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QUANTUM CONDITIONAL PROBABILITY: IMPLICATIONS FOR CONCEPTUAL CHANGE OF SCIENCE.

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June 2009 Isabel Guerra Phd Thesis

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Implications for Conceptual Change in Science

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Phd Thesis, June 2009

Quantum Conditional Probability

Implications for Conceptual Change in Science

A Thesis submitted in partial fulfillment of the requirements for the Degree of *Doctor por la Universidad Complutense de Madrid*.

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I hereby declare that this submission is my own original work and that, to the best of my knowledge, it contains no material previously published or written by another person, except where due acknowledgment has been made in the text. Research toward this thesis has been carried out thanks to an FPI Scholarship of the Spanish Ministry of Science and Education (MEC) associated to the Research Project Causation, Propensities and Causal Inference in Quantum Physics within the DGICYT Research Network HUM2005-01787: 2005- 2008 Classical and Causal Concepts in Science, and to the Complutense Research Group MECISR.

Munich, June 2009

To Arthur Fine,
for making quantum mechanics &
philosophy become alive for me.

Y a Jose.

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Giving birth and life to this dissertation has mostly occurred during the past five years. The story begins on the morning of Thursday July 1st 2004, in which I met Mauricio Suárez at a cafe in Madrid. I had decided at that time that this was the last meeting I would have in the search for a PhD, for I was not finding anything I really wanted to pursue. At that meeting, I heard the name 'Philosophy of Physics' and 'London School of Economics' for the first time. Interestingly enough, these two words, were soon part of my everyday vocabulary. With the incredible help of Mauricio, whom, at that time, I hardly knew, on Friday July 2nd I decided I would apply for a one-year Masters program at LSE. I am really thankful to Mauricio for having opened the door for me into this world.

The year in London was a *very* special year. Eneritz, Stefania, Giacomo y Clara: life is a miracle! At the intellectual level, it was an incredible, but, at the same time, a very demanding and difficult experience. I remember reading Hume's essay 'An enquiry concerning human understanding' – my first philosophical reading ever! – and having no clue *whatsoever* as to what 'this guy' was trying to say. Stephan Hartmann was my anchor point during the first term at LSE. To him I am very grateful for his understanding and listening; I still remember sitting at his office in November and shedding some tears. In the Lent term, I interacted a lot with Roman Frigg, who later became my (master's) dissertation supervisor. Since then, we have kept in touch and I have received very helpful feedback from him at many stages. Thank you very much, Roman. Finally, I thank Jon Williamson for encouraging me with my difficulty in writing essays and for introducing me to philosophy of probability, albeit with some reluctance on my part – after feeling frustrated in many classes, I finally allowed myself ask: 'but *why* are we considering *seriously* that probability is the degree of belief of a person? It just doesn't make any sense!'.

After the year at LSE I decided to start a PhD in philosophy of physics, although, already at that time, many doubts had started dancing around me: is this really what I want to do? Now, looking back, I am happy I decided to start, and I am happy to have continued till the end. *Much* has occurred during this process.

I decided to go back to Madrid – I did not want to (permanently) live far away from my family – and again Mauricio was crucial for making this possible. He agreed to be my supervisor, which I was very happy about for he had continued to be a very important reference point and source of encouragement throughout the year in London. And so I entered the PhD program 'Entre Ciencia y Filosofía' at UCM. From this first year at UCM, I wish to thank Julian Reiss, Antonio Blanco and Javier Vilanova. Ana Rioja's course on the Heisenberg's relations of uncertainty was particularly interesting. Muchas gracias Ana. In September 2006 I received the FPI scholarship associated with Mauricio's research project, which since then has allowed me to be economically independent. If this had not been so, I probably wouldn't have continued my PhD. Again, I wish to thank Mauricio for this.

The FPI scholarship has the incredible feature of allowing one to spend six months each year at a foreign university. I here want to thank the former Ministry of Science and Technology for developing such a wonderful program. This possibility, alongside with Mauricio's help, has allowed me to take advantage of many otherwise inaccessible opportunities which have enormously enriched my education. I have been able to interact with many different people and follow courses at other universities. Arthur Fine, Seattle and the University of Washington have played such a major role in my life and work during the last three years, that it's difficult to convey it in words. Arthur, having received so much from you, and of such fine quality, both at a human and at an intellectual level, makes me feel incredibly fortunate, happy and grateful. I would not have reached the point of writing the acknowledgments of my dissertation without you.

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Now, I am in Munich, at the Carl von Linde Academy (TUM). And again luck seems to follow me. For working with Wolfgang and Fred has turned out to be a *really* wonderful and unexpected treat. I am very grateful I have two great new friends! Wolfgang, thank you also for making this research stay possible and for giving me extremely helpful comments on my dissertation; and thank you Fred for our wonderful morning routine. I am also grateful to the Carl von Linde Academy for allowing me to be part of it – Jörg Wernecke, Klaus Mainzer and Rainhard Bengez. Finally, I want to thank the audiences at different congresses – especially, at LSE, Cambridge University, UCM, University of Washington, UA Barcelona, and APA Pasadena – for the very helpful opportunity of preparing a talk, giving it, and getting feedback.

And in my mind, I now go back to Madrid. For there is Albert, with whom I've been with during the four years at UCM. Me ha encantado encontrar un lugar para estar juntos estos últimos meses de escritura. Mil gracias. Thanks also to the people in our research group: Iñaki, Fernanda, Carlos, Pedro, Carl, Henrik, Mari Cruz Boscá. Y a Mari Cruz, que me encanta tener de referencia del departamento en la UCM. Y gracias al departamento por haberme dado la oportunidad de impartir docencia y haber proporcionado el marco en el que se desarrolla este programa de doctorado; en particular, muchísimas gracias a Luis Fernández por todo su apoyo y ayuda.

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Finally, I'd like to end by talking about my research project, how it started going, how it evolved and what I have learned. I found myself involved with philosophy of physics because of quantum mechanics; actually, I didn't quit after my third year of physics precisely because of quantum mechanics. It had a strong enough pull to make me keep on going. Till now. This pull is related, I think, to my wanting to really understand what is going

on (with whatever). And quantum physics is indeed difficult to understand! But now I can say more clearly what it is that I don't understand, why exactly I think that things turn perplexing (for they remain so!). And I am really happy to have arrived at this point.

Working on this dissertation, on this particular topic, has allowed me to come to terms with many of the questions that have nagged me for a long time now. And it is a great feeling! Mauricio was the one who suggested getting involved in thinking about quantum Bayesianism and was crucial for the first part of the project. Developing it with Arthur has been of invaluable help; it was Arthur's insight which guided my research to focus on the conceptual analysis of the notion of conditional probability, something which has turned out to be extremely fruitful in tackling the conceptual problems of quantum mechanics. And working by myself after my return from Seattle in September 2008, has allowed me to further pursue the questions and answer them in ways I found satisfactory. Here, Arthur's incredible conceptual clarity and simplicity, from which I have learned so much during these years, have played a major role in allowing my own thought to arise.

And whither then? I cannot say.[1]

1. This line comes from one of Tolkien's poems, which I often have in mind, and LOVE reciting with the love of my life, my little sister Marta.

The Road goes ever on and on
Down from the door where it began.
Now far ahead the Road has gone,
And I must follow, if I can,
Pursuing it with weary feet,
Until it joins some larger way
Where many paths and errands meet.
And whither then? I cannot say.

Still round the corner there may wait
A new road or a secret gate
And though I oft have passed them by
A day will come at last when I
Shall take the hidden paths that run
West of the Moon, East of the Sun.

'all the paradoxes of quantum theory arise from the implicit or explicit application of Bayes' axiom [...] to the statistical data of quantum theory. This application being unjustified both physically and mathematically.'

([Accardi, 1984a], pp.298 - 299)

Abstract

In this dissertation we argue against the possibility of defining a notion of conditional probability in quantum theory, both at a mathematical and physically meaningful level. We defend that the probability defined by the Lüders rule, the only possible candidate to play such a role, cannot be interpreted as such. This claim holds whether quantum events are interpreted as projection operators in an abstract Hilbert space, as the physical values associated to them, or as measurement outcomes, both from a synchronic and a diachronic perspective. The only notion of conditional probability the Lüders rule defines is a purely instrumental one. In addition, we show that the *un*conditional quantum probabilities can also be interpreted as probabilities only under a purely instrumental perspective, where the difficulties in interpreting them non-instrumentally are, ultimately, the same as those we encounter in giving a non-instrumental conditional interpretation of the probability defined by the Lüders rule.

We frame this discussion within the general issue of conceptual change in science and show how, generally, the fact that two concepts are co-extensive in their shared domain of application – as the probability defined by the Lüders rule and classical conditional probability are for compatible events – does not guarantee that the more general concept is a conceptual extension of the more limited one. To give an appropriate account of concept extension, we show that concepts present an 'open texture' that does not allow for a set of jointly necessary and sufficient conditions to characterize an extended concept, and thus formulate a new account, namely the 'Cluster of Markers account', in terms of a cluster of markers which are expected to hold for the extended concept. This account, we argue, can capture the complexity involved in actual cases of conceptual change in science and can account for the fact that there are concepts which, even if co-extensive in their shared domain of application, do not share enough meaning to justify regarding them as defining the same concept.

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Chapter 1
Introduction

1.1 Quantum Probability: a peculiar kind of probability

From about the beginning of the twentieth century experimental physics amassed an impressive array of strange phenomena which demonstrated the inadequacy of classical physics.[1.1] The attempt to discover a theoretical structure for the new phenomena was resolved in 1926 and 1927 in the theory called quantum mechanics. This new theory is, by its very nature, a statistical or stochastic theory; that is, it only yields probabilistic predictions for the values of physical quantities. Traditionally, this feature of quantum mechanics has been taken as showing that the exact outcome of an experiment is fundamentally unpredictable, and that one has to be satisfied with merely computing the probabilities of various outcomes.

In addition, quantum mechanics determines that the laws of combining these probabilities are not those of the classical probability theory of Laplace. As Feynman remarks,

> 'Nature with her infinite imagination has found another set of principles for determining probabilities; a set other than that of Laplace, which nevertheless does not lead to logical inconsistencies.' ([Feynman, 1945] p.533)

The quantum mechanical laws approach very closely the laws of Laplace as the size of the objects involved in the experiments increases, but differ considerably when dealing with objects of atomic dimensions. Therefore, the laws of probabilities which are conventionally applied are quite satisfactory in analyzing the behaviour of the roulette wheel but not the behavior of a single electron or a photon of light.

In this introduction, we illustrate the probabilistic laws of quantum mechanics by describing an experiment dealing with a single electron.[1.2] We focus on the all time favourite: the two-slit experiment (see figure 1.1.). In this experiment a source emits identically prepared electrons; all the electrons have the same energy but come out in different directions to impinge on a detecting screen (S_2). Between them is another screen with two slits (S_1), call them A and B, through which the electrons may pass; they are then detected one by one as they 'hit' the detecting screen. The electrons are emitted at a

1.1. This introduction draws on [Feynman, 1945] and [Fine, 1972].

1.2. One can just as well use light instead of electrons in this experiment. The same points would be illustrated.

steady rate slow enough to ensure that no more than one electron passes through the apparatus at the same time,[1.3] and the experiments are run long enough to have a large number of electrons detected. What one measures for various positions R on the detector screen is the mean number of pulses per second. In other words, one determines experimentally the (relative) probability p that the electron passes from the source to R as a function of R.

When one runs the experiment with both slits open, the graph of the probability that the electron hits S_2 at R, $p_{AB}(R)$, is the complicated curve illustrated qualitatively in figure 1.1.(a). It has several maxima and minima, and there are locations near the center of the screen at which electrons hardly ever arrive. Quantum physics yields precisely the laws governing the structure of this curve.

To understand this curve, one might at first suppose that each electron which passes from the source to the detecting screen S_2 must go either through slit A or slit B. As a consequence, one expects that the chance of arrival at R is the sum of two parts, namely, $p_A(R)$, the chance of arrival at R coming through slit A, plus $p_B(R)$, the chance of arrival at R coming through slit B. However, one can show by direct experiment that this is *not* the case. Indeed, each of the component probabilities is easy to determine: to determine the probability $p_A(R)$, we simply close slit B and measure the chance of arrival at R with only slit A open; and similarly, by closing B, we find the chance $p_B(R)$ of arrival through slit B. These probabilities are given in figure 1.1.(b).

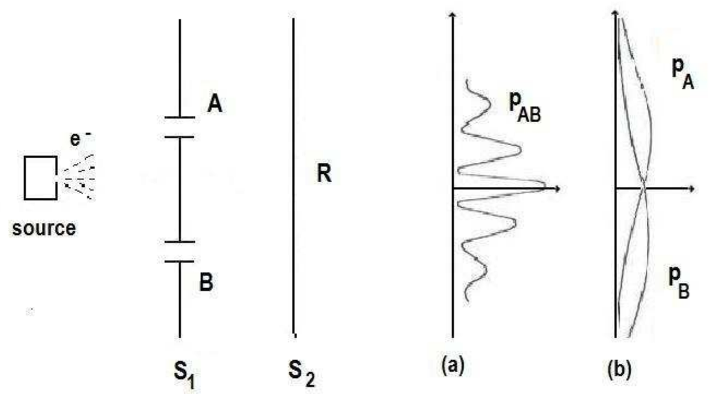

Figure 1.1. Double slit experiment.

1.3. Indeed, if the detectors are extremely sensitive (such as a Geiger counter), one finds that the current arriving at S_2 is not continuous, but corresponds to a rain of particles. If the intensity of the source is very low the detector will record pulses representing the arrival of a particle, separated by gaps in time during which nothing arrives. If we had detectors simultaneously all over the screen S_2, with a very weak source, only one detector would respond, then, after a little time, another would record the arrival of an electron, etc. There would never be a half response of the detector, either an entire electron arrives or nothing happens. And two detectors would never respond simultaneously (except for the coincidence that the source emits two electrons within the resolving time of the detectors – a coincidence whose probability can be decreased by further decreasing the source's intensity).

As one can clearly see, the sum of $p_A(R)$ and $p_B(R)$ does not agree with the probability $p_{AB}(R)$. Hence, experiment tells us definitely that $p_{AB}(R) \neq p_A(R) + p_B(R)$; that is, that the chance of arrival at R with both holes open is not the sum of the chance with just hole A open plus that with just hole B open, i.e. an additive pattern. In fact, the complicated curve $p_{AB}(R)$ is exactly the intensity of distribution of an interference pattern, i.e. the pattern one would expect if waves were to start from the source and, after passing through the two slits, were to impinge on the screen S_2. The additive and interference pattern are substantially different: there are places, for example, where the interference pattern shows a light patch of few electron hits but where the additive pattern shows a dark patch of many electron hits. And conversely, there are places where the interference pattern shows a dark patch of many hits but where the additive pattern shows a light one.

How is the interference pattern then to be understood? One might be tempted to say that, given that it is not true that $p_{AB}(R) = p_A(R) + p_B(R)$, we must conclude that when both slits are open it is not true that the particle goes through one slit or the other. For if it had gone through one or the other we could classify all the arrivals at R into two disjoint classes, namely, those arriving via slit A and those arriving through slit B, and the frequency of arrival at R would be surely the sum of the frequency of those coming through A and of those coming through slit B.

However, it is easy to perform an experiment which speaks against this conclusion. One has to merely place a source of light behind the slits and watch to see through which slit the electron passes. For electrons scatter light, so that if light is scattered behind slit A we may conclude that an electron passed through slit A; and if it is scattered in the neighborhood of slit B, then the electron has passed through slit B. When one runs this experiment, one finds, in effect, that for every electron which arrives at the screen S_2 light is scattered either behind slit A or behind slit B, and never (if the source is very weak) at both places. Thus, one verifies that the electron *does* pass through either slit A or slit B.

Moreover, the fact that when these which-slit measurements are performed no interference pattern is found – in fact, one retrieves the classical additive pattern – does not alter this conclusion. For if observation is to be an objective guide to reliable information, then what we observe must correspond to how things are, either simultaneous with or just prior to our observation. Thus, when both slits are open, just prior to our observation of an electron at the outlet of slit A, the electron *must* have been passing through slit A, regardless of actually measuring or not which slit the particle goes through. And this is, of course, compatible with a possible disturbance of the electrons by our observation of them that would subsequently result in retrieving the additive instead of the interference pattern.[1.4]

In fact, Niels Bohr and Werner Heisenberg, among others, offered the following reasoning as an explanation of these results. Their basic idea is that just by 'watching' the electrons one changes their chance of arrival at R. Indeed, to observe them one needs to use light, and the light in collision with the electron alters its motion and thus its chance of arrival at R. And the difficulty is that, for objects of atomic dimensions, one cannot get rid of this disturbance (by direct measurement). In effect, since the momentum carried by the light is h/λ, where λ is the wavelength associated to the photon, weaker effects could be produced by using light of longer wave length λ. However, there is a limit to this. For if light of too long a wave length is used, one will not be able to tell whether it was scattered from behind slit A or slit B (given that a source of light of wave length λ cannot be located in space with precision greater than that of order λ). Thus, any physical agency designed to determine through which slit the electron passes produces enough disturbance to alter the distribution from $p_{AB}(R)$ to $p_A(R) + p_B(R)$.

In addition, Bohr and Heisenberg claimed that the consistency of quantum mechanics *requires* a limitation to the subtlety to which experiments can be performed. In the case of the double-slit experiment it says that *any* attempt to determine which slit the electron passed through without deflecting the electron, and thus changing its momentum and destroying the interference pattern, *must necessarily* fail. Note that this is different from saying that any attempt to design an apparatus to determine which slit the electron passed through, while being delicate enough so as not to deflect the electron sufficiently to destroy the interference pattern, turns out to actually fail. Indeed, while the latter statement implies that one cannot in fact make a precise *direct* simultaneous measurement of the position and momentum of the electron passing through the double slit screen, the former implies that no such (precise simultaneous) measurement *whatsoever* – neither direct nor indirect – can be *in principle* performed.[1.5]

1.4. See [Fine, 1972], section 6, for further discussion of this point.

1.5. Actually, this is the content of Heisenberg's uncertainty relations [Heisenberg, 1927], whose interpretation is a rather intricate issue. What is uncontroversial is that, in the case of, say, position q and momentum p, they imply that there is no way to make a *precise direct simultaneous* measurement of position and momentum. Indeed, if one measures position on half the copies of an identically prepared system in state ψ, and momentum on the other half, there is a statistical scatter such that the product of the standard deviation of position and of momentum is always greater or equal to $\hbar/2$. That is, $\Delta p_\psi \, \Delta q_\psi \geqslant \hbar/2$, where $\Delta p_\psi = \langle \psi, p^2 \, \psi \rangle - |\langle \psi, p \, \psi \rangle|^2$ and $\Delta q_\psi = \langle \psi, q^2 \, \psi \rangle - |\langle \psi, q \, \psi \rangle|^2$ are the standard deviations of position and momentum in the state vector ψ.

However, what these relations imply at an interpretive level is controversial. Some hold that the uncertainty to which incompatible quantities can be determined is only a restriction on our simultaneously *knowing* their values by means of *direct* measurements (one could then, in principle, come to know them by means of indirect measurements, namely, by observing one of them directly and then inferring the value of the other one), while others, take a stronger view, and hold that the uncertainty relations restrict what *is* or can be simultaneously real.

Moreover, according to Bohr and Heisenberg's view and their so-called Copenhagen interpretation,[1.6] *all* the puzzling features of quantum mechanics can be traced back to this inevitable and uncontrollable physical disturbance brought about by the act of measurement. Presented with this situation, the practicing physicist takes the following view. When no attempt is made to determine which hole the electron passes through, one cannot say that it must have passed through one hole or the other. Only in a situation where an apparatus is operating to determine which hole the electron goes through is it permissible to say that it passes through one or the other. That is, when one watches, one observes, and thus *can* say, that the electron goes either through one or the other hole, but if one is not looking, one does not observe, and thus *cannot* say, that it either goes one way or the other.

But is this all we can say about the quantum mechanical image of the world? Should we be satisfied with taking the practicing physicist view which remains silent about whatever it is not directly observing? Should we also hold, along with Bohr and Heisenberg, that quantum mechanics implies that the act of observation necessarily alters the phenomenon being observed, that by the very act of watching the observer necessarily affects the observed reality? And, maybe, as the popular interpretation of Bohr has it, slip into saying that quantum mechanics is 'subjective', that some of the data quantum physics provides depend on the subjectivity of this or that particular experiencing subject?

We think not. Although, ultimately, we will conclude that we do not understand the quantum mechanical image of the world, we will, at least, understand much better the precise difficulties which give rise to this perplexing situation. Moreover, we will show that the Copenhagen doctrine is mistaken in that not all the conceptual problems of quantum mechanics can be traced back to the alleged irreducible and uncontrollable disturbance of the system measured by a measuring instrument. In addition, hopefully, we may move a little step further in our understanding of the picture of the world our best science offers.

1.2 Overview

In this dissertation we consider the puzzling phenomena described by quantum mechanics (such as the double-slit experiment) and try to understand what picture of the world quantum mechanics might provide. To do so, we undertake a conceptual (or philosophical) investigation of the concept of quantum probability. In particular, we focus on the notion of conditional probability, for it turns out to be a particularly beautiful and encompassing way of tackling many of the conceptual difficulties of quantum theory.

1.6. We do not go into the intricacies of the differences between Bohr and Heisenberg's interpretation of quantum mechanics and simply refer to this roughly described view as the Copenhagen interpretation. A somewhat more detailed account of Bohr's view is given in section 9.1.2.

Consider again the double slit experiment with the two slits open. An analysis in terms of conditional probabilities is not correct since it does not yield the interference pattern that is found experimentally. Indeed, let A be the event that the electron passes through slit A, B the event that it passes through slit B, and R the event that the electron strikes the region R of the detecting screen. Given that the notion of conditional probability is defined as the pro rata increase of a joint probability distribution, i.e. for two classical events A and B, the probability of A conditional on B with respect to the probability p, is given by

$$\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{1.1}$$

one can write the following conditional probabilities:

- $\mathbb{P}_p(R|A) = \frac{p(R \cap A)}{p(A)}$ is the probability that the electron strikes at R given that it passes through slit A,
- $\mathbb{P}_p(R|B) = \frac{p(R \cap B)}{p(B)}$ is the probability that the electron strikes at R given that it passes through slit B, and
- $\mathbb{P}_p(R|A \cup B) = \frac{p[R \cap (A \cup B)]}{p(A \cup B)}$ is the probability that the electron strikes at R given that it passes through either slit A or slit B.

A simple calculation shows that $\mathbb{P}_p(R|A \cup B)$ can be expressed in terms of $\mathbb{P}_p(R|A)$ and $\mathbb{P}_p(R|B)$ as [1.7]

$$\mathbb{P}_p(R|A \cup B) = \frac{1}{2}\,\mathbb{P}_p(R|A) + \frac{1}{2}\,\mathbb{P}_p(R|B) \tag{1.2}$$

for $p(A) = p(B)$ corresponding to the most simple experimental arrangement.

An analysis in terms of conditional probabilities thus yields an additive distribution pattern which, as we have seen, is not what we obtain experimentally. The two slit experiment, and more generally quantum mechanical phenomena, cannot, therefore, be analyzed in terms of classical conditional probabilities. And hence the question arises as to whether and, if so how, an appropriate notion of conditional probability can be introduced in quantum mechanics.

A long-standing literature claims that the answer is 'yes'; that it is in fact possible to define an appropriate extension of conditional probability with respect to an event in quantum mechanics, namely the probability defined by the so-called Lüders rule. This rule yields the correct probabilistic predictions for the quantum phenomena as, for example, the double slit experiment. Indeed, it predicts that the probability to arrive at R when the two slits are open, is not, as in the classical case, the weighted sum of the probabilities

1.7. By distributivity: $\mathbb{P}_p(R|A \cup B) = \frac{p(R \cap (A \cup B))}{p(A \cup B)} = \frac{p((R \cap A) \cup (R \cap B))}{p(A \cup B)}$. Since A and B are two mutually exclusive events, $\mathbb{P}_p(R|A \cup B) = \frac{p(R \cap A) + p(R \cap B)}{p(A) + p(B)}$. And if we set $p(A) = p(B)$ corresponding to the most simple experimental arrangement, then $\mathbb{P}_p(R|A \cup B) = \frac{1}{2}\frac{p(R \cap A)}{p(A)} + \frac{1}{2}\frac{p(R \cap B)}{p(B)}$.

when each slit is open; rather, the characteristic quantum interference terms are present in this probability, namely[1.8]

$$\mathbb{P}_{\psi}(R|A\cup B)=\frac{1}{2}\,\mathbb{P}_{\psi}(R|A)+\frac{1}{2}\,\mathbb{P}_{\psi}(R|B)+I \tag{1.3}$$

where,

$$\begin{aligned}\mathbb{P}_{\psi}(R|A)&=\langle\psi'_A\,,P_R\,\psi'_A\rangle\\ \mathbb{P}_{\psi}(R|B)&=\langle\psi'_B\,,P_R\,\psi'_B\rangle\\ I&=\frac{1}{2}\,\langle\psi'_A\,,P_R\,\psi'_B\rangle+\frac{1}{2}\,\langle\psi'_B\,,P_R\,\psi'_A\rangle\end{aligned} \tag{1.4}$$

More generally, the Lüders rule states that for two quantum events, represented by projection operators P and Q on the Hilbert space $\mathcal{H}$ associated to the system, the probability of the quantum event P conditional on the quantum event Q is given by

$$\mathbb{P}_W(P|Q)=\frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W)} \tag{1.5}$$

where W is a density operator on $\mathcal{H}$. In the context of quantum probability theory, rule (1.5) satisfies the formal condition of specifying the only probability measure on the state space that reduces to a pro rata conditional probability for compatible events. Moreover, this formal condition is analogous to an existence and uniqueness property of classical conditional probability. Thus, several authors have argued for interpreting the Lüders rule as defining an appropriate notion of conditional probability in quantum mechanics.

In addition, the Lüders rule appears in the orthodox interpretation of quantum mechanics. Indeed, it is the generalized version of the so-called 'Projection Postulate', which determines uniquely the state of the system after a measurement of a certain physical quantity. The new density matrix representing this state can then be used to calculate probability assignments for subsequent measurements. In effect, imagine we perform a measurement of a certain observable, where Q belongs to its spectral decomposition, on a system in state W, and find measurement outcome q. The Lüders rule determines that the new state is $W_q=\frac{Q\,W Q}{\mathrm{Tr}(Q\,W Q)}$. If we then perform a measurement of a second observable, where P belongs to its spectral decomposition, the probability to find measurement outcome p in this second measurement is given by this new density operator as

$$\mathbb{P}_W(p|q)=p_{W_q}(p)=\mathrm{Tr}\left(\frac{Q\,W Q}{\mathrm{Tr}(Q\,W Q)}\,P\right) \tag{1.6}$$

Thus, in these cases, it (seemingly) becomes meaningful to speak of the probability distribution of a physical quantity given the result of a previous measurement of another physical quantity. Indeed, it seems that the probability given by (1.6) can be interpreted as the probability of measurement outcome p conditional on measurement outcome q.

1.8. A detailed derivation of this result is given in section 4.4.2. and 7.6.

Hence, the proposal is that the Lüders rule defines the notion of conditional probability in quantum mechanics both for quantum events represented by projection operators and for measurement results. The quantum notion agrees with its classical counterpart when it applies to compatible events (those represented by commuting projection opertors) but differs from it when incompatible events (those represented by *non*-commuting projection opertors) are involved. In these cases it cannot be interpreted as a classical conditional probability but rather is seen as providing an extension of this notion appropriate for the quantum context.

In our dissertation we first argue that, even if the probabilities defined by the Lüders rule are the only probabilities which are co-extensive with conditional probabilities for compatible events, we have no reason to assimilate them to conditional ones for incompatible events, neither for physical values nor at a formal level for projection operators, both from a synchronic and a diachronic perspective. Rather, we give many reasons against this assimilation. Second, we argue that the orthodox interpretation of quantum mechanics also does not justify the understanding of the probability defined by the Lüders rule as a conditional-on-measurement-outcome probability (again both from a synchronic and a diachronic perspective). The only notion of conditional probability the Lüders rule defines is a purely instrumental one which reduces quantum theory to a mere algorithm for generating the statistical predictions of the outcomes of measurements.

We develop these arguments in Chapters 5 and 7. In sections 5.3 and 5.4 we show why the probability defined by the Lüders rule cannot be understood as a synchronic conditional probability for physical values. In section 7.3 we show why it also cannot be understood as a synchronic nor diachronic conditional probability for measurement results, nor as a diachronic conditional probability for physical values. This allows us to further establish the inadequacy of the formal notion of conditional probability for projection operators, both from a synchronic and a diachronic perspective (sections 5.2 and 7.4). Finally, in section 7.5, we argue that the only notion of conditional probability offered by the Lüders rule is a purely instrumental one. Indeed, if when one says the probability of a certain measurement outcome p given a previous measurement which has outcome q is $\mathbb{P}_W(p|q)$ one only means that if these two measurements are repeated many times, one after the other, one expects that the fraction of those which give the outcome p is roughly $\mathbb{P}(p|q)$, then no problems arise. But as soon as one attempts to say *anything* else, then all the problems we consider in sections 5.2, 5.3, 5.4, 7.3 and 7.4 appear.

Thus, we conclude that, contrary to the standard view, the probability defined by the Lüders rule does *not* acquire a precise meaning, in the sense of synchronic or diachronic conditional probability, when quantum mechanics is interpreted as a generalized probability space or as probability space for measurement results. While establishing this result, we also show that the puzzles of quantum mechanics cannot be traced back to an inevitable and uncontrollable physical disturbance brought about by the act of measurement.

It is important to note that these questions do not apply to another type of (purportedly) conditional probability which also arises in the context of measurements. Indeed, it is not uncommon to hear that all quantum probabilities are conditional probabilities for measurement outcomes conditional on measurements. However, in section 6.4 we argue that these conditional-on-measurement probabilities (not conditional-on-measurement-*outcome* probabilities) are not really conditional probabilities. For there is an important distinction between the role of background conditions which specify the conditions in effect at the assessment of a probability function – in this case, the measurement procedure – and the propositions that can really be conditioned on.

In Chapter 8, we consider the interpretation of the *un*conditional quantum probabilities. We show that, similarly to the probabilities defined by the Lüders rule, these can only be interpreted as probabilities under a purely instrumental view of quantum mechanics. And we argue that the difficulties in giving a (non-instrumental) interpretation of quantum unconditional probability are ultimately the same as those we encountered in giving a (non-instrumental) interpretation of the probability defined by the Lüders rule. More concretely, we argue that quantum Bayesianism is *not* a viable interpretation of quantum mechanics, both from a subjective and an objective perspective; and that a (non-instrumental) frequency interpretation of the quantum probabilities is not possible either.

Finally, in Chapter 9, we frame this discussion within the general issue of the dynamics of conceptual change in science. We first show that the standard account of conceptual generalization or extension, based on co-extension of the 'extended' and the old concept in their shared domain of application (as for example, that presented by the logical positivists, by Imre Lakatos or by Albert Einstein), is inadequate. We then argue that concepts present an 'open texture' that does not allow for a set of jointly necessary and sufficient conditions to characterize an extended concept, and propose a new account of concept extension in terms of a 'cluster of markers' which, though not fast-holding conditions, do provide an appropriate rationale to evaluate conceptual extension. This account, we argue, provides a more adequate analysis of when two concepts, even if co-extensive in their shared domain of application, do not share enough meaning to justify regarding them as defining the same concept, and comes closer to capturing the actual relations between concepts which appear in different theoretical contexts.

In Chapter 10 we bring this dissertation to an end by briefly summarizing our main conclusions.

Chapter 2

Classical Conditional Probability

In classical probability theory the probability of an event A conditional on another event B is defined as the probability of their joint event $A \cap B$, divided by the probability of the conditioning event B. This ratio is supposed to capture the notion of conditional probability, namely the probability of an event, qualified or informed by some body of evidence. In this chapter we consider whether this is in fact so.

We first argue that the ratio $p(A \cap B)/p(B)$ should not be seen as a definition of conditional probability but rather as an analysis of this notion (section 2.3). Then we show why ratio can in fact capture such notion, both from an intuitive understanding of probability and from the perspective of two particular interpretations of probability, namely the subjective Bayesian and the frequency interpretation of probability. Finally, we give two formal characterizations of the conditional measure defined by the ratio $p(A \cap B)/p(B)$ (section 2.4); first, as the only probability measure defined on the whole classical event space such that for events A contained in B, conditionalizing on B just involves a renormalization of the initial probability measure; and second, as the only measure which is necessarily additive with respect to conditioning events.

2.1 Classical Probability Theory

The theory of probability has a mathematical and a foundational or philosophical aspect. Whereas there is a significant consensus about its mathematics, there is much disagreement about the philosophy. In this section we only briefly introduce the main formal elements of classical probability theory. The aim is to quickly lay out the formalism in which to consider interpretive questions about conditional probability and establish the notation we use.

Let us start with the definition of a *classical* probability space and the concepts in terms of which it is defined.[2.1]

Definition 2.1. ***Classical Probability Space***. *A classical probability space consists of a triple* $\langle S, \mathcal{F}(S), p\rangle$ *where*

i. S *is a space of points* w *called the sample space and sample points*

ii. $\mathcal{F}(S)$ *is a* σ*-field of subsets of* S*. These subsets are called events.*

iii. $p(\cdot)$ *is a probability measure on* $\mathcal{F}(S)$*.*

Definition 2.2. ***σ-Field***. *A set of subsets* $\mathcal{F}(S)$ *of a space* S *is a* σ*-field if it is closed under complementation (*^{c}*), and countable unions (* $\cup$ *) and intersections (* $\cap$ *). The complement of* S *is the empty set* $\emptyset$*.*

With these operations the set of subsets of a real space form a Boolean algebra $\mathcal{B}$. In full generality a Boolean algebra is a set A together with binary operations ' $+$ ' and ' $\cdot$ ', a unary operation ' $-$ ', and elements '0', '1' of A for which the following laws hold: commutative and associative laws for addition and multiplication, the distributive laws both for multiplication over addition and for addition over multiplication, and the special laws $x + (x \cdot y) = x$, $x \cdot (x + y) = x$, $x + (-x) = 1$, and $x \cdot (-x) = 0$. In a classical event structure, in which events are represented by subsets of S, the set A consists of the set $\mathcal{F}(S)$ of subsets of S, ' $+$ ' corresponds to the union of subsets, ' $\cdot$ ' corresponds to their intersection, and ' $-$ ' corresponds to the complementation with respect to S, with members '$\emptyset$' and 'S' playing the role of '0' and '1' respectively.

The standard axiomatization of probability is the following. It was first provided by [Kolmogorov, 1950].

Definition 2.3. ***Classical Probability***. *A set function* $p(\cdot)$ *defined on a* σ*-field* $\mathcal{F}(S)$ *of subsets of* S *is a classical probability measure if*

i. (Non-negativity) $p(A) \geqslant 0$ *for all* $A \in \mathcal{F}(S)$*.*

ii. (Normalization) $p(S) = 1$

iii. (σ*-additivity) for every finite or countable collection* $\{A_i\}$ *of sets in* $\mathcal{F}(S)$ *such that* A_i *is disjoint from* A_j*,* $i \neq j$*,*

$$p\left(\bigcup_i A_i\right) = \sum_i p(A_i) \tag{2.1}$$

2.1. We mostly follow [Breiman, 1968].

Notice that additivity is really the essential constraint for a probability measure: non-negativity simply establishes a scale and normalization says that the whole sample space is maximally probable, which seems almost self-evident.

Denote the class of Borel subsets of $\mathbb{R}$, i.e. the smallest family of subsets of $\mathbb{R}$ that includes the open sets and is closed under complements and under countable intersections, by $\mathbb{B}(\mathbb{R})$. A *random variable* is a measurable function $f: S \rightarrow \mathbb{R}$ with the following special features:[2.2]

Definition 2.4. ***Random Variable***. *A real function $f(w)$ defined on S is called a random variable if for every Borel set $\mathbb{b}$ in the real line $\mathbb{R}$, the set $\{w; f(w) \in \mathbb{b}\}$ is in $\mathcal{F}(S)$. For $\mathbb{b} \in \mathbb{B}(\mathbb{R})$ and random variable f, $f^{-1}(\mathbb{b})$ is the event that f has a value in $\mathbb{b}$.*

In a random experiment, the elements of S correspond to the possible outcomes of the experiment, the sets in $\mathcal{F}(S)$ correspond to random events, and the measure $p(A)$ for $A \in \mathcal{F}(S)$ gives the probability that the event A occurs. Random variables correspond to measurable quantities for the random experiment. In effect, we can associate with each quantity A a function f_A such that for every point w of the sample space S, $f_A(w)$ yields a real number, namely, the value of A. Thus the possible values of A will correspond to the range of the function f_A: A will take a value in the Borel set $\mathbb{b}$ for the set of sample points w for which $f_A(w) \in \mathbb{b}$. That is, A will take a value in $\mathbb{b}$ for all sample points $w \in f_A^{-1}(\mathbb{b})$. Therefore the event $(A, \mathbb{b})$, namely 'quantity A has a value in $\mathbb{b}$' is represented in classical theory by the subset of the phase-space $f_A^{-1}(\mathbb{b}) \subseteq S$.

One can associate probabilities to the events $f^{-1}(\mathbb{b})$ in the usual way: $p[f^{-1}(\mathbb{b})]$ is the probability of the event that the random variable f has value in $\mathbb{b}$.

Definition 2.5. ***Classical Probability Distribution***. *For a random variable f defined on S, the probability measure p_f on $\mathbb{B}(\mathbb{R})$ defined by*

$$p_f(\mathbb{b}) = p[f^{-1}(\mathbb{b})] \tag{2.2}$$

is called the distribution of f.

If f, g are random variables, one can also define the probability of the simultaneous occurrence of events such as $f^{-1}(\mathbb{a}) \cap g^{-1}(\mathbb{b})$, $\mathbb{a}, \mathbb{b} \in \mathbb{B}(\mathbb{R})$.

2.2. We restrict ourselves to the family of Borel subsets of $\mathbb{R}$ because it is not possible to construct a probability measure defined on all subsets of $\mathbb{R}$.

Definition 2.6. ***Classical Joint Distribution***. *The joint distribution of* f, g *is defined as the probability measure* $p_{f,g}$ *on* $\mathbb{B}(\mathbb{R}^2)$ *satisfying, for all* $\mathbb{a}, \mathbb{b} \in \mathbb{B}(\mathbb{R})$,

$$p_{f,g}(\mathbb{a} \times \mathbb{b}) = p[f^{-1}(\mathbb{a}) \cap g^{-1}(\mathbb{b})] \tag{2.3}$$

$p_{f,g}(\mathbb{a} \times \mathbb{b})$ is naturally interpreted as the probability that f has a value in $\mathbb{a}$ and g has a value in $\mathbb{b}$. It can be shown that $p_{f,g}$ always exists and satisfies the consistency conditions as to the marginal distributions $p_f(\mathbb{a})$ and $p_g(\mathbb{b})$, i.e.

$$p_{f,g}(\mathbb{a} \times \mathbb{R}) = p_f(\mathbb{a}) \tag{2.4}$$

$$p_{f,g}(\mathbb{R} \times \mathbb{b}) = p_g(\mathbb{b}) \tag{2.5}$$

Thus, the joint distribution $p_{f,g}$ determines the marginal distributions p_f and p_g. The converse, however, does not hold: one can give examples of cases in which the individual distributions p_f, p_g do not determine the joint distribution $p_{f,g}$. Nevertheless the distributions of $x_1 f + x_2 g$ for all $x_1, x_2 \in \mathbb{R}$ do determine $p_{f,g}$. In fact, it can be shown that $p_{f,g}$ is the unique measure on $\mathcal{B}(\mathbb{R}^2)$ that satisfies

$$p_{f,g}\{(y_1, y_2): x_1 y_1 + x_2 y_2 \in \mathbb{b}\} = p\{\omega \in S: x_1 f(\omega) + x_2 g(\omega) \in \mathbb{b}\} \tag{2.6}$$

for every $E \in \mathbb{B}(\mathbb{R})$, $x \in \mathbb{R}^2$.

The definition of joint distribution can be easily generalized for a finite set of n random variables. The *joint distribution* of $f_1, ..., f_n$ is defined as the probability measure $p_{f,...,f_n}$ on $\mathbb{B}(\mathbb{R}^n)$ satisfying:

$$p_{f_1,...,f_n}(\mathbb{a}_1 \times ... \times \mathbb{a}_n) = p[f_1^{-1}(\mathbb{a}_1) \cap ... \cap f_n^{-1}(\mathbb{a}_n)] \tag{2.7}$$

for all $\mathbb{a}_1, ..., \mathbb{a}_n \in \mathbb{B}(\mathbb{R})$. $p_{f_1,...,f_n}$ always exists and satisfies the consistency conditions as to the marginal distributions $p_{f_1,...,f_n}(\mathbb{a} \times ... \times \mathbb{R}) = p_{f_1}(\mathbb{a})$.

2.2 Conditional Probability: A Definition

Conditional probability is, roughly, probability given some body of evidence or information. In classical probability theory this notion is defined by the so-called ratio formula, which stipulates that the probability of an event A conditional on another event B, $\mathbb{P}_p(A|B)$, is given by the ratio of two unconditional probabilities, namely their joint probability $p(A \cap B)$ divided by the probability of B.

Definition 2.7. ***Conditional Probability with Respect to an Event.*** *Given a classical probability space $\langle S, \mathcal{F}(S), p\rangle$, for sets $A, B \in \mathcal{F}(S)$, such that $p(B) > 0$, the probability of event A conditional on event B is defined as*

$$\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{2.8}$$

This new function $\mathbb{P}_p$ is indeed a probability function. It is non-negative given that p is non-negative; it is also normalized, i.e.

$$\mathbb{P}_p(B|B) = \frac{p(B \cap B)}{p(B)} = 1 \tag{2.9}$$

And it is additive: for every finite or countable collection $\{A_i\}$ of sets in $\mathcal{F}(S)$ such that A_i is disjoint from A_j, $i \neq j$, it satisfies $\mathbb{P}_p(\cup_i A_i|B) = \sum_i \mathbb{P}_p(A_i|B)$. In effect,

$$\mathbb{P}_p(\cup_i A_i|B) = \frac{p[(\cup_i A_i) \cap B]}{p(B)} = \frac{p[\cup_i (A_i \cap B)]}{p(B)} = \frac{\sum_i p(A_i \cap B)}{p(B)} = \sum_i \mathbb{P}_p(A_i|B) \tag{2.10}$$

What is essential for additivity to hold is first, that the distributive law, i.e. $(\cup_i A_i) \cap B = \cup_i (A_i \cap B)$, holds in $\mathcal{F}(S)$; and second, given that the sets $A_i \cap B$ are disjoint, that the probability of their union is simply the sum of the probability of each set.

Following [Beltrametti & Cassinelli, 1981] and [Hughes, 1989], we use the notation $\mathbb{P}_p(\cdot|\cdot)$ for conditional probability rather than the standard notation $p(\cdot|\cdot)$ to emphasize the distinction between the conditional probability function $\mathbb{P}_p$ and the unconditional one p. For even if $\mathbb{P}_p$ is defined in terms of p, they are conceptually distinct notions.

2.3 Justification of the Ratio Analysis

It is part of the orthodoxy to take (2.8) as the *definition* of conditional probability: unconditional probability is taken as the basic notion, and conditional probability is taken as a subsidiary one mathematically defined in terms of it (and is thus taken as the fourth axiom of classical probability by adding it to the axioms of definition 2.3). However, thought of this way, there is nothing conditional about conditional probability – it is just one mathematical function of two variables defined in terms of a function of one variable. But this abstract function is of interest precisely because it comes close to capturing some other intuitive notion: the probability of A given B; that is, the probability that A has, given that certain conditions obtain, among others, that a probability of 1 is assigned to event B.

Conditional probability is thus not just a technical term devoid of any associated intuitions; it is meant to capture the notion of 'the probability of A, qualified by or informed by some condition B', words which are loaded with philosophical and commonsensical associations. E. J. Lowe writes

> '... we can only make clear sense of the notion of 'conditional probability' if we attempt to explain it [...] in conditional terms – not, that is, as a new *kind* of probability, but rather as the (ordinary!) probability that a proposition has *if* certain conditions obtain. In short: talk about conditional probability is properly construed not as *talk about a conditional kind of probability*, but rather as *talk of a conditional kind about probability*.'([Lowe, 2008] p.222)

Some authors, therefore, prefer to denote the conditional-on-B probability function of A by $p_B(A)$ rather than by $\mathbb{P}_p(A|B)$.

Now the problem is that the ratio in (2.8) may or may not express our associations adequately. So while we are free to stipulate that $\mathbb{P}_p(A|B)$ is merely shorthand for the ratio $p(A\cap B)/p(B)$, we are not free to stipulate that 'the conditional probability of A, given B' should be identified with this ratio. Hence the ratio formula (2.8) should *not* be regarded as a stipulative definition, but rather as an *analysis* of the notion of conditional probability in need of justification.[2.3] We refer to this analysis as the ratio analysis, or simply by ratio.

What is then the rationale for the identification of conditional probability with ratio? That is, why is the probability of an event qualified or informed by some condition captured by the ratio analysis?

2.3.1 General Rationale

Consider the following example. Imagine a fair die is about to be tossed. The probability that it lands with '1' showing up, i.e. $p(1)$, is one sixth; this is an *un*conditional probability. But the probability that it lands with '1' showing up *conditional* on or given that the outcome is an odd number, i.e. $\mathbb{P}_p(1|\text{odd})$, is one third. Intuitively, this conditional probability is one third because the possible outcomes are narrowed to the three equally possible odd ones, and '1' is one of them. And this number agrees with what the ratio formula delivers, namely

$$\mathbb{P}_p(1|\text{odd}) = \frac{p(1\cap \text{odd})}{p(\text{odd})} = \frac{1/6}{1/2} = \frac{1}{3} \tag{2.11}$$

Let us spell out the underlying rationale in more detail. First, if we know that the outcome of the throw is an odd number, then the appropriate sample space is not $S = \{1, 2, 3, 4, 5, 6\}$ anymore; rather S gets replaced by a new one, namely the set of odd outcomes

2.3. [Hájek, 2003, 2008] and [Easwaran, 2008] defend this view. See also [Mellor] Chp.7.

$S_{\text{odd}} = \{1, 3, 5\}$. There are however many probability measures on this new sample space. For example, the outcome '1' could be assigned a probability of one half while '3' and '5' a probability of one fourth each. Or both '1' and '3' could have a zero probability, while '5' have a probability one.

What specifically defines the conditional probability measure is that the sample space changes from S to S_{odd} and nothing else. That is, the conditional probability given 'odd', by definition, differs from the original one solely by taking into account the qualification of an odd outcome. This means that one has to eliminate the points in S that are not in S_{odd} (2, 4 and 6), without altering the relative probability of the points which remain (1, 3 and 5), i.e. by increasing the latter's value 'pro rata'. Thus, $\mathbb{P}_p(1|\text{odd})$ is derived from the initial probability measure by dividing the initial measure by the initial probability of odd, i.e. $\mathbb{P}_p(1|\text{odd}) = \frac{p(1)}{p(\text{odd})} = \frac{1/6}{1/2} = \frac{1}{3}$, which agrees with what the ratio formula delivers. Indeed, in this example $A = \{1\}$ is a subset of $B = \{\text{odd}\}$; hence, $A \cap B = A$ and the ratio formula reduces simply to $\mathbb{P}(A|B) = \frac{p(A)}{p(B)}$.

For general subsets A that are not necessarily subsets of B, as for example $A = \{1, 2, 3\}$ and $B = \{\text{odd}\}$, one has to consider only the probability of the sample points in A that are also in B and disregard the rest. For the sample points in A that are not also in B will not be possible outcomes in the new event space S_B and, therefore, will be assigned zero probability. Hence for any set A, not generally included in B, its conditional-on-B probability is the sum of the initial probability of the sample points that are both in A and in B, i.e. $p(A \cap B)$, increased pro rata. In other words, the conditional probability of A given B is the probability of that part of A lying in B increased pro rata. Just what the ratio analysis stipulates.

2.3.2 Ratio and Interpretations of Probability

The ratio analysis also captures the notion of conditional probability under specific *interpretations* of probability. It is standard to assume that probability comes in at least two varieties: *epistemic* and *physical*. Epistemic interpretations take probabilities to be related to our knowledge of the world, whereas physical interpretations regard probabilities as features of the objective material world, unrelated and independent of our knowledge of it. Physical probabilities are thus necessarily *objective* in the sense of being agent-independent, whereas epistemic probabilities can be either *subjective* or *objective* depending on whether or not prior degrees of belief are taken to be uniquely determined by the agent's background knowledge.

Under the subjective epistemic view, probabilities measure how strongly one believes certain propositions and are, therefore, features of the people who hold those beliefs; they are neither features of the world nor features of what the credences are about and are generally referred to as *credence*s or *degrees of belief*. In contrast, under the physical view, probabilities exist heedless of our beliefs and interests, and of our ever coming to conceive or know about them; they are neither relative to evidence nor matters of opinion and are generally referred to as *chances*. Finally, under the objective epistemic view, probabilities measure how far evidence confirms (or disconfirms) a certain hypothesis and are neither real features of the world nor matters of opinion.

There are different particular interpretations of these three kinds of probabilities. We here focus and develop one particular interpretation of two of them, namely the frequency interpretation of chances and the subjective Bayesian interpretation of credences.[2.4] We present both of them in turn, first, specifying how the notion of conditional probability is understood under each of them, and then showing how the ratio analysis agrees with such understanding. Appendix A provides a more detailed presentation on the subjective Bayesian interpretation of probability.

2.3.2.1 Frequency Interpretation

Long run relative frequency is typically a good guide to determining chances. Some, e.g. [Reichenbach, 1949], [von Mises, 1957], think that, more than being a good guide, such relative frequency should be identified with objective chance. This view is normally referred to as the 'Frequency Interpretation'. Frequentism applies to chances but not to credences. In addition, frequentists may deny that credences exist; that is, frequentists may deny either that belief comes by degrees or that, in case they do, these degrees have a probability measure.

Frequencies do not measure possibilities of outcomes but just how often the outcomes occur in a large number of (identical) experiments. Indeed, frequentism provides a non-modal surrogate for the idea of chance as a measure of physical possibility. Probabilities are generally taken as measuring possibilities, where possibility is further (standardly) seen as coming in degrees. And hence, frequentism, by identifying 'how possible something is' with 'how frequently something occurs', can interpret probabilities as measuring physical possibilities. Note, however, that it does not explain possibilities as such, it just explains them away.

2.4. On the various interpretations of probability see, for example, [Gillies, 2000a], [Mellor, 2005].

Frequentism is also closely related to the 'Humean view of causation', namely the view in which all it takes for causes to be sufficient for their effects is that they always produce them. Similarly, causes are necessary for their effects if the latter never occur without the presence of the former (i.e. effects only occur in the presence of their causes). Frequentism about chances then gives a Humean reading of this idea of sufficiency and necessity: causes are sufficient for their effects if there is a zero chance (relative frequency) for them not to occur. Similarly, a cause is necessary for its effect if there is a zero relative frequency of the effect in the absence of its cause.

Let us see how the frequency interpretation justifies the ratio analysis of conditional probability. Suppose that we run a long sequence of n trials, on each of which B might or might not occur. On a simple frequency interpretation the probability of B is identified with the relative frequency of trials on which it occurs, that is, the number of trials on which B appears divided by the total number of trials:

$$p(B) \equiv \frac{\mathrm{nb}(B)}{n} \tag{2.12}$$

Consider among those trials in which B occurs the proportion of those on which A also occurs, namely $\mathrm{nb}(A \cap B)/\mathrm{nb}(B)$. This is by definition the conditional probability of A given B, that is,

$$\mathbb{P}(A|B) = \mathrm{nb}(A \cap B)/\mathrm{nb}(B) \tag{2.13}$$

Now divide both terms by the total number of trials n. Under a simple frequentist interpretation $\mathrm{nb}(A \cap B)/n$ is identified with the probability of the joint occurrence of A and B, that is,

$$p(A \cap B) \equiv \frac{\mathrm{nb}(A \cap B)}{n} \tag{2.14}$$

And $\mathrm{nb}(B)/n$ is identified with the probability of B as (2.12) shows. Hence, $\mathbb{P}(A|B) = \frac{p(A \cap B)}{p(B)}$, as the ratio analysis stipulates.

Similarly, in terms of the die example, to conditionalize on 'odd' under the frequency reading is to select the subsequence of throws with results 1, 3 and 5. This selection leaves unaltered the numbers of throws with each of these three results, and hence the ratios of the relative frequencies of these results are also unaltered. Thus the conditional probability of '1' given 'odd' agrees with the ratio analysis.

2.3.2.2 Subjective Bayesian Interpretation of Probability

Consider now the *subjective Bayesian* interpretation wherein probabilities are defined as the subjective degrees of belief of a coherent agent. Degrees of belief are measured through so-called betting quotients and coherence requires that the agent will not accept a series of bets that will make her lose whatever happens. This ensures that degrees of belief satisfy the standard axioms of probability.

The most usual approach to subjective conditional probability is the so-called *Ramsey test*, which takes the subjective conditional probability $\mathbb{P}(A|B)$ as given by the degree of belief one has in A when supposing B (or hypothetically adding B to one's stock of beliefs).[2.5] The notion of supposition is crucial for it allows one's conditional degree of belief to differ from how one's beliefs would actually change were one to learn B with certainty. (See Appendix A for further detail.) However, regardless of what exactly conditional degrees of belief are – or whether they can be reduced to some notion of supposition – betting behavior, as with the notion of degree of belief, sheds important light on this notion. Indeed, it seems that $\mathbb{P}_p(A|B)$ ought to have some connection to the agent's disposition to accept bets on A that will be called off if B is not true.

It turns out there is a standard Dutch book argument suggesting that under this interpretation, one ought to set $\mathbb{P}_p(A|B)$ to what the ratio analysis stipulates. In effect, one can show that an agent would be incoherent, i.e. be 'Dutch Booked', if she does not set her conditional degree of belief in A given B to her degree of belief in their joint occurrence divided by the degree of belief in B, i.e. if she does not set $\mathbb{P}_p(A|B) = \frac{p(A\cap B)}{p(B)}$. Thus the coherent agent will set her conditional degree of belief to precisely what the ratio formula requires.

More intuitively using the die example, if the probabilities are read as degrees of belief and all we know is that 'odd' is true but nothing about which particular result of the throw actually made 'odd' true, then we should leave the relative values of our degrees of belief in the three odd results unaltered. And hence our degrees of belief conditional on 'odd' are derived from our initial unconditional ones by increasing their value 'pro rata', again in agreement with the ratio analysis.

We end this section by emphasizing the distinction between conditional probability and conditionalization. Subjectivists typically recognize no constraints on initial or prior subjective probabilities beyond the coherence condition. But they typically advocate a learning rule for updating probabilities in the light of new evidence. This rule is the so-called *principle of conditionalization* which states that, when one acquires new evidence B at time t_f, one should systematically transform one's initial probability assignment $p_i(A)$ to generate a final or posterior probability assignment $p_f(A)$ by conditioning on B, that is,

$$p_i(A) \quad\longrightarrow\quad p_f(A) = \mathbb{P}_{p_i}(A|B) \tag{2.15}$$

2.5. One also finds in the literature the attempt to define subjective conditional probability as the probability of a conditional, that is, as the degree of belief assigned unconditionally to an indicative conditional. However, this account does not work. See [Easwaran, 2008], [Eells and Skyrms, 1994].

It is important to realize that the notions of conditional probability and conditionalization are distinct notions: while conditionalization is a diachronic notion – it applies to probabilities held at a time prior to learning of evidence B and to probabilities held at a time posterior to this learning, conditional probability is a synchronic notion – it applies only to probabilities at one time. And arguments in favour of the synchronic notion do not necessarily support the diachronic rule. (See Appendix A for further detail)

But regardless of whether or not conditionalization is the appropriate rule for updating probability assignments it is clear that one can always associate a diachronic dimension to the notion of conditional probability. For one can straightforwardly use the conditional probability function to update a probability assignment. And this is all we need to have in mind when evaluating whether it is conceptually possible to define either a quantum notion of conditional probability or a quantum conditionalization rule.

2.3.3 Problems for the Ratio Analysis

In this section we have emphasized that the ratio formula (2.8) should not be regarded as a stipulative definition of conditional probability, but rather as an analysis of that notion. And we have seen various justifications for why this should be so. However, [Alan Hájek, 2003] has forcefully argued against the adequacy of the ratio formula as an analysis of conditional probability. Briefly, he argues that conditional probabilities can be well defined in many and important cases in which the ratio analysis goes silent. However, given that Hájek's arguments give rise to difficulties which are not particularly problematic for the project of defining a quantum notion of conditional probability, we will simply bracket his arguments in our discussion. We provide a brief summary of them in Appendix B.

2.4 Two Characterizations of Conditional Probability

We now consider two formal characterizations of the conditional measure $\mathbb{P}_p(\,\cdot\,|\,\cdot\,)$ defined by the ratio $p(A \cap B)/p(B)$. First, we show that the conditional probability measure thus defined is the only probability measure defined on the whole event space $\mathcal{F}(S)$ such that for sets A contained in B, conditionalizing on B just involves a renormalization of the initial probability measure $p(A)$ to $\mathbb{P}_p(A|B)$, where $\mathbb{P}_p(B|B)=1$. In other words, starting with a probability p defined over S, if for all $A \subseteq B$ one defines a probability measure m_p in terms of the initial probability measure p as $m_p(A) = \frac{p(A)}{p(B)}$, then m_p can be extended to all $\mathcal{F}(S)$; the extension is a new probability measure which is unique and is precisely given by the usual ratio. We refer to this characterization of conditional probability as '*the existence and uniqueness characterization*' (section 2.4.1).

Second, we present a characterization of conditional probability which arises given that not only is the conditional probability measure necessarily additive with respect to the conditioning events, but, conversely, any measure additive with respect to the conditioning events is necessarily a conditional probability measure. More specifically, if a probability measure m has the structure of a mixture of the conditional probability measures $\mathbb{P}_p(\,\cdot\,|B_i)$ with weights $\frac{p(B_i)}{p(B)}$ for $B_i \cap B_j = \emptyset$, that is, $m(A) = \sum_i \frac{p(B_i)}{p(B)}\, \mathbb{P}_p(A|B_i)$, then m is the conditional probability with respect to $B = \cup_i B_i$. We refer to this characterization of conditional probability as '*additivity with respect to conditioning events*' (section 2.4.2).

2.4.1 Existence and Uniqueness

Let $\langle S, \mathcal{F}(S), p\rangle$ be a classical probability space. To every subset $A \in \mathcal{F}(S)$, $p(\,\cdot\,)$ assigns A a value in the interval $[0,1]$. We have seen that every probability measure $p(\,\cdot\,)$ defines a conditional probability measure $\mathbb{P}_p(\,\cdot\,|\,\cdot\,)$, such that the conditional probability of A given B is given by $\mathbb{P}_p(A|B) = \frac{p(A\cap B)}{p(B)}$ (with $p(B) \neq 0$). The following theorem shows that the conditional probability measure thus defined from $p(\,\cdot\,)$ is the *only* probability measure on the set of all events $\mathcal{F}(S)$ such that for every event A in $\mathcal{F}(B)$, $\mathbb{P}_p(A|B) = \frac{p(A)}{p(B)}$.[2.6]

Theorem 2.1. ***Existence and Uniqueness***. *Given a non-empty set S and a field $\mathcal{F}(S)$ of subsets of S, let B belong to $\mathcal{F}(S)$ and $p(\,\cdot\,)$ be any probability measure on $\mathcal{F}(S)$ such that $p(B) \neq 0$. For any A in $\mathcal{F}(B)$ – the subsets of B that are in $\mathcal{F}(S)$ – define the function*

$$m_p(A) = \frac{p(A)}{p(B)} \tag{2.16}$$

Then:

1. *$m(\,\cdot\,)$ is a probability measure on $\mathcal{F}(B)$,*
2. *there exists an extension $\mathbb{P}_p(\,\cdot\,|B)$ of $m(\,\cdot\,)$ to all $\mathcal{F}(S)$,*
3. *the extended probability measure $\mathbb{P}_p(\,\cdot\,|B)$ is unique and, for all C in $\mathcal{F}(S)$, is given by*

$$\mathbb{P}_p(C|B) = m(C\cap B) = \frac{p(C\cap B)}{p(B)} \tag{2.17}$$

This theorem, hence, states that there is only one way of extending a conditional probability measure defined for sets A in $\mathcal{F}(B)$ as $\mathbb{P}_p(A|B) = \frac{p(A)}{p(B)}$, to all sets C in $\mathcal{F}(S)$ which is precisely given by the ratio formula $\mathbb{P}_p(C|B) = \frac{p(C\cap B)}{p(B)}$. The extended probability measure of any measurable subset A of B agrees with the original probability of that subset, i.e. $\mathbb{P}_p(A|B) = m_p(A)$; and for general subsets C in $\mathcal{F}(S)$ simply assigns C the probability that corresponds to that part of C which is contained in B, and zero value to the remaining part, i.e. $\mathbb{P}_p(C|B) = m_p(C\cap B)$.

2.6. See [Cassinelli & Zanghí, 1983], [Teller & Fine, 1975], [Hughes, 1989].

This uniqueness result may seem to distinguish conditional probability formally. In fact it does not. For in classical probability every probability measure on a subspace is uniquely extendable to the full space; there is nothing special about a conditional probability measure defined by the ratio rule. Indeed the following theorem holds:[2.7]

Theorem 2.2. ***Extendability of Classical Probability***. *Given a non-empty set S and a field $\mathcal{F}$ of subsets of S, let B belong to $\mathcal{F}(S)$ and $m(\,\cdot\,)$ be a probability measure on $\mathcal{F}(B)$. Then:*

1. *there exists an extension $p(\,\cdot\,)$ of $m(\,\cdot\,)$ to all of $\mathcal{F}(S)$,*
2. *this extension is unique and, for all C in $\mathcal{F}(S)$, is given by*

$$p(C) = m(C \cap B) \tag{2.18}$$

Hence, it is a simple consequence of this general fact about classical extendability that in the special case of conditional probabilities the extension will satisfy the ratio formula. That is, theorem 2.1 is just a particular instance of the more general theorem 2.2 when $m(\,\cdot\,)$ for A in $\mathcal{F}(B)$ is defined as $m_p(A) = \frac{p(A)}{p(B)}$. There is nothing special about conditional probability: all other measures $m(\,\cdot\,)$ on $\mathcal{F}(B)$, which also assign B probability one but which need not be defined as the ratio $\frac{p(A)}{p(B)}$, will be likewise extendable to the full space $\mathcal{F}(S)$.

Note that both points 1. and 2. of theorem 2.2 depend *only* on the probability measure $m(\,\cdot\,)$ restricted to $\mathcal{F}(B)$: as we have seen, the extended probability measure $p(C) = m(C \cap B)$ simply assigns C the probability that corresponds to that part of C which is contained in B, and zero value to the remaining part. To anticipate, the situation will not be analogous in the quantum case: although an analogue version of theorem 2.1 holds, points 2. and 3. in that case will depend critically on the initial measure $p(\,\cdot\,)$ defined on the *full* space. Thus it will not be possible to see the quantum version of theorem 2.1 as a consequence of an analogue version of theorem 2.2; in fact the quantum version of theorem 2.2 is false.

2.4.2 Additivity with Respect to Conditioning Events

The conditional probability measure with respect to an event $B = \cup_i B_i$, with $B_i \cap B_j = \emptyset$ for $i \neq j$, is necessarily additive with respect to the (disjoint) conditioning events B_i. In effect,

$$\mathbb{P}_p(A | \cup B_i) = \frac{p[A \cap (\cup_i B_i)]}{p(B)} = \frac{p[\cup_i (A \cap B_i)]}{p(B)} = \frac{\sum_i p(A \cap B_i)}{p(B)} \tag{2.19}$$

2.7. This theorem was formulated by Arthur Fine. The proof is simple. Imagine there is another measure $p'(\,\cdot\,)$, where $p'(\,\cdot\,) \neq p(\,\cdot\,)$ that also satisfies the condition $p'(A) = m(A)$. For all $C \in F(S)$, $p'(C) = p'(C \cap B) + p'(C \cap (S - B))$. Now $p'(B) = m(B) = 1$ and so $p'(S - B) = 0$; and since $0 \leqslant p'(C \cap (S - B)) \leqslant p'(S - B)$, we get $p'(C) = p'(C \cap B)$. But $C \cap B \in F(B)$ and so $p'(C) = p'(C \cap B) = m(C)$, which is equal to $p(C)$. And thus $p'(C) = p(C)$.

And multiplying (2.19) by $\frac{p(B_i)}{p(B_i)}$, we get that

$$\mathbb{P}_p(A|\cup B_i) = \sum_i \frac{p(B_i)}{p(B)} \mathbb{P}_p(A|B_i) \tag{2.20}$$

One can also show that the converse also holds. That is, that any measure which is additive with respect to the (disjoint) conditioning events B_i is necessarily the conditional probability measure given $B = \cup_i B_i$. The following theorem establishes this result.[2.8]

Theorem 2.3. *Let $\langle S, \mathcal{F}(S), p\rangle$ be a classical probability space, B an element of $\mathcal{F}(S)$ such that $p(B) \neq 0$ and $\{B_i\}$ a countable disjoint set of elements of $\mathcal{F}(S)$ such that $B = \cup_i B_i$. Let m be a probability measure such that for each B_i and A, an element of $\mathcal{F}(S)$,*

$$m(A) = \frac{p(A)}{p(B)} \tag{2.21}$$

Then m is the conditional probability with respect to B.

Let us go through the proof of this theorem to see exactly how it establishes that additivity with respect to conditioning events characterizes conditional probability.

Proof.

To prove theorem 2.3, it suffices to show that it holds for a set $A \subseteq B$. For if $m(A) = \frac{p(A)}{p(B)}$ coincides with $\mathbb{P}_p(A|B)$ for $A \subseteq B$, then it coincides with $\mathbb{P}_p$ on the whole $\mathcal{F}(S)$ because of theorem 2.1. Let A be any element of $\mathcal{F}(S)$ such that $A \subseteq B$, then

$$A = A \cap B = A \cap (\cup_i B_i) = \cup_i (A \cap B_i) \tag{2.22}$$

Hence $m(A) = m[\cup_i (A \cap B_i)]$, and since m is a probability measure:

$$m(A) = \sum_i m(A \cap B_i) \tag{2.23}$$

Now for each index i, $A \cap B_i \subseteq B_i$, so that, by definition (2.21) of m,

$$m(A \cap B_i) = \frac{p(A \cap B_i)}{p(B)} \tag{2.24}$$

And inserting (2.24) into (2.23),

$$m(A) = \sum_i \frac{p(A \cap B_i)}{p(B)} = \frac{\sum_i p(A \cap B_i)}{p(B)} = \frac{p(\sum_i A \cap B_i)}{p(B)} = \frac{p(A \cap B)}{p(B)} = \frac{p(A)}{p(B)} \tag{2.25}$$

Hence $m(A) = \mathbb{P}_p(A|B)$ □

Additivity with respect to the B_i's can be now made explicit by multiplying (2.25) by $\frac{p(B_i)}{p(B_i)}$:

$$m(A) = \sum_i \frac{p(A \cap B_i)}{p(B)} \frac{p(B_i)}{p(B_i)} = \sum_i \frac{p(B_i)}{p(B)} \frac{p(A \cap B_i)}{p(B_i)} = \sum_i \frac{p(B_i)}{p(B)} \mathbb{P}_p(A|B_i) \tag{2.26}$$

2.8. See [Cassinelli & Zanghí, 1984].

Thus we can reformulate theorem 2.3 in the following way:

Theorem 2.4. ***Additivity with Respect to Conditioning Events****. Let $\langle S, \mathcal{F}(S), p\rangle$ be a classical probability space, B an element of $\mathcal{F}(S)$ such that $p(B) \neq 0$ and $\{B_i\}$ a countable disjoint set of elements of $\mathcal{F}(S)$ such that $B = \cup_i B_i$. Let m be a probability measure such that for each B_i and A, an element of $\mathcal{F}(S)$, $m(A)$ is the convex combination of the conditional measures $\mathbb{P}_p(\,\cdot\,|B_i)$, i.e.*

$$m(A) = \sum_i \frac{p(B_i)}{p(B)} \mathbb{P}_p(A|B_i) \tag{2.27}$$

Then m is the conditional probability with respect to B, i.e.

$$m(A) = \mathbb{P}(A|B) = \sum_i \frac{p(B_i)}{p(B)} \mathbb{P}_p(A|B_i) \tag{2.28}$$

Hence, in addition to the characterization of conditional probability provided by theorem 2.1, conditional probability can also be characterized by its being additive with respect to conditioning events.

Note that additivity with respect to conditioning events is another way of stating the so-called *theorem of compound probabilities* or *law of total probability*, namely, for a partition $\{B_i\}$ of B – B_i disjoint and $\sum B_i = \mathbb{I}$ – the total probability of A is given by

$$p(A) = \sum \mathbb{P}_p(A|B_i)p(B_i) \tag{2.29}$$

Indeed from (2.28) it follows that

$$\mathbb{P}_p(A|\sum B_i)\, p(B) = \sum p(B_i)\mathbb{P}_p(A|B_i) \tag{2.30}$$

And given that $\sum B_i = \mathbb{I}$ and $p(B) = 1$, (2.30) reduces to (2.29).

To anticipate, the situation will not be analogous in the quantum case. In effect, unlike the classical case, there is not a unique probability measure on the quantum event structure which coincides with the quantum analogue of $\mathbb{P}_p(\,\cdot\,|B)$ for each B_i. And quantum conditioning, when conditioning is taken over a set of quantum events that mutually exclude each other, does not in general return a classical convex mixture over the components in the sum.

Chapter 3
Quantum Probability Theory

By quantum mechanics we will refer to the Hilbert space formalism, including the dynamical rule for the quantum state given by the Schrödinger equation, Born's rule for calculating probabilities, and the association of physical magnitudes with Hermitian operators. These elements seem to be the core of the (non relativistic) theory. There are many mathematical formulations of quantum mechanics: the standard formulation in terms of Hilbert spaces and operators, Feynman path integrals, axiomatic and algebraic approaches, C^* algebra formalism, etc. In addition, the quantum mechanical formalism can be expressed as a theory of probability, an approach which is traditional and goes back at least to [von Neumann, 1932].[3.1] In this chapter we present the formalism of quantum probability theory in detail. This approach is useful in general to study the mathematical structure of quantum theory. For us the motivation is straightforward: it is the natural framework in which to consider whether it is possible (and, if so how) to define a quantum notion of conditional probability.

The probability theory underlying quantum mechanics is phrased in terms of operators on a Hilbert space. In general these operators do not commute and hence quantum probability is sometimes called a non-commutative probability theory. This non-commutativity is the main difference between the classical and quantum probability theories and has far-reaching consequences. At the formal level, this claim is uncontroversial: quantum mechanics simply uses a method for calculating probabilities which is different from that of classical probability theory. However, whether this also implies that the interpretation of quantum probability is fundamentally different from classical probability requires further investigation. We do this in chapters 5 to 8. In the present chapter we simply lay out the basic formalism of quantum probability theory.

Traditionally a theory of probability distinguishes between the set of possible events (called the algebra of events, or the set of possible outcomes) and the probability measure defined on them. In section 3.1, we consider the mathematical entities which represent

3.1. More modern formulations can be found, for example, in the following books: [Mackey, 1963], [Bub, 1974], [Gudder, 1979], [Beltrametti & Cassinelli, 1981], [Varadarajan, 1985], [Gudder, 1988], and [Pitowsky, 1989].

quantum events and their algebraic structure, and, in section 3.2, the probability functions which can be defined over this structure. Then, in section 3.3, we consider the standard semantic rule for ascribing values in quantum mechanics, namely the eigenstate-eigenvalue link, a term coined by Arthur Fine.[3.2] Finally, in section 3.4, we present the connection between the non-commutativity of the quantum mechanical operators and the joint probability distributions ascribed to these.

3.1 Quantum Events and their Structure

Standard presentations of quantum mechanics go as follows. Each physical system is associated with a *Hilbert space* $\mathcal{H}$. A Hilbert space is a vector space on which an inner product has been defined and which is complete.[3.3] This stands in striking contrast to the classical case in which a physical system is not associated with a vector space but with a real space. Observables are also not represented by real-valued functions on a real space as for a classical system but by Hermitian operators acting on the Hilbert space associated with a system.[3.4] The possible values of the observable are given by the spectrum of the operator which represents it.

For operators with a discrete spectrum, the possible values of the corresponding observable are restricted to a discrete set of eigenvalues. An observable A will take a certain value a_i, where a_i is a discrete eigenvalue of the operator A, for a system whose state lies in the eigenspace $L^A_{a_i}$ associated with the eigenvalue a_i; the quantum event 'A takes the value a_i' is thus represented by the subspace $L^A_{a_i}$ of the relevant Hilbert space. For general operators that do not admit eigenvectors and have a continuous spectrum, to each Borel set $\mathbb{b}$ on $\mathbb{R}$ there corresponds a closed subspace $L^A(\mathbb{b})$ of $\mathcal{H}$ such that the value of A is within $\mathbb{b}$; the quantum event 'A takes the value $\mathbb{b}$' is thus represented by the subspace $L^A(\mathbb{b})$ of the relevant Hilbert space.[3.5] Hence, in quantum theory events are not represented by subsets of the real phase space as for a classical system, but by closed *subspaces* of a Hilbert space.

3.2. See for example, [Fine, 1970].

3.3. $\mathcal{V}$ is a vector space if for any vectors $u, v \in \mathcal{V}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 u + \lambda_2 v \in \mathcal{V}$. A vector space is complete if any converging sequence of vectors in the space converges to a vector in the vector space.

3.4. An operator A on $\mathcal{H}$ is said to be *linear* if $\forall \psi, \varphi_1, \varphi_2 \in \mathcal{H}, \lambda \in \mathbb{C}$, $\langle \psi, A \lambda (\varphi_1 + \varphi_2) \rangle = \lambda \langle \psi, A \varphi_1 \rangle + \lambda \langle \psi, A \varphi_2 \rangle$, where $\langle , \rangle$ represents the inner product in $\mathcal{H}$. A linear operator A on $\mathcal{H}$ is said to be a *Hermitian operator* if $\forall \psi, \varphi \in \mathcal{H}, \langle \psi, A \varphi \rangle = \langle A \psi, \varphi \rangle$.

3.5. The identification between quantum events and subspaces of a Hilbert space assumes the so-called eigenstate-eigenvalue link, which we present in detail in section 3.3 after defining quantum states in section 3.2.

There is a natural one-to-one correspondence between the set of *orthogonal projection operators* P^A of $\mathcal{H}$ and the set of closed subspaces L^A of $\mathcal{H}$.[3.6] Projection operators have the property of 'projecting onto a subspace': given a subspace L, we can decompose any vector ψ of $\mathcal{H}$ into two parts, $\psi = \psi_L + \psi_{L^\perp}$, such that ψ_L lies in L and $\psi_{L^\perp}$ is orthogonal to ψ_L, and the projection operator P over subspace L is then defined by its action over the arbitrary vector ψ as $P\psi = \psi_L$. Given the bijection between the set of closed subspaces and the set of orthogonal projection operators, we can use projectors and closed subspaces interchangeably and represent the event that the observable A has a value a_i [in the set $\mathbb{b}$], i.e. (A, a_i) [$(A, \mathbb{b})$], both as the closed subspace $L_{a_i}^A$ [$L_A(\mathbb{b})$] or as the projection operator $P_{a_i}^A$ [$P^A(\mathbb{b})$].

The usual operations of (1) set-containment $A \subseteq B$, (2) set-union $A \cup B$, (3) set-intersection $A \cap B$, and (4) set-complementation A^c in S have their natural counterparts for subspaces and projectors. In subspace language these are (1) $M \subseteq N$, (2) $\overline{\text{span}}\, M \cup N = M \oplus N$ – where span means the closed span, (3) $M \cap N$, and (4) orthogonal complement $M^\perp$. And, in projection language, (1) $P \leqslant Q$ (so that $PQ = QP = P$), (2) the orthogonal projection onto the closed subspace spanned by the ranges of P and Q, i.e. $P \vee Q$, (3) the orthogonal projection onto the intersection of the ranges of P and Q, i.e. $P \wedge Q$, and (4) the orthogonal projection onto the complement of the closed subspace spanned by the range of P, i.e. $P^\perp = I - P$, where I is the identity operator on $\mathcal{H}$. If P and Q are orthogonal, then $P \wedge Q = PQ$ and $P \vee Q = P + Q$.

With these operations, the algebraic structure of the set of quantum events is not a Boolean algebra; rather is a complete orthocomplemented lattice.[3.7] We denote the set of orthogonal projections of $\mathcal{H}$ as $\mathcal{L}(\mathcal{H})$; $\mathcal{L}(\mathcal{H})$ is thus the set of possible quantum mechanical events. For our present purposes it suffices to note that in $\mathcal{L}(\mathcal{H})$ the distributive property does not hold. This can easily be seen in the following example.

Example 3.1. Non-Distributivity. Consider a spin-1 particle. The system's associated space is a three-dimensional Hilbert space, for which one complete orthonormal basis is given by the set of unit vectors $\{\psi_{z_{-1}}, \psi_{z_0}, \psi_{z_{+1}}\}$, corresponding to the eigenvalues $\{-1, 0, 1\}$ of the spin observable in the z-direction S_z. Each of these vectors spans a subspace of the Hilbert space $\mathcal{H}$ $\{L_{s_{z_{-1}}}, L_{s_{z_0}}, L_{s_{z_{+1}}}\}$, with its corresponding projector operator $\{P_{s_{z_{-1}}}, P_{s_{z_0}}, P_{s_{z_{+1}}}\}$. Consider a nontrivial linear combination ψ_a of $\psi_{z_{-1}}$ and ψ_{z_0} and denote by L_a the subspace spanned by ψ_a, so that $L_a \subseteq L_{s_{z_{-1}}} \oplus L_{s_{z_0}}$.

3.6. A linear operator P on $\mathcal{H}$ is said to be a *projection operator* if P is Hermitian and Idempotent, i.e. $P^2 = P$.

3.7. For a detailed account see, for example, [Beltrametti & Cassinelli, 1981], [Hughes, 1989].

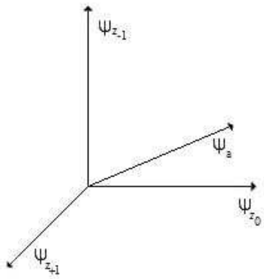

Figure 3.1. Non-distributive quantum event structure.

For the lattice of projectors $\{P_{s_{z_{-1}}}, P_{s_{z_0}}, P_{s_{z_{+1}}}\}$ to be Boolean the condition of distributivity must hold, i.e.

$$P_a \wedge (P_{s_{z_{-1}}} \vee P_{s_{z_0}}) = (P_a \wedge P_{s_{z_{-1}}}) \vee (P_a \wedge P_{s_{z_0}}) \tag{3.1}$$

(or equivalently, $L_a \cap (L_{s_{z_{-1}}} \oplus L_{s_{z_0}}) = (L_a \cap L_{s_{z_{-1}}}) \oplus (L_a \cap L_{s_{z_0}})$). However, while the left-hand side of the equation is

$$P_a \wedge (P_{s_{z_{-1}}} \vee P_{s_{z_0}}) = P_a \tag{3.2}$$

the right-hand side is

$$(P_a \wedge P_{s_{z_{-1}}}) \vee (P_a \wedge P_{s_{z_0}}) = 0 \tag{3.3}$$

And thus equality (3.1) is violated.

It is important to note that although the algebra $\mathcal{L}(\mathcal{H})$ as a whole is not Boolean, it does contain Boolean sub-lattices: each observable considered separately can be identified with a Boolean algebra and so can every set of *compatible* observables. (We call two physical quantities compatible when they are represented by commuting operators; and two Hermitian operators A, B in $\mathcal{H}$ are said to *commute* whenever $AB = BA$.[3.8]) Indeed, for P and Q projection operators on the Hilbert space $\mathcal{H}$, P and Q commute if and only if the sublattice of $\mathcal{L}(\mathcal{H})$ generated by P, Q, $P^{\perp}$ and $Q^{\perp}$ is Boolean.

To sum up, in quantum probability theory the measurable space $\langle S, \mathcal{F}(S)\rangle$ of classical probability theory is replaced with the pair $\langle \mathcal{H}, \mathcal{L}(\mathcal{H})\rangle$.

3.2 Quantum Probability

Quantum theory can be essentially regarded as a theory of probability defined over the projection lattice $\mathcal{L}(\mathcal{H})$. This probability measure is a map $p(\,\cdot\,)$ from the projection operators into the real numbers in the closed interval $[0,1]$ which is normalized and additive for orthogonal projection operators.

3.8. Note also that only if two physical quantities are compatible is their 'product' a physical quantity: indeed, the product of two Hermitian operators A, B is Hermitian if and only if $AB = BA$.

Definition 3.1. ***Quantum Probability***. *A real function $p(\,\cdot\,)$ defined from the lattice $\mathcal{L}(\mathcal{H})$ of projection operators on a Hilbert space $\mathcal{H}$ is a quantum probability measure if*

i. (Non-negativity) $p(P) \geqslant 0$ for all $P \in \mathcal{L}(\mathcal{H})$.

ii. (Normalization) $p(P_I) = 1$, where I stands for the identity operator on $\mathcal{H}$

iii. (Additivity) for every countable set of mutually orthogonal projection operators $\{P_i\}$ in $\mathcal{L}(\mathcal{H})$,

$$p\Big(\sum_i P_i\Big) = \sum_i p(P_i), \ \ (\text{for } P_i\, P_j = 0 \text{ if } i \neq j) \tag{3.4}$$

Unit vectors ψ on $\mathcal{H}$ and, more generally, density operators W on $\mathcal{H}$, define all the possible quantum probability measures. Indeed, if ψ is a unit vector in $\mathcal{H}$ and P a projection operator of $\mathcal{L}(\mathcal{H})$, the function defined by the inner product

$$p_\psi(P) = \langle \psi, P\psi \rangle \tag{3.5}$$

defines a probability function on $\mathcal{L}(\mathcal{H})$, where additivity holds given the additivity property of the inner product.[3.9] And any given density operator W defines a probability function p_W on $\mathcal{L}(\mathcal{H})$ by

$$p_W(P) = \mathrm{Tr}(WP) \tag{3.6}$$

for $P \in \mathcal{L}(\mathcal{H})$. A density operator W on $\mathcal{H}$ is a positive trace-class operator – an operator such that its trace is positive for all $\psi \in \mathcal{H}$ – of trace one, where the trace of a positive operator is defined as the quantity

$$\mathrm{Tr}\, A = \sum_i \langle \psi_i, A\, \psi_i \rangle \tag{3.7}$$

for an orthonormal basis $\{\psi_i\}$ of $\mathcal{H}$. (Note that this quantity is independent of the basis $\{\psi_i\}$.)

The probability measures p_W defined by density operators correspond to classical statistical mixtures of the probability measures p_ψ defined by vectors. Indeed, any density operator can be expressible as a weighted sum of orthogonal projection operators P_i, i.e. $W = \sum_i a_i\, P_i$ where $a_i \geqslant 0$ and $\sum_i a_i = 1$, so that expression (3.6) can be written as the weighted sum of the traces $\mathrm{Tr}(P_i\, P)$, where $\mathrm{Tr}(P_i\, P) = \langle \psi_i, P\psi_i \rangle$, for an orthonormal basis $\{\psi_j\}$ of $\mathcal{H}$ such that $P_i\, \psi_j = \delta_{ij}\, \psi_j$.[3.10]

3.9. Indeed, $p_\psi(\sum P_i) = \langle \psi, \sum_i P_i\, \psi \rangle = \sum_i \langle \psi, P_i\, \psi \rangle = \sum_i p_\psi(P_i)$.

3.10. In detail, $\mathrm{Tr}(P_i\, P) = \mathrm{Tr}(P\, P_i) = \sum_j \langle \psi_j, P\, P_i\, \psi_j \rangle$, with $\{\psi_j\}$ any orthonormal basis of $\mathcal{H}$. And we can choose $\{\psi_j\}$ such that $P_i\, \psi_j = \delta_{ij}\, \psi_j$ so that $\sum_j \langle \psi_j, P P_i\, \psi_j \rangle = \langle \psi_i, P\psi_i \rangle = p_{\psi_i}(P)$.

In contrast to classical mechanics in which the state of a system is determined by its position and momentum, in quantum theory the physical state of a system is given by a probability function on $\mathcal{L}(\mathcal{H})$. If ψ is a unit vector of $\mathcal{H}$, we call the probability function $p_\psi(P) = \langle\psi, P\psi\rangle$, where $P \in \mathcal{L}(\mathcal{H})$, a *pure state*. If W is a density operator in $\mathcal{H}$, we call the probability function $p_W(P) = \mathrm{Tr}(WP)$, where $P \in \mathcal{L}(\mathcal{H})$, a *mixed state*.[3.11] Quantum states can thus be represented in full generality by density operators W on $\mathcal{L}(\mathcal{H})$.

Remarkably, the converse of this results holds as well. [Gleason, 1957] proved that if the dimension of $\mathcal{H}$ is equal or greater than 3, the probability measures on $\mathcal{L}(\mathcal{H})$ representable by density operators on $\mathcal{H}$ *exhaust* the set of *all* probability measures on $\mathcal{L}(\mathcal{H})$, where each probability measure corresponds uniquely to a density operator.[3.12]

Theorem 3.1. ***Gleason's Theorem.*** *If the dimension of $\mathcal{H}$ is no less than 3, then every probability measure p on $\mathcal{L}(\mathcal{H})$ arises from a density operator W in $\mathcal{H}$, according to the rule*

$$p_W(P) = \mathrm{Tr}(WP) \tag{3.8}$$

for every projection operator $P \in \mathcal{L}(\mathcal{H})$.

Gleason's theorem thus characterizes the set of all *possible* states on set $\mathcal{L}(\mathcal{H})$ of subspaces of $\mathcal{H}$: it contains just those states which are representable by density operators on $\mathcal{H}$. Hence once one assumes the algebraic structure of the set of quantum events is $\mathcal{L}(\mathcal{H})$, Gleason's theorem dictates the probabilistic structure. An important consequence of Gleason's theorem is that it rules out all discontinuous measures over $L(\mathcal{H})$ when $\dim(\mathcal{H}) \geqslant 3$. This is because for any given density operator W the map $P \rightarrow \mathrm{Tr}(WP)$ is continuous on the unit sphere of $\mathcal{H}$. Thus, non-trivial probability measures having only the values 0 and 1 are not admitted. This is one way of putting the no-go results of the Bell-Kochen-Specker theorem.[3.13]

There is a simplified version of Gleason's theorem for the case in which the density operator is a one-dimensional projection operator. It is a considerably weaker form of Gleason's theorem but requires a less sophisticated proof.[3.14] Following [Malley, 1998, 2004], we refer to it as 'micro-Gleason'.

3.11. In effect, a mixed state is a function p: $\mathcal{H} \rightarrow [0,1]$ of the form $p = \sum_i \lambda_i p_{\psi_i}$, where ψ_i are unit vectors, and $\lambda_i \geqslant 0$, $\sum \lambda_i = 1$. Mixed states correspond to the fact that convex combinations of probability measures are again probability measures. One can show that the ψ_i can always be chosen to be orthogonal and that the mixed state $p = \sum_i \lambda_i p_{\psi_i}$ can always be represented by the density operator $W = \sum_i \lambda_i P_i$, where P_i is the orthogonal projection onto the span of ψ_i.

3.12. [Gleason, 1957]. See also [Beltrametti & Cassinelli, 1981], p.115.

3.13. See [Beltrametti & Cassinelli, 1981] p.267ff.

3.14. [Gudder, 1979], p.129 corollary 5.17.

Theorem 3.2. ***Micro-Gleason's Theorem.*** *Let dim $\mathcal{H} \geqslant 3$ and let p be a probability measure on the lattice of projectors $\mathcal{L}(\mathcal{H})$ which assigns probability one to any one-dimensional projector P_ψ, i.e. $p(P_\psi) = 1$, where $\|\psi\| = 1$. Then p must be such that*

$$p_\psi(P) = \mathrm{Tr}(P_\psi P) = \langle \psi, P\psi \rangle \tag{3.9}$$

for all projectors $P \in \mathcal{L}(\mathcal{H})$.

It is instructive to see why equality (3.9) holds. Given $\mathrm{Tr}(P_\psi P) = \mathrm{Tr}(P P_\psi)$ and $\sum_j \langle \phi_j, \phi_j \rangle = 1$, we have

$$\mathrm{Tr}(P P_\psi) = \sum_j \langle \phi_j, P P_\psi \phi_j \rangle = \sum_j \langle \phi_j, P\,\psi \rangle \langle \psi\,, \phi_j \rangle = \langle \psi, P\,\psi \rangle \sum_j \langle \phi_j, \phi_j \rangle \tag{3.10}$$

and thus $\mathrm{Tr}(P_\psi P) = \langle \psi, P\psi \rangle$.

To sum up, we may identify quantum probability theory with the quantum probability space $(\mathcal{H}, \mathcal{L}(\mathcal{H}), W)$, which is defined as follows:

Definition 3.2. ***Quantum Probability Space.*** *A quantum probability space consists of a triple $\langle \mathcal{H}, \mathcal{L}(\mathcal{H}), W \rangle$ where*

i. $\mathcal{H}$ is a closed complex Hilbert space.

ii. $\mathcal{L}(\mathcal{H})$ is the set of projection operators on $\mathcal{H}$. These projectors represent quantum events.

iii. W are the density operators which generate all the possible probability functions according to the rule $p_W(P) = \mathrm{Tr}(WP)$, for every projection operator $P \in \mathcal{L}(\mathcal{H})$.

3.3 Eigenstate-Eigenvalue Link

Recall that in section 3.1 we identified the quantum event 'observable A takes the value a_i', with the projection operator P_{a_i} by saying that 'A will take the value a_i for all systems whose state lies in the eigenspace L_{a_i} associated with the eigenvalue a_i'. This identification rests upon an assumption that has come to be known as the *eigenvalue-eigenstate link*, (or e-e link in short). This link is the standard rule for ascribing values in quantum mechanics, although many interpretations of quantum mechanics actually deny it.

The e-e link states that an observable A has a well-defined value for a quantum system in state ψ *if and only if* ψ is an eigenstate of A, that is, $A\psi = a_i\psi$, in which case a_i is the value of A in state ψ. It can be analyzed as the conjunction of two rules which correspond to the 'if' and 'only if' part:

1. **Rule of Law**. The *eigenstate to eigenvalue* rule says that if ψ is an eigenstate of A with eigenvalue a_i, then the system, whose state is ψ, has the value a_i for the observable A. Fine calls this the 'Rule of Law': if for some eigenvalue a_i of the operator A, the state ψ of a system is an eigenstate of A, then the 'law' requires that we attribute the value a_i to the system.

2. **Rule of Silence**. The *eigenvalue to eigenstate* rule says that if the system, whose state is ψ, has the value a_i for the observable A, then ψ is an eigenstate of A with eigenvalue a_i. Fine, considering the contrapositive formulation of this rule, refers to it as the 'Rule of Silence': if there is *no* eigenvalue a_i of A such that ψ is an eigenstate of A, then we must be silent about saying that the system has the value a_i for the observable A.

Note that when we identify the quantum event 'observable A takes the value a_i' with P_{a_i} by saying that 'A will take the value a_i for all systems whose state lies in the eigenspace L_{a_i}' we are using the eigenstate to eigenvalue rule (or the rule of law). And that when we only allow eigenstates to take determinate values, we employ the eigenvalue to eigenstate rule (or the rule of silence)

A more general way to formulate the e-e link is as follows.[3.15]

Definition 3.3. ***Eigenstate-Eigenvalue Link***

1. *Eigenstate to eigenvalue link: if $p_W(P_{a_i}) = 1$, then the system, whose state is W, takes value a_i for observable A.*

2. *Eigenvalue to eigenstate link: if the system, whose state is W, takes value a_i for observable A, then W is such that $p_W(P_{a_i}) = 1$.*

3.4 Joint Probability Distributions

As (3.6) prescribes, the probability that observable A takes a value in the Borel set $\mathbb{b}$ is given by the function $\mathrm{Tr}(WP^A(\mathbb{b}))$. Thus in quantum probability theory observables play the role of random variables and the projection operators $P^A(\mathbb{b})$ correspond to events. By analogy with classical probability theory, we call the probability measure $\mathbb{b} \to \mathrm{Tr}(WP^A(\mathbb{b}))$ the *distribution* of the observable A in the state W.

3.15. [Dickson, 1998], pp.18-19.

Definition 3.4. ***Quantum Probability Distribution***. *The probability measure*

$$\mathbb{b} \to \mathrm{Tr}(W P^A(\mathbb{b})) \tag{3.11}$$

is the probability distribution of observable A in state W.

In classical probability theory the joint distribution for random variables f, g is defined by (2.3) as the probability measure $p_{f,g}$ on $\mathcal{B}(\mathbb{R}^2)$ satisfying $p_{f,g}(\mathbb{a} \times \mathbb{b}) = p[f^{-1}(\mathbb{a}) \cap g^{-1}(\mathbb{b})]$ for all $\mathbb{a}, \mathbb{b} \in \mathbb{B}(\mathbb{R})$. In an analogue way, one could try to define a joint probability distribution for two self-adjoint operators A, B and a system in state W as the function $p_{W;A,B}$ on the subsets of $\mathbb{R}^2$ of the form $\mathbb{a} \times \mathbb{b}$, with $\mathbb{a}, \mathbb{b} \in \mathbb{B}(\mathbb{R})$, such that

$$p_{W;A,B}(\mathbb{a} \times \mathbb{b}) = \mathrm{Tr}[W P^A(\mathbb{a})\, P^B(\mathbb{b})] \tag{3.12}$$

However, it is not possible to define a joint probability distribution in this way when dealing with non-commuting observables because (3.12) may not be a real number. Only when A and B commute does there exist a third observable C and two Borel functions f, g such that $A = f(C)$ and $B = g(C)$ [3.16] and hence $p_{W;A,B}$ takes the form

$$p_{W;A,B}(\mathbb{a} \times \mathbb{b}) = \mathrm{Tr}[W P^C(f^{-1}(\mathbb{a}) \cap g^{-1}(\mathbb{b})] \tag{3.13}$$

analogous to the classical joint distribution. In this way commuting observables act like random variables and their stochastic properties can be found using classical probability theory.

A better way of defining a joint distribution of two observables in the quantum setting is by taking the lead from a fact we pointed out in section 2.1, namely that even if p_f, p_g do not determine $p_{f,g}$, the distributions of $x_1 f + x_2 g$ for all $x_1, x_2 \in \mathbb{R}$ do determine $p_{f,g}$ as the unique measure on $\mathbb{B}(\mathbb{R}^2)$ which satisfies $p_{f,g}\{(y_1, y_2)\colon x_1 y_1 + x_2 y_2 \in \mathbb{a}\} = p\{w \in S\colon x_1 f(w) + x_2 g(w) \in \mathbb{a}\}$ for every $\mathbb{a} \in \mathbb{B}(\mathbb{R})$, $x \in \mathbb{R}^2$. Motivated by this fact, we give the following definition of a joint probability distribution over $\mathcal{L}(\mathcal{H})$.

Definition 3.5. ***Quantum Joint Probability Distribution***. *Let A_1 and A_2 be observables such that $x\,A = x_1 A_1 + x_2 A_2$ are self-adjoint for every $x = (x_1, x_2) \in \mathbb{R}^2$. A_1 and A_2 have a joint distribution in state W if there exists a measure p_{A_1,A_2} on $\mathbb{B}(\mathbb{R}^2)$ such that for every $\mathbb{a} \in \mathbb{B}(\mathbb{R}^2)$*

$$p_{A_1,A_2}\{y \in \mathbb{R}^2\colon x\, y \in \mathbb{a}\} = \mathrm{Tr}(W P^{xA}(\mathbb{a})) \tag{3.14}$$

There have been other proposals for definitions of joint distributions in the quantum mechanics literature [3.17] but we will henceforth only consider that given by (3.14).

3.16. [von Neumann, 1955].

3.17. [Gudder, 1968], [Margenau, 1963a], [Urbanik, 1961], [Varadarajan, 1962].

3.4.1 Joint Distributions and Commutativity

The joint distribution of two observables A_1, A_2 as defined by (3.14) need not always exist. In fact, while it always exists for compatible observables and agrees with that defined by (3.13), it does not generally exist for incompatible observables. However, the relation between the non-existence of the joint distribution of two observables and their incompatibility is subtle and depends critically on the fact that a joint distribution is defined in terms of a particular state W. In this section we present the connection in detail.

We begin with a result which was first established by [Nelson, 1967] (pp.117-119) and then reproved by [Gudder, 1979] (pp.18-19). It establishes that a pair of observables may be treated as random variables if and only if they commute.

Theorem 3.3. ***Nelson-Gudder***. *Let A_1, A_2 be self-adjoint operators on a Hilbert space $\mathcal{H}$. Then A_1, A_2 commute if and only if they have a joint distribution in every state.*

It is critical for the above bi-conditional to hold that it applies to *every* state rather than to a particular state. In effect, it is not the case that if A_1, A_2 have a joint distribution in a *particular* state W, then A_1, A_2 commute. Or contra-positively, it is false that if A_1, A_2 do not commute, then necessarily their joint distribution in state W does not exist. To emphasize, whereas if the joint distribution of two observables in a particular state W does not exist, then the observables do not commute, it is *not* true that if two observables do not commute that their joint probability distribution does not exist for any W. In symbols, where joint distribution is abbreviated as *j.d.*,

$$A_1, A_2 \text{ commute} \underset{\nLeftarrow}{\Rightarrow} \text{there exists } j.d. \text{ of } A_1, A_2 \text{ in state } W \tag{3.15}$$

or contra-positively:

$$\text{there does not exist } j.d. \text{ of } A_1, A_2 \text{ in state } W \underset{\nLeftarrow}{\Rightarrow} A_1, A_2 \text{ do not commute} \tag{3.16}$$

There have been several recent attempts in the literature to achieve a result that, while remaining valid, is as 'close' as possible to the invalid implications given in (3.15) and (3.16). The conditions for these results were originally formulated in terms of conditions for hidden variable models of quantum mechanics. However, they can be easily reformulated in terms of joint probability distributions since, as [Fine, 1982a, 1982b] proves in detail, 'the idea of deterministic hidden variables is just the idea of a suitable joint probability function.' We here present a brief overview of them in terms of joint probability distributions.

James Malley proves the following result.[3.18]

Theorem 3.4. ***Malley 2004***. *If all observables have joint distributions in every state, then all observables must commute.*

3.18. [Malley, 2004], p. 5, Theorem 2 and its proof on pp. 6-7, [Malley & Fine, 2005].

which is a generalization of theorem 3.3. Malley and Fine further proved the following stronger result.[3.19]

Theorem 3.5. ***Malley & Fine 2005***. *If a given set of observables do not commute in a particular state, then not all observables have joint distributions in that state.*

In effect, it might be the case that if A_1, A_2 do not commute, their joint distribution does exist in a particular ψ.

In 2006 Malley then showed that any attempt to augment with joint probabilities for pairs of incompatible observables leads to contradictions. In more detail, he proved that the existence of a joint distribution for all observables imply that the space of projectors collapses to a single one-dimensional projector. That is, if one insists on defining joint distributions for all observables, then there can at most be a single one-dimensional projector acting on the Hilbert space $\mathcal{H}$.

Finally, [Malley & Fletcher, 2008], in a still unpublished article, have proved an even stronger result.[3.20] Suppose a given projector pair $\{P, Q\}$ with a joint distribution in state W (note that the projectors need not be orthogonal for this to be the case). Call the pair *ortho-consistent* if $p(P = 1, Q = 1) = 0$. That is, the probability for their simultaneous occurrence is zero, but one can still ascribe them a properly defined joint distribution. (Note that every orthogonal pair of projectors is necessarily ortho-consistent.) Then, we have the following result:

Theorem 3.6. ***Malley & Fletcher 2008***. *Suppose an arbitrary nonorthogonal projector pair $\{P, Q\}$ has a properly defined joint distribution in state W. Then there exists a finitely constructible set of projectors $S = \{P, Q, R, .., Z\}$ such that*

1. *there exists at least one joint distribution for S that is consistent with that for $\{P, Q\}$;*
2. *given any joint distribution on S, consistent with that for $\{P, Q\}$, there exists at least one orthogonal pair in S that fails to be ortho-consistent.*

Given that the second result contradicts quantum mechanics – an orthogonal pair of projectors is necessarily ortho-consistent – one can conclude that even in those cases in which a non-commuting projector pair $\{P, Q\}$ can be ascribed a joint distribution in a particular W, this ascription will always result in a contradiction with the quantum mechanical probabilistic ascriptions for a finitely constructible set of projectors $S = \{P, Q, R, .., Z\}$ to which they belong. And this certainly draws us very close to the desired result, namely if P and Q do not commute then their joint distribution does not exist for *any* W.

3.19. [Malley & Fine, 2005], p.53, Theorem 1.

3.20. I here follow Arthur Fine's reformulation of their result.

Hence, the requirement that each pair of statistical variables in quantum theory have a joint distribution in every state is at variance with the most fundamental and distinctive feature of quantum theory: the use of non-commuting observables. It is not simply that joint distributions happen to be undefined; rather the fact that some joints are undefined points to *the* essential feature of the theory: without this feature, quantum theory would simply reduce to a classical probability theory.

Chapter 4
Quantum Conditional Probability

We have just seen that, because of its non-commutative structure, quantum mechanics does not assign joint probabilities to all pairs of quantum events; and that, moreover, the fact that some joints are undefined points to *the* essential feature of quantum theory. We also saw in Chapter 2 how the notion of conditional probability is standardly analyzed as the pro rata increase of a joint probability distribution. Hence, the question arises as to whether and, if so how, an appropriate notion of conditional probability can be introduced in quantum mechanics.

A long-standing literature claims that the answer to both questions is 'yes'; that it is in fact possible to define an appropriate extension of conditional probability with respect to an event in quantum mechanics, and that it is given by the probability defined by the so-called Lüders rule. This rule states that for all projectors P and Q of $\mathcal{L}(\mathcal{H})$, the probability of the quantum event represented by projector P conditional on the event represented by projector Q is given by

$$\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)} \tag{4.1}$$

In the context of quantum probability theory this rule satisfies the formal condition of specifying the only probability measure on the state space that reduces to a pro rata conditional probability for compatible events. Moreover, this formal condition is analogous to the existence and uniqueness property of classical conditional probability captured by theorem 2.1. Thus, several authors have argued for interpreting the Lüders rule as defining the appropriate notion of conditional probability in quantum mechanics.

Explicit arguments for this view are found in [Bub, 1979] and in [Cassinelli & Truni, 1979], which have then been expounded in [Cassinelli & Zanghí, 1983, 1984], [Bub, 1979a, 1979b] and [Beltrametti & Cassinelli, 1981]. Modern textbooks in the Philosophy of Quantum Mechanics presenting this view are, among others, [Hughes, 1989] and [Dickson, 1998].[4.1] These authors claim that

4.1. In addition, in his 1979 book Gudder presents it as the standard view for 'quantum conditional expectation' with references that go back to at least to 1954 with H. Umegaki's paper 'Conditional Expectations on an Operator Algebra I' *Tohoku Mathematics Journal 6*, pp. 177-181.

> '[The Lüders rule] acquires a precise meaning, in the sense of conditional probabilities, when quantum mechanics is interpreted as a generalized probability space' ([Cassinelli & Zanghí, 1983], p.245).

> 'I want to suggest that the Lüders rule is to be understood as the quantum mechanical rule for conditionalizing an initial probability assignment [...] with respect to an element in the non-Boolean possibility structure of the theory' ([Bub, 1979b], p.218)

Hence, in their view, the probabilities dictated by the Lüders rule are to be properly interpreted as the quantum conditional probabilities. Busch and Lahti in the 'Compendium of Quantum Physics' state:

> 'The Lüders rule is directly related to the notion of conditional probability in quantum mechanics, conditioning with respect to a single event.' ([Busch & Lahti, 2009], p.1)[4.2]

Another argument is also standardly invoked in favour of this same conclusion. Recall that classical conditional probability, in addition to being characterized by the existence and uniqueness theorem 2.1, is also characterized by being additive with respect to conditioning events (as theorem 2.4 shows). It turns out that the probability defined by the Lüders rule lacks this additive property and it is precisely because of this that the Lüders rule can account for the specifically quantum interference effects. In effect, when quantum conditioning is taken over a set of quantum events that mutually exclude each other, the probability defined by the Lüders rule yields the interference of probabilities that is typical of some quantum situations, as for example in the two-slit experiment we considered in the introduction. Thus, Bub writes:

> 'The natural generalization of the classical conditionalization rule appropriate to non-Boolean possibility structure is the Lüders rule. Thus, the 'paradox' involved in the two-slit experiment is resolved by showing precisely how the assumption of a non-Boolean possibility structure explains the existence of the 'anomalous' interference effects' ([Bub, 1979b], p.224).

In this chapter we present in detail the arguments in favour of interpreting the probabilities defined by the Lüders rule as conditional probabilities. We begin in sections 4.1 and 4.2 by motivating the need for an extended or generalized notion of conditional probability in quantum mechanics; to do so we consider the difficulties that arise when one attempts to define conditional probability by ratio or by a quantum analogue of ratio within the structure of quantum theory. In section 4.3 we present the argument for the conditional

4.2. The page numbering refers to the paper on the web; the book in which it appears is still unpublished.

interpretation of the probabilities defined by the Lüders rule based on its uniqueness and existence characterization. Then, in section 4.4, we show how these probabilities fail to be additive with respect to conditioning events and why this is taken to support their conditional interpretation.

4.1 No Ratio Analysis in Quantum Theory

We saw in chapter 3 that, because of its non-commutative structure, quantum mechanics does not assign joint probabilities to all pairs of quantum events; it does so for commuting pairs, but not necessarily for non-commuting ones. We also saw that the requirement that each pair of statistical variables in quantum theory have a joint distribution in every state is at variance with the most fundamental and distinctive feature of quantum theory: the use of non-commuting observables. Hence, the ratio analysis of conditional probability is wildly at odds with the most fundamental and distinctive aspects of quantum theory.

What are the implications of this failure of the ratio analysis to provide an analysis of conditional probability in quantum theory? Two different perspectives seem available. On the one hand, one can hold, as we mentioned in section 2.3.3 (and developed in appendix B), that the ratio formula is not a definition of conditional probability but an analysis of the notion and that, moreover, it is only a partially successful analysis. Hence the failure of ratio in quantum theory could be seen as a defect of the analysis itself. One could then consider whether a modified version or a different analysis might adequately capture the quantum notion of conditional probability.

On the other hand, one might argue that, regardless of ratio being a definition or an analysis of conditional probability, to demand that the same definitions or analyses of classical notions hold in quantum theory is utterly unreasonable – after all, if this were the case, quantum probability theory and, hence, quantum mechanics, would not present any novelties with respect to classical theory! And that hence, the failure of ratio in quantum theory has no implications for a quantum notion of conditional probability. One could then consider whether such a notion does in fact exist. A notion which, while being different in some aspects from its classical counterpart, is sufficiently similar to it to justify calling it an extension or a generalization of the latter.

Although both perspectives are logically possible, we focus on the latter one since we bracket the arguments against the adequacy of the ratio formula as an analysis of conditional probability. Moreover, as we will soon see, the characterizations of conditional probability given in section 2.5 – in which conditional probability was defined in accordance to the ratio analysis – will be important in evaluating whether there is a quantum notion of conditional probability. So let us see whether such a notion exists.

4.2 A Quantum Analogue of Ratio?

At first sight one might be tempted to define the conditional probability function in a quantum probability space $\langle\mathcal{H},\mathcal{L}(\mathcal{H}),W\rangle$ in strict analogy with the classical case as

$$\mathbb{P}_{p_W}(P|Q)\equiv\frac{p_W(P\wedge Q)}{p_W(Q)} \tag{4.2}$$

for all projection operators $P,Q\in\mathcal{L}(\mathcal{H})$ and $p_W(P)=\mathrm{Tr}(WP)$. For, just as the classical probability of an event A conditional on another event B is given by their joint probability – the probability of their intersection $A\cap B$ – divided by the probability of B, the quantum probability of a quantum event P conditional on another quantum event Q should be given by the probability of the event $P\wedge Q$ – the orthogonal projection onto the intersection of the ranges of P and Q – divided by the probability of Q. In this way, one would circumvent the difficulty with the non-existence of joint probability distributions – $p_W(P\wedge Q)$ is defined for all projectors in $\mathcal{L}(\mathcal{H})$ – and hope to capture the core features of conditional probability.

However, the function defined by (4.2) is not a probability measure on $\mathcal{L}(\mathcal{H})$. This is easily seen by noticing that it is not an additive function as definition 3.1 requires. Indeed, for every countable set of mutually orthogonal projection operators $\{P_i\}$ in $\mathcal{L}(\mathcal{H})$, additivity requires that

$$p(\sum_i P_i)=\sum_i p(P_i)\quad (P_i\,P_j=0\,\mathrm{if}\,i\neq j) \tag{4.3}$$

But for two orthogonal projectors P_1 and P_2, $\mathbb{P}_{p_W}$ as defined by (4.2) is not in general additive; that is,

$$\mathbb{P}_{p_W}(P_1+P_2|Q)\neq\mathbb{P}_{p_W}(P_1|Q)+\mathbb{P}_{p_W}(P_2|Q) \tag{4.4}$$

(Note, however, that since P_1 and P_2 are orthogonal, this poses no problem for their unconditional probability; that is $p(P_1\vee P_2)=p(P_1+P_2)$.) This failure of additivity can be easily seen in the following case. Let φ_1 and φ_2 be two orthogonal vectors in $\mathcal{H}$, and ψ an element of the subspace spanned by φ_1 and φ_2. Then,

$$(P_{\varphi_1}+P_{\varphi_2})\wedge P_\psi=P_\psi \tag{4.5}$$

$$P_{\varphi_i}\wedge P_\psi=0,\ \mathrm{for}\,i=1,2 \tag{4.6}$$

So that,

$$\mathbb{P}_{p_W}[(P_{\varphi_1}+P_{\varphi_2})|P_\psi]\equiv\frac{p_W[(P_{\varphi_1}+P_{\varphi_2})\wedge P_\psi]}{p_W(P_\psi)}=\frac{p_W(P_\psi)}{p_W(P_\psi)}=1 \tag{4.7}$$

$$\mathbb{P}_{p_W}(P_{\varphi_i}|P_\psi)\equiv\frac{p_W(P_{\varphi_i}\wedge P_\psi)}{p_W(P_\psi)}=0 \tag{4.8}$$

But then,

$$\mathbb{P}_{p_W}(P_{\varphi_1}+P_{\varphi_2}|P_\psi)\neq\mathbb{P}_{p_W}(P_{\varphi_1}|P_\psi)+\mathbb{P}_{p_W}(P_{\varphi_2}|P_\psi) \tag{4.9}$$

And additivity fails.

As we pointed out in section 2.1, additivity is *the* critical feature of a probability function. In effect, for an event that is made up of sub-events which have no overlap, the probability of the event must be the sum of the probabilities of the components. End of story: this is what characterizes a probability function. Moreover, this requirement seems to be fully justified for a quantum probability function since it is defined only for orthogonal events. The non-additivity problem arises because in (4.4) $\mathbb{P}_W(P|Q)$, as defined by equation (4.2), applies to a further event Q, which need not be orthogonal to P_1 and P_2. And hence, the differences that exist between Boolean and non-Boolean event structures arise. Recall that we showed in example 3.1 that $\mathcal{L}(\mathcal{H})$ is not distributive; this is precisely what precludes the function defined by equation (4.2) from being a probability function. Indeed, if Q is not orthogonal to two orthogonal events P_1 and P_2

$$(P_1+P_2)\wedge Q \neq P_1\wedge Q+P_2\wedge Q \tag{4.10}$$

and hence

$$p_W[(P_1+P_2)\wedge Q] \neq p_W(P_1\wedge Q)+p_W(P_2\wedge Q) \tag{4.11}$$

which then implies (4.9). In contrast, in a classical event structure,

$$(A_1\cup A_2)\cap B=(A_1\cap B)\cup(A_2\cap B) \tag{4.12}$$

so that the conditional probability measure, as defined by the ratio $p(A\cap B)/p(B)$, is additive.

4.3 The Lüders Rule

Hence, to define a conditional probability function in quantum theory, one needs, to begin with, a function linking events P and Q in an additive way; it will then also have to link them in a way which allows its interpretation precisely as a conditional probability. The existence and uniqueness characterization of classical conditional probability (theorem 2.1) provides the key for finding this function.

4.3.1 Existence and Uniqueness Theorem

Let us start then by defining a conditional function for projectors $P \leqslant Q$; since the sub-lattice of projectors $P \leqslant Q$, i.e. $\mathcal{L}(Q)$, is Boolean, this function will be defined analogously to the classical one. Hence, define a new probability function $m_{p_W}(P)$ over the sub-lattice $\mathcal{L}(Q)$ as

$$m_{p_W}(P)=\frac{p_W(P)}{p_W(Q)}, \qquad \text{for } P\leqslant Q \tag{4.13}$$

The measure $m_{p_W}(P)$ is the probability of the event P conditional on the event Q. Note that $m_{p_W}(\,\cdot\,)$ is defined in terms of a more general probability measure $p_W(P)$, which is in turn defined over the whole set of quantum events $\mathcal{L}(\mathcal{H})$ as $p_W(P) = \mathrm{Tr}(WP)$. (This is the only possibility as fixed by Gleason for $\dim\mathcal{H} \geqslant 3$.)

We now ask whether the function just defined can be extended to all $\mathcal{L}(\mathcal{H})$, that is, whether this new probability measure can be defined over all projectors P which are not included in Q. It just so happens that, as in the classical case, it *can* be extended. And, in addition, also in a unique way.[4.3] The following theorem is an existence and uniqueness result analogue to theorem 2.1 (A proof of it is given in appendix C.)

Theorem 4.1. ***Existence and Uniqueness***. *Let Q be any projector in the lattice $\mathcal{L}(\mathcal{H})$ of projectors of a Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) \geqslant 3$. Let $p(\,\cdot\,)$ be any probability measure on $\mathcal{L}(\mathcal{H})$, with corresponding density operator W, such that $p_W(Q) \neq 0$. For any P in $\mathcal{L}(Q)$ define*

$$m_{p_W}(P) = \frac{p_W(P)}{p_W(Q)} \tag{4.14}$$

where $p_W(P) = \mathrm{Tr}(WP)$, as fixed by Gleason's theorem. Then,

1. *$m_{p_W}(\,\cdot\,)$ is a probability measure on $\mathcal{L}(Q)$*
2. *there is an extension $\mathbb{P}_W(\,\cdot\,|Q)$ of $m_{p_W}(\,\cdot\,)$ to all $\mathcal{L}(\mathcal{H})$*
3. *the extended probability $\mathbb{P}_W(\,\cdot\,|Q)$ is unique and, for all P in $\mathcal{L}(\mathcal{H})$, is given by the density operator*

$$W_Q = \frac{Q\,W Q}{\mathrm{Tr}(Q\,W Q)} \tag{4.15}$$

so that

$$\mathbb{P}_W(P|Q) = \mathrm{Tr}(W_Q\,P) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)} \tag{4.16}$$

This expression is referred to as the Lüders rule.

This theorem tells us that if one begins with a probability measure p on the whole event space (defined through the trace rule by a density operator W), and defines a new probability function m_{p_W} for an event P whose range is included in the range of another event Q as the ratio of the probabilities of each event, then this restricted probability function can be extended to apply to *all* quantum events (one makes no restriction on the projectors to which it applies, in particular, one does not require that the range of projector P be

4.3. See [Beltrametti&Cassinelli, 1981], p.288, [Cassinelli & Zanghí, 1983], [Malley, 2004], pp.13-15.

included in the range of projector Q), this extension is unique, and it is given by the Lüders rule.

The uniqueness here derives from the uniqueness built into the trace-density operator rule of Gleason's theorem, and would apply to any probability measure on $\mathcal{L}(Q)$, not just the conditional one. Also note that the result depends on the use of Gleason's theorem to extract a density operator from the unrestricted probability measure with which we start. That is, points 2. and 3. depend critically on the initial measure $p_W(\,\cdot\,)$ defined on *all* projectors of $\mathcal{L}(\mathcal{H})$. This is how we get to use Gleason's theorem: not only to achieve uniqueness but also to define an extension to all $\mathcal{L}(\mathcal{H})$ via the density operator W that Gleason's theorem associates with the new probability measure $\mathbb{P}_W(\,\cdot\,|Q)$, namely $W_Q = \frac{Q\,W Q}{\mathrm{Tr}(Q\,W Q)}$. If p_W, and hence W, were only defined in $\mathcal{L}(Q)$, then in some cases one could not define a density operator on all $\mathcal{L}(\mathcal{H})$.[4.4]

Recall that this was not so in the classical case. Points 1. and 2. of theorem 2.2 on the extendability of classical probability measures depend *only* on the probability measure $m(\,\cdot\,)$ restricted to $\mathcal{F}(B)$ (the extended probability measure $p(C) = m(C \cap B)$ simply assigns C the probability that corresponds to that part of C which is contained in B, and zero value to the remaining part.) However, given that points 2. and 3. of theorem 4.2 depend critically on the initial measure $p_W(\cdot)$ defined on the *full* space, the quantum analogue of theorem 2.2 is false.

In addition, while in the classical case both the restricted conditional probability $m_p(A) = \frac{p(A)}{p(B)}$ and the extended probability $\mathbb{P}_p(A|B) = \frac{p(A\cap B)}{p(B)}$ are defined as a ratio of two unconditional probabilities, this is not so in the quantum case. The restricted probability function $m_{p_W}(P) = \frac{p_W(P)}{p_W(Q)}$ is defined in perfect analogy to its classical counterpart. However, the extended function $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)}$ is defined directly in terms of W since it cannot be defined as the ratio of two probabilities p_W. Indeed, when P and Q do not commute the operator $Q\,W Q$ is not a density operator and hence the quantity $\mathrm{Tr}(Q\;W\,Q\;P)$ is not a probability. Rather, $W_Q = \frac{Q\,W Q}{\mathrm{Tr}(Q\,W Q)}$ is the density operator that generates the extended probability function $\mathbb{P}_W(P|Q)$ through the trace rule $\mathrm{Tr}(W_Q\,P)$.

Finally, note that contrary to the function $\frac{p_W(P\wedge Q)}{p_W(Q)}$ considered in the previous section, the function $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)}$ *is* additive, and hence a probability function. In effect, for P_1 and P_2 orthogonal projectors and $Q \in \mathcal{L}(\mathcal{H})$, by additivity of the trace, i.e. $\mathrm{Tr}[Q\,W Q\,(P_1 + P_2)] = \mathrm{Tr}(Q\,W Q\,P_1) + \mathrm{Tr}(Q\,W Q\,P_2)$, we have

$$\mathbb{P}_W(P_1 + P_2|Q) = \mathbb{P}_W(P_1|Q) + \mathbb{P}_W(P_2|Q) \tag{4.17}$$

4.4. For example, a non-trivial probability measures having only the values 0 and 1 in $\mathcal{H}$ of dimensionality 2, cannot be extended to a higher dimensionality space since Gleason's theorem rules out all discontinuous measures for $\dim\mathcal{H} \geqslant 3$. Recall our discussion on Gleason's theorem in section 3.2.

4.3.2 Quantum Conditional Probability with Respect to an Event

The formal results of theorem 4.1 are standardly invoked to support an interpretation of the Lüders rule as defining the appropriate notion of conditional probability on the quantum event structure $\mathcal{L}(\mathcal{H})$. The reasoning given is as follows:

> '... as in the classical case, the Lüders rule gives the only probability measure that, for events $P \leqslant Q$, just involves a renormalization of the [initial] generalized probability function [p_W] given by the operator W. This offers strong grounds for regarding it as the appropriate conditionalization rule for generalized probability functions on $\mathcal{L}(\mathcal{H})$' ([Hughes, 1989], p.224, notation adapted).

Hence the claim is that the Lüders rule

> 'is the appropriate rule for conditionalizing probabilities in the non-Boolean possibility structure of quantum mechanics.' ([Bub, 1977] p.381)

The proposal is thus that the probabilities defined by the Lüders rule define the notion of conditional probability in quantum mechanics:

Definition 4.1. ***Quantum Conditional Probability with Respect to an Event****. The probability given by the Lüders rule for two quantum events $P, Q \in \mathcal{L}(\mathcal{H})$*

$$\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)} \tag{4.18}$$

is the probability of the event P conditional on event Q with respect to the probability measure p_W.

Grounds for thinking of the probabilities dictated by the Lüders rule as the natural extension of classical conditional probabilities are also taken to appear from their behavior in two special cases.[4.5] Consider first the case when P and Q are compatible. Then the Lüders rule straightforwardly defines classical conditional probabilities. Indeed if events P and Q are compatible then the corresponding projection operators P and Q commute so that $PQ = Q\,P = R$, where R projects onto the intersection of the subspaces associated with P and Q, i.e. $L_P \cap L_Q$. Inserting this into the Lüders rule, and using the invariance of the cyclic permutations of the trace operation, we obtain:

$$\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)} = \frac{\mathrm{Tr}(W R)}{\mathrm{Tr}(W Q)} = \frac{p(R)}{p(Q)} = \frac{p(L_P \cap L_Q)}{p(L_Q)} \tag{4.19}$$

which is the ratio analysis of classical conditional probability. Commutativity of the projection operators is thus a *sufficient* condition for the probability defined by the Lüders rule to be equal to classical conditional probability as given by the ratio analysis.

4.5. [Hughes, 1989] pp. 224-225 explicitly points this out.

It has also been claimed that commutativity is a necessary condition for the probability defined by the Lüders rule to be equal to classical conditional probability.[4.6] However, this does not seem correct. Indeed, given that in some cases joint distributions exist for non-commuting operators (see section 3.4), it seems possible to equate the probability defined by the Lüders rule to the classical definition of conditional probability in those cases. And in these cases, the function $\mathrm{Tr}(Q\,W Q\,P)$ would turn out (presumably) to be equivalent to the definition of quantum joint probability distribution as given by definition 3.5.

As a second case, or rather as a particular case of the previous one, consider a composite system with two components 1 and 2; the states of the composite system will be represented in the tensor-product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.[4.7] Let P_1 be a projector on $\mathcal{H}_1$ representing a quantum event associated with system 1, P_2 a projector on $\mathcal{H}_2$ representing a quantum event associated with system 2, and W the density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that represents the state of the composite system. Here P_1 and P_2 commute and the joint probability of P_1 and P_2 is given by

$$p(P_1, P_2) = \mathrm{Tr}[W\,(P_1 \otimes P_2)] \tag{4.20}$$

and the probabilities of the individual events are given by

$$p(P_1) = \mathrm{Tr}[W\,(P_1 \otimes I_2)] \quad \text{and} \quad p(P_2) = \mathrm{Tr}[W\,(I_1 \otimes P_2)] \tag{4.21}$$

One can then show[4.8] that the probabilities dictated by the Lüders rule behave exactly as in the classical case:

$$\mathbb{P}_W(P_1|P_2) = \frac{p_W(P_1, P_2)}{p_W(P_2)} \tag{4.22}$$

To finish this section, we consider the probability defined by the Lüders rule for a system in a pure state represented by the vector ψ. We show how (4.18) reduces to

$$\mathbb{P}_\psi(P|Q) = \left\langle \frac{Q\,\psi}{||Q\,\psi||}, P\,\frac{Q\,\psi}{||Q\,\psi||} \right\rangle \tag{4.23}$$

where writing $\psi_Q = \frac{Q\,\psi}{||Q\,\psi||}$, we have

$$\mathbb{P}_\psi(P|Q) = p_{\psi_Q}(P) = \langle \psi_Q, P\psi_Q \rangle \tag{4.24}$$

Indeed, if the initial state of the system is in a pure state ψ, then (4.18) yields $W_Q = \frac{Q\,P_\psi\,Q}{\mathrm{Tr}(Q\,P_\psi\,Q)}$. Now, for any vector ϕ

$$(Q\,P_\psi\,Q)\,\phi = Q\,P_\psi\,(Q\,\phi) = Q\,\psi\,\langle \psi, Q\,\phi \rangle = Q\,\psi\,\langle Q\,\psi, \phi \rangle = P_{Q\psi}\,\phi = P_{\psi'_Q}\,\phi \tag{4.25}$$

with $\psi'_Q = Q\,\psi$. In addition,

$$\mathrm{Tr}(Q\,P_\psi\,Q) = \mathrm{Tr}(P_{Q\psi}) = \sum_j \langle \phi_j, P_{Q\psi}\,\phi_j \rangle = \langle Q\psi, Q\,\psi \rangle = \|Q\,\psi\|^2 = \|\psi'_Q\|^2 \tag{4.26}$$

4.6. [Butterfield, 1987] p.219

4.7. For the quantum formalism for composite systems see, for example, [Ballentine, 1998].

4.8. $\mathbb{P}(P_1|P_2) = \frac{\mathrm{Tr}[(I_1 \otimes P_2) W\,(I_1 \otimes P_2)\,(P_1 \otimes I_2)]}{\mathrm{Tr}[W\,(I_1 \otimes P_2)]} = \frac{\mathrm{Tr}[W\,(I_1 \otimes P_2)\,(I_1 \otimes P_2)\,(P_1 \otimes I_2)]}{\mathrm{Tr}[W\,(I_1 \otimes P_2)]} = \frac{\mathrm{Tr}[W\,(P_1 \otimes P_2)]}{\mathrm{Tr}[W\,(I_1 \otimes P_2)]} = \frac{p(P_1, P_2)}{p(P_2)}$

Hence, $W_Q = P_{\psi'_Q}/\|\psi'_Q\|^2$. This expression can be further simplified. For any vector ϕ

$$W_Q\,\phi = \frac{P_{\psi'_Q}}{\|\psi'_Q\|^2}\,\phi = \frac{\psi'_Q}{\|\psi'_Q\|}\left\langle \frac{\psi'_Q}{\|\psi'_Q\|}, \phi\right\rangle = \psi_Q\,\langle\psi_Q,\phi\rangle = P_{\psi_Q}\,\phi \text{ with } \psi_Q = \frac{\psi'_Q}{\|\psi'_Q\|} \tag{4.27}$$

So $\mathrm{Tr}(W_Q\,P) = \mathrm{Tr}(P_{\psi_Q}\,P)$. But $\mathrm{Tr}(P_{\psi_Q}\,P) = \langle\psi_Q, P\psi_Q\rangle$ – see (3.11) – and hence (4.24).

Definition 4.2. ***Quantum Conditional Probability with Respect to an Event for Pure States***. *The probability given by the Lüders rule for two quantum events $P, Q \in \mathcal{L}(\mathcal{H})$*

$$\mathbb{P}_\psi(P|Q) = \left\langle \frac{Q\,\psi}{||Q\,\psi||}, P\,\frac{Q\,\psi}{||Q\,\psi||}\right\rangle \tag{4.28}$$

is the probability of the event P conditional on event Q with respect to the probability measure p_ψ given by a pure state ψ.

As before, if $W = P_\psi$, for P_1 and P_2 orthogonal projectors and $Q \in \mathcal{L}(\mathcal{H})$, additivity holds for $\mathbb{P}_\psi(P|Q)$. By linearity of the scalar product, i.e. $\langle\psi_Q, (P_1 + P_2)\ \psi_Q\rangle = \langle\psi_Q, P_1\,\psi_Q\rangle + \langle\psi_Q, P_2\,\psi_Q\rangle$, we have

$$\mathbb{P}_\psi(P_1 + P_2|Q) = \mathbb{P}_\psi(P_1|Q) + \mathbb{P}_\psi(P_2|Q) \tag{4.29}$$

4.4 Non-Additivity and Interference

In section 2.4 we showed that classical conditional probability with respect to an event is characterized both by an existence and uniqueness theorem – theorem 2.1 – and by its being additive with respect to conditioning events – theorems 2.3 and 2.4. In the previous section we saw that the quantum analogue of theorem 2.1, namely theorem 4.1, serves to characterize quantum conditional probability with respect to an event. However, trouble arises if we try to carry the additivity characterization to a quantum probability space.

Indeed, let $\{Q_i\}$ be a countable orthogonal set of elements of $\mathcal{L}(\mathcal{H})$ such that $\sum_i Q_i = Q$, that is, event Q is split into a set of physical events that mutually exclude each other. Then, unlike the classical case, there is not a unique probability measure on $\mathcal{L}(\mathcal{H})$ which coincides with $\mathbb{P}_W(\,\cdot\,|Q)$ for each Q_i. Only if the density operator W commutes with each Q_i, can one single out quantum conditional probabilities that behave classically with respect to the splittings of the conditioning event.[4.9] And hence the probabilities defined by the Lüders rule are in general non-additive when we consider conditioning with respect to an event that is the sum of orthogonal events. That is,

$$\mathbb{P}_W(P|Q) \neq \sum_i \mathbb{P}_W(P|Q_i) \tag{4.30}$$

4.9. [Cassinelli & Zanghí, 1984] theorem 2, p.144.

However, defenders of the conditional interpretation of the probabilities defined by the Lüders rule do not see any problem in this non-additive feature. On the contrary, they hold that quantum conditional probabilities differ from their classical counterparts exactly in the way they should. For precisely because of their non-additive character one can replace classical conditional probabilities by quantum conditional probabilities and obtain the quantum statistics. Indeed, in (4.30) there appears an extra term which is responsible for the non-additivity, and which precisely yields the interference of probabilities that is typical of some quantum situations in which the occurrence of the event Q is not drawn back to the occurrence of the single events Q_i that compose it.

Suppose, for simplicity, that the quantum system is the pure state $W = P_\psi$, with $\psi \in \mathcal{H}$, and let $Q = \sum_i Q_i$, with $\|Q_i\|^2 \neq 0$ for all i and $Q_i Q_j = 0$, if $i \neq j$. For a pure state, the probability of event P conditional on Q is given by definition 4.2. Setting $\frac{Q_i \psi}{\|Q_i \psi\|} = \psi_{Q_i}$, a straightforward calculation yields:

$$\mathbb{P}_W(P|\sum_i Q_i) = \sum_i \left(\frac{\|Q_i \psi\|}{\|Q \psi\|}\right)^2 \mathbb{P}_W(P|Q_i) + \sum_{i \neq j} \frac{\|Q_i \psi\| \, \|Q_j \psi\|}{\|Q \psi\|^2} \langle \psi_{Q_i}, P \psi_{Q_j} \rangle \tag{4.31}$$

Contrary to the classical case, expression (4.31) says that the conditioned state $\mathbb{P}_W(P| \sum_i Q_i)$ is not a mixture of the probability measures $\mathbb{P}_W(P|Q_i)$. Rather $\mathbb{P}_W(P| \sum_i Q_i)$ is the sum of two parts: the first part contains the diagonal terms and is the exact transcription of the classical form (2.27); the second part contains the off-diagonal terms, which are the typical quantum interference terms, and is responsible of the fact that the state $\mathbb{P}_W(P| \sum_i Q_i)$ is not a mixture. Note that for the interference term to be zero P and Q have to commute in state ψ. If this is the case, then $P Q_i \psi = Q_i P \psi$ and there exists a common basis of eigenvectors for Q and P so that $\left\langle \psi_{Q_i}, P \psi_{Q_j} \right\rangle = 0$.[4.10]

To conclude, quantum conditioning, when conditioning is taken over the orthogonal decomposition of the conditioning event, yields interference terms, thus sharply distinguishing it from classical conditioning. Precisely because of their non-additive character one can replace classical conditional probabilities by quantum conditional probabilities and obtain the quantum statistics. Thus Cassinelli and Zanghí write:

> '... the generalized conditional probability maintains all the characterizing features of the classical one and, at the same time, it introduces typical quantum effects. The essential point is that, in the non-commutative case the «theorem of compound probabilities» [or, equivalently, additivity with respect to conditioning events,] does not hold' ([Cassinelli & Zanghí, 1984], p.244)

4.10. If P and Q commute then $\left\langle \psi_{Q_i}, P \psi_{Q_j} \right\rangle \sim \langle Q_i \psi, P Q_j \psi \rangle = \langle Q_i \psi, Q_j P \psi \rangle$, and there exists a common basis of eigenvectors for Q and P so that $P\psi \sim \psi$. Thus $\langle Q_i \psi, Q_j P \psi \rangle \sim \langle Q_i \psi, Q_j \psi \rangle = 0$ for $i \neq j$.

In the next subsection, we present two concrete physical examples of how the Lüders rule yields the correct probabilistic predictions when interference effects are present.[4.11]

4.4.1 Stern-Gerlach Series Experiment

Consider a spin-1 particle and two Stern-Gerlach devices that separate the possible values of the spin component, viz. $-1,0,1$, along the x- and y- axis, as given in figure 4.1.

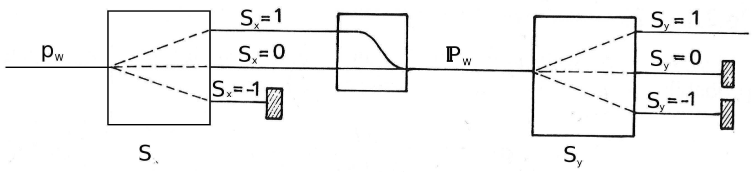

Figure 4.1. A Stern-Gerlach series experiment with interference.

Let Q be the event 'the x-component is 0 or 1' and let P be the event 'the y-component is $+1$'. We have $Q=Q_1+Q_2$, where Q_1 and Q_2 are the events 'the x-component is $+1$' and 'the x-component is 0', respectively, i.e. $Q_1=S_{x+1}$ and $Q_2=S_{x0}$, and $P=S_{y+1}$. Then

$$\mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}} + P_{s_{x0}}) = \left(\frac{\|P_{s_{x+1}}\psi\|}{\|(P_{s_{x+1}}+P_{s_{x0}})\,\psi\|}\right)^2 \mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}}) + \left(\frac{\|P_{s_{x0}}\psi\|}{\|(P_{s_{x+1}}+P_{s_{x0}})\,\psi\|}\right)^2 \mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x0}}) + \frac{\|P_{s_{x+1}}\psi\|\,\|P_{s_{x0}}\psi\|}{\|(P_{s_{x+1}}+P_{s_{x0}})\,\psi\|^2}\,\mathrm{Re}\left\langle\frac{P_{s_{x+1}}\psi}{\|P_{s_{x+1}}\psi\|}, P_{s_{y+1}}\frac{P_{s_{x0}}\psi}{\|P_{s_{x0}}\psi\|}\right\rangle \tag{4.32}$$

gives precisely the empirical probability of getting the system in the S_{y+1} channel after having passed through the S_x Stern-Gerlach device. Thus, $\mathbb{P}_\psi(P|Q)=\mathbb{P}_\psi(S_{y+1}|S_{x+1}+S_{x0})$ is interpreted as the probability that 'the y-component of spin is $+1$' *conditional* on the event that 'the x-component of spin is 0 or 1'. Quantum conditioning as defined by the Lüders rule thus yields the appropriate interference terms and allows one to replace classical conditional probabilities by quantum conditional probabilities.

Note, however, that if one were to draw back the occurrence of event Q to the occurrence of the single events Q_i that compose it, the interference terms would vanish. Indeed, imagine that the channels emerging from the first Stern-Gerlach apparatus are made totally independent. That is, consider the experiment given in figure 4.2.

4.11. [Beltrametti & Cassinelli, 1981] pp. 281-285.

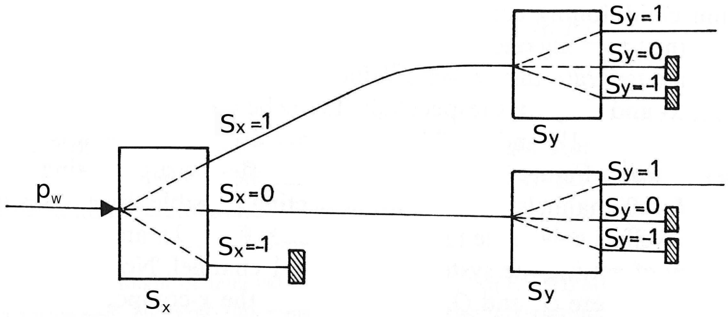

Figure 4.2. A Stern-Gerlach series experiment with no interference.

Then only the first part of (4.32), that is, the quantum transcription of the classical conditional probability (2.27), would result. In effect

$$\mathbb{P}_\psi(S_{y+1}|S_{x+1} + S_{x0}) = \left(\frac{\|S_{x+1}\psi\|}{\|(S_{x+1}+S_{x0})\psi\|}\right)^2 \mathbb{P}_\psi(S_{y+1}|S_{x+1}) + \left(\frac{\|S_{x0}\,\psi\|}{\|(S_{x+1}+S_{x0})\,\psi\|}\right)^2 \mathbb{P}_\psi(S_{y+1}|S_{x0}) \tag{4.33}$$

gives the probability of getting the particle in the $S_{y_{+1}}$ channel in this experiment. And hence (4.33) gives the probability that 'the y-component of spin is $+1$' *conditional* on the event that 'the x-component of spin is 0' or 'the x-component of spin is 1'.

4.4.2 The Double Slit Experiment

The double-slit experiment provides another physical example of such a situation. In this experiment the event with respect to which conditioning occurs is the passage of the particle through the double slit and no attempt is made to verify which slit the particle has passed through. Let us consider again this experiment. Recall that we showed in the introduction how an analysis in terms of pro rata conditional probabilities does not provide an adequate description of the experiment if the two slits are open. Indeed, it yields that

$$\mathbb{P}_p(R|A\cup B) = \frac{1}{2}\,\mathbb{P}_p(R|A) + \frac{1}{2}\,\mathbb{P}_p(R|B) \tag{4.34}$$

(for $p(A) = p(B)$ corresponding to the most simple experimental arrangement). In contrast, an analysis in terms of the Lüders rule does yield the correct probabilistic predictions.

Consider first the experiment with only one slit open. Imagine a free particle traveling toward the S_1 screen in the direction of the x-axis with constant velocity $\boldsymbol{v}$, which then reaches the detecting screen S_2 (see figure 4.3). A common simplifying approximation is to treat classically the motion along the x-axis. Let $t = 0$ the instant at which the particle reaches S_1, and $t = \tau$ the instant at which it reaches S_2; adopt an inertial frame of reference in which the particle has no velocity along the x-axis. Thus we have just to consider the position along the y axis.

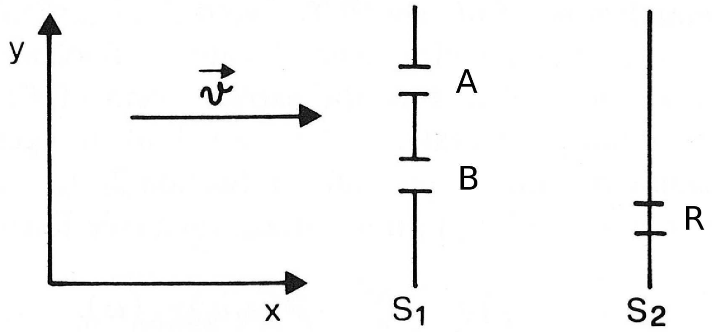

Figure 4.3. Reference frame for double slit experiment.

The conditional probability that the y-coordinate of the particle has a value in the (Borel) set R on the screen S_2 at time $t=\tau$, given that it was localized in the interval A on the screen S_1 at time $t=0$, is given by the Lüders rule as

$$\mathbb{P}_W(P_R, t=\tau | P_A, t=0) \tag{4.35}$$

where W is the state of the incoming particle. We suppose that W is a pure state with density operator P_ψ, and that the y-coordinate of the particle has a non-vanishing probability of having a value in A at time $t=0$:

$$\langle \psi, P_A \psi \rangle = \|P_A \psi\|^2 \neq 0 \tag{4.36}$$

Using the expression for quantum conditional probability for pure states given by definition 4.2, we get:[4.12]

$$\mathbb{P}_\psi(P_R, t=\tau | P_A, t=0) = \langle U_\tau \psi_A\,, P_R U_\tau \psi_A \rangle = \int_R |U_\tau \psi_A(y)|^2 \, d\, y \tag{4.37}$$

where U_τ is the free evolution operator in $\mathcal{L}(\mathcal{H})$ from $t=0$ to $t=\tau$ (i.e. from screen S_1 to S_2), and $\psi_A = \frac{P_A \psi}{\|P_A \psi\|}$. Equation (4.37) gives the probability that the particle arrives at the the region R on the detecting screen when only slit A is open. The claim is that $\mathbb{P}_\psi(P_R, \tau | P_A, 0)$ is the probability that the particle arrives at R on S_2 conditional on localization to the range A on S_1.

Similarly, if only slit B is open, we have that

$$\mathbb{P}_\psi(P_R, t=\tau | P_B, t=0) = \langle U_\tau \psi_B\,, P_R U_\tau \psi_B \rangle = \int_R |U_\tau \psi_B(y)|^2 \, d\, y \tag{4.38}$$

4.12. $\mathbb{P}_\psi(P_R, t=\tau | P_A, t=0) = \mathrm{Tr}\Big(\Big[U_\tau \frac{P_A P_\psi P_A}{\mathrm{Tr}(P_\psi P_A)} U_\tau^{-1}\Big] P_R\Big) = \frac{\mathrm{Tr}(P_A P_\psi P_A U_\tau^{-1} P_R U_\tau)}{\mathrm{Tr}(P_\psi P_A)}$. Using that $P_A P_\psi P_A = P_{P_A \psi}$, $\mathrm{Tr}(P_\psi P_A) = \|P_A \psi\|^2$ and that $\mathrm{Tr}(P_{P_A \psi} U_\tau^{-1} P_R U_\tau) = \langle P_A \psi, U_\tau^{-1} P_R U_\tau P_A \psi \rangle$, as well as writing $\psi_A = \frac{P_A \psi}{\|P_A \psi\|}$ we get $\mathbb{P}_\psi(P_R, t=\tau | P_A, t=0) = \langle U_\tau \psi_A\,, P_R U_\tau \psi_A \rangle$.

with $\psi_B = \frac{P_B \psi}{\|P_B \psi\|}$. Equation (4.38) gives the probability that the particle arrives at the the region R on the detecting screen when only slit B is open. The claim is that $\mathbb{P}_\psi(P_R, \tau | P_B, 0)$ is the probability that the particle arrives at R on S_2 conditional on localization to the range B on S_1.

Let us now turn to the third experiment in which the two slits are open. We make the same simplifying assumptions about the motion along x; let P_ψ be the density operator of the initial pure state ψ, and let $\|P_A \psi\|^2 \neq 0$, $\|P_B \psi\|^2 \neq 0$. We are interested in the conditional probability that the y-coordinate of the particle has values in the (Borel) set R of the screen S_2 at time $t = \tau$, given that it was localized in the set $A \cup B$ of the screen S_1 at time $t = 0$. That is,

$$\mathbb{P}_\psi(P_R, t = \tau | P_{A \cup B}, t = 0) \tag{4.39}$$

Noting that $A \cap B = \emptyset$ and thus that $P_{A \cup B} = P_A + P_B$, we get[4.13]

$$\mathbb{P}_\psi(P_R, t = \tau | P_{A \cup B}, t = 0) = \int_R |U_\tau (C_A \psi_A + C_B \psi_B)|^2 \, d\, y \tag{4.40}$$

where

$$C_A = \frac{\|P_A \psi\|}{\|(P_A + P_B) \psi\|}, C_B = \frac{\|P_B \psi\|}{\|(P_A + P_B) \psi\|} \tag{4.41}$$

Equation (4.40) thus gives the probability that the particle arrives at the the region R on the detecting screen when both slits are open. Again, the claim is that $\mathbb{P}_\psi(P_R, \tau | P_{A \cup B}, 0)$ is the probability of arrival at R on S_2 conditional on localization to the range $A \cup B$ on S_1.

The probability of arrival at R with both slits open can be expressed in terms of the probabilities of arrival at R with only one slit open as

$$\mathbb{P}_\psi(P_R, \tau | P_{A \cup B}, 0) = \mathbb{P}_\psi(P_R, \tau | P_A, 0) + \mathbb{P}_\psi(P_R, \tau | P_B, 0) + I \tag{4.42}$$

where the interference term I given by

$$I = 2\, C_A\, C_B \operatorname{Re} \int_R \overline{U_\tau \psi_A(y)}\, U_\tau \psi_B(y) \, dy \tag{4.43}$$

which is different from zero if $\tau \neq 0$.

4.13. First, $\mathbb{P}_\psi(P_R, t = \tau | P_{A \cup B}, t = 0) = \mathbb{P}_\psi(P_R, t = \tau | (P_A + P_B), t = 0) = \mathrm{Tr}\Big(\Big[U_\tau \frac{(P_A + P_B) P_\psi (P_A + P_B)}{\mathrm{Tr}[P_\psi (P_A + P_B)]} U_\tau^{-1}\Big] P_R\Big) = \frac{\mathrm{Tr}[(P_A + P_B) P_\psi (P_A + P_B) U_\tau^{-1} P_R U_\tau)}{\mathrm{Tr}[P_\psi (P_A + P_B)]}$. Using that $(P_A + P_B) P_\psi (P_A + P_B) = P_{(P_A + P_B)\psi}$ and that $\mathrm{Tr}[P_\psi (P_A + P_B)] = \|(P_A + P_B)\psi\|^2$, we have $\frac{\mathrm{Tr}[(P_A + P_B) P_\psi (P_A + P_B) U_\tau^{-1} P_R U_\tau)}{\mathrm{Tr}[P_\psi (P_A + P_B)]} = \frac{\mathrm{Tr}(P_{(P_A + P_B)\psi} U_\tau^{-1} P_R U_\tau)}{\|(P_A + P_B) \psi\|^2}$, which, in turn, is equal to $\frac{\langle (P_A + P_B) \psi, U_\tau^{-1} P_R U_\tau (P_A + P_B) \psi \rangle}{\|(P_A + P_B) \psi\|^2}$. Now $\frac{(P_A + P_B) \psi}{\|(P_A + P_B) \psi\|} = \frac{\|P_A \psi\|}{\|(P_A + P_B) \psi\|} \frac{P_A \psi}{\|P_A \psi\|} + \frac{\|P_B \psi\|}{\|(P_A + P_B) \psi\|} \frac{P_B \psi}{\|P_B \psi\|} = C_A \psi_A + C_B \psi_B$, with $C_A = \frac{\|P_A \psi\|}{\|(P_A + P_B) \psi\|}$, $C_B = \frac{\|P_B \psi\|}{\|(P_A + P_B) \psi\|}$ so that $\Big\langle \frac{(P_A + P_B) \psi}{\|(P_A + P_B) \psi\|}, U_\tau^{-1} P_R U_\tau \frac{(P_A + P_B) \psi}{\|(P_A + P_B) \psi\|} \Big\rangle$ is equal to $\Big\langle C_A \psi_A + C_B \psi_B, U_\tau^{-1} P_R U_\tau (C_A \psi_A + C_B \psi_B) \Big\rangle$. And hence $\mathbb{P}_\psi(P_R, t = \tau | P_{A \cup B}, t = 0) = \Big\langle U_\tau (C_A \psi_A + C_B \psi_B), P_R U_\tau (C_A \psi_A + C_B \psi_B) \Big\rangle$.

The distribution pattern one obtains for the experiment when both slits are open, i.e. $\mathbb{P}_\psi(P_R, \tau | P_{A\cup B}, 0)$, is thus not the sum of the patterns when only one or the other slit is open, i.e. $\mathbb{P}_\psi(P_R, \tau | P_A, 0) + \mathbb{P}_\psi(P_R, \tau | P_B, 0)$, but also contains the interference term I, which is a peculiar quantum effect and is responsible for the empirical fact that the probability of finding the particle in R is not the sum of the probabilities that one would have for each slit separately. Note that the occurrence of the quantum superposition $C_A\psi_A + C_B\psi_B$ in (4.40) is here a clear consequence of the calculus of the probabilities by means of the Lüders rule, and is directly responsible for the existence of the interference term.

4.5 Conclusion

In this chapter we have considered the main arguments in favour of the interpretation of the probability given by the Lüders rule as defining conditional probability with respect to an event in quantum probability theory. The claim is that $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W\,Q\,P)}{\mathrm{Tr}(Q\,W\,Q)}$, for $P, Q \in \mathcal{L}(\mathcal{H})$, is to be interpreted as the probability of the event P conditional on event Q with respect to the initial probability measure p_W. The two main arguments for this interpretation are, first, that it is the only probability measure over the whole quantum event structure $\mathcal{L}(\mathcal{H})$ which agrees with classical conditional probabilities for compatible events. And second, that in the cases in which quantum interference effects are present, the probabilities defined by the Lüders rule differ from their classical counterparts exactly in the way they should. Precisely because of their non-additive character, one can replace classical conditional probabilities by quantum conditional probabilities and obtain the quantum statistics. In the next chapters we evaluate the validity of these arguments.

Chapter 5

Interpreting Quantum Conditional Probability I

In the previous chapter we showed why the Lüders rule is standardly taken as defining the notion of conditional probability with respect to an event in quantum probability theory. The quantum notion agrees with its classical counterpart when it applies to compatible events but differs from it when incompatible events are involved. In these cases, it cannot be interpreted as a classical conditional probability but is rather seen as providing an extension of this notion appropriate to the quantum context. As such, it presents features which are different from those of classical conditional probability.

For example, suppose we have a spin $\frac{1}{2}$ particle in a state corresponding to a positive value of spin along the z-axis, i.e. $\psi_{s_{+z}}$. Then, the probability for the event $P = P_{s_{+z}}$ corresponding to a positive value of spin along the z-axis, as given by (3.5), is one, i.e.

$$p_\psi(P_{s_{+z}}) = \langle \psi_{s_{+z}}, P_{s_{+z}} \psi_{s_{+z}} \rangle = |\langle \psi_{s_{+z}} | \psi_{s_{+z}} \rangle|^2 = 1 \tag{5.1}$$

What is the probability for this same event conditional on another event Q such that $p_W(Q) \neq 0$, say $Q = P_\phi$, with $\phi = a\, \psi_{s_{+z}} + b\, \psi_{s_{-z}}$?[5.1] Intuitively, it should also be equal to one. For since the unconditional probability of P is already one, then considering Q, where $p_W(Q) \neq 0$, should leave this value unaltered. This intuition is preserved by the classical notion of conditional probability and, in particular, is secured by the ratio analysis. Indeed, in a classical probability space, if $p(A) = 1$ then any other event B such that its intersection with A is zero – and hence such that $p(B) = 0$ – is ruled out. In all other cases – those in which $p(B) \neq 0$ – the ratio analysis yields $\mathbb{P}_p(A|B) = 1$.[5.2]

5.1. For $Q = P_\phi$, $\phi = a\, \psi_{s_{+z}} + b\, \psi_{s_{-z}}$ ($|a|^2 + |b|^2 = 1$) and $\psi = \psi_{s_{+z}}$, (3.5) yields $p_\psi(Q) = |\langle \phi | \psi_{s_{+z}} \rangle|^2 = |a|^2$. Hence, $p_\psi(Q) \neq 0$ if $a \neq 0$.

5.2. There are three possible cases:

(i) if $B \subseteq A$ then $A \cap B = B$ and $\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(B)}{p(B)} = 1$.

(ii) if $A \subset B$ then $A \cap B = A$ and $\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(A)}{p(B)}$. But given $p(A) = 1$, then $p(B) = 1$ (where the part of B that is different from A , i.e $A^c - B^c$, is assigned a zero probability). So $\mathbb{P}_p(A|B) = 1$.

(iii) if $A \cap B = C \neq 0$ then $\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(C)}{p(B)}$. But given $p(A) = 1$, then $p(C) = p(B)$. So $\mathbb{P}_p(A|B) = 1$.

However, this is not so for the probabilities defined by the Lüders rule. The probability $\mathbb{P}_{\psi}(P_{s_{+z}}|P_{\phi})$ given by definition 4.2, is not assigned the value one as the unconditional probability $p_{\psi}(P_{s_{+z}})$. Rather it can range from 0 to 1 depending on the value of a given that[5.3]

$$\mathbb{P}_{\psi}(P_{s_{+z}}|P_{\phi}) = |a|^2 \tag{5.2}$$

Moreover, the probability $\mathbb{P}_{\psi}(P_{s_{+z}}|P_{\phi})$ should seemingly not take any value different from zero, be it one or any other value. For $P_{s_{+z}}$ and P_{ϕ} seem to have nothing in common since the intersection of their ranges is zero. And hence the conditional probability of $P_{s_{+z}}$ given P_{ϕ} should be zero.

How do we then understand these new features of the quantum notion of conditional probability? That is, what does it mean to say that the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule gives the probability of the quantum event P *conditional* on the quantum event Q for incompatible events? Perhaps it may not even be possible to interpret it, after all, as a conditional probability. For the fact that it agrees with its classical counterpart in their shared domain of application, i.e. compatible events, does not necessarily guarantee that outside this domain, i.e. incompatible events, the two notions will have the same meaning; and this regardless of the fact that the probability defined by the Lüders rule is the only possible candidate for a quantum notion of conditional probability (recall theorem 4.1). Indeed, when extending concepts, it is important to keep in mind that, while there may be some similarities between the old and the extended concept, it is critical to evaluate whether these similarities can provide enough interpretive content so as to justify regarding the concept in the new domain as an extension or a generalization of the old one.

Hence, the question arises as to whether the probability defined by the Lüders rule can be interpreted as a genuine extension of the notion of classical conditional probability to the quantum context. In this chapter we consider this question and thus evaluate whether the arguments presented in the previous chapter provide enough interpretive content for its reading as a conditional probability.

In section 5.1 we argue that, if the probability defined by the Lüders rule is to be understood as a conditional probability for incompatible quantum events, it cannot rely on the classical notion of commonality in terms of subspace intersection. Rather, one needs a notion of commonality which can first, cope with the fact that $\mathbb{P}_W(P|Q)$ is in general non-zero for events P and Q such that the intersection of their ranges is zero, and second,

5.3. Given $\psi_Q = \frac{Q\,\psi}{\|Q\,\psi\|} = \frac{P_{\phi}\,\psi_{s_{+z}}}{\|P_{\phi}\,\psi_{s_{+z}}\|} = \phi$, $\mathbb{P}_{\psi}(P|Q) = \langle \psi_Q, P\psi_Q\rangle = \langle \phi, P_{s_{+z}}\phi\rangle = |\langle \phi|\psi_{s_{+z}}\rangle|^2 = |a|^2$.

determine the particular non-zero value which the Lüders rule actually assigns $\mathbb{P}_W(P|Q)$. In section 5.2 we provide such a rationale, although, we argue, the notion of conditional probability it yields – what we call 'synchronic projective quantum conditional probability' – is rather weak and counterintuitive.

Then, in sections 5.3 and 5.4, we consider whether this same rationale can be translated for the eigenvalues p and q associated with the projection operators P and Q. For, we argue, that the projective reading, however poor and unsatisfactory, is a physically adequate interpretation of the probability defined by the Lüders rule only in so far as it can underwrite a quantum notion of conditional probability in terms of the physically meaningful values p and q. We show that this is *not* possible; that is, that the mathematical notion of quantum conditional probability afforded by the projective reading – a notion which applies to mathematical projection operators P and Q – does *not* translate into a physical notion of quantum conditional probability – a notion which applies to their corresponding physical values p and q – when P and Q are incompatible projectors. And hence we conclude that the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be understood (from a physically meaningful perspective) as an extension of the notion of conditional probability to the quantum context.

5.1 A First Look I

Let us begin by considering some requirements which seem to be essential for any conditional probability function to be regarded as such. We write this conditional probability as $\mathbb{P}(A|B)$ but do not commit ourselves to the ratio analysis, the Lüders rule or any other analysis of this notion; the idea is to characterize the intuitive notion of conditional probability, irrespective of how the notion should be analyzed. These basic requirements are the following:

1. The probability of any event given itself must be one, i.e. $\mathbb{P}(A|A) = 1$.

2. The probability of the complement of any event given the event itself must be zero, i.e. $\mathbb{P}(A^c|A) = 0$.

3. The probability of a necessarily true event T (e.g. the whole event space) given any event A must be one, i.e. $\mathbb{P}(T|A) = 1$.

4. The probability of a necessarily false event F (e.g. the null event) given any event A must be zero, i.e. $\mathbb{P}(F|A) = 0$.

Alan Hájek, considering different intuitions concerning the values of conditional probabilities, writes:

> '1-4 are alike in being extreme cases, and in being not merely true but necessarily true. That is, all [conditional] probability functions should agree on them. They are non-negotiable in the strongest sense.' ([Hájek, 2008] p.4)

It is easy to show that the previous four basic requirements hold for the probabilities defined by the Lüders rule as $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(QWQP)}{\mathrm{Tr}(QWQ)}$. (Of course, they also hold for the conditional probabilities defined by the standard ratio formula.) In effect,

1. $\mathbb{P}_W(P|P) = 1$: the probability of any event P given itself is one given that $\mathrm{Tr}(PWPP) = \mathrm{Tr}(PWP)$, and hence $\mathbb{P}_W(P|P) = \frac{\mathrm{Tr}(PWP)}{\mathrm{Tr}(PWP)} = 1$ for all $P \in \mathcal{L}(\mathcal{H})$.

2. $\mathbb{P}_W(P^{\perp}|P) = 0$: the probability of the orthogonal complement $P^{\perp}$ of any event is zero given that $PP^{\perp} = 0$ and hence $\mathrm{Tr}(PWPP^{\perp}) = 0$ for all $P \in \mathcal{L}(\mathcal{H})$.

3. $\mathbb{P}_W(P_I|P) = 1$: the probability of the necessarily true event P_I, i.e. the identity event, given any event P is one given that $\mathrm{Tr}(PWPP_I) = \mathrm{Tr}(PWP)$, and hence $\mathbb{P}_W(P_I|P) = \frac{\mathrm{Tr}(PWP)}{\mathrm{Tr}(PWP)} = 1$ for all $P \in \mathcal{L}(\mathcal{H})$.

4. $\mathbb{P}_W(P_{\emptyset}|P) = 0$: the probability of the necessarily false event $P_{\emptyset}$, i.e. the null event, given any event P is zero given that $PP_{\emptyset} = 0$ and hence $\mathrm{Tr}(PWPP_{\emptyset}) = 0$ for all $P \in \mathcal{L}(\mathcal{H})$.

Note, however, the fact that these requirements hold for the quantum probabilities $\mathbb{P}_W(P|Q)$ should come as no surprise. For in all four cases the events involved are compatible and hence the quantum probabilities are identical to classical conditional probabilities. Indeed, the possible differences between classical and quantum conditional probability arise precisely for *in*compatible events, which are the distinctively quantum events. And it is in these cases that we need to consider how – or whether – quantum conditional probability can be thought of as a conditional probability.[5.4]

5.4. Note that this remark also applies to the two special cases we saw in section 4.3.2 in which the probabilities defined by Lüders rule behave like classical conditional probabilities and which (supposedly) gave additional grounds for thinking of them as conditional probabilities. Note also that that Fuchs uses the latter case in [Fuchs, 2002a] pp.34-35 to seemingly show how neatly quantum updating works in the EPR type cases. But this comes as no surprise given that the projectors are compatible: in a case of commuting operators the Lüders' rule straightforwardly becomes Bayesian updating. The challenge is when the operators don't commute! See section 7.7 for a detailed discussion of Fuchs' analysis.

So let us consider the Lüders rule for incompatible events. Take again the example of a spin $\frac{1}{2}$ particle in state $\psi_{s_{+z}}$. The quantum events $P = P_{s_{+z}}$, and $Q = P_\phi$, where $\phi = a\ \ \psi_{s_{+z}} + b\, \psi_{s_{-z}}$, are incompatible if a and b are different form zero. As before, the conditional probability of P given Q should seemingly be zero because these projectors have nothing in common given the intersection of their ranges is zero. And yet the Lüders rule yields a value which in general is different from zero, namely

$$\mathbb{P}_\psi(P|Q) = |a|^2 \tag{5.3}$$

Only if a or b are zero – in which case P and Q are compatible – will the intuitive notion of conditionality agree with the probability dictated by the Lüders rule: if $a = 0$, $Q = P_{s_{-z}}$, $P\,Q = Q\ P = 0$, and hence $\mathbb{P}_\psi(P|Q) = 0$; and if $b = 0$, $P = Q = P_{s_{+z}}$ and $\mathbb{P}_\psi(P|Q) = \mathbb{P}_\psi(P|P) = 1$.

However, note that we have so far assumed that if the intersection of the ranges of two events P and Q is zero, then these events should be regarded as having nothing in common. But to what extent is this claim really justified? Indeed, in a classical probability space two events have nothing in common if their intersection is the empty set, i.e. $A \cap B = \emptyset$, in which case their conditional probability $\mathbb{P}(A|B)$ is necessarily zero. Thus, the second basic requirement for a conditional probability function, namely $\mathbb{P}(A^c|A) = 0$, is simply a particular case of this general fact when $B = A^c$. In general, when $A \cap B = \emptyset$, the event B will be a subset of A^c, i.e $B \subseteq A^c$, and their conditional probability will be zero.

In contrast, in quantum probability theory the fact that the second basic requirement holds, namely $\mathbb{P}_W(P^\perp|P) = 0$, is *not* a particular case of a more general situation in which $\mathbb{P}_W(P|Q) = 0$ if the intersection of the ranges of P and Q is zero with $P = Q^\perp$. Rather, $\mathbb{P}_W(P|Q)$ is in general non-zero for events P and Q such that the intersection of their ranges is zero; in our example the probability given by equation (5.3) is simply a particular example of this general fact. Hence, if $\mathbb{P}_W(P|Q)$ is to be generally understood as a conditional probability, we cannot appeal to a notion of commonality tied to the intersection of the ranges of P and Q.

Similarly, consider the intuition that if the probability of an event A is one, then the probability of that same event given any other event (whose probability is not zero) must also be one. In classical probability theory this is true because $p(A) = 1$ automatically rules out any other event B that has nothing in common with A, i.e. any B such that $A \cap B = 0$. And hence $p(A|B) = 1$ for every possible B. Or equivalently, if the probability of an event A is one, events B such that $A \cap B = 0$ are not ruled out but are assigned a zero probability. And hence $\mathbb{P}(A|B) = 0$ for B such that $p(B) = 0$, and $\mathbb{P}(A|B) = 1$ for B such that $p(B) \neq 0$.[5.5]

However, again, this is not so in quantum probability theory. There can be events P which are assigned a probability one without this implying that all other events Q such that the intersection of their ranges with that of P is zero, are assigned a zero probability. As we have already seen, for a spin $\frac{1}{2}$ particle in state $\psi_{s_{+z}}$ and events $P = P_{s_{+z}}$, $Q = P_\phi$, where $\phi = a\ \psi_{s_{+z}} + b\,\psi_{s_{-z}}$, even if $p_\psi(P) = 1$ and $P \wedge Q = 0$, the probability assigned to Q, namely $p_\psi(Q) = |a|^2$, is different from zero for $a \neq 0$. One cannot, therefore, appeal to the intuition that if $p_W(P) = 1$, any event Q such that $P \wedge Q = \emptyset$ should be assigned a zero probability (on account of having nothing in common with P), and thus ensure that $\mathbb{P}_W(P|Q) = 1$ for the Q's such that $P \wedge Q \neq \emptyset$. This only holds if P is the identity operator: in this case there are no Q such that $Q \wedge P_I = \emptyset$ (because $Q \wedge P_I = Q$ for all $Q \in \mathcal{L}(\mathcal{H})$ (excluding $Q = \emptyset$)); and thus $\mathbb{P}_W(P_I|Q) = 1$, which is the third basic requirement for a conditional probability function.

To conclude, if the probability defined by the Lüders rule is to be understood as a conditional probability for general quantum events, one cannot think of a notion of commonality between projectors in terms of the intersection of their subspaces. One needs a new notion of commonality which can cope with the fact that $\mathbb{P}_W(P|Q)$ is in general non-zero for events P and Q such that the intersection of their ranges is zero. Moreover, not only does this notion need to explain why one should assign $\mathbb{P}_W(P|Q)$ a non-zero value; it should also determine the particular non-zero value which the Lüders rule actually assigns it.

As a final remark, note that the fact that a notion of commonality based on subspace intersection cannot underwrite a quantum notion of conditional probability was to be expected. Indeed, this notion corresponds to the definition of conditional probability given in (4.3) as the ratio $\frac{p_W(P \wedge Q)}{p_W(Q)}$, which we showed does not define a probability function over the quantum event structure $\mathcal{L}(\mathcal{H})$ due to its non-additive character.

We also showed that this non-additivity is a direct consequence of the non-Boolean structure of $\mathcal{L}(\mathcal{H})$. Hence, what we did in the above examples was to consider particular pairs of events P and Q which exhibit this non-Boolean character, namely incompatible events, and which thus raise difficulties for the interpretation of the ratio $\frac{p_W(P \wedge Q)}{p_W(Q)}$ as a

5.5. This reading may seem somewhat counterintuitive from a diachronic perspective. For example, if for a throw of a die the result '1' is assigned a probability one, then '1' is the case and the result '2' cannot occur. Hence, the probability of '1' given '2' might seem to make no sense. However this is not so; $\mathbb{P}(1|2) = 0$, where $p(2) = 0$, can be understood in terms of the counter-factual *had* '2' been the case – which it has not since $p(2) = 0$ – then the probability of '1' given '2' *would have* been zero. Note that the ratio analysis cannot yield this value for it is undefined if $p(B) = 0$. This is an instance of the *zero denominator problem* we considered in appendix B: contingent propositions – such as B = '2' – may be assigned probability zero, and yet it is legitimate to form conditional probabilities with them as conditionals.

conditional probability. In other words, the value we assigned to the conditional probability of the various pairs of P's and Q's appealed to intuitions based on Boolean relations between events; this value was thus necessarily in conflict with the value assigned to these pairs by the Lüders rule, a probability assignment for two events which is tailor-made for a non-Boolean event structure.

In the next section we look for a new notion of commonality between quantum events, one which is more appropriate for their non-Boolean structure, and which thus might serve to underwrite a notion of conditional probability fitting the probabilities defined by the Lüders rule.

5.2 Quantum Conditional Probability

To consider whether such a notion of commonality exists, let us begin by examining the probabilities defined by the Lüders rule in a concrete example.

Example 5.1. Incompatible Observables. Suppose a system in a generic state given by

$$\psi = c_1\,\beta_1 + c_2\,\beta_2 + c_3\,\beta_3 \tag{5.4}$$

where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, and consider two observables given by operators:

$$A = a_1\,P_{\alpha_1} + a_2\,P_{\alpha_2} + a_3\,P_{\alpha_3} \tag{5.5}$$

$$B = b_1(P_{\beta_1} + P_{\beta_3}) + b_2\,P_{\beta_2} \tag{5.6}$$

where

$$\alpha_1 = \frac{\beta_1 + \beta_2}{\sqrt{2}},\, \alpha_2 = \frac{\beta_1 - \beta_2}{\sqrt{2}},\, \alpha_3 = \beta_3 \tag{5.7}$$

Initially state ψ assigns probabilities to the quantum projectors P_{α_i} as in (3.7), i.e.

$$p_\psi(P_{\alpha_i}) = \langle \psi, P_{\alpha_i}\,\psi\rangle = |\langle \alpha_i|\psi\rangle|^2 \tag{5.8}$$

Hence,

$$p_\psi(P_{\alpha_1}) = \frac{1}{2}\,(|c_1|^2 + |c_2|^2)$$

$$p_\psi(P_{\alpha_2}) = \frac{1}{2}\,(|c_1|^2 + |c_2|^2)$$

$$p_\psi(P_{\alpha_3}) = |c_3|^2 \tag{5.9}$$

Given the e-e link, each projector P_{α_i} corresponds to a value of observable A, namely the eigenvalue a_i, and we can think of the probabilities $p_\psi(P_{\alpha_i})$ as the probabilities for each value a_i. In contrast, not every projector P_{β_i} corresponds to a possible value of observable B; rather projector $P_{\beta_1} + P_{\beta_3}$ corresponds to the value b_1 and projector P_{β_2} corresponds to the value b_2. We calculate the probabilities for these two projectors as $p_\psi(P_{\beta_i}) = \langle\psi, P_{\beta_i}\psi\rangle = |\langle\beta_i|\psi\rangle|^2$, i.e.

$$\begin{aligned} p_\psi(P_{\beta_1} + P_{\beta_3}) &= |c_1|^2 + |c_3|^2 \\ p_\psi(P_{\beta_2}) &= |c_2|^2 \end{aligned} \tag{5.10}$$

Given the e-e link, theses probabilities are, respectively, interpreted as the probability for value b_1 and for value b_2.

The probability assignments dictated by the Lüders rule for $\mathbb{P}_\psi(P_{\alpha_i}|P_{\beta_1} + P_{\beta_3})$ are given by definition 4.2 as $p_{\psi_{P_{\beta_1}+P_{\beta_3}}}(P_{\alpha_i}) = |\langle\alpha_i|\psi_{P_{\beta_1}+P_{\beta_3}}\rangle|^2$ with $\psi_{P_{\beta_1}+P_{\beta_3}} = \frac{c_1}{\sqrt{|c_1|^2+|c_3|^2}}\beta_1 + \frac{c_1}{\sqrt{|c_1|^2+|c_3|^2}}\beta_3$, i.e.

$$\begin{aligned} \mathbb{P}_\psi(P_{\alpha_1}|P_{\beta_1} + P_{\beta_3}) &= \frac{1}{2}\frac{|c_1|^2}{|c_1|^2 + |c_3|^2} \\ \mathbb{P}_\psi(P_{\alpha_2}|P_{\beta_1} + P_{\beta_3}) &= \frac{1}{2}\frac{|c_1|^2}{|c_1|^2 + |c_3|^2} \\ \mathbb{P}_\psi(P_{\alpha_3}|P_{\beta_1} + P_{\beta_3}) &= \frac{|c_3|^2}{|c_1|^2 + |c_3|^2} \end{aligned} \tag{5.11}$$

The question we want to consider is how to understand the probabilities $\mathbb{P}_\psi(P_{\alpha_i}|P_{\beta_1} + P_{\beta_3})$ given in (5.11) as the conditional probabilities of the various events P_{α_i} conditional on event $P_{\beta_1} + P_{\beta_3}$.

Let us begin with an easy case, that of $\mathbb{P}_\psi(P_{\alpha_3}|P_{\beta_1} + P_{\beta_3})$. Given that $P_{\alpha_3} \leqslant (P_{\beta_1} + P_{\beta_3})$ – recall $\alpha_3 = \beta_3$ – the probability of P_{α_3} conditional on $P_{\beta_1} + P_{\beta_3}$ is simply the pro rata increase of the initial probability of P_{α_3}. Thus, given (5.9) and (5.10), we have that

$$p[P_{\alpha_3}\,\text{given}\,(P_{\beta_1} + P_{\beta_3})] = \frac{p_\psi(P_{\alpha_3})}{p_\psi(P_{\beta_1} + P_{\beta_3})} = \frac{|c_3|^2}{|c_1|^2 + |c_3|^2} \tag{5.12}$$

This value is the same as that prescribed by the Lüders rule in (5.11). Hence $\mathbb{P}_\psi(P_{\alpha_3}|P_{\beta_1} + P_{\beta_3})$ can be straightforwardly interpreted as the conditional probability of a_3 given b_1. This was, however, to be expected, for events $(P_{\beta_1} + P_{\beta_3})$ and P_{α_3} are compatible, and hence the probabilities dictated by the Lüders rule are simply classical conditional probabilities.

Now what about $\mathbb{P}_\psi(P_{\alpha 1}|P_{\beta_1}+P_{\beta_3})$ and $\mathbb{P}_\psi(P_{\alpha 2}|P_{\beta_1}+P_{\beta_3})$? This is a trickier situation since it is not prima facie clear what either P_{α_1} or P_{α_2} have in common with $P_{\beta_1}+P_{\beta_3}$. Indeed, the classical intuitions do not work given that the quantum events $P_{\beta_1}+P_{\beta_3}$ and P_{α_1} or P_{α_2} are incompatible. If, as in the previous section, we consider that two events have nothing in common if the intersection of their ranges is zero, then, given $P_{\alpha_i}\wedge(P_{\beta_1}+P_{\beta_3})=0$ $i=1,2$, the conditional probability of P_{α_i} given $P_{\beta_1}+P_{\beta_3}$ is zero. And we know that this result does no coincide with the probability assigned by the Lüders rule.

However, P_{α_1}, where $\alpha_1=\frac{\beta_1+\beta_2}{\sqrt{2}}$, does seem to have something more in common with $P_{\beta_1}+P_{\beta_3}$ than, say, P_{β_2}, even if both $P_{\alpha_1}\wedge(P_{\beta_1}+P_{\beta_3})$ and $P_{\beta_2}\wedge(P_{\beta_1}+P_{\beta_3})$ are zero. For while P_{β_2} and $(P_{\beta_1}+P_{\beta_3})$ are orthogonal[5.6] – the vector β_2 forms a 90° angle with the β_1-β_3 plane – this is not so for P_{α_1}[5.7] – the vector $\alpha_1=\frac{\beta_1+\beta_2}{\sqrt{2}}$ forms an angle with the β_1-β_3 plane which is different from 90°. Hence, if we consider that two quantum events have nothing in common *only* if they are orthogonal (and not if the intersection of their ranges is zero), we can account for the fact that $\mathbb{P}_\psi(P_{\beta_2}|P_{\beta_1}+P_{\beta_3})=0$ and $\mathbb{P}_\psi(P_{\alpha 1}|P_{\beta_1}+P_{\beta_3})\neq 0$. Indeed, given $P_{\beta_2}(P_{\beta_1}+P_{\beta_3})=0$, P_{β_2} has nothing in common with $P_{\beta_1}+P_{\beta_3}$, and hence the conditional probability of P_{β_2} given $P_{\beta_1}+P_{\beta_3}$ is zero, in accordance with $\mathbb{P}_\psi(P_{\beta_2}|P_{\beta_1}+P_{\beta_3})=0$. And, given $P_{\alpha_1}(P_{\beta_1}+P_{\beta_3})\neq 0$, P_{α_1} does have something in common with $P_{\beta_1}+P_{\beta_3}$, and hence the conditional probability of P_{α_1} given $P_{\beta_1}+P_{\beta_3}$ is different from zero, again in accordance with $\mathbb{P}_\psi(P_{\alpha 1}|P_{\beta_1}+P_{\beta_3})\neq 0$.[5.8] And similarly for P_{α_2}.

Nevertheless, these intuitions are simply the first step in understanding the probabilities defined by the Lüders rule as conditional probabilities. We still need to find a rationale to explain why the Lüders rule assigns $\mathbb{P}_\psi(P_{\alpha 1}|P_{\beta_1}+P_{\beta_3})$ and $\mathbb{P}_\psi(P_{\alpha 2}|P_{\beta_1}+P_{\beta_3})$ precisely the value $\frac{1}{2}\frac{|c_1|^2}{|c_1|^2+|c_3|^2}$. Consider the following one. First, take as the common 'event' of events P_{α_1} and $P_{\beta_1}+P_{\beta_3}$ the 'projector' onto the (non-normalized) vector $(P_{\beta_1}+P_{\beta_3})\,\alpha_1=\frac{1}{\sqrt{2}}\beta_1$, i.e. $P_{\frac{1}{\sqrt{2}}\beta_1}$. Then assign this common 'projector' a new 'probability' by means of the state vector ψ through the standard trace rule, i.e $p_\psi(P_{\frac{1}{\sqrt{2}}\beta_1})=\mathrm{Tr}(P_\psi\ P_{\frac{1}{\sqrt{2}}\beta_1})=|\langle\frac{1}{\sqrt{2}}\ \beta_1|\psi\rangle|^2=\frac{1}{2}\ |c_1|^2$. This 'probability' can thus be read as something like the joint or common 'probability' of projectors P_{α_1} and $(P_{\beta_1}+P_{\beta_3})$. Finally, increase this number pro rata, i.e. divide it by $p_\psi(P_{\beta_1}+P_{\beta_3})=|c_1|^2+|c_3|^2$, i.e.

$$p[P_{\alpha_1}\,\text{given}\,(P_{\beta_1}+P_{\beta_3})]=\frac{1}{2}\frac{|c_1|^2}{|c_1|^2+|c_3|^2} \tag{5.13}$$

5.6. $P_{\beta_2}(P_{\beta_1}+P_{\beta_3})=P_\emptyset$

5.7. $P_{\alpha_1}(P_{\beta_1}+P_{\beta_3})=|\frac{\beta_1+\beta_2}{\sqrt{2}}\rangle\langle\frac{\beta_1+\beta_2}{\sqrt{2}}|\beta_1+\beta_3\rangle\langle\beta_1+\beta_3|=\frac{1}{\sqrt{2}}|\beta_1+\beta_2\rangle\langle\beta_1+\beta_3|\neq 0$.

5.8. Note that $p_\psi(P_{\beta_1}+P_{\beta_3})\neq 0$. Otherwise $\mathbb{P}_W(P|Q)$ would be zero.

The probability of a_1 conditional on b_1 is thus given by (5.13). Which is exactly the value assigned to $\mathbb{P}_\psi(P_{\alpha_1}|P_{\beta_1}+P_{\beta_3})$ by the Lüders rule. And similarly for P_{α_2}.

More generally, take the definition of quantum conditional probabilities 4.2. By the invariance of the trace under cyclic permutations,

$$\mathbb{P}_W(P|Q)=\frac{\mathrm{Tr}(Q\,W Q\,P)}{\mathrm{Tr}(Q\,W Q)}=\frac{\mathrm{Tr}(W Q\,P Q)}{\mathrm{Tr}(Q\,W Q)} \tag{5.14}$$

If we now substitute P by P_{α_1} and write $Q\,P_{\alpha_1}\,Q$ as $P_{Q\alpha_1}$,[5.9] i.e. the 'projector' onto the (non-normalized) vector $Q\,\alpha_1$, we obtain

$$\mathbb{P}_W(P_{\alpha_1}|Q)=\frac{\mathrm{Tr}(W P_{Q\alpha_1})}{p_W(Q)}=\frac{p'_W(P_{Q\alpha_1})}{p_W(Q)} \tag{5.15}$$

Thus, it seems that the probabilities defined by the Lüders rule can be read in general as conditional probabilities. In effect, (5.15) defines quantum conditional probability analogously to classical conditional probability as given by the ratio analysis. The 'projector' $P_{Q\alpha_1}$ represents the common quantum 'projector' of Q and P_{α_1} which corresponds to the classical event $A\cap B$. If P is not one-dimensional we cannot manipulate (5.14) to yield (5.15) and then it is the 'projector' $Q\,PQ$ which represents the common 'projector' of P and Q. This 'projector' is assigned a 'probability' value by means of the trace rule, i.e. $p'_W(Q\,PQ)$, and is then increased pro rata, i.e. divided by $p_W(Q)$.

We thus have seemingly not only explained why one should assign a non-zero conditional probability to projectors P and Q such that $P\wedge Q=0$, but also why it takes the particular non-zero value the Lüders rule assigns it. Hence, by appealing to a notion of commonality of quantum projectors based on their projective geometry, it looks like the probability defined by the Lüders rule *can* be interpreted as the probability of a quantum projector conditional on another quantum projector. Note that it is a synchronic notion.

Definition 5.1. ***Synchronic Quantum Conditional Probability***. *The probability given by the Lüders rule for two quantum projectors $P,Q\in\mathcal{L}(\mathcal{H})$*

$$\mathbb{P}_W(P|Q)=\frac{\mathrm{Tr}[W\,(Q\,PQ)]}{\mathrm{Tr}(WQ)}=\frac{p'_W(Q\,PQ)}{p_W(Q)} \tag{5.16}$$

is the probability of the projector P conditional on projector Q with respect to the probability measure p_W. The operator $Q\,PQ$ represents the common operator of projectors P and Q.

Let us pause for a moment. Does this notion really yield a notion of conditional probability? First, notice that when giving the conditional reading of $\mathbb{P}_W(P|Q)=\frac{p'_W(Q\,PQ)}{p_W(Q)}$ we say that the common quantum 'projector' $Q\,P\,Q$ is assigned a 'probability' p'_W (not p_W) by means of the trace rule. And all these quotation marks are not here by accident.

5.9. As in (4.24), for any vector ϕ: $(Q\,P_{\alpha_1}\,Q)\,\phi=Q\,P_{\alpha_1}\,(Q\,\phi)=Q\,\alpha_1\,\langle\alpha_1,Q\,\phi\rangle=Q\,\alpha_1\,\langle Q\,\alpha_1,\phi\rangle=P_{Q\,\alpha_1}\,\phi$.

For the operator $Q\,P\,Q$ is *not* a projector, and hence $p'_W(Q\,P\,Q)$ is *not* a probability function. We already pointed out this fact in section 4.3.1: $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W\,Q\,P)}{\mathrm{Tr}(Q\,W\,Q)}$ cannot be defined as the ratio of two probabilities p_W because, when P and Q do not commute, the operator $Q\,W\,Q$ is not a density operator and hence the quantity $\mathrm{Tr}(Q\,W\,Q\,P)$ is not a probability.[5.10] And second, why should we take $Q\,P\,Q$, or $P_{Q\alpha}$ when $P = P_\alpha$, as the *common* quantum 'projector'? It seems to be a blatantly ad hoc manoeuvre.

The first objection to the notion of quantum conditional probability does not seem conclusive, for the function $\mathrm{Tr}(W\,Q\,P\,Q)$ is not a probability function only in the sense that it is not normalized. Indeed, it is non-negative, it takes values which are smaller than one and, moreover, it is additive, which, as we have already emphasized, is the substantial requirement for a probability function. Hence, it does not seem so inadequate to regard the function $\mathrm{Tr}(WQ\,PQ)$ as giving the probability of $Q\,PQ$. The second objection cannot, however, be so easily dismissed.

Indeed, one could bite the bullet and simply stipulate that $Q\,PQ$ is by definition the common 'projector' of P and Q in the projective lattice $\mathcal{L}(\mathcal{H})$. This option, however, would not yield a very satisfactory notion of commonality between projectors and would thus provide a somewhat feeble notion of conditional probability. Moreover, it seems counterintuitive to regard $Q\,P\,Q$ as the common 'projector' of Q and P. For in $\mathbb{P}_W(P|Q)$ the common 'projector' is the operator $Q\,P\,Q$, while in $\mathbb{P}_W(Q|P)$ it is $P\,Q\,P$, which are in general different from each other. And yet, why should they be different if they are both supposed to represent what P and Q have in common? The operator $Q\,P\,Q$, therefore, cannot be so straightforwardly taken as the common quantum 'projector' for the probability of P conditional on Q.

And thus it seems that, after all, definition 5.1 does *not* provide an adequate notion of quantum conditional probability. Contrary to the standard view, the probabilities defined by the Lüders rule do *not* seem to acquire a precise meaning, in the sense of conditional probabilities, when quantum mechanics is interpreted as a generalized probability space.

Some caution seems to be, however, recommended. Indeed, the claim that $Q\,PQ$ does not adequately represent the common operator of P and Q, solely rests on our intuitions. And one could easily reply that these are not reliable when considering projection operators which have a non-Boolean structure. Hence, one cannot conclude that the quantum notion conditional probability as given by definition 5.1 does not provide an understanding of why the probability defined by the Lüders rule should be read as a conditional probability. It

5.10. In the case of $Q\,P\,Q = P_{Q\alpha}$ we can easily check this: $P_{Q\alpha}$ is not a projection operator given that it is not idempotent: $(P_{Q\alpha})^2 \neq P_{Q\alpha}$ since $Q\,\alpha$ is in general not a normalized vector. Indeed, $P_{Q\alpha} = |Q\,\alpha\rangle\langle Q\,\alpha|$ and thus $(P_{Q\alpha})^2 = \langle Q\,\alpha|Q\,\alpha\rangle|Q\,\alpha\rangle\langle Q\,\alpha| \neq |Q\,\alpha\rangle\langle Q\,\alpha| = P_{Q\alpha}$.

does provide one, although one which is rather poor – it needs to stipulate that the common projector of P and Q is $Q\,P\,Q$ – and counterintuitive –it takes $Q\,P\,Q$ as the common quantum 'projector' for the probability of P conditional on Q, and $PQ\,P$ as the common quantum 'projector' for the probability of Q conditional on P, something for which it provides no understanding.

5.3 No Physical Quantum Conditional Probability

Nevertheless, the crucial question still remains to be raised: does the notion of quantum conditional probability given by definition 5.1 provide a notion of conditional probability for the *values of physical quantities* of a quantum system? Indeed, it applies directly to projection operators on a Hilbert space $\mathcal{H}$ (we thus henceforth refer to this notion as synchronic *projective* quantum conditional probability) but theses are only physically meaningful through their associated eigenvalues. Hence, the projective reading, however poor and unsatisfactory, is a physically adequate interpretation of the probability defined by the Lüders rule only in so far as it can underwrite a quantum notion of conditional probability in terms of physically relevant values. That is, a reading of $\mathbb{P}_W(P|Q)$ as the probability for value p – the eigenvalue associated with P – conditional on value q – the eigenvalue associated with Q.

However, we now argue that this is *not* possible; that is, that the mathematical notion of quantum conditional probability afforded by the projective reading – a notion which applies to mathematical projection operators P and Q – does *not* translate into a physical notion of quantum conditional probability – a notion which applies to their corresponding physical values p and q – when P and Q are incompatible projectors. And hence we conclude that the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be understood, from a physically meaningful perspective, as an extension of the notion of conditional probability to the quantum context.

Consider again example 5.1. We argued that P_{α_1} has more in common with $P_{\beta_1}+P_{\beta_3}$ than P_{β_2} – even if both $P_{\alpha_1}\wedge(P_{\beta_1}+P_{\beta_3})=0$ and $P_{\beta_2}\wedge(P_{\beta_1}+P_{\beta_3})=0$ – because whereas the vector β_2 forms a 90° angle with the plane β_1-β_3, the vector $\alpha_1=\frac{\beta_1+\beta_2}{\sqrt{2}}$ forms an angle with the plane β_1-β_3 which is different from 90°. Thus the projection of β_2 onto β_1-β_3 is zero while the projection of α_1 onto β_1-β_3 is different form zero. And hence P_{β_2} is seen as having nothing in common with $P_{\beta_1}+P_{\beta_3}$, while P_{α_1} does have something in common with $P_{\beta_1}+P_{\beta_3}$. In addition, to explain why the Lüders rule assigns a particular number to the conditional probability of P given Q, it was crucial to take the 'projector' $Q\,PQ$ (P_{Q_α} if P is the one-dimensional projector P_α) as the common quantum event.

But how can these rationales, which are crucial in understanding the notion of quantum conditionality for projection operators, even get off the ground for *scalar* values such as the eigenvalues associated with these projectors? Indeed, the notion of commonality between projectors and the notion of common 'event' $Q\,PQ$ relies critically on the projective geometry of a Hilbert space $\mathcal{H}$. This geometry has a non-Boolean structure which, as Gleason's theorem dictates, determines the probabilistic structure that can be defined over it. And the problem is that, this event structure, and the ensuing quantum probabilistic structure, do not allow the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule to be interpreted as the probability of value p conditional on value q.

First, the event 'p and q' cannot always be represented in terms of the projection operators P and Q when the projectors are incompatible; that is, the projector $P\wedge Q$ cannot always be understood as the event 'p and q'. And second, in the cases in which such a correspondence does exist, the probability assigned to the event $P\wedge Q$ does not always correspond to the probability assigned to the common operator of P and Q employed by the projective notion of quantum conditional probability, i.e. $Q\,PQ$. Thus, the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be understood as the pro rata increase of the probability of 'p and q'; and hence, it cannot be interpreted as the probability of value p conditional on value q. Only if P and Q are compatible projectors do all these notions line up – 'p and q' can be represented by the projector $P\wedge Q$, which in turn is equivalent to the common operator $Q\,PQ$ – and the probability $\mathbb{P}_W(P|Q)$ can be read as the pro rata increase of 'p and q'.

Let us see this in detail by considering some examples. We start with an example involving compatible observables A and B, and we show how one can derive the probabilities defined by the Lüders rule by thinking of them as conditional probabilities for the possible values of A and B.

Example 5.2. Compatible Observables[5.11]. Let ψ be the state of a system represented by a state in a three dimensional Hilbert space $\mathcal{H}$, spanned by one-dimensional vectors α_1, α_2, α_3. Consider the operators

$$A = a_1\,P_{\alpha_1} + a_2\,P_{\alpha_2} + a_3\,P_{\alpha_3}, \text{with}\, a_1 \neq a_2 \neq a_3 \tag{5.17}$$

$$B = b_1\,P_{\alpha_1} + b_2(P_{\alpha_2} + P_{\alpha_3}), \text{with}\, b_1 \neq b_2 \tag{5.18}$$

where P_{α_i} projects onto the α_i 1-D subspace. For instance, if we are considering a spin 1 system, A could be the observable for spin in the z direction S_z, with $a_1 = 0$, $a_2 = 1$, and $a_3 = -1$; and B would then be the observable A^2, i.e. $(S_z)^2$, with $b_1 = 0$ and $b_2 = 1$. Clearly A and B are compatible, i.e. $A\,B = B\,A$.

5.11. This example is taken from [Teller, 1983], p.414.

If the initial state is given by the vector

$$\psi = c_1\,\alpha_1 + c_2\,\alpha_2 + c_3\,\alpha_3 \tag{5.19}$$

where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, the probabilities for the events P_{α_i} are given by

$$p_\psi(P_{\alpha_i}) = |c_i|^2 \tag{5.20}$$

The Lüders rule yields the following values for $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_1})$[5.12]

$$\begin{aligned} \mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_1}) &= 1\\ \mathbb{P}_\psi(P_{\alpha_2}|P_{\alpha_1}) &= 0\\ \mathbb{P}_\psi(P_{\alpha_3}|P_{\alpha_1}) &= 0 \end{aligned} \tag{5.21}$$

and $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2} + P_{\alpha_3})$[5.13]

$$\begin{aligned} \mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_2} + P_{\alpha_3}) &= 0\\ \mathbb{P}_\psi(P_{\alpha_2}|P_{\alpha_2} + P_{\alpha_3}) &= \frac{|c_2|^2}{|c_2|^2 + |c_3|^2}\\ \mathbb{P}_\psi(P_{\alpha_3}|P_{\alpha_2} + P_{\alpha_3}) &= \frac{|c_3|^2}{|c_2|^2 + |c_3|^2} \end{aligned} \tag{5.22}$$

Can these probabilities be interpreted as conditional probabilities for the physical values of A and B for a spin-1 particle? We begin with the unconditional probabilities assigned to the various a_i and b_i's. First, since the e-e link assigns each P_{α_i} the eigenvalue a_i, the probabilities $p_\psi(P_{\alpha_i})$ can be directly understood in terms of the probabilities of the possible values of observable A. And hence

$$p_\psi(P_{\alpha_i}) = p(a_i) = |c_i|^2 \tag{5.23}$$

In contrast, P_{α_2} and P_{α_3} are not each associated a particular value of B; rather the e-e links assigns the quantum event $(P_{\alpha_2} + P_{\alpha_3})$ the value b_2. Thus, the probabilities for the possible values of B are

$$\begin{aligned} p(b_1) &= p_\psi(P_{\alpha_1}) = |c_1|^2\\ p(b_2) &= p_\psi(P_{\alpha_2} + P_{\alpha_3}) = p_\psi(P_{\alpha_2}) + p_\psi(P_{\alpha_3}) = |c_2|^2 + |c_3|^2 \end{aligned} \tag{5.24}$$

Turn now to the interpretation of the probabilities $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_1})$ given in (5.21). Since A and B are compatible, it should be possible to interpret $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_1})$ as the probability that observable A takes the value a_i conditional on B taking the value b_1, i.e. $\mathbb{P}(a_i|b_1)$. This is particularly simple to see in the case of $A = S_z$, with $a_1 = 0$, $a_2 = 1$, $a_3 = -1$, and $B = (S_z)^2$, with $b_1 = 0$ and $b_2 = 1$. In effect, if the particle's squared value of spin along the

5.12. $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_1}) = p_{\psi_{P_{\alpha_1}}}(P_{\alpha_i}) = \langle\psi_{P_{\alpha_1}}, P_{\alpha_i}\,\psi_{P_{\alpha_1}}\rangle = |\langle\alpha_i|\alpha_1\rangle|^2$.

5.13. $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2} + P_{\alpha_3}) = p_{\psi_{P_{\alpha_2}+P_{\alpha_3}}}(P_{\alpha_i}) = \langle\psi_{P_{\alpha_2}+P_{\alpha_3}}, P_{\alpha_i}\,\psi_{P_{\alpha_2}+P_{\alpha_3}}\rangle = |\langle\alpha_i|\psi_{P_{\alpha_2}+P_{\alpha_3}}\rangle|^2$ with $\psi_{P_{\alpha_2}+P_{\alpha_3}} = \frac{(P_{\alpha_2} + P_{\alpha_3})\,\psi}{\|(P_{\alpha_2} + P_{\alpha_3})\,\psi\|} = \frac{c_2}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_2 + \frac{c_3}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_3$.

z-axis is zero, i.e. $b_1 = (s_{z_0})^2 = 0$, then its value of spin along the z-axis has to be zero, i.e. $a_1 = s_{z_0} = 0$, but cannot be one nor minus one, i.e. not $a_2 = s_{z_{+1}} = +1$ nor $a_3 = s_{z_{-1}} = -1$. The probability of a_1 conditional on b_1 should thus be one, i.e.

$$p(a_1 \,\text{given}\, b_1) = 1 \tag{5.25}$$

which is precisely the value the Lüders rule assigns $\mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_1})$. And the probability of a_2 conditional on b_1, and of a_3 conditional on b_1, should be zero, again in accordance with the value the Lüders rule assigns $\mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_2})$ and $\mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_3})$.

Consider next the probabilities $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2} + P_{\alpha_3})$ given in (5.22). Again, since A and B are compatible, it should be possible to interpret $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2} + P_{\alpha_3})$ as the probability that observable A takes the value a_i conditional on B taking the value b_2, i.e. $\mathbb{P}(a_i|b_2)$. In effect, using the spin example, if the particle's squared value of spin along the z-axis is one, i.e. $b_2 = (s_z)^2 = 1$, then its value of spin along the z-axis can either be one or minus one, i.e. $a_2 = s_{z_{+1}}$ and $a_3 = s_{z_{-1}}$, but not zero, i.e. not $a_1 = s_{z_0}$. The probability of a_1 conditional on b_2 should thus be zero

$$p(a_1 \,\text{given}\, b_2) = 0 \tag{5.26}$$

which is precisely the value the Lüders rule assigns $\mathbb{P}_\psi(P_{\alpha_1}|P_{\alpha_2} + P_{\alpha_3})$. In contrast, the probability of a_2 conditional on b_1, and that of a_3 conditional on b_1, should not be zero; rather their conditional-on-b_2 probability should simply be the pro rata increase of their unconditional probability. That is,

$$\begin{aligned} p(a_2 \,\text{given}\, b_2) &= \frac{p_\psi(a_2)}{p_\psi(b_2)} = \frac{|c_2|^2}{|c_2|^2 + |c_3|^2} \\ p(a_3 \,\text{given}\, b_2) &= \frac{p_\psi(a_3)}{p_\psi(b_2)} = \frac{|c_3|^2}{|c_2|^2 + |c_3|^2} \end{aligned} \tag{5.27}$$

Again these values are the same as those assigned by the Lüders rule to $\mathbb{P}_\psi(P_{\alpha_2}|P_{\alpha_2} + P_{\alpha_3})$ and $\mathbb{P}_\psi(P_{\alpha_3}|P_{\alpha_2} + P_{\alpha_3})$.

Hence, for compatible observables A and B, one can derive the probabilities defined by the Lüders rule by thinking of them as conditional probabilities for the possible values of A and B. As we pointed out in section 3.4, commuting observables act like random variables whose stochastic properties can be found using classical probability theory. The probability for the joint occurrence of values a_i and b_j, i.e. $p(a_i \cap b_j)$, corresponds to the probability for the corresponding projection operators, i.e. $p(P_{a_i} \wedge P_{b_j})$; and, for commuting projectors, $P_{a_i} \wedge P_{b_j}$ is simply the product $P_{a_i} P_{b_j}$, which is also equal to the common quantum event employed by the Lüders rule, i.e. $P_{b_j} P_{a_i} P_{b_j}$. Thus, the probabilities defined by the Lüders rule are straightforwardly classical conditional probabilities.

Turn now to consider incompatible observables; in this case, one *cannot* derive the probabilities defined by the Lüders rule by thinking of them as conditional probabilities for the possible values of A and B. Consider, for instance, in example 5.1, the probability $\mathbb{P}_W(P_{\alpha_i}|P_{\beta_1} + P_{\beta_3})$. Even if the relation '$a_i$ and b_1' can be understood as the relation of conjunction between the associated projectors of these eigenvalues, i.e. $P_{\alpha_i} \wedge (P_{\beta_1} + P_{\beta_3})$ (we elaborate on this in section 5.4.2), the probability assigned to this event does not correspond to the 'probability' assigned to the common operator employed by the projective notion of quantum conditional probability, i.e. $(P_{\beta_1} + P_{\beta_3})\, P_{\alpha_i}\, (P_{\beta_1} + P_{\beta_3})$.

In effect, the probability of the projector associated with 'a_i and b_1', i.e. $\mathrm{Tr}[P_{\alpha_i} \wedge (P_{\beta_1} + P_{\beta_3})]$, is not equivalent to the 'probability' of the common projector employed by the Lüders rule, i.e. $\mathrm{Tr}[(P_{\beta_1} + P_{\beta_3})\, P_{\alpha_i}\, (P_{\beta_1} + P_{\beta_3})]$. This is due to the fact that, since P_{α_i} does not commute with P_{β_1} nor with P_{β_3}, the projector $P_{\alpha_i} \wedge (P_{\beta_1} + P_{\beta_3})$ is not equal to the projector $P_{\alpha_i}\, P_{\beta_1} + P_{\alpha_i}\, P_{\beta_3}$ (for $i = 1, 2$); and thus, its trace is not equal to $\mathrm{Tr}[P_{\alpha_i} P_{\beta_1} + P_{\alpha_i} P_{\beta_3}]$, which is precisely the 'probability' of the common projector employed by the Lüders rule. Indeed, by the cyclic property of the trace and the fact that the P_{β_i}'s are orthogonal, $\mathrm{Tr}[(P_{\beta_1} + P_{\beta_3})\, P_{\alpha_i}\, (P_{\beta_1} + P_{\beta_3})] = \mathrm{Tr}[P_{\beta_1} P_{\alpha_i} P_{\beta_1} + P_{\beta_1} P_{\alpha_i} P_{\beta_3} + P_{\beta_3} P_{\alpha_i} P_{\beta_1} + P_{\beta_3} P_{\alpha_i} P_{\beta_3}] = \mathrm{Tr}[P_{\alpha_i} P_{\beta_1}^2 + P_{\alpha_i} P_{\beta_3}^2] = \mathrm{Tr}[P_{\alpha_i} P_{\beta_1} + P_{\alpha_i} P_{\beta_3}]$. Thus, $\mathbb{P}_W(P_{\alpha_i}|P_{\beta_1} + P_{\beta_3})$ cannot be understood as the pro rata increase of the probability of a_i and b_2, and, therefore, it cannot be interpreted as the probability of value a_i conditional on b_2.

Moreover, sometimes the projector $P \wedge Q$ cannot even be understood as (the standard) conjunction of the associated eigenvalues, i.e. 'p and q', let alone be equal to the common operator $Q\,P\,Q$. For instance, as in section 4.4, consider the case in which Q is the sum of two orthogonal projection operators Q_1 and Q_2, i.e. $Q = Q_1 + Q_2$, where both Q_1 and Q_2 are associated distinct eigenvalues q_1 and q_2. Then, it turns out that projector $P \wedge Q = P \wedge (Q_1 + Q_2)$, although (somehow) understandable as the event 'p and q_1 or q_2', cannot be read as the event 'p and q_1 or p and q_2' (we elaborate on this in section 5.4.2). Thus, when projector Q is decomposed into the sum of the orthogonal projectors Q_i, i.e. $Q = \sum_i Q_i$, or, in other words, the event represented by Q is split into a set of physical events that mutually exclude each other, the situation is even worse for the interpretation of the probability $\mathbb{P}_W(P|Q)$ as a conditional probability in terms of physical values.

To conclude, the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be read as the probability of the physical value p conditional on the physical value q associated with projectors P and Q (for P and Q incompatible events). It can only be interpreted as a conditional probability at a formal or mathematical level for projection operators – and then, only under a weak and counterintuitive construal of such a notion. To assign a number to two values of incompatible quantities one has to introduce a special rule such as the Lüders rule. One might then call this an 'extension' of conditional probability, but it is really a *different* concept that bears no resemblance with conditionality at any level that is not purely formal.

The uniqueness argument of section 4.3 can only underwrite this formal notion, but is mute as to its interpretation for physical values. Even if the probabilities defined by the Lüders rule are co-extensive with classical conditional probabilities in their shared domain of application, this formal argument does not provide any interpretive content so as to justify regarding them as an extension or a generalization of classical conditional probabilities to the quantum context at a physical level. Thus, although the premises of the uniqueness argument are correct, its interpretive conclusion only follows at a formal level. That is, it is a valid argument at a formal level – although with important reservations for the formal notion it secures is rather poor and counterintuitive – but is *in*valid at physically meaningful level.

Similarly, the argument of section 4.4 based on the non-additive character of the probabilities defined by the Lüders rule, can also only work at a formal level. In addition, precisely because of this non-additivity with respect to conditioning events, the event 'p and q_1 or q_2' cannot be represented in terms of the projection operators P and Q as the projector $P \wedge (Q_1 + Q_2)$, thus making their interpretation as conditional-on-physical-values probabilities even more difficult. (Because not only is $P \wedge (Q_1 + Q_2)$ not equivalent to the common projector employed by the Lüders rule, but $P \wedge (Q_1 + Q_2)$ cannot even be interpreted as p and q_1 or q_2).

5.4 Disengaging Formal and Interpretive Features

Our claim that the probabilities defined by the Lüders rule cannot be interpreted as conditional probabilities for physical values relies crucially on the premise that formal and interpretive aspects need to be kept distinct when considering the meaning or interpretation of a concept. And hence, the claim that the projective reading, even if poor and counterintuitive, is a physically adequate interpretation of the probability defined by the Lüders rule as a conditional probability *only* in so far as it translates to a reading in terms of physically relevant values. In general, even if formal features are a good guide when extending established concepts to new contexts, formal features *alone* can never justify that the formally extended concept is also conceptually extending the notion to the new domain.

This need to disengage formal and interpretive features is even more critical when extending concepts to the quantum domain. We here consider the case of the so-called 'logic' of quantum mechanics which provides a particularly illuminating illustration of this fact. Similarly to quantum 'conditional probability', although the 'logical' relations between quantum events are in many ways formally analogue to ordinary logical relations between classical events, their meaning is so different from that of the latter, that quantum 'logical' notions *cannot* be interpreted as extensions of our ordinary (physically meaningful) logical notions. This will further shed light on the difficulties to interpret the probabilities defined by the Lüders rule as quantum conditional probabilities.

5.4.1 Quantum Logic

As we saw in chapter 3, in the quantum event structure $\mathcal{L}(\mathcal{H})$ one can define algebraic relations between the projection operators representing quantum events which are the counterparts of set-union, set-intersection, and set-complementation. These are, respectively, the orthogonal projection onto the closed subspace spanned by the ranges of P and Q, i.e. $P \vee Q$; the orthogonal projection onto the intersection of the ranges of P and Q, i.e. $P \wedge Q$; and the orthogonal projection onto the complement of the closed subspace spanned by the range of P, i.e. $P^{\perp} = I - P$. Many of the relations between the quantum algebraic relations are similar to those between the classical one. For example, just as $(A^c)^c = A$ holds for set-complementation, $(P^{\perp})^{\perp} = P$ holds for subspace-complementation; or just as $A \cup A^c = S$ in $\mathcal{F}(S)$, $P \vee P^{\perp} = I$ in $\mathcal{L}(\mathcal{H})$; or similarly to $(A \cap A^c)^c = S$, we have $(P \wedge P^{\perp})^{\perp} = I$.

Now, in classical logic, the logical relations between events (or propositions representing those events) correspond naturally to the algebraic relations between the subsets that represent those events. To the *disjunction* of events 'or', there corresponds the set-union $(A \cup B)$; to the *conjunction* of events 'and', there corresponds the set-intersection $(A \cap B)$; and to the *negation* of events 'not', there corresponds the set-complementation (A^c). The suggestion is, thus, that, given the similarities between the classical and the quantum algebraic relations, the algebraic relations in $\mathcal{L}(\mathcal{H})$ correspond to 'quantum logical relations', where it is assumed that these provide some kind of extension of our ordinary logical notions in the quantum context. Thus, the algebraic relation '$\vee$' between quantum projectors is interpreted as the quantum logical 'or' for quantum events. Similarly, the algebraic relation '$\wedge$' is taken to correspond to the quantum logical 'and', and '$\perp$' is read as the quantum logical 'not'.[5.14]

5.14. [Putnam, 1969]

However, just as with quantum 'conditional probability', these analogies turn out to hold only at a purely formal level. That is, the algebraic relations ' $\vee$ ', ' $\wedge$ ', and ' $\perp$ ' for quantum projectors cannot be understood, respectively, as generalized or extended notions of the ordinary logical concepts of disjunction ('or'), conjunction ('and') – when the algebraic relation ' $\wedge$ ' appears in an expression along with relations ' $\vee$ ' and/or ' $\perp$ ' – and negation ('not') from any physically meaningful perspective. And hence, for example, even if formally one has $(P^{\perp})^{\perp} = P$, one should not interpret this equation as double negation; nor should one interpret $P \vee P^{\perp}$ or $(P \wedge P^{\perp})^{\perp}$ as the logical laws of excluded middle and non-contradiction respectively.

5.4.1.1 A Toy Model

[Arthur Fine, 1972] argues for this conclusion by constructing the analogue to quantum logic for a simple, two-dimensional system. In this logic it is clear that the meaning of the algebraic relations differs substantially from the meaning of the ordinary logical relations. And hence, the conclusion that the former cannot be regarded as extensions of the latter. These conclusions then carry over to quantum logic. Let us consider this toy model.

Consider the location of a certain point P on a given circle C. Suppose that for the location of P there are three accessible regions: (1) the center of the circle, (2) the entire area of the circle and (3) any diameter of the circle. Any sentence of the form 'P is on X', where X is one of the accessible regions, corresponds to an elementary sentence. The idea is to construct a logic from the elementary sentences by introducing sentential connectives and truth conditions.

Let us first introduce the binary connective ' $\wedge$ ' such that for elementary sentences 'P is on X' and 'P is on Y', the conjunction

'P is on X' $\wedge$ 'P is on Y' $\equiv$ 'P is on Z'

where Z describes the region of the circle that is the intersection of the X and Y regions. One can readily verify that the intersection of two accessible regions is again an accessible region and, therefore, that conjunction is well-defined. The functor ' $\wedge$ ' is just the usual sentential conjunction with regard to the interpretation of sentences as locating the particle on the circle.

It is also the usual conjunction with regard to truth conditions. In effect, each possible location L for the particle P that is on the circle but not at the center yields an assignment of truth values according to the prescription:

'P is on X' is true under L iff under L, P is on X.

Hence the sentence 'P is on the center of the circle' is false under all truth assignments (and will play the role of 'the false' in this system.) And the semantic rule for conjunction is thus defined as follows. If ϕ, ψ are elementary sentences, an assignment L of truth values to the elementary sentences automatically assigns truth values to conjunctions according to the rule

'$\phi \wedge \psi$' is true under L iff 'ϕ' and 'ψ' are true under L

The functor '$\wedge$' is thus also the usual conjunction with regard to truth conditions. Note that the semantic notions of validity and logical equivalence are defined as usual

ϕ is valid iff ϕ is true under all assignments of truth values

ϕ is logically equivalent to ψ iff ϕ and ψ have the same truth value under all assignments

The situation with negation is, however, quite different. If one wanted to introduce the usual negation, then one should introduce a unary functor '$\sim$' as

$\sim(P$ is on $X) \;\equiv\; P$ is in the circle but not on the regions described by X

The problem with the '$\sim$' definition of negation is that the set of elementary sentences is not closed under it. For example, if X describes a diameter, then $\sim(P$ is on $X)$ describes the circle minus a diameter, which is not an accessible region. For the elementary sentences to be closed under negation '$\sim$' one can either expand the list of accessible regions so as to include with each region on the list its complement relative to the circle (and then introduce ordinary negation as above), or retain the previous list of accessible regions by introducing a unary functor under which the elementary sentences are closed. The new functor will, therefore, be different from the ordinary sentential negation.

Consider the second option and define the unary functor '$\neg$' as

$\neg(P$ is on $X) \;\equiv\; P$ is on $X^{\perp}$

where if R is the region described by X, then $X^{\perp}$ describes (1) the center of the circle if R is the whole circle, (2) the whole circle if R is the center of the circle, and (3) the diameter perpendicular to R if R is a diameter. Note that '$\neg$' satisfies the desired involutary property, namely $\neg(\neg P) = P$. Also, an assignment L of truth values to the elementary sentences automatically assigns truth values to $\neg\phi$ according to the rule

If 'ϕ' is true under L, then '$\neg\phi$' is false under L

However, contrary to conjunction, the functor '$\neg$' is *not* the usual logical negation. Both with regard to the interpretation of sentences as locating the particle on the circle and with regard to truth conditions.

First, to deny that point P is in diameter X is *not* to assert that it is in the diameter perpendicular to X, as 'negation' $\neg$ prescribes. Indeed, the point could be anywhere in the circle! Second, whereas the above semantic rule holds for 'negation' '$\neg$', its converse – while true for ordinary negation – does not hold here. For example, suppose that the assignment L derives from P being on diameter X. If ϕ is the sentence 'P is on Y', where Y describes a diameter not perpendicular to the X diameter, then both 'P is on Y' and 'P is on $Y^{\perp}$' are false under L; that is, both 'ϕ' and '$\neg\phi$' are false under L. The trouble arises because if it is false that P is on a certain diameter, it does not follow that P is on the perpendicular diameter. Thus, even though the set of elementary sentences is closed under functor '$\neg$', it is *not* ordinary negation nor an extension of it.

Finally, given conjunction and negation, one can introduce disjunction by the De Morgan Laws

$$(\phi \vee \psi) \;\equiv\; \neg(\neg\phi \wedge \neg\psi)$$

The semantics forced on disjunction by this definition are as follows:

If 'ϕ' is true under L or 'ψ' is true under L, then '$\phi \vee \psi$' is true under L

The converse, however, does not hold, that is, the disjunction can be true although neither disjunct is true. For example, if ϕ, ψ locate P in distinct diameters, then the disjunction $(\phi \vee \psi)$ is true under all assignments of truth values, since it merely says that P is somewhere on the circle. And under an assignment in which P is on neither of the mentioned diameters, each disjunct will be false, whereas the disjunction as a whole will be true. Also notice that if ϕ, ψ locate P in distinct diameters, the conjunction $(\phi \wedge \psi)$ is false under all assignments, since it would place P on the center of the circle. Thus, the algebraic relation '$\vee$' cannot be understood as disjunction nor as an extension of it.

To finish, let us look at the distributive law. Suppose ϕ_1, ϕ_2, ϕ_3 locate P on distinct diameters R_1, R_2, R_3 respectively. The conjunction $(\phi_1 \vee \phi_2) \wedge \phi_3$ locates P on R_3 while the disjunction $(\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3)$ locates P on the center of the circle. Thus the latter disjunction is false under every assignment of truth values while the former conjunction is true under the assignment where P is on R_3. Hence

$$(\phi_1 \vee \phi_2) \wedge \phi_3 \neq (\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3) \tag{5.28}$$

and the distributive law does not hold in this 'circular logic'. The distributive law fails due to the oddities of disjunction, which in turn derive from the nonstandard 'negation' '$\neg$'. But given that the latter differs in meaning (with regard to both interpretation and truth conditions) form ordinary negation, the failure of the distributive law for this system does *not* illustrate how the ordinary law of distributivity might be false. To assert the distributive law in this circular logic is *not* to assert the ordinary distributive law at all.

5.4.1.2 Quantum Logic

Similarly to the definition of 'negation' '$\neg$' in the 'circular logic', quantum logic also chooses the second option when defining quantum 'negation', or *nequation*, as Fine calls it, where the 'q' reminds us of quantum theory and the difference in spelling helps us to keep in mind the difference between negation and nequation. The features of 'circular logic' hence have their corresponding analogues in quantum 'logic'. Let us consider them.

In quantum logic the elementary sentences are of the form 'observable A takes a value in the Borel set $\mathbb{b}$' – what we have been calling quantum events $P^A(\mathbb{b})$. For operators with discrete spectrum, the elementary sentences are of the form 'observable A takes a value a_i', where a_i is an eigenvalue of the operator A, and are represented by the projector $P^A_{a_i}$. (Note that the sentences or events are referred to a fixed system.) The assignments of truth values are simply the various states ψ of the system.[5.15] Indeed, for an elementary sentence P_{a_i} and state ψ

> P_{a_i} is true under an assignment V_ψ (i.e. in state ψ) iff ψ is an eigenstate of P_{a_i} [5.16]

The unary functor $\perp$ is defined on the quantum event $P \in \mathcal{L}(\mathcal{H})$ as

> $P^\perp \equiv$ orthogonal projection onto the complement of the closed subspace spanned by the range of P

An assignment V_ψ of truth values to the elementary sentences automatically assigns truth values to the nequation of P_{a_i}, i.e. $(P_{a_i})^\perp$, according to the rule

> If 'P_{a_i}' is true under V_ψ, then '$(P_{a_i})^\perp$' is false under V_ψ

Similarly to '$\neg$' in the 'circular logic', nequation '$\perp$' *cannot* be interpreted as logical negation nor an extension of it. For example, consider a two dimensional Hilbert space and an observable A with a discrete and non-degenerate spectrum $A = a_1 P_{a_1} + a_2 P_{a_2}$. For P_{a_1}

5.15. For ease of exposition we will stick to pure states ψ and operators with discrete spectrum.

5.16. We use V_ψ to note an assignment of truth values rather than L_ψ, which can be confused with subspace L_ψ.

and P_{a_2} (which are orthogonal projectors), nequation prescribes that to deny that event P_{a_1}, i.e. observable A takes value a_1, is the case is to assert that event P_{a_2}, i.e. observable A takes value a_2, is the case. But again this is crazy. For the event could be any of the possible combinations of P_{a_1} and P_{a_2}, i.e. $c_1 P_{a_1} + c_2 P_{a_2}$ with $b \neq 0$ and $|c_1|^2 + |c_2|^2 = 1$, in which case A would simply take no value for state ψ.

Moreover, whereas the above semantic rule holds for nequation '$\perp$', its converse – while true for ordinary negation – does not hold here. For example, suppose that the assignment V_ψ derives from A taking no value, e.g. $\psi = c_1 \alpha_1 + c_2 \alpha_2$, with c_1 and c_2 different from zero. If P_{a_j} is the sentence 'A takes value a_j', then both 'A takes value a_j' and 'A takes a value a_i different form a_j' are false under V_ψ (because ψ is not an eigenstate of either). That is, both 'P_{a_j}' and '$(P_{a_j})^\perp$' are false under V_ψ and thus the semantic rule 'if '$(P_{a_i})^\perp$' is true under V_ψ, then 'P_{a_i}' is false under V_ψ' does not hold. The trouble arises because nequation of (P_ψ), i.e. it is false that A takes no value, implies $(P_\psi)^\perp$, i.e. A takes no value, and not P_{a_i}, i.e. A takes a determinate value (any of the eigenvalues of A), as it would intuitively do if it could be interpreted as negation.

If one wanted to introduce ordinary negation, then one would define the negation of 'A takes the value a_i' as the assertion that 'either A takes no value or it takes a value corresponding to an eigenvalue different from a_i'. This negation is true under an assignment V_ψ just in case ψ is either not an eigenstate of A, i.e. ψ is a superposition of eigenstates of A with distinct eigenvalues, or ψ is an eigenstate of A but with eigenvalue different from a_i, i.e. ψ lies in the subspace $(L_{a_i})^\perp$ orthogonal to the space spanned by P_{a_i}. Thus the negation of 'A takes the value a_i' is true under V_ψ iff either ψ is a superposition of eigenstates of A with distinct eigenvalues, or V_ψ lies in $(L_{a_i})^\perp$.

Both alternatives of defining negation are perfectly meaningful and experimentally verifiable. Nevertheless, as we have seen, quantum logic does not use this last negation. It instead focuses on only one of the alternatives above and takes the quantum 'negation' to be nequation, and thus takes the 'negation' of 'A takes the value a_i' to be 'A takes a value corresponding to an eigenvalue different from a_i', which is true under V_ψ just in case ψ lies in $(L_{a_i})^\perp$. Notice that nequation corresponds to negation for compatible observables, but is completely different from it for incompatible events. Indeed, the nequation of 'A takes no value under V_ψ', i.e. $(P_\psi)^\perp$, is also 'A takes no value under V_ψ', and thus has nothing to do with negation. Hence, similarly to quantum conditional probability, the fact that nequation is co-extensive with negation in their shared domain of application, does not guarantee that outside that domain the nequation can be regarded as an extension or a generalization of negation.

Conjunction is defined in quantum 'logic' analogously to that of the 'circular logic':

'$P_{a_i} \wedge P_{a_j}$' is true under V_ψ iff 'P_{a_i}' and 'P_{a_j}' are true under V_ψ.

As we can see, the functor '$\wedge$' is just the ordinary relation 'and'. Disjunction is also defined in quantum 'logic' analogously to disjunction in the 'circular logic':

If 'P_{a_i}' is true under V_ψ or 'P_{a_j}' is true under V_ψ, then '$P_{a_i} \vee P_{a_j}$' is true under V_ψ

and thus presents analogue problems for its interpretation as an extension of the logical 'or'. Indeed, the converse of this semantic rule for disjunction does not hold; that is, the disjunction can be true although neither disjunct is true. For example, for an assignment of truth values V_ψ with $\psi = c_1\,\alpha_1 + c_2\,\alpha_2$ and observable $A = a_1\,P_{a_1} + a_2 P_{a_2}$, the disjunct $P_{a_1} \vee P_{a_2}$ is true in ψ (because ψ is an eigenstate of P_I), while neither P_{a_1} nor P_{a_2} are true in ψ (because if both c_1 and c_2 are different from zero ψ is not an eigenstate of P_{a_1} nor P_{a_2}). And hence, for the system in state ψ, A does *not* take value a_1 *nor* does it take value a_2, yet A *does* take some value. This certainly precludes understanding disjunction '$\vee$' as an extension of the logical relation 'or'.

Another particularly relevant example is the following. For incompatible quantities A and B, the conjunction $P_{a_i} \wedge P_{b_j}$ is false under *all* truth assignments. For example, for a spin $\frac{1}{2}$ particle and an assignment of truth values V_ψ with $\psi = c_1\ \psi_{s_{+z}} + c_2\,\psi_{s_{-z}}$, the conjunct $P_{s_{+z}} \wedge P_{s_{+x}}$ is false for any c_1, c_2. And hence the spin $\frac{1}{2}$ particle can never take *both* a positive value of spin along the z-axis and a positive value of spin along the x-axis.[5.17] This is the famous non-simultaneity of incompatible observables (in this example S_x and S_z). Similarly, the non-simultaneity of position and momentum of a quantum mechanical particle, i.e. the non-localizability of such a particle in arbitrarily regions of both position and momentum, is a consequence of the fact that the conjunction $P_{\delta x} \wedge P_{\delta p}$ is false under *all* truth assignments.

5.4.2 Quantum Conditional Probability

In section 5.3 we argued that the quantum probabilities defined by the Lüders rule are conditional probabilities for values of physical quantities only if the quantities are compatible. Our discussion on quantum logic helps us understand why $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2} + P_{\alpha_3})$ in example 5.2, for A and B compatible quantities, can be so interpreted, and why $\mathbb{P}_\psi(P_{\alpha_i}|P_{\beta_1} + P_{\beta_3})$

5.17. Notice these two examples are analogue to the two particular cases we considered at the end of section 5.4.1.1.

in example 5.1, for A and B incompatible quantities, *cannot*. Briefly, even though both $[P_{\alpha_i} \wedge (P_{\alpha_2} + P_{\alpha_3})]$ in example 5.2 and $P_{\alpha_i} \wedge (P_{\beta_1} + P_{\beta_3})$ in example 5.1 correspond, respectively, to 'A takes the value a_i and B takes the value b_2' and 'A takes the value a_i and B takes the value b_1', where 'and' is the ordinary conjunction relation, the latter event is false under *all* truth assignments and thus cannot serve as the common quantum projector to which a rationale for why the probability defined by the Lüders rule for P_{α_i} and $P_{\beta_1}+P_{\beta_3}$ is non-zero could resort to.

In more detail, consider first the example with incompatible observables. The event $P_{\alpha_i} \wedge (P_{\beta_1}+P_{\beta_3})$ corresponds to 'A takes the value a_i and B takes the value b_1', with 'and' the ordinary logical relation. However, it is false under *all* truth assignments V_ψ since there is no state ψ which is an eigenstate of both P_{α_i} and $(P_{\beta_1}+P_{\beta_3})$. Hence the probability of a_i conditional on b_1 should be zero for all ψ, which we know is *not* what the Lüders rule prescribes. Thus the probability $\mathbb{P}_\psi(P_{\alpha_i}|P_{\beta_1}+P_{\beta_3})$ cannot be understood as the probability of a_i conditional on b_1. Only if either c_1 or c_3 are zero, will the Lüders rule also assign a zero value to this probability and will one be able to understand it as a conditional probability. Indeed, if $c_1=0$, $p(a_1)=0$ and thus $p(a_1|b_1)=0$; and if $c_3=0$, $p(a_3)=0$ and thus $p(a_3|b_1)=0$.

Turn now to the example with compatible observables. Event $[P_{\alpha_i} \wedge (P_{\alpha_2} + P_{\alpha_3})]$ corresponds to 'A takes the value a_i and B takes the value b_2' – 'a_i and b_2' in short – where 'and' is the ordinary conjunction relation. But now, depending on the truth value assignment V_ψ, 'a_i and b_2' will either be true – in which case it will be assigned a probability one – or be false – in which case it will be assigned a zero probability, or have no determinate truth value – in which case its probability will be in the open interval $(0, 1)$. Increasing pro rata these joint probabilities yields the same value as that of $\mathbb{P}_\psi(P_{\alpha_i}|P_{\alpha_2}+P_{\alpha_3})$.

Indeed, if $a_i=a_1$ then 'a_1 and b_2' is false under all truth assignments V_ψ because there is no ψ which is an eigenstate of both P_{α_1} and $(P_{\alpha_2}+P_{\alpha_3})$. Hence, $p(a_1 \text{ and } b_2)=0$ and $p(a_1|b_2)=0$. Second, if $a_i=a_2$, then 'a_2 and b_2' can either be true, false or indeterminate depending on the truth value assignment V_ψ. It is false only under a truth assignment V_ψ for which $\psi=0$ $(c_1=c_2=c_3=0)$, in which case $p(a_2|b_2)=0$. It is true if $\psi=\alpha_2$ $(c_2=1)$ since α_2 is an eigenstate of both P_{α_2} and $(P_{\alpha_2}+P_{\alpha_3})$, and hence $p(a_2|b_2)=1$. Under all other truth value assignments, 'a_1 and b_2' is not either true nor false but rather will not take a determine value since ψ cannot be an eigenstate of both P_{α_2} and $(P_{\alpha_2}+P_{\alpha_3})$. In this case its probability will be determined by the trace rule as $\mathrm{Tr}[W\,(P_{\alpha_2} \wedge (P_{\alpha_2}+P_{\alpha_3}))]=\mathrm{Tr}(WP_{\alpha_2})=p_\psi(a_2)$. And increasing this value pro rata, we get $\mathbb{P}(a_2|b_2)=\frac{|c_2|^2}{|c_2|^2+|c_3|^2}$. (Note that the same reasoning applies when $a_i=a_3$.)

Lastly, consider the case in which Q is the sum of two orthogonal projection operators Q_1 and Q_2, i.e. $Q = Q_1 + Q_2$, where both Q_1 and Q_2 are associated distinct eigenvalues q_1 and q_2. For example, as in section 4.3.1, take a spin-1 particle and let Q_1 and Q_2 be the events 'the x-component is $+1$' and 'the x-component is 0', respectively, i.e. $Q_1 = P_{s_{x+1}}$ and $Q_2 = P_{s_{x0}}$. And let P be the event 'the y-component is $+1$', i.e. $P = P_{s_{y+1}}$. Recall that the probability defined by the Lüders rule in this case is given by

$$\mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}} + P_{s_{x0}}) = \left(\frac{\|P_{s_{x+1}}\psi\|}{\|(P_{s_{x+1}} + P_{s_{x0}})\,\psi\|}\right)^2 \mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}}) + \left(\frac{\|P_{s_{x0}}\psi\|}{\|(P_{s_{x+1}} + P_{s_{x0}})\,\psi\|}\right)^2 \mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x0}}) + \frac{\|P_{s_{x+1}}\psi\|\,\|P_{s_{x0}}\psi\|}{\|(P_{s_{x+1}} + P_{s_{x0}})\,\psi\|^2}\ \mathrm{Re}\ \left\langle \frac{P_{s_{x+1}}\psi}{\|P_{s_{x+1}}\psi\|}, P_{s_{y+1}}\frac{P_{s_{x0}}\psi}{\|P_{s_{x0}}\psi\|}\right\rangle \tag{5.29}$$

The question is whether $\mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}} + P_{s_{x0}})$ can be interpreted as the probability that the y-component of spin is $+1$ *conditional* on the x-component being $+1$ or 0, i.e. the probability of physical value 's_{y+1}' conditional on physical value 's_{x+1} or s_{x0}'.

The problem here is two-fold. First, the event 'the y-component of spin is $+1$ and the x-component of spin is $+1$ or 0', i.e. the event 's_{y+1} and s_{x+1} or s_{x0}', cannot be represented as $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$. And second, even if $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ could be so understood, the probability assigned to it does not correspond to the probability assigned to the common operator of P and Q employed in the Lüders rule, i.e. QPQ. Thus, its probability would not be equal to that given by (5.29). Hence, $\mathbb{P}_\psi(P_{s_{y+1}}|P_{s_{x+1}} + P_{s_{x0}})$ *cannot* be interpreted as the probability that the y-component of spin is $+1$ conditional on the x-component being $+1$ or 0.

Consider the first problem is some more detail. The fact that 's_{y+1} and s_{x+1} or s_{x0}', cannot be represented as $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ is due to the fact that the algebraic relations '$\vee$' and '$\wedge$' in this expression cannot be understood, respectively, as ordinary disjunction and conjunction (nor as extensions of these notions). Indeed, $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ can seemingly be read as 's_{y+1} and s_{x+1} or s_{x0}', because '$\wedge$' defines the usual notion of conjunction, and '$\vee$' applies here to two orthogonal and, therefore, compatible events.

However, if $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ could really be thus read, then its reading as 's_{y+1} and s_{x+1} or s_{y+1} and s_{x0}' should also be possible. For otherwise '$\wedge$' and '$\vee$' could not be interpreted as the logical relations of conjunction and disjunction. And the problem is that $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ *cannot* be read as 's_{y+1} and s_{x+1} or s_{y+1} and s_{x0}' because, given that $P_{s_{y+1}}$ and $P_{s_{x+1}}$ and $P_{s_{x0}}$ are incompatible, $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ is *not* equal to $(P_{s_{y+1}} \wedge P_{s_{x+1}}) \vee (P_{s_{y+1}} \wedge P_{s_{x0}})$. Thus even if the algebraic relation '$\vee$' applies to two orthogonal and, therefore, compatible, events in $P_{s_{y+1}} \wedge (P_{s_{x+1}} \vee P_{s_{x0}})$ it cannot be understood as ordinary disjunction nor as an extension of it.[5.18] Once again, we see that, when incompatible events are involved, the conceptual difficulties associated with the non-Boolean character of the quantum event structure arise.

To sum up, for incompatible quantities P and Q, the event 'p and q' cannot always be represented in terms of the projection operators P and Q as $P \wedge Q$, and in the cases in which such a correspondence does exist, the probability assigned to the event $P \wedge Q$ does not correspond to the probability assigned to the common operator of P and Q employed by the projective notion of quantum conditional probability, i.e. $Q\,PQ$. Thus, the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be understood as the pro rata increase of the probability of 'p and q'; and hence, it cannot be interpreted as the probability of value p conditional on value q for P and Q incompatible quantities.

There is thus no physically meaningful quantum notion of conditional probability. Only at mathematical level, in which we cannot appeal to Boolean intuitions, does it seem possible to interpret the probabilities defined by the Lüders rule as such.

5.5 A new concept?

As we have shown, when extending a concept into a new domain, it is not sufficient to show that there are some formal analogies between the old and the extended concept. In addition, it is essential to evaluate whether these analogies can provide enough interpretive content so as to justify regarding the concept in the new domain as an extension or a generalization of the old one. In the case of the probabilities defined by the Lüders rule we have argued that the formal analogies do *not* provide enough interpretive content so as to justify regarding them as an extension or a generalization of classical conditional probabilities to the quantum context from a physically meaningful perspective; they can only, and with great difficulty, be interpreted as extensions of classical conditional probabilities at a purely formal level.

5.18. This same conclusion can also be reached in the following way. The disjunct $P_{s_{x+1}} \vee P_{s_{x0}}$ is true in $\psi = c_{+1}\,\psi_{s_{x+1}} + c_0\,\psi_{s_{x0}} + c_{-1}\,\psi_{s_{x-1}}$ (because ψ is an eigenstate of P_I), while neither $P_{s_{x+1}}$ nor $P_{s_{x0}}$ are true in ψ (because if both c_{+1} and c_0 are different from zero, ψ is not an eigenstate of $P_{s_{x+1}}$ nor of $P_{s_{x0}}$). And hence, for the system in state ψ, S_x does *not* take value $+1$ *nor* does it take value 0, yet S_x *does* take some value, something which certainly precludes understanding '$\vee$' as logical disjunction.

But a question remains. Does the Lüders rule define any concept which can be understood from a physically meaningful perspective? After all, quantum theory is *not* just a formal mathematical theory, but a theory which purportedly describes the physical world. To try to answer this question we first need to make a small detour and consider how measurement was introduced into quantum theory in the attempt to make it a little more comprehensible.

Chapter 6
Orthodox Quantum Theory

We have so far been concerned with the conceptual understanding of quantum probability theory. This theory is a perfectly consistent mathematical theory. The problem is that it is not at all clear how to get from it to a consistent and satisfactory physical theory. In this chapter we present in some detail how the orthodox account of quantum mechanics manages to get a consistent, though not very satisfactory physical theory if one wants to interpret it as something more than a mere algorithm for generating the statistical predictions of the outcomes of measurements. A central element of the orthodox interpretation is its adherence to the so-called *Projection (or Collapse) Postulate* which prescribes that every measurement, represented by some suitably chosen observable, leads to non-unitary reduction of the total state vector to an eigenstate of the measured observable.

In section 6.1 we present the so-called quantum measurement problem, namely the problem of reconciling the fact that quantum mechanics predicts no definite outcomes for measurements and the fact that (we perceive) measurements do have definite outcomes. Then, in section 6.2, we present how the orthodox account, as presented by [von Neumann, 1932], solves this problem by changing the dynamics of quantum mechanics when measurements are performed: states always evolve in accordance with the linear dynamics of the Schrödinger equation except when measurements are performed, for which a nonlinear collapse dynamics, explicitly probabilistic, takes over. Under this view, when a measurement of a physical observable is performed upon a system, the system's state will instantaneously, and non-linearly, 'jump' or 'collapse', with a certain probability, to one of the eigenstates of the observable being measured. This is the so-called 'Projection Postulate', which ensures that after the measuring interaction the measurement device *does* have a definite outcome.

The Lüders rule appears in the orthodox interpretation of quantum mechanics as the generalized version of the Projection Postulate. It determines uniquely the state of the system after a measurement of a certain physical quantity with degenerate eigenvalues. This new density matrix W can then be used to calculate probability assignments for subsequent measurements. Hence, as we discuss in section 6.3, it seems that the probability given by this new density operator can be interpreted as the probability of a measurement outcome of the second measurement conditional on a measurement outcome of the first one. We refer to this interpretation of the probability defined by the Lüders rule as 'conditional-on-measurement-outcome probability'.

Then, in section 6.4, we distinguish this conditional-on-measurement-outcome probability from another type of (purportedly) conditional probability which also arises in the context of quantum measurements. Indeed, it is not uncommon to hear that all quantum probabilities are conditional probabilities for measurement outcomes conditional on measurements. However, we argue that these conditional-on-measurement probabilities are not really conditional probabilities. Finally, in section 6.5, we argue that the orthodox interpretation, with its reliance on the Projection Postulate, does not provide an adequate interpretation of quantum theory unless one thinks of quantum theory instrumentally.

6.1 The Consistency – or Measurement – Problem

As we have seen, in quantum mechanics each system is associated to a Hilbert space $\mathcal{H}$ and the state of a system is represented by a density operator W defined on $\mathcal{H}$. Observables are represented by Hermitian operators acting on $\mathcal{H}$, where their possible values are given by the spectrum of the operator which represents them. The evolution in time of a system which begins in state W_{t_0} at the initial time t_0 is given by the time-evolution unitary operator $U_{t-t_0} = e^{-iH(t-t_0)}$, where H is the Hamiltonian operator for a given system, according to the equation

$$W_t = U_{t-t_0} W_{t_0} U_{t-t_0}^{-1} \tag{6.1}$$

If the system's state is a pure state represented by the vector ψ_t, then (6.1) may be written as

$$\frac{d\psi_t}{dt} = -i\hbar H \psi_t \tag{6.2}$$

This expression of the dynamical equation for quantum systems is usually referred to as the Schrödinger equation. The dynamics given by (6.1) or (6.2) is linear and deterministic.

A problem arises if we attempt to describe measurement interactions by a unitary operator for, as we now show, measurements then generally turn out to *not* have definite outcomes. However, it is an empirical fact that measurements *do* have definite outcomes (or, at the very least, we perceive definite outcomes at the conclusion of a measurement). The difficulty of reconciling these two facts, i.e. that quantum mechanics predicts no definite outcomes for measurements and that (we perceive) measurements do have definite outcomes, is generally known as the 'measurement problem'. Let us consider this problem in some detail.

A familiar idealization of the measurement process amounts to supposing that measuring instruments are *non-destructive*. This means that when a measurement is performed the physical system and the measuring instrument, initially separated, will first form a compound system, both preserving their own identity during the mutual interaction, and then, become separated again after a certain time. This enables one to ascribe both the system and the measuring instrument a final state after the measurement. It is important to realize that many actual measurements instruments violate this idealization: for example, a photon spectrometer measures the energy of a photon by absorbing it and hence destroying it by the very act of measurement.

A further idealization is that of an *ideal* measurement. To establish a correlation between the values of a certain property of the system and the values of another property of the measuring device requires some sort of interaction between the system and the measuring apparatus. For some measuring interactions one can suppose that the magnitude of the interaction can be made so small that the system's state remains totally unaffected. An ideal measurement is precisely a measurement in which the state of the system remains unchanged after the measurement interaction.

Quantum theory would initially describe a measurement interaction as any other physical interaction by first, ascribing a quantum state to the measuring device and then, treating the interaction between the system and the measuring device as a quantum interaction, i.e. one that obeys the Schrödinger equation or, more generally, one that is described by a unitary operator. Let us see how an ideal measurement works in the quantum context. We employ pure states and observables with discrete spectrum to keep the formulation simple.

Suppose that a quantum system begins in the state α_1, an eigenvector of observable A with eigenvalue a_1. We perform an ideal measurement of A: the measuring device begins in a ready-to-measure state M_0, i.e. an eigenstate of an observable M, and after the measurement is perfectly correlated with the value of A possessed by the system:

$$\alpha_1\, M_0 \longrightarrow \alpha_1\, M_1 \tag{6.3}$$

where M_1 is the state of the apparatus that indicates the value of a_1. The interaction in (6.3) can be rewritten in terms of a unitary time evolution operator $U(t)$ as $U_t(\alpha_1\, M_0) = \alpha_1\, M_1$. Since the states before and after the interaction are simultaneous eigenstates of $A \otimes I$ and $I \otimes M$, the eigenstate-eigenvalue link allows an ascription of the value a_1 to the system, both before and after the interaction, and of the value m_0 to the measuring device before the measurement, and m_1 after the measurement.

Similarly, if the system begins in the state α_2, then the interaction of the system with the measuring device will be:

$$\alpha_2 M_0 \longrightarrow \alpha_2 M_2 \tag{6.4}$$

and the e-e link allows an ascription of the value a_2 to the system, both before and after the interaction, and of the value m_0 to the measuring device before the measurement, and m_2 after the measurement. Similarly, (6.4) can be rewritten in terms of a unitary time evolution operator $U(t)$ as $U_t(\alpha_2 M_0) = \alpha_2 M_2$.

So far so good. But consider what happens now if the system is in a superposition of the states α_1 and α_2, i.e. $\psi = c_1 \alpha_1 + c_2 \alpha_2$. This interaction, given that physical interactions are described by some unitary operator which is linear, is [6.1]

$$\psi M_0 = (c_1 \alpha_1 + c_2 \alpha_2) M_0 \longrightarrow c_1 \alpha_1 M_1 + c_2 \alpha_2 M_2 \tag{6.5}$$

The initial state ψ, although not an eigenstate of $A \otimes I$, was an eigenstate of $I \otimes M$. But the final state $c_1 \alpha_1 M_1 + c_2 \alpha_2 M_2$ is not an eigenstate of either $A \otimes I$ or $I \otimes M$; indeed, the measurement interaction has left the joint system still in a non-eigenstate of $A \otimes I$ and has changed the joint system into a non-eigenstate of $I \otimes M$. Hence, after the measurement interaction, the eigenstate-eigenvalue link, implies that no single, definite state, and hence no definite outcome, can be attributed to the measuring device.[6.2] So how is it then that at the conclusion of a measurement we always observe a definite outcome? This is the famous measurement problem of quantum mechanics.[6.3]

The foundations and philosophy of quantum mechanics literature is loaded with articles considering this problem and trying to provide adequate answers to it. The different solutions proposed give rise to the myriad of interpretations of quantum mechanics. We here only consider the orthodox (textbook) interpretation with its famous collapse or projection postulate. To name a few others: the relative-state interpretations, introduced by [Everett, 1957] and further developed as 'many-worlds' and 'many-minds' interpretations, propose alternative readings of the formalism of standard quantum mechanics; the class of modal

6.1. Indeed, by linearity of the time evolution operator $U(t)$, we have that $U_t[(c_1 \alpha_1 + c_2 \alpha_2) M_0] = U_t(c_1 \alpha_1 M_0) + U_t(c_2 \alpha_2 M_0) = c_1 \alpha_1 M_1 + c_2 \alpha_2 M_2$.

6.2. Note that no single, definite state, and hence no definite outcome, can be attributed to the system either, both before and after the measurement. This also presents an interpretive problem. However, given that the systems which quantum mechanics describes are generally too small to be observed by 'ordinary' means (although not all those which it presumably describes are small!; see section 6.5), this issue becomes more pressing for macroscopic systems such as measuring devices.

6.3. The 'measurement problem' is actually a more complex problem which contains two separate questions, namely, why we perceive a single outcome for the determinate variable –this is the problem we have presented and is referred to as 'the problem of outcomes' – and why a particular quantity (usually position) is always selected as the determinate variable. The latter is known as 'the preferred-basis problem'. For a more detailed exposition see, for example, [Schlosshauer & Fine, 2007], [Maudlin, 1995].

interpretations, first suggested by [van Fraassen, 1991], modify the rules that connect the formalism to the actual physical properties (they reject the 'rule of silence'); physical collapse theories like the Ghirardi-Rimini-Weber (GRW) approach [GRW, 1986] change the dynamics and postulate new physical mechanisms; and the de Broglie-Bohm pilot-wave theory, is a highly nonlocal hidden-variable interpretation [Bohm, 1952] which changes the state space by introducing additional variables – the so-called 'hidden variables' – and also introduces additional governing equations.

6.2 The Projection Postulate

6.2.1 Von Neumann's Projection Postulate

The orthodox account, as presented by [von Neumann, 1932] (pp. 347-349), solves the measurement problem by changing the dynamics of quantum mechanics and postulating the existence of 'collapses': states always evolve in accordance with the linear dynamics of the Schrödinger equation except when measurements are performed, for which a nonlinear collapse dynamics, explicitly probabilistic, takes over. Under this view, when a measurement of a physical observable A is performed upon a system in state ψ, the system's state will instantaneously, and non-linearly, 'jump' or 'collapse' to one of the eigenstates α_i of the observable being measured. This ensures that after the measuring interaction the measurement device will have a definite outcome: ψ will collapse to a particular eigenstate α_i of A – rather than remain a superposition of the different α_i's – and hence $\psi\, M_0$ will evolve to a particular $\alpha_i\, M_i$.

The orthodox account further postulates that the probability with which the state's system collapses onto each particular α_i after the measurement of observable A is performed is given by the *Born rule*, namely,

$$p_\psi(a_i) = |\langle \alpha_i, \psi \rangle|^2 \tag{6.6}$$

where a_i is the eigenvalue associated with the eigenvector α_i. This rule can be more generally expressed in terms of the projection operator P_{a_i} onto the subspace α_i as

$$p_\psi(a_i) = \langle \psi, P_{a_i} \psi \rangle \tag{6.7}$$

Or, even more generally, in terms of the density operator $W = P_\psi$, as

$$p_W(a_i) = \mathrm{Tr}(W P_{a_i}) \tag{6.8}$$

This new kind of time evolution, which is explicitly probabilistic, is the content of the projection postulate.

Proposition 6.1. ***von Neumann's Projection Postulate***. *Upon a measurement of an observable $A = \sum_i a_i P_{a_i}$ on a system in state W, the state of the system 'collapses' to the state W', where*

$$W' = P_{a_i} \tag{6.9}$$

for some eigenvalue a_i of A. The probability that the state collapses to P_{a_i} is

$$p_W(a_i) = \mathrm{Tr}(W P_{a_i}) \tag{6.10}$$

Note that (6.10) is the general expression for a probability function we saw in section 3.2. But within the orthodox approach it is interpreted in a very specific way, namely as the probability for finding *measurement result* a_i when a *measurement* of observable A is performed on a system in state W. Also note that the orthodox interpretation changes radically the classical way of conceiving measurements: they are not ideal processes for merely learning something; they are invariably processes which drastically change the measured system, (in addition to being unlike any other interactions since they are not and cannot be represented by a unitary time evolution operator – more on this in section 6.5).

The projection postulate is usually justified because it ensures repeatability of measurement results. That is, it guarantees that when we repeat a measurement, the result of the second measurement always matches the result of the first. The projection postulate is certainly sufficient to guarantee this matching. In effect, according to it, a measurement must necessarily change the state of the measured system – it makes it 'collapse', it makes it 'jump' – from whatever it may have been just prior to the measurement into an appropriate eigenstate of the measured observable operator, namely the eigenstate whose eigenvalue matches the outcome of the measurement. After the measurement, the system remains in that particular eigenstate so that the probability of finding that same measurement result upon a second measurement at a later time is 1.

Von Neumann's postulate only determines uniquely the final state of the system if a_i is a non-degenerate eigenvalue of A, for in this case the corresponding eigenspace is one-dimensional and the final state is the projection operator P_{a_i}; but when a_i is degenerate, so that the corresponding eigenspace is at least two dimensional, the final state of the system is left undetermined. If one tries to generalize von Neumann's postulate in the obvious way, namely as prescribing a state change from W to $\frac{P_{a_i}}{\mathrm{Tr}\, P_{a_i}}$, where P_{a_i} is the projection operator onto the eigenspace of dimension greater or equal to two associated with a_i,[6.4] the resulting change does not satisfy the repeatability requirement for *degenerate* eigenvalues. Indeed the von Neumann measurement interaction represents a degeneracy-breaking measurement: it

6.4. P_{a_i} appears divided by its trace $\mathrm{Tr}(P_{a_i})$ because only when a projection operator projects onto a 1-dimensional space is it of trace one and hence a density operator.

makes a particular selection of eigenstates within each degeneracy subspace and thus does not leave the degenerate eigenspaces invariant.

6.2.2 Lüders' Projection Postulate

To overcome this problem, and thus be parallel with the requirement that the measurement of a non-degenerate eigenvalue be repeatable, [Lüders, 1951] proposed that the change of state upon measurement of an observable A when the eigenvalue a_i is found, be from W to $W_{a_i} = \frac{P_{a_i} W P_{a_i}}{\mathrm{Tr}(P_{a_i} W)}$. This measurement interaction distinguishes the eigenstates belonging to each eigenvalue from those belonging to any other eigenvalue, but all those belonging (degenerately) to a single eigenvalue are left indistinguishable.

Proposition 6.2. ***Lüders' Projection Postulate***. *Upon a measurement of an observable $A = \sum_i a_i P_{a_i}$ on a system in state W, the state of the system 'collapses' to the state W_{a_i}, where*

$$W_{a_i} = \frac{P_{a_i} W P_{a_i}}{\mathrm{Tr}(P_{a_i} W)} \tag{6.11}$$

for some eigenvalue a_i of A. The probability that the state collapses to W_{a_i} is

$$p_W(a_i) = \mathrm{Tr}(W P_{a_i}) \tag{6.12}$$

If A does not have a purely discrete spectrum and if the observed value is in a subset $\mathbb{b}$ of the spectrum of A, then formula (6.11) admits the obvious generalization:

$$W^A(\mathbb{b}) = \frac{P^A(\mathbb{b})\, W P^A(\mathbb{b})}{\mathrm{Tr}(W P^A(\mathbb{b}))} \tag{6.13}$$

We briefly note that for continuous observables the Lüders rule violates the condition that repeated measurements should be stable. It appears that the only fix for this is to alter the notion of conditional expectation.[6.5]

To gain a better understanding of how the Lüders' projection postulate works, let us consider the case in which the initial state of the system is in a pure state ψ. In this case, (6.11) reads:

$$P_\psi \longrightarrow \frac{P_{a_i} P_\psi P_{a_i}}{\mathrm{Tr}(P_\psi P_{a_i})} \tag{6.14}$$

This expression can be simplified as follows. Let $P_{a_i} \psi = \psi'_{a_i}$ be the *non-normalized* projection of ψ onto the subspace spanned by the eigenvectors associated with the eigenvalue a_i. For any vector ϕ, we have that

$$(P_{a_i} P_\psi P_{a_i})\phi = P_{a_i} P_\psi (P_{a_i} \phi) = P_{a_i} \psi \, \langle \psi, P_{a_i} \phi \rangle = P_{a_i} \psi \, \langle P_{a_i} \psi, \phi \rangle = P_{P_{a_i} \psi} \phi = P_{\psi'_{b_i}} \phi \tag{6.15}$$

Since this holds for any ϕ, the numerator of (6.14) is $P_{a_i} P_\psi P_{a_i} = P_{\psi'_{a_i}}$. In addition,[6.6]

$$\mathrm{Tr}(P_\psi P_{a_i}) = \mathrm{Tr}(P_{a_i} P_\psi P_{a_i}) = \mathrm{Tr}(P_{P_{a_i}\psi}) = \langle P_{a_i}\psi, P_{a_i} \psi \rangle = \|P_{a_i}\psi\|^2 = \|\psi'_{a_i}\|^2 \tag{6.16}$$

6.5. See [Valente, 2007] and references therein.

Thus the change of state given by (6.14) is equivalent to the change

$$P_\psi \longrightarrow \frac{P_{\psi'_{a_i}}}{\|\psi'_{a_i}\|^2}, \text{with } \psi'_{a_i} = P_{a_i}\psi \tag{6.17}$$

But $\frac{P_{\psi'_{a_i}}}{\|\psi'_{a_i}\|^2} = P_{\psi_{a_i}}$, with $\psi_{a_i} = \frac{\psi'_{a_i}}{\|\psi'_{a_i}\|}$. Indeed, similarly to how we derived (6.15), for any vector ϕ, we have

$$\frac{P_{\psi'_{a_i}}}{\|\psi'_{a_i}\|^2}\,\phi = \frac{\psi'_{a_i}}{\|\psi'_{a_i}\|}\,\langle\frac{\psi'_{a_i}}{\|\psi'_{a_i}\|}, \phi\rangle = \psi_{a_i}\,\langle\psi_{a_i}, \phi\rangle = P_{\psi_{a_i}}\,\phi \tag{6.18}$$

Hence, the Lüders' projection postulate for pure states prescribes that when a measurement of observable A is performed on a system in state P_ψ and the observed value is a_i, its state P_ψ changes to $P_{\psi_{a_i}}$ or, equivalently, its state changes from ψ to the normalized projection of ψ onto the subspace spanned by the eigenvectors associated with the eigenvalue a_i, i.e.

$$\psi \longrightarrow \psi_{a_i} = \frac{\psi'_{a_i}}{\|\psi'_{a_i}\|}\ , \text{with } \psi'_{a_i} = P_{a_i}\psi \tag{6.19}$$

Proposition 6.3. *Lüders' Projection Postulate for Pure States*. *Upon a measurement of an observable $A = \sum_i a_i P_{a_i}$ on a system in state ψ, the state of the system 'collapses' to the state ψ_{a_i} for some eigenvalue a_i of A, namely,*

$$\psi_{a_i} = \frac{P_{a_i}\psi}{\|P_{a_i}\psi\|} = \frac{\psi'_{a_i}}{\|\psi'_{a_i}\|} \tag{6.20}$$

where ψ_{a_i} is the normalized projection of ψ onto the eigenspace belonging to a_i.

The probability that the state collapses to ψ_{a_i} is given by

$$p_\psi(a_i) = |\langle\alpha_i, \psi\rangle|^2 \tag{6.21}$$

where α_i is the eigenstate associated to the eigenvalue a_i.

The general version of Lüders' projection postulate given by proposition 6.2 can be recovered from its version for pure states given by proposition 6.3 by adding the assumption that the non-pure initial states of the system are affected by the measuring instrument in such a way that the convex structure is preserved.

We draw this section to an end by seeing how examples 5.1 and 5.2 are seen from the perspective of orthodox quantum mechanics.

Example 6.1. Incompatible Observables. In example 5.1 we considered a system in state $\psi = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$, where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, and two incompatible observables $A = a_1 P_{\alpha_1} + a_2 P_{\alpha_2} + a_3 P_{\alpha_3}$ and $B = b_1(P_{\beta_1} + P_{\beta_3}) + b_2 P_{\beta_2}$, where $\alpha_1 = \frac{\beta_1 + \beta_2}{\sqrt{2}}$; $\alpha_2 = \frac{\beta_1 - \beta_2}{\sqrt{2}}$; $\alpha_3 = \beta_3$.

6.6. $\mathrm{Tr}(P_{P_{a_i}\psi}) = \sum_j\ \langle\phi_j, P_{P_{a_i}\psi}\,\phi_j\rangle = \sum_j\ \langle\phi_j, P_{a_i}\psi\rangle\,\langle P_{a_i}\psi\ , \phi_j\rangle = \langle P_{a_i}\psi, P_{a_i}\psi\rangle$.

Suppose that we perform an ideal first-class measurement of observable B and find result b_1. Then the Lüders projection postulate (proposition 6.3) tells us that the new state will be given by the normalized projection of ψ onto the subspace spanned by β_1 and β_3, namely

$$\psi_{b_1} = \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\,\beta_1 + \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\,\beta_3 \tag{6.22}$$

as is illustrated in figure 6.1.

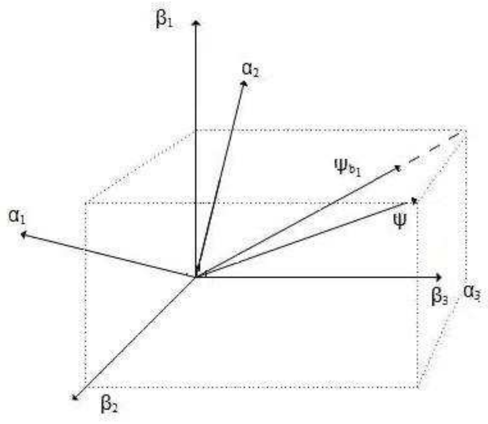

Figure 6.1. Lüders' Projection Postulate for Incompatible Observables

Example 6.2. Compatible Observables. In example 5.2 we considered a system in state $\psi = c_1\,\alpha_1 + c_2\,\alpha_2 + c_3\,\alpha_3$ and two commuting operators $A = a_1\,P_{\alpha_1} + a_2\,P_{\alpha_2} + a_3\,P_{\alpha_3}$, and $B = b_1\,P_{\alpha_1} + b_2(P_{\alpha_2} + P_{\alpha_3})$.

Imagine we perform an ideal first-class measurement of observable B, getting result b_2. The Lüders projection postulate (proposition 6.3) then tells us that the resultant state after this measurement is ψ_{b_2}, that is the (normalized) projection of ψ onto the $\alpha_2 - \alpha_3$ plane. The post-measurement state will thus be given by

$$\psi_{b_2} = \frac{c_2}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_2 + \frac{c_3}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_3 \tag{6.23}$$

Note that the Lüders projection postulate rule is only one rule among many possibilities specifying the state of the system after the measurement interaction. Another rule might, for example, specify that, after a measurement of observable B in which one finds result b_2, in the situation described in example 6.2, the resultant state lies halfway between ψ_{b_2} and α_2 or α_3, or whichever is nearer to ψ_{b_2}. The fact that the Lüders projection postulate yields ψ_{b_2} as given by (6.23) is the correct post-measurement state is, as we saw, to ensure repeatability of measurement results. Indeed, the Lüders projection postulate is chosen precisely because it leaves indistinguishable all the eigenstates belonging (degenerately) to a single eigenvalue.

6.3 Conditional-on-Measurement-Outcome Probability

Von Neumann and Lüders' projection postulates allow one to specify uniquely the state of the system after a measurement of a quantity with a given result. The new density matrix can then be used to calculate probability assignments for subsequent measurements. (Henceforth we will only discuss Lüders's projection postulate given that it generalizes von Neumann's for degenerate eigenvalues.)

In effect, imagine we first measure an observable A on a system in state W and obtain measurement result a_i; subsequently, we measure a second observable B. The probability to find the value b_i upon measuring B will then be given by the trace rule using the density operator W_{a_i} after the measurement of A given by the Lüders projection postulate (proposition 6.2), i.e.

$$p_{W_{a_i}}(b_i) = \mathrm{Tr}(W_{a_i} P_{b_i}) = \frac{\mathrm{Tr}(P_{a_i} W P_{a_i} P_{b_i})}{\mathrm{Tr}(P_{a_i} W)} \tag{6.24}$$

Thus in these cases it appears to be meaningful to speak of the probability distribution of a physical quantity (B) *given* the result (a_i) of a previous measurement of another physical quantity (A).

The Lüders rule thus seems to allow the introduction and interpretation, within quantum theory, of the concept of conditional probabilities: it seems possible to interpret the probability given by (6.24) as the probability of *measurement outcome* b_i conditional on *measurement outcome* a_i. This interpretation leads naturally to writing (6.24) as

$$p_{W_{a_i}}(b_i) = \mathbb{P}_W(b_i|a_i) = \frac{\mathrm{Tr}(P_{a_i} W P_{a_i} P_{b_i})}{\mathrm{Tr}(P_{a_i} W)} \tag{6.25}$$

When considering conditional probabilities we usually talk of the probability of an event a given another event b – rather than that of b given a. Thus, we henceforth exchange the order of the b_i and a_i's in (6.25), with the resultant change in the order in which the measurements are performed: first a measurement of B and then a measurement of A. We refer to this interpretation of (6.25) as *Conditional-on-Measurement-Outcome interpretation*.

Definition 6.1. ***Conditional-on-Measurement-Outcome Probability***. *When an observable B is measured on a system in state W, followed by a second measurement of an observable A, the probability*

$$\mathbb{P}_W(a_i|b_i) = \mathbb{P}_{W_{b_i}}(a_i) = \frac{\mathrm{Tr}(P_{b_i} W P_{b_i} P_{a_i})}{\mathrm{Tr}(P_{b_i} W)} \tag{6.26}$$

is the probability of measurement outcome a_i conditional on measurement outcome b_i.

If the system is in a pure state ψ, then the probability (6.28) can be expressed as

$$\mathbb{P}_\psi(a_i|b_i) = p_{\psi_{b_i}}(a_i) = |\langle \alpha_i, \psi_{b_i}\rangle|^2 \tag{6.27}$$

with $\psi_{b_i} = \frac{P_{b_i}\psi}{\|P_{b_i}\psi\|}$.

Consider these probabilities in example 6.1. With the new state vector prescribed by the Lüders rule after a measurement of observable B with measurement outcome b_1, i.e. $\psi_{b_1} = \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\beta_1 + \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\beta_3$, one can calculate the probabilities assigned to subsequent measurements. If one performs a measurement of observable A, the probabilities to find the results a_i are given by $p_{\psi_{b_1}}(a_i) = |\langle \alpha_i|\psi_{b_1}\rangle|^2$, namely,

$$\begin{aligned}
p_{\psi_{b_1}}(a_1) &= \mathbb{P}_\psi(a_1|b_1) = \frac{1}{2}\frac{|c_1|^2}{|c_1|^2 + |c_3|^2} \\
p_{\psi_{b_1}}(a_2) &= \mathbb{P}_\psi(a_2|b_1) = \frac{1}{2}\frac{|c_1|^2}{|c_1|^2 + |c_3|^2} \\
p_{\psi_{b_1}}(a_3) &= \mathbb{P}_\psi(a_3|b_1) = \frac{|c_3|^2}{|c_1|^2 + |c_3|^2}
\end{aligned} \tag{6.28}$$

These probabilities thus seem to allow a conditional-on-measurement-outcome probability interpretation. Indeed, $\mathbb{P}_\psi(a_i|b_1)$ is read as the probability of finding measurement outcome a_i when observable A is measured upon a system in state ψ, conditional on having found measurement outcome b_1 upon a previous measurement of observable B.

Similarly, in example 6.2, using the new state vector prescribed by the Lüders rule after a measurement of observable B with measurement outcome b_2, i.e. $\psi_{b_2} = \frac{c_2}{\sqrt{|c_2|^2 + |c_3|^2}}\alpha_2 + \frac{c_3}{\sqrt{|c_2|^2 + |c_3|^2}}\alpha_3$, one can calculate the probabilities assigned to subsequent measurements. If one performs a measurement of observable A, the probabilities to find the results a_i are given by $p_{\psi_{b_2}}(a_i) = |\langle \alpha_i|\psi_{b_2}\rangle|^2$:

$$\begin{aligned}
p_{\psi_{b_2}}(a_1) &= \mathbb{P}_\psi(a_1|b_2) = 0 \\
p_{\psi_{b_2}}(a_2) &= \mathbb{P}_\psi(a_2|b_2) = \frac{|c_2|^2}{|c_2|^2 + |c_3|^2} \\
p_{\psi_{b_2}}(a_3) &= \mathbb{P}_\psi(a_3|b_2) = \frac{|c_3|^2}{|c_2|^2 + |c_3|^2}
\end{aligned} \tag{6.29}$$

$\mathbb{P}_\psi(a_i|b_2)$ is, in accordance with definition 6.1, interpreted as the probability of finding measurement outcome a_i when observable A is measured upon a system in state ψ, conditional on having found measurement outcome b_2 upon a previous measurement of observable B.

At this point we do not evaluate whether the conditional-on-measurement-outcome interpretation is an adequate interpretation of the probabilities defined by the Lüders projection postulate – we discuss this issue at length in the next chapter. What we want to do now is to clearly distinguish the conditional-on-measurement-outcome probability from another type of (purportedly) conditional probability which also arises in the context of quantum measurements.

6.4 Conditional-on-Measurement Probability

It is not uncommon to hear that all quantum probabilities are conditional probabilities since they are given by the Born rule and this rule yields conditional probabilities for particular measurement outcomes conditional on measurements.[6.7] The projection postulate of the orthodox interpretation, in both its von Neumann and Lüders' version, does indeed lead to this reading: $p_W(a_i) = \mathrm{Tr}(WP_{a_i})$ is interpreted as the probability of measurement result a_i conditional on a measurement of observable A on a system in state W). The quantum probabilities under the Orthodox account of quantum mechanics are thus interpreted as probabilities for measurement outcomes conditional on measurements performed. We refer to this interpretation of the quantum probability $p_W(a_i) = \mathrm{Tr}(W P_{a_i})$ as the conditional-on-measurement interpretation (and not conditional-on-measurement-*outcome*).

Definition 6.2. ***Conditional-on-Measurement Probability****. When an observable A is measured on a system in state W, the probability*

$$p_W(a_i) = \mathrm{Tr}(WP_{a_i}) \tag{6.30}$$

is the probability of measurement outcome a_i conditional on a measurement of observable A.

But is the conditional-on-measurement probability just defined really a conditional probability? This probability seems, if at all, a very strange species of conditional probability. For it is evidently not a probability for an event given another event, nor, in particular, for a measurement outcome conditional on another measurement outcome. These conditional-on-measurement probabilities really seem to be *un*conditional probabilities for finding certain measurement outcomes *when* measurements are performed. One can stretch the use of 'conditional' and say that these probabilities are probabilities 'conditional' on performing measurements. But this reading seems to retain little of the notion of conditionality.

6.7. For example, Hájek says 'Quantum mechanics apparently tells us that certain chances, conditional on free acts, are defined, and it even purports to tell us their values. For example, it tells us that the chance that a certain particle is measured to be spin-up, given that it is measured for spin in a given direction, is 1/2.' ([Hájek, 2003a], p-305)

Indeed, there seems to be an important distinction between the role of 'background conditions' which specify the conditions in effect at the assessment of the probability function – in this case, the measurement procedure – and the propositions that can really be conditioned on. It is not clear that the physical situation of the measurement can be given any sort of propositional form that could resemble an event nor, if it could, whether it is a type of event to which one could ascribe (or be interested in ascribing) a probability. Moreover, quantum mechanics does not itself provide such a probability ascription: it would need to give probabilities to propositions of the form 'measurement M is performed by experimenter X', which it does not, and most likely cannot, do. Hence, the conditional-on-measurement probabilities are better understood as *un*conditional probabilities.

The question of whether the conditional-on-measurement probability is a genuine conditional probability, is in fact very similar to another debate about probabilities in the propensity account literature. On the propensity interpretation probabilities measure the disposition or tendency of a particular experimental set-up to produce a certain frequency of outcomes in the long run; similarly, $p_W(a_i)$ in (6.30) can be thought of as the tendency of the measuring device to produce a frequency of outcome a_i equal to the value $p_W(a_i)$ when measuring observable A. And the question arises as to whether propensities actually play the role of conditional or unconditional probabilities.

We will not go into this discussion in any detail[6.8] for we here take the view that, irrespective of the particular interpretation of probability, the distinction between background conditions and regular events does indicate the distinction between conditional and *un*conditional probabilities. Hence, we regard conditional-on-background-conditions probabilities, such as the conditional-on-measurement probability of definition 6.2, as *un*conditional probabilities, and probabilities conditional on events as (genuine) conditional probabilities.

This distinction is particularly relevant for us since the classical theorem 2.1 and its analogue in quantum probability theory (theorem 4.2), which are precisely supposed to characterize conditional probability, could otherwise not fulfill their task. Indeed, for the probability $\mathbb{P}_p(A|B)$ to be a (genuine) conditional probability and be defined in terms of the unconditional probability measure p, it is necessary that the measure p applies to both A and B. For example, $\mathbb{P}(1|\text{odd})$ in the die example is a (genuine) conditional probability since p assigns both the events '$1 \cap \text{odd}$' and 'odd' a probability which serves to calculate $\mathbb{P}(1|\text{odd})$. In contrast, in the example of the radioactive particle, p(particle decays $\cap$

6.8. See, for example, [Easwaran, 2008]'s reconstruction of the debate, [McCurdy, 1996], [Gillies, 2000], and [Humphreys, 1985].

particle has the relevant features) is undefined and hence the conditional probability that the particle decays given it has the relevant features is also undefined. And thus the probability for a radioactive particle to decay is better conceived of as an unconditional probability.[6.9] Of course, that the particle is of the relevant isotope, or has the relevant atomic structure, is indeed relevant for the particle to decay, but not in the way in which a (genuine) conditional probability requires.

Similarly, the conditional-on-measurement probability of definition 6.2 is not a (genuine) conditional probability. The event 'measurement M of observable A is performed by experimenter X' is not relevant to the event 'measurement result a_i' in the right way. There is no projector operator which represents it as P_{a_i} represents 'measurement result a_i', and hence the density operator W cannot assign it any probability similarly to how it assigns it to P_{a_i}. Hence their conditional probability is not determined. And thus, again, the conditional-on-measurement probabilities are better understood as *un*conditional probabilities – albeit ones which only apply to measurement results.

To conclude, the conditional-on-measurement probability given by definition 6.2 is really an *un*conditional probability which, as such, does not need to be considered as a possible candidate for a quantum notion of conditional probability.

6.5 Non-Adequacy of the Orthodox Interpretation

We finally turn to evaluate whether the orthodox interpretation, with its reliance on the Projection Postulate, provides an adequate interpretation of quantum theory. First, as we saw in section 6.2, the projection postulate is usually justified because it ensures repeatability of measurement results. That is, it guarantees that when we repeat a measurement, the result of the second measurement always matches the result of the first. However, while sufficient to ensure repeatability, the projection postulate is by no means necessary. Indeed, nothing more than the quantum probability theory and the Schrödinger equation are required to guarantee the matching.[6.10]

Consider, similarly to equation (6.5), a measurement interaction described by a unitary dynamical evolution (in particular, we make no use of the projection postulate), i.e.

$$\sum_i c_i\,\alpha_i\,M_0 \longrightarrow \sum c_i\,\alpha_i\,M_i \tag{6.31}$$

6.9. Note that Hájek sees this as a failure of the ratio analysis while we see it as indicating that the probability involved is not really a conditional probability. See footnote B.2 and [Hájek, 2003a].

6.10. [Dickson, 1998], pp.28-29

And introduce a second apparatus, whose eigenstates are N_i. The second measurement interaction, again under unitary dynamics, gives

$$\sum_i c_i\,\alpha_i\,M_0\,N_0 \longrightarrow \sum c_i\,\alpha_i\,M_i\,N_i \tag{6.32}$$

Calculate now the probability of two states of the measuring devices P_j^M and P_i^N, i.e.

$$p_\psi(P_j^M \otimes P_i^N) = \left|\left(\sum_i c_i\langle\alpha_i|\langle M_i|\langle N_i|\right) P_j^M \otimes P_k^N \left(\sum_i c_i|\alpha_i\rangle|M_i\rangle|N_i\rangle\right)\right|^2 = \delta_{jk}\,|c_j|^2 \tag{6.33}$$

where $\delta_{ij}=0$ if $i \neq j$ and $\delta_{ij}=1$ if $i=j$. The probability for finding the two measuring devices in non-matching states given by (6.33) is already zero, with no need to invoke von Neumann's or the Lüders projection postulate. Hence, the projection postulate can only be partly justified by empirical evidence. As Lüders says,

> 'The [von Neumann] ansatz – at least so far – is justified only partly through experiment, but mainly by its compelling simplicity' ([Lüders, 1951], p.664)[6.11]

In addition, the projection postulate faces other difficulties that cast serious doubt on its adequacy. First, the projection postulate introduces an extra-dynamics for the act of measurement and thus makes measurement interactions unlike *any other* interactions. Indeed, measurement interactions cannot count as regular physical process because the projection postulate, which gives the evolution of the system upon measurement, cannot be derived by considering a Schrödinger evolution for the composite system containing the measuring apparatus – this is what the insolubility proofs of the measurement show.[6.12] But what else could measurement interactions be?

Moreover, even if one admits this special status for measurements, the orthodox view does not say what kinds of interactions qualify as measurements. It introduces the notion of measurement into the statement of the fundamental physical laws without providing an explicit definition of measurement nor what is it about measurements that causes such a collapse. And to make things worse, the solution it gives for the 'measurement problem' is a non-starter as soon as one realizes that the problem is not only restricted to the context of a measurement but to *all* macroscopic objects: all sorts of interactions involving macroscopic systems will evolve by Schrödinger's law into states that are not eigenstates of ordinary physical properties. The orthodox interpretation is hence on very bad grounding unless one thinks of quantum theory instrumentally, that is, as merely providing an algorithm for generating the statistical predictions of the outcomes of measurements.

6.11. Note that Lüders' statement equally applies to his own rule for change of state upon measurement.

6.12. [Wigner, 1963], [Fine, 1970], [Shimony, 1974]

Von Neumann's and Lüders' projection postulates are thus, at best, seen as nothing more than definitions of special classes of measuring instruments (with interactions unlike any other physical interactions!). And though the majority of the measuring instruments used in practice do not satisfy either of them, there is no definite example of a physical quantity not admitting at least one measuring instrument satisfying, even if only approximately, these postulates.[6.13] Following a terminology proposed by Wolfgang Pauli, the measuring instruments that obey von Neumann's projection postulate are referred to as 'first-kind instruments'. And those that match the stronger postulate of Lüders are often called 'ideal and of first kind'.

6.13. [Beltrametti & Cassinelli, 1981], p.79.

Chapter 7
Interpreting Quantum Conditional Probability II

In Chapter 5 we argued that the Lüders rule does not define the quantum extension of the notion of conditional probability at a physically meaningful level, and we asked ourselves what concept it could possibly define. It appears that the previous chapter provides an answer to this question: when an observable B is measured on a system in state W (with an ideal first-kind measuring device), followed by a second measurement of an observable A, the probability $\mathbb{P}_W(a_i|b_i) = \mathbb{P}_{W_{b_i}}(a_i) = \frac{\mathrm{Tr}(P_{b_i} W P_{b_i} P_{a_i})}{\mathrm{Tr}(P_{b_i} W)}$ given by definition 6.1 seems to define the probability of measurement outcome a_i conditional on measurement outcome b_i. But is this really so? In this chapter we argue against this claim.

We also argue that introducing a diachronic perspective for the conditional-on-measurement-outcome probability interpretation, and thus interpreting $\mathbb{P}_W(a_i|b_i)$ as the probability for finding measurement outcome a_i when observable A is measured *at time* t_f, conditional on a measurement of measurement of observable B with measurement outcome b_i *at time* t_i, provides little help in understanding the probabilities defined by the Lüders rule as conditional-on-measurement-outcomes probabilities (section 7.3). Moreover, this discussion sheds further light on why the formal projective notion of conditional probability cannot, after all, yield an adequate reading, both interpreted synchronically or diachronically (section 7.4). Hence, we conclude that the probability defined by the Lüders rule cannot be interpreted as a conditional probability neither for measurement results, nor at a formal level for projection operators (nor for physical values), both from a synchronic and a diachronic perspective.

In section 7.5 we use our discussion on the diachronic projective notion of conditional probability to show explicitly that the only possible reading of the probability defined by the Lüders rule as a conditional probability is a purely instrumental one. Indeed, this rule can only define the probability for measurement outcome a_i at time t_2 *immediately before* the measurement of observable A conditional on measurement outcome b_i at time t_0 *immediately before* the measurement of B, namely

$$\mathbb{P}_W(a_i, t_2|b_i, t_0) = \frac{\mathrm{Tr}[W P_{b_i}(t_0)\, P_{a_i}(t_2)\, P_{b_i}(t_0)]}{\mathrm{Tr}[W P_{b_i}(t_0)]} \tag{7.1}$$

therefore saying nothing about what happens to the system during the first measurement nor between both measurements. Thus, it offers a purely instrumental interpretation with a strong 'black-box' character which is unsatisfactory unless one reduces quantum theory to an algorithm for generating the statistical predictions of the outcomes of measurements.

Indeed, if when one says the probability of a certain measurement outcome a_i at time t_2 given a previous measurement at time t_0 which has outcome b_i is $\mathbb{P}_W(a_i, t_2|b_i, t_0)$ one only means that if these two measurements are repeated many times, one after the other, one expects that the fraction of those which give the outcome a_i is roughly $\mathbb{P}_W(a_i, t_2|b_i, t_0)$, then no problems arise. But as soon as one attempts to say *anything* else, then all the problems we consider in sections 5.2, 5.3, 5.4, 7.3 and 7.4 appear.

We end the chapter by reconsidering the two-slit experiment in the light of our discussion (section 7.6) and by evaluating two further arguments, namely those presented in [Bub, 1979a, 1979b, 2007] and [Fuchs, 2002a, 2002b], for the interpretation of the probability defined by the Lüders rule as a conditional probability (section 7.7).

7.1 A First Look II

So far we have seen two possible definitions of a quantum notion of conditional probability. First, in Chapter 5, we argued that the probability defined by the Lüders for projectors P and Q with respect to the initial probability measure p_W, namely

$$\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(WQ\,PQ)}{\mathrm{Tr}(Q\,WQ)} \tag{7.2}$$

can be interpreted, albeit under a quite feeble and counterintuitive reading, as the quantum extension of conditional probability at the formal level of projection operators (definition 5.1). Indeed, since $\mathbb{P}_W(P|Q)$ is the pro rata increase of the common 'projector' $Q\,PQ$ of P and Q, it is to be understood as the probability of projector P conditional on projector Q, with respect to the initial probability measure p_W.

We also argued in Chapter 5 that this probability cannot be understood as a conditional probability for the eigenvalues p and q associated with P and Q. For if P and Q are incompatible, the event 'p and q' cannot always be represented in terms of the projection operators P and Q as $P \wedge Q$; and in the cases in which such a correspondence does exist, the probability assigned to the event 'p and q' does not correspond to the probability assigned to the common operator of P and Q employed by the Lüders rule, i.e. $Q\,P\,Q$. $\mathbb{P}_W(P|Q)$ cannot, therefore, be read as the pro rata increase of the probability of 'p and q'

and, hence, cannot be interpreted as a quantum conditional probability from a physically meaningful perspective.

Second, in chapter 6, we defined the notion of quantum conditional-on-measurement-outcome probability (definition 6.1). For an ideal first-kind measurement of an observable B (with possible eigenvalues b_i) performed on a system in state W, followed by a measurement of an observable A (with possible eigenvalues a_i), the probability of measurement outcome a_i conditional on measurement outcome b_i is

$$\mathbb{P}_W(a_i|b_i) = \mathbb{P}_{W_{b_i}}(a_i) = \frac{\mathrm{Tr}(P_{b_i} W P_{b_i} P_{a_i})}{\mathrm{Tr}(P_{b_i} W)} \tag{7.3}$$

Or, for a pure state ψ, $\mathbb{P}_\psi(a_i|b_i) = p_{\psi_{b_i}}(a_i) = |\langle \alpha_i, \psi_{b_i}\rangle|^2$, with $\psi_{b_i} = \frac{P_{b_i}\psi}{\|P_{b_i}\psi\|}$.

The probability given by (7.3) can also be expressed in terms of the values p and q associated with P and Q. Imagine we perform an ideal first-kind measurement of a certain observable, where Q belongs to its spectral decomposition, and find measurement outcome q. We then perform a measurement of a second observable, where P belongs to its spectral decomposition. According to definition 6.1, the probability to find measurement outcome p in this second measurement, conditional on having found q in the first one, is

$$\mathbb{P}_W(p|q) = p_{W_q}(p) = \mathrm{Tr}\left(\frac{Q W Q}{\mathrm{Tr}(Q W Q)} P\right) \tag{7.4}$$

for a system in state W.

Now, the right hand side of equations (7.2) and (7.4) are formally the same. But in (7.4) it is interpreted as giving the probability of measurement outcome p conditional on measurement outcome q, i.e $\mathbb{P}_W(p|q)$, whereas in (7.2) it is interpreted as the probability of the projector P conditional on the projector Q, i.e. $\mathbb{P}_W(P|Q)$ (albeit under a weak construal of such a notion). We want to evaluate whether $\mathbb{P}_W(p|q)$ can really be interpreted as the probability of measurement outcome p conditional on measurement outcome q, when P and Q are incompatible. It seems that it cannot, and for reasons that go beyond those we presented in sections 6.5 when discussing the inadequacy of the orthodox interpretation.

Indeed, under the projective interpretation, the probability $\mathbb{P}_W(P|Q)$ is read as the 'probability' of the common 'projector' $Q\,P\,Q$ increased pro rata, i.e. divided by $p_W(Q)$. The state W determines the 'probability' of both the common 'projector' $Q\,P Q$ – an operator which only depends on the projectors P and Q – and the probability of projector Q, and hence determines the probability of P conditional on Q. However, for the conditional-on-measurement-outcome reading of $\mathbb{P}_W(p|q)$, there seems to be no possible notion of commonality between measurement outcomes p and q which would underwrite interpreting it as a conditional-on-measurement-outcome probability when P and Q are incompatible observables. For one cannot seemingly perform simultaneous measurements of incompatible quantities and thus cannot read $\mathbb{P}_W(p|q)$ as the probability for finding measurement result p and q, increased pro rata.[7.1]

The only possible reading of $\mathbb{P}_W(p|q)$ seems to be that of a *transition* probability: changes of state are called transitions, and the probabilities associated with them are called transition probabilities. In effect, upon finding q as the measurement outcome of the first measurement, the state of the system changes from W to $W_q = \frac{QWQ}{\mathrm{Tr}(QWQ)}$, where the probability for finding q is given by $p_W(q) = \mathrm{Tr}(WQ)$. Then, when the system is measured a second time and measurement result p is found, its state changes again, this time from state W_q to state W_p, where $W_p = \frac{PW_qP}{\mathrm{Tr}(W_qP)}$ if the second measurement is also ideal and of the first-kind. The probability for finding the outcome p is given by the state W_q as $p_{W_q}(p) = \mathrm{Tr}(W_q P)$ in accordance with (7.4). Thus, $\mathbb{P}_W(p|q)$ is the probability that the state of the system changes from W_q to W_p, that is, it is a transition probability.

However, the situation is far more involved that this. First, even if one cannot seemingly perform simultaneous measurement of incompatible quantities, one can still consider the probability for finding measurement result p and q at different times, i.e. $p_W(p_{t_f} \& q_{t_i})$. And thus one can try to interpret $\mathbb{P}_W(p|q)$ as the probability for finding measurement outcome p at a certain time t_f, conditional on having found measurement outcome q at an earlier time t_i, i.e. $\mathbb{P}_W(p_{t_f}|q_{t_i})$. This would yield a notion of conditional probability different both from a synchronic notion of conditional probability $\mathbb{P}_W(p_{t_i}|q_{t_i})$ – wherein measurement results p and q are both considered at the same initial time t_i – and from a diachronic notion of conditionalization $p_W(p_{t_f}) = \mathbb{P}_W(p_{t_i}|q_{t_i})$ – wherein the probability of measurement outcome p at time t_f is updated by equating it to the synchronic conditional probability of p given q at time t_i. We might call this notion, in general, diachronic conditional probability; and in our particular case, diachronic conditional-on-measurement-outcome probability.

And second, while it is true that one cannot measure simultaneously two incompatible quantities *directly*, one can measure them simultaneously by means of *indirect* measurements (when one allows the system of interest to interact with another system on which one can also perform measurements). Thus, the possibility of a simultaneous measurement of incompatible quantities cannot be ruled out so lightly and, therefore, $\mathbb{P}_W(p|q)$ might still allow a synchronic conditional-on-measurement-outcome interpretation. Let us consider these different possibilities in detail.

7.1. See for example [Margenau, 1963a, 1963b], [Parker & Margenau, 1968], and [Varadarajan, 1962].

7.2 Transition Probabilities

7.2.1 Classical Transition Probability

Classical transition probabilities appear in the theory of Markov chains.[7.2] A *Markov chain*, is a stochastic process with the *Markov property*, namely that, given the present state, future states are independent of the past states; in other words, that the description of the present state fully captures all the information that could influence the future evolution of the process. Formally, a Markov chain is a sequence of random variables $X_1, X_2, \ldots., X_n$ which satisfy the Markov property

$$\mathbb{P}(X_{n+1}=x|X_n=x_n, \ldots, X_1=x_1)=\mathbb{P}(X_{n+1}=x|X_n=x_n) \tag{7.5}$$

At each step the system may change its state from the current state to another state, or remain in the same state, according to the probability distribution given by (7.5).[7.3] The changes of state are called transitions, and the probabilities associated with the various state-changes are called transition probabilities.

Now *only* for reversible Markov chains, which are necessarily stationary, can transition probabilities be read as conditional probabilities. In effect, time-homogeneous or stationary Markov chains are processes where, for all n,

$$\mathbb{P}(X_{n+1}=x|X_n=y)=\mathbb{P}(X_n=x|X_{n-1}=y) \tag{7.6}$$

So that if X_0 has a certain distribution p, then X_n at any subsequent time has the same distribution. Reversible Markov chains are those in which one can 'invert' a transition probability using Bayes' rule, i.e.

$$\mathbb{P}(X_n=i|X_{n+1}=j)=\frac{p(X_n=i, X_{n+1}=j)}{p(X_{n+1}=j)}=\frac{p(X_{n+1}=j|X_n=i)\,p(X_n=i)}{p(X_{n+1}=j)} \tag{7.7}$$

Intuitively, a reversible chain is one in which given a movie of the chain run forward and the same movie run backward, one cannot tell which is which.

7.2. We give here a very basic exposition of the theory of Markov chains. See [Doob, 1953] for a detailed exposition.

7.3. Note that chains with a certain 'memory' can also be regarded as Markov chains. In effect, for a Markov chain with memory m, where m is finite, i.e. $p(X_n = x_n|X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_1 = x_1) = p(X_n = x_n|X_{n-1}=x_{n-1}, X_{n-2}=x_{n-2}, \ldots, X_{n-m}=x_{n-m})$, it is possible to construct a chain (Y_n) from (X_n) which has the Markov property. Indeed, let $Y_n = (X_n, X_{n-1}, \ldots, X_{n-m+1})$, the ordered m-tuple of X values. Then Y_n is a Markov chain that has the Markov property.

In general, thus, transitions probabilities do not satisfy equation (7.7). Only for reversible processes does it hold and therefore

$$\begin{aligned} p(X_n = i, X_{n+1} = j) &= \mathbb{P}(X_n = i | X_{n+1} = j)\, p(X_{n+1} = j) \\ &= \mathbb{P}(X_{n+1} = j | X_n = i)\, p(X_n = i) \end{aligned} \tag{7.8}$$

That is, both products are equal to the joint probability $p(X_n = i, X_{n+1} = j)$ and hence the transition probabilities $\mathbb{P}(X_{n+1} = j | X_n = i)$ and $\mathbb{P}(X_n = i | X_{n+1} = j)$ can also be read as conditional probabilities, i.e.

$$\begin{aligned} \mathbb{P}(X_{n+1} = j | X_n = i) &= \frac{p(X_n = i, X_{n+1} = j)}{p(X_n = i)} \\ \mathbb{P}(X_n = i | X_{n+1} = j) &= \frac{p(X_n = i, X_{n+1} = j)}{p(X_{n+1} = j)} \end{aligned} \tag{7.9}$$

Note, however, that these conditional probabilities, are different both from the synchronic notion of conditional probability and from the diachronic notion of conditionalization we saw in chapter 2. Expressing them in terms of our usual notation for conditional probabilities, with the events indexed in time, namely

$$\begin{aligned} \mathbb{P}(A_{t_2} | B_{t_1}) &= \frac{p(B_{t_1} \cap A_{t_2})}{p(B_{t_1})} \\ \mathbb{P}(B_{t_1} | A_{t_2}) &= \frac{p(B_{t_1} \cap A_{t_2})}{p(A_{t_2})} \end{aligned} \tag{7.10}$$

we can clearly see that they are different from a synchronic notion of conditional probability

$$\mathbb{P}_{t_1}(A|B) = \mathbb{P}(A_{t_1} | B_{t_1}) = \frac{p(B_{t_1} \cap A_{t_1})}{p(B_{t_1})} \tag{7.11}$$

wherein events A and B are both considered at the same initial time t_1, and from a diachronic notion of conditionalization

$$p_{t_2}(A) = \mathbb{P}_{t_1}(A|B) \tag{7.12}$$

wherein the probability of A at time t_2 is updated by equating it to the synchronic conditional probability of A given B at time t_1. Indeed, A and B are considered at different times in the joint event of A and B in the conditional probabilities (7.10). As we suggested in the previous section, we can call this new notion of conditional probability, diachronic conditional probability.

To sum up, in the classical case, if a process is reversible then the equality

$$\mathbb{P}(A_{t_2} | B_{t_1})\, p(B_{t_1}) = \mathbb{P}(B_{t_1} | A_{t_2})\, p(A_{t_2}) = p(B_{t_1} \cap A_{t_2}) \tag{7.13}$$

is satisfied for the transition probabilities $\mathbb{P}(A_{t_2}|B_{t_1})$ and $\mathbb{P}(B_{t_1}|A_{t_1})$. And hence $\mathbb{P}(A_{t_2}|B_{t_1})$ and $\mathbb{P}(B_{t_1}|A_{t_2})$ can also be interpreted as diachronic conditional probabilities. The reverse implications also hold. Hence, a process is reversible if and only if equality (7.13) holds; and the probabilities $\mathbb{P}(A_{t_2}|B_{t_1})$, $\mathbb{P}(B_{t_1}|A_{t_2})$ can be interpreted as (classical) conditional probabilities if and only if equality (7.13) holds.

7.2.2 Quantum Transition Probability

As we have just seen, if $\mathbb{P}(A_{t_2}|B_{t_1})\, p(B_{t_1}) \neq \mathbb{P}(B_{t_1}|A_{t_2})\, p(A_{t_2})$, then the classical process that these probabilities represent is non-reversible and the transition probabilities $\mathbb{P}(A_{t_2}|B_{t_1})$ and $\mathbb{P}(B_{t_1}|A_{t_2})$ cannot be interpreted as conditional probabilities. However, if A and B are considered at the same time, the ratio definition of synchronic conditional probability implies that $\mathbb{P}(A|B)\ p(B) = \mathbb{P}(B|A)\ p(A)$ always holds. Hence, the failure of equality (7.13) only makes sense in the classical case if events A and B occur at different times. In contrast, in the quantum case, the equality analogue to (7.13) for the probabilities $\mathbb{P}_W(p|q)$ and $\mathbb{P}_W(q|p)$ defined by the Lüders rule, namely

$$\mathbb{P}_W(p|q)\, p_W(q) = \mathbb{P}_W(q|p)\, p_W(p) \tag{7.14}$$

can fail even if p and q are considered at the same time. Indeed, only if P and Q are compatible, is (7.14) satisfied. We can easily show that this is the case for our previous examples involving compatible and incompatible observables.

Example 7.1. Incompatible Observables. In example 6.1 we considered a system in state $\psi = c_1\,\beta_1 + c_2\,\beta_2 + c_3\,\beta_3$, where $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, and two incompatible observables $A = a_1\,P_{\alpha_1} + a_2\,P_{\alpha_2} + a_3\,P_{\alpha_3}$ and $B = b_1(P_{\beta_1} + P_{\beta_3}) + b_2\,P_{\beta_2}$, where $\alpha_1 = \frac{\beta_1+\beta_2}{\sqrt{2}}$; $\alpha_2 = \frac{\beta_1-\beta_2}{\sqrt{2}}$; $\alpha_3 = \beta_3$. Initially state ψ assigns probabilities to the various measurements results a_i of observable A

$$p_\psi(a_1) = \frac{1}{2}\,(|c_1|^2 + |c_2|^2); \quad p_\psi(a_2) = \frac{1}{2}\,(|c_1|^2 + |c_2|^2); \quad p_\psi(a_3) = |c_3|^2 \tag{7.15}$$

And to the various measurements results b_i of observable B

$$p_\psi(b_1) = |c_1|^2 + |c_3|^2; \quad p_\psi(b_2) = |c_2|^2 \tag{7.16}$$

Let us first calculate $\mathbb{P}_\psi(a_1|b_1)\ p_\psi(b_1)$ and then $\mathbb{P}_\psi(b_1|a_1)\ p_\psi(a_1)$. If we perform an ideal first-class measurement of observable B and find result b_1, the Lüders projection postulate tells us that the new state just after the measurement of B is

$$\psi_{b_1} = \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\,\beta_1 + \frac{c_1}{\sqrt{|c_1|^2 + |c_3|^2}}\,\beta_3 \tag{7.17}$$

With this new state vector:

$$p_{\psi_{b_1}}(a_1) = \mathbb{P}_\psi(a_1|b_1) = \frac{1}{2}\,\frac{|c_1|^2}{|c_1|^2 + |c_3|^2} \tag{7.18}$$

and thus

$$\mathbb{P}_\psi(a_1|b_1)\, p_\psi(b_1) = \frac{1}{2}|c_1|^2 \tag{7.19}$$

Now, if we perform first an ideal first-class measurement of observable A and find result a_1, the Lüders projection postulate tells us that the new state is

$$\psi_{a_1} = \alpha_1 = \frac{\beta_1 + \beta_2}{\sqrt{2}} \tag{7.20}$$

With this new state vector:[7.4]

$$p_{\psi_{a_1}}(b_1) = \mathbb{P}_\psi(b_1|a_1) = \frac{1}{2} \tag{7.21}$$

and thus

$$\mathbb{P}_\psi(b_1|a_1)\, p_\psi(a_1) = \frac{1}{4}\,(|c_1|^2 + |c_2|^2) \tag{7.22}$$

which is different form the result of (7.19). Hence,

$$\mathbb{P}(b_1|a_1)\, p(a_1) \neq \mathbb{P}(a_1|b_1)\, p(b_1) \tag{7.23}$$

Example 7.2. Compatible Observables. In example 6.2 we considered a system in state $\psi = c_1\,\alpha_1 + c_2\,\alpha_2 + c_3\,\alpha_3$ and two commuting operators $A = a_1\,P_{\alpha_1} + a_2\,P_{\alpha_2} + a_3\,P_{\alpha_3}$, and $B = b_1\ P_{\alpha_1}\ +\ b_2(P_{\alpha_2}\ +\ P_{\alpha_3})$. Initially state ψ assigns probabilities to the various measurements results a_i of observable A

$$p_\psi(a_i) = |c_i|^2 \tag{7.24}$$

and to the possible measurement results b_i of observable B

$$p_\psi(b_1) = |c_1|^2; \quad p_\psi(b_2) = |c_2|^2 + |c_3|^2 \tag{7.25}$$

If we perform an ideal first-class measurement of observable B and find result b_2, the Lüders projection postulate tells us that the resultant state after this measurement

$$\psi_{b_2} = \frac{c_2}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_2 + \frac{c_3}{\sqrt{|c_2|^2 + |c_3|^2}}\,\alpha_3 \tag{7.26}$$

With this new state vector

$$p_{\psi_{b_2}}(a_2) = \mathbb{P}_\psi(a_2|b_2) = \frac{|c_2|^2}{|c_2|^2 + |c_3|^2} \tag{7.27}$$

7.4. $\langle \psi_{a_1}, (P_{\beta_1} + P_{\beta_3})\,\psi_{a_1}\rangle = \langle \psi_{a_1}, P_{\beta_1}\,\psi_{a_1}\rangle = |\langle \beta_1, \psi_{a_1}\rangle|^2 = \frac{1}{2}$

and thus

$$\mathbb{P}_\psi(a_2|b_2)\, p_\psi(b_2) = |c_2|^2 \tag{7.28}$$

Now, if we perform an ideal first-class measurement of observable A getting result a_2, the Lüders projection postulate tells us that the resultant state after this measurement is

$$\psi_{a_2} = \alpha_2 \tag{7.29}$$

With this new state vector

$$p_{\psi_{a_2}}(b_2) = \mathbb{P}_\psi(b_2|a_2) = 1 \tag{7.30}$$

and thus

$$\mathbb{P}_\psi(b_2|a_2)\, p_\psi(a_2) = |c_2|^2 \tag{7.31}$$

Hence,

$$\mathbb{P}(a_2|b_2)\, p(b_2) = \mathbb{P}(b_2|a_2)\, p(a_2) \tag{7.32}$$

So what are the interpretive consequences of the failure of equality (7.14) for incompatible observables? Orthodox interpreters claim that the key to understanding this failure lies in that, analogously to the classical case, the probabilities $\mathbb{P}_W(p|q)$ for incompatible observables are not really conditional-on-measurement-outcome probabilities, but rather transition probabilities for irreversible processes, and hence cannot be given a conditional interpretation. Indeed, the probability $\mathbb{P}_W(p|q)$ only makes sense if it is indexed in time as $\mathbb{P}_W(p_{t_2}|q_{t_1})$ and then it is to be interpreted as a transition probability; that is, as the probability that a measurement changes the state of the quantum system from state W_q at time t_1 to state W_p at time t_2. The underlying reasoning would go something like this.

Given that one cannot perform simultaneous measurements of incompatible observables, the probability $\mathbb{P}_W(p|q)$ (for P and Q incompatible) only makes sense if it is indexed in time. In addition, given that quantum measurements are necessarily disturbing, they correspond to irreversible processes. Indeed, quantum measurements are invariably processes which drastically change the measured system and which introduce an irreducible disturbance to the quantum system. And thus, transition probabilities for incompatible observables cannot be understood as classical conditional probabilities. That is why equation (7.14), appropriately indexed in time, i.e. $\mathbb{P}_W(p_{t_2}|q_{t_1})\, p_W(q_{t_1}) = \mathbb{P}_W(q_{t_1}|p_{t_2})\, p_W(p_{t_2})$, does not hold for incompatible ones. In orthodox quantum mechanics measurements are thus not ideal processes for merely learning something. Only if the quantities involved are compatible is quantum measurement like classical measurement.

The reasoning afforded by the orthodox interpreter for why the probabilities defined by the Lüders rule for incompatible events are transition probabilities for the irreversible process of a quantum measurement, and hence cannot be given a classical conditional-on-measurement-outcome interpretation is, however, flawed. Schematically, it is the following:

Premise 1: One cannot perform simultaneous measurements of incompatible observables, and hence the probability $\mathbb{P}_W(p|q)$ (for P and Q incompatible) only makes sense if it is indexed in time. For P and Q incompatible, $\mathbb{P}_W(p_{t_f}|q_{t_i})$ is thus a transition probability.

Premise 2: Quantum measurements are necessarily disturbing, and hence correspond to irreversible processes.

Premise 3: Transitions probabilities for irreversible processes cannot be interpreted as conditional probabilities.

Conclusion: The transition probability $\mathbb{P}_W(p_{t_2}|q_{t_1})$ for P and Q incompatible observables cannot be understood as a classical diachronic conditional-on-measurement-outcome probability.

The problem with this argument is two-fold. First, premise 1's claim on the non-simultaneous measurability of incompatible observables is incorrect. Indeed, even if one cannot measure simultaneously two incompatible quantities *directly*, and, seemingly, cannot interpret $\mathbb{P}_W(p|q)$ as the pro rata increase of the joint probability of measurement outcome p and q, one *can* measure them simultaneously by means of *indirect* measurements (when one allows the system of interest to interact with another system on which one can also perform measurements). And thus it is not correct to hold that $\mathbb{P}_W(p|q)$ (for P and Q incompatible) only makes sense if it is indexed in time. In effect, $\mathbb{P}_W(p|q)$ might still allow a *synchronic* conditional-on-measurement-outcome interpretation.

Second, the orthodox interpreter does not present any justification for the assumption of premise 2, namely that quantum measurements are necessarily disturbing. Moreover, even if this assumption does hold in practice – there is no argument to the effect that quantum measurements are necessarily disturbing as a matter of principle, but it in practice quantum measurements *are* disturbing– the conclusion of premise 2 does not follow from it. For disturbing processes need not always lead to irreversible processes. Hence, one can still consider the probability for finding measurement result p and q at different times, i.e. $p_W(p_{t_2}\&q_{t_1})$, and try to interpret $\mathbb{P}_W(p_{t_2}|q_{t_1})$ as the probability for finding measurement outcome p at time t_2 conditional on having found measurement outcome q at an earlier time t_1; that is, as a diachronic-conditional-on-measurement-outcome probability.

The rationale given by the orthodox interpreter to the effect that the probability $\mathbb{P}_W(p|q)$ only makes sense if it is indexed in time as $\mathbb{P}_W(p_{t_2}|q_{t_1})$, and that then it is to be interpreted as a transition probability, thus, is not correct. Unlike the classical case, the fact that (7.14) does not hold for P and Q incompatible at the same and at different times, i.e. $\mathbb{P}_W(p|q)\, p_W(q) \neq \mathbb{P}_W(q|p)\, p_W(p)$ and $\mathbb{P}_W(p_{t_2}|q_{t_1})\, p_W(q_{t_1}) \neq \mathbb{P}_W(q_{t_1}|p_{t_2})\, p_W(p_{t_2})$, does not rule out the interpretation of the probability $\mathbb{P}_W(p|q)$ and $\mathbb{P}(p_{t_2}|q_{t_1})$ as a synchronic or diachronic conditional-on-measurement-outcome probability.

However, as it turns out, this is not so. But for reasons which are not, contrary to what the orthodox interpreter holds, related to the (allegedly) disturbing character of measurements of incompatible quantities. The reasons for why $\mathbb{P}_W(p|q)$ and $\mathbb{P}(p_{t_2}|q_{t_1})$ cannot be so interpreted, i.e. as synchronic or a diachronic conditional-on-measurement-outcome probabilities for incompatible quantities, lie elsewhere. We tackle this issue in the next section.

7.3 Conditional-on-Measurement-Outcome Probability

Let us then see why the probability defined by the Lüders projection postulate $\mathbb{P}_W(p|q)$ cannot be interpreted as a (synchronic nor diachronic) conditional-on-measurement-outcome probability. Recall that to motivate this possibility, we appealed to the fact that while one cannot measure simultaneously two incompatible quantities *directly* (and thus cannot interpret $\mathbb{P}_W(p|q)$ as the pro rata increase of the joint probability of measurement outcome p and q), one *can* measure them simultaneously by means of *indirect* measurements when one allows the system of interest to interact with another system on which one can also perform measurements.

Discussions on indirect measurements are intimately tied with discussions on locality, and these are rather intricate and involved. However, we do not need to go into them. For, as we already mentioned in the previous section, $\mathbb{P}_W(p|q)$ cannot be interpreted as a synchronic nor diachronic conditional-on-measurement-outcome probability for reasons that have nothing to do with measurements. Hence, we show that the conceptual problems of quantum mechanics *cannot* be traced back to the alleged irreducible and uncontrollable disturbance of the system measured by a measuring instrument.

Let us begin by considering the synchronic case. An interpretation of $\mathbb{P}_W(p|q)$ as a synchronic conditional probability for measurement results, simply considers a particular interpretation of the physical values of the probability $\mathbb{P}_W(p|q)$ of sections 5.3 and

5.4, namely physical values as measurement outcomes. Thus, the same difficulties we encountered there come into play here.

Indeed, recall that we there argued that, for incompatible quantities P and Q, the event 'p and q' cannot always be represented in terms of the projection operators P and Q as $P \wedge Q$, and in the cases in which such a correspondence does exist, the probability assigned to the event 'p and q' does not correspond to the probability assigned to the common operator of P and Q employed by the projective notion of quantum conditional probability, i.e. QPQ. Thus, the probability $\mathbb{P}_W(P|Q)$ defined by the Lüders rule cannot be understood as the pro rata increase of the probability of 'p and q'; and hence, it cannot be interpreted as the probability of value p conditional on value q for P and Q incompatible quantities.

Let us rehearse this argument again for the spin-1 particle example of section 4.4.1. in terms of measurement results. The probability

$$\mathbb{P}_\psi(s_{y+1}|s_{x_{+1}}) = \left\langle \frac{P_{s_{x_{+1}}}\psi}{||P_{s_{x_{+1}}}\psi||}, P_{s_{y+1}} \frac{P_{s_{x_{+1}}}\psi}{||P_{s_{x_{+1}}}\psi||} \right\rangle \tag{7.33}$$

cannot be interpreted as the probability of measurement outcome s_{y+1} conditional on measurement outcome $s_{x_{+1}}$. Indeed, the event $P_{s_{x_{+1}}} \wedge P_{s_{y_{+1}}}$ corresponds to 'S_x takes the value $+1$ and S_y takes the value $+1$', with 'and' the ordinary logical relation. However, it is false under *all* truth assignments V_ψ since there is no state ψ which is an eigenstate of both $P_{s_{x_{+1}}}$ and $P_{s_{y_{+1}}}$. Hence the probability of s_{x+1} conditional on s_{y+1} should be zero for all ψ. Which we know is *not* what the Lüders rule prescribes in (7.33). Hence, $\mathbb{P}_\psi(s_{y+1}|s_{x_{+1}})$ cannot be interpreted as a the probability for measurement outcome s_{y+1} conditional on measurement outcome $s_{x_{+1}}$.

The same applies for the probability $\mathbb{P}_\psi(P_{s_{y_{+1}}}|P_{s_{x_{+1}}} + P_{s_{x_0}})$, although here it is even more difficult to interpret it as a conditional-on-measurement-outcome probability. For the projector $P_{s_{y_{+1}}} \wedge (P_{s_{x_{+1}}} \vee P_{s_{x_0}})$ cannot even be interpreted as measurement outcome s_{y+1} and measurement outcome $s_{x_{+1}}$ or s_{x_0} as we discussed at length in section 5.4.2.

In addition, notice that nothing in the above argument hinges on time; we could run this same argument for measurement results p and q considered at different times. Hence, the probability $\mathbb{P}_W(p|q)$ defined by the Lüders rule *cannot* be interpreted either as the synchronic or the diachronic probability of measurement outcome p conditional on measurement outcome q. Thus, it seems that, after all, we cannot interpret the probabilities defined by the Lüders rule as conditional-on-measurement-outcome probabilities. Their only possible interpretation as conditional probabilities appears to be the, weak and counterintuitive, formal notion of conditional-on-quantum-event probability.

But is the latter interpretation viable after all? In the next section we argue that this formal notion of synchronic conditional probability is even less plausible than what we argued in section 5.2.

7.4 Diachronic Projective Conditional Probability

Similarly to the conditional-on-measurement-outcome probabilities $\mathbb{P}_W(p|q)$ and $\mathbb{P}_W(q|p)$, the conditional-on-quantum-event probabilities $\mathbb{P}_W(P|Q)$ and $\mathbb{P}_W(Q|P)$ are such that for P and Q incompatible

$$\mathbb{P}_W(P|Q)\, p_W(Q) \neq \mathbb{P}_W(Q|P)\, p_W(P) \tag{7.34}$$

(recall that $\mathbb{P}_W(P|Q)$ is formally equivalent to $\mathbb{P}_W(p|q)$), where this inequality fails both for P and Q considered at the same and at different times. i.e.

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})\, p_W(Q_{t_1}) = \mathbb{P}_W(Q_{t_1}|P_{t_2})\, p_W(Q_{t_2}) \tag{7.35}$$

In section 5.2 we considered its interpretive consequences for the former case. We argued that $\mathbb{P}_W(P|Q)$ and $\mathbb{P}_W(Q|P)$ can be understood as extended conditional probabilities – $\mathbb{P}_W(P|Q)$ as the pro rata increase of the common 'projector' $Q\,PQ$, and $\mathbb{P}_W(Q|P)$ as the pro rata increase of the common 'projector' $PQ\,P$ – even though the notion of commonality it relies on is somewhat counterintuitive – the common projector for P and Q in $\mathbb{P}_W(P|Q)$ is different from that of Q and P in $\mathbb{P}_W(Q|P)$ – and weak – the common projector of P and Q is taken to be $Q\,PQ$ (or $PQ\,P$) without any explanation.

However, this notion of commonality would not be so counterintuitive if the events P and Q were considered at different times. Indeed, the operator $Q_{t_1}\,P_{t_2}\,Q_{t_1}$ would be the common 'projector' of P at time t_2 given Q at time t_1 as given by $\mathbb{P}_W(P_{t_2}|Q_{t_1})$, which, prima facie, need not be equal to $P_{t_2}\,Q_{t_1}\,P_{t_2}$, i.e. the common 'projector' of Q at time t_1 given P at time t_2 as given by $\mathbb{P}_W(Q_{t_1}|P_{t_2})$ for the time-reverse process. Indeed, already in the classical context, there are physical processes which are not reversible in which this is so. One would then calculate the (diachronic) joint 'probability' of P_{t_2} and Q_{t_1} through the trace rule, i.e. $\mathrm{Tr}(W\,Q_{t_1}\,P_{t_2}\,Q_{t_1})$, and thus arrive at a less counterintuitive reading of $\mathbb{P}_W(P|Q)$.

Hence, the correct interpretation of $\mathbb{P}_W(P|Q)$ is not as a synchronic conditional probability, namely as the probability at time t_1 of P conditional on Q, but rather as a diachronic conditional probability, namely as the probability of projector P at time t_2 conditional on projector Q at time t_1, i.e. $\mathbb{P}_W(P_{t_2}|Q_{t_1})$. This notion would be somewhat weak – it would still need to postulate that precisely $Q_{t_1}\,P_{t_2}\,Q_{t_1}$ represents the common 'projector' of P_{t_2} given Q_{t_1} – but it would not be as counterintuitive as its synchronic counterpart.

Moreover, this same result would also hold in the presence of interference terms. Indeed, consider the particular example of section 4.4.1 of the spin-1 particle for events $Q = P_{s_{x_{+1}}} + P_{s_{x_0}}$ and $P = P_{s_{y_{+1}}}$. The existence of interference terms in $\mathbb{P}_\psi(P_{s_{y_{+1}}}|P_{s_{x_{+1}}} + P_{s_{x_0}})$ would raise no additional difficulties: again, under the diachronic perspective, there is no reason to suppose that the common 'projector' of $(P_{s_{y_{+1}}})_{t_2}$ given $(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}$ should be the same as that of $(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}$ given $(P_{s_{y_{+1}}})_{t_2}$. In addition, one would understand the origin of the interference terms given that each term of the common 'projector' gives rise to a term in the probability defined by the Lüders rule. In effect, the probability $\mathbb{P}_\psi\Big[(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}\Big|(P_{s_{y_{+1}}})_{t_2}\Big]$ is the sum of two terms because $(P_{s_{y_{+1}}})_{t_2}(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}\ (P_{s_{y_{+1}}})_{t_2}$ is composed of only two common 'projectors'; whereas $\mathbb{P}_\psi\Big[(P_{s_{y_{+1}}})_{t_2}\Big|(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}\Big]$ is the sum of four terms because $(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}\ (P_{s_{y_{+1}}})_{t_2}(P_{s_{x_{+1}}} + P_{s_{x_0}})_{t_1}$ is composed of four common 'projectors'.

However, notice that under this diachronic reading of $\mathbb{P}_W(P_{t_2}|Q_{t_1})$, the projectors P and Q evolve in time. Is this possible? Thus far, we have been working in the 'Schrödinger picture', according to which states evolve in time (according to the Schrödinger equation) and any given observable is at all times represented by the same operator. But to interpret $\mathbb{P}_W(P_{t_2}|Q_{t_1})$ as a diachronic conditional probability for projectors in $\mathcal{L}(\mathcal{H})$, P and Q need to evolve in time. Thus, can $\mathbb{P}_W(P_{t_2}|Q_{t_1})$ really be interpreted as a diachronic conditional probability for projection operators?

Yes, it seems that it can. For there is an equivalent formulation of quantum mechanics, namely the so-called 'Heisenberg picture', which is equivalent to the 'Schrödinger picture' – in the sense that they generate the same probability measures over the values of observables at all times – but which employs the reverse time evolution dependence. That is, in the Heisenberg picture, contrary to what happens in the Schrödinger one, states are constant in time and observables evolve in time. In effect, if A_{t_1} represents a given observable at time t_1, then, in the Heisenberg time-picture, A_t represents the same observable at time t, with

$$A_t = U_{t-t_1}^{-1} A_{t_1} U_{t-t_1} \tag{7.36}$$

Note that this time evolution is different from that of the quantum states, namely $W_t = U_{t-t_1}\ W_{t_1}\ U_{t-t_1}^{-1}$ as given by equation (6.1) because of the order in which the unitary evolution operator U_{t-t_1} and its inverse appear.

And if one now writes $\mathbb{P}_W(P_{t_2}|Q_{t_1})$ in the Heisenberg time picture, i.e. $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$, one gets

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{\mathrm{Tr}(W Q_{t_1} P_{t_2} Q_{t_1})}{\mathrm{Tr}(Q_{t_1} W Q_{t_1})} = \frac{\mathrm{Tr}(W Q_{t_1} U_{t_2-t_1}^{-1} P_{t_1} U_{t_2-t_1} Q_{t_1})}{\mathrm{Tr}(Q_{t_1} W Q_{t_1})} \tag{7.37}$$

Hence, it seems that $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$, as given by (7.37), can actually be read as the probability of P at time t_2 given Q at time t_1: it is the pro rata increase of the 'probability' of the common 'projector' $Q_{t_1}\,(U^{-1}_{t_2-t_1}\,P_{t_1}\,U_{t_2-t_1})Q_{t_1}$. The state of the system W, which does not evolve in time in the Heisenberg picture, generates both the 'probability' of the common 'projector' $Q_{t_1}\,P_{t_2}\,Q_{t_1}$ and the probability of Q_{t_1}. Thus, it seems that, after all, we do have a sound interpretation of the probability defined by the Lüders rule as a conditional probability for projection operators. We refer to this interpretation as the Heisenberg diachronic projective notion of conditional probability.

Definition 7.1. ***Heisenberg Diachronic Projective Quantum Conditional Probability***. *The probability given by (the Heisenberg analogue of) the Lüders rule for projectors $P_{t_2}, Q_{t_1} \in \mathcal{L}(\mathcal{H})$ in the Heisenberg picture, namely*

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{\mathrm{Tr}[W\,(Q_{t_1}\,P_{t_2}\,Q_{t_1})]}{\mathrm{Tr}(WQ_{t_1})} = \frac{p'_W(Q_{t_1}\,P_{t_2}\,Q_{t_1})}{p_W(Q_{t_1})} \tag{7.38}$$

is the probability of projector P at time t_2 conditional on projector Q at time t_1 with respect to the probability measure p_W. The operator $Q_{t_1}\,P_{t_2}\,Q_{t_1}$ represents the common operator of projectors P_{t_2} and Q_{t_1}, where the evolution from P_{t_1} to P_{t_2} is given by the unitary evolution operator $U_{t_2-t_1}$ as

$$P_{t_2} = U^{-1}_{t_2-t_1}\,P_{t_1}\,U_{t_2-t_1} \tag{7.39}$$

To repeat, even if the Heisenberg diachronic projective notion, similarly to the synchronic one, is somewhat weak – it gives no understanding of why precisely the operator $Q_{t_1}\,P_{t_2}\,Q_{t_1}$ represents the common 'projector' of Q_{t_2} and P_{t_1} – it is *not* counterintuitive – there is nothing wrong with the fact that the common projector of P_{t_2} given Q_{t_1} is not equal to that of Q_{t_1} given P_{t_2}. Moreover, this diachronic projective notion appeals to a time evolution which is perfectly acceptable from a physical perspective since it is dictated by a unitary operator (unlike the one which the diachronic conditional-on-measurement-outcome notion would need to invoke).

In effect, the evolution given by (7.39) is from event P at time t_1 to event P at time t_2, i.e. $P_{t_2} = U^{-1}_{t_2-t_1}\,P_{t_1}\,U_{t_2-t_1}$, and then P_{t_2} is used to calculate the common projector of Q_{t_1} and P_{t_2}. This evolution is indeed quite different from the evolution dictated by the Lüders projection postulate from state W before the first measurement to state $W_Q = \frac{Q\,W\,Q}{\mathrm{Tr}(WQ)}$ after the first measurement. Indeed, while the former *can* be represented by a unitary evolution operator, the latter *cannot*. And thus, while the evolution from W to W_Q is generated by an interaction which is unlike any other physical interaction, the evolution from P_{t_1} to P_{t_2} is a perfectly normal time evolution.

This last remark, however, throws cold water on the Heisenberg diachronic projective interpretation. It is indeed quite dubious that, simply by a change of time picture, one gets *both* the probability for event P after a perfectly normal time evolution process represented by the unitary evolution operator $U_{t_2-t_1}$, and the probability for measurement result p after a measurement interaction governed by the projection postulate. Indeed, these two pictures should be equivalent in the sense that they generate the same probability measures over the values of observables at all times. Thus, maybe, the change from the Schrödinger to the Heisenberg picture which led to definition 7.1 is not as innocent as it looks. Indeed, upon a closer look, it turns out to be incorrect: the probability given by (7.38), i.e. $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{\mathrm{Tr}[W\,(Q_{t_1}P_{t_2}Q_{t_1})]}{\mathrm{Tr}(W Q_{t_1})}$, is not equal to the probability given by the Lüders rule, i.e. $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\,W\,Q\,P)}{\mathrm{Tr}(W Q)}$.

To see why our derivation of (7.38) is incorrect, we begin with the (inadequate) interpretation of $\mathbb{P}_W(p_{t_2}|q_{t_1})$ as a diachronic conditional-on-measurement-outcome probability under the Schrödinger picture. At time t_1 an ideal first-kind measurement is performed on a system in state W, and measurement outcome q is found; thus the system's state changes from W to $(W_q)_{t_1} = \frac{Q\,W\,Q}{\mathrm{Tr}(WQ)}$. The system then evolves freely until time t_2: $(W_q)_{t_2} = U_{t_2-t_1}\,(W_q)_{t_1}\,U_{t_2-t_1}^{-1}$. Finally, a second measurement is performed on the system, where P belongs to the spectral decomposition of the observable measured. If it is an ideal first class measurement, then the state after this measurement is again given by the Lüders projection postulate as $(W_p)_{t_2} = \frac{P\,(W_q)_{t_2}\,P}{\mathrm{Tr}[(W_q)_{t_2}\,P]}$. This process is, schematically, the following:

$$W \xrightarrow{\textbf{First Measurement}} (W_q)_{t_1} = \frac{Q\,W\,Q}{\mathrm{Tr}(WQ)} \xrightarrow{\textbf{Free Evolution}} (W_q)_{t_2} = U_{t_2-t_1}\,(W_q)_{t_1}\,U_{t_2-t_1}^{-1} \xrightarrow{\textbf{Second Measurement}} (W_p)_{t_2} = \frac{P\,(W_q)_{t_2}\,P}{\mathrm{Tr}[(W_q)_{t_2}\,P]} \tag{7.40}$$

Now, if the system's state does not change in its free evolution between measurements, as for example in the spin-1 example of section 4.4.1, we have that $(W_q)_{t_2} = (W_q)_{t_1}$. In this case, the state only changes due to the measurement interaction as described by the projection postulate. The probability given by the Lüders rule for measurement outcome p at time t_2 is then given by $(W_q)_{t_2} = (W_q)_{t_1} = \frac{Q\,W\,Q}{\mathrm{Tr}(WQ)}$. Hence,[7.5]

$$\mathbb{P}_W(p, t = t_2|q, t = t_1) = p_{(W_q)_{t_2}}(p) = p_{(W_q)_{t_1}}(p) = \frac{\mathrm{Tr}[(W_q)_{t_1}\,P]}{\mathrm{Tr}[(W_q)_{t_1}\,Q]} = \frac{\mathrm{Tr}(Q\,W\,Q\,P)}{\mathrm{Tr}(WQ)} \tag{7.41}$$

Now write (7.41) in terms of projection operators instead of measurement outcomes, i.e.

$$\mathbb{P}_W(P, t = t_2|Q, t = t_1) = p_{(W_Q)_{t_2}}(P) = p_{(W_Q)_{t_1}}(P) = \frac{\mathrm{Tr}[(W_Q)_{t_1}\,P]}{\mathrm{Tr}[(W_Q)_{t_1}\,Q]} = \frac{\mathrm{Tr}(Q\,W\,Q\,P)}{\mathrm{Tr}(WQ)} \tag{7.42}$$

7.5. Note that probability 'p' and measurement outcome 'p', appear together in (7.39). Though this is not particularly appropriate, their difference should be clear by the context in which they appear.

And then switch to the Heisenberg picture as we did to derive (7.38): first, we said that $\mathbb{P}_{W_{t_2}}(P|Q)$ in the Schrödinger picture, which is equal to $\mathbb{P}_{W_{t_1}}(P|Q)$, was equivalent to $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$ in the Heisenberg picture; second, we developed P_{t_2} as $U_{t_2-t_1}^{-1}P_{t_1}U_{t_2-t_1}$; and we, thus, obtained $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{\mathrm{Tr}(WQ_{t_1}U_{t_2-t_1}^{-1}P_{t_1}U_{t_2-t_1}Q_{t_1})}{\mathrm{Tr}(Q_{t_1}WQ_{t_1})}$, and its understanding as the pro rata increase of the common projector of Q at time t_1 and P at time t_2 (under an ordinary dynamical evolution of projector P_t).

However, this derivation is flawed. For even though $\mathbb{P}_{W_{t_2}}(P|Q)$ is equal to $\mathbb{P}_{W_{t_1}}(P|Q)$ in the Schrödinger time picture – given that the system's state does not change between t_1 and t_2 – neither of them is equivalent to $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$ in the Heisenberg picture. Indeed, in the Heisenberg equivalent of the Schrödinger $\mathbb{P}_{W_{t_2}}(P|Q)$, the projector P does *not* evolve freely from P_{t_1} to $P_{t_2}=U_{t_2-t_1}^{-1}P_{t_1}U_{t_2-t_1}$. For, just as the system's state does not change in between measurements, there is no free evolution of projector P between t_1 and t_2. And hence, $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$ given by (7.38) in the Heisenberg time picture does not describe the same process as $\mathbb{P}_{W_{t_2}}(P|Q)$ in the Schrödinger time picture.

If the Schrödinger and Heisenberg pictures are to be equivalent – in the sense that they generate the same probability measures over the values of observables at all times – then Q_{t_1} must be $Q_{t_1} = \frac{QWQ}{\mathrm{Tr}(WQ)}$. And hence, the probability defined by the Lüders rule in the Heisenberg picture should be

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{☼}} = \mathrm{Tr}\left(\frac{QWQ}{\mathrm{Tr}(WQ)}P\right) \tag{7.43}$$

where the sun symbol ☼ emphasizes that $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{☼}}$ is different from the previous one, i.e. $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$, and is the correct one. Thus, the relevant time evolution for $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{☼}}$ is that of Q_t from Q to $Q_{t_1} = \frac{QWQ}{\mathrm{Tr}(WQ)}$, so that

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{☼}} = \mathrm{Tr}(Q_{t_1}P_{t_2}) = \mathrm{Tr}\left(\frac{QWQ}{\mathrm{Tr}(WQ)}P\right) = \frac{p'_W(QPQ)}{p_W(Q)} \tag{7.44}$$

We refer to this interpretation as the Heisenberg☼ diachronic projective notion of quantum conditional probability.

Definition 7.2. ***Heisenberg☼ Diachronic Projective Quantum Conditional Probability***. *The probability given by (the Heisenberg☼ analogue of) the Lüders rule for projectors $P_{t_2}, Q_{t_1} \in \mathcal{L}(\mathcal{H})$ in the Heisenberg picture, namely*

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{☼}} = \mathrm{Tr}(Q_{t_1}P_{t_2}) = \frac{p'_W(QPQ)}{p_W(Q)} \tag{7.45}$$

is the probability of projector P at time t_2 conditional on projector Q at time t_1 with respect to the probability measure p_W, where the evolution from Q to Q_{t_1} is given by the Heisenberg analogue of the Lüders projection postulate as

$$Q_{t_1} = \frac{Q\,W Q}{\mathrm{Tr}(W Q)} \tag{7.46}$$

The probabilities defined by the Heisenberg* notion, in contrast to those defined by the Heisenberg one, *do* agree with the probabilities defined by the Lüders rule. Thus, the Heisenberg* projective diachronic reading of (7.45), even if rather weak – it gives no understanding of why precisely the operator $Q\,PQ$ represents the common 'projector' of Q_{t_1} and P_{t_2} – does seem to provide a correct non-counterintuitive understanding of why the probability defined by the Lüders rule should be read as a diachronic conditional probability for quantum projectors.

However, a more careful analysis shows that again this is not so; that, in fact, the Heisenberg* notion faces serious and, as we now argue, unsurmountable difficulties. Indeed, first, it is clear that it relies on a physically unacceptable time evolution, for the evolution it invokes is not a unitary one from event P at time t_1 to event P at time t_2. Rather, to calculate $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}}$, one needs to consider the time evolution from Q to $Q_{t_1} = \frac{Q\,W Q}{\mathrm{Tr}(W Q)}$ as if the observable to whose spectral decomposition Q belongs to, say $B = \sum_i q_i\, Q_i$, were subject to a measurement and the measurement interaction were governed by the Heisenberg analogue of the Lüders projection postulate. Indeed, one needs to consider the following time evolution

$$W, Q, P \xrightarrow{\textbf{First Measurement}\; B=\sum_i q_i Q_i} W, Q_{t_1} = \frac{Q\,W Q}{\mathrm{Tr}(W Q)}, P \tag{7.47}$$

The relevant change for $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}}$ is that of Q_t when the first 'measurement' is performed, i.e. from Q to Q_{t_1}. Indeed, in the Schrödinger picture – see diagram (7.40) – it is the evolution from W to $(W_q)_{t_2}$ which is relevant to calculate $\mathbb{P}_W(p|q)$, and not that from $(W_q)_{t_2}$ to $(W_p)_{t_2}$, as would need to be if the relevant time evolution for $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}}$ were $P_{t_2} = U^{-1}_{t_2-t_1}\,P_{t_1}\,U_{t_2-t_1}$.

Now the fact that the evolution from Q to $Q_{t_1} = \frac{Q\,W Q}{\mathrm{Tr}(W Q)}$ is given by an extra-dynamics which cannot be derived from a unitary-type dynamics may not be particularly troublesome, since the projective notion can, at most, only work at a mathematical level for projection operators. So what sense would it make to consider non-measurement unitary-type interactions as physically acceptable and measurement non-unitary-type interactions as unacceptable? Moreover, even if this is so, a problem does arise as to when and why one should consider unitary and non-unitary time-evolutions. Indeed, to make these choices and to justify them (and thus retrieve the correct probabilistic predictions), one needs to supplement the Heisenberg* diachronic projective reading with too many physical intuitions, i.e. 'imagine' process (7.47) as taking place, which are, moreover, not even adequate in the case of the evolution from Q to $Q_{t_1} = \frac{QWQ}{\mathrm{Tr}(WQ)}$ due to all the problems related to the quantum notion of measurement (section 6.5). The resulting notion thus presents, at best, a strange mixture of mathematical and physical components.

However, regardless of whether or not the preceding arguments provide grounds to fully dismiss this second Heisenberg diachronic projective reading (definition 7.2), it turns out that this notion is, after all, not even correct. Indeed, the problem is that for the general case in which the projection operator Q also 'evolves freely' after the first 'measurement', i.e. it 'evolves' from $Q_{t_1} = \frac{QWQ}{\mathrm{Tr}(WQ)}$ (just after the first 'measurement') to $Q_{t_2} = U_{t_2-t_1}^{-1} Q_{t_1} U_{t_2-t_1}$ (just before the second 'measurement'), the probabilities provided by the Heisenberg* picture do not agree with the probabilities defined by the Lüders rule in the Schrödinger picture. And the latter are the correct empirical probabilities.

Let us see this in detail. When considering the probability of projector P at time t_2 given projector Q at time t_1 in the Heisenberg* picture, one has to *imagine* a time evolution process similar to that given in (7.40), namely

$$[W, \quad Q_{t_0}, \quad P_{t_0}] \text{ --- } \xrightarrow{\text{'Measurement'} B_{t_0}} \left[W, \quad Q_{t_1} = \frac{Q_{t_0} W Q_{t_0}}{\mathrm{Tr}(W Q_{t_0})}, P_{t_0}\right] \text{ --- } \xrightarrow{\text{'Free Evolution' of } Q_t} [W, Q_{t_2} = U_{t_2-t_1}^{-1} Q_{t_1} U_{t_2-t_1}, P_{t_0}] \text{ --- } \xrightarrow{\text{'Measurement'} A_{t_0}} \left[W, Q_{t_2}, P_{t_2} = \frac{P_{t_0} W P_{t_0}}{\mathrm{Tr}(W P_{t_0})}\right] \tag{7.48}$$

After the first 'measurement' Q_t changes from Q_{t_0} to $Q_{t_1} = \frac{Q_{t_0} W Q_{t_0}}{\mathrm{Tr}(W Q_{t_0})}$. If no further evolution of Q_t occurs, then the probability of P at time t_2 is given by definition 7.2 as

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}} = \mathrm{Tr}\left(\frac{Q_{t_0} W Q_{t_0}}{\mathrm{Tr}(W Q_{t_0})} P_{t_0}\right) = \mathrm{Tr}(Q_{t_1} P_{t_0}) \tag{7.49}$$

(Note that (7.49) employs both projection operators P_t and Q_t before the second 'measurement' is performed, i.e. $P_{t_1} = P_{t_0}$ and $Q_{t_1} = \frac{Q_{t_0} W Q_{t_0}}{\mathrm{Tr}(W Q_{t_0})}$.) But if the operator Q_t *does* evolve freely from the first to the second 'measurement', namely from Q_{t_1} to $Q_{t_2} = U_{t_2-t_1}^{-1} Q_{t_1} U_{t_2-t_1}$, then to calculate the probability of P at time t_2 in the Heisenberg* picture one now needs to consider this freely evolved projection operator. Analogously to (7.49), this probability would be given by

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}} = \mathrm{Tr}(Q_{t_2} P_{t_0}) = \mathrm{Tr}\left(U_{t_2-t_1}^{-1} \frac{Q_{t_0} W Q_{t_0}}{\mathrm{Tr}(W Q_{t_0})} U_{t_2-t_1} P_{t_0}\right) \tag{7.50}$$

And the problem is that the probability given by (7.50) does *not* agree with the probability given by the Lüders rule in the Schrödinger picture.

Indeed, consider again the process given by (7.40), namely

$$W_{t_0} \;\text{---}\; \textbf{First Measurement} \longrightarrow (W_q)_{t_1} = \frac{Q\,W_{t_0}\,Q}{\mathrm{Tr}(W_{t_0}\,Q)} \;\text{---}\; \textbf{Free Evolution} \longrightarrow (W_q)_{t_2} =$$

$$U_{t_2-t_1}\,(W_q)_{t_1}\,U^{-1}_{t_2-t_1} \;\text{---}\; \textbf{Second Measurement} \longrightarrow (W_p)_{t_2} = \frac{P\,(W_q)_{t_2}\,P}{\mathrm{Tr}[(W_q)_{t_2}\,P]} \tag{7.51}$$

where measurements are now really existing processes. Consider the probabilities defined by the Lüders rule for measurement outcomes p and q when, as described by (7.51), the system's state does change in its free evolution between the two measurements, i.e. $(W_q)_{t_2} = U_{t_2-t_1}\,(W_q)_{t_1}\,U^{-1}_{t_2-t_1}$ (rather than the case we considered before with $(W_q)_{t_2} = (W_q)_{t_1}$). The probability for measurement outcome p at time t_2 is then given by the standard trace rule by using the state of the system at time t_2, i.e. $(W_q)_{t_2} = U_{t_2-t_1}\,\frac{Q\,W_{t_0}\,Q}{\mathrm{Tr}(W_{t_0}\,Q)}\,U^{-1}_{t_2-t_1}$:

$$\mathbb{P}_{W_{t_0}}(p, t_2|q, t_1) = p_{(W_q)_{t_2}}(p) = \mathrm{Tr}[(W_q)_{t_2}\,P] = \mathrm{Tr}\left(U_{t_2-t_1}\,\frac{Q\,W_{t_0}\,Q}{\mathrm{Tr}(W_{t_0}\,Q)}\,U^{-1}_{t_2-t_1}\,P\right) \tag{7.52}$$

Written in terms of projection operators P and Q, instead of measurement outcomes p and q, we get

$$\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1) = \mathbb{P}_{(W_Q)_{t_2}}(P) = \mathrm{Tr}\left(U_{t_2-t_1}\,\frac{Q\,W_{t_0}\,Q}{\mathrm{Tr}(W_{t_0}\,Q)}\,U^{-1}_{t_2-t_1}\,P\right) \tag{7.53}$$

Now compare this probability with the (supposedly) same probability calculated in the Heisenberg picture as given by (7.50), namely

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}} = \mathrm{Tr}\left(U^{-1}_{t_2-t_1}\,\frac{Q_{t_0}\,W\,Q_{t_0}}{\mathrm{Tr}(W\,Q_{t_0})}\,U_{t_2-t_1}\,P_{t_0}\right) \tag{7.54}$$

One can see that $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\star}}$ given by (7.54) does not agree with the empirically adequate $\mathbb{P}_W(P,\, t = t_2|Q,\, t = t_1)$ given by (7.53), where the difference lies in the order in which the unitary evolution operators appear.[7.6] Hence, the Heisenberg* projective reading of the probability defined by the Lüders rule as a diachronic conditional probability (definition 7.2) is not correct.

The Heisenberg projective reading (definition 7.1) is also incorrect when Q_t evolves freely between both measurements (in addition to, as we showed before, when Q_t does not evolve). Indeed, in this picture, the probability of projector P_{t_2} given projector Q_{t_1} is given by

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{\mathrm{Tr}[W\,(Q_{t_2}\,P_{t_2}\,Q_{t_2})]}{\mathrm{Tr}(W\,Q_{t_1})} = \frac{p'_W(Q_{t_2}\,P_{t_2}\,Q_{t_2})}{p_W(Q_{t_1})} \tag{7.55}$$

with $Q_{t_2} = U^{-1}_{t_2-t_1}\,Q_{t_1}\,U_{t_2-t_1} = U^{-1}_{t_2-t_1}\,\frac{Q_{t_0}\,W\,Q_{t_0}}{\mathrm{Tr}(W\,Q_{t_0})}\,U_{t_2-t_1}$ and $P_{t_2} = P_{t_0}$. And (7.55) is also different from (7.53).[7.7]

7.6. This order cannot be altered by invoking the invariance property of the trace under cyclic permutations. Also note that that P_{t_0} in the Heisenberg picture is just P in the Schrödinger picture, i.e. $[P_{t_0}]_H = [P]_S$, and similarly, $[Q_{t_0}]_H = [Q]_S$ and $[W]_H = [W_{t_0}]_S$.

We thus seem to have arrived at the end of our story. For we have considered all the seemingly possible ways of giving an understanding of the probability defined by the Lüders rule as a conditional probability. In sections 5.3 and 5.4 we showed why it cannot be understood as a synchronic conditional probability for physical values. In section 7.3 we showed why it cannot be understood as a synchronic nor diachronic conditional probability for measurement results, (nor as a diachronic conditional probability for physical values). And in this section we have shown why it cannot either be understood as a diachronic conditional probability for projection operators.

Furthermore, acknowledging the inadequacy of the diachronic projective notion, weakens even more the synchronic projective notion of conditional probability we discussed in section 5.2. Indeed, it shows that, what seems to be a possible intuitive explanation of the fact that the common projector of P and Q for $\mathbb{P}_W(P|Q)$ is different than that for $\mathbb{P}_W(Q|P)$, namely that the projectors are to be considered at different times, is not satisfactory. Hence, even if one cannot fully dismiss the synchronic projective reading of the probabilities defined by the Lüders rule, one can safely conclude that it is unlikely that it can provide an adequate understanding of a quantum notion of conditional probability. Indeed, it gives no understanding of why the operator $Q\ P\ Q$ represents the common 'projector' of Q and P; it relies on a counterintuitive notion of commonality since the common projector of P given Q is not equal to that of Q given P; and a possible way, if not the only, of making sense of this counterintuitive property is inadequate.

To conclude, we have seen nothing so far which justifies the understanding of the probability defined by the Lüders rule as a conditional probability, but quite, on the contrary, have given many arguments against this understanding at different levels. Indeed, it cannot be interpreted as a conditional probability for physical values, nor for measurement results, nor at a formal level for projection operators, both from a synchronic and a diachronic perspective. Even if the probabilities defined by the Lüders rule are the only probabilities which are co-extensive with conditional probabilities for compatible events, we have no reason to assimilate them to conditional ones for incompatible events and many reasons against this assimilation.

Contrary to the standard view, the probability defined by the Lüders rule does *not* acquire a precise meaning, in the sense of synchronic or diachronic conditional probability, when quantum mechanics is interpreted as a generalized probability space or as probability space for measurement results. Nothing comparable to the classical way of generating the conditional probability measure works in the Hilbert space when incompatible events are involved.

7.7. This is explicitly shown in the next section.

7.5 So Why Is It Seemingly a Conditional Probability?

However, there remains a tension. For in spite of the validity of the rather intricate and long argument developed in this dissertation (against the conditional reading of the probability defined by the Lüders rule), the Lüders rule still defines a probability which is, in *some* sense, a conditional probability. Indeed, look again at the examples of section 4.4. For example, in the double slit experiment it does yield the probability that the electron hits the detecting screen given that either one or both slits are open. So isn't this enough to regard the probabilities it defines as conditional probabilities? How do we reconcile this intuition with our arguments? Incredibly enough, seeing explicitly why the Heisenberg probability $\mathbb{P}_W(P_{t_2}|Q_{t_1})_H$ given by (7.55) does not agree with the empirically adequate (Schrödinger) probability defined by the Lüders rule, i.e. $\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1)$, will allow us to resolve this tension.

Take the (Schrödinger) probability $\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1)$ defined by the Lüders rule as given by (7.53). Using first the invariance of the trace rule,[7.8] then switching to the Heisenberg picture,[7.9] and finally rearranging this expression taking into account that $P_{t_0} = P_{t_1}$,[7.10] we get

$$\left[\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1)\right]_S = \left[\frac{p'_W(Q_{t_0}\, P_{t_2}\, Q_{t_0})}{p_W(Q_{t_0})}\right]_H \tag{7.56}$$

where we emphasize by an under-script the time picture used in each expression. And hence we can clearly see that the probability defined by the Lüders rule $\left[\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1)\right]_S$ is not equal to either the (alleged) Heisenberg H projective reading of it (definition 7.1) given by (7.55), i.e.

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_H = \frac{p'_W(Q_{t_1}\, P_{t_2}\, Q_{t_1})}{p_W(Q_{t_1})} \tag{7.57}$$

nor to the (alleged) Heisenberg $H^{\varstar}$ projective reading of it given by (7.54), i.e.

$$\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\varstar}} = \frac{\mathrm{Tr}(Q_{t_0}\, W\, Q_{t_0}\; U_{t_2-t_1}\, P_{t_1}\, U^{-1}_{t_2-t_1})}{\mathrm{Tr}(W Q_{t_0})} \neq \left[\frac{p'_W(Q_{t_0}\, P_{t_2}\, Q_{t_0})}{p_W(Q_{t_0})}\right]_H \tag{7.58}$$

(because, given the order in which the evolution operators appear, one *cannot* equate $U_{t_2-t_1}\, P_{t_1}\, U^{-1}_{t_2-t_1}$ with P_{t_2}, and hence write $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^{\varstar}}$ as $\left[p'_W(Q_{t_0}\, P_{t_2}\, Q_{t_0})/p_W(Q_{t_0})\right]_H$). Thus, as we concluded in the last section, the probability defined by the Lüders rule *cannot* be interpreted as the probability of projector P at time t_2 given projector Q at time t_1.

7.8. $\mathbb{P}_{W_{t_0}}(P, t_2|Q, t_1) = \mathrm{Tr}\Big(U_{t_2-t_1} \frac{Q\, W_{t_0}\, Q}{\mathrm{Tr}(W_{t_0}\, Q)} U^{-1}_{t_2-t_1}\, P\Big) = \frac{\mathrm{Tr}(Q\, W_{t_0}\, Q\, U^{-1}_{t_2-t_1}\, P\, U_{t_2-t_1})}{\mathrm{Tr}(W_{t_0}\, Q)}$.

7.9. $\left[\frac{\mathrm{Tr}(Q\, W_{t_0}\, Q\, U^{-1}_{t_2-t_1}\, P\, U_{t_2-t_1})}{\mathrm{Tr}(W_{t_0}\, Q)}\right]_S = \left[\frac{\mathrm{Tr}(Q_{t_0}\, W\, Q_{t_0}\, U^{-1}_{t_2-t_1}\, P_{t_0}\, U_{t_2-t_1})}{\mathrm{Tr}(W\, Q_{t_0})}\right]_H$.

7.10. $\left[\frac{\mathrm{Tr}(Q_{t_0}\, W\, Q_{t_0}\, U^{-1}_{t_2-t_1}\, P_{t_0}\, U_{t_2-t_1})}{\mathrm{Tr}(W\, Q_{t_0})}\right]_H = \left[\frac{\mathrm{Tr}(Q_{t_0}\, W\, Q_{t_0}\, U^{-1}_{t_2-t_1}\, P_{t_1}\, U_{t_2-t_1})}{\mathrm{Tr}(W\, Q_{t_0})}\right]_H = \left[\frac{\mathrm{Tr}(Q_{t_0}\, W\, Q_{t_0}\, P_{t_2})}{\mathrm{Tr}(W\, Q_{t_0})}\right]_H = \left[\frac{p'_W(Q_{t_0}\, P_{t_2}\, Q_{t_0})}{p_W(Q_{t_0})}\right]_H$

However, simply by reading (7.56), it appears that we can now finally say how it *can* be interpreted. Indeed, the probability defined by the Lüders is simply the probability of projector P at time t_2 – immediately before the second measurement – given projector Q at time t_0 – immediately before the first measurement. The problem with the Heisenberg H and $H^{\star}$ readings of the probability defined by the Lüders rule is that they invoke incorrect Heisenberg time pictures. Indeed, guided by the attempt to interpret this probability as the probability of projector Q after the first measurement, i.e. event Q_{t_1}, given projector P after the second measurement, i.e. P_{t_2}, we have derived incorrect Heisenberg expressions of it.

The probability defined by the Lüders rule is defined for projection operator Q_{t_0} immediately prior to the first measurement, and not for Q_{t_1} immediately posterior to the first measurement. That is, projector Q_t needs to be considered immediately before the first measurement rather than immediately after it. (Note that, since the actual result of the second measurement plays no role in $\mathbb{P}_{W_{t_0}}(P,\, t_2|Q,\, t_1)$ – just as the actual result of a measurement plays no role in the 'unconditional' probability given by the trace rule $p_W(P)$ – we have not needed to introduce a further time t_3 to distinguish between P_{t_2} immediately before the second measurement and P_{t_2} immediately after the second measurement). We refer to this (almost last!) interpretation of the probabilities defined by the Lüders rule as Heisenberg diachronic$_{(t_0-t_2)}$ projective notion of conditional probability.

Definition 7.3. ***Heisenberg Diachronic$_{(\boldsymbol{t_0-t_2})}$ Projective Quantum Conditional Probability*** *The probability given by the Lüders rule for projectors P_t, $Q_t \in \mathcal{L}(\mathcal{H})$ in the Heisenberg picture, namely*

$$\mathbb{P}_W(P_{t_2}|Q_{t_0}) = \frac{\mathrm{Tr}(Q_{t_0}\, W Q_{t_0}\, P_{t_2})}{\mathrm{Tr}(Q_{t_0}\, W)} = \frac{p'_W(Q_{t_0}\, P_{t_2}\, Q_{t_0})}{p_W(Q_{t_0})} \tag{7.59}$$

is the probability of projector P_t at time t_2 – immediately before the second measurement of the observable to whose spectral decomposition P_t belongs to – conditional on projector Q_t at time t_0 – immediately before the first measurement of the observable to whose spectral decomposition Q_t belongs to – with respect to the probability measure p_W.

The operator $Q_{t_0}\, P_{t_2}\, Q_{t_0}$ represents the common operator of projectors P_{t_2} and Q_{t_0}, where, given that $P_{t_0} = P_{t_1}$, the evolution from P_{t_0} to P_{t_2} is given by the unitary evolution operator $U_{t_2-t_1}$ as

$$P_{t_2} = U^{-1}_{t_2-t_1}\, P_{t_0}\, U_{t_2-t_1} \tag{7.60}$$

Note that, since $[Q_{t_0}]_H = [Q]_S$ and $[P_{t_0}]_H = [P]_S$, (7.59) simply reduces the usual expression of the probability defined by the Lüders rule, i.e. $\mathbb{P}_W(P|Q) = \frac{\mathrm{Tr}(Q\, W Q\, P)}{\mathrm{Tr}(Q\, W)}$.

It is not strange that, after all, the Heisenberg picture does provide the appropriate interpretation of the probability defined by the Lüders rule. Indeed, the notion of conditional probability is that of the probability of an event conditional on another event, and given that quantum events are represented by projection operators and that these need to be considered at different times, one needs to resort to the time picture in which the time evolution is carried by the projection operators, namely the Heisenberg picture.

Note that we have arrived at this conclusion after a long argument. Indeed, to derive expression (7.59) we have been guided by our discussion in section 5.2 and in this chapter, which has first, made us search for a common quantum event of P and Q given by $Q\,PQ$ and second, consider these projection operators at different times (and hence in the Heisenberg picture).[7.11] But expression (7.59) is very easy to derive from the probability defined by the Lüders rule in the Schrödinger picture. Indeed, it takes, at most, two lines of trivial calculations (see footnotes 7.8-7.10). However, as we have seen, adequately interpreting such an expression and seeing why each attempted reading goes wrong has not been as trivial.

However, the Heisenberg diachronic$_{t_0-t_2}$ reading is again not wholly satisfactory. First, it relies on a strange mixture of physical and mathematical notions. For it is defined for projection operators, and hence gives an interpretation of a mathematical notion, while it invokes time and physical and measurement processes, and hence, gives a physical interpretation of such a notion. Indeed, both time t_0 and t_2 are defined as immediately prior to both the first and the second measurement, respectively, and the time evolution process of P_t, namely $P_{t_0} = P_{t_1}$ and $P_{t_2} = U_{t_2-t_1}^{-1}\,P_{t_0}\,U_{t_2-t_1}$, needs to be considered. Second, it still provides no explanation of why $Q_{t_0}\,P_{t_2}\,Q_{t_0}$ represents the common quantum event of Q_{t_0} and P_{t_2} in $\mathbb{P}_W(P_{t_2}|Q_{t_0})$.

Finally, and most importantly, the Heisenberg diachronic$_{t_0-t_2}$ projective reading, however weak and unsatisfactory, is physically adequate only in so far as it can underwrite a quantum notion of conditional probability in terms of physically relevant values. That is, a reading of $\mathbb{P}_W(P_{t_2}|Q_{t_0})$ as the probability for value p at time t_2 conditional on value q at t_0 associated with projection operators P_{t_2} and Q_{t_0}, respectively, where p and q can both be interpreted as physical values or, in particular, as measurement results. Given that $\mathbb{P}_W(P_{t_2}|Q_{t_0})$, as defined by definition 7.3, makes explicit use of measurements, we focus on the latter interpretation of p and q. We refer to this interpretation as the diachronic$_{(t_2-t_0)}$ conditional-on-measurement-outcome reading of the probability defined by the Lüders rule.

7.11. Note that this has led us to, contrary to what we intended, define a Heisenberg notion which is not even calculated through the standard trace rule. Indeed, definition 7.2 reads $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^\circ} = \mathrm{Tr}(Q_{t_1}\,P_{t_2})$, something which should have warned us against the adequacy of such a reading before developing the general expression of $\mathbb{P}_W(P_{t_2}|Q_{t_1})_{H^\circ}$ when Q_t evolves freely between measurements.

Definition 7.4. ***Diachronic*$_{(t_2-t_0)}$ *Conditional-on-Measurement-Outcome Quantum Probability*.** *When an ideal first-kind measurement of a certain observable (where Q belongs to its spectral decomposition), is performed at time t_1 on a system in state W, followed by a second measurement of an observable (where P belongs to its spectral decomposition) at time t_2, the probability given by the Lüders rule for measurement outcome q_{t_0} at time t_0 – immediately before the first measurement – associated to projector $Q_{t_0} \in \mathcal{L}(\mathcal{H})$ – and measurement outcome p_{t_2} at time t_2 – immediately before the second measurement – associated to projector $P_{t_2} \in \mathcal{L}(\mathcal{H})$, namely*

$$\mathbb{P}_W(p_{t_2}|q_{t_0}) = \frac{\mathrm{Tr}(W Q_{t_0} P_{t_2} Q_{t_0})}{\mathrm{Tr}(W Q_{t_0})} \tag{7.61}$$

is the probability of measurement outcome p_{t_2} conditional on measurement outcome q_{t_0}. The evolution from P_{t_0} to P_{t_2} is given by the unitary evolution operator $U_{t_2-t_1}$ as

$$P_{t_2} = U^{-1}_{t_2-t_1} P_{t_0} U_{t_2-t_1} \tag{7.62}$$

Note that this is so because $P_{t_0} = P_{t_1}$.

So can the probability given by (7.61) be interpreted as the probability for measurement outcome p at time t_2, immediately prior to the second measurement, conditional on measurement outcome q at time t_0, immediately prior to the first measurement?

There are several problems with this interpretation. First, neither q_{t_0} nor p_{t_2} are really measurement outcomes, for they are both considered *prior* to the actual performance of both measurements. One could try to solve this by thinking of them as *potential* measurement outcomes rather than *actual* measurement outcomes. Moreover, this potential interpretation would allow one to escape the difficulty in the cases in which the occurrence of the event Q_{t_0} is not drawn back to the occurrence of the single events $(Q_i)_{t_0}$ that compose it. Indeed, consider the Stern-Gerlach or the double slit experiments we presented in sections 4.4.1 and 4.4.2. There, we have $Q_{t_0} = (P_{s_{x_{+1}}})_{t_0} + (P_{s_{x_0}})_{t_0}$ and $Q_{t_0} = (P_A)_{t_0} + (P_B)_{t_0}$, respectively, where no actual measurement result can be ascribed to them since no actual measurement of $(P_{s_{x_{+1}}})_{t_0}$ or $(P_{s_{x_0}})_{t_0}$, and of $(P_A)_{t_0}$ or $(P_B)_{t_0}$, is performed at time t_0. But since q_{t_0} is interpreted as a potential measurement outcome, one does not need to consider actually performed measurements; q_{t_0} can potentially be either $(q_1)_{t_0}$ or $(q_2)_{t_0}$, without this implying that either of them has to be actual.

But regardless of whether this potential versus actual talk is acceptable, the notion of conditional probability afforded by the diachronic$_{(t_2-t_0)}$ conditional-on-measurement-outcome reading is utterly unsatisfactory. Indeed, it says nothing about what happens to the system in the first measurement since the notion strictly applies to q at time t_0 immediately before the first measurement. This is the way in which this reading of the probability defined by the Lüders rule avoids introducing an extra-dynamics for the act of measurement.

The diachronic$_{t_2-t_0}$ conditional-on-measurement-outcome notion thus offers a purely instrumental interpretation with a strong 'black-box' character which is unsatisfactory unless one thinks of quantum theory as merely providing an algorithm for generating the statistical predictions of the outcomes of measurements. Moreover, even if one admits that this is in fact so, there is no determinate criteria as to what kinds of interactions qualify as measurements.

To conclude, if when one says the probability of a certain measurement outcome p given a previous measurement which has potential outcome q is $\mathbb{P}$, one only means that if these two measurement are repeated many times one expects that the fraction of those which give the outcome p is roughly $\mathbb{P}$, then no problems arise. But as soon as one attempts to say anything else, then all the problems we saw in sections 5.3, 5.4 and 7.3 appear.

7.6 Revisiting The Two-Slit Experiment

Let us revisit the two-slit experiment in the light of all our discussion. We begin by describing it from the perspective of an orthodox interpreter. A particle first leaves the source in a state described by ψ. The two-slit screen S_1 then performs a position measurement of the particle in its plane: it localizes the particle to a certain range of values of the position observable Y_1, namely A when only slit A is open, B when only slit B is open, and $A \cup B$ when both slits are open. Immediately after this measurement the state of the particle is given by the Lüders projection postulate. When only slit A is open, the new state is the normalized projection of the initial state ψ onto the subspace which is the range of the projector P_A belonging to the spectral decomposition of position observable Y_1. That is,

$$\psi_A = \frac{P_A \psi}{\|P_A \psi\|} \tag{7.63}$$

Similarly, when only slit B is open, the new state of the system immediately after the electron has passed through slit B is

$$\psi_B = \frac{P_B \psi}{\|P_B \psi\|} \tag{7.64}$$

And when both slits are open, the new state is the normalized projection of the initial state ψ onto the subspace which is the range of the projector $P_A + P_B$. Indeed, given $A \cap B = \emptyset$, P_A and P_B are orthogonal and hence $P_{A \cup B} = P_A \vee P_B = P_A + P_B$. And, therefore, the state of the particle immediately after the measurement of the double-slit screen is

$$\psi_{AB} = C_A \psi_A + C_B \psi_B \tag{7.65}$$

where $C_A = \frac{\|P_A\,\psi\|}{\|(P_A + P_B)\,\psi\|}$, $C_B = \frac{\|P_B\,\psi\|}{\|(P_A + P_B)\,\psi\|}$. If we set $p_\psi(P_A) = p_\psi(P_B)$, i.e. $\|P_A\,\psi\|^2 = \|P_B\,\psi\|^2$, corresponding to the most simple experimental arrangement, then $C_A = C_B = \frac{1}{\sqrt{2}}$ and thus

$$\psi_{AB} = \frac{\psi_A + \psi_B}{\sqrt{2}} \tag{7.66}$$

The particle then evolves freely between the double slit screen S_1 and the detecting screen S_2. Letting $t=0$ be the instant at which the particle reaches S_1, $t=\tau$ the instant at which it reaches S_2, and U_τ the free evolution operator in $\mathcal{L}(\mathcal{H})$ between the two slits, the particle reaches the detecting screen in state $\psi'_A = U_\tau \psi_A$ when only slit A is open, $\psi'_B = U_\tau \psi_B$ when only slit B is open, and $\psi'_{AB} = U_\tau\,\psi_{AB}$ when both slits are open. Finally, upon reaching the latter is localized in the set R in S_2's plane.

As we saw in section 4.4.2, using the time evolution of the states $\psi' = U_\tau\,\psi$ one can calculate the probability that the particle is measured at R on the detecting screen. This probability is given by the trace rule using the time evolution of the collapsed state as given by the Lüders projection postulate. When only slit A is open, this probability is thus given by

$$\mathbb{P}_\psi(R, t=\tau\,|A, t=0) = p_{\psi'_A}(R) = \langle \psi'_A\,, P_R\,\psi'_A\rangle \tag{7.67}$$

Similarly, when only slit B is open, the probability that the particle is measured at R on the detecting screen at $t=\tau$ is

$$\mathbb{P}_\psi(R, t=\tau\,|B, t=0) = p_{\psi'_B}(R) = \langle \psi'_B\,, P_R\,\psi'_B\rangle \tag{7.68}$$

And when both slits are open, the probability that the particle is measured at R on the detecting screen at $t=\tau$ is

$$\mathbb{P}_\psi(R, t=\tau\,|A\cup B, t=0) = p_{\psi'_{AB}}(R) = \left\langle \psi'_{AB}, P_R\,\psi'{}_{AB} \right\rangle \tag{7.69}$$

The probability to arrive at R when the two slits are open (7.69) , is not, as in the classical case, the weighted sum of the probabilities when each slit is open (7.67)-(7.68). Rather we have the characteristic quantum interference terms. That is,

$$\mathbb{P}_\psi(R, \tau\,|A\cup B, 0) = \frac{1}{2}\,\mathbb{P}_\psi(R, \tau\,|A, 0) + \frac{1}{2}\,\mathbb{P}_\psi(R, \tau\,|P_B, 0) + I \tag{7.70}$$

with

$$I = \frac{1}{2}\,\langle \psi'_A\,, P_R\,\psi'_B\rangle + \frac{1}{2}\,\langle \psi'_B\,, P_R\,\psi'_A\rangle \tag{7.71}$$

We now verify that in these three experiments the corresponding equalities given by $\mathbb{P}(A_{t_2}|B_{t_1})\,p(B_{t_1}) = \mathbb{P}(B_{t_1}|A_{t_2})\,p(A_{t_2})$ fail, that is,

$$\mathbb{P}_\psi(R, \tau\,|A, 0)\,p_\psi(A, 0) \neq \mathbb{P}_\psi(A, 0|R, \tau)\,p_\psi(R, \tau) \tag{7.72}$$

$$\mathbb{P}_\psi(R, \tau\,|B, 0)\,p_\psi(B, 0) \neq \mathbb{P}_\psi(P, 0|R, \tau)\,p_\psi(R, \tau) \tag{7.73}$$

$$\mathbb{P}_\psi(R, \tau\,|A\cup B, 0)\,p_\psi(A\cup B, 0) \neq \mathbb{P}_\psi(A\cup B, 0|R, \tau)\,p_\psi(R, \tau) \tag{7.74}$$

Consider first the experiment in which only slit is A open. The left-hand side of inequality (7.72), i.e.[7.12]

$$\mathbb{P}_\psi(R,\tau|A,0)\, p_\psi(A,0) = \langle \psi, U_\tau^{-1}\, P_R\, U_\tau P_A\, \psi\rangle \tag{7.75}$$

and the right-hand side, i.e.[7.13]

$$\mathbb{P}_\psi(A,0|R,\tau)\, p_\psi(R,\tau) = \frac{1}{\|P_R\,\psi\|^2}\, \langle \psi, U_\tau P_R\, U_\tau^{-1}\, P_A\, \psi\rangle\, \langle \psi, {U_\tau}^{-1}\; P_R\, U_\tau \psi\rangle \tag{7.76}$$

are, in general, not equal. Second, similarly to the previous case, when only slit B is open we can verify inequality (7.73). Finally, consider the case in which both slits A and B are open. We show this for the case in which $\|P_A\,\psi\|^2 = \|P_B\,\psi\|^2$. The left-hand side of inequality (7.74), i.e.[7.14]

$$\mathbb{P}_\psi(R,\tau|A\cup B,0)\, p_\psi(A\cup B,0) = \frac{1}{2}\,\langle \psi, U_\tau^{-1}\, P_R\, U_\tau P_A\, \psi\rangle + \frac{1}{2}\,\langle \psi, U_\tau^{-1}\, P_R\, U_\tau P_B\, \psi\rangle + \frac{1}{2}\,\langle \psi, P_A\, U_\tau^{-1}\, P_R\, U_\tau P_B\, \psi\rangle + \frac{1}{2}\,\langle \psi, P_B\, U_\tau^{-1}\, P_R\, U_\tau P_A\, \psi\rangle \tag{7.77}$$

and the right-hand side, i.e.[7.15]

$$\mathbb{P}_\psi(A\ \cup\ B,\ 0|R,\ \tau)\ p_\psi(R,\ \tau) = \frac{1}{\|P_R\,\psi\|^2}\left[\langle P_R\ \psi,\ U_\tau^{-1}\ P_A\ U_\tau\ P_R\ \ \psi\rangle + \langle P_R\ \psi, U_\tau^{-1}\, P_B\, U_\tau P_R\ \psi\rangle\right] \langle \psi, {U_\tau}^{-1}\; P_R\, U_\tau \psi\rangle \tag{7.78}$$

are not equal to each other, even in the case in which the interference term, i.e.

$$I = \frac{1}{2}\,\langle \psi'_A\,, P_R\,\psi'_B\rangle + \frac{1}{2}\,\langle \psi'_B\,, P_R\,\psi'_A\rangle \tag{7.79}$$

vanishes.

In addition, note that the interference term I is equal to zero only if τ is zero, and hence there is a non-zero distance between the two screens. Indeed, for $\tau = 0$, we have that $\psi'_A = \psi_A$ and $\psi'_B = \psi_B$, and thus

$$I = \frac{1}{2}\,\langle \psi_A\,, P_R\,\psi_B\rangle + \frac{1}{2}\,\langle \psi_B\,, P_R\,\psi_A\rangle \tag{7.80}$$

7.12. $\mathbb{P}_\psi(P_R,\ \tau|P_A,\ 0)\ p_\psi(P_A,\ 0) = \langle U_\tau\ \frac{P_A\,\psi}{\|P_A\,\psi\|},\ P_R\ U_\tau\ \frac{P_A\,\psi}{\|P_A\,\psi\|}\rangle\ \langle \psi,\ P_A\ \psi\rangle = \langle U_\tau\ P_A\ \psi,\ P_R\ U_\tau\ P_A\ \psi\rangle = \langle \psi, P_A\, U_\tau^{-1}\, P_R\, U_\tau P_A\, \psi\rangle = \langle \psi, U_\tau^{-1}\, P_R\, U_\tau P_A\, \psi\rangle$.

7.13. $\mathbb{P}_\psi(P_A,0|P_R,\tau)\, p_\psi(P_R,\tau) = \langle U_\tau\, \frac{P_R\,\psi}{\|P_R\,\psi\|}, P_A\, U_\tau\, \frac{P_R\,\psi}{\|P_R\,\psi\|}\rangle\, \langle U_\tau \psi, P_R\, U_\tau \psi\rangle = \frac{1}{\|P_R\,\psi\|^2}\, \langle U_\tau P_R\,\psi, P_A\, U_\tau P_R\,\psi\rangle\, \langle \psi, {U_\tau}^{-1}\ \ P_R\ U_\tau\ \psi\rangle = \frac{1}{\|P_R\,\psi\|^2}\ \langle \psi,\ P_R\ U_\tau^{-1}\ P_A\ U_\tau\ P_R\ \psi\rangle\ \langle \psi,\ {U_\tau}^{-1}\ \ P_R\ U_\tau\ \psi\rangle = \frac{1}{\|P_R\,\psi\|^2}\ \langle \psi,\ U_\tau\ P_R\ U_\tau^{-1}\ P_A\ \psi\rangle\ \langle \psi, {U_\tau}^{-1}\ P_R\, U_\tau \psi\rangle$.

7.14. $\mathbb{P}_\psi(P_R,\tau|P_{A\cup B},0)\, p_\psi(P_{A\cup B},0) = \frac{1}{2}\left[\|P_A\,\psi\|^2 + \|P_B\,\psi\|^2\ \right]\left[\frac{1}{\|P_A\,\psi\|^2}\,\langle \psi, U_\tau^{-1}\, P_R\, U_\tau P_A\,\psi\rangle + \frac{1}{\|P_B\,\psi\|^2}\,\langle \psi, U_\tau^{-1}\ P_R\ U_\tau\ P_B\ \psi\rangle + \frac{1}{\|P_A\,\psi\|}\ \frac{1}{\|P_B\,\psi\|}\ \langle \psi,\ P_A\ U_\tau^{-1}\ P_R\ U_\tau\ P_B\ \psi\rangle + \frac{1}{\|P_A\,\psi\|}\ \frac{1}{\|P_B\,\psi\|}\ \langle \psi,\ P_B\ U_\tau^{-1}\ P_R\ U_\tau\ P_A\ \psi\rangle\right]$. If $\|P_A\,\psi\|^2 = \|P_B\,\psi\|^2$, then we get (7.77).

7.15. $\mathbb{P}_\psi(P_A + P_B, 0|P_R, \tau) = \mathrm{Tr}\Big(\Big[U_\tau\,\frac{P_R\,P_\psi\,P_R)}{\mathrm{Tr}(P_\psi\,P_R)}\,U_\tau^{-1}\Big]\,(P_A + P_B)\Big) = \frac{\mathrm{Tr}[P_R\,P_\psi\,P_R\ U_\tau^{-1}\,(P_A + P_B)\,U_\tau]}{\mathrm{Tr}(P_\psi\,P_R)}$. Using that $P_R\,P_\psi\,P_R = P_{P_R\psi}$, it is equal to $\frac{\mathrm{Tr}(P_{P_R\psi}\ U_\tau^{-1}\,(P_A + P_B)\,U_\tau)}{\|P_R\,\psi\|^2}$, which, in turn, is equal to $\frac{\langle P_R\,\psi, U_\tau^{-1}\,(P_A + P_B)\,U_\tau\,P_R\ \psi\rangle}{\|P_R\,\psi\|^2} = \frac{1}{\|P_R\,\psi\|^2}\Big[\langle P_R\,\psi, U_\tau^{-1}\,P_A\,U_\tau P_R\ \psi\rangle + \langle P_R\,\psi, U_\tau^{-1}\,P_B\,U_\tau P_R\ \psi\rangle\Big]$, we get (7.78).

But since

$$\langle \psi_A\,, P_R \psi_B \rangle = \frac{1}{\|P_A \psi\|\,\|P_B \psi\|}\,\langle \psi, P_A\,P_R\,P_B\,\psi_B \rangle = 0 \tag{7.81}$$

(given that P_A commutes with P_R – because Y_1 and Y_2 are compatible observables – and that P_A and P_B are orthogonal), and similarly for $\langle \psi_B\ , P_R\,\psi_A \rangle$, the interference term I vanishes. In contrast, I is different from zero if $\tau \neq 0$. Indeed, for $\tau \neq 0$ both terms in (7.79) are different from zero. Consider the first one, i.e.

$$\langle \psi'_A\,, P_R\,\psi'_B \rangle = \frac{1}{\|P_A \psi\|\,\|P_B \psi\|}\,\langle \psi, P_A\,U_\tau^{-1}\,P_R\,U_\tau P_B\,\psi_B \rangle \tag{7.82}$$

Given P_A does not commute with $P'_R = U_\tau^{-1}\ P_R\ U_\tau$ (since the evolution $\psi_A \to \psi'_A$ and $\psi_B \to \psi'_B$ will be generated by a Hamiltonian involving the momentum operator which is incompatible with the position observables Y_1 and Y_2), we have that $P_A\,U_\tau^{-1}\ P_R\,U_\tau\,P_B \neq U_\tau^{-1}\,P_R\,U_\tau\,P_A\,P_B$. And hence (7.82), and therefore I, are non-zero.

This last analysis makes clear the role played by (i) the initial quantum state, namely a superposition of states ψ_A and ψ_B, and (ii) the non-zero distance between the slit screen and the detecting screen, i.e. a non-zero time evolution between the two screens, for the presence of interference terms. Indeed, because of the former, inequality (7.74) – when both slits are open – holds even more strongly than (7.72) and (7.73). Indeed, the superposition $\psi_{AB} = \frac{\psi_A + \psi_B}{\sqrt{2}}$ after the measurement of the double-slit is crucial to get the interference terms in (7.79). And because of the the non-zero distance between the slit screen and the detecting screen, the position observables Y_1 and Y_2 are incompatible – unlike S_x and S_y in the S-G example which are always incompatible, Y_1 and Y_2 are only incompatible if they are considered at different times.

Let us now look at the interpretation of the probabilities defined by the Lüders rule for the three experiments and show why they cannot be interpreted as diachronic conditional-on-measurement-outcome probabilities. That is, why they do not allow a *non-instrumental* conditional-on-measurement-outcome interpretation. Recall that in section 7.3. we considered this interpretation for the probabilities defined by the Lüders rule in general and dismissed it as not applicable. We now develop these arguments for the present case.

First, the probability defined by the Lüders rule with only one slit open, e.g. $\mathbb{P}_\psi(R,\tau\,|\,A,0)$, cannot be understood as the pro-rata increase of the probability of measurement result A at time $t = 0$ and measurement result R at time $t = \tau$, and hence as a conditional-on-measurement-outcome probability. For even though the event 'measurement result A_0 and measurement result R_τ' is represented by $(P_A)_0 \wedge (P_R)_\tau$, where ' $\wedge$ ' can be interpreted as the ordinary 'and', the pro rata increase of the probability assigned to this event does not coincide with the value assigned to $\mathbb{P}_\psi(R,\tau\,|\,A,0) = \langle U_\tau \psi_A\,, P_R\,U_\tau \psi_A \rangle$ by the Lüders rule.

By an analogue reasoning, one concludes that $\mathbb{P}_\psi(R,\tau|B,0)$ cannot be interpreted as the pro-rata increase of the probability of measurement result B at time $t=0$ and measurement result R at time $t=\tau$, and hence as a conditional-on-measurement-outcome probability.

Second, it is even more difficult to understand the probability defined by the Lüders rule for the case in which both slits open, i.e. $\mathbb{P}_\psi(R,\tau|A\cup B,0)=\langle\psi'_{AB},P_R\,\psi'_{AB}\rangle$, as a conditional-on-measurement-outcome probability. For now $(P_A\vee P_B)_0\wedge(P_R)_\tau$ cannot even be interpreted as measurement result A or measurement result B at time $t=0$, and measurement result R at time $t=\tau$. Indeed, recall our discussion in section 5.4.2. In addition, the probability assigned to $(P_A\vee P_B)_0\wedge(P_R)_\tau$ does not coincide with that assigned to the common event $(P_A+P_B)_0\,(P_R)_\tau\,(P_A+P_B)_0$ employed by the Lüders rule.

To conclude, the probabilities $\mathbb{P}_\psi(R,\tau|A,0)$, $\mathbb{P}_\psi(R,\tau|B,0)$ and $\mathbb{P}_\psi(R,\tau|A\cup B,0)$ defined by the Lüders rule for the three situations considered in the double slit experiment *cannot* be interpreted as conditional-on-measurement-outcome probabilities. Contrary to the standard view, they *cannot* be interpreted as

> 'the probability that the [particle] will arrive at a certain region in the detecting screen [R], conditional on localization to a certain range of values of Y_1, (A, B, or $A\cup B$)' on the double-slit screen. ([Bub, 1977], p.387)

As a final remark, note that the orthodox interpreter would try to resist our conclusion. Take the experiment in which both slits are open. The orthodox interpreter would say that the measurement performed by the double slit need not be interpreted in terms of the measurement when only A is open and the measurement when only B is open. That is, she would hold that the fact that $(P_R)_\tau\wedge(P_A\vee P_B)_0$ cannot be interpreted as measurement result R and measurement result A or B is irrelevant. For the measurement performed by the double slit screen when both slits are open does *not* measure which slit the particle actually goes through. Thus, according to her, one should not say anything about measurements which have not been performed. Moreover, she would continue, if one were to perform a which-slit experiment, then one *could* talk about the results of both measurements. But since, in this case, no experimental interference terms would be obtained, this would pose no further problem.

Now even if one accepts this reasoning, the orthodox interpreter would still have to explain why the probability given by the pro rata increase of $(P_R)_\tau\wedge(P_A\vee P_B)_0$ – where now $P_A\vee P_B$ would simply be interpreted as a measurement result localizing the particle to the range of values $A\cup B$ – does not coincide with the probability $\mathbb{P}_\psi(P_R,t=\tau|P_{A\cup B},t=0)$. And thus, the orthodox interpreter cannot provide an answer to the interpretive difficulties by not speaking about results of measurement which have not been performed.

7.7 Two further arguments

One finds in the literature two further arguments for the claim that the probabilities defined by the Lüders rule should be interpreted as conditional probabilities. First, [Bub, 1979a, 1979b, 2007] presents an argument based on a formal analogy between the classical rule of conditionalization and the Lüders rule for this interpretation. He also claims that, given this interpretation, one can show that the difference between classical and quantum conditionalization cannot be regarded as grounds for interpreting the quantum-mechanical conditionalization rule as reflecting the irreducible and uncontrollable disturbance of the system measured by a measuring instrument. Rather, for him,

> 'the peculiar features of the quantum-mechanical conditionalization rule relative to the classical rule reflect solely the non-Boolean character of the possibility structures of quantum-mechanical systems.' ([Bub, 1979b] p.90)

And second, [Fuchs, 2002a, 2002b] presents an argument which goes beyond interpreting the Lüders rule as a conditionalization rule; he argues that, in contrast to the classical picture in which gathering new information simply refines the agent's old degrees of belief through conditionalization,

> 'quantum measurement is [...] *a refinement and a readjustment* of one's initial state of belief' ([Fuchs, 2002a], p.34; emphasis added).

Let us, and with this finish this chapter, consider these arguments in detail.

7.7.1 Bub's Analogy Argument

Bub considers a countable classical probability space $\langle S, \mathcal{F}(S), p\rangle$, where he denotes by $x_1, x_2, ...$ the elementary events, associated with the characteristic functions $\chi_1, \chi_2, ...$ He denotes by $a, b, ...$ other non-elementary events. He then gives the following formal expression of a classical conditional probability assignment.[7.16] For any probability measure p defined by an assignment of probabilities p_i to the elementary events x_i, it is possible to introduce a density operator $\rho = \sum_i p_i \chi_i$ (where $\sum_i p_i = 1$, $p_i \geqslant 0$, for all i) in terms of which the probability of an event a can be represented as

$$p_\rho(a) = \sum_j \Big(\sum_i p_i \chi_i(x_j)\Big)\chi_a(x_j) = \sum_j \rho(x_j)\chi_a(x_j) \qquad (7.83)$$

7.16. The notation and numbering of equations are adapted to maintain uniformity.

(7.83) can also be simply expressed as $p_\rho(a) = \sum \rho\chi_a$, where the summation sign without an index is understood as summing over all the atomic events. In terms of this density operator ρ, the conditional probability of an event a, given an event b, can be expressed as

$$\mathbb{P}_{p_\rho}(a|b) = \frac{\sum_j \rho(x_j)\chi_b(x_j)\chi_a(x_j)}{\sum_j \rho(x_j)\chi_b(x_j)} = \frac{\sum \rho\chi_b\chi_a}{\sum \rho\chi_b} \tag{7.84}$$

He then considers the transition from the probability measure given by p to the probability measure given by $\mathbb{P}_{p_\rho}$, and writes it in terms of the corresponding density operators. That is, he considers the transition from ρ to $\rho_b = \frac{\rho\chi_b}{\sum \rho\chi_b}$ or, equivalently in the symmetrized form,

$$\rho \rightarrow \rho_b = \frac{\chi_b\,\rho\,\chi_b}{\sum \rho\,\chi_b} \tag{7.85}$$

so that $\mathbb{P}_{p_\rho}(a|b) = \sum \rho_b\,\chi_a$.

Bub then notes the formal similarity between equation (7.85) and the change of state described by the Lüders rule and claims:

> 'Now, equation (7.85) is just the classical analogue of the Von Neumann-Lüders projection postulate in quantum mechanics! [...] After a measurement of an observable B with outcome b [associated with the eigenvalue corresponding to the projector P_b], the conditional probability of an event a, relative to an initial probability assignment given by W, is:
>
> $$\mathbb{P}_W(a|b) = \mathrm{Tr}(W_b\,P_a) \tag{7.86}$$
>
> where,
>
> $$W_b = \frac{P_b W P_b}{\mathrm{Tr}(P_b\,W P_b)} \tag{7.87}$$
>
> That is, since the projection operators P_b are the non commutative analogues of the characteristic functions χ_b, the transition $W \rightarrow W_b$ in (7.87), which is the quantum projection postulate, is just the Bayesian rule (7.85) for updating a probability distribution on new information.' ([Bub, 2007], pp. 245-246, notation adapted)

Bub thus takes the formal analogy between (7.85) and (7.87) to show that the projection postulate describes the conditionalization of the statistics of a quantum system.

In addition, he uses this same conclusion to argue that the difference between classical and quantum conditionalization cannot be regarded as grounds for interpreting the quantum-mechanical conditionalization rule as reflecting the irreducible and uncontrollable disturbance of the system measured by a measuring instrument. For him, this difference only reflects the non-Boolean character of the 'possibility structures' of quantum mechanical systems. He gives the following argument.

In the classical case, the rule (7.85) $\rho \rightarrow \rho_b = \frac{\chi_b \rho \chi_b}{\sum \rho \chi_b}$ represents conditionalization with respect to the event b, in the sense that ρ_b preserves all initial 'statistical information' specified by χ_b concerning the system, consistent with the event b. That is, the transition $\rho \rightarrow \rho_b$ preserves the relative probabilities of events a such that $a \subseteq b$. In contrast, the rule

$$\rho \rightarrow \rho'_b = \frac{\chi_b}{\sum \chi_b} \tag{7.88}$$

represents conditionalization with respect to the event b *and* randomization of the initial probability measure corresponding to ρ on the subset b, so that the initial 'information' specified by ρ or p_ρ concerning the relative probabilities of the events $a \subseteq b$ is eliminated. That is, the initial measure is not merely renormalized to include the new information (that the value is b), but replaced by a uniform measure over the set b, so that the information contained in the initial measure concerning the relative probabilities of properties represented by subsets a in b is lost.

Now in quantum theory, the transition $W \rightarrow W_b$ dictated by the Lüders rule also preserves the relative probabilities of events $P_a \leqslant P_b$, and hence, in his view, W_b retains all initial statistical information specified by W concerning the system consistent with b. Thus, it should be regarded as describing the conditionalization of the statistics of a quantum system. In contrast, the Von Neumann rule

$$W \rightarrow W'_b = \frac{P_b}{\mathrm{Tr}(W P_b)} \tag{7.89}$$

is the analogue of the classical rule (7.85) representing a conditionalization and *randomization* of the initial measure within the subsets b.

Thus, Bub argues, that whereas the change of state prescribed by von Neumann's projection postulate $W \rightarrow W'_b$ is conditionalization *and* randomization of the initial measure, the transition prescribed by the Lüders rule $W \rightarrow W_b$ is *only* conditionalization on the non-Boolean possibility structure. And hence, that the difference between classical conditionalization and quantum conditionalization, as given by the Lüders rule, should not be regarded as grounds for interpreting the quantum-mechanical conditionalization rule as reflecting the irreducible and uncontrollable disturbance of the system measured by any measuring instrument. In his own words,

> 'On the usual interpretation, the projection postulate is a rule representing the effect of the necessarily finite and uncontrollable disturbance of a system involved in any quantum mechanical measurement process. My point is that the projection postulate in its corrected Lüders version is properly

> understood as *mere conditionalization on a non-Boolean possibility structure*, since it is the analogue of mere conditionalization on new information in the Boolean case. The effect of a measurement disturbance involved in obtaining this information would be represented as an *additional* change in the statistical operator, over and above the change defined by the Lüders rule. Such a measurement disturbance may be more or less violent. The von Neumann rule corresponds to the most violent disturbance possible, in which *all* initial information concerning the system is lost, and only information represented by the measurement result is retained.' ([Bub, 1977], p.389)

7.7.2 Fuchs' Two Process Interpretation

In the classical case, updating through Bayesian conditionalization can be seen as just involving a *refinement* of the agent's degrees of beliefs. This interpretation is grounded on the fact that the degree of belief in A, $q(A)$, can be expressed as a linear sum of the various conditional degrees of belief $q(A|B_i)$, weighted by the degree of belief in each possible B_i, i.e. $q(A) = \sum_i q(A|B_i)\, q(B_i)$. When the agent learns that B_i is the case, she transforms her initial or prior degree of belief $q(A)$ to generate a final or posterior degree of belief $q_{B_i}(A)$, by conditioning on B_i, that is, $q_{B_i}(A) = q(A|B_i)$. As Fuchs somewhat mysteriously puts it,

> 'It is not as if the new state $[q_{B_i}(A)]$ is incommensurable with the old $[q(A)]$. *It was always there*; it was just initially averaged in with various other potential beliefs.' ([Fuchs, 2002a], p.30)

However the situation is more complex in the quantum case. When the agent learns that b_i is the case, the density operator generating the agent's old degrees of belief W cannot be expressed as a linear sum of the various W_{b_i}'s due to the non-commutativity structure of the quantum events. And hence, Fuchs argues, the change of state given by the Lüders rule from W to $W_{b_i} = \frac{P_{b_i} W P_{b_i}}{\mathrm{Tr}(P_{b_i} W P_{b_i})}$ does not lend itself to be interpreted as a refinement of the agent's degrees of belief. Instead he claims that one can achieve a proper interpretation by considering not only a refinement but also a 'readjustment'.

Fuchs bases his interpretation on the possibility of formally breaking up the quantum transition from W to W_{b_i} into two distinct processes, which are subsequently understood (respectively) as refinement and readjustment of an agent's degrees of belief:[7.17]

1. 'Process 1': $W \rightarrow \tilde{W}_{b_i}$, where $W = \sum_i \tilde{W}_{b_i}\, q(P_{b_i})$ and $\tilde{W}_{b_i} = \frac{1}{\mathrm{Tr}(W P_{b_i})} W^{1/2} P_{b_i} W^{1/2}$.

2. 'Process 2': $\tilde{W}_{b_i} \to W_{b_i}$, where $W_{b_i} = \frac{P_{b_i} W P_{b_i}}{\mathrm{Tr}(P_{b_i} W P_{b_i})}$.

Given that $\tilde{W}_{b_i}$ and W_{b_i} have the same eigenvalues and that W can be expressed as a linear sum of the various $\tilde{W}_{b_i}$ with weights $q(P_{b_i})$, Fuchs interprets the first process as

> 'an observer refining his initial state of belief and simply plucking out a term corresponding to the "data" collected.' ([Fuchs, 2002a], p.34)

To make fully explicit the comparison between the classical and the quantum case we rewrite the classical expression

$$q(A) = \sum_i q(A|B_i)\, q(B_i) \tag{7.90}$$

in terms of density operators as $\rho = \sum_i \rho_{B_i} q(B_i)$, where $q(A) = \sum \rho\, \chi_A$ and

$$q(A|B_i) = \sum \rho_{B_i} \chi_A, \text{with } \rho_{B_i} = \frac{\chi_{B_i} \rho\, \chi_{B_i}}{\sum \rho\, \chi_{B_i}} \tag{7.91}$$

Thus the refinement form $q(A)$ to $q(A|B_i)$ is formally equivalent to the change from ρ to ρ_{B_i}, which is what Fuchs compares to the change from W to $\tilde{W}_{b_i}$.

Fuchs then sees the second process as

> 'a further "mental readjustment" of the observer's beliefs, which takes into account details both of the measurement interaction and the observers initial quantum state.' ([Fuchs, 2002a], p.34)

Hence, he concludes that

> 'one can think of quantum collapse as a non commutative variant of Bayes' rule.'([Fuchs, 2002a], p.35) where '[t]aking into account the idea that quantum measurements are 'invasive' or 'disturbing' alters the classical Bayesian picture only in introducing a further outcome-dependent readjustment.' ([Fuchs, 2002a], p.38)

7.7.3 Evaluation

We now argue that both Bub's analogy argument for the claim that the Lüders rule is the appropriate conditionalization rule in quantum mechanics and Fuchs' interpretation that quantum measurement is a refinement and a readjustment of one's initial state of belief are incorrect.

7.17. Note that [Palge and Konrad, 2008] argue that the expression for the unitary re-adjustment operator in [Fuchs, 2002a] is not correct.

Bub shows nicely that for a countable space we can rewrite the classical Bayes' rule as a state transition that mimics the transition achieved by applying the Lüders rule. The relevant equations can be manipulated to display the same form. But in addition to a formal analogy, what one would require is an interpretive analogy. As we have already emphasized, even if formal features are a good guide when extending established concepts to new contexts, formal features alone can never justify that the formally extended concept is also a conceptual extension of the relevant notion to the new domain. Thus our main reservation with his argument is that, by failing to distinguish formal and interpretive aspects, he draws an unwarranted interpretive conclusion. Bub owes us an argument for why his formal analogy about change of state would sanction the conclusion that the meaning of this change in the quantum case is analogous to that of the classical case.

Moreover, given our discussion in Chapters 5 and 7, Bub's argument is not merely inconclusive but incorrect: the Lüders rule cannot be interpreted as quantum conditionalization. And, therefore, his (incorrect) conclusion can provide no grounds for his further claim that the Lüders projection postulate, which describes the change of state of a quantum system upon measurement, does not reflect the irreducible and uncontrollable disturbance of the system measured by a measuring instrument. This claim may turn out to be true, but Bub's argument cannot support it.

Fuchs' proposal also turns out to be inadequate. At first glance Fuchs' suggestion for the interpretation of 'process 1' seems to be reasonable. After all, the change from W to $\tilde{W}_{B_i}$ looks equivalent to the classical conditionalization change from ρ to ρ_{B_i}. So why should it not be interpreted in the same fashion? However, as we have already emphasized, formal similarity is not enough. We would like an argument for why 'plucking out' $\mathrm{Tr}(\tilde{W}_{B_i}\, P_A)$ bears the same interpretation as $q(A|B_i)$, which Fuchs says 'was always there'. It is hard to see in what sense what corresponds to a term in an operator equation would be 'there' at all, and especially if one is building a subjective view.

Fuchs' treatment of 'process 2' fares worse. Here too Fuchs provides no rationale for the interpretation he gives to the process; namely as a change of state arising from the disturbance character of quantum measurements. Moreover, this reading of the state transition cannot be correct in general; that is, the change *cannot* be understood as due to an ordinary physical interaction between the measurement apparatus and the quantum system. This is clearly seen in his own example of a probability function given by a pure state ψ. The corresponding density operator $W = |\psi\rangle\langle\psi|$ can already be expressed as a linear sum of the different W_{B_i}'s and hence he maintains that the only change that can come about is a transition of the readjustment type ('process 2') according to the operator $U_i = |i\rangle\langle\psi|$. In Fuchs' words,

> 'we learn nothing new; we just change what we can predict as a consequence of our experimental intervention. [...] there is a sense in which the measurement is solely disturbance' ([Fuchs, 2002a], p.34).

However the transition from $W = \tilde{W}_{B_i}$ to W_{B_i} is just the usual transition given by Lüders' rule, which in this case is simply the old projection postulate.[7.18] We know that it *cannot* be derived from the Schrödinger equation and thus it *cannot* be interpreted as the effect of an ordinary physical interaction.

Perhaps a more sophisticated treatment of Process 2 using decoherence might help Fuchs here. But one needs to be careful because the state transitions used to demonstrate decoherence effects already make use of the projection (or the Lüders) rule in tracing over the environment-system to get the reduced state.[7.19] In any case, the interpretive moves that Fuchs does make with respect to the two processes seem unsupported by his formal analysis. Finally nothing in Fuchs' treatment hinges on taking a subjective view, either of probability or the quantum state. Indeed the language Fuchs uses (beliefs tracking 'data', and 'invasive' disturbances) readily lends itself to a realist view.

To sum up, both Bub and Fuchs' arguments are not valid because, while their conclusion is supposed to work at an interpretive level, their arguments are merely formal.

7.18. It is a transition from ψ to i enacted via the operator $U_i = |i\rangle\langle\psi|$: $U_i|\psi\rangle = |i\rangle\langle\psi|\psi\rangle = |i\rangle$. U simply acts as a projection operator on ψ.

7.19. [Schlosshauer, 2007], section 8.1, especially p.333.

Chapter 8

Implications for the Interpretations of Quantum Probability

In this chapter we consider the interpretation of the *un*conditional quantum probabilities defined by the trace rule. We show that, similarly to the conditional probabilities defined by the Lüders rule, these can only be interpreted as probabilities under a purely instrumental view of quantum mechanics. Indeed, if when one says the probability of a certain outcome of an experiment is p, one only means that if the experiment is repeated many times one expects that the fraction of those which give the outcome in question is roughly p, then no problems arise. But as soon as one tries to give a more satisfactory interpretation, then problems start cropping up. The difficulties in giving a (non-instrumental) interpretation of quantum unconditional probability, we show, are ultimately the same as those we encountered in giving a (non-instrumental) interpretation of quantum conditional probability.

We show this by focusing on two interpretations of probability, namely quantum Bayesianism and Frequentism. In sections 8.1.1 and 8.1.2 we consider, respectively, whether a subjective or an objective Bayesian interpretation of the quantum probabilities is possible. We argue that neither of them can provide an adequate interpretation. In section 8.2, we turn to the frequency interpretation of probability and consider whether the empirically found quantum frequencies can be thought of, as in the classical frequency interpretation, as arising from an ensemble of similarly prepared systems. We argue that this is not so and that the quantum probabilities cannot be thus thought of as revealing an underlying distribution of properties of quantum objects.

8.1 Quantum Bayesianism

Bayesianism interprets the concept of probability as 'a measure of a state of knowledge', where the 'state of knowledge' concept is, broadly speaking, interpreted in two different ways. For the subjectivist school, the state of knowledge corresponds to a 'personal belief', and is in that respect subjective. In particular, the claim is that the choice of the prior probability is necessarily subjective. We considered this interpretation in Chapter 2. In contrast, other Bayesians state that such subjectivity can be avoided, and claim that the prior state of knowledge uniquely defines a prior probability distribution for well posed problems. For the objectivist school, the rules of Bayesian statistics can be justified by desiderata of rationality and consistency.

We now show that the quantum probabilities do not allow either a subjective or an objective Bayesian interpretation.

8.1.1 Subjective Quantum Bayesianism: a Quantum Dutch Book

The quantum (subjective) Bayesian interpretation maintains that probabilities in quantum mechanics are subjective, *across the board*.[8.1] As Timpson writes in 'Quantum Bayesianism: a Study',

> 'Considered as an interpretation of quantum mechanics, the characteristic feature of quantum [subjective] Bayesianism is [...] its non-realist view of the quantum state. This takes a distinctive form: the quantum state ascribed to an individual system is understood to represent a compact summary of an agent's degrees of belief about what the results of measurement interventions on a system will be, **and nothing more**.' ([Timpson, 2008], p.583)

That is, on this view, quantum states are a matter of what degrees of belief one has about what the outcomes of measurement will be. The probability ascriptions arising from a particular state assignment are understood in a purely subjective, Bayesian manner, in the mold of de Finetti, and are assigned to individual systems.

In *Quantum Chance and Non-Locality* (pp.10-14), Michael Dickson considers whether the quantum probabilities can be interpreted in such subjective manner by considering whether a Dutch Book can be made against an agent whose degrees of belief are dictated by the quantum probabilities. He first argues that it can be made, but then, casting doubt on the adequacy of the assumptions on which the Dutch Book relies, does not reach a definite conclusion. We first present his Dutch Book argument and his reservations about it, and then argue why a Dutch Book *can* in fact be made against an agent who sets his degrees of belief to the quantum probabilities.

8.1. This particular form of quantum Bayesianism can be found in the writings of Caves, Fuchs and Schack, especially in [Fuchs, 2002a, 2002b]; [Caves, Fuchs, & Schack, 2002, 2007].

Consider a quantum system in a pure state ψ and two quantum events, namely P_χ, where $\chi = c_1\,\psi + c_2\,\varphi$, $|c_1| > |c_2|$, and φ is perpendicular to ψ, and P_ξ, where $\xi = d_1\,\psi + d_2\,\varphi$ and $|d_1| > |d_2|$. Also, c_1 and d_1 are such that $c_1 \neq d_1$. Note first, that P_χ and P_ξ are non-orthogonal, i.e. $\langle\chi|\xi\rangle = c_1\,d_1 + c_2\,d_2 \neq 0$, and hence incompatible; and second, that the intersection of their ranges is zero, i.e. $P_\chi \wedge P_\xi = \emptyset$. The probabilities of these two events are given, according to the trace rule (3.6), by

$$p_\psi(P_\chi) = \mathrm{Tr}(W_\psi\,P_\chi) = |\langle\psi,\chi\rangle|^2 = |c_1|^2 \tag{8.1}$$

$$p_\psi(P_\xi) = \mathrm{Tr}(W_\psi P_\xi) = |\langle\psi,\xi\rangle|^2 = |d_1|^2 \tag{8.2}$$

We now show that if one sets one's degrees of belief on events P_χ and P_ξ to the quantum probabilities (8.1) and (8.2), that is, if one accepts to pay $|c_1|^2$ for a wager that pays 1 if P_χ is occurrent and, similarly, one accepts to pay $|d_1|^2$ for a wager that pays 1 if P_ξ is occurrent, then one is subject to a Dutch Book.

The situation is the following. There are three possible cases. First, if P_χ occurs and P_ξ does not occur, i.e. the event $P_\chi \wedge \neg P_\xi$ occurs, then one wins $|c_2|^2$ from the first bet,[8.2] and loses $|d_1|^2$ from the second one. So that the net gain if event $P_\chi \wedge \neg P_\xi$ occurs is $|c_2|^2 - |d_1|^2 < 0$. Second, if P_χ does not occur and P_ξ does occur, i.e. the event $\neg P_\chi \wedge P_\xi$ occurs, then one wins $|d_2|^2$ from the second bet and looses $|c_1|^2$ from the first one, so that the net gain is $|d_2|^2 - |c_1|^2 < 0$. Finally, if neither of them occur, i.e. the event $\neg P_\chi \wedge \neg P_\xi$ occurs, then one looses $|c_1|^2$ from the first bet and $|d_1|^2$ from the second one, and the net gain is $-|c_1|^2 - |d_1|^2 < 0$. Given that these three net gains are negative, one is guaranteed to lose no matter what happens. And hence, a Dutch Book can be made against an agent whose degrees of belief are dictated by the quantum probabilities. Thus, if one sets one's degrees of belief on quantum events to the quantum probabilities, then these degrees cannot be given a subjective interpretation.

Let us evaluate this argument. To begin with, note that it rests on the following assumptions. First, that one's degrees of belief on P_χ and P_ξ individually should be set to the probabilities dictated by quantum mechanics. Second, that if one's degrees of belief on P_χ and P_ξ individually are $|c_1|^2$ and $|d_1|^2$ respectively, then these degrees of belief also hold for P_χ and P_ξ at the same time. That is, if one is committed individually to the fairness of both bets, then one is committed to both bets taken together. And third, that P_χ and P_ξ cannot co-occur, i.e. the event $P_\chi \wedge P_\xi$ cannot occur.

The first assumption seems justified for, in both cases, the degrees of belief are coherent. In effect, one's degrees of belief on the two mutually exclusive outcomes P_χ and $\neg P_\chi$ should be coherent, i.e. $p(P_\chi) + p(\neg P_\chi) = 1$, which is satisfied if $p(P_\chi) = |c_1|^2$ and $p(\neg P_\chi) = |c_2|^2$.

8.2. Given one's degree of belief in P_χ is $|c_1|^2$, if P_χ occurs then one wins $1 - |c_1|^2 = |c_2|^2$

Precisely what quantum theory dictates.[8.3] And similarly for $p(P_\xi)$ and $p(\neg P_\xi)$. It is not clear, however, whether the second and the third assumptions are justified.

Consider first the third one. If one denies that $P_\chi \wedge P_\xi$ cannot occur, that is, if one holds that $P_\chi \wedge P_\xi$ *can* occur, then the alleged Dutch Book is ruined. For then a fourth case is possible, namely $P_\chi \wedge P_\xi$, and in this case the bettor wins – he wins $|c_2|^2$ from the first bet and $|d_2|^2$ from the second one, so that the net gain is $|c_2|^2 + |d_2|^2 > 0$. Hence, no sure loss is guaranteed and no Dutch Book can be made against the bettor.

Now the fact that P_χ and P_ξ *can* co-occur seems justified by appealing to the non-existence of joint distributions for incompatible events. Indeed, given that the joint distribution of P_χ and P_ξ is not zero (because it is undefined), there is no reason to deny the occurrence of P_χ and P_ξ. On the other hand, precisely because their joint probability is undefined, a story about how exactly it is possible for them to co-occur seems to be wanting. Moreover, there seem to be other reasons for denying the occurrence of $P_\chi \wedge P_\xi$. Indeed, it may *not* be possible for P_χ and P_ξ to co-occur because P_χ and P_ξ have nothing in common, given that the intersection of their ranges is zero, i.e $P_\chi \wedge P_\xi = \emptyset$. Thus, Dickson writes:

> 'Any proponent of the epistemic interpretation who wishes to avoid the Dutch Book by allowing that P_χ and P_ξ can co-occur must have a story to tell about *how* they can co-occur given that (1) their lattice-theoretic meet is the zero subspace (i.e. they are *distinct* simple events), and (2) their joint probability is undefined. ([Dickson, 1998] p.12)

The appeal to the fact that the joint distribution of P_χ and P_ξ is not defined, can also be seen as providing grounds for denying the second assumption, thus again ruining the Dutch Book argument and allowing the subjective interpretation of the quantum probabilities. In effect, given that the joint distribution of P_χ and P_ξ is not defined, it seems plausible to agree to each of the bets individually, while refusing to agree to both of them together. In addition, one could argue that, because one cannot (seemingly) verify the occurrence or non-occurrence of each of a pair of incompatible events, propositions involving incompatible events are not well-defined. And thus that to agree to both of them together is to take a stand on the joint occurrence or non-occurrence of each of a pair of non-orthogonal events. Something, which one might reasonably not want to do. In Dickson's words,

> 'To agree to both bets together is to take a stand on a statement about non-orthogonal events. Therefore, agreeing to both bets together amounts to betting on the truth or falsity of a statement whose meaning is undefined, and to refuse such a bet seems completely reasonable.' ([Dickson, 1998], p.13)

8.3. Or as Dickson puts it, this bet is fair for its expected value, i.e. $p(P_\chi)$ (amount won) $+\, p(\neg P_\chi)$ (amount lost), is zero, i.e. $|c_1|^2\, |c_2|^2 - |c_2|^2\, |c_1|^2 = 0$.

The difficulty, hence, seems to be the following. The third assumption of the Dutch Book argument, namely that P_χ and P_ξ *cannot* co-occur, seems to be justified because the joint probability of P_χ and P_ξ is undefined and because the intersection of their ranges is zero. But does *not* seem justified because the joint distribution of P_χ and P_ξ is not zero. And the second assumption, namely that if one agrees to each of the bets individually, then one must also agree to both of them together, seems to *not* be justified because first, the joint distribution of P_χ and P_ξ is not defined; and second, because given that one *cannot* (seemingly) verify the occurrence or non-occurrence of each of a pair of incompatible events, propositions involving them are not well-defined and thus should not be assigned a probability. And the problem is that if either of these assumptions are not in place, then one cannot make a Dutch Book against an agent who sets his degrees of belief to the quantum probabilities, and hence the quantum probabilities would still allow a subjective Bayesian interpretation. Maybe it would, in the end, turn out that they cannot be so interpreted, but a new argument would need to be provided.

Ultimately, the crucial question is whether one can, and if so, what it means to have a degree of belief in the joint outcome of two incompatible quantum events. Indeed, one can hold a perfectly adequate degree of belief in a certain event P_ξ, and another perfectly adequate degree of belief in P_χ. But if these two events are incompatible, then it is not clear if one should assign a degree of belief to the joint occurrence of both events, i.e. to the event $P_\chi \wedge P_\xi$; and if one *can* assign a value to $P_\chi \wedge P_\xi$, what the appropriate value is. Again, the fact that the structure of quantum events is non-Boolean gives rise to this situation. For if one is concerned only with the results of a single observable (or with a compatible set of them), then no problems arise: the quantum probabilities would simply agree with the classical probabilities, and would thus allow a subjective Bayesian interpretation. The interpretive difficulties precisely appear when one considers two or more incompatible quantum events.

Let us reconsider the third assumption, i.e. that $P_\chi \wedge P_\xi$ cannot occur, but now bringing in what we learned form the previous chapters. First, the fact that the intersection of the ranges of P_χ and P_ξ is zero, does *not* seem to justify it. For it rests on the unjustified assumption that if the intersection of the ranges of two events is zero, then they have nothing in common, and thus cannot co-occur. Indeed, as we discussed in section 5.2, this only holds for orthogonal events. For non-orthogonal events, we suggested that there is, albeit under a somewhat lax reading, a measure of commonality between any two projectors P and Q given by the 'probability' of their common 'event', namely $\mathrm{Tr}(WQ\,PQ)$. (Recall that this function is not normalized, but is non-negative and additive.) Hence, even if $P_\chi \wedge P_\xi = \emptyset$, P_χ and P_ξ *can* be seen as having something in common which, in general, will not be assigned a zero 'probability'.

However, the denial that P_χ and P_ξ cannot co-occur, i.e. the claim that P_χ and P_ξ *can* co-occur, on the grounds that their 'joint probability' *is* defined by $\mathrm{Tr}(W\,P_\chi\,P_\xi\,P_\chi)$ and is generally non-zero, needs to be appropriately qualified. First, this 'joint probability' $\mathrm{Tr}(W\,P_\chi\,P_\xi\,P_\chi)$ we have resorted to can only be so interpreted at a mathematical level for projection operators; that is, it cannot be interpreted as the 'joint probability' for the physical values nor for the measurement outcomes associated to P_χ and P_ξ. And second, it is not uniquely specified – for it can be either given by $\mathrm{Tr}(W\,P_\chi\,P_\xi\,P_\chi)$ or $\mathrm{Tr}(W\,P_\xi\,P_\chi\,P_\xi)$ [8.4] – and it is quite weak – for it postulates that these traces give the 'joint probability' of P_χ and P_ξ without giving any explanation of why this is so.

Moreover, the (seemingly) only possible way of uniquely specifying the 'joint probability' (and thus have a more adequate notion of joint probability distribution at the mathematical level for projection operators), namely by interpreting it as a diachronic probability – $\mathrm{Tr}[W\,(P_\chi)_{t_1}\,(P_\xi)_{t_2}\,(P_\chi)_{t_1}]$ would then represent the probability of P_χ at time t_1 and P_ξ at time t_2, i.e. $(P_\chi)_{t_1}\wedge(P_\xi)_{t_2}$, and $\mathrm{Tr}[W\;(P_\xi)_{t_1}\;(P_\chi)_{t_2}\;(P_\xi)_{t_1}]$ would represent the probability of P_ξ at time t_1 and P_χ at time t_2, i.e. $(P_\xi)_{t_1}\wedge(P_\chi)_{t_2}$ – does not yield an adequate notion of joint probability.

Indeed, similarly to what we argued in section 7.4, the Heisenberg diachronic projective notion of joint probability of P_χ at time t_1 and P_ξ at time t_2 as given by $\mathrm{Tr}[W\;(P_\chi)_{t_1}\;(P_\xi)_{t_2}\;(P_\chi)_{t_1}]$, does not coincide with the empirical 'probability' of P_χ at time t_1 and P_ξ at time t_2 as given by $\mathrm{Tr}\Big(U_{t_2-t_1}\;P_\chi\;W_{t_0}\;P_\chi\;\;U^{-1}_{t_2-t_1}\;P_\xi\Big)$.[8.5] The only adequate interpretation of the latter expression is as the probability for measurement outcome p_χ at time t_0 – immediately before the first measurement of a certain observable where P_χ belongs to its spectral decomposition – and measurement outcome p_ξ at time t_2 – immediately before the second measurement of an observable where P_ξ belongs to its spectral decomposition. And, as we argued in section 7.5, this reading yields a notion of joint probability only under a purely instrumental interpretation of quantum mechanics.

But then the claim that the event $P_\chi\wedge P_\xi$ *cannot* occur, i.e. the second assumption of the Dutch Book argument, is fully justified whether it is interpreted at a formal level for projection operators P_χ and P_ξ, or in terms of the physical values or measurement results associated to these, both from a synchronic and a diachronic perspective. And, therefore, a Dutch Book *can* indeed be made against an agent who sets his/hers degrees of belief to the quantum probabilities.

8.4. Note how the counterintuitive feature of the (synchronic) projective notion of conditional probability translates into the non-uniqueness of the (synchronic) projective notion of joint probability.

8.5. See expression (7.53).

Consider now the second assumption of the Dutch Book argument, namely that if one agrees to each of the bets individually, then one must also agree to both of them together. Recall that it was seemingly *not* justified because the joint distribution of P_χ and P_ξ is not defined. However, this does not seem correct. For one *can* take the joint distribution of P_χ and P_ξ to be defined for all incompatible events as $\mathrm{Tr}(WP_\xi P_\chi P_\xi)$.

To sum up, given that both the second and third assumptions are justified, a Dutch Book can be made against an agent who sets his/hers degrees of belief to the quantum probabilities. Indeed, given that the net gains are negative in the only three possible cases, one is guaranteed to loose no matter what happens. And, therefore, if ones degrees of belief on quantum events are dictated by the quantum probabilities, then they are not coherent and, hence, cannot be given a (non-instrumental) subjective interpretation.

Note that this argument does not need to invoke the supposed impossibility of the joint measurement of two incompatible quantities. In particular, one does not need to evaluate whether the fact that one can (seemingly) not verify the occurrence or non-occurrence of each of a pair of incompatible events justifies regarding propositions involving incompatible events as not well-defined, and thus not worthy of assigning a degree of belief. (So that one can agree to each of the bets individually, without also agree to both of them together). As in section 7.3, we have shown that the Bohr-Heisenberg doctrine according to which puzzling features of quantum mechanics can be traced back to an inevitable and uncontrollable physical disturbance brought about by the act of measurement is not correct.

To conclude, the quantum probabilities dictated by Gleason's theorem *cannot* be interpreted as subjective probabilities, for physical values, nor for measurement results, nor at a formal level for projection operators; both from a synchronic and a diachronic perspective. Hence, subjective quantum Bayesianism is *not* a viable interpretation of quantum mechanics.

8.1.2 Objective Quantum Bayesianism

Up till now we have been focusing on the subjectivist view of credences, in which probability is interpreted as the degree of belief (or strength of belief) an individual has in the truth of a proposition. However, as we already noted, there are two different views within the Bayesian approach to probability. For the subjectivist school, the state of knowledge corresponds to a 'personal belief', and is in that respect subjective. In contrast, for the objectivist school, the rules of Bayesian statistics can be justified by desiderata of rationality and consistency. This was actually the position taken in by the first followers of the Bayesian view, beginning with Laplace. In the Bayesian revival in the 20th century, the chief proponents of this objectivist school were E. T. Jaynes and Harold Jeffreys.

Objective Bayesians resort to several principles for the objective construction of the prior distribution as, for example, the maximum entropy principle, transformation group analysis and reference analysis. Objective Bayesianism, as developed by [Jaynes, 1957, 1983] and [Williamson, 2004, 2009] focuses on the maximum entropy principle. Briefly, as defined by Shannon's Information Theory, entropy is a function of a probability distribution which measures the amount of uncertainty that a certain probability distribution represents. Hence, on this view, it is a measure of the lack of knowledge. According to objective Bayesianism, an agent's prior probability distribution should be given by the one which maximizes entropy because it is the only possible unbiased assignment; that is, according to this interpretation, to use any other probability distribution would amount to an arbitrary assumption on information that we do not have. In Jaynes' words,

> 'The distribution that maximizes H [the entropy] subject to constraints which represent whatever information we have, provides the most honest description of what we know.' ([Jaynes, 1983], p.109).

Our prior distribution should thus be the one which, while satisfying the constraints imposed by our knowledge, is otherwise as non-committed as possible with regard to missing information.

Now, an analogue objective Bayesian interpretation of the quantum probabilities can be found in the recent literature. For example, Bub argues that

> 'a quantum theory is best understood as a theory about the possibilities and impossibilities of information transfer, as opposed to a theory about the mechanics of non classical waves or particles' ([Bub, 2004] p.241)

And one can find similar ideas in the writings of Chris Fuchs. For example, he claims

> 'I myself see no alternative but to contemplate deep and hard the tasks, the techniques, and the implications of quantum information theory. The reason is simple, and I think inescapable. Quantum mechanics has always been about information. It is just that the physics community has somehow forgotten this.' ([Fuchs, 2002a] p.4)

> '[T]he quantum state is solely an expression of subjective information – the information one has about a quantum system. It has no objective reality in and of itself.' ([Fuchs, 2002a] p.7)

In addition, [Fuchs, 2002a] gives an expression for a quantum entropy which also, upon maximization, yields the quantum probabilities.

We do not give a detailed presentation of either of these approaches and simply consider the general idea behind an objective Bayesian view of quantum mechanics.[8.6] For an evaluation of whether 'objective quantum Bayesianism' can provide an adequate interpretation of the quantum probabilities seems to be possible without considering each different approach in depth.

Indeed, prima facie, it seems unlikely that objective quantum Bayesianism is capable of presenting a viable interpretation of the quantum probabilities for it faces the same challenge as subjective Bayesianism, namely, that there is no (satisfactory, i.e. non-instrumental) notion of the joint distribution of incompatible events, or, equivalently, no notion of quantum conditional probability. And this precludes the interpretation of the quantum probabilities as degrees of belief, regardless of whether or not the prior probability is determined by appealing to principles for the objective construction of the prior distribution. Hence, while at a formal level the analogies between classical and quantum information theory might be fruitful and interesting to study, they do not seem capable of providing any new conceptual or interpretive insight.

8.2 Quantum Frequentism

Let us now turn to the frequency interpretation of probability. As we pointed out in the introduction, the quantum probabilities allow a frequency interpretation under an instrumental perspective. Indeed, if when one says the probability of a certain outcome of an experiment is p, one only means that if the experiment is repeated many times one expects that the fraction of those which give the outcome in question is roughly p, then no problems arise. However, if one wants to interpret these frequencies as revealing an underlying distribution of properties of quantum objects, as in the classical ensemble interpretation, then problems arise.

The discussion in this section draws heavily on Pitowsky's work 'Quantum Probability. Quantum Logic' and various works of Arthur Fine (an integrative summary of them is provided in [Fine, 1986]). We thus provide a detailed exposition of their results in Appendix D and present here only a brief overview of them.

8.6. See [Timpson, 2007] for a detailed analysis.

8.2.1 Classical Correlation Experiments

As we have seen, the differences between classical and quantum probabilities stem from the fact that in quantum mechanics observables are generally represented by non-commutative operators. Thus a good place to study the peculiarities of quantum mechanics is by looking at scenarios which involve probabilities for more than a single observable. The experiments considered in section 4.4 provided one such scenario. Correlation experiments provide another such scenario, and a more appropriate one to consider *un*conditional probabilities.

In correlation experiments one performs simultaneous measurements of pairs of different physical quantities on a system and studies the relations that hold between the various single and joint probabilities. In the most simple case, one performs simultaneous measurements of two physical quantities, call them 1 and 2, and one finds that the relation $p_1 + p_2 - p_{12} \leqslant 1$ always holds between the various single and joint probabilities, i.e. p_1, p_2, and p_{12}, for classical systems. This is well understood by thinking of these probabilities as having their source in an ensemble of systems with well-defined properties; that is, as reflecting the distributions of properties of the systems.

To give an example, consider randomly selecting atoms of a gas and simultaneously measuring their velocity and position. The number of atoms which either have a certain velocity – property 1 – or a certain position – property 2 – or both, is simply the number of atoms with property 1, plus those with property 2, minus the number with property 1 and 2 (in order to not count the atoms with properties 1 and 2 twice). In symbols, $N_{1\,\mathrm{or}\,2} = N_1 + N_2 - N_{1\&2}$. And given that the number of atoms which have either property 1 or 2 is at most equal to the total number of atoms in the gas, the inequality $N_1 + N_2 - N_{1\&2} \leqslant N$ must hold. Now if probabilities reflect the distributions of the atom's properties, then they will be simply given by proportions; that is, the probability for selecting an atom with property 1 is given by the proportion of atoms with property 1, i.e. $p_1 = \frac{N_1}{N}$; and similarly $p_2 = \frac{N_2}{N}$ and $p_{1\&2} = \frac{N_{1\&2}}{N}$. And hence the previous inequality translates into an inequality for probabilities, namely $p_1 + p_2 - p_{12} \leqslant 1$.

In addition to this objective reading which regards probabilities as reflecting the frequency distributions of the properties of the various systems in the ensemble, one can also provide a subjective one and hence view the probabilities as reflecting degrees of belief of a rational agent. Both interpretations turn out to be formally equivalent. The objective view leads to constraints on the probabilities in terms of linear inequalities, e.g. $p_1 + p_2 - p_{12} \leqslant 1$, while the subjective view leads to constraints as convex sums of certain vectors (see section D.1.1), and both constraints can be shown to be mathematically equivalent. These constraints are, in turn, equivalent to requiring that the various single and joint probabilities allow an ensemble representation (see theorem D.1). These results can be generalized for the single and joint probabilities in a general correlation experiment (section D.1.2).

Two particularly relevant cases are the so-called Bell-Wigner and Clauser-Horne correlation experiments (sections D.1.3 and D.1.4, respectively). In the former, the single and joint probabilities p_1, p_2, p_3, p_{12}, p_{13}, p_{23} admit an ensemble representation if and only if they satisfy the 'Bell inequalities', namely

$$\begin{aligned} p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} &\leqslant 1 \\ p_1 - p_{12} - p_{13} + p_{23} &\geqslant 0 \\ p_2 - p_{12} - p_{23} + p_{13} &\geqslant 0 \\ p_3 - p_{13} - p_{23} + p_{12} &\geqslant 0 \end{aligned} \tag{8.3}$$

And in the latter, p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24} admit an ensemble representation if and only if they satisfy the 'Clauser-Horne inequalities'.

$$\begin{aligned} -1 \leqslant p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leqslant 0 \\ -1 \leqslant p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leqslant 0 \\ -1 \leqslant p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leqslant 0 \\ -1 \leqslant p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leqslant 0 \end{aligned} \tag{8.4}$$

8.2.2 Quantum Correlation Experiments

It turns out that in quantum correlation experiments the empirically found (and theoretically predicted) single and joint frequencies (probabilities) do not generally satisfy the above relations. (In sections D.2.1 and D.2.2 we give particular examples.) The quantum probabilities do not, therefore, generally admit an ensemble representation, and thus their understanding as revealing an underlying distribution of properties of quantum objects or as subjective degrees of belief is precluded. That is, they cannot in general be given an objective reading in terms of frequencies revealing proportions of properties nor a subjective reading in terms of degrees of belief. (Note that the events can here be interpreted as physical values or as measurement results.)

One can, however, try to modify the classical construal of an ensemble interpretation so as to make it a viable interpretation for the quantum probabilities. In section D.3 we consider this possibility in detail in the case of a Clauser-Horne experiment and show that the problem in giving an ensemble representation appears when we consider a unique probability function defined over an ensemble of systems in which all four observables together, call them $AA'BB'$, take determinate pre-measurement values. For it is then that we cannot ignore the incompatibility between the values given for the BB' correlations by the ABB' distribution and those given by the $A'BB'$ distribution (see theorem D.2). Hence, if one could somehow get rid of this incompatibility, the quantum statistics might be understood as having their source in an ensemble of similarly prepared systems with well-defined properties.

One way of avoiding this incompatibility is by employing Fine's 'Prism Models' [Fine, 1986]. The main idea is that measurement results are not restricted to the two possible eigenvalues corresponding to the measured observable, but can also yield no result at all (so for example, if we measure A, the measurement result need not be a_1 or a_2, but also no value at all). If one allows this possibility then not all measurements give determinate values for the four observables, and the incompatible probabilistic predictions for the incompatible observables disappear. In this way one can build statistical models that reproduce the quantum statistics successfully.

The probabilities that arise from these statistical models are by construction compatible with an ensemble representation. They can thus be seen as having their source in an ensemble of similarly prepared systems taking values for the different observables but without the requirement that every system has a definite value for *all* four observables: some types of particles would have determinate values for $AA'B$ ($a_1a'_1b_1$ or $a_1a'_1b_2$ or any of the remaining six combinations) but would be 'B'-defective', i.e. yielding no value upon a B' measurement, others would have determinate values for $A'BB'$ but would be A-defective, and so on. The various single and joint probabilities arise by averaging over the appropriate non-defective results. The probabilities could then be understood as reflecting the properties of these 'some times for some observables'-defective systems and could thus be given this modified frequency reading.

What about a subjective interpretation? Our agent would assign degrees of belief to these different possibilities but would in some way have to take into consideration the possibility of defective systems. This would result in some modification of the coherence condition imposed on her degrees of belief. Indeed, it is the requirement of coherence that lies at the heart of the Ramsey-de Finetti theorem ensuring that degrees of belief satisfy the classical axioms of probability. Hence if one is to give a consistent subjective interpretation of the quantum probabilities it is this requirement that needs, in some way, to be modified. How exactly the coherence assumption is to be changed would need to be developed.

We will not attempt here to give an evaluation of this particular proposal – Fine himself wonders whether

> 'it really contributes to our understanding of nature to suppose that quantum systems have built-in properties that predetermine their suitability for measurements' ([Fine, 1986], p.56).

The point we want to make is that it is possible to relax some of the conditions on the statistical models of the quantum probabilities and explore how these modifications open up new possibilities for understanding the quantum probabilities. Of course, ultimately one needs to provide some rationale backing these modifications. But both an elaboration of these changes and their detailed discussion lie outside the scope of the present essay.

8.3 Conclusion

In this chapter, we have considered the impossibility of providing an interpretation of the quantum probabilities in terms of a frequency and a Bayesian interpretation. Ultimately, the reason for the failure of these interpretations lies in the difficulty to define a quantum notion of joint probability, or equivalently, a quantum notion of conditional probability, which can be interpreted both at a mathematical level for projection operators and at a physical level for the physical values associated with these projectors from a non-instrumental perspective.

For the notion of conditional probability, on the one hand, if one takes the joint probability of projectors P and Q (or physical values p and q) to be defined by $p(P \wedge Q)$, then the ensuing conditional probability is inconsistent with the probability defined by the Lüders rule. Moreover if either P or Q is the sum of two or more orthogonal events, say $Q = Q_1 + Q_2$, then $P \wedge (Q_1 + Q_2) = (P \wedge Q_1) \vee (P \wedge Q_2)$ cannot be generally interpreted as the value 'p and q_1 or q_2'. And, on the other hand, if one takes the joint 'probability' of P and Q to be defined by the probability of their 'common projector', i.e. $\mathrm{Tr}(W Q P Q)$, then the ensuing conditional probability only works at a mathematical level for projection operators in a weak and counterintuitive way from a synchronic perspective. When arguing in section 8.1. that a Dutch Book can be made against an agent whose degrees of belief are dictated by the quantum probabilities we relied on both these reasonings.

For the quantum correlations experiments this difficulty appears in a different way. Take the Clauser-Horne case. There we calculated the joint probabilities by employing equation (D.15), namely $p_{ij} = \mathrm{Tr}[W_S\ (P_i \wedge P_j)]$. These probabilities are well-defined for, even though they involve incompatible quantities for a single electron, they apply here to different electrons.[8.7] Now choosing the P_i and P_j's in an appropriate way, these probabilities fail to comply with the Clauser-Horne inequalities and can thus not be seen as arising from an ensemble of similarly prepared systems with well-defined properties. Which, in turn, precludes the Bayesian and the frequency interpretations.

8.7. For example, in example of section D.2.1, the projector $P_1 \wedge P_3 = P_{+x} \otimes P_{+z}$ corresponds to the left electron having spin up in the x direction and the right electron to have spin up in the z direction.

Putting it somewhat differently, the difficulty is that the axioms of classical probability are not compatible with the quantum probabilities. The axioms of non-negativity and normalization do not pose any problems, and the axiom of additivity is respected for the quantum probabilities since it is defined for orthogonal (i.e. compatible) projection operators. But the fourth axiom of classical probability, which takes conditional probability to be defined as a joint probability increased pro rata, simply does not work for the quantum probabilities (if by joint occurrence we try to take the direct quantum analogue of the classical joint distribution, i.e. $p(P \wedge Q)$). In this way, we can now understand why we chose to use Accardi's quote at the beginning of this dissertation. For

> 'all the paradoxes of quantum theory arise from the implicit or explicit application of Bayes' axiom (or of the theorem of composite probabilities, which is an equivalent form of it [...]) to the statistical data of quantum theory. This application being unjustified both physically and mathematically.' ([Accardi, 1984a], pp.298 - 299).

Thus, it seems that the quantum probabilities cannot be interpreted as measuring our degrees of belief on the quantum events as the Bayesian interpretation holds, nor as frequencies revealing an objective distribution of the properties of quantum systems. Maybe they need to be understood as part of the physical 'furniture' of the world described by quantum theory, and, in particular, as dispositions that get manifested upon measurement. Indeed, many different ways of fleshing out dispositional notions from a propensity interpretation perspective have been used in the attempt to solve the quantum paradoxes.[8.8] However, an evaluation of these proposals and their adequacy goes beyond the scope of this dissertation.

8.8. Some of the best known are Heisenberg's potentialities, Margenau's latencies, and Maxwell's propensitons. See [Suárez, 2007] for a review and references therein. More recently, Mauricio Suárez [Suárez, 2004a, 2004b, 2007] has developed the selective propensities interpretation of quantum mechanics.

Chapter 9
Concept Extension

In Chapter 4, we showed that the probabilities defined by the Lüders rule, while not additive with respect to 'conditioning' events, are characterized by an analogue existence and uniqueness theorem (theorem 4.2) to the classical one (theorem 2.1), which states that the Lüders rule gives the only probability measure that, for events $P \leqslant Q$ in $\mathcal{L}(\mathcal{H})$, just involves a renormalization of the initial probability function p_W. Combining this result with the fact that, if P and Q are compatible, the probabilities defined by the Lüders rule reduce to classical conditional probabilities, we arrive at the result that the Lüders rule gives the only probability measure on the quantum event space $\mathcal{L}(\mathcal{H})$ which reduces to classical conditional probability in their shared domain of application, i.e. compatible events. In Chapter 4 we explained why this result is standardly taken as justifying the interpretation of the Lüders rule as defining the quantum extension of conditional probability in quantum probability theory.

However, as we have argued in detail in Chapters 5 and 7, this is not so. Even if the probabilities defined by the Lüders rule are the only probabilities which are co-extensive with conditional probabilities for compatible events, we have no reason to assimilate them to conditional ones for incompatible events, neither for physical values, nor at a formal level for projection operators, both from a synchronic and a diachronic perspective (except under a purely instrumental perspective). Rather, we have given many reasons against this assimilation.

In this chapter we claim that this result holds in general. Indeed, in sections 9.1 and 9.2, we argue that the fact that a concept appearing in a certain theoretical context is co-extensive with another concept of a different theoretical context in their shared domain of application, does *not* guarantee that the former will be the conceptual extension of the latter.[9.1] Thus the standard philosophical view that concept extension can be characterized by co-extension of two concepts in their shared domain of application – as for example, that presented by the logical positivists, by Imre Lakatos or by Albert Einstein – is shown to be inadequate.

9.1. Notice the different senses in which the word 'extension' is being used, namely as an enlargement in scope – concept *extension* – and as the total range over which something extends – co-*extension* of two concepts in the shared domain.

We then argue that concepts present an 'open texture' that does not allow for a set of jointly necessary and sufficient conditions to characterize an extended concept (section 9.3), and develop a scheme and methodology – the 'Cluster of Markers' account – for the problem of tracing conceptual lineages so as to judge when one concept truly extends another (section 9.4).[9.2] This new account, we argue, can capture the complexity involved in actual cases of conceptual change in science and can account for the fact that there are concepts which, even if co-extensive in their shared domain of application, do not share enough meaning so as to justify regarding them as defining one and the same concept.

9.1 Concept 'Refinement'

Physicists and mathematicians have tended to focus on limits as the appropriate criterion to characterize concept extension: a concept in a new context is taken to be an extension of an old one if it reduces to it when taking the appropriate limit. For example, so-called 'relativistic mass' $m\,\gamma = m/\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}$ is standardly seen as the extension of classical mass because it reduces to it in the Newtonian limit, namely for speeds small compared to the speed of light.

Similarly, philosophers and philosophically minded physicists have focused on co-extension in the shared domain of application as the adequate requirement for conceptual extension. Most extremely, the logical positivist tradition (e.g. philosophers such as Carnap, Reichenbach and Hempel), tried to characterize extended concepts as logical extensions of previous ones. But co-extension has also been defended as the appropriate requirement for conceptual extension under less stringent accounts. Albert Einstein, for example, explicitly defended this view in his 'method of conceptual refinement'.[9.3]; [Lakatos, 1976] developed a somewhat similar account which he termed 'conceptual stretching'; and [Fine, 1978, 1986], building on Einstein and criticizing Lakatos' account, further spelled out the method of conceptual refinement.[9.4]

9.2. Credit for the cluster of markers account should be jointly given to Arthur Fine and Isabel Guerra. It was born from our going back and forth with each other over this problem.

9.3. We follow Fine's interpretation of Einstein's thought as presented in 'The Shaky Game. Einstein, Realism and the Quantum Theory' [Fine, 1986]. The reader should, therefore, look for reference to Einstein's own works in Fine's book. Regarding the method of conceptual refinement, Fine writes:

> 'I try to tease out of Einstein's scientific papers a general method of his, *the method of conceptual refinement*, that actually requires significant conceptual change as the vehicle for scientific development.' ([Fine, 1986] pp. 3-4)

In Lakatos' view, science grows by making adjustments in the concepts employed in a certain scientific argument so as to accommodate counterexamples or empirical anomalies; and these adjustments result in the 'stretching' of concepts.[9.5] In more detail, the concept to be stretched figures centrally in a scientific argument (a law, a hypothesis or a theorem), whose conclusion, as well as some stage of the argument itself, is challenged by a counterexample. In order for progress to be made, what happens is that the argument is essentially preserved by altering the concept so as to accommodate the counterexample. That is, the concept is molded or stretched so as to avoid the counterexample and preserve the validity of the argument. [Fine, 1978] argues that the program to construe conceptual change as concept stretching is idle for it is seriously confused about the role of argument (both in empirical sciences and mathematics). The method of conceptual refinement suggests why this is so. So let us first look at this method.

In Einstein and Fine's view, new concepts arise as a refinement of old ones: they agree with the old ones in their shared range of application, but have a broader range of application which enables a new and 'deeper' way of understanding and organizing experience. Conceptual refinement can be seen as a process occurring in two-stages. The first stage consists in a mapping of the boundary of a concept. One examines the limits of the concept-to-be-changed so as to find out its range of application, i.e. where the concept clearly applies, where it does not clearly apply and any possible middle grounds (which will mostly cover unclear or indeterminate applications).

The second stage consists in the extension of the concept to its indeterminate or unclear range of application. A new theory is in charge of determining how the refinement of the concepts exactly takes place. It will employ the refined concepts satisfying the following two constraints:

i. the refined concepts are co-extensional with the unrefined ones, at least approximately, in the central region where the unrefined concepts clearly apply. This is supposed to ensure that the refined concepts generalize or extend the originals.

ii. the refined concepts apply in a determinate way beyond this central region. This constraint represents the progress of science: the new theory employing the refined concept

9.4. We present here a rather simplified account of Einstein and Fine's method of conceptual refinement. These authors do not defend explicitly that co-extension in the shared domain of application is a necessary and sufficient condition for conceptual extension. However, co-extension seems to be implicitly assumed to be the only condition in their account since no other is considered.

9.5. Lakatos focused mainly on the development of mathematical science but took much of his conclusion to also apply to empirical science.

> 'will penetrate into nature more deeply than its predecessors; it should advance our understanding by allowing us to put to nature certain questions which were not clearly expressible on the basis of the old concepts and it should lead us to expect that some such questions have determinate answers.' ([Fine, 1978], p.331)

Thus, the first stage prepares the way for the second one:

> 'In the first stage we show that there is room for conceptual innovation, and in the second stage we do it.' ([Fine, 1978], p.332).

Lakatos' concept stretching account can, from this perspective, be seen as follows. First, the role of counterexamples is to mark out the limits of where the concept of interest applies, and second, the 'good' argument around which our concept is stretched functions like the new theory. Indeed, the good argument forces our stretched concept to apply to the clear cases, for otherwise it would not be a *good* argument at all; and it extends the application of our concept by making the concept fit the more general constraints of the proof. But, as Fine argues,

> 'We are not playing "save the proof" (by stretching the concept), the way the medievals played "save the phenomena" (by stretching the hypothesis). We are playing the game of advancing our science, by developing new theories. If it happens, as in mathematics especially it sometimes does, that to develop the theory involves focusing on an especially interesting line of argument, then conceptual refinement may come dressed as conceptual stretching.
>
> She wears those clothes well. And because Imre had an excellent eye for finery, he may have mistaken the persona for the person. Had he got to know her more intimately, I think it unlikely that he would have continued his misapprehension.' ([Fine, 1978] pp.339-340)

9.1.1 An Example: Cardinality

The extension of the notion of 'the number of elements of a set' from the finite to the infinite provides an example of conceptual extension.[9.6] For a finite set this notion applies straightforwardly: we simply count the number of elements of the set. But what about for

9.6. [Fine, 1978] pp.335-338 and [Buzaglo, 2002], pp.42-44.

an infinite set? Talk of the number of elements is not clear; indeed, one could plausibly argue that counting only makes sense for a finite collection of objects. Does the set of, say, the real numbers have the same number of elements as that of the natural numbers given that they both have infinite elements? Or rather are there more real numbers than natural numbers? Or to take another example, does the set of even numbers have fewer elements than the set of the natural numbers, given that the former is a proper subset of the latter? Or, on the contrary, do they have the same number of elements?

The notion of 'the number of elements of a set' of the finite domain does not afford a clear answer to these questions involving infinite sets. Georg Cantor, in the late 19th century proposed one. He argued that the concept 'number of elements of a set' is rooted in the process of counting and that this is what allows defining its coherent extension in the infinite domain. Indeed, establishing one-to-one correspondences is what we do when we count the number of elements in the case of finite sets: we establish or follow an ordering of its elements that maps isomorphically to an initial segment of the positive integers; then the number of elements of a set is just the positive integer that corresponds to the last element counted.

Therefore, Cantor proposed that two infinite sets have the same number of elements if they can be put into a one-to-one correspondence with each other. The extended notion is known as the cardinality of a set. The cardinality of set A is, thus, equal to the cardinality of B just in case there is a one-to-one correspondence between A and B. Hence, the set of natural numbers and the set of even numbers have the same number of elements or cardinality given that there is a one-to-one correspondence between their elements – simply associate 1 in N to 2 in E, 2 in N to 4 in E, 3 in N to 6 in E, etc. And this regardless of the fact that the even numbers are a proper subset of the natural numbers. In contrast, the set of natural numbers and the set of the real numbers do not have the same cardinality since there is no way of establishing a one-to-one correspondence between them. In effect, there are 'many more' real numbers than natural numbers even if both have an infinite number of elements.

Two final remarks. First, notice that these counterintuitive properties of the extended concept result from the very project of conceptual extension. Indeed, if the extended concept is to adequately apply in a new domain, which the old concept was unable to capture, then it cannot retain all the features of the concept it extends. Rather, it will retain the essential features of the old concept, and let go of those which would not enable it to capture the phenomena of the new domain. Second, notice that the concept of cardinality reduces to the number of elements in the case of finite sets, in satisfaction of the conceptual refinement analysis' first constraint, but takes over for infinite sets, according to the second constraint. Here it enables one to raise questions which will now have determinate answers.

9.1.2 Necessity of Classical Concepts: the Bohr-Einstein Debate

Both Einstein and Bohr shared the view that, in constructing new theories, it is necessary to seek for the precise limits of application of a concept. However they disagreed on how exactly this information should be used; that is, they did not agree on the second stage of the conceptual refinement method. In particular, they held opposite views on the implications of the limits of application of the classical concepts. While Einstein thought that in general the construction of a good theory calls for a refinement of the old concepts – one looks to a series of revisions of the classical concepts by first seeking the limits of their application in the experimental situations and then by building a theory to refine them – Bohr thought that there could be no concepts which would extend the range of application of the classical ones in a uniform way.[9.7]

In more detail, Bohr claimed that one must view the world through the old classical concepts; that is, that we have only the classical concepts with which to organize experience for only these seem to be linked with the human capacity for conceptualization. New contexts, though presenting new phenomena, may not be novel at a conceptual level because new concepts are, as a matter of principle, ruled out.[9.8] Thus, when looking into the quantum domain one must first select a particular set of classical concepts to employ; and what one then sees, once this choice is made, will depend on the chosen set of concepts, i.e. on our view point, in such a way that different perspectives cannot be pieced together in one unitary picture of the quantum world. This view became known as Bohr's doctrine of complementarity (recall section 6.6).

The most distinctive feature of the Bohr's Copenhagen interpretation (compared to the orthodox interpretation) is that, in addition to the projection postulate, it postulates of the necessity for classical concepts to describe quantum phenomena. Instead of deriving classicality from the quantum world, e.g., by considering the macroscopic limit, the requirement for a classical description of the 'phenomena', which comprise the whole experimental arrangement, is taken to be a fundamental and irreducible element of a complete quantum theory.[9.9]

9.7. This section follows [Fine, 1986] chapter 2. See the latter for further detail.

9.8. Bohr is usually seen as taking the lead from Immanuel Kant, who is in turn frequently interpreted as a defender of this view.

Einstein strongly disagreed with this view. In a letter to Schrödinger – who had also expressed the view that, given the limitation on the applicability of the concepts of position and momentum embodied in the Heisenberg indeterminacy relations, these concepts would eventually be replaced by new ones that apply not approximately but exactly – he writes:

> 'Your claim that the concepts p, q [momentum, position] will have to be given up, if they can only claim such 'shaky' meaning, seems to me to be fully justified. The Heisenberg-Bohr tranquilizing philosophy – or religion? – is so delicately contrived that, for the time being, it provides a gentle pillow for the true believer from which he cannot very easily be aroused' (cited in [Fine, 1986], p.18)

Thus, whereas Bohr thought that the concepts of classical physics should be segregated in the manner of his complementarity doctrine, Einstein defended the project of seeking new concepts that would replace the classical ones in the quantum context. He thought that holding classical concepts as necessary significantly obstructs any kind of scientific progress. In his own words,

> '... concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as 'conceptual necessities', 'a priori situations', etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyze familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little from the data of experience. In this way they are deprived from their excessive authority.' (cited in [Fine, 1986], p.15-16)

Thus Einstein did not conceal the difficulty in the interpretation of quantum theory by appealing to the complementarity doctrine and continued searching for some concepts that would yield an appropriate understanding of the quantum realm.

9.9. This introduces a quantum-classical dualism into the description of nature and requires the assumption of an essentially non-movable boundary (the famous 'Heisenberg cut') between the 'microworld', containing the objects that are to be treated as quantum systems, and the 'macroworld' that has to be described by classical physics. However, the studies of decoherence phenomena demonstrate that quasiclassical properties, across a broad range from microscopic to macroscopic sizes, can emerge directly from the quantum substrate through environmental interactions. This makes the postulate of an a priori existence of classicality seem unnecessary, if not mistaken, and it renders unjustifiable the placement of a fixed boundary to separate the quantum from the classical realm on a fundamental level. See [Schlosshauer, 2004, 2007] and [Schlosshauer & Fine, 2008].

Fine describes the debate between Bohr and Einstein over the conceptualization of quantum theory with a beautiful analogy:

> 'According to Bohr the system of classical concepts splits into mutually exclusive packages if one attempts to use this system outside the region of macroscopic physics, the region where all the concepts of the system have a determinate, simultaneous application. To get beyond this central core one must select which package of concepts to use. Different selections will enable one to get beyond the core in different ways. But the results of these different explorations do not combine into some unified picture of a region beyond the core.
>
> Bohr thus views the product of conceptual refinement as a wheel-like structure: a central hub from which there extends a number of disjoint spokes. Different explorers can move out separately along different spokes but, according to Bohr, the reports they send back will not enable one to piece together an account of some region between the spokes of a rim that connects them. Thus the new conceptual structure for Bohr looks like the steering wheel of an old-fashioned ship. The beauty of this steering mechanism and the aspect that Einstein saw as a 'gentle pillow' is that it enables one to navigate into the quantum domain using only a classical chart in any given direction.
>
> Einstein's dispute with Bohr (and others) is a dispute over this wheel-like structure. Einstein asks whether the spokes must really be disconnected, could there not at least be a rim? This is the question as to whether the quantum theory allows a realist interpretation, a picture of the world as a single entity with a rich set of simultaneously determinate properties that are observer-independent. And Einstein asks whether the spoke must be made of the same material as the hub. Must we, that is, stick with just the classical concepts?' ([Fine, 1986], p.21)

Two final remarks. First, note that the 'Heisenberg-Bohr tranquilizing philosophy' cannot provide an adequate recourse for the difficulties in the interpretation of probability defined by the Lüders rule. For, from this perspective, one would need to argue that this probability only makes sense for compatible events, and only then as conditional probabilities. But then, what are we to do in all the other cases in which it applies *de facto* to pairs of incompatible events and gives the correct probabilistic predictions? A Bohrian would need to claim that in all these cases, which are precisely the ones in which the quantum predictions differ from the classical ones, the probability defined by the Lüders rule makes no sense. But this claim is indeed quite difficult to defend in any satisfactory way.

Second, note that Einstein did not believe that the new concepts would come from quantum theory. Indeed, already in 1936 he describes quantum theory as

> 'an incomplete representation of real things, although it is the only one which can be built out of the fundamentals concepts of force and material points.' (quoted in [Fine, 1986], p.24)

He saw quantum theory as essentially statistical, unable to predict the simplest phenomena unless one understood the theory as only treating statistical aggregates of individual systems, and not the individual systems themselves. He thus felt that it could not be fundamental in the sense of providing a framework for all physics. Rather, he hoped that the framework of general relativity would provide the new fundamental concepts and the theoretical basis from which the quantum theory would emerge as a statistical approximation. He devoted his last years to exploring the possibility for thus finding an account that would penetrate the quantum domain more deeply than does the present theory. Unfortunately, he did not succeed.

9.2 Inadequacy of the Concept Refinement Account

The refinement account thus characterizes conceptual extension by means of its first requirement, namely that of co-extension of the refined concept with the unrefined one in the shared domain. This is taken to ensure that the refined concepts generalize or extend the original ones. However, we have seen that in the case of the probabilities defined by the Lüders rule, co-extension in the shared domain is not sufficient to characterize conceptual extension: they reduce to classical conditional probabilities for compatible events and yet cannot be understood in terms of conditionality for incompatible events.

In this section, we argue that this failure is not an isolated instance due to the particularly difficult task of interpreting the quantum formalism. That, in fact, co-extension in the shared domain of application does not in general guarantee that the concept of the new domain is an extension of the concept it is co-extensive with. In effect, co-extension of two notions in a certain domain by no means guarantees that outside the shared domain the new notion will have the same (core) meaning. And hence, it is not a sufficient condition for a conceptual extension. The conceptual refinement account is, thus, shown to not be an adequate account of conceptual extension.

We argue for this claim by considering a particularly interesting example of a concept which, while satisfying the co-extension requirement, cannot be seen as a proper extension of the concept it formally reduces to, namely, the notion of 'relativistic mass'.[9.10] Indeed, there are two possible formal functions in relativistic physics which are co-extensive with classical mass for speeds small compared to the speed of light, i.e. in the shared domain of classical and relativistic physics. However, only one of them can in fact be interpreted as an extension of classical mass, something which, given both satisfy the co-extension requirement, the method of conceptual refinement cannot account for. Let us see this in some detail.

The Newtonian equations of motion are empirically correct only if the speed of the object under description is considerably smaller than the speed of light. Otherwise they are replaced by the equations of relativistic physics which involve mass in a new way. We will focus on the fundamental equations of special relativity, which, for a free body, are

$$E^2 - \boldsymbol{p}^2 c^2 = m^2 c^4 \tag{9.1}$$

$$\boldsymbol{p} = \boldsymbol{v}\,\frac{E}{c^2} \tag{9.2}$$

where E is the energy, $\boldsymbol{p}$ the momentum and $\boldsymbol{v}$ the velocity of the particle, and c is the speed of light.

What is the proper interpretation of the symbol m that appears in (9.1)? Is it just our ordinary classical notion of mass? Or is it an extension of that concept to the relativistic domain? Or, rather, is it a new concept altogether? Prima facie, m seems to define the relativistic extension of the classical notion of mass. For the relativistic equations (9.1) and (9.2) reduce to the classical expressions for momentum and energy involving classical mass when the speed of the body is small relative to the speed of light. In effect, equation (9.1) can be rewritten as $\frac{E}{m\,c^2} = \sqrt{1 + \frac{\boldsymbol{p}^2}{m^2\,c^2}}$ which in the Newtonian limit reduces to:[9.11]

$$E = m\,c^2 + \frac{\boldsymbol{p}^2}{2\,m} \tag{9.3}$$

Here we can identify $\frac{\boldsymbol{p}^2}{2m}$ as the classical kinetic energy of a particle. Similarly, writing equation (8.2) as $\frac{\boldsymbol{p}}{m} = v\,\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2\,c^2}}$,[9.12] we obtain the classical expression for momentum:

$$\boldsymbol{p} = m\,\boldsymbol{v} \tag{9.4}$$

9.10. This discussion mainly follows [Lange, 2002], pp.224-240. See also [Adler, 1987], [Earman & Fine, 1977], [Field, 1973], [Okun, 1989], [Okun, 2001], [Okun, 2002] and [Sandin, 1991].

9.11. Considering the case in which $v \ll c$ so that $x = \frac{\boldsymbol{p}^2}{m^2 c^2}$ is very small, one can use the expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x^2)$ for small x's.

The problem is that the quantity symbolized by m is not standardly taken to define relativistic mass. Rather many standard textbooks on relativity take the quantity $m\gamma$ to be the appropriate notion of mass in relativity. Indeed, equations (9.1) and (9.2) can be rewritten as:[9.13]

$$\boldsymbol{p} = m\gamma\boldsymbol{v} \tag{9.5}$$

$$E = m\gamma c^2 \tag{9.6}$$

where $\gamma = 1/\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}$, and then it is $m\gamma$, rather than m alone, that seems to be the proper extension of mass in relativity theory. For $m\gamma$ reduces directly to the Newtonian notion of mass when the speed of the particle is small compared to the speed of light. In effect, when $v \ll c$, γ reduces to one,[9.14] and $m\gamma$ directly reduces to classical mass. Moreover, the relativistic equation (9.5) is analogous to the Newtonian expression of momentum $\boldsymbol{p} = m\boldsymbol{v}$. Hence, $m\gamma$ is generally taken to be the proper relativistic extension of mass and, as such, is usually referred to as 'relativistic mass'.

But is this really so? Is $m\gamma$, rather than m, the appropriate extension of classical mass in relativity? No, it turns out, that the function m, and not $m\gamma$, is the proper extension of the notion of mass in special relativity because it is the one which is Lorentz invariant. Indeed, the quantity m is Lorentz invariant, that is, it does not depend or change upon the transition from one inertial reference frame to another. To see this, one need only substitute the Lorentz transformations for E and p, namely

$$E \to (E' + \boldsymbol{v}\boldsymbol{p}')\gamma \tag{9.7}$$

$$\begin{aligned} p_x &\to \left(p'_x + \frac{v E'}{c^2}\right)\gamma \\ p_y &\to p'_y \\ p_z &\to p'_z \end{aligned} \tag{9.8}$$

where $\boldsymbol{v}$ is the velocity of one reference frame relative to another and $v = |\boldsymbol{v}|$ (we assume that vector $\boldsymbol{v}$ is directed along the x axis), into equation (9.1). After some straightforward calculation, one obtains the same equation for the transformed quantities E' and $\boldsymbol{p}'$, namely $E'^2 - \boldsymbol{p}'^2 c^2 = m^2 c^4$, where mass m appears unchanged.

9.12. Using the same expansion as before, only the first term survives for the second one is already of second order.

9.13. Substituting in (9.2) the expression of energy from (8.1) we get $\boldsymbol{p}^2 = \frac{\boldsymbol{v}^2}{c^2}(m^2 c^4 + \boldsymbol{p}^2 c^2)$. And rearranging this expression, $\boldsymbol{p} = \frac{m}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}\boldsymbol{v}$, which taking $\gamma = \sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}$, is normally expressed as $\boldsymbol{p} = m\gamma\boldsymbol{v}$. Substituting this expression into (9.3) we get $E = m\gamma c^2$.

9.14. Indeed, only the first term survives: $\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} = 1 - \frac{1}{2}\frac{v^2}{c^2} + \ldots = 1 + o(\frac{v^2}{c^2})$

Lorentz invariant quantities are taken to be the physically relevant quantities because, given that they do not depend on the reference frame respect to which they are described, they seem to capture what is objective. Given that a body's mass m is independent of its speed and of any change of reference frame, it is thus the appropriate extension of mass in relativity theory. In contrast, a body's so-called 'relativistic mass' $m\,\gamma$ can change simply due to a change of the reference frame in respect to which it is described, and depending on the reference frame, its variation will be different;[9.15] this 'change' does not represent an objective physical process. As Marc Lange observes:

> 'Because 'relativistic mass' [$m\,\gamma$] is not an invariant quantity, the best thing to do in order to avoid confusing frame-dependent quantities with invariant ones is just avoid using the term 'relativistic mass' [...] We should just use the term 'mass', applying to the invariant quantity symbolized m' ([Lange, 2002], pp.224-225)

Or in Einstein's words,

> 'I find it not very good to say that the mass of a body in movement is increased by the speed. It is better to use the word mass exclusively for [Lorentz-invariant] rest mass.[9.16] This rest mass, f.i. for a molecule of copper, always the same, independent from the speed of the molecule.'
>
> 'It is not good to introduce the concept of a mass which depends from its velocity for this is not a clear concept.'
>
> 'One should always introduce as 'mass' m a quantity independent of motion.' (quoted in [Earman & Fine, 1977], p.538)

In addition, relativistic mass m is, as classical mass, a conserved quantity. Indeed, given that E and p are conserved and together determine m by equation (9.1), mass conservation holds in relativity theory.

To sum up, the notion of 'relativistic mass' $m\ \gamma$ is not appropriate and is, at best, highly misleading: the proper extension of the concept of mass to relativity is the Lorentz invariant property symbolized by m in equation (9.1). Thus, we see that, in this case, co-extension in the shared domain of application is again not sufficient to characterize conceptual extension: it cannot determine which, if either, of the two purported notions of relativistic mass appropriately extends the notion of classical mass. Given this example of

9.15. See [Lange, 2002] ,pp. 236-238 for several examples.

9.16. See the next section on why Einstein, inappropriately, or so we argue, terms Lorentz-invariant mass m as 'rest mass'.

relativistic physics and the case of conditional probability in quantum physics, we contend that in general co-extension is not sufficient to characterize conceptual extension.

As a final remark, notice that the function $m\,\gamma$ allows the relativistic equation (9.5) $\boldsymbol{p} = m\,\gamma\,\boldsymbol{v}$ to take the classical form 'momentum = mass × velocity'. However, this formal analogy does not guarantee the interpretation of '$m\ \gamma$' as a relativistic extension of the classical notion of mass since, as we have seen, $m\ \gamma$ is not Lorentz invariant. Indeed, this formal analogy constitutes only a mnemonic device that leads to much conceptual confusion. Hence, similarly to the case of the probabilities defined by the Lüders rule, we see the need to disengage formal and interpretive features when evaluating whether a concept is a genuine extension of the concept it is co-extensive with. Any satisfactory theory of conceptual extension should be able to account for this feature.

9.3 No Fast-Holding Conditions for Concept Extension

Co-extension in the shared domain hence cannot adequately characterize conceptual extension. What seems to really matter for a notion to be the conceptual extension of another is that there is some 'core' meaning which both concepts share that carries over the boundary between the old and the new context.[9.17] When this happens co-extension will (usually) be satisfied, but not (necessarily) vice versa. However, evaluating whether there is in fact some shared core meaning is not a simple matter. In general, the concept of the new theory will have some features in common with the old concept and some completely new features but there does not seem to be a clear cut criteria to determine when the overlapping features justify calling the concept an extension and when they will not. To take a provocative example, think of the enormous controversy on whether or not the concept of 'abortion' is an extension of the concept of 'murder'.

Or think again about mass in relativistic physics. The Lorentz-invariant quantity symbolized by m cannot, as in classical physics, be interpreted as the amount matter of which a body is made of. This is because mass m is not additive in relativity theory: the total matter of a whole system is not the sum of the matter of the system's parts (where those parts are non-overlapping and together include the entire system.) Take for example the mass of a system composed of two subsystems. Given that energy and momentum are additive we have that $E = E_1 + E_2$ and $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$. But substituting this into expression

9.17. Meaning is here, and throughout the text, to be understood intuitively as the concept or sense marked out by a word and not in a technical philosophical sense.

(9.1) we get that $m^2 = \frac{(E_1+E_2)^2}{c^4} - \frac{(\boldsymbol{p}_1+\boldsymbol{p}_2)^2}{c^2}$, which is not equal to the sum of the individual masses $(m_1+m_2)^2$. Only at $v_1 = v_2 = 0$ does the second term vanish and is mass additive.

Nor is mass m a measure of inertia. That is, we cannot think of it as the property a body possesses that determines its resistance to a force in the vein of 'the more massive the body, the more force is required to give it a certain acceleration'. Indeed, in the special relativity framework the formula $\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}$ is valid and substituting in it equation (9.1) we get $\boldsymbol{a} = \frac{(\boldsymbol{F} - \boldsymbol{F}\boldsymbol{\beta})\boldsymbol{\beta}}{m\gamma}$, where $\boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}$. The acceleration is not parallel to the force as in the Newtonian situation and hence we cannot cling onto the Newtonian relation of proportionality between $\boldsymbol{a}$ and $\boldsymbol{F}$. The Newtonian equation '$\boldsymbol{F} = m\,\boldsymbol{a}$' cannot be used in the relativistic context, and hence m cannot be interpreted as a measure of inertia.

Nevertheless, there seems to be a surrogate quantity which both classical and Lorentz-invariant relativistic mass m are measures of. For even though energy is *not* generally a Lorentz invariant quantity, in the special case of a body at rest, it is invariant given that it is proportional to relativistic mass m – for $\boldsymbol{p} = 0$ equation (9.1) yields $m = E_0/c^2$. And it is the rest energy E_0, 'dormant' in massive bodies, which Lorentz-invariant mass m, along with classical mass, measures. Rest energy thus seems to be the core feature of the notion of mass which is present both in the classical and relativistic domains, allowing us to appropriately think of mass m in relativity as an extension of the classical notion of mass.

Prima facie, one could argue that this is not really so; that is, that rest energy is really *not* the core feature of the notion of mass, present both in the classical and relativistic domains, which allows us to appropriately think of mass m in relativity as an extension of the classical notion of mass. For when $\boldsymbol{p} = 0$, Lorentz-invariant mass $m = \sqrt{\frac{E^2}{c^4} - \frac{\boldsymbol{p}^2}{c^2}}$ reduces to so-called rest mass $m_0 = E_0/c^2$, and can be interpreted, analogously to Newtonian mass, as the 'dormant' energy in massive bodies. But what does Lorentz-invariant mass $m = \sqrt{\frac{E^2}{c^4} - \frac{\boldsymbol{p}^2}{c^2}}$ measure when $\boldsymbol{p} \neq 0$? So far, we have only said negative things, namely that it is not a measure of the amount of matter nor of inertia. What is it then a measure of?

This reasoning is, however, misguided for it is confused about the role that $\boldsymbol{p} = 0$ plays in interpreting mass in relativistic physics, or, put somewhat differently, it is mistaken in calling $m = E_0/c^2$ rest mass m_0. Indeed, $\boldsymbol{p} = 0$ is not a physically relevant condition because momentum is not a physical property in relativity – in contrast to mass, neither energy nor momentum are Lorentz-invariant and, therefore, are not objective properties of the system. Thus, the condition $\boldsymbol{p} = 0$ simply selects a particular frame of reference, one which allows understanding mass as the rest energy E_0, 'dormant' in massive bodies. It is the only frame in which the combination of energy and momentum only reflects the body's mass; in general, a body's combination of energy and momentum in a given frame reflects both its mass and that frame.

It is thus correct to consider rest energy as the core feature of the notion of mass; it is this 'dormant' energy in bodies which allows us to appropriately think of mass m in relativity as an extension of the classical notion of mass. (Note, that calling E_0 rest energy is, in contrast to calling m_0 rest mass, fully justified.) The Lorentz-invariant mass m is generally referred to as 'proper mass'. Hence, the notion of proper mass in relativity theory has some features in common with classical mass – both measure the rest energy in bodies – and some completely new features – it is not a measure of the amount of matter nor of inertia.

To determine which, if any, of the overlapping features are *the* critical features of the notion of mass, we have needed to engage in a detailed study of the particular science involved. This brings out the highly topic specificity in the evaluation of concept extension, and the ensuing difficulty in giving a general account of it. The evaluation of concept extension needs to be applied alongside with all the scientific knowledge of particular cases. As Buzaglo points out in 'The Logic of Concept Expansion',

> 'Modern logic, founded by Frege, gives us no tools for understanding concept development, for it forces us to claim of the developed concept that either it is identical to the old one or it is completely different from it. When we get to philosophy, we feel that this sharp division is insufficient to get to the bottom of the problem. [...] I suggest we should escape the narrow dichotomy of the new concepts being identical or totally different from the old. [...]
>
> I propose to take examples of expansions from mathematics and science, preferably as simple as possible, and to analyze them. Thus [...] we can see what is happening in a clearer way.' ([Buzaglo, 2002], p.169-170)

However, the project of giving an account of concept extension is not as hopeless as it looks. Indeed, even if concepts have an 'open texture' that does not allow for a set of jointly necessary and sufficient conditions to characterize an extended concept, one can try to formulate a 'cluster of markers' which are expected to hold for the extended concept. If too many of these markers fail then one would hesitate to regard the concept of the new theory as an extension of an old one rather than a new concept altogether. And though this account would be far from giving fast-holding conditions for concept extension, this need not be considered as a vice but, rather, as a virtue. Indeed, we think that it can more appropriately capture the complexity of the cases of conceptual change in actual scientific practice.

9.4 Cluster of Markers Account

So let us try to formulate this 'cluster of markers' which would characterize – in the loose sense explained above – concept extension. First, we have seen that the requirement of co-extension in the shared domain of two concepts is not sufficient to characterize conceptual extension. For example, the probabilities defined by the Lüders rule are co-extensive with conditional probabilities for compatible events, but this does not ensure that they can be interpreted as conditional probabilities for general quantum events. Similarly, 'relativistic mass' $m\,\gamma$ reduces to classical mass in the Newtonian limit, yet it does not define an acceptable notion of mass in the relativistic domain. Nevertheless, co-extension of two concepts in their shared domain does seem to be a necessary feature of concept extension; at least approximate co-extension is. Hence, we propose to take co-extension of the concepts of the new and old theory in their shared domain of application as the first marker in our cluster-of-markers account of concept extension.

Second, we have also emphasized that what seems to really matter for a notion to be the conceptual extension of another is that there is some 'core' meaning which both concepts share which thus carries over the boundary between the old and the new context. The requirement of 'teachability' is, we think, a particularly useful way of capturing this core meaning. Indeed, for a concept to extend an old one into a new context, it seems crucial that teaching standard applications of the concept in the the old context allows its application in the new one. This is so because it affords a way of focusing on the role that inferences and explanations play in determining the meaning of a concept.[9.18] Teachability is, therefore, the second marker in our cluster-of-markers account of concept extension.

Consider, for example, the notion of cardinality. Cantor's idea is that the extension of the concept of 'the number of elements of a set' to encompass infinite numbers lies in extending the idea of an ordering, like that associated with counting, to infinite sets. The notion 'number of' is bound up with the process of counting both in the finite and infinite domains thus making cardinality a genuine extension of the notion 'number of'. Indeed, two sets have the same number of elements just in case there is a one-to-one correspondence between them both in the finite and infinite domain. This constitutes the central feature of

9.18. This is similar to the conceptual role semantics (CRS) approach. The basic idea of CRS (also called functional role semantics) is that the content of syntactic entities and mental representations is at least partially constituted by the cognitive or inferential role they have for a thinker or community. Concepts have a specific role in thought, perception, decision making, and action. As CRS focuses on how content figures in reasoning and rational behavior, it conforms to the idea that the crucial purpose of the ascription of concepts and thought content is to explain behavior, including verbal behavior. The study of conceptual change is thus about the change of the inferences and explanations supported by concepts, focusing on something that matters for scientific change and progress. See [Brigandt, 2004] and references therein (especially footnote 4).

the notion and as such, if taught in standard applications of the finite domain, will allow its application in the infinite domain.

Similarly, if one teaches the concept of mass in classical mechanics as the measure of 'dormant' energy in massive bodies, one can readily think of proper mass m in relativity as an extension of the classical notion of mass. In contrast, even if one learns the notion of conditional probability for compatible quantum events, one is left clueless as to how to apply it for incompatible ones. The difficulty lies in that the notion of conditionality – in contrast to the notion of the 'number of elements of' or the notion of mass which extend smoothly from finite to infinite cases and from low velocities to high velocities, respectively – does not extend smoothly from compatible cases in which the notion clearly applies to incompatible ones.

Indeed, the quantum analogue of the classical rationale for obtaining conditional probabilities is simply not available. And thus there seems to be no feature of conditionality which, if taught for compatible events, will allow one to make an assignment of conditional probabilities for incompatible events. To assign a number to two incompatible events one has to introduce a special rule; one such rule is the Lüders rule. One might then call this an 'extension' of conditional probability, but it is really a *different* concept that bears no resemblance with conditionality.

This is certainly so if quantum events are interpreted as physical values or as measurements results (both synchronically and diachronically). And if quantum events are interpreted as projection operators, then, as we have seen, the same conclusion holds for the diachronic notion and can be arguably shown to hold for the synchronic one. Indeed, it is difficult to see why one would assign the probability of projector P conditional on projector Q the pro-rata increase of the probability of the operator QPQ.

Third, an extended concept usually deepens or adds more resolution to an old concept. This generally enhances our understanding both of the original and the new context. Indeed, the extended concept helps us distinguish between critical and context-dependent features of a concept; and it helps us acquire familiarity and understanding of the new context by allowing us to think of the new area as if it were the old one. Buzaglo describes it nicely

> the extended concept usually leads to making distinctions 'as if it had taken an unpainted surface and painted it with a variety of colors, giving us a way of demonstrating the differences between objects in the newly painted area.' ([Buzaglo, 2002], p.66)

Or, as Fine puts it,

> the extended concept 'penetrate[s] into nature more deeply than its predecessors; it [advances] our understanding by allowing us to put to nature certain questions which were not clearly expressible on the basis of the old concepts and it should lead us to expect that some such questions have determinate answers.' ([Fine, 1986], p.20).

We refer to this feature of concept extension as 'conceptual fruitfulness', a fruitfulness which works both from the old to the new context and vice versa, and take it as the third marker in our cluster-of-markers account of concept extension.

To illustrate it, take first the notion of cardinality. Learning this notion enhances our understanding of the finite domain by showing that what is crucial about the notion of 'number of elements' is the process of counting. Other features present in the finite domain but not in the infinite one, as say, having fewer number of elements if a set is a proper subset of another set, are now seen as accessory to the notion of 'number of', only applying in the finite domain. In addition, the notion of cardinality enables to answer questions pertaining to the infinite domain that did not have determinate answers in the finite one, as say, the number of elements in the even set is the same as that of the set of natural numbers.

Similarly, the relativistic notion of proper mass m deepens or adds more resolution to the classical one. It leads to making further distinctions as, for example requiring to decouple mass from matter or showing how mass is an objective property, in the sense of being Lorentz invariant, whereas energy is not. Thus, it helps us distinguish between critical and context-dependent features of the notion of mass – 'dormant' energy in massive bodies versus a measure of the amount of matter or inertia – and it helps us acquire familiarity and understanding of the relativistic domain.

Finally, a number of important results that are formulated in terms of the old concept usually carry over to the extended domain. 'Conservativeness' from the old to the new context will thus be the fourth and final marker in our cluster-of-markers account of concept extension. For example, the extended concept of cardinality is extremely fruitful at a formal level. Indeed, an important class of results as, for example, the results of finite arithmetic, carry over from the finite to the infinite. This allows us to navigate the infinite domain with an ease otherwise difficult to acquire.

However, special care must be taken with this marker. For formal fruitfulness is not always a reliable indicator of concept extension: while formal features are a useful guide for conceptual extension, these can easily lead one astray when drawing their interpretive conclusions. For example, we have seen that the fact that $m\,\gamma$ allows $p = m\,\gamma\,v$ to take the classical form 'momentum = mass × velocity' gives no additional grounds to regard it as the appropriate extension. Even if regarding $m\,\gamma$ as the relativistic extension of mass is formally fruitful, this formal analogy constitutes only a mnemonic device that leads to much conceptual confusion. Thus, one has to make a detailed evaluation of each particular case. As we already emphasized, the evaluation of conceptual extension needs to be applied alongside with all the scientific knowledge of particular cases.

Consider also the case of conditional probability. As we have seen, there are two ways of formally characterizing conditional probability in a classical probability space, namely by its additivity property (theorem 2.4) and by the existence and uniqueness theorem (theorem 2.1). Now the probabilities defined by the Lüders rule, while also being characterized by an analogue existence and uniqueness theorem (theorem 4.2), do not retain the additivity property. And we have argued that, even if the formal characterization of existence and uniqueness holds both for classical conditional probabilities and the probabilities defined by the Lüders rule, it does not provide enough interpretive content for the latter to be read as a conditional probability.

In analogue fashion, the notion of the 'number of elements' has two defining features in the finite domain, namely two sets have the same number of elements if one can establish a one-to-one correspondence between its elements, and one set has fewer elements than another if it is a proper subset of it; and only the former property is retained in the infinite domain. However, whereas in the case of the probabilities defined by the Lüders rule the extended feature *does not* yield a genuine extension of conditional probability, in the number case the extended feature *does* yield a genuine extension of the notion 'number of'.

One could also wonder what would happen if the additivity-with-respect-to-conditioning-events characterization of conditional probability were to be retained in the quantum case. Maybe, one could hope, the now additive probabilities defined by the (somehow modified) Lüders rule, would genuinely extend the notion of conditional probability to the quantum context. However, these hopes would be unfounded. For one encounters difficulties in interpreting the probabilities defined by the Lüders rule as conditional probabilities even in the absence of interference terms. Indeed, for our spin 1 particle, both $\mathbb{P}_\psi(P_{y_{+1}}|P_{x_{+1}})$ – or $\mathbb{P}_\psi(s_{y_{+1}}|s_{x_{+1}})$ – and $\mathbb{P}_\psi(P_{s_{y_{+1}}}|P_{s_{x_{+1}}} + P_{s_{x_0}})$ – or $\mathbb{P}_\psi(s_{y_{+1}}|s_{x_{+1}}$ or $s_{x_0})$ – cannot be interpreted as conditional probabilities. Hence, once more, we see the intricate interplay between formal and interpretive features.

To sum up, the general markers we propose to characterize an extended concept are the following:

1. *Co-extension* of the concepts of the new and old theory in their shared domain of application.

2. *Teachability*: teaching standard applications of the concept in the the old context allow its application in the new context.
3. *Conceptual fruitfulness*: the extended concept enhances our understanding of both the original and the new context.
4. *Conservativeness*: a number of important formal results that are formulated in terms of the old concept carry over to the extended domain.

This cluster of markers is expected to hold if the concept of the new theory is to be regarded as an extension of a concept of the old domain. However, they do *not* constitute a set of jointly necessary conditions for conceptual extension. Indeed, the notion of proper mass in relativity m, unlike $m\,\gamma$, fails to satisfy the conservativeness requirement and yet it is a genuine extension of mass. And they are not individually sufficient either: for example, 'relativistic mass' $m\,\gamma$ satisfies conservativeness and co-extension and yet does not define a proper concept of mass. Nevertheless, taken all together they do seem to provide an appropriate rationale for concept extension.

To finish, note that this account is yet to be further developed and would greatly benefit from a detailed study of more particular mathematical and physical concepts. As Buzaglo suggests, we should engage in a more promising case-by-case investigation so that 'we can see what is happening in a clearer way.' ([Buzaglo, 2002], p.169) Moreover, this study need not restrict itself to physical and mathematical concepts. For example, [Fine, 1978] studies in some detail how the concept of sexuality underwent a significant expansion in psychoanalytic theory.

9.5 Implications for Conceptual Change in Science

When new scientific theories are developed the concepts of the new theories may or may not extend those of the older theories in a uniform way. Our attempt in this chapter has been to provide a rationale to evaluate when conceptual change can be viewed as concept extension. However, this perspective stands in contrast with much of the literature on conceptual change in science, in which the emphasis has been more on the discontinuous character of conceptual change. Indeed, the 'radical' historicist philosophers – such as Norman R. Hanson, Paul Feyerabend and Thomas S. Kuhn – saw scientific change as abrupt and discontinuous, and thus sought to characterize conceptual change in terms of new conceptual structures as completely replacing the previous ones with inconsistent or, more radically, with incommensurable ones.[9.19]

9.19. For the classical theses of incommensurability see [Kuhn, 1962], [Feyerabend, 1965, 1975].

Under this view, the concepts employed in one theory are seen as radically different and not comparable or translatable to those used in the other theory: it is as if different observers of the same world see it from radically different and not comparable points of view. For example, for Kuhn, theoretical changes are scientific revolutions, where

> 'Scientific revolutions are here taken to be those non-cumulative developmental episodes in which an older paradigm is replaced in whole or in part by an incompatible new one' ([Kuhn, 1962], p.92).

Kuhn does not simply define scientific revolutions as non-cumulative; rather, scientific revolutions must be non-cumulative because of certain presumed truths about the nature of paradigms, observation, and meaning. And when during these periods one scientific theory is replaced by another, there are changes in the standards of governing permissible problems, concepts, and explanations.

We do not wish to enter here the enormous literature surrounding the issue incommensurability and meaning change within theoretical change. What we want to do is emphasize that claims of incompatibility and incommensurability between concepts of different scientific theories should be handled with great care. Indeed, as we have already stressed, evaluating whether there is in fact some shared core meaning between two concepts which carries over the boundary between the old and the new context is not a simple matter. There are no clear cut criteria to determine when the overlapping features justify calling the concept an extension and when they will not, and much less to determine when the new concept is (supposedly) not even comparable with the old one. The evaluation of conceptual change is a very subtle and intricate issue that needs to be applied alongside with all the scientific knowledge of particular cases.

Take again the notion of mass in the classical and relativistic contexts. While we have argued that in relativistic physics the notion defined by Lorentz-invariant proper mass m *is* an extension of the classical notion of mass, Kuhn claims that it is *not*. In his view, or better in Field's reconstruction of his view [Field, 1973], there are three different notions of mass, namely Newtonian mass, proper mass m, and 'relativistic mass' $m\,\gamma$. Newtonian mass is neither proper mass m nor 'relativistic mass' $m\,\gamma$. Indeed, like 'relativistic mass' $m\,\gamma$ but unlike proper mass m, Newtonian mass is equal to momentum divided by velocity. Like proper mass m but unlike 'relativistic mass' $m\,\gamma$, Newtonian mass has the same value in all reference frames and is conserved in all interactions. And, unlike both proper mass m and 'relativistic mass' $m\,\gamma$, Newtonian mass is a measure of the amount of matter and of inertia. Under this perspective, Newtonian mass does not, thus, denote the same physical quantity that either proper mass m or 'relativistic mass' $m\,\gamma$ denote.

That is, Kuhn argued that Newtonian mechanics cannot even be seen as a limiting case, even, of special relativity: although one can derive laws that look like the Newtonian ones from the laws of special relativity under appropriate assumptions, namely low speeds, the significance of these laws is vastly different in the two theories given that the concepts marked by the terms shared by the two theories, i.e. mass, space, time, will have changed radically in the move from one theory to another. In Kuhn's words,

> 'Though the N_i's [i.e. the equations derived from the set of statements E_1, E_2,..., E_n, which together embody the laws of relativity theory, in the special case of $(\frac{v}{c})^2 \ll 1)$] are a special case of the laws of relativistic mechanics, they are not Newton's Laws. [...] The variables and parameters that in Einstein's E_i's represented spatial position, time, mass, etc., still occur in the N_i's; and they there still represent Einsteinian space, time and mass. But the physical referents of these Einsteinian concepts are by no means identical with those of the Newtonian concepts that bear the same name. (Newtonian mass is conserved; Einsteinian is convertible with energy. [...]) Unless we change the definitions of the variables in the N_i's, the statements we have derived are not Newtonian. If we do change them, we cannot say to have properly *derived* Newton's Laws [...].
>
> [T]he transition from Newtonian to Einsteinian mechanics illustrates with particular clarity the scientific revolution as a displacement of the conceptual network through which scientists view the world.'
>
> ([Kuhn, 1962], pp. 100-101).[9.20]

The problem is that this view relies heavily on the assumption that the concepts marked by the terms shared by the two theories, e.g. mass, space, time, have changed radically in the move from one theory to another. But, as we have argued, this need not be the case. Indeed, first, $m\,\gamma$ should be dismissed as defining a valid concept of mass in relativistic physics because it is not Lorentz-invariant, and thus not an objective property. And second, the concepts marked by Newtonian and proper mass m have *not* radically changed as Kuhn claims: they have a common feature which both share, namely they both measure the 'dormant' energy in massive bodies. Indeed, it is not by coincidence that the symbol m appearing in the relativistic equations (9.1) has been termed 'proper mass'. As John Earman remarks,

> ' 'Proper mass' is not a misnomer!' ([Earman & Fine, 1977], p.537)

9.20. See Appendix E for some clarifications on the (supposed) mass-energy equivalence.

Regarding proper mass m as an extension of classical mass in relativity theory allows one to capture an evolution of the concept of mass which is closer to actual scientific practice. Indeed, as we acquire more empirical data, and accordingly modify our scientific theories, the concepts employed in them suffer changes. But these changes do not necessarily give rise to completely unrelated, inconsistent or incommensurable concepts; rather the concepts of the different theories have retained some relation, and this relation can be stable enough to allow understanding the concept of the new theory as an extension of that of the old theory. In the case of mass, the concept of proper mass m in relativity can indeed be seen as an extension of the classical concept of mass.

Chapter 10
Conclusion

Describing the conceptual changes from the classical to the relativistic notion of mass as conceptual extension, one can account for the fact that the transition from classical to relativistic physics is in fact marked by very significant conceptual innovation and change, *without* making the process of scientific development degenerate into a series of irrational choices, as the historicist philosophers view on radical conceptual change entails. But what about in quantum theory? Have the concepts of quantum mechanics retained a stable enough relation with respect to the concepts of classical mechanics to allow their understanding as extensions of the classical ones?

We have argued that in the case of the probability defined by the Lüders rule, the only possible candidate for a definition of a quantum notion of conditional probability, this is not the case. That is, we have argued that the change of meaning from conditional probability to the probability defined by the Lüders rule is so substantial that, even if these probabilities are co-extensive in their shared domain of application, namely compatible events, we cannot talk anymore about the same concept. We have shown this claim holds for physical values, for measurement results, and at a formal level for projection operators, both from a synchronic and a diachronic perspective.

In addition, in Chapter 8, we have argued that neither quantum Bayesianism nor quantum Frequentism can provide an adequate interpretation of the quantum *un*conditional probabilities. We have concluded that, ultimately, the reason for the failure of these interpretations is equivalent to the failure of interpreting the probability defined by the Lüders rule as a conditional probability. Indeed, we have argued that there is no quantum notion of joint probability, or equivalently, no quantum notion of conditional probability, which can be interpreted at any level which is not thoroughly instrumental. We have brought into this evaluation a very detailed analysis of (what now seems as) the relevant scientific knowledge. Quantum mechanics remains conceptually as puzzling as ever.

Now the problem is that the quantum probabilities defined by the trace rule – both using a general density operator W or the density operator given by the Lüders rule – coincide, and to an incredible degree of accuracy, with the empirically found frequencies.

Moreover, these frequencies provide the sole connection between quantum mechanics and the empirical world. Does this then imply, as is widely held, that the classical and quantum concepts are incommensurable? That the quantum world represents a radical conceptual break with the classical one?

Maybe yes. And again maybe not. A conclusive answer does not seem to be yet forthcoming. Perhaps, as Einstein hoped, there are new fundamental concepts and a new theoretical basis from which the quantum theory will emerge as a statistical approximation. Or perhaps one of the (many) interpretations of quantum mechanics will turn out to present a particularly appropriate reading of the quantum formalism. Who knows? In this sense, we think that it would be interesting to apply our analysis of 'conditional' and 'unconditional' quantum probability within each particular interpretation of quantum mechanics and that, hopefully, this will provide a fruitful guide for future research.

Appendix A

Subjective Bayesian Interpretation of Probability

Bayesianism takes probability to be a measure of ignorance, reflecting our state of knowledge about the world. In this approach, what constitutes a possible event is dictated by Nature, i.e. is an objective fact of reality, but the probability assigned to that event is not determined by objective features of the world; rather this probability represents our uncertainty about facts. In particular, for the subjective Bayesian interpretation the probability of an event is nothing more than a rational (later to be defined) agent's degree of belief on its occurrence.[A.1] Further, it does not assume that rationality leads to consensus: different individuals, although all perfectly reasonable and having the same evidence, may have different degrees of belief in an event. Probability is thus defined as the degree of belief of a particular individual, that is, as a *subjective* degree of belief.

A.1 Betting Quotients and Ramsey-de Finetti Theorem

Probabilities are numbers in the interval $[0,1]$ which satisfy a certain mathematical definition (definition 2.3). Hence, if the subjective theory is to provide an adequate interpretation of the mathematical calculus, one must first find a way of measuring the degrees of belief of an individual so that these are assigned numerical values, and then show that these degrees of belief satisfy the standard axioms of probability. Betting behavior provides an answer to the first task and the Ramsey-de Finetti theorem fulfills the second requirement. We will proceed to present both in turn.[A.2]

A.1. We here develop the account of subjective Bayesianism associated to [de Finetti, 1937], wherein credences are regarded as capturing all information about the bets that you are prepared to enter into. Other accounts of credences are the following. According to [van Fraassen, 1991], credences encapsulate all information about the judgments that one makes. [Ramsey, 1926], [Savage, 1954] and [Jeffrey, 1983] derive both probabilities and utilities (desirabilities) from rational preferences. And, in a similar spirit, [Lewis, 1986] analyzes credences as the probability function belonging to the utility/probability function pair that best rationalizes ones behavioral dispositions.

A.2. This presentation of the subjective Bayesian interpretation follows [Gillies, 2000a], Chapter 4.

Suppose that Ms A wants to measure the degree of belief of Mr B in some event E. One way she can do this is by getting Mr B to agree on a bet on it under the following conditions. Mr B has to choose a number q (called his *betting quotient* on E) and then Ms A chooses the *stake* S. Mr B will pay Ms A $q\,S$ in exchange for S if E occurs. Thus his total gain will be $S - q\,S$ if E occurs but he will loose $q\,S$ if E does not occur. S can be positive or negative but Mr B does not know this when choosing q for this guarantees that he will adjust q to his actual belief.[A.3] Under this betting set-up, q is taken to be a measure of Mr B's degree of belief in E.

Now it does not yet follow that these betting quotients are a probability measure. Indeed it seems easy to imagine an individual whose degrees of belief are quite arbitrary and do not satisfy the standard axioms of probability. The subjectivists solve this problem and derive the axioms of probability by making the assumption of *coherence*. The coherence assumption says that an agent will not accept terms for bets he wants to win that will make him certain to loose whatever happens. More precisely,

Definition A.1. ***Coherence****. If a bettor (Mr B) has to bet on a number of events $E_1, \ldots,$ E_n, his betting quotients are said to be coherent if and only if the stake-maker (Ms A) cannot choose stakes $S_1, \ldots, S_n$ such that she wins whatever happens.*

If Ms A can choose stakes so that she wins whatever happens, she is said to have made a *Dutch Book* against Mr B. Mr B will obviously want his bets to be coherent to avoid the possibility of losing whatever happens. An agent that holds coherent degrees of belief is said to be rational.

It is a remarkable fact that the coherence condition is both necessary and sufficient for betting quotients to satisfy the axioms of probability. This is the content of the so-called Ramsey-de Finetti theorem:

Theorem A.1. ***Ramsey-de Finetti Theorem****. A set of betting quotients is coherent if and only if they satisfy the axioms of probability.*

If your degrees of belief violate the probability axioms, then there exists a Dutch Book against you, and if your degrees of belief do not violate the probability axioms, then there does not exist a Dutch Book against you. Thus in the subjective theory the axioms of

A.3. If Mr B knew that the stake is positive then he would set q as low as possible so as to get more money if E occurs (remember he gains $S\,(1-q)$ if E occurs) and loose less money if E does not occur (remember he looses $q\,S$ if E does not occur). Whereas if Mr B knows that the stake is negative then it will be in his interest to set q as high as possible.

probability can be proved rigorously from the very plausible condition of coherence. We will not provide a full proof of this result but we will consider parts of it and indicate the reference to the missing ones.

Let us begin by showing how coherence implies $0 \leqslant p(A) \leqslant 1$ for any event $A \in \mathcal{F}(S)$ and $p(B) = 1$ in case of the certain event B. Imagine Mr B were to set his degree of belief in an arbitrary event A to $q(A) > 1$. He would have a Dutch Book made against him for Ms A would win whatever happens by choosing $S > 0$: if A occurs then Mr B's total gain $S\,(1-q)$ would be negative, which simply means that he looses $S\,(1-q)$; and if A does not occur then Mr B would also loose, since $q\,S$ is positive. And similarly if he set $q(A) < 0$ for now Ms A would win no matter what happens by choosing $S < 0$. Hence to be coherent, Mr B must choose $0 \leqslant q(A) \leqslant 1$. One can prove that $q(B) = 1$ in an analogue way: if $q(B) > 1$ Ms A can win by choosing $S > 0$ and if $q(B) < 1$ Ms A can win by choosing $S < 0$. Hence to be coherent, Mr B must choose $q(B) = 1$. The reverse implications also hold, that is if $0 \leqslant p(A) \leqslant 1$ for any event $A \in \mathcal{F}(S)$ and $p(B) = 1$ in case of the certain event B, then coherence holds.[A.4]

Let us now show that coherence ensures that (finite) additivity holds, namely that for events A and B mutually exclusive $p(A \cup B) = p(A) + p(B)$.[A.5] Assume that Mr B assigns a degree of belief $q(A)$ to the occurrence of event A, $q(B)$ to the occurrence of event B and $q(C)$ to the occurrence of event $C = A \cup B$, and Ms A chooses the stakes $S(A), S(B)$ and $S(C)$ respectively. What we want to prove is that coherence implies that $q(C) = q(A) + q(B)$. To do so let us consider the possible outcome events and the net gain of Mr B in each case:[A.6]

i. A and $\neg B$: $S_A\,(1-q_A) - S_B\,q_B + S_C\,(1-q_C)$

ii. $\neg A$ and B: $-S_A\,q_A + S_B\,(1-q_B) + S_C\,(1-q_C)$

iii. $\neg A$ and $\neg B$: $-S_A\,q_A - S_B\,q_B - S_C\,q_C$

(Note that we haven't considered the event A and B because A and B are mutually exclusive events.) Now for Ms A to not be able to chose stakes so that Mr B looses whatever happens (net gain<0 for all S_A, S_B, S_C) nor for Mr A to loose whatever happens (net gain>0 for all S_A, S_B, S_C), the net gain in each of these three cases must be 0. This corresponds to setting the determinant of the set of equations

$$S_A\,(1-q_A) - S_B\,q_B + S_C\,(1-q_C) = 0$$

A.4. [Gillies, 2000a], p.61.

A.5. See [Williamson, 1999] on whether countable additivity is an acceptable axiom of subjective probability.

A.6. To make the notation easier we will simply set $q(A) = q_A$ and similarly for the rest.

$$- S_A\, q_A + S_B\, (1 - q_B) + S_C\, (1 - q_C) = 0$$
$$- S_A\, q_A - S_B\, q_B - S_C\, q_C = 0$$

to zero. The value of the determinant is $q_A + q_B - q_C$, so it then follows that the probability assignment is incoherent unless $q_A + q_B = q_C$. And hence, (finite) additivity holds for coherent degrees of belief. In reverse fashion, the addition law implies coherence.[A.7] Thus under a *subjective Bayesian* interpretation probabilities are defined as the subjective degrees of belief of a coherent agent. Let us consider now the notion of subjective conditional probability.

A.2 Subjective Conditional Probability

As we saw in Chapter 2, the most usual usual approach to subjective conditional probability is the so-called *Ramsey test*, which takes the subjective conditional probability $\mathbb{P}_p(A|B)$ as given by the degree of belief one has in A when supposing B (or hypothetically adding B to one's stock of beliefs). The notion of supposition is crucial for it allows one's conditional degree of belief to differ from how one's beliefs would actually change were one to learn B with certainty.[A.8] Indeed, if one defines $\mathbb{P}_p(A|B)$ as the degree of belief an agent would have (or ought to have) if she were to learn (with certainty) that B is in fact the case, then one faces several problems that are easily overcome by the supposition account.

For one, even in ordinary cases, it takes a lot of idealization to claim that there is a single proposition that an agent learns between one time and another. And in many cases, it seems there are infinitely many such propositions, and it's not clear that an agent's algebra of events will always be closed under such infinite conjunctions. Furthermore, there are also many cases in which an agent's degrees of belief change by loss of certainty, rather than gaining new knowledge. But supposition generally features a single event (or a finite conjunction of them) rather than an infinite set of premises. And worries about lack of certainty or the loss of information are irrelevant for there is no such thing as 'partially supposing' a proposition, or 'negatively supposing' something.

However, care must be taken with the notion of supposition for it can lead to conclude that one is omniscient – that is, one should believe 'if p then I believe that p, and if I believe that p, then p'. Both a *subjunctive truth* notion of supposition – on which $\mathbb{P}_p(A|B)$ measures how strongly the agent believes that A would have been the case, if B had been true – and a *subjunctive belief* notion of supposition – on which $\mathbb{P}_o(A|B)$ measures how

A.7. See [Gillies, 2000a], p.61

A.8. We here follow [Easwaran, 2008].

strongly the agent would believe A, if she believed B – face their own particular challenges. (To give an example of these two notions of supposition. If A is the event that someone wrote Macbeth, and B is the event that Shakespeare did not write Macbeth, then for most people, on the subjunctive truth notion $\mathbb{P}_p(A|B)$ is quite low, while on the subjunctive belief notion $\mathbb{P}_p(A|B)$ is fairly high. I would not stop believing that Macbeth had been written, even if I learned that Shakespeare had not written it.)

The subjunctive truth notion can't be the correct account since there are many cases in which it violates the ratio analysis. For example, if A is the event that I exist, and B is the event that humans have been in Europe for more than 100 years, then my subjunctive value for $\mathbb{P}_p(A|B)$, is fairly low (if humans had not been in Europe, presumably history would have been so radically different that I would never have come to exist), but the ratio account says that $\mathbb{P}_p(A|B) = 1$ because my degree of belief in $A \cap B$ is the same as my degree of belief in B. The subjunctive belief account can't be correct either, as was already acknowledged by Ramsey:

> 'The conditional probability $\mathbb{P}_p(A|B)$ is not the same as the degree to which [the agent] would believe A, if he believed B for certain; for knowledge of B might for psychological reasons profoundly alter his whole system of beliefs.' ([Ramsey, 1926], p. 180; notation adapted)

[van Fraassen, 1980] gives a particular example of this problem in raising worries for Brian Ellis' account of conditionals and [Chalmers & Hájek, 2007] also analyze this problem.

But regardless of what exactly conditional degrees of belief are – or whether they can be reduced to some notion of supposition – betting behavior, as with the notion of degree of belief, sheds important light on this notion. Indeed, it seems that $\mathbb{P}_p(A|B)$ ought to have some connection to the agent's disposition to accept bets on A, that will be called off if B is not true; and there is a standard Dutch book argument suggesting that under this interpretation, one ought to set $\mathbb{P}_p(A|B)$ to what the ratio analysis stipulates.

In effect, coherence also implies that subjective conditional probabilities agree with the probabilities defined by the ratio formula; that is, an agent would be incoherent, i.e. be 'Dutch Booked', if she does not set her conditional degree of belief in A given B to her degree of belief in their joint occurrence divided by the degree of belief in B, i.e. if she does not set $\mathbb{P}_p(A|B) = \frac{p(A \cap B)}{p(B)}$. Hence the coherent agent will set his conditional degree of belief to precisely what the ratio analysis requires. Let us consider this argument.[A.9]

A.9. See [Gillies, 2000a], pp. 62-64, and [Howson & Urbach, 1996], pp.63-64.

The conditional betting quotient for A given B, $q_{A|B}$, is defined as the betting quotient which the bettor would give for A on the understanding that the bet is called off and all stakes returned if B does not occur. Assume that Mr B assigns a degree of belief q_B to the occurrence of event B, $q_{A\&B}$ to the occurrence of event A and B and $q_{A|B}$ to the occurrence of event A given that B has occurred. And the stake-maker chooses the stakes S_B, $S_{A|B}$ and $S_{A\&B}$ respectively. What we want to prove is that coherence implies that $q_{A|B} = q_{A\&B}/q_B$. To do so, we consider again the possible outcome events and the total gain of Mr B in each case:

i. B and A: $S_B\,(1-q_B) + S_{A\&B}\,(1-q_{A\&B}) + S_{A|B}(1-\,q_{A|B})$

ii. B and $\neg A$: $S_B(1-\,q_B) - S_{A\&B}\,q_{A\&B} - S_{A|B}\,q_{A|B}$

iii. $\neg B$: $-\,S_B\,q_B - S_{A\&B}\,q_{A\&B}$

Note that when $\neg B$ occurs the bet concerning the degree of belief on A conditional on the occurrence of B is called off. Again, for Ms A to not be able to chose stakes so that Mr B looses whatever happens (net gain<0 for all $S_{A\&B}, S_B, S_{A|B}$) nor for Mr A to loose whatever happens (net gain>0 for all $S_{A\&B}, S_B, S_{A|B}$), the net gain in each of these three cases must be 0. This corresponds to setting the determinant of the set of equations

$$
\begin{aligned}
&S_B\,(1-q_B) + S_{A\&B}\,(1-\,q_{A\&B}) + S_{A|B}\,(1-q_{A|B}) = 0\\
&S_B\,(1-\,q_B) - S_{A\&B}\,q_{A\&B} - S_{A|B}\,q_{A|B} = 0\\
&-\,S_B\,q_B - S_{A\&B}\,q_{A\&B} = 0
\end{aligned}
$$

to zero. The value of this determinant is $-\,q_{A|B}\,q_B + q_{A\&B}$ and equating it to zero it follows that the probability assignment is incoherent unless $q_{A|B} = q_{A\&B}/q_B$. Thus an agent would be incoherent, i.e. be ‘Dutch Booked’, if she does not set her conditional degree of belief in A given B to her degree of belief in their joint occurrence divided by the degree of belief in B. Hence the coherent agent will set his conditional degree of belief to precisely what the ratio analysis requires.

A.3 Conditionalization and Conditional Probability

Subjectivists typically recognize no constraints on initial or prior subjective probabilities beyond the coherence condition or, equivalently, their conformity to Kolmogorov’s axioms. But they typically advocate a learning rule for updating probabilities in the light of new evidence. Suppose that you initially have a probability function, and that you become certain of B (and of nothing more). What should be your new probability function? The favoured updating rule among Bayesians is the so-called *principle of conditionalization.*

Proposition A.1. ***Principle of Conditionalization***. *If at time t_i one assigns an event A an initial or prior probability $p_i(A)$, and one acquires new evidence B at a later time t_f, then one should systematically transform one's initial assignment $p_i(A)$ to generate a final or posterior probability assignment $p_f(A)$ at time t_f by conditioning on B, that is,*

$$p_i(A) \quad \longrightarrow \quad p_f(A) = \mathbb{P}_{p_i}(A|B) \tag{A.1}$$

Conditionalization derives probabilities posterior to B by redistributing the prior probabilities of all the sample points that B rules out pro rata over all the points that remain. So, for example, the difference between $\mathbb{P}_p(1|\text{odd})$ and $p(1)$ represents the extent to which the probability attached to '1' changes on receipt of the knowledge that the outcome was an odd number.

It is important to realize that the notions of conditional probability and conditionalization are distinct notions: while conditionalization is a diachronic notion – it applies to probabilities held at a time prior to learning of evidence B and to probabilities held at a time posterior to this learning, conditional probability is a synchronic notion – it applies only to probabilities at one time. And arguments in favour of the synchronic notion do not necessarily support the diachronic rule. Indeed, the Dutch Book argument which proved that $\mathbb{P}_i(A|B) = \frac{p_i(A \cap B)}{p_i(B)}$ only deals with probabilities at the initial time t_i but can say nothing as to whether the degree of belief in B at time t_f, $p_f(B)$, should be equated with the conditional on A degree of belief in B, $\mathbb{P}_i(A|B)$.

The principle of conditionalization is allegedly supported by a 'diachronic' Dutch Book argument: one is subject to a Dutch book (with bets placed at different times) if one does not conditionalize and conversely, if one does conditionalize, then one is immune to such a Dutch Book.[A.10] What is involved in this argument is the so-called *diachronic* coherence condition, i.e. coherence over time, in contrast to the previous Dutch Book arguments that involved only *synchronic* coherence.

There is, however, not a wide consensus on the validity of this argument. The main criticism is that the allegedly diachronic Dutch Book argument relies on the unjustified assumption that the probability the agent attaches to A if he were to know that B is true does not change after he in fact learns that B is true. That is, that if at t_f the agent learns that B is true (and nothing more), then at t_f he *still maintains* the value $\mathbb{P}_i(A|B)$, i.e. $\mathbb{P}_i(A|B) = \mathbb{P}_f(A|B)$. For then, under this assumption, the condition of *synchronic* coherence is enough to guarantee that the agent will infer the unconditional fair betting quotient on A, $p_f(A)$, to be $\mathbb{P}_i(A|B)$. But why should the agent be obliged to say in advance how he is going to bet on A in the event of B's being true? Moreover he will not

A.10. See [Teller, 1976] and [Lewis, 1997] be incoherent if he changes his mind between t_i and t_f. And in this case no Dutch Book can be made against him.[A.11] In Hacking's words,

> if 'the man announces his post-B rates only after B is discovered, and simultaneously cancels his pre-B rates, [then] there is no system for betting with him which is guaranteed success in the sense of a Dutch Book.' ([Hacking, 1967], p.315)

Thus the allegedly diachronic Dutch Book argument is criticized as being merely a re-proof of the synchronic Dutch Book proof of ratio, that is, as begging the question. However, even if the diachronic Dutch Book argument fails to show that conditionalization is always right, this failure does not show that it is often or even ever wrong: even if it does not follow from some general principle of rationality, the plausibility of its prescriptions may still recommend it as a general rule.

A.11. See [Hacking, 1967], [Howson & Franklin, 1994], [Mellor, 2005].

Appendix B
Problems for the Ratio Analysis

In section 2.3 we emphasized that the ratio formula (2.8) should not be regarded as a stipulative definition of conditional probability, but rather as an analysis of that notion. And we saw various justifications for why this should be so. However, [Alan Hájek, 2003] has forcefully argued against the adequacy of the ratio formula as an analysis of conditional probability. Briefly, he argues that conditional probabilities can be well defined in many and important cases in which the ratio analysis goes silent.

In more detail, Hájek shows that every probability assignment has uncountably many 'trouble spots' that come in three varieties which can create serious problems for the ratio analysis. First, there is the so-called *zero denominator problem.* The ratio analysis is mute whenever the condition has probability zero – if $p(B) = 0$ then $\frac{p(A \cap B)}{p(B)}$ is undefined so the ratio analysis delivers no verdict – and yet conditional probabilities may nevertheless be well defined in such cases. Indeed, contingent propositions may be assigned probability 0 – hence, probability 0 does not imply impossibility – and hence it should be legitimate to form conditional probabilities with probability zero conditionals.[B.1]

The second problem arises when conditional probabilities are sharp, i.e. determine a single probability function, and the corresponding unconditional probabilities are vague, something which the ratio analysis cannot respect. For example, the probability that the Democrats win in the next election is vague; but the probability that the Democrats win, given that the Democrats win is not vague: the answer is clearly 1. Similarly, p(the Democrats do not win, given the Democrats win) is not vague and is 0; and $p(T,$

B.1. Kolmogorov himself was well aware of this problem and elaborated the ratio analysis to handle cases such as these while preserving the guiding idea behind the simpler ratio analysis ([Hájek, 2003] p.291). However this extended analysis also turns out to be inadequate for it also falls prey to the other two problems of the ratio analysis. Indeed, Kolmogorov's analysis equates a certain integral in which the relevant conditional probability figures, to the probability of a conjunction; but when this latter probability is either vague or fails to exist at all, as in my cases of undefined unconditional probabilities, the analysis goes silent and yet the corresponding conditional probabilities are defined. In addition, the extended analysis delivers some conditional distributions that fail to comply to the requirement that the probability of anything consistent, given *itself*, is 1, a requirement that is self-evident if an analysis is to adequately capture the pre-theoretical notion of conditional probability. See [Hájek, 2008].

given the Democrats win) = 1, and $p(F$, given the Democrats win) = 0, where T and F, respectively, stand for necessary and impossible events.

Finally, the third problem lies in that various conditional probabilities are defined even when the corresponding unconditional probabilities appearing in the ratio analysis are undefined, and indeed cannot be defined. For example, the probability that a fair coin lands heads (H), given that I toss it fairly (F_T) is $1/2$. But the terms that appear in the ratio formula, i.e. $p(H \cap F_T)$ and $p(F_T)$, are undefined; and 'undefined' divided by 'undefined' does not equal $1/2$.[B.2] Hájek claims that 'the trouble spots are inescapable, and that they are, to put it mildly, plentiful' ([Hájek, 2003], p.281) and thus concludes that the ratio analysis is not an adequate analysis of conditional probability.

Nevertheless, he concedes that one might instead regard the ratio analysis as providing a constraint on conditional probability:

> '... the ratio might be thought of as a successful partial analysis, one that works for an important sub-class of conditional probabilities, in which the conditions are met. A sufficient condition for a conditional probability to equal a particular value is for the corresponding ratio to equal that value. However, it is not a necessary condition: a conditional probability can equal a particular value without the corresponding ratio equating that value.' ([Hájek, 2003], p.314)

Hájek then takes a further step and casts doubt on the very project of analyzing conditional probability itself, and claims that conditional probability, and not *un*conditional probability, should be in fact taken as *the* primitive notion. He says:

> 'At best, this leaves unfinished the project of giving a correct analysis of conditional probability [...] But perhaps the very project of analyzing conditional probability was misguided from the start. [...] We should regard conditional probability as conceptually prior to unconditional probability. So I suggest that we reverse the traditional direction of analysis: regard conditional probability to be the primitive notion, and unconditional probability as the derivative notion.' ([Hájek, 2003], p.314-315).

B.2. Note that this example is not an appropriate counterexample because the conditional probability involved is not a (genuine) conditional-on-event probability but rather is a 'conditional'-on-background-conditions probability (see section 6.5). However Hájek gives another example that does involve an event conditional probability. Briefly, imagine throwing an infinitely fine dart at the [0,1] interval, with you assigning a uniform distribution (Lebesgue measure) over the points at which it could land. The probability you give to its landing in C is undefined (see [Hájek, 2003a] p.279). However, the probability of the dart's landing in C, given that it lands in C is 1. So we have *p(C, given C) = 1* while ratio is undefined.

Be this as it may – actually, in a subsequent paper, Hájek says: 'even primitive conditional probabilities give at best an incomplete account' ([Hájek, 2008] p.7) – it suffices for our purposes to keep to Hájek's conclusion on thinking of the ratio formula as a successful partial analysis and simply concentrate on those cases in which it *does* work. For we are interested in evaluating whether or not quantum theory admits a notion of conditional probability; and the difficulties in defining such a notion are not related to the problems we have considered in this appendix.

As a final remark note that, even in Hájek's arguments against the adequacy of the ratio analysis of conditional probability were directed toward the general ratio $p(A\cap B)/p(B)$, they apply likewise to the ratio $p(A)/p(B)$ for $A\subseteq B$ which, as we showed in section 2.4, suffices to characterize conditional probability. Indeed, theorem 2.1 shows that the ratio $p(A)/p(B)$ for $A\subseteq B$ extends uniquely to the ratio $p(A\cap B)(p(B)$, and theorem 2.4 is proved by first showing that additivity with respect to conditioning events holds for $A\subseteq B$ and then applying theorem 2.1 for all A and B in $\mathcal{F}(S)$.

We give one example of each of the three problems Hájek considers for the ratio $p(A)/p(B)$ for $A\subseteq B$. First, using the dart example of footnote B.2, the probability that $A=$the dart lands on the point $1/4$, given that $B=$it lands on either $1/4$ or $3/4$, where $A\subseteq B$, is one half, and yet the probability that the point lands on $1/4$ or $3/4$ is zero according to the uniform measure. Second, in the examples of the probability that $A=$the Democrats win, given that $B=$the Democrats win, A is included in B. And finally, in the more involved example of the third-case – the probability that $A=$the dart lies in C given that it $B=$lies in C [B.3] – A is also included in B.

B.3. The question of A being a subset of B cannot even arise for the simple example – probability that $A=$a fair coin lands heads, given that $B=$I toss it fairly – since it is not a (genuine) conditional probability (see footnote B.2).

Appendix C
Proof of theorem 4.2

We prove that the following bi-conditional holds in $\mathcal{L}(\mathcal{H})$.

$$\forall P, Q \in \mathcal{L}(\mathcal{H})\, \mathbb{P}(P|Q) = \frac{\mathrm{Tr}(Q\,WQ\,P)}{\mathrm{Tr}(Q\,WQ)} \Leftrightarrow \mathbb{P}(P|Q) = \frac{p(P)}{p(Q)} \text{ for } P \leqslant Q \tag{C.1}$$

This proof mostly follows ([Malley, 2004], pp.13-15) and ([Beltrametti & Cassinelli, 1981], p.288).

Let us first prove the implication reading from left to right

$$\text{if } \mathbb{P}(P|Q) = \frac{\mathrm{Tr}(Q\,WQ\,P)}{\mathrm{Tr}(Q\,WQ)} \text{ for all } P \text{ and } Q, \text{then } \mathbb{P}(P|Q) = \frac{p(P)}{p(Q)} \text{ for } P \leqslant Q \tag{C.2}$$

Assume that $P \leqslant Q$. Then $Q\,P = PQ = P$. By the cyclic property of the trace, we have that $\mathrm{Tr}(Q\,WQ\,P) = \mathrm{Tr}(WQ\,PQ)$ and hence $\mathrm{Tr}(Q\,WQ\,P\,) = \mathrm{Tr}(WQ\,Q\,P) = \mathrm{Tr}(W\,Q\,P) = \mathrm{Tr}(WP)$ for $P \leqslant Q$. Introducing these results in the general expression of the Lüders rule we get

$$\mathbb{P}(P|Q) = \frac{\mathrm{Tr}(Q\,WQ\,P)}{\mathrm{Tr}(Q\,WQ)} = \frac{\mathrm{Tr}(WP)}{\mathrm{Tr}(WQ)} = \frac{p(P)}{p(Q)} \tag{C.3}$$

which is the desired result for $P \leqslant Q$.

Let us now prove the (much more difficult) implication reading from right to left

$$\text{if } \mathbb{P}(P|Q) = \frac{p(P)}{p(Q)} \text{ for } P \leqslant Q, \text{then } \mathbb{P}(P|Q) = \frac{\mathrm{Tr}(Q\,WQ\,P)}{\mathrm{Tr}(Q\,WQ)} \text{ for all } P \text{ and } Q \tag{C.4}$$

By Gleason's theorem, which shows that there is a one-to-one correspondence between the set of probability measures on $\mathcal{L}(\mathcal{H})$ and the set of density operators on $\mathcal{H}$, $\mathbb{P}(\,\cdot\,|Q)$ must be of the form $\mathrm{Tr}(W\,\cdot\,)$ for some density operator W. We will first show that this density operator W is unique and then give its expression.

i. Assume there exist two density operators W_1 and W_2 such that for all $P \leqslant Q$ $\mathrm{Tr}(W_1\,P) = \frac{p(P)}{p(Q)}$ and $\mathrm{Tr}(W_2\,P) = \frac{p(P)}{p(Q)}$. Hence $\mathrm{Tr}(W_1\,Q) = 1$ and $\mathrm{Tr}(W_2\,Q) = 1$. And so

$$\mathrm{Tr}(W_1\,Q^{\perp}) = \mathrm{Tr}(W_2\,Q^{\perp}) = 0 \tag{C.5}$$

for $Q^{\perp}$ the orthogonal complement of projector Q.

Consider now the unit vectors φ_i in the range of projector $Q^\perp$ which are the eigenvectors of $Q^\perp$. We may express $Q^\perp$ as the sum of the one dimensional projection operators onto the 1-D subspaces spanned by its eigenvectors. That is, $Q^\perp = \sum_i P_{\varphi_i}$, where $P_{\varphi_i} = |\varphi_i\rangle\langle\varphi_i|$. For any density operator W it then follows $\mathrm{Tr}(W\,Q^\perp) = \sum_i \mathrm{Tr}(W\,P_{\varphi_i})$. Given that W is positive (by definition of density operator) we have that $\mathrm{Tr}(W\,|\varphi_i\rangle\langle\varphi_i|) = \varphi_i^* W \varphi_i \geqslant 0$. And hence

$$\mathrm{Tr}(W Q^\perp) = \sum_i \varphi_i^* W \varphi_i \tag{C.6}$$

where each term in the sum is positive or equal to zero. Now given (B.5) and (B.6)

$$W_1 \varphi_i = W_2 \varphi_i = 0 \tag{C.7}$$

for all vectors φ_i in the range of $Q^\perp$.

Next let P_φ by any 1-D projector in $L(\mathcal{H})$ so that $P_\varphi = |\varphi\rangle\langle\varphi|$ for some unit vector φ. Then $\mathrm{Tr}(W_1 P_\varphi) = \langle\varphi, W_1\varphi\rangle$ and $\mathrm{Tr}(W_2 P_\varphi) = \langle\varphi, W_1\varphi\rangle$. We can always uniquely decompose φ into $\varphi = \varphi_Q + \varphi_{Q^\perp}$ where $\varphi_Q \in \{v|v = Q\,u, u \in \mathcal{H}\}$ and $\varphi_{Q^\perp} \in \{v|v = Q^\perp u, u \in \mathcal{H}\}$. Consequently, for $i = 1, 2$ we have

$$\mathrm{Tr}(W_i P_\varphi) = \langle\varphi, W_i\varphi\rangle = \langle\varphi_Q, W_i\varphi_Q\rangle + \langle\varphi_{Q^\perp}, W_i\varphi_{Q^\perp}\rangle + 2\,\mathrm{Re}\,\langle\varphi_Q, W_i\varphi_{Q^\perp}\rangle \tag{C.8}$$

But given that by (B.7) $W_i\varphi_{Q^\perp} = 0$, we have that $\mathrm{Tr}(W_i\;P_\varphi) = \langle\varphi_Q,\;W_i\;\varphi_Q\rangle$. Now $\langle\varphi_Q,\;W_i\;\varphi_Q\rangle = |\varphi_Q|^2\;\mathrm{Tr}(W_i\;P_{\varphi_Q})$, where P_{φ_Q} is the projector onto the 1-D subspace generated by φ_Q. Note that P_{φ_Q} commutes with Q so that by assumption $\mathrm{Tr}(W_1\,P_{\varphi_Q}) = \mathrm{Tr}(W_2\,P_{\varphi_Q})$. From this it follows that $\mathrm{Tr}(W_1\,P) = \mathrm{Tr}(W_2\,P)$ for all one-dimensional projectors P_f. And hence

$$\mathrm{Tr}[(W_1 - W_2)\,P_\varphi] = 0 \tag{C.9}$$

As $W_1 - W_2$ is a normal operator and a one-dimensional projection operator is always positive, we can conclude from (C.9) that $W_1 - W_2 = 0$, and so $W_1 = W_2$.

ii. The operator $W_Q = \frac{Q\,W\,Q}{\mathrm{Tr}(Q\,W\,Q)}$ is a density operator, i.e. a trace class operator of unit trace:

a) W_Q is of the trace class: for two general operators A and B, if A is a trace-class operator and B is a bounded linear operator, then $A\,B$ and $B\,A$ are both in the trace class. Thus, for any density operator W, the operator $Q\,W$ is of the trace class as is also the operator $Q\,W\,Q$. $Q\,W\,Q$ divided by the c-number $\mathrm{Tr}(Q\,W\,Q)$ is also of the trace class.

b) W_Q is of trace one: $\mathrm{Tr}(W_Q) = \mathrm{Tr}(\frac{Q\,W\,Q}{\mathrm{Tr}(Q\,W\,Q)}) = \frac{\mathrm{Tr}(Q\,W\,Q)}{\mathrm{Tr}(Q\,W\,Q)} = 1$

As we was shown in the first part of the proof, $W_Q = \frac{Q\,W\,Q}{\mathrm{Tr}(Q\,W\,Q)}$ is such that $p(P|Q) = \mathrm{Tr}(W_Q\,P) = \frac{p(P)}{p(Q)}$ for $P \leqslant Q$. Given that by i. W is unique, it follows that $W = W_Q$.

Appendix D
Quantum Frequentism

The differences between classical and quantum probabilities stem from the fact that in quantum mechanics observables are generally represented by non-commutative operators. Thus a good place to study the peculiarities of quantum mechanics is by looking at scenarios which involve probabilities for more than a single observable. The experiments considered in section 4.4 provided one such scenario; correlation experiments provide another such scenario, and a more appropriate one to consider *un*conditional probabilities. In correlation experiments one performs simultaneous measurements of pairs of different physical quantities on a system and studies the relations that hold between the various single and joint probabilities.

In section D.1, we show that, for the classical case, these relations are well understood by thinking of these probabilities as having their source in an ensemble of systems with well-defined properties; that is, as reflecting the distributions of properties of the systems. In addition to this objective reading of the classical probabilities, one can also provide a subjective one and hence view the probabilities as reflecting degrees of belief of a rational agent. Both interpretations turn out to be formally equivalent.

However, in quantum correlation experiments the empirically found (and theoretically predicted) single and joint frequencies (probabilities) do not generally satisfy the classical relations. In sections D.2 we show that the quantum probabilities do not, therefore, generally admit an ensemble representation, and that, thus, their understanding as revealing an underlying distribution of properties of quantum objects or as subjective degrees of belief is precluded. Finally, in section D.3, we explore one way in which one might modify the classical construal of an ensemble interpretation so as to make it a viable interpretation for the quantum probabilities.

The discussion in this appendix draws heavily on [Pitowsky, 1989] and [Fine, 1986].

D.1 Classical Correlations

We begin our discussion of *classical* correlations by focusing on the simplest classical correlation experiment (section D.1.1). This experiment involves only two physical quantities and is thus defined by the probabilities for each of the two individual physical quantities and the probability of their joint occurrence. We assume that the probabilities involved can be thought of as arising from an ensemble of systems with well-defined properties and study how this view constrains the relations between them.

We then consider a general correlation experiment (section D.1.2). Here we proceed in reverse manner: we do not suppose that probabilities reflect proportions of properties and then consider how this constrains the various single and joint probabilities; rather we consider the different single and joint probabilities of a general correlation experiment as uninterpreted numbers and ask when these frequencies allow an ensemble representation. We give a formal answer to this question by means of a theorem that states the conditions under which they admit (what we will define in mathematical precise terms as) an ensemble representation. The discussion in these section draws on Pitowsky's work 'Quantum Probability. Quantum Logic'.

D.1.1 A simple example

Consider the typical understanding of probabilities in the simple scenario of an urn filled with balls. Each ball has a set of different properties: color, material, size, etc. Now imagine we randomly select one of the balls from the urn. We cannot know for certain what properties the ball will have, but we can make some assertions as to the probabilities for those properties to obtain. So if we want to know the probability that a randomly selected ball is red we simply count the number of red balls in the urn and divide this number by the total number of balls, i.e. $p(\text{red}) = \frac{N_{\text{red}}}{N}$. The probabilities thus simply reflect the distribution of the different properties of the balls in the urn.

If we do not know the values of $N_r, N_w, N_{r\&w}$ and N, then to ascertain the values of the various probabilities we would need to perform a large number of draws and subsequently identify the probabilities with the relative frequency of the various results. This experiment would constitute the simplest case of a correlation experiment since we only consider the probability of the occurrence of two individual events and probability of their joint occurrence.

Let us concentrate on two of these properties, namely color and material, each of which can take only two values: either the ball is red (r) or it is blue (b), and it is either made out of wood (w) or of plastic (p). Now imagine we randomly select one of the balls from the urn. As we noted before, we cannot know for certain what properties the ball will have, but we can make some assertions as to the probabilities that certain properties obtain. The probabilities are simply given by proportions: the probability for selecting a red ball will be given by the proportion of red balls in the urn, i.e., $p_r = \frac{N_r}{N}$; the probability for selecting a wooden ball will be given by the proportion of wooden balls in the urn, i.e., $p_w = \frac{N_w}{N}$; the probability of selecting a ball that is both red and wooden will be given by the proportion of red wooden balls in the urn $p_{r\&w} = \frac{N_{r\&w}}{N}$; and so forth.

Let us consider the relations that obtain between these numbers. First, we know that the number of red wooden balls must be at most equal to that of red balls (in which case there are no plastic red balls), and this number is in turn at most equal to the total number of balls (in which case there are no blue balls). That is, $0 \leqslant N_{r\&w} \leqslant N_r \leqslant N$. So the following relation between the probabilities must hold (we simply divide by N):

$$0 \leqslant p_{r\&w} \leqslant p_r \leqslant 1 \tag{D.1}$$

And similarly, the number of red wooden balls must be at most equal to that of wooden balls (in which case there are no blue wooden balls), and this number is in turn at most equal to the total number of balls (in which case there are no plastic balls). In symbols, $0 \leqslant N_{r\&w} \leqslant N_w \leqslant N$. So the following relation between the probabilities must hold:

$$0 \leqslant p_{r\&w} \leqslant p_w \leqslant 1 \tag{D.2}$$

Thirdly, the number of balls which are either red or wood (or both) is the number of red balls plus the number of wooden balls minus the number of red wooden balls (in order to not count the red wooden balls twice). In symbols, $N_{r\,\mathrm{or}\,w} = N_r + N_w - N_{r\&w}$. Now the number of balls that are either red or wooden is at most equal to the total number of balls (in which case there are no blue plastic balls) so the following inequality must hold: $N_r + N_w - N_{r\&w} \leqslant N$. Translating this into probabilities:

$$p_r + p_w - p_{r\&w} \leqslant 1 \tag{D.3}$$

We have derived these three inequalities by considering a priori constraints on proportions of properties. That is, we have shown that if the probabilities $p_r, p_w, p_{r\&w}$ reflect proportions of well-defined properties of the balls in the urn then they must satisfy inequalities (D.1)-(D.3). It turns out these three inequalities are not only necessary for the numbers p_r, $p_w, p_{r\&w}$ to represent proportions of properties but also sufficient.[D.1] That is, inequalities (D.1)-(D.3) are both necessary and sufficient for the numbers p_r, p_w, $p_{r\&w}$ to represent probabilities of two events and their joint respectively when these have their source in an urn filled with balls with well-defined properties. Hence if in a correlation experiment the probabilities of two events and their joint do not satisfy these inequalities, their understanding as reflecting proportions of properties will be forbidden. As we will see in section D.2, this is precisely what happens for the quantum probabilities.

D.1. The references to this and similar results are given in the following section.

We can also analyze the correlation experiment from a subjective Bayesian point of view. Call r the proposition that the selected ball is red and w the proposition that the selected ball is wooden, and consider the four different possibilities for the draw of the ball:

1. a blue plastic ball is drawn, in which case both r and w are false (so $r\&w$ is false)
2. a red plastic ball is drawn, in which case r is true and w is false (so $r\&w$ is false)
3. a blue wooden ball is drawn, in which case r is false and w is true (so $r\&w$ is false)
4. a red wooden ball is drawn, in which case both r and w are true (so $r\&w$ is true)

We are asked to place bets on these four different possibilities. We denote by q_1 our degree of belief in the first case, by q_2 our degree of belief in the second case and so forth. Given that our degrees of belief ought to be coherent and that these four possibilities are mutually exclusive and jointly exhaustive, these degrees of belief must add to 1, that is $q_1 + q_2 + q_3 + q_4 = 1$. Our degrees of belief in propositions r, w and $r\&w$ can then be expressed in terms of the previous q_i's as follows: given that proposition r is true in the second and the fourth case, we have $p_r = q_2 + q_4$; given that proposition w is true in the third and the fourth case, we have $p_w = q_3 + q_4$; and given that proposition $r\&w$ is true only in the fourth case, we have $p_{r\&w} = q_4$.

Let us express this result in the formal language of vectors. We can represent the numbers p_r, p_w, $p_{r\&w}$ as the components of a vector in $\mathbb{R}^3$, namely $(p_r,\, p_w,\, p_{r\&w})$ and write the relation of these p_i's to the q_i's as $(p_r, p_w, p_{r\&w}) = (q_2 + q_4,\, q_3 + q_4,\, q_4)$. This is equivalent to writing $(p_r, p_w, p_{r\&w})$ as the following sum: $(p_r, p_w, p_{r\&w}) = q_1\,(0,0,0) + q_2(1, 0,0) + q_3(0,1,0) + q_4(1,1,1)$ which, given that $q_1 + q_2 + q_3 + q_4 = 1$, is simply the convex sum[D.2] or weighted average of the four previous vectors. Letting 1 stand for 'true' and 0 stand for 'false', these vectors can be interpreted as the truth values assigned to the propositions r, w and $r\&w$ in the four possible cases: $(0,0,0)$ corresponds to the first case wherein r, w and $r\&w$ are all false; $(1,0,0)$ corresponds to the second case wherein r is true but both w and $r\&w$ are false; and so forth. Hence if p_r, p_w, $p_{r\&w}$ represent our degrees of belief in two propositions and their conjunction then they will be expressed as a convex combination of all truth value assignments.

Now it turns out that the set of vectors $(p_r,\, p_w,\, p_{r\&w})$ which can be expressed as a convex sum of the vectors $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$ is equivalent to the set of vectors $(p_r, p_w, p_{r\&w})$ whose components $p_r, p_w, p_{r\&w}$ satisfy inequalities (D.1)-(D.3). That is, p_r, p_w, $p_{r\&w}$ satisfy the inequalities (D.1)-(D.3) if and only if $(p_r,\, p_w,\, p_{r\&w}) = q_1\,(0,0, 0) + q_2(1,0,0) + q_3(0,1,0) + q_4(1,1,1)$, where $\sum_{i=1}^4 q_i = 1$. Hence the numbers p_r, p_w, $p_{r\&w}$ of our correlation experiment can only represent degrees of belief of a rational agent

D.2. By definition a *convex* sum of vectors $\{v_1, \ldots, v_n\}$ is another vector $v = \sum_{i=1}^n \lambda_i\, v_i$ such that $\forall i, \lambda_i \in \mathbb{R}$ and $\sum_{i=1}^n \lambda_i = 1$.

in two propositions and their conjunction or proportions of two properties and their joint in a given sample if they satisfy inequalities (D.1)-(D.3) or, equivalently, if they can be expressed as the convex sum of vectors $(0,0,0),(1,0,0),(0,1,0)$ and $(1,1,1)$.

To conclude, we have considered two different understandings of the probabilities in our simple correlation experiment: an objective reading that regards probabilities as reflecting distributions of properties and a subjective reading that views them as reflecting degrees of belief of a rational agent. Both of these readings have been grounded in thinking of the ensemble of balls in the urn as their source. We have also seen how both interpretations are formally equivalent: the objective view leads to constraints on the probabilities in terms of linear inequalities while the subjective view leads to constraints as convex sums of certain vectors, where both constraints are mathematically equivalent. We turn now to generalizing our previous results for a general correlation experiment.

D.1.2 General Correlation Polytopes & Ensemble Representations

The structure of this section is as follows: we first present the general way of talking about the single and joint probabilities in a general correlation experiment, then we define what it is to give an ensemble or classical representation for these probabilities and last we consider the conditions under which this representation can be given.

Let us first name the various single and joint probabilities of a general correlation experiment. Let N be a non-empty subset of the set of pairs of numbers (i,j) such that for a given $n\geqslant 2$, $1\leqslant i<j\leqslant n$. That is $N\subseteq\{(i,j)\colon 1\leqslant i<j\leqslant n\}$. To give some examples,

- for $n=2$: $N\subseteq\{(1,2)\}$
- for $n=3$: $N\subseteq\{(1,2),(1,3),(2,3)\}$
- for $n=4$: $N\subseteq\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

We denote by $|N|$ the cardinality of N, that is, the number of elements of N. Note that N is a subset of $\{(i,j)\}$ so it does not necessarily contain all the possible pairs.

We are given $n+|N|$ numbers as follows:

- p_i $\quad i=1,\ldots,n$
- p_{ij} $\quad (i,j)\in N$

These numbers are intended to represent the single and joint probabilities of the different quantities involved in a correlation experiment. We give some particular instances:

- for $n=2$, are given $2+1$ numbers, namely p_1,p_2,p_{12}.
- for $n=3$ and $N=\{(1,2),(1,3),(2,3)\}$, are given $3+3$ numbers, namely p_1,p_2,p_3, p_{12},p_{13},p_{23}.
- for $n=4$ and $N=\{(1,3),(1,4),(2,3),(2,4)\}$, are given $4+4$ numbers, namely p_1, $p_2,p_3,p_4,p_{132},p_{14},p_{23},p_{24}$.

We can think of these numbers as determining an $(n+|N|)$-tuple which we denote $(p_1, ..., p_n, ...p_{ij}....)$ where the p_{ij} are ordered by their indices ij ordered lexicographically. Each of these tuples defines a correlation experiment.

One formally defines what it means to give an ensemble representation of the p_1, ..., $p_n, ...p_{ij}...$ numbers as follows.[D.3]

Definition D.1. ***Ensemble Representation****. The $(n+|N|)$-tuple $(p_1, ..., p_n, ..., p_{ij}, ...)$ admits an ensemble or classical space representation if*

i. there exists a probability space $\langle S, \mathcal{F}(S), p\rangle$.

ii. there exist (not necessarily distinct) sets $A_1, ..., A_n \in \Sigma$ such that for all $i \in \{1, 2, ..., n\}$ and all $(i, j) \in N$ we have

$$p_i = \mu(A_i) \quad \text{and}\, p_{ij} = \mu(A_i \cap A_j) \tag{D.4}$$

Let us now consider the conditions under which the $(n + |N|)$-tuple $(p_1, ..., p_n, ..p_{ij}...)$ admits an ensemble representation. To give an answer we will first need to introduce the notion of a classical correlation polytope. And to do so at this level of generality we again need to introduce some more notation. Let $\{0, 1\}^n$ denote the set of all n-tuples of 0's and 1's. To give some particular cases:

- for $n=2$, $\{0,1\}^2 = \{(1,1), (1,0), (0,1), (0,0)\}$
- for $n=3$, $\{0,1\}^3 = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$

The cardinality of $\{0,\ 1\}^n$ is 2^n. Now denote by ε any such n-tuple. That is, $\varepsilon \in \{0, 1\}^n$, which we can write as $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ where the ε_i can be either a 0 or a 1. Clearly there are 2^n possible ε's. With this notation we can express the $(n + |N|)$-tuple $(p_1, ..., p_n, ...p_{ij}....)$ when the p_i's and p_{ij}'s take only values 0 or 1 as $p^\varepsilon = (\varepsilon_1, ..., \varepsilon_n, ..., \varepsilon_i \varepsilon_j, ...)$, where the term $\varepsilon_i \varepsilon_j$ appears only if $(i, j) \in N$. For example,

- for $n=2$ and $N = \{(1,2)\}$ there are 2^2 possible ε's: (1,1), (1,0), (0,1) and (0,0), and the 4 corresponding $p^\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_{12})$'s are:
$$p^{(1,1)} = (1,1,1) \qquad p^{(1,0)} = (1,0,0) \qquad p^{(0,1)} = (0,1,0) \qquad p^{(1,1)} = (0,0,0)$$
- for $n=3$ and $N = \{(1,2), (2,3)\}$, the p^ε's take th form $p^\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{12}, \varepsilon_{23})$, which for a particular ε, say $\varepsilon = (1,1,0)$, yields the tuple $p^{(1,1,0)} = (1,1,0,1,0)$.

The classical correlation polytope $c(n,\ N)$ is the set of all $n\ +\ |N|$ vectors that can be expressed as a convex or weighted sums of the 2^n vectors of the form p^ε, where $\varepsilon \in \{0,1\}^n$.

D.3. One can show that definition 8.1 of a classical space representation is equivalent to a deterministic hidden variable model that explicitly constructs observables as random variables over a classical probability space, i.e. the space of hidden variables. For a proof in a simple case see [Malament,1], pp.7-9. See also [Fine,1982a], p.165.

Definition D.2. ***Convex Polytope.*** *$c(n, N)$ is the closed, convex polytope in $\mathbb{R}^{(n+|N|)}$ whose vertices are the 2^n vectors of the form p^{ε}, where $\varepsilon \in \{0,1\}^n$.*

This characterizes the polytope as the convex hull of its vertices. Thus a given vector (p_1, ..., p_n, ..p_{ij}...) is an element of the polytope if and only if it can be expressed as a convex combination of the vertices p^{ε}. Consider the $n = 2$ case where $N = \{(1, 2)\}$. The polytope $c(2, N)$ is the set of all vectors (p_1, p_2, p_{12}) that can be expressed as a convex sum of $p^{(1,1)} = (1,1,1)$, $p^{(1,0)} = (1,0,0)$, $p^{(0,1)} = (0,1,0)$ and $p^{(1,1)} = (0,0,0)$. So a given vector (p_1, p_2, p_{12}) is an element of the polytope $c(2, N)$ if and only if it can be expressed as a convex combination of the vertices $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,0)$.

Now every convex polytope in $\mathbb{R}^{(n+|N|)}$ has another description. Under this second description a vector is an element of the polytope if and only if its coordinates satisfy a set of linear inequalities which represent the supporting hyperplanes of the polytope. However the characterization of the general correlation polytope as the set of vectors whose components satisfy a particular set of linear inequalities turns out to be extremely complex (in fact it is practically impossible since it would require too much, i.e. exponential, time[D.4]). In the simple case of $c(2, N)$ the vector (p_1, p_2, p_{12}) belongs to $c(2, N)$ if and only if the following inequalities hold:[D.5]

$$\begin{aligned} 0 \leqslant p_{12} \leqslant p_1 \leqslant 1 \\ 0 \leqslant p_{12} \leqslant p_2 \leqslant 1 \\ p_1 + p_2 - p_{12} \leqslant 1 \end{aligned} \tag{D.5}$$

This set of inequalities correspond to the inequalities (D.1)-(D.3). We will also give the inequalities that characterize two other important polytopes in sections D.1.3 and D.1.4.

We are now ready to answer the main question of this section, namely when does the $(n + |N|)$-tuple (p_1, ..., p_n, ...p_{ij}....) admit an ensemble representation? We motivate the answer by reasoning on the $c(2, N)$ case of the previous section.

Recall that $(p_r, p_w, p_{r\&w})$ can be seen as representing proportions of two properties and their joint in a given sample only if these numbers satisfy inequalities (D.1)-(D.3). And that they can be seen as the degrees of belief of a rational agent on two propositions and their conjunction only if $(p_r, p_w, p_{r\&w})$ can be expressed as the convex sum of the vectors $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,0)$. Now inequalities (D.1)-(D.3) and the convex sum of vectors $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,0)$ are precisely the two ways of characterizing the polytope $c(2, N)$; and giving an ensemble representation for $p_r, p_w, p_{r\&w}$ is what allows their reading as proportions of properties or as degrees of belief. Thus we see that (p_r, p_w, $p_{r\&w}$) admits an ensemble representation only if it belongs to the polytope $c(2, N)$.

D.4. See [Pitowsky, 1989], pp.33-46 for an account of the intractability of this problem.

D.5. For a proof see [Pitowsky, 1989], p. 24 or [Malament, 1], p.5.

It turns out that this result holds generally as the following theorem states:[D.6]

Theorem D.1. *For all n and N, the $(n + |N|)$-tuple $(p_1, ..., p_n, ...p_{ij}....)$ admits an ensemble representation if and only if it belongs to the polytope $c(n, N)$.*

We now focus on two particular polytopes which will be important for our discussion of the quantum probabilities.

D.1.3 The Bell-Wigner polytope

Let $n = 3$ and $N = \{(1,2), (1,3), (2,3)\}$. Theorem D.1 tells us that a vector $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ admits a classical representation if an only if it belongs to the the polytope $c(3, N)$. This is the so-called *Bell-Wigner* polytope. Let us see the two characterizations of this polytope.

a) As a convex sum of its vertices: $c(3, N)$ is the closed, convex polytope in $\mathbb{R}^6$ whose vertices are the 2^3 vectors of the form $p^\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$, where $\varepsilon \in \{0, 1\}^3 = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$

   Thus a vector $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ belongs to the polytope $c(3, N)$ if and only if it can be expressed as a convex sum of the following eight p^ε's:

   $$(0,0,0,0,0,0), (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (1,1,0,1,0,0), (1,0,1,0,1,0), (0,1,1,0,0,1), (1,1,1,1,1,1) \tag{D.6}$$

b) In terms of linear inequalities[D.7]: a vector $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ belongs to the *Bell-Wigner* polytope $c(3, N)$ if and only if for all $i, j \in \{1, 2, 3\}$ and $(i, j) \in R$,

   $$\begin{gathered} 0 \leqslant p_{ij} \leqslant p_i \leqslant 1 \\ 0 \leqslant p_{ij} \leqslant p_j \leqslant 1 \\ p_i + p_j - p_{ij} \leqslant 1 \\ p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leqslant 1 \\ p_1 - p_{12} - p_{13} + p_{23} \geqslant 0 \\ p_2 - p_{12} - p_{23} + p_{13} \geqslant 0 \\ p_3 - p_{13} - p_{23} + p_{12} \geqslant 0 \end{gathered} \tag{D.7}$$

   The last 4 inequalities go by the name of the 'Bell inequalities' and as we shall see in section D.2.2 they are violated by the quantum probabilities.

Thus the vector $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ admits an ensemble representation if and only if it can be expressed as a convex sum of the eight vectors in (8.8) or if its components satisfy the inequalities (8.9).

D.6. For a proof see [Pitowsky, 1989], p.23 or [Malament, 1], pp.2-3.

D.7. For a proof see [Pitowsky, 1989], pp.25-27 or [Malament, 1], pp.5-6.

We could have arrived at this same result by the same route that we used in section D.1.1 First consider the subjective view on probability and how it leads naturally to a description of the correlation polytope $c(3,\ N)$ in terms of its vertices. Consider three propositions a_1, a_2, a_3 and their possible three conjunctions $a_1\&a_2, a_1\&a_3, a_2\&a_3$. In terms of our urn experiment we can think of a_1 as the proposition that the selected ball is red, a_2 as the proposition that the selected ball is wooden, and a_3 as the proposition that the selected ball is small, where there are only two sizes (small or large).

There are eight possible cases when a ball is drawn: 1) a big blue plastic ball is drawn, in which case a_1, a_2, a_3 and all the possible conjunctions are false; 2) a big red plastic ball is drawn, in which case a_1 is true but a_2 and a_3 and all the possible conjunctions are false; 3) a big blue wooden ball is drawn, in which case a_2 is true and a_1, a_3 and all the possible conjunctions are false; 4) a small blue plastic ball is drawn, in which case a_3 is true and a_1, a_2 and all the possible conjunctions are false; 5) a big red wooden ball is drawn, in which case $a_1, a_2, a_1\&a_2$ are true and $a_3, a_1\&a_3, a_2\&a_3$ are false; 6) a small red plastic ball is drawn, in which case a_1, a_3, $a_1\&a_3$ are true and a_2, $a_1\&a_2$, $a_2\&a_3$ are false; 7) a small blue wooden ball is drawn, in which case a_2, a_3, $a_2\&a_3$ are true and a_1, $a_1\&a_2$, $a_1\&a_3$ are false; and 8) a small red wooden ball is drawn, in which case a_1, a_2, a_3 and all the possible conjunctions are true. These eight possible truth value assignments are exactly what the vectors in equation (8.8) correspond to.

Now we are asked to place bets on these eight different possibilities. We denote by q_i our degree of belief in the i case, with $i=1,\ldots,8$. Given that our degrees of belief ought to be coherent and that these eight possibilities are mutually exclusive and jointly exhaustive, we have that $\sum_{i=1}^{8} q_i = 1$. We can then express our degrees of belief in propositions a_1, a_2, a_3, $a_1\&a_2, a_1\&a_3, a_2\&a_3$, namely $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$, in terms of these q_i's. This will result in expressing $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ as the convex sum of the vectors in (8.8). Thus if $(p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ are to be regarded as coherent degrees of belief then they must belong the Bell-Wigner polytope. And the converse also holds true.

The frequency view is most naturally connected to the description of the correlation polytope in terms of linear inequalities. Consider three events with probabilities p_1, p_2, p_3 and the joints p_{12}, p_{13}, p_{23}. In terms of our urn example we can think of p_1 as expressing the proportion of red balls, p_2 that of wooden balls, p_3 that of small balls, p_{12} that of red wooden balls, p_{13} that of small red balls, and p_{23} that of small wooden balls. Just as in our urn example, each pair out of the three events must satisfy the inequalities for pair of events and their joint. So we have for $1 \leqslant i < j \leqslant 3$:

$$0 \leqslant p_{ij} \leqslant p_i \leqslant 1 \qquad 0 \leqslant p_{ij} \leqslant p_j \leqslant 1 \qquad p_i + p_j - p_{ij} \leqslant 1 \tag{D.8}$$

Now these inequalities are not sufficient for the numbers $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$ to represent proportions of properties; we have to add some constraints on all three events and not just pairs. First, we know that the number of balls which are either red or wood or small is the number of red balls, plus the number of wooden balls, plus the number of small balls, minus the number of red wooden balls, minus the number of small red balls and minus the number of small wooden balls. In symbols, $N_{1\,\mathrm{or}\,2\,\mathrm{or}\,3} = N_1 + N_2 + N_3 - N_{12} - N_{13} - N_{23}$. Now the number of balls that are either red or wooden or small is at most equal to the total number of balls (in which case there are no big blue plastic balls) so the following inequality must hold:

$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leqslant 1 \tag{D.9}$$

We also know that the number of balls which are either *not* red (so blue) or wood or small is the number of blue balls ($N_{\neg 1} = N - N_1$), plus the number of wooden balls, plus the number of small balls, minus the number of blue wooden balls ($N_{\neg 12} = N_2 - N_{12}$), minus the number of small blue balls ($N_{\neg 13} = N_3 - N_{13}$) and minus the number of small wooden balls. In symbols, $N_{\neg 1\,\mathrm{or}\,2\,\mathrm{or}\,3} = N_{\neg 1} + N_2 + N_3 - N_{\neg 12} - N_{\neg 13} - N_{23} = N - N_1 + N_{12} + N_{13} - N_{23}$. Now again the number of balls that are either blue or wooden or small is at most equal to the total number of balls so the following inequality must hold:

$$p_1 - p_{12} - p_{13} + p_{23} \geqslant 0 \tag{D.10}$$

By considering that the number of balls which are red or plastic or small is at most equal to the total number of balls, and that the number of balls which are red or wooden or big is at most equal to the total number of balls we deduce the remaining two inequalities, namely

$$p_2 - p_{12} - p_{23} + p_{13} \geqslant 0 \qquad\qquad p_3 - p_{13} - p_{23} + p_{12} \geqslant 0 \tag{D.11}$$

D.1.4 The Clauser-Horne polytope

Let $n = 4$ and $N = \{(1,3),(1,4),(2,3),(2,4)\}$. Theorem D.1 tells us that a vector $(p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24})$ admits a classical representation if an only if it belongs to the polytope $c(4, N)$. This is the so-called *Clauser-Horne* polytope. We won't characterize this polytope in terms of its vertices (which by now is straightforward), but will give its characterization in terms of inequalities. A vector $(p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24})$ belongs to $c(4, N)$ if and only if for all $i, j \in \{1, 2, 3\}$ and $(i, j) \in N$,[D.8]

$$\begin{gathered} 0 \leqslant p_{ij} \leqslant p_i \leqslant 1 \\ 0 \leqslant p_{ij} \leqslant p_j \leqslant 1 \\ p_i + p_j - p_{ij} \leqslant 1 \\ -1 \leqslant p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leqslant 0 \end{gathered}$$

D.8. For a proof see [Pitowsky, 1989], pp.28-30 or [Malament, 1], pp.6-8.

$$
\begin{aligned}
-1 &\leqslant p_{23}+p_{24}+p_{14}-p_{13}-p_{2}-p_{4}\leqslant 0\\
-1 &\leqslant p_{14}+p_{13}+p_{23}-p_{24}-p_{1}-p_{3}\leqslant 0\\
-1 &\leqslant p_{24}+p_{23}+p_{13}-p_{14}-p_{2}-p_{3}\leqslant 0
\end{aligned}
\tag{D.12}
$$

The last 4 inequalities go by the name of the 'Clauser-Horne inequalities' and as we shall see in section D.2.1 they are violated by the quantum probabilities.

Following the procedure of the previous section we could deduce these inequalities by assuming that the probabilities represent proportions or relative frequencies. Similarly, we can also deduce the representation of $c(4, N)$ in terms of its vertices by assuming that the probabilities represent degrees of belief on four propositions and their conjunctions.

To conclude, the probabilities p_1, ..., p_n, ... p_{ij} ... of a general correlation experiment can only represent proportions of single properties and their joints in a given sample or the degrees of belief of a rational agent on various propositions and their conjunctions if the vector $(p_1, ..., p_n, ... p_{ij} ...)$ belongs to the polytope $c(n, N)$. The frequency view is most naturally connected to the description of the correlation polytope in terms of linear inequalities while the subjective view leads naturally to a description of the correlation polytopes in terms of its vertices. But both views lead to the same set of constraints on correlations.

D.2 Quantum Correlations

In this section we consider several quantum correlation experiments and show how in many cases the quantum probabilities do not admit an ensemble representation. We begin in section D.2.1 by considering a correlation experiment involving the measurement of spin on a two-electron system in four different directions. We show that for certain directions the single and joint probabilities of this experiment violate the Clauser-Horne inequalities, thus posing difficulties for their interpretation. In section D.2.2 we briefly consider a quantum correlation experiment involving three physical quantities and how for certain situations this leads to violations of the Bell inequalities.

D.2.1 Violations of the Clauser-Horne inequalities

Consider the following experiment performed on a composite system of two electrons which briefly interact and then become widely separated in space. When a pair is emitted the two electrons travel in opposite directions 'left' and 'right'. On each particle we perform one of two incompatible measurements: we measure the spin of the left electron either in the x direction (S_x) or in the y direction (S_y); and we measure the spin of the right electron either in the z direction (S_z) or in the w direction (S_w). Measurements on the pair are made simultaneously.

The quantum mechanical analysis of this experimental situation determines the probabilities for any measurement outcome for each observable separately and the joint probabilities for any pair of outcomes in a simultaneous measurement of either S_x or S_y on the left electron with either S_z or S_w on the right electron. These probabilistic predictions are in accordance with the experimental results, where these probabilities can be interpreted as physical values or measurement results. Let us consider this analysis.

Quantum mechanics describes the space of the spin states of each individual electron by a two-dimensional Hilbert space $\mathcal{H}$. We denote by S_x the operator corresponding to a measurement of the spin of the electron in the x direction which has (normalized) eigenstates ψ_{+x} and ψ_{-x} corresponding to 'spin up' and 'spin down' in the x direction. P_{+x} and P_{-x} are the projection operators on the one-dimensional subspaces spanned by ψ_{+x} and ψ_{-x} respectively; by the spectral theorem we have $S_x = \frac{1}{2}(P_{+x} - P_{-x})$. Analogue results hold for S_y, S_z and S_w corresponding to measurement of spin in the y, z and w direction respectively. Taking spherical coordinates (r, θ, ϕ) in physical space, the following relations hold between the eigenvectors corresponding to the spin in the $z = (1, 0, 0)$ direction S_z and those corresponding to the spin in a general $u = (1, \theta, \phi)$ direction S_u:

$$
\begin{aligned}
\psi_{+u} &= \cos(\frac{\theta}{2})e^{-\frac{i\phi}{2}}\psi_{+z} + \sin(\frac{\theta}{2})e^{\frac{i\phi}{2}}\psi_{-z} \\
\psi_{-u} &= -\sin(\frac{\theta}{2})e^{-\frac{i\phi}{2}}\psi_{+z} + \cos(\frac{\theta}{2})e^{\frac{i\phi}{2}}\psi_{-z}
\end{aligned}
\tag{D.13}
$$

The space of spin states of a two-electron system is described by the tensor product of the Hilbert spaces of each individual electron, i.e., $\mathcal{H}^{(L)} \otimes \mathcal{H}^{(R)}$. We shall perform our experiment on electron pairs in the so-called singlet state; this state corresponds to a state of total spin zero and is expressed as:

$$
\psi_S = \frac{1}{\sqrt{2}}[\psi_{+\xi} \otimes \psi_{-\xi} - \psi_{-\xi} \otimes \psi_{+\xi}] \tag{D.14}
$$

for an arbitrary direction ξ. We denote by W_S the corresponding density operator which is simply the projection operator in $\mathcal{H}^{(L)} \otimes \mathcal{H}^{(R)}$ onto the one-dimensional subspace spanned by ψ_S, i.e., $W_S = P_{\psi_S}$.

We are interested in calculating the probabilities of certain 'properties' of the two-electron system:

– single probabilities: for the left electron to have spin up in the x direction (p_1); for the left electron to have spin up in the y direction (p_2); for the right electron to have spin up in the z direction (p_3); and for the right electron to have spin up in the w direction (p_4).

– joint probabilities: for the left electron to have spin up in the x direction and the right electron to have spin up in the z direction (p_{13}); for the left electron to have spin up in the x direction and the right electron to have spin up in the w direction (p_{14}); for the left electron to have spin up in the y direction and the right electron to have spin up in the z direction (p_{23}); and for the left electron to have spin up in the y direction and the right electron to have spin up in the w direction (p_{24}).

Note that this situation corresponds to setting $n = 4$ and $N = \{(1,3),(1,4),(2,3),(2,4)\}$ in the formal description of correlation experiments. These single and joint probabilities are given by quantum probability theory as:

$$\begin{aligned} p_i &= \mathrm{Tr}(W_S P_i) \\ p_{ij} &= \mathrm{Tr}[W_S (P_i \wedge P_j)] \end{aligned} \tag{D.15}$$

In order to proceed to their calculation we first need to identify the projection operators P_i on $\mathcal{H}^{(L)} \otimes \mathcal{H}^{(R)}$ that correspond to the 'properties' we are interested in. Consider the projector associated with the left electron having spin up in the x direction: for the left electron the projection operator is simply P_{+x}, defined on $\mathcal{H}^{(L)}$; given that we are not considering any property of the right electron we may identify the projection operator on $\mathcal{H}^{(R)}$ with the identity, which can be expressed as $P_{+x} \vee P_{-x}$. Hence, the projector on $\mathcal{H}^{(L)} \otimes \mathcal{H}^{(R)}$ corresponding to the left electron having spin up in the x direction is

$$P_1 = (P_{+x} \otimes P_{+x}) \vee (P_{+x} \otimes P_{-x}) \tag{D.16}$$

Similarly, the projector corresponding to the left electron having spin up in the y direction is:

$$P_2 = (P_{+y} \otimes P_{+y}) \vee (P_{+y} \otimes P_{-y}) \tag{D.17}$$

the projector corresponding to the right electron having spin up in the z direction is:

$$P_3 = (P_{-z} \otimes P_{+z}) \vee (P_{+z} \otimes P_{+z}) \tag{D.18}$$

and the projector corresponding to the right electron having spin up in the w direction is:

$$P_4 = (P_{-w} \otimes P_{+w}) \vee (P_{+w} \otimes P_{+w}) \tag{D.19}$$

Let us now identify the projections corresponding to the joint properties of the composite system, namely

$P_1 \wedge P_3 = P_{+x} \otimes P_{+z}$ corresponds to the left electron having spin up in the x direction and the right electron to have spin up in the z direction,

$P_1 \wedge P_4 = P_{+x} \otimes P_{+w}$ corresponds to the left electron having spin up in the x direction and the right electron to have spin up in the w direction,

$P_2 \wedge P_3 = P_{+y} \otimes P_{+z}$ corresponds to the left electron having spin up in the y direction and the right electron to have spin up in the z direction, and

$P_2 \wedge P_4 = P_{+y} \otimes P_{+w}$ corresponds to the left electron having spin up in the y direction and the right electron to have spin up in the w direction.

Note that the spin observables for the left electron commute with those of the right electron, i.e., $[P_1, P_3] = [P_1, P_4] = [P_2, P_3] = [P_2, P_4] = 0$, so that all the joint probabilities we are considering correspond to well-defined measurements. Also note that the spin observables for each particle are incompatible, i.e., $[P_1, P_2] \neq 0$ and $[P_3, P_4] \neq 0$.

We are now ready to calculate the various probabilities predicted by quantum theory for this correlation experiment. For the single probabilities $p_i = \mathrm{Tr}(W_S P_i)$ the calculation yields:[D.9]

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{2} \tag{D.20}$$

We can calculate the joint probability $p_{13} = \mathrm{Tr}(W_S\,(P_1 \wedge P_3)$ as $|\langle \psi_{+x} \otimes \psi_{+z} | \psi_S \rangle|^2$. Taking spherical coordinates (r, θ, ϕ) in physical space with $z = (1, 0, 0)$ and $u = x = (1, \theta, \phi)$,

$$\psi_{+x} = e^{\frac{i\varphi}{2}} \cos(\frac{\theta}{2}) \psi_{+z} + e^{-\frac{i\varphi}{2}} \sin(\frac{\theta}{2}) \psi_{-z} \tag{D.21}$$

where $\theta = \widehat{xz}$, which yields:[D.10]

$$p_{13} = \frac{1}{2} \sin^2(\frac{\widehat{xz}}{2}) \tag{D.22}$$

Similarly,

$$p_{14} = \frac{1}{2} \sin^2(\frac{\widehat{xw}}{2}); \qquad p_{23} = \frac{1}{2} \sin^2(\frac{\widehat{yz}}{2}); \qquad p_{24} = \frac{1}{2} \sin^2(\frac{\widehat{yw}}{2}) \tag{D.23}$$

This concludes the calculation of the single and joint probabilities for this quantum correlation experiment. We can now easily show that for certain choices of x, y, z, w the vector $p = (p_1, p_2, p_3, p_4, p_{13}, p_{14}, p_{23}, p_{24})$ does not belong to the correlation polytope $c(4, N)$. Set for example $\widehat{xz} = \widehat{xw} = \widehat{yw} = 120°$ and $z = y$ so that $\widehat{yz} = 0°$. Substituting these values in equations (D.20)-(D.23) yields the following values $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, 0, \frac{3}{8})$. Given that

$$p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 = \frac{1}{8} > 0 \tag{D.24}$$

these probabilities do not satisfy one of the Clauser-Horne inequalities (D.12), namely

$$-1 \leqslant p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leqslant 0 \tag{D.25}$$

The maximal violations of the Clauser-Horne inequalities occur for $\widehat{xz} = \widehat{xw} = \widehat{yz} = 135°$ and $\widehat{yw} = 45°$.

D.9. $p_1 = |\langle \psi_{+x} \otimes \psi_{+x} \vee \psi_{+x} \otimes \psi_{-x} | \psi_S \rangle|^2 = \frac{1}{2} |\langle \psi_{+x} \otimes \psi_{+x} \vee \psi_{+x} \otimes \psi_{-x} | \psi_{+x} \otimes \psi_{-x} - \psi_{-x} \otimes \psi_{+x} \rangle|^2 = \frac{1}{2} |\langle \psi_{+x} \otimes \psi_{-x} | \psi_{+x} \otimes \psi_{-x} \rangle|^2 = \frac{1}{2}$. Similarly for p_2, p_3, p_4.

D.10. $p_{13} = |\langle \psi_{+x} \otimes \psi_{+z} | \psi_S \rangle|^2 = |\langle \psi_{+x} \otimes \psi_{+z} | \frac{1}{\sqrt{2}} [\psi_{+z} \otimes \psi_{-z} - \psi_{-z} \otimes \psi_{+z}] \rangle|^2 = \frac{1}{2} |\langle \psi_{+x} \otimes \psi_{+z} | \psi_{+z} \otimes \psi_{-z} \rangle|^2 + \frac{1}{2} |\langle \psi_{+x} \otimes \psi_{+z} | \psi_{-z} \otimes \psi_{+z}] \rangle|^2 = 0 + \frac{1}{2} |\langle \psi_x | \psi_{-z} \rangle|^2 = \frac{1}{2} |\langle e^{\frac{i\varphi}{2}} \cos(\frac{\theta}{2}) \psi_{+z} + e^{-\frac{i\varphi}{2}} \sin(\frac{\theta}{2}) \psi_{-z} | \psi_{-z} \rangle|^2 = \frac{1}{2} \sin^2(\frac{\theta}{2})$

Now as we showed in section D.1.4, $p = (p_1,\, p_2,\, p_3,\, p_4,\, p_{13},\, p_{14},\, p_{23},\, p_{24})$ admits an ensemble representation if and only if it satisfies equations (D.12). Hence the quantum probabilities of this correlation experiment cannot be given an ensemble representation. That is, they cannot be given an objective reading in terms of frequencies or proportions of properties nor a subjective reading in terms of degrees of belief, both if the events are interpreted as physical values or measurement results.

D.2.2 Violations of the Bell Inequalities

We now show how the quantum probabilities also violate the Bell Inequalities. The experimental situation is similar to the one in the previous section, but now we consider the following projections on $\mathcal{H}^{(L)} \otimes \mathcal{H}^{(R)}$ for three direction x, y, z:

$$P_1 = (P_{+x} \otimes P_{+x}) \vee (P_{+x} \otimes P_{-x}) \tag{D.26}$$

the projector corresponding to the left electron having spin up in the x direction,

$$P_2 = (P_{+y} \otimes P_{+y}) \vee (P_{+y} \otimes P_{-y}) \vee (P_{-y} \otimes P_{-y}) \tag{D.27}$$

the projector corresponding to the left electron having spin up in the y direction or the right electron having spin down in the y direction, and

$$P_3 = (P_{-z} \otimes P_{-z}) \vee (P_{+z} \otimes P_{-z}) \tag{D.28}$$

the projector corresponding to the right electron having spin down (not up as before) in the z direction. The projections corresponding to the joint properties of the composite system are:

$$\begin{aligned} P_1 \wedge P_2 &= P_{+x} \otimes P_{-y} \\ P_1 \wedge P_3 &= P_{+x} \otimes P_{-z} \\ P_2 \wedge P_3 &= P_{+y} \otimes P_{-z} \end{aligned} \tag{D.29}$$

As before we perform this experiment on an electron pair system in the singlet state, so the quantum probabilities are given by equation (D.15). Similarly we find:

$$p_1 = p_2 = p_3 = \frac{1}{2} \tag{D.30}$$

$$p_{12} = \frac{1}{2} \cos^2(\frac{\widehat{xy}}{2}); \qquad p_{13} = \frac{1}{2} \cos^2(\frac{\widehat{xz}}{2}); \qquad p_{23} = \frac{1}{2} \cos^2(\frac{\widehat{yz}}{2}) \tag{D.31}$$

We can now easily show that for certain choices of x, y, z the vector $p = (p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ does not belong to the correlation polytope $c(3, N)$, with $N = \{(1,2), (1,3), (2,3)\}$. Set for example $\widehat{xy} = \widehat{xz} = \widehat{yz} = 120°$. Substituting these values in equations (8.29) and (8.30) yields the following values $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. Given that

$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} = \frac{9}{8} > 1 \tag{D.32}$$

these probabilities do not satisfy one of the Bell inequalities (D.7), namely

$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leqslant 1 \tag{D.33}$$

D.3 Modifying the Ensemble Interpretation

In this section we briefly explore one way in which one might modify the classical construal of an ensemble interpretation so as to make it a viable interpretation for the quantum probabilities. It is provided by Fine's 'prism models' [Fine, 1986].

To discuss this issue we focus on the Clauser-Horne case. Fine considers a general version of this correlation experiment. As before we perform the experiment on a composite system of two particles, call them α and β, which briefly interact and then become widely separated in space. Let ψ denote the state function of this coupled system. On each particle we perform one of two incompatible measurements: we can measure either A or A' on α, and either B or B' on β. Each of these observables has only two possible values: a_1 or a_2 for A , a_1' or a_2' for A', b_1 or b_2 for B and b_1' or b_2' B'. Observable A commutes with B and B', as does A'. Measurements on the pair are made simultaneously.

The quantum mechanical analysis of this experimental situation determines the probabilities for any measurement outcome for each observable separately and the joint probabilities for any pair of outcomes in a simultaneous measurement of A or A' with either B or B'. For example, for the composite system in state ψ quantum mechanics determines the following single probabilities $p_\psi(a_1)$, $p_\psi(a_1')$, $p_\psi(b_1)$, $p_\psi(b_1')$ and the following joint probabilities $p_\psi(a_1\&b_1)$, $p_\psi(a_1\&b_1')$, $p_\psi(a_1'\&b_1)$, $p_\psi(a_1'\&b_1')$.[D.11] The question again is whether these quantum probabilities can be seen as arising from an ensemble of two-particle systems all prepared in state ψ.

We know from our previous section that one can give an ensemble representation for $AA'\ BB'$ in state ψ if and only if the Clauser-Horne inequalities are satisfied in state ψ. Fine shows that in this situation of correlated systems the existence of an ensemble representation is fully equivalent to the existence of a well-defined joint distribution for the incompatible observables of each particle. That is, there exists an ensemble interpretation only if one can interpolate a well-defined joint distribution for the incompatible observables (the $AA'\&BB'$ pair) among the given quantum distributions for the compatible ones (the AB pairs). The following theorem states this result:[D.12]

D.11. In terms of our previous notation we have: $p_1 = p^\psi(a_1)$, $p_2 = p^\psi(a_1')$, $p_3 = p^\psi(b_1)$, $p_4 = p^\psi(b_1')$, $p_{13} = p^\psi(a_1\ \&b_1)$, $p_{14} = p^\psi(a_1\ \&b_1')$, $p_{23} = p^\psi(a_1'\&b_1)$, $p_{24} = p^\psi(a_1'\&b_1')$. Note that it is enough to consider the single and joint probabilities for one of the outcomes of the observables, say, a_1, a_1', b_1, b_1', for we can calculate the probabilities for the rest of the outcomes form these: (i) we can calculate the rest of the single probability distributions $p(a_2)$,$p(a_2')$,$p(b_2)$ and $p(b_2')$. For example, $p(a_2)$ is simply $p(a_2) = 1 - p(a_1)$; (ii) we can calculate the various joints. For example, given that $p(a_1) = p(a_1b_1) + p(a_1b_2)$ we get $p(a_1b_2) = p(a_1) - p(a_1b_1)$. And analogously for $p(a_1b_2')$, $p(a_1'b_2)$ and $p(a_1'b_2')$; Also, given that $p(b_1) = p(a_1b_1) + p(a_2b_1)$ we get $p(a_2b_1) = p(b_1) - p(a_1b_1)$. And analogously for $p(a_2b_1')$, $p(a_2'b_1)$ and $p(a_2'b_1')$; and finally we can calculate, say, $p(a_2b_2)$ since we know that $p(a_1b_1) + p(a_1b_2) + p(a_2b_1) + p(a_2b_2) = 1$. Analogously for $p(a_2b_2')$, $p(a_2'b_2)$ and $p(a_2'b_2')$.

Theorem D.2. *There exists an ensemble representation for the observables $AA'BB'$ in state ψ if and only if there is a well-defined distribution for BB'* (relative to AA').

There is a well-defined distribution for B and B' (relative to A and A') in state ψ if

i. there exists a joint distribution for ABB' that returns as marginals the quantum mechanical distributions for AB and for AB' and analogously for $A'BB'$.

ii. Each of these distributions $p(A'BB')$ and $p(A'BB')$ returns one and the same joint distribution for BB'

Hence we can conclude that an ensemble representation can be given, or, equivalently, the Clauser-Horne inequalities are satisfied for $AA'BB'$ in ψ, if and only if there exists this well-defined joint distribution for BB' and AA' in state ψ. Thus, if the Clauser-Horne inequalities are violated, there is no joint distribution for BB' and AA'.

The problem in giving an ensemble representation thus appears when we consider a unique probability function defined over an ensemble of systems in which all four observables $AA'BB'$ together take determinate pre-measurement values. For it is then that we cannot ignore the incompatibility between the values given for the BB' correlations by the ABB' distribution and those given by the $A'BB'$ distribution. Hence if one could somehow get rid of this incompatibility then the quantum statistics might be understood as having their source in an 'urn' of systems with well-defined properties.

Fine presents a clever way of avoiding this incompatibility which he puts to work in his 'Prism Models'.[D.13] We considered this possibility in section 8.2.2.

D.12. See [Fine, 1986] p.44, or [Fine, 1982b], p.292, Proposition 1.

D.13. See [Fine, 1986], pp. 51-57 and references therein.

Appendix E
The Mass-Energy 'Equivalence'

In the passage cited in section 9.5, when comparing classical and relativistic mass, Kuhn writes that Newtonian mass is conserved while Einsteinian is convertible with energy. Both these claims are incorrect. First, as we saw in section 9.3, proper mass m, like classical mass, *is* a conserved quantity. Indeed, given that E and p are conserved and together determine m by equation (9.1), mass conservation holds in relativity theory. And second, mass is *not* convertible into energy. Indeed, the supposed conversion of mass into energy (or vice versa) is not a physical process: whether and when a 'conversion' occurs depends on the frame respect to which we describe the real physical process. As Lange says:

> 'To think of the 'conversion' of mass into energy (or vice versa) as a process that really occurs in nature, like the conversion of a caterpillar into a butterfly, would make sense only if energy and mass were (or measured the quantities of) real [i.e. objective] stuff.' ([Lange, 2002] p.240)

Let us see why this is so. Sometimes physics textbooks present examples in which mass is 'converted' into energy, and the equation $E = m\,c^2$ is then used to determine how much energy is 'equivalent' to the mass that disappears. For example, after a radioactive nucleus decays, there is a 'mass defect': the sum of the masses of the daughter bodies is less (by Δm) than the mass of the original nucleus. Some of the original mass (Δm) is said to have been converted into the kinetic energy E of the daughter bodies, where $E = (\Delta m)\,c^2$. (Since c is so large, a very small mass can be 'turned into' a great deal of energy).

The equation $E = m\,c^2$ is thus generally taken as saying that mass and energy are two forms of the same thing, and that one can be converted into the other. For example, Edwin F. Taylor and John Archibald Wheeler, in their famous book 'Space time Physics', write

> [J]oules and kilograms are two units – different only because of historical accident – for one and the same kind of quantity, mass-energy [...]. The conversion factor c^2, like the factor of conversion from [...] miles to feet, can today be counted, if one wishes, as a detailed of convention, rather than as a new deep principle' (quoted in [Lange, 2002] p.227)

And physicist-philosopher of physics Max Jammer, in his book 'Concepts of Mass in Classical and Modern Physics', says

> 'Are not the two entities [i.e. mass and energy] which are interchangeable essentially the same? Is not what is generally spoken of as an equivalence relation in reality an identity? Are therefore not 'mass' and 'energy' merely synonyms for the same physical reality, which [...] may perhaps be termed 'massergy'?' ([Jammer, 1961], p.184)

whose answers he clearly intends to be 'yes'. Even Einstein himself writes

> 'Mass and energy are therefore essentially alike; they are only different expressions for the same thing.' (quoted in [Lange, 2002], p.227)

But is this mass-energy identification correct? In relativistic physics mass is an objective property (recall that mass is Lorentz invariant) whereas energy is not, and hence mass should *not* be viewed as a form of energy nor as converting into it (or vice versa).

To understand the conceptual mistake, let us look in detail at the following example of a radioactive nucleus decay. When a tritium nucleus – one proton, two neutrons (^3_1T) – decays into a helium-3 nucleus – two protons, one neutron ($^3_2\mathrm{He}$) – along with an electron (e^-) and an anti-neutrino ($\bar{v}_e$), i.e.

$$^3_1T \longrightarrow {}^3_2\mathrm{He} + e^- + \bar{v_e}$$

the tritium's mass exceeds the sum of the product's masses by a small quantity.[E.1] In this decay there is a 'mass defect' in that the masses of a helium-3 nucleus, an electron, and an anti-neutrino add up to less than the mass of a tritium nucleus. The 'missing mass' is said to have been 'converted' into kinetic energy of the resulting bodies.

Now, while the transformation of the tritium's neutron into a proton, an electron, and an anti-neutrino is a real occurrence, this 'conversion' of mass into energy is *not*, in contrast, real. The 'mass defect' appears to rise from the fact that the sum of the three masses after the decay is less than the system's mass before the decay, the difference reflecting the three bodies' kinetic energies in the $\boldsymbol{p} = 0$ frame. (Recall that equation $E = m\,c^2$ only holds in the $\boldsymbol{p} = 0$ frame). But the sum of the three masses after the decay is also less than the system's mass *after* the decay, that is, it is less not only *before* the decay. This is because, as we showed in section 9.3, mass is not additive. It is our mistaken expectation that it is additive (arising because we expect it to measure the amount of matter forming the bodies) that leads us to characterize the system as suffering from a 'mass defect' and to ask where the 'missing mass' has gone. There is simply no 'missing mass' and no 'conversion' into energy.

E.1. The tritium releases 18.6 keV of energy in the decay process. The electron has an average kinetic energy of 5.7 keV, while the remaining energy is carried off by the electron anti-neutrino.

The supposed 'conversion' of mass into energy is an illusion produced by a subtle shift in our perspective. Indeed, we treated the system as initially forming a single body: a tritium nucleus; but we treated the system after the decay as consisting of three bodies, each with its own mass. This change of perspective is what led us to characterize the system as suffering from a 'mass defect'. But the system's mass after the decay is the same as the system's mass before the decay. There is no 'mass defect' here for mass is conserved given that E and p are conserved and together determine m by equation (9.1). As Lange explains,

> The 'mass defect' results not from some physical transformation of matter-stuff into energy-stuff, but rather from our illicitly trying to view the system from two different 'perspectives' at the same time. It is produced by our treating the post-decay system as a collection of bodies though we treated the pre-decay system as a single body. The fact that Δm of the system's initial mass 'becomes' energy $(\Delta m)\, c^2$ when we think of the post-decay system as a collection of bodies, each with its own mass, does not mean that mass is really nothing but energy or that mass and energy are different ways of measuring the same property (like distance in feet and in miles). The 'conversion' of mass into energy occurs because we have shifted our perspective, not because the nucleus has decayed.' ([Lange, 2002] pp.238-239)

Chapter 11
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