , and  $\alpha^a \beta^b \gamma^i$ ,  $a+b < g-k-i$ , are a basis for  $\gamma^i T_{g,k}/\gamma^{i+1} T_{g,k}$ . This completes the proof.  $\square$

Lemma 2.3 gives that, as  $\mathbb{C}[[t]][\alpha, \beta]$ -algebras,

$$(8) \quad \text{Gr}_\gamma T_{g,k} \cong \bigoplus_{i=0}^{g-k-1} \gamma^i \cdot \frac{T_{g,k+i}}{\gamma T_{g,k+i}}.$$

Thus in order to describe (8) we only need to understand the  $\mathbb{C}[[t]][\alpha, \beta]$ -module

$$(9) \quad \bar{T}_{g,k} = T_{g,k}/\gamma T_{g,k},$$

for  $0 \leq k \leq g-1$ . In [8] we have a description of  $\bar{T}_{g,k}$  (where it is denoted by  $\bar{\mathcal{F}}_{g-k}$ ).

**Proposition 2.4.** ([8, proposition 5.11]) *For all  $0 \leq k \leq g-1$ ,  $\bar{T}_{g,k}$  is a free  $\mathbb{C}[[t]]$ -module of rank  $\text{rk}_{\mathbb{C}[[t]]} \bar{T}_{g,k} = \binom{g-k+1}{2}$ , with basis  $\alpha^a \beta^b$ ,  $a+b < g-k$ . Moreover  $\bar{T}_{g,k} = \mathbb{C}[[t]][\alpha, \beta]/\bar{J}_{g-k}$ , with  $\bar{J}_r = (\bar{R}_r^1, \bar{R}_r^2)$ , where  $\bar{R}_0^1 = 1$ ,  $\bar{R}_0^2 = 0$ , and for all  $0 \leq r \leq g-1$ ,*

$$\begin{cases} \bar{R}_{r+1}^1 = (\alpha + \bar{f}_{12}) \bar{R}_r^1 + r^2(1 + \bar{f}_{22}) \bar{R}_r^2 \\ \bar{R}_{r+1}^2 = (\beta + (-1)^{r+1} 8 + \bar{f}_{21}) \bar{R}_r^1 + \bar{f}_{22} \bar{R}_r^2 \end{cases}$$

for some (unknown)  $\bar{f}_{ij} \in t\mathbb{C}[[t]][\alpha, \beta]$ , dependent on  $r$  and  $g$ .  $\square$

By proposition 2.2 and equation (9) there is a decomposition

$$(10) \quad \bar{T}_{g,k} = \bigoplus_{r=-(g-k-1)}^{g-k-1} \bar{R}_{g,k,r}, \quad \text{where } \bar{R}_{g,k,r} = \frac{R_{g,k,r}}{\gamma R_{g,k,r}}.$$

Here  $\bar{R}_{g,k,r}$  is an algebra over  $\mathbb{C}[[t]]$ , free of finite rank as  $\mathbb{C}[[t]]$ -module. Indeed  $\bar{R}_{g,k,r}$  is characterized as the subset of  $\bar{T}_{g,k}$  where the eigenvalues of  $(\alpha, \beta)$  are of the form  $(4r + O(t), -8 + O(t))$  if  $r$  is odd,  $(4ri + O(t), 8 + O(t))$  if  $r$  is even. In  $\bar{R}_{g,k,r}$ , we shall put  $\bar{\beta} = \beta + (-1)^{r+1} 8$ . We may consider  $\bar{R}_{g,k,r}$  only as a  $\mathbb{C}[[t]][\bar{\beta}]$ -algebra. Then we have the following result

**Lemma 2.5.** *The rank of  $\bar{R}_{g,k,r}$  is  $d+1 = \left\lfloor \frac{g-k-1-|r|}{2} \right\rfloor + 1$ , for  $-(g-k-1) \leq r \leq g-k-1$ . That is,  $\bar{R}_{g,k,r} = \mathbb{C}[[t]][\bar{\beta}]/(P_{d+1,t}(\bar{\beta}))$ , where  $P_{d+1,t}(\bar{\beta}) \in \mathbb{C}[[t]][\bar{\beta}]$  is a monic polynomial of degree  $d+1$ , dependent on  $g$  and  $r$ , such that  $P_{d+1,0} = \bar{\beta}^{d+1}$ .*



*Proof.* Decomposing (6) according to the eigenvalues of  $\alpha$  we get the exact sequence

$$(11) \quad R_{g-k+1,r} \hookrightarrow \bar{R}_{g,k-1,r} \twoheadrightarrow \bar{R}_{g,k,r},$$

for  $-(g-k-1) \leq r \leq g-k-1$ , where we put  $R_{g-k+1,r} = 0$  if  $r \not\equiv g-k \pmod{2}$ . If  $r \equiv g-k \pmod{2}$  then  $R_{g-k+1,r}$  is a free  $\mathbb{C}[[t]]$ -module of rank 1 such that  $\beta = (-1)^r 8 + O(t)$ . Therefore

$$\begin{cases} \text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,k-1,r} &= \text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,k,r} + 1, & r \equiv g-k \pmod{2} \\ \text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,k-1,r} &= \text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,k,r}, & r \not\equiv g-k \pmod{2} \end{cases}$$

Also (11) for  $r = \pm(g-k)$  gives that  $\text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,g-|r|-1,r} = 1$ , so we get by descending induction on  $k$  that  $\text{rk}_{\mathbb{C}[[t]]} \bar{R}_{g,k,r} = \left\lfloor \frac{g-k-1-|r|}{2} \right\rfloor + 1$ , for  $0 \leq k \leq g-|r|-1$ . To prove the second assertion, suppose that we have already  $\bar{R}_{g,k,r} = \mathbb{C}[[t]][\bar{\beta}]/(P_{d+1,t}(\bar{\beta}))$ , where  $P_{d+1,t}(\bar{\beta}) \in \mathbb{C}[[t]][\bar{\beta}]$  is a monic polynomial of degree  $d+1$  such that  $P_{d+1,0}(\bar{\beta}) = \bar{\beta}^{d+1}$ , and let us prove the result for  $\bar{R}_{g,k-1,r}$ . If  $r \not\equiv g-k \pmod{2}$  then  $\bar{R}_{g,k-1,r} \cong \bar{R}_{g,k,r} = \mathbb{C}[[t]][\bar{\beta}]/(P_{d+1,t}(\bar{\beta}))$ , so we are done. If  $r \equiv g-k \pmod{2}$  then let  $(-1)^r 8 + f(t)$ , with  $f(t) \in t\mathbb{C}[[t]]$ , be the eigenvalue of  $\beta$  in  $R_{g-k+1,r}$ , so that  $R_{g-k+1,r} = \mathbb{C}[[t]][\bar{\beta}]/(\bar{\beta} + f(t))$ . If the exact sequence (11) is not split (over  $\mathbb{C}[[t]]$ ) then  $\bar{R}_{g,k-1,r} = \mathbb{C}[[t]][\bar{\beta}]/(P_{d+2,t}(\bar{\beta}))$ , where  $P_{d+2,t}(\bar{\beta}) = (\bar{\beta} + f(t))P_{d+1,t}(\bar{\beta})$ , and the result follows.

Let us see that (11) does not split. Consider as in [8, proposition 4.3], the spaces  $F_r = (H F_r^*)_I$  and  $\bar{F}_r = F_r/\gamma F_r$ , so that the map given by equating  $t = 0$  is (see [8, proposition 5.11]),

$$(12) \quad \bar{T}_{g,k-1} = \bar{\mathcal{F}}_{g-k+1} \twoheadrightarrow \bar{F}_{g-k+1}.$$

By [7, proposition 20], there is a decomposition  $\bar{F}_{g-k+1} = \bigoplus_{-(g-k) \leq i \leq g-k} A_i$ , where  $\alpha - 4ii$  and  $\beta - 8$  are nilpotent for  $i$  even,  $\alpha - 4i$  and  $\beta + 8$  are nilpotent for  $i$  odd. Also from the proof of [7, proposition 20] we have that

$$\begin{cases} \bar{F}_g/(\beta + 8) = \mathbb{C}[\alpha]/((\alpha^2 + (2[\frac{g-1}{2}])^2 16) \cdots (\alpha^2 + 2^2 16)\alpha) \\ \bar{F}_g/(\beta - 8) = \mathbb{C}[\alpha]/((\alpha^2 - (2[\frac{g}{2}] - 1)^2 16) \cdots (\alpha^2 - 1^2 16)) \end{cases}$$

So in  $A_i$ ,  $\alpha - 4ii$  is a polynomial in  $\beta - 8$  (with no independent term) for  $i$  even,  $\alpha - 4i$  is a polynomial in  $\beta + 8$  (with no independent term) for  $i$  odd. In particular  $A_r = \mathbb{C}[\bar{\beta}]/(\bar{\beta}^{d+2})$  as  $\mathbb{C}[\bar{\beta}]$ -algebra. By (12), we have  $\bar{R}_{g,k-1,r} \twoheadrightarrow A_r$  and it follows that  $\bar{R}_{g,k-1,r}$  cannot be a direct sum.  $\square$

We summarize what we have in the following

**Theorem 2.6.** *The Fukaya-Floer homology of  $Y = \Sigma \times \mathbb{S}^1$ , where  $\Sigma$  is a surface of genus  $g \geq 1$ , is*

$$HFF_g^* = \bigoplus_{\substack{0 \leq k \leq g-1 \\ -(g-k-1) \leq r \leq g-k-1}} \Lambda_0^k(\psi_1, \dots, \psi_{2g}) \otimes R_{g,k,r},$$

where  $R_{g,k,r}$  is a  $\mathbb{C}[[t]][\alpha, \beta, \gamma]$ -algebra, free as  $\mathbb{C}[[t]]$ -module, characterized by the condition that the eigenvalues of  $\alpha$  in  $R_{g,k,r}$  are  $4r + O(t)$  if  $r$  is odd,  $4ri + O(t)$  if  $r$  is even. In  $R_{g,k,r}$  put  $\bar{\beta} = \beta + (-1)^{r+1}8$ . Then as  $\mathbb{C}[[t]][\bar{\beta}]$ -module,

$$\mathrm{Gr}_\gamma(R_{g,k,r}) \cong \bigoplus_{i=0}^{g-k-|r|-1} \gamma^i \cdot \frac{\mathbb{C}[[t]][\bar{\beta}]}{(P_{d(g,r,k,i)+1,t}(\bar{\beta}))},$$

where  $d = d(g, r, k, i) = \lfloor \frac{g-k-|r|-i-1}{2} \rfloor$  and where  $P_{d+1,t}(\bar{\beta}) \in \mathbb{C}[[t]][\bar{\beta}]$  is a monic polynomial of degree  $d+1$  dependent on  $g$  and  $r$  such that  $P_{d+1,0}(\bar{\beta}) = \bar{\beta}^{d+1}$ .  $\square$

We may decompose the Fukaya-Floer homology with respect to the eigenvalues of  $\alpha$  alone,

$$(13) \quad HFF_g^* = \bigoplus_{r=-(g-1)}^{g-1} \mathcal{H}_r,$$

where the eigenvalues of  $\alpha$  in  $\mathcal{H}_r$  are of the form  $4r + O(t)$  if  $r$  is odd,  $4ri + O(t)$  if  $r$  is even. Note that (13) is actually an orthogonal decomposition. Theorem 2.6 implies that

$$(14) \quad \mathcal{H}_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes R_{g,k,r},$$

which can be considered as a  $\mathbb{C}[[t]][\bar{\beta}, \gamma]$ -module. So there is a  $\mathrm{Sp}(2g, \mathbb{Z})$ -equivariant epimorphism

$$(15) \quad \tilde{\mathbb{A}}(\Sigma) \otimes \mathbb{C}[[t]] \cong \Lambda^*(\psi_1, \dots, \psi_{2g}) \otimes \mathbb{C}[[t]][\bar{\beta}] \twoheadrightarrow \mathcal{H}_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes R_{g,k,r},$$

where the isomorphism of the left hand side is given by  $\gamma_i \mapsto \psi_i$ ,  $\wp \mapsto \bar{\beta}$ , and endows  $\Lambda^* \otimes \mathbb{C}[[t]][\bar{\beta}]$  with a grading  $d$  with  $d(\bar{\beta}) = 2$  and  $d(\psi_i) = 1$ ,  $1 \leq i \leq 2g$  (the elements of  $\mathbb{C}[[t]]$  are the coefficients, with degree 0).  $\mathcal{H}_r$  is not graded in general, as the kernel of (15) is not a graded ideal.

**Corollary 2.7.** *For  $-(g-1) \leq r \leq g-1$  we have a presentation*

$$\mathcal{H}_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes \frac{\mathbb{C}[[t]][\bar{\beta}, \gamma]}{\mathcal{I}_k},$$

where  $\bar{\beta} = \beta + (-1)^r 8$ . A basis for  $\mathbb{C}[[t]][\bar{\beta}, \gamma]/\mathcal{I}_k$  is given by  $\bar{\beta}^i \gamma^j$ ,  $2i + j < g - k - |r|$ . For  $0 \leq k \leq g - |r|$ , there are polynomials (depending on  $g$  and  $r$ )

$$R_k = P_{d+1,t}(\bar{\beta}) - \sum_{\substack{2i+j \leq g-k-|r| \\ j > 0}} c_{ij}^k \bar{\beta}^i \gamma^j,$$

where  $d = \lfloor \frac{g-k-|r|-1}{2} \rfloor$ ,  $P_{d+1,t}(\bar{\beta}) \in \mathbb{C}[[t]][\bar{\beta}]$  is a monic polynomial of degree  $d+1$  such that  $P_{d+1,0}(\bar{\beta}) = \bar{\beta}^{d+1}$ , and  $c_{ij}^k \in \mathbb{C}[[t]]$ , satisfying  $\mathcal{I}_k = (R_k, \gamma R_{k+1}, \gamma^2 R_{k+2}, \dots, \gamma^{g-k-|r|})$ . (Note that  $R_{g-|r|} = 1$ .)

*Proof.* Theorem 2.6 implies that the elements  $\gamma^i \bar{\beta}^j$ , with  $j \leq d(g, r, k, i) = \lfloor \frac{g-k-|r|-i-1}{2} \rfloor$  form a basis for  $R_{g,k,r}$ . This condition is equivalent to  $2j + i < g - k - |r|$ . The ideal  $\mathcal{I}_k$  is generated by polynomials

$$(16) \quad P_{n,t}(\bar{\beta})\gamma^m - \sum_{\substack{2i+j \leq g-k-|r| \\ j > m}} c_{ij}^{nm} \bar{\beta}^i \gamma^j,$$

for some  $c_{ij}^{nm} \in \mathbb{C}[[t]]$ , where  $0 \leq m \leq g - k - |r|$  and  $n = \lfloor \frac{g-k-|r|-m-1}{2} \rfloor + 1$ . Now let  $R_k$  be the relation (16) with  $m = 0$ . The inclusions  $\gamma \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \mathcal{I}_{k+1}$  yield that the relation (16) in  $\mathcal{I}_k$  equals  $\gamma^m R_{k+m}$ . The result follows.  $\square$

### 3. FLOER HOMOLOGY AND SYMMETRIC PRODUCTS OF $\Sigma$

Let  $HF_g^* = HF^*(Y)$  be the Floer homology of  $Y = \Sigma \times \mathbb{S}^1$ , where  $\Sigma$  is a surface of genus  $g \geq 1$ , and for the  $SO(3)$ -bundle with  $w_2 = \text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$ . This has been determined by the works of Dostoglou and Salamon [2] and the ring structure by the author in [7]. The Floer homology  $HF_g^*$  is a finite dimensional algebra over  $\mathbb{C}$  (with a graduation modulo 4) and there is a natural epimorphism  $HF_g^* \twoheadrightarrow HF_g^*$  given by equating  $t = 0$  (see [8]). As a consequence of theorem 2.6 we have

**Proposition 3.1.** *The Floer homology of  $Y = \Sigma \times \mathbb{S}^1$ , where  $\Sigma$  is a surface of genus  $g \geq 1$ , is*

$$HF_g^* = \bigoplus_{\substack{0 \leq k \leq g-1 \\ -(g-k-1) \leq r \leq g-k-1}} \Lambda_0^k(\psi_1, \dots, \psi_{2g}) \otimes \hat{R}_{g,k,r},$$

where  $\hat{R}_{g,k,r}$  is a  $\mathbb{C}[\alpha, \beta, \gamma]$ -algebra, such that the eigenvalue of  $\alpha$  in  $\hat{R}_{g,k,r}$  is  $4r$  if  $r$  is odd,  $4ri$  if  $r$  is even. In  $\hat{R}_{g,k,r}$  put  $\bar{\beta} = \beta + (-1)^{r+1}8$ . Then as  $\mathbb{C}[\bar{\beta}]$ -module,

$$Gr_\gamma(\hat{R}_{g,k,r}) \cong \bigoplus_{i=0}^{g-k-|r|-1} \gamma^i \cdot \frac{\mathbb{C}[[t]][\bar{\beta}]}{(\bar{\beta}^{d(g,r,k,i)+1})},$$

where  $d(g, r, k, i) = \lfloor \frac{g-k-|r|-i-1}{2} \rfloor$ .  $\square$

We have an artinian decomposition  $HF_g^* = \bigoplus_{r=-(g-1)}^{g-1} H_r$ , where the eigenvalue of  $\alpha$  in  $H_r$  is  $4r$  if  $r$  is odd,  $4ri$  if  $r$  is even. For  $-(g-1) \leq r \leq g-1$  there is a  $\text{Sp}(2g, \mathbb{Z})$ -equivariant epimorphism

$$\tilde{\mathbb{A}}(\Sigma) \cong \Lambda^*(\psi_1, \dots, \psi_{2g}) \otimes \mathbb{C}[\bar{\beta}] \twoheadrightarrow H_r,$$

where  $\bar{\beta} = \beta + (-1)^r 8$ . The isomorphism of the left hand side is given by  $\gamma_i \mapsto \psi_i$ ,  $\wp \mapsto \bar{\beta}$ , and endows  $\Lambda^* \otimes \mathbb{C}[\bar{\beta}]$  with a grading  $d$  with  $d(\bar{\beta}) = 2$  and  $d(\psi_i) = 1$ ,  $1 \leq i \leq 2g$ . Let us see the interesting fact that the product of two elements of  $H_r$  is a sum of elements of the same or higher degree.

**Lemma 3.2.** *Consider the local ring  $\hat{R}_{g,k,r}$  and put  $\bar{g} = g - k - |r|$ . Then the elements  $\bar{\beta}^n \gamma^m$ ,  $2n + m < \bar{g}$ , form a basis of  $\hat{R}_{g,k,r}$ . Any  $w = \bar{\beta}^n \gamma^m$  with  $2n + m \geq \bar{g}$  can be written as*

$$(17) \quad w = \sum_{\substack{2i+j < \bar{g}, j \geq m \\ i+j \geq n+m}} c_{ij} \bar{\beta}^i \gamma^j,$$

for some  $c_{ij} \in \mathbb{C}$ , i.e.  $w$  is a linear combination of monomials of the basis of the same or higher degree.

*Proof.* Let  $w = \bar{\beta}^n \gamma^m$  with  $2n + m \geq \bar{g}$ . We shall prove that  $w$  can be written as (17), by descending induction on  $k$ . For  $k = g - |r| - 1$ , it is  $\bar{g} = 1$ ,  $\hat{R}_{g,k,r} = \mathbb{C}[\bar{\beta}, \gamma]/(\bar{\beta}, \gamma)$  and the statement is true. Now let  $0 \leq k < g - |r| - 1$  and suppose that the statement is proved for  $k + 1$ . Note that the inclusion  $\gamma J_{g-k-1} \subset J_{g-k}$  yields a well defined map  $T_{g,k+1} \xrightarrow{\gamma} T_{g,k}$ , which in turn gives maps  $R_{g,k+1,r} \xrightarrow{\gamma} R_{g,k,r}$  and  $\hat{R}_{g,k+1,r} \xrightarrow{\gamma} \hat{R}_{g,k,r}$ . So if  $m > 0$ , the inductive hypothesis implies that

$$\bar{\beta}^n \gamma^{m-1} = \sum_{\substack{2i+j < \bar{g}-1, j \geq m-1 \\ i+j \geq n+m-1}} a_{ij} \bar{\beta}^i \gamma^j$$

in  $\hat{R}_{g,k+1,r}$ . Now multiplying by  $\gamma$  we have the equation (17) in  $\hat{R}_{g,k,r}$  with  $c_{ij} = a_{i,j-1}$  for  $j \geq 1$ ,  $c_{i0} = 0$ .

For the case  $m = 0$ , we see that it is enough to prove the statement for  $\bar{\beta}^{d+1}$ , where  $d = \lfloor \frac{\bar{g}-1}{2} \rfloor$ . This is so since once we have equation (17) for  $w = \bar{\beta}^{d+1}$ , we may multiply by  $\bar{\beta}$  and use recurrence to get equation (17) for any  $w = \bar{\beta}^n$  with  $n > d + 1$ .

Now let  $w = \bar{\beta}^{d+1}$ , and write it in terms of the basis as

$$(18) \quad \bar{\beta}^{d+1} = \sum_{\substack{2i+j < \bar{g} \\ j \geq 1}} c_{ij} \bar{\beta}^i \gamma^j.$$

Note that it must be  $j \geq 1$  from proposition 3.1. Multiplying both sides of (18) by  $\gamma$  we get  $\bar{\beta}^{d+1} \gamma = \sum c_{ij} \bar{\beta}^i \gamma^{j+1}$ . Using the case already proved above (for  $m > 0$ ), we get that

$$\sum_{\substack{2i+j < \bar{g}, j \geq 1 \\ i+j+1 \leq d+1}} c_{ij} \bar{\beta}^i \gamma^{j+1}$$

is expressible as a linear combination of monomials of the basis of degree bigger or equal than  $d+2$ . This is impossible since  $\bar{\beta}^i \gamma^{j+1}$ ,  $i+j+1 \leq d+1$ ,  $j \geq 1$ , are themselves monomials of the basis. Indeed,  $2i + (j+1) \leq 2d + 1 - j \leq 2d = 2\lfloor \frac{\bar{g}-1}{2} \rfloor < \bar{g}$ . So it must be  $c_{ij} = 0$  in (18) whenever  $i + j \leq d$ .  $\square$

Analogously to corollary 2.7 we have in the case of the Floer homology  $HF_g^*$  the following result

**Proposition 3.3.** *We have an artinian decomposition  $HF_g^* = \bigoplus_{r=-(g-1)}^{g-1} H_r$ , where the eigenvalue of  $\alpha$  in  $H_r$  is  $4r$  if  $r$  is odd,  $4ri$  if  $r$  is even. For  $-(g-1) \leq r \leq g-1$  there is a  $Sp(2g, \mathbb{Z})$ -equivariant epimorphism  $\tilde{\mathbb{A}}(\Sigma) \cong \Lambda^*(\psi_1, \dots, \psi_{2g}) \otimes \mathbb{C}[\bar{\beta}] \rightarrow H_r$ , where  $\bar{\beta} = \beta + (-1)^r 8$ . There is a presentation*

$$H_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes \frac{\mathbb{C}[\bar{\beta}, \gamma]}{I_k}.$$

Also for  $0 \leq k \leq g - |r|$ , there are polynomials

$$R_k = \bar{\beta}^{d+1} - \sum_{\substack{2i+j < g-k-|r| \\ i+j \geq d+1, j > 0}} c_{ij}^k \bar{\beta}^i \gamma^j,$$

where  $d = \lfloor \frac{g-k-|r|-1}{2} \rfloor$  and  $c_{ij}^k \in \mathbb{C}$ . Then  $I_k = (R_k, \gamma R_{k+1}, \gamma^2 R_{k+2}, \dots, \gamma^{g-k-|r|})$ .  $\square$

*Remark 3.4.* In proposition 3.3 we have not taken any effort in looking for a minimal set of generators of the ideals  $I_k$ . We conjecture that the first two generators suffice.

It is our intention to relate the pieces  $H_r$  with the cohomology of the symmetric product  $s^{g-|r|-1}\Sigma$  of the surface  $\Sigma$ , as predicted in [7, conjecture 24]. Fix  $0 \leq d \leq g-1$ . We shall review the description of  $H^*(s^d\Sigma)$  from [5], putting it into the right form for our purposes. We interpret  $s^d\Sigma$  as the moduli space of degree  $d$  effective divisors on  $\Sigma$  (for this we need to put a complex structure on  $\Sigma$ , but at the end this will be irrelevant for the description of the cohomology). Let  $D \subset s^d\Sigma \times \Sigma$  be the universal divisor. Then we define

$$\begin{cases} \psi_i = c_1(D)/\gamma_i \in H^1(s^d\Sigma), & 1 \leq i \leq 2g \\ \eta = c_1(D)/x \in H^2(s^d\Sigma) \end{cases}$$

These elements generate  $H^*(s^d\Sigma)$ . More precisely, there is a (graded!)  $Sp(2g, \mathbb{Z})$ -equivariant epimorphism

$$(19) \quad \tilde{\mathbb{A}}(\Sigma) \cong \Lambda^*(\psi_1, \dots, \psi_{2g}) \otimes \mathbb{C}[\eta] \rightarrow H^*(s^d\Sigma).$$

In particular (for  $d > 0$ )  $H^1(s^d\Sigma)$  is freely generated by  $\psi_1, \dots, \psi_{2g}$ . We put  $\Lambda_0^k = \Lambda_0^k(\psi_1, \dots, \psi_{2g})$ . Note that since  $H^*(s^d\Sigma)$  is graded from 0 to  $2d$ ,  $\Lambda_0^k$  goes to zero under (19) for  $k > d$ . Finally note that  $\eta$  and  $\theta = \sum \psi_i \psi_{g+i}$  generate the invariant part  $H^*(s^d\Sigma)_I$ .

**Proposition 3.5.** *For  $-(g-1) \leq r \leq g-1$  there is a presentation*

$$H^*(s^d\Sigma) = \bigoplus_{k=0}^d \Lambda_0^k \otimes \frac{\mathbb{C}[\eta, \theta]}{J_k},$$

with  $J_k = (R_k, \theta R_{k+1}, \theta^2 R_{k+2}, \dots, \theta^{d+1-k})$  and

$$R_k = \sum_{i=0}^{\alpha} \frac{\binom{d-k-\alpha+1}{i}}{\binom{g-k}{i}} \frac{(-\theta)^i}{i!} \eta^{\alpha-i},$$

where  $\alpha = \lfloor \frac{d-k}{2} \rfloor + 1$ , for  $0 \leq k \leq d$ , and  $R_{d+1} = 1$ .

*Proof.* The relations of  $H^*(s^d\Sigma)$  are given in [5] and are the following

$$(20) \quad \eta^r \prod_{i \in I} (\eta - \psi_i \psi_{g+i}) \prod_{j \in J} \psi_j \prod_{k \in K} \psi_{g+k},$$

for  $I, J, K \subset \{1, \dots, g\}$  disjoint and  $r + 2|I| + |J| + |K| \geq d + 1$ . Suppose  $d \not\equiv k \pmod{2}$ , so  $d + 1 = 2\alpha + k$ . Take the relation  $\psi_1 \cdots \psi_k \prod_{i \in I} (\eta - \psi_i \psi_{g+i})$ , with  $I \subset \{k + 1, \dots, g\}$ ,  $|I| = \alpha$ . Let  $\text{Sp}(2g - 2k, \mathbb{Z})$  act on the relation, acting in the standard way on  $\{\psi_{k+1}, \dots, \psi_g, \psi_{g+k+1}, \dots, \psi_{2g}\}$ . This produces the relation

$$\psi_1 \cdots \psi_k \sum_{i=0}^{\alpha} \frac{\binom{\alpha}{i}}{\binom{g-k}{i}} \eta^{\alpha-i} \frac{(-\theta')^i}{i!},$$

where  $\theta' = \sum_{i=k+1}^g \psi_i \psi_{g+i}$ . Using  $\alpha = d - k - \alpha + 1$  this is equivalent to the relation  $\psi_1 \cdots \psi_k R_k$ , i.e.  $R_k \in J_k$ . For  $d \equiv k \pmod{2}$  take (20) with  $r = 1$ ,  $J = \{1, \dots, k\}$ ,  $K = \emptyset$ ,  $d + 1 = 2(\alpha - 1) + k + 1$ , and proceed as above to get

$$\psi_1 \cdots \psi_k \sum_{i=0}^{\alpha} \frac{\binom{\alpha-1}{i}}{\binom{g-k}{i}} \eta^{\alpha-1-i} \frac{(-\theta)^i}{i!}.$$

Then we use  $\alpha - 1 = d - k - \alpha + 1$  to get  $R_k \in J_k$ .

From the inclusions  $\theta J_{k+1} \subset J_k \subset J_{k+1}$  we get that  $\theta R_{k+1}, \dots, \theta^{d-k} R_d, \theta^{d+1-k} \in J_k$ . So  $(\theta R_{k+1}, \dots, \theta^{d-k} R_d, \theta^{d+1-k}) \subset J_k$  and there is an epimorphism

$$(21) \quad A^* = \frac{\mathbb{C}[\eta, \theta]}{(R_k, \theta R_{k+1}, \dots, \theta^{d+1-k})} \twoheadrightarrow B^* = \frac{\mathbb{C}[\eta, \theta]}{J_k}.$$

Both  $A^*$  and  $B^*$  are (evenly) graded rings and (21) is graded. As  $H^*(s^d\Sigma)$  has Poincaré duality with  $\dim s^d\Sigma = 2d$ , we conclude that  $B^*$  has Poincaré duality  $B^* \otimes B^{2d-2k-*} \rightarrow \mathbb{C}$ . Also all relations in (20) have degrees bigger than or equal to  $d + 1$ , so all elements in  $J_k$  have degrees bigger than or equal to  $d - k + 1$ . The generators of the ideal of relations of  $A^*$  have also degrees bigger than or equal to  $d - k + 1$ , so  $\dim A^{2i} = \dim B^{2i}$ , for  $0 \leq 2i \leq d - k$ . Now

$$\text{Gr}_{\theta} A^* = \bigoplus_{i=0}^{d-k} \theta^i \cdot \frac{\mathbb{C}[\eta]}{(\eta^{\lfloor (d-k-i)/2 \rfloor + 1})}.$$

Hence for  $d - k \leq 2i \leq 2d - 2k$ , we have  $\dim A^{2i} = d - k - i + 1 = \dim A^{2d-2k-2i} = \dim B^{2d-2k-2i} = \dim B^{2i}$ . Therefore (21) is an isomorphism.  $\square$

*Remark 3.6.* In proposition 3.5, the first two generators of  $J_k$  suffice, i.e.  $J_k = (R_k, \theta R_{k+1})$ . This follows from

$$\begin{cases} R_k = R_{k+1} - \frac{g-k-\alpha}{(g-k)(g-k-1)} \theta R_{k+2}, & d \not\equiv k \pmod{2} \\ R_k = \eta R_{k+1} + \frac{\alpha-1}{(g-k)(g-k-1)} \theta R_{k+2}, & d \equiv k \pmod{2} \end{cases}$$

where  $\alpha = \lfloor \frac{d-k}{2} \rfloor + 1$ .

As a consequence of proposition 3.3 we have

$$\mathrm{Gr}_\gamma H_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes \left( \bigoplus_{i=0}^{g-|r|-1-k} \gamma^i \cdot \frac{\mathbb{C}[\bar{\beta}]}{(\bar{\beta}^{d(g,r,k,i)+1})} \right),$$

where  $d(g, r, k, i) = \lfloor \frac{g-k-|r|-i-1}{2} \rfloor$ . And as a consequence of proposition 3.5,

$$\mathrm{Gr}_\theta H^*(s^d \Sigma) = \bigoplus_{k=0}^d \Lambda_0^k \otimes \left( \bigoplus_{i=0}^{d-k} \theta^i \cdot \frac{\mathbb{C}[\eta]}{(\eta^{[(d-k-i)/2]+1})} \right).$$

This proves the following

**Corollary 3.7.** *There is an isomorphism  $\mathrm{Gr}_\gamma H_r \cong \mathrm{Gr}_\theta H^*(s^{g-|r|-1} \Sigma)$ , for  $-(g-1) \leq r \leq g-1$ .  $\square$*

In particular  $H_r$  and  $H^*(s^{g-|r|-1} \Sigma)$  are isomorphic as  $\mathrm{Sp}(2g, \mathbb{Z})$ -representations, which proves [7, conjecture 24]. We believe that in fact  $H_r$  is isomorphic to the quantum cohomology  $QH^*(s^{g-|r|-1} \Sigma)$  (see [1] for a partial computation of the latter).

#### 4. NEW RELATIONS FOR THE FUKAYA-FLOER HOMOLOGY

In this section we are going to get more information on the shape of the relations for  $\mathcal{H}_r$  given in corollary 2.7 in order to prove our main theorems. Our next result is a weaker version of lemma 3.2 for the Fukaya-Floer homology, telling us that the degrees homogeneous components of the relations for  $\mathcal{H}_r$  cannot be too low.

**Lemma 4.1.** *With the notations of corollary 2.7, let  $e$  be the multiplicity of the root  $\bar{\beta} = 0$  in  $P_{d+1,t}(\bar{\beta})$ . Then we have*

$$R_k = P_{d+1,t}(\bar{\beta}) - \sum_{\substack{2i+j < g-k-|r| \\ i+j \geq e, j > 0}} c_{ij}^k \bar{\beta}^i \gamma^j.$$

*Proof.* Let  $d = d(g, r, k, 0)$  and  $\bar{g} = g - k - |r|$ . Write  $P_{d+1,t}(\bar{\beta}) = \bar{\beta}^e Q_t(\bar{\beta})$ , where  $Q_t(0) \neq 0$ . We shall prove by descending induction on  $k$  the following two statements:

- (1)  $P_{d+1,t}(\bar{\beta}) = \sum_{\substack{2i+j < \bar{g} \\ i+j \geq e, j > 0}} c_{ij} \bar{\beta}^i \gamma^j \in R_{g,k,r}$ , for some  $c_{ij} \in \mathbb{C}[[t]]$ .
- (2)  $R_{g,k,r} = \bigoplus_{i+j < e} \langle \bar{\beta}^i \gamma^j \rangle \oplus \langle \bar{\beta}^i \gamma^j | i+j \geq e \rangle$ , i.e. all non zero monomials of all relations in  $R_{g,k,r}$  have degrees bigger or equal than  $e$ .

For  $k = g - |r| - 1$  the statement is obvious. Now let  $0 \leq k < g - |r| - 1$  and suppose that the statement is proved for any number strictly bigger than  $k$ . By corollary 2.7 we may write  $P_{d+1,t}(\bar{\beta}) = \sum_{2i+j < \bar{g}, j > 0} a_{ij} \bar{\beta}^i \gamma^j$ , in  $R_{g,k,r}$ . We have to prove that  $a_{ij} = 0$  for  $i+j < e$ . We have the following three cases:

- $d(g, r, k + 1, 0) = d$ , i.e.  $\bar{g}$  even. Take the relation  $P_{d+1,t}(\bar{\beta}) = \sum_{\substack{2i+j < \bar{g}-1 \\ i+j \geq e, j > 0}} c_{ij} \bar{\beta}^i \gamma^j$  in  $R_{g,k+1,r}$ . The map  $\gamma : R_{g,k+1,r} \hookrightarrow R_{g,k,r}$  gives that

$$\gamma P_{d+1,t}(\bar{\beta}) = \sum_{\substack{2i+j < \bar{g}-1 \\ i+j \geq e, j > 0}} c_{ij} \bar{\beta}^i \gamma^{j+1} = \sum_{2i+j < \bar{g}, j > 0} a_{ij} \bar{\beta}^i \gamma^{j+1},$$

in  $R_{g,k,r}$ . Using that  $\bigoplus_{i+j < e} \langle \bar{\beta}^i \gamma^{j+1} \rangle \oplus \langle \bar{\beta}^i \gamma^{j+1} | i+j \geq e \rangle = \gamma R_{g,k+1,r} \subset R_{g,k,r}$ , we get that  $a_{ij} = 0$  for  $i+j < e$ , as required. The second item also follows, for all the relations  $\gamma^m R_{k+m}$ ,  $m > 0$ , in  $R_{g,k,r}$ , involve terms of degrees bigger or equal than  $e$  by induction hypothesis, and the new relation,  $\bar{\beta}^e Q_t(\bar{\beta}) = \sum_{i+j \geq e, j > 0} a_{ij} \bar{\beta}^i \gamma^j$ , also involves terms of degrees bigger or equal than  $e$ .

- $d(g, r, k + 1, 0) = d - 1$  and  $P_{d+1,t}(\bar{\beta}) = (\bar{\beta} + f(t)) P_{d,t}(\bar{\beta})$ , with  $f(t) \in t\mathbb{C}[[t]]$  non zero. As above, the relation in  $R_{g,k+1,r}$  produces  $\gamma P_{d+1,t}(\bar{\beta}) = \sum_{i+j \geq e, j > 0} c_{ij} \bar{\beta}^i \gamma^{j+1}$  in  $R_{g,k,r}$ . Multiplying by  $\bar{\beta} + f(t)$ ,

$$\gamma P_{d+1,t}(\bar{\beta}) = \sum_{i+j \geq e, j > 0} c'_{ij} \bar{\beta}^i \gamma^{j+1} = \sum_{2i+j < \bar{g}, j > 0} a_{ij} \bar{\beta}^i \gamma^{j+1}$$

in  $R_{g,k,r}$ . The rest is as above.

- $d(g, r, k + 1, 0) = d - 1$  and  $P_{d+1,t}(\bar{\beta}) = \bar{\beta} P_{d,t}(\bar{\beta})$ . Now the relation in  $R_{g,k+1,r}$  is  $P_{d,t}(\bar{\beta}) = \sum_{i+j \geq e-1, j > 0} c_{ij} \bar{\beta}^i \gamma^j$ , since the multiplicity of the root  $\bar{\beta} = 0$  in  $P_{d,t}(\bar{\beta})$  is  $e - 1$ . Multiplying by  $\bar{\beta}$ ,

$$(22) \quad \gamma P_{d+1,t}(\bar{\beta}) = \sum_{i+j \geq e, j > 0} c'_{ij} \bar{\beta}^i \gamma^{j+1} = \sum_{2i+j < \bar{g}, j > 0} a_{ij} \bar{\beta}^i \gamma^{j+1},$$

in  $R_{g,k,r}$ . This time we need to use the inclusion  $\gamma^2 : R_{g,k+2,r} \hookrightarrow R_{g,k,r}$ , which yields that  $\bigoplus_{i+j < e-1} \langle \bar{\beta}^i \gamma^{j+2} \rangle \oplus \langle \bar{\beta}^i \gamma^{j+2} | i+j \geq e-1 \rangle = \gamma^2 R_{g,k+2,r} \subset R_{g,k,r}$ . Applying this to (22) we get that  $a_{ij} = 0$  for  $i+j < e$ . The second item follows as above.

□

Now we aim to prove the vanishing of polynomials of high degree on  $\bar{\beta}$  and  $\gamma$ , giving a explicit bound on such degree. If we want to have this for  $\mathcal{H}_r$  we must prove first that  $P_{d+1,t}(\bar{\beta}) = \bar{\beta}^{d+1}$ , this being the only case in which  $\bar{\beta}$  is nilpotent in  $\mathcal{H}_r$ . This is equivalent to proving that the only eigenvalues of  $\beta$  on  $HFF_g^*$  are  $\pm 8$ . (We note also that if this is the case then the proof of lemma 3.2 carries over to the Fukaya-Floer setting giving a simpler method for proving lemma 4.1.) On the other hand, from the point of view of Donaldson invariants we may well restrict attention to the effective Fukaya-Floer homology, where the eigenvalues of  $\beta$  are already  $\pm 8$  (see [8, theorem 5.13]). We recall its definition

**Definition 4.2.** ([8, definition 5.10]) We define the effective Fukaya-Floer homology as the sub- $\mathbb{C}[[t]]$ -module  $\widetilde{HFF}_g^* \subset HFF_g^*$  generated by all  $\phi^w(X_1, z_1 e^{tD_1})$ , for all 4-manifolds  $X_1$



with boundary  $\partial X_1 = Y = \Sigma \times \mathbb{S}^1$  such that  $X = X_1 \cup_Y A$  has  $b^+ > 1$ ,  $z_1 \in \mathbb{A}(X_1)$ ,  $D_1 \subset X_1$  with  $\partial D_1 = \mathbb{S}^1$  and  $w \in H^2(X_1; \mathbb{Z})$  with  $w|_Y = w_2 = \text{P.D.}[\mathbb{S}^1]$ .

Clearly there is a decomposition  $\widetilde{HFF}_g^* = \bigoplus_{r=-(g-1)}^{g-1} \tilde{\mathcal{H}}_r$ , where  $\tilde{\mathcal{H}}_r \subset \mathcal{H}_r$ . By [8, theorem 5.13] the only eigenvalues of  $(\bar{\beta}, \gamma)$  on  $\tilde{\mathcal{H}}_r$  are  $(0, 0)$ . Decompose  $\tilde{\mathcal{H}}_r = \bigoplus_{k=0}^{g-|r|-1} \Lambda_0^k \otimes \tilde{R}_{g,k,r}$ , where  $\tilde{R}_{g,k,r} \subset R_{g,k,r}$ . By lemma 4.1 and corollary 2.7,

$$(23) \quad \gamma^m P_{d+1,t}(\bar{\beta}) = \gamma^m \bar{\beta}^e Q_t(\bar{\beta}) = \sum_{\substack{2i+j < g-k-|r|-m \\ i+j \geq e, j > 0}} c_{ij} \bar{\beta}^i \gamma^{j+m}$$

in  $R_{g,k,r}$ , where  $d = \lfloor \frac{g-k-|r|-m-1}{2} \rfloor$ ,  $e$  is the multiplicity of the root  $\bar{\beta} = 0$  in  $P_{d+1,t}(\bar{\beta})$  and  $Q_t(\bar{\beta})$  does not have  $\bar{\beta} = 0$  as root. On  $\tilde{R}_{g,k,r}$  the only eigenvalue of  $\bar{\beta}$  is zero, so  $Q_t(\bar{\beta})$  is invertible and hence (23) produces

$$\gamma^m \bar{\beta}^e = \sum_{i+j \geq e, j > 0} c_{ij} \bar{\beta}^i \gamma^{j+m},$$

for some (different from the previous ones) coefficients  $c_{ij} \in \mathbb{C}[[t]]$ , as endomorphisms acting on  $\tilde{R}_{g,k,r}$ . As  $e \leq d+1$ , this implies that

$$(24) \quad \gamma^m \bar{\beta}^{d+1} = \sum_{i+j \geq d+1, j > 0} c_{ij} \bar{\beta}^i \gamma^{j+m},$$

acting on  $\tilde{R}_{g,k,r}$ .

**Corollary 4.3.** *Let  $b = \bar{\beta}^n \gamma_{i_1} \cdots \gamma_{i_m} \in \tilde{\mathbb{A}}(\Sigma)$ . If  $d(b) = 2n + m > 2(g - |r| - 1)$  then  $b = 0$  acting on  $\tilde{\mathcal{H}}_r$ .*

*Proof.* Let us first prove that any  $b = \bar{\beta}^n \gamma^m$  with  $n + m \geq g - k - |r|$  is zero acting on  $\tilde{R}_{g,k,r}$ . As  $\gamma^{g-k-|r|} = 0$  in  $R_{g,k,r}$  we may suppose  $m < g - k - |r|$  and  $n > 0$ . Clearly  $n \geq d + 1 = \lfloor \frac{g-k-|r|-m-1}{2} \rfloor + 1$ , so we use (24) to write the endomorphism  $\bar{\beta}^n \gamma^m$  as a  $\mathbb{C}[[t]]$ -linear combination of endomorphisms  $\bar{\beta}^i \gamma^j$  with  $i + j \geq n + m$  and  $j > m$ . By recursion the claim follows.

Now let  $b = \bar{\beta}^n \gamma_{i_1} \cdots \gamma_{i_m}$  with  $2n + m > 2g - 2|r| - 2$ . We have the simple fact

$$\gamma_{i_1} \cdots \gamma_{i_m} \Lambda_0^k \subset \Lambda_0^{k+m} \oplus \gamma \Lambda_0^{k+m-2} \cdots \oplus \gamma^m \Lambda_0^{k-m},$$

where a negative exponent is understood as that the corresponding term does not appear. Hence to see that  $b$  kills  $\Lambda_0^k \tilde{R}_{g,k,r}$  it is enough to prove that  $\bar{\beta}^n \gamma^i = 0$  on  $\tilde{R}_{g,k+m-2i,r}$ , for any  $0 \leq i \leq \lfloor \frac{k+m}{2} \rfloor$ . This follows from the above.  $\square$

If we introduce extra relations in  $HFF_g^*$ , we can expect that elements of less degree may become zero acting on  $\tilde{\mathcal{H}}_r$ . The next result deals with the case where we kill all elements  $\psi_i$ ,  $1 \leq i \leq 2g$ .

**Proposition 4.4.** *We have that  $\bar{\beta}^{d+1}$  acts as zero on  $\tilde{\mathcal{H}}_r/(\psi_1, \dots, \psi_{2g})$ , where  $d = \lfloor \frac{g-|r|-1}{2} \rfloor$ .*

*Proof.* From equation (24) one has that

$$\bar{\beta}^{d+1} = \sum_{i+j \geq d+1, j > 0} c_{ij} \bar{\beta}^i \gamma^j,$$

on  $\tilde{R}_{g,0,r}$ , with  $d = \lfloor \frac{g-|r|-1}{2} \rfloor$ . This means that  $\bar{\beta}^{d+1}$  sends  $\tilde{\mathcal{H}}_r$  into the ideal  $(\psi_1, \dots, \psi_{2g}) \subset \tilde{\mathcal{H}}_r$ , which is equivalent to the statement.  $\square$

Now we shall weaken the conditions in proposition 4.4 in the form we need them to prove theorem 1.8. The following result is parallel to [11, proposition 6.12]. In fact the proof of [11, proposition 6.12] can be done along the lines of the proof herein (see [10, section 8]).

**Proposition 4.5.** *Let  $l = g - |r| - 1$ . Consider the ideal  $\mathcal{I} = (\psi_1, \dots, \psi_l)$  in  $\tilde{\mathcal{H}}_r$ . Then any element  $b \in \hat{\mathbb{A}}(\Sigma)$  of degree bigger or equal than  $l + 1$ , sends  $\tilde{\mathcal{H}}_r$  into  $\mathcal{I}$ .*

*Proof.* It is enough to consider elements of degree  $l + 1$ . A basic monomial of degree  $l + 1$  is of the form

$$z = \bar{\beta}^a \prod_{i \in I} (\psi_i \psi_{g+i}) \prod_{j \in J} \psi_j \prod_{k \in K} \psi_{g+k},$$

where  $I, J, K$  are disjoint subsets of  $\{1, \dots, g\}$  with  $2a + 2|I| + |J| + |K| = l + 1$ . If any  $i \in I$  or  $j \in J$  lies in  $\{1, \dots, l\}$  then  $z \in \mathcal{I}$ . Therefore we may suppose that  $I$  and  $J$  are disjoint with  $\{1, \dots, l\}$ . Moreover if  $a = |I| = |J| = 0$  then  $z = \prod_{k \in K} \psi_{g+k} \in \Lambda_0^{g-|r|}$  and by (14) then  $z = 0 \subset \mathcal{H}_r$ . So we also suppose  $a + |I| + |J| > 0$ . Hence  $|\{1, \dots, l\} - K| \geq l - |K| = 2a + 2|I| + |J| - 1 \geq |I|$ . This means that we may associate (in an injective way) to every  $i \in I$  an  $r(i) \in \{1, \dots, l\} - K$ , so that  $\{r(i) | i \in I\}$ ,  $I$ ,  $J$  and  $K$  are mutually disjoint. Therefore  $z \in \mathcal{H}_r$  is congruent (modulo  $\mathcal{I}$ ) to

$$z' = \bar{\beta}^a \prod_{i \in I} (\psi_i \psi_{g+i} - \psi_{r(i)} \psi_{g+r(i)}) \prod_{j \in J} \psi_j \prod_{k \in K} \psi_{g+k}.$$

We conclude that  $z' = \bar{\beta}^a z_0$ , with  $z_0 \in \Lambda_0^{l+1-2a}$ . Now equation (24) tells us that

$$\bar{\beta}^a = \sum_{i+j \geq a, j > 0} c_{ij} \bar{\beta}^i \gamma^j,$$

acting on  $\tilde{R}_{g,l+1-2a,r}$ . So we rewrite  $z'$  as a polynomial with all homogeneous components of degree bigger or equal than  $l + 1$ , and such that all its monomials contain powers  $\bar{\beta}^i$ , with  $i < a$ . By induction we are done, since when  $a = 0$ ,  $z' = z_0 \in \Lambda_0^{g-|r|}$ , whose image is zero in  $\mathcal{H}_r$ .  $\square$

## 5. PROOF OF MAIN RESULTS

*Proof of theorem 1.1.* Suppose that  $X$  is a 4-manifold with  $b^+ > 1$ . In the first place we shall reduce to the case of self-intersection  $\Sigma^2 = 0$  with  $\Sigma$  representing an odd homology class. Suppose that  $N = \Sigma^2 > 0$ . Take the  $N$ -th blow-up of  $X$  and call it  $\tilde{X} = X \# N\overline{\mathbb{CP}}^2$ . Let  $E_1, \dots, E_N$  denote the cohomology classes of the exceptional divisors, which are represented by embedded spheres of self-intersection  $-1$ . Consider the proper transform  $\tilde{\Sigma}$ . This is an embedded surface of genus  $g$  in  $\tilde{X}$  representing the homology class  $\Sigma - E_1 - \dots - E_N$ . It is obtained by tubing together  $\Sigma$  with the exceptional spheres with reversed orientation inside  $\tilde{X}$ . Then  $\tilde{\Sigma}^2 = 0$  and  $\tilde{\Sigma}$  represents an odd homology class, since if we take  $w = E_1 \in H^2(X; \mathbb{Z})$  then  $\tilde{\Sigma} \cdot w = 1$ . On the other hand, the blow-up formula [3] implies that the order of finite type of  $\tilde{X}$  is the same as that of  $X$ , so it is enough to prove the theorem for  $\tilde{X}$  and  $\tilde{\Sigma}$ .

Once we have done this reduction, choose  $w \in H^2(X; \mathbb{Z})$  with  $w \cdot \Sigma \equiv 1 \pmod{2}$  (from [9] we know that the order of finite type does not depend on  $w \in H^2(X; \mathbb{Z})$ ). What we shall prove is the following: for any  $b \in \tilde{\mathbb{A}}(\Sigma)$  with  $d(b) \geq 2g - 1$ , we have  $D_X^w(bz) = 0$ , for any  $z \in \mathbb{A}(X)$ . This is slightly stronger than the statement of the theorem.

We can suppose without loss of generality that  $b$  is homogeneous, i.e. of the form  $b = \wp^n \gamma_{i_1} \cdots \gamma_{i_m}$ , where  $d(b) = 2n + m$ . Now consider a small tubular neighbourhood  $A = \Sigma \times D^2$  of  $\Sigma$  inside  $X$  and let  $X_1$  be the complement of the interior of  $A$ . Therefore  $\partial X_1 = Y = \Sigma \times \mathbb{S}^1$  and  $X = X_1 \cup_Y A$ . Let  $D \in H_2(X)$  with  $D \cdot \Sigma = 1$ . Represent  $D$  by a 2-cycle of the form  $D = D_1 + \Delta$ , where  $D_1 \subset X_1$  with  $\partial D_1 = \mathbb{S}^1$  and  $\Delta = \text{pt} \times D^2 \subset A$ . Take  $z \in \mathbb{A}(X_1)$ . Then

$$\phi^w(X_1, ze^{tD_1}) \in \widetilde{HFF}_g^*$$

may be decomposed according to components

$$\phi^w(X_1, ze^{tD_1})_r \in \tilde{\mathcal{H}}_r,$$

with  $-(g-1) \leq r \leq g-1$ . With the notations of section 2,  $\bar{\beta} = \beta + (-1)^{r+1}8$  in  $\mathcal{H}_r$ . By corollary 4.3,  $\bar{\beta}^n \psi_{i_1} \cdots \psi_{i_m}$  is zero acting on  $\tilde{\mathcal{H}}_r$ , since it has degree  $d(b) = 2n + m \geq 2g - 1 \geq 2(g - |r|) - 1$ . Hence

$$0 = \frac{1}{16^n} (\beta^2 - 64)^n \psi_{i_1} \cdots \psi_{i_m} \phi^w(X_1, ze^{tD_1})_r = \phi^w(X_1, \wp^n \gamma_{i_1} \cdots \gamma_{i_m} ze^{tD_1})_r,$$

for any  $r$ . So  $\phi^w(X_1, bze^{tD_1}) = 0$  and

$$D_X^{(w, \Sigma)}(bze^{tD}) = \langle \phi^w(X_1, bze^{tD_1}), \phi^w(A, e^{t\Delta}) \rangle = 0.$$

In particular  $D_X^w(bze^{tD}) = 0$  for any  $z \in \mathbb{A}(X_1)$  and any  $D \in H_2(X)$  with  $D \cdot \Sigma = 1$ . This implies the result for a 4-manifold with  $b^+ > 1$ .  $\square$

*Proof of theorem 1.6.* First we shall reduce to the case of  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd homology class. Let  $X$  be a 4-manifold with  $b^+ > 1$  and consider its blow-up  $\tilde{X} = X \# \overline{\mathbb{CP}}^2$  at one point, where  $E$  stands for the cohomology class of the exceptional divisor. Now if  $b \in \tilde{\mathbb{A}}(X) = \tilde{\mathbb{A}}(\tilde{X})$  and  $K$  is a basic class for  $D_X^w(b \bullet)$ , then there exists  $z \in \mathbb{A}(X)$  such that  $D_X^w(bze^{tD+\lambda x}) = e^{Q(tD)/2+2\lambda+K \cdot tD}$ , for all  $D \in H_2(X)$ . The blow-up formula [3] tells us that  $D_{\tilde{X}}^w(bze^{tD+\lambda x}) = e^{Q_{\tilde{X}}(tD)/2+2\lambda+K \cdot tD} \cosh(E \cdot tD)$ , for  $D \in H_2(\tilde{X})$  (compare [9, proposition 16]). Therefore  $K \pm E$  are basic classes for  $D_{\tilde{X}}^w(b \bullet)$ . If  $\Sigma$  is an embedded surface in  $X$  of genus  $g$  with self-intersection  $\Sigma^2 > 0$  then its proper transform  $\tilde{\Sigma}$  is an embedded surface in  $\tilde{X}$  of genus  $g$ ,  $\tilde{\Sigma}^2 = \Sigma^2 - 1$  and representing the homology class  $\Sigma - E$ . Put  $\varepsilon$  for the sign of  $K \cdot \Sigma$ . Then  $K - \varepsilon E$  is a basic class for  $\tilde{X}$  and

$$|(K - \varepsilon E) \cdot \tilde{\Sigma}| + \tilde{\Sigma}^2 = |K \cdot \Sigma| + \Sigma^2.$$

So if we prove the theorem for  $\tilde{X}$  then it will be proved for  $X$ . This means that we can reduce to the case of self-intersection  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd element in homology.

As in the proof of theorem 1.1, we choose  $w \in H^2(X; \mathbb{Z})$  with  $w \cdot \Sigma \equiv 1 \pmod{2}$  (since  $\Sigma$  represents an odd homology class). Also we can suppose  $b = \wp^n \gamma_{i_1} \cdots \gamma_{i_m}$ , where  $d(b) = 2n + m$ . Consider a small tubular neighbourhood  $A = \Sigma \times D^2$  of  $\Sigma$  inside  $X$  and let  $X_1$  be the complement of the interior of  $A$ . Therefore  $\partial X_1 = Y = \Sigma \times \mathbb{S}^1$  and  $X = X_1 \cup_Y A$ . Let  $D \in H_2(X)$  with  $D \cdot \Sigma = 1$ . Represent  $D$  by a 2-cycle of the form  $D = D_1 + \Delta$ , where  $D_1 \subset X_1$  with  $\partial D_1 = \mathbb{S}^1$  and  $\Delta = \text{pt} \times D^2 \subset A$ . Fix an homogeneous  $z_0 \in \Lambda^* H_1(X) \subset \mathbb{A}(X)$ . Then  $\phi^w(X_1, bz_0 e^{tD_1}) \in \widetilde{HFF}_g^*$  and

$$(25) \quad D_X^{(w, \Sigma)}(bz_0 e^{tD+\lambda x+s\Sigma}) = \langle \phi^w(X_1, bz_0 e^{tD_1}), \phi^w(A, e^{t\Delta+\lambda x+s\Sigma}) \rangle.$$

Decomposing according to components  $\widetilde{HFF}_g^* = \oplus \tilde{\mathcal{H}}_r$ , we have

$$D_X^{(w, \Sigma)}(bz_0 e^{tD+\lambda x+s\Sigma}) = \sum_{r=-(g-1)}^{g-1} \langle \phi^w(X_1, bz_0 e^{tD_1})_r, \phi^w(A, e^{t\Delta+\lambda x+s\Sigma})_r \rangle.$$

Put  $d_0 = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$ , so

$$(26) \quad \begin{aligned} D_X^w(bz_0 e^{tD+\lambda x+s\Sigma}) &= \frac{1}{2} \left( D_X^{(w, \Sigma)}(bz_0 e^{tD+\lambda x+s\Sigma}) + i^{-d_0+d(b)+d(z_0)} D_X^{(w, \Sigma)}(bz_0 e^{itD-\lambda x+is\Sigma}) \right) = \\ &= \frac{1}{2} \left( \sum_{r=-(g-1)}^{g-1} \langle \phi^w(X_1, bz_0 e^{tD_1})_r, \phi^w(A, e^{t\Delta+\lambda x+s\Sigma})_r \rangle + \right. \\ &\quad \left. + i^{-d_0+d(b)+d(z_0)} \sum_{r=-(g-1)}^{g-1} \langle \phi^w(X_1, bz_0 e^{itD_1})_r, \phi^w(A, e^{it\Delta-\lambda x+is\Sigma})_r \rangle \right). \end{aligned}$$

By the definition of  $\mathcal{H}_r$  in (13) through the eigenvalues of  $\alpha$ , we have that  $\phi^w(A, e^{t\Delta+\lambda x+s\Sigma})_r$  is killed by some product of differential operators of the form  $\frac{\partial}{\partial s} - (2r + f_i(t))$ ,  $f_i(t) \in t\mathbb{C}[[t]]$ , if  $r$  is odd, and by some product of differential operators of the form  $\frac{\partial}{\partial s} - (2ri + g_i(t))$ ,

$g_i(t) \in t\mathbb{C}[[t]]$ , if  $r$  is even. Also  $\phi^w(A, e^{it\Delta - \lambda x + is\Sigma})_r$  is killed by some product of differential operators  $\frac{\partial}{\partial s} - (2ri + f_i(it)i)$  if  $r$  is odd, and by some product of differential operators  $\frac{\partial}{\partial s} - (-2r + g_i(it)i)$  if  $r$  is even. So the existence of an exponential term of the form  $e^{(2r+f(t))s}$ ,  $f(t) \in t\mathbb{C}[[t]]$ , in (26) is equivalent to the non-vanishing of the summand corresponding to  $r$ .

Put  $2r = K \cdot \Sigma$  (it is an even number since  $K$  is an integral lift of  $w_2(X)$  by [9, theorem 6]). That  $K$  is a basic class for  $D_X^w(b\bullet)$  means that there is  $z \in \mathbb{A}(X)$  (which we may suppose to be of the form  $z = z_0 z_1$ , with  $z_0 \in \Lambda^* H_1(X)$  homogeneous and  $z_1 \in \text{Sym}^*(H_0(X) \oplus H_2(X))$ ) with  $D_X^w(bz_0 z_1 e^{tD' + \lambda x}) = e^{Q(tD')/2 + 2\lambda + K \cdot tD'}$ , for any  $D' \in H_2(X)$ . So

$$D_X^w(bz_0 z_1 e^{tD + \lambda x + s\Sigma}) = e^{Q(tD)/2 + 2\lambda + K \cdot tD + (2r+t)s}.$$

Hence an exponential term of the form  $e^{(2r+t)s}$  appears in (26) and thus  $\phi^w(X_1, bz_0 e^{tD_1})_r \neq 0$ . As  $b = \wp^n \gamma_{i_1} \cdots \gamma_{i_m}$ , we deduce that  $\tilde{\beta}^n \gamma_{i_1} \cdots \gamma_{i_m}$  acts as non-zero on  $\tilde{\mathcal{H}}_r$  (note that  $\beta + (-1)^r 8$  is an isomorphism in  $\tilde{\mathcal{H}}_r$ ). By corollary 4.3,  $d(b) = 2n + m \leq 2(g - |r| - 1)$ . So  $|2r| + d(b) \leq 2(g - 1)$ .  $\square$

*Proof of theorem 1.7.* In principle theorem 1.7 follows from the more general theorem 1.8, but we shall give an independent proof because of its simplicity. It is similar to the proof of theorem 1.6. We reduce to consider the case  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd element in homology. Now by definition 1.4, and since  $X$  has  $b_1 = 0$ , we have that  $K$  is a basic class for  $D_X^w(b\bullet)$  where  $b = \wp^n$  and  $2n = d(K)$ .

We perform the splitting  $X = X_1 \cup_Y A$  as in the proof of theorem 1.6. Then  $\phi^w(X_1, e^{tD_1}) \in \widetilde{HFF}_g^*$  lies in the kernels of all  $\psi_i$ ,  $1 \leq i \leq 2g$ , since  $X$  has  $b_1 = 0$ . By proposition 4.4,  $\tilde{\beta}^{d+1}$  sends  $\tilde{\mathcal{H}}_r$  to the ideal  $(\psi_1, \dots, \psi_{2g}) \subset \tilde{\mathcal{H}}_r$ , where  $d = \lfloor \frac{g-|r|-1}{2} \rfloor$ . In particular,  $\tilde{\beta}^{d+1} \phi^w(X_1, e^{tD_1})_r = 0$ .

Put  $2r = K \cdot \Sigma$ . Again  $K$  being a basic class for  $D_X^w(b\bullet)$  implies that  $\phi^w(X_1, be^{tD_1})_r \neq 0$ . Therefore  $\tilde{\beta}^n \neq 0$  and hence  $n \leq d = \lfloor \frac{g-|r|-1}{2} \rfloor$ . So  $2n \leq g - |r| - 1$ , i.e.  $|2r| + 2d(K) \leq 2(g - 1)$ .  $\square$

*Proof of theorem 1.8.* As above we reduce to the case  $\Sigma^2 = 0$  and  $\Sigma$  representing an odd element in homology. We may suppose without loss of generality that  $b = \wp^n \gamma_{i_1} \cdots \gamma_{i_m}$  with  $d(b) = 2n + m$ . Keeping the notations of the proof of theorem 1.7, we have that

$$\phi^w(X_1, be^{tD_1})_r \in \tilde{\mathcal{H}}_r$$

is non-zero, where  $2r = K \cdot D$ .

If  $l+1 \leq g-1-|r|$  then obviously  $|2r| + 2d(b) \leq 2(g-1)$  is true. Otherwise  $g-1-|r| \leq l$ . Then  $\phi^w(X_1, be^{tD_1})_r$  lives in the kernels of  $\psi_1, \dots, \psi_l$ , since  $i_*(\gamma_j) = 0 \in H_1(X)$  for  $j =$

$1, \dots, l$ . Therefore we conclude that  $\bar{\beta}^n \psi_{i_1} \cdots \psi_{i_m} \neq 0$  acting on  $\mathcal{H}_r / (\psi_1, \dots, \psi_{g-|r|-1})$ . By proposition 4.5, we have that  $d(b) \leq g - |r| - 1$ , i.e.  $|2r| + 2d(b) \leq 2(g - 1)$ .  $\square$

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DEPARTAMENTO DE ÁLGEBRA GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE MÁLAGA, 29071 MÁLAGA, SPAIN

*E-mail address:* vmunoz@agt.cie.uma.es