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INTEGRAL MULTILINEAR FORMS ON $C(K, X)$ SPACES

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Abstract. We use polymeasures to characterize when a multilinear form defined on a product of $C(K, X)$ spaces is integral.

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1. INTRODUCTION AND NOTATION

Given a compact Hausdorff space K with Borel σ -algebra Σ , and Banach spaces X and Y , it is well known that an operator $T: C(K, X) \rightarrow Y$ can be represented in terms of a measure $m: \Sigma \rightarrow \mathcal{L}(X; Y^{**})$ verifying certain properties (see for instance [5, § 19]).

In a series of papers (see [7], [8] and the references therein), Dobrakov developed a theory of *polymeasures* (set functions defined on a product of σ -algebras which are separately measures) that can be used to extend the classical Riesz representation theorem to a multilinear setting. With this theory, multilinear operators from a product of $C(K, X)$ spaces into Y can be represented as operator valued polymeasures. This representation theorem can be found in [12, Theorem 1.1]. The theory of polymeasures has been used by different authors, see, f.i., [1], [9], [10], [6] and the references therein.

In [12] we used the above mentioned representation theorem to obtain necessary and sometimes sufficient conditions on the polymeasure Γ representing a multilinear operator T for T to be completely continuous or unconditionally converging. In this note, which can be thought of as a continuation to [12], we use some techniques

developed in [3] to characterize the integral multilinear forms (see definition below) $T: C(K_1, X_1) \times \dots \times C(K_n, X_n) \longrightarrow \mathbb{K}$ in terms of their representing polymeasures Γ .

In this note we follow the notation of [12]. However we recall some basic notation. K, K_i will always be compact Hausdorff spaces and Σ, Σ_i will be their Borel σ -algebras. If X is a Banach space, $C(K, X)$ is the Banach space of the X -valued continuous functions, endowed with the supremum norm. $S(\Sigma, X)$ is the space of the X -valued Σ -simple functions defined on K and $B(\Sigma, X)$ is the completion of $S(\Sigma, X)$ under the supremum norm. It is well known that $C(K, X)^* = \text{bvrca}(\Sigma; X^*)$, the space of regular measures with bounded variation defined on Σ with values in X^* , endowed with the variation norm. We write $\text{bv}(\Sigma; X)$ for the measures from Σ into X with bounded variation and similarly we write $\text{bv}(\Sigma_1, \dots, \Sigma_n; X)$ for the polymeasures from $\Sigma_1 \times \dots \times \Sigma_n$ into X with bounded variation.

For notation and basic facts concerning polymeasures we refer to [12] and the references therein.

The following two definitions go back to Grothendieck.

Definition 1.1. A multilinear form $T \in \mathcal{L}^k(X_1, \dots, X_n)$ is integral if \hat{T} (i.e., its linearization) is continuous for the injective (ε) topology on $X_1 \otimes \dots \otimes X_n$. Its norm (as an element of $(X_1 \hat{\otimes}_{\varepsilon} \dots \hat{\otimes}_{\varepsilon} X_n)^*$) is the *integral* norm of T , $\|T\|_{\text{int}} := \|\hat{T}\|_{\varepsilon}$.

Definition 1.2. An operator $T \in \mathcal{L}(X; Y)$ is integral if the associated bilinear form

$$\begin{aligned} B_T: X \times Y^* &\longrightarrow \mathbb{K}, \\ (x, y) &\mapsto y(T(x)) \end{aligned}$$

is integral. In that case the integral norm of T , $\|T\|_{\text{int}} := \|B_T\|_{\text{int}}$. $\mathcal{I}(X; Y)$ denotes the Banach space of the integral operators from X into Y , endowed with the integral norm.

We will use the fact that a bilinear form $T \in \mathcal{L}^2(E_1, E_2)$ is integral if and only if any of the two associated linear operators $T_1 \in \mathcal{L}(E_1; E_2^*)$ and $T_2 \in \mathcal{L}(E_2; E_1^*)$ is integral in the above sense (see, f.i., [4, Chapter VI]).

We will also need the following result from [11].

Proposition 1.3. Let $T \in \mathcal{L}(C(K, X); Y)$ and let m be its representing measure. Then T is integral if and only if m is $\mathcal{I}(X; Y)$ -valued and it has bounded variation when considered with values in this space.

We will later need the following well known lemma, which can be found, for instance, in [2].

Lemma 1.4. Let Σ be a σ -algebra, X a Banach space and $Y \subset X^*$ a subspace norming X . If $m: \Sigma \rightarrow X$ is a strongly additive and $\sigma(X, Y)$ -regular measure, then m is regular.

If $\Gamma: \Sigma_1 \times \dots \times \Sigma_n \rightarrow X$ is a polymasure, we define its *variation*

$$v(\Gamma): \Sigma_1 \times \dots \times \Sigma_n \rightarrow [0, +\infty]$$

by

$$v(\Gamma)(A_1, \dots, A_n) = \sup \left\{ \sum_{j_1=1}^{m_1} \dots \sum_{j_n=1}^{m_n} \|\Gamma(A_{1,j_1}, \dots, A_{n,j_n})\| \right\}$$

where $(A_{i,j_i})_{j_i=1}^{m_i}$ is a Σ_i -partition of A_i ($1 \leq i \leq n$).

The following lemma can be found in [3].

Lemma 1.5. Let X be a Banach space, $\Omega_1, \dots, \Omega_n$ sets and $\Sigma_1, \dots, \Sigma_n$ σ -algebras defined on them. Let now $\gamma: \Sigma_1 \times \dots \times \Sigma_n \rightarrow X$ be a polymasure and let $\varphi_1: \Sigma_1 \rightarrow \text{pm}(\Sigma_2, \dots, \Sigma_n; X)$ be the measure given by $\varphi_1(A_1)(A_2, \dots, A_n) = \gamma(A_1)(A_2, \dots, A_n)$. Then $v(\gamma) < \infty$ if and only if φ_1 takes values in $\text{bvp}\text{m}(\Sigma_2, \dots, \Sigma_n; X)$ and $v(\varphi_1) < \infty$ when we consider the variation norm in the image space. In that case, $v(\varphi_1)(A_1) = v(\gamma)(A_1, \Omega_2, \dots, \Omega_n)$ and $v(\varphi_1(A_1))(A_2, \dots, A_n) \leq v(\gamma)(A_1, A_2, \dots, A_n)$. Of course the role played by the first variable could be played by any of the other ones.

2. THE RESULT

We can present now our main result. In the following we write $B_0(K_1 \times \dots \times K_n)$ for the σ -algebra of the Borel sets of $C(K_1 \times \dots \times K_n)$.

Proposition 2.1. Let $T \in \mathcal{L}^k(C(K_1, X_1), \dots, C(K_n, X_n))$ and let Γ be its representing polymasure. Then the following are equivalent:

- The polymasure $\Gamma: \Sigma_1 \times \dots \times \Sigma_n \rightarrow (X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_n)^*$ can be extended to a measure $m \in \text{bvrca}(B_0(K_1 \times \dots \times K_n); (X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*)$ (which implies in particular that Γ is $(X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*$ -valued).
- Γ is $(X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*$ -valued and $v(\Gamma) < \infty$, when we consider the integral norm in the image space.
- T is integral.

Moreover, in that case $v(\Gamma) = v(\mu) = \|T\|_{\text{int}}$, so there is an isometric isomorphism between $(C(K_1, X_1) \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon C(K_n, X_n))^*$ and the space of separately regular polymeasures with bounded variation defined on $\Sigma_1 \times \dots \times \Sigma_n$ with values in $(X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*$, endowed with the variation norm.

Proof. (c) \Rightarrow (a): If

$$T: C(K_1, X_1) \times \dots \times C(K_n, X_n) \longrightarrow \mathbb{K}$$

is integral, then we can consider the continuous linear operator

$$T': C(K_1) \hat{\otimes}_\varepsilon X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon C(K_n) \hat{\otimes}_\varepsilon X_n \longrightarrow \mathbb{K}$$

and, using the associativity of the injective tensor product and the fact that $C(K_1) \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon C(K_n) \approx C(K_1 \times \dots \times K_n)$, we can define the integral operator

$$T_1: C(K_1 \times \dots \times K_n) \longrightarrow (X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*.$$

Let $\mu \in \text{bvrca}(B_0(K_1 \times \dots \times K_n); (X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*)$ be the representing measure of T_1 .

From regularity it follows that, for every $(A_1, \dots, A_n) \in \Sigma_1 \times \dots \times \Sigma_n$,

$$\mu(A_1 \times \dots \times A_n) = \Gamma(A_1, \dots, A_n),$$

i.e., that μ extends Γ .

(a) \Rightarrow (b) is clear (observe that $v(\Gamma) \leq v(m)$).

(b) \Rightarrow (c): We reason by induction on n . If $n = 1$ there is nothing to prove. Let us consider $n = 2$ and let T and Γ be as in the hypothesis.

Let $\varphi_1: \Sigma_1 \longrightarrow \text{bv}(\Sigma_2; (X_1 \hat{\otimes}_\varepsilon X_2)^*)$ be the measure associated to Γ given by $\varphi_1(A_1)(A_2) = \Gamma(A_1, A_2)$. Since $v(\Gamma) < \infty$, Lemma 1.5 assures that φ_1 is indeed $\text{bv}(\Sigma_2; (X_1 \hat{\otimes}_\varepsilon X_2)^*)$ -valued and has bounded variation with values in this space.

Claim 1. For every $A_1 \in \Sigma_1$, $\varphi_1(A_1)$ is a regular measure.

Every measure of bounded variation is strongly additive ([4, Proposition I.1.15]). So, by Lemma 1.4, to prove the claim we just need to check that, for every $g \in X_1 \otimes X_2$, $g \circ \varphi_1(A_1)$ is regular. So, let us first suppose that $g = x_1 \otimes x_2$. Then

$$(x_1 \otimes x_2) \circ \varphi_1(A_1)(A_2) = \Gamma(A_1, A_2)(x_1, x_2).$$

Since Γ is weak*-separately regular (see [12, Theorem 1.1]) we get that $(x_1 \otimes x_2) \circ \varphi_1(A_1)$ is regular. From here the result follows easily for a general $g \in X_1 \otimes X_2$ and the claim is established. As a consequence of it we obtain that φ_1 is $\text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*)$ -valued.

Claim 2. *The measure*

$$\varphi_1: \Sigma_1 \longrightarrow \text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*)$$

is regular.

We have that $C(K_2, X_1 \hat{\otimes}_\varepsilon X_2)$ is isometrically isomorphic to a subspace of $B(\Sigma_2, X_1 \hat{\otimes}_\varepsilon X_2)$, which in turn is isometrically isomorphic to a subspace of $C(K_2, X_1 \hat{\otimes}_\varepsilon X_2)^{**} \approx (\text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*))^*$. Moreover, $S(\Sigma_2, X_1 \hat{\otimes}_\varepsilon X_2)$ is a dense subspace of $B(\Sigma_2, X_1 \hat{\otimes}_\varepsilon X_2)$ and $S(\Sigma_2, X_1 \otimes X_2)$ is a dense subspace of $S(\Sigma_2, X_1 \hat{\otimes}_\varepsilon X_2)$. So, $S(\Sigma_2, X_1 \otimes X_2) \subset C(K_2, X_1 \hat{\otimes}_\varepsilon X_2)^{**}$ is a subspace norming $\text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*)$. Therefore, by reasonings analogous to the proof of Claim 1, we just need to prove that, for every $s \in S(\Sigma_2, X_1 \otimes X_2)$, $s \circ \varphi_1$ is regular. This follows again from the separate weak*-continuity of Γ , considering first $s = \chi_A(x_1 \otimes x_2)$, then $s = \chi_A g$ for any $g \in X_1 \otimes X_2$ and finally $s = \sum_{m=1}^n \chi_{A_m} g_m$ for any $A_m \in \Sigma_2$ and $g_m \in X_1 \otimes X_2$.

Therefore $\varphi_1: \Sigma_1 \longrightarrow \text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*)$ is regular and $v(\varphi_1) = v(\Gamma) < \infty$. Observe that $\text{bvrca}(\Sigma_2, (X_1 \hat{\otimes}_\varepsilon X_2)^*) = C(K_2, X_1 \hat{\otimes}_\varepsilon X_2)^* = (X_1 \hat{\otimes}_\varepsilon C(K_2, X_2))^* = I(X_1; C(K_2, X_2)^*)$.

So, we can consider the operator $T_{\varphi_1}: C(K_1, X_1) \longrightarrow C(K_2, X_2)^*$ defined by $T_{\varphi_1}(f) = \int f d\varphi_1$ and, according to Proposition 1.3, T_{φ_1} is integral (and $\|T_{\varphi_1}\| = v(\varphi_1) = v(\gamma)$). Hence, the bilinear form

$$\widetilde{T_{\varphi_1}}: C(K_1, X_1) \times C(K_2, X_2) \longrightarrow \mathbb{K}$$

is integral.

Let

$$\overline{T_{\varphi_1}}: B(\Sigma_1, X_1) \times B(\Sigma_2, X_2) \longrightarrow \mathbb{K}$$

be the extension of $\widetilde{T_{\varphi_1}}$ given by [12, Theorem 1.1]. Then, for every $x_i \in X_i$, $A_i \in \Sigma_i$ ($1 \leq i \leq 2$), we have

$$\begin{aligned} \overline{T_{\varphi_1}}(x_1 \chi_{A_1}, x_2 \chi_{A_2}) &= \varphi_1(A_1)(A_2)(x_1 \otimes x_2) \\ &= \Gamma(A_1, A_2)(x_1 \otimes x_2) = \overline{T}(x_1 \chi_{A_1}, x_2 \chi_{A_2}), \end{aligned}$$

where \overline{T} is the extension of T given by [12, Theorem 1.1].

Therefore $\overline{T_{\varphi_1}} = \overline{T}$, so $\widetilde{\overline{T_{\varphi_1}}} = T$ and $\|T\|_{\text{int}} = \|\overline{T_{\varphi_1}}\|_{\text{int}} = v(\varphi_1) = v(\gamma)$, which finishes the proof in the case $n = 2$.

Let us now suppose the result to be true for $n = 1$, consider

$$T: C(K_1, X_1) \times \dots \times C(K_n, X_n) \longrightarrow \mathbb{K}$$

and let its associated polymeasure Γ be as in the hypothesis. Let

$$\varphi_1: \Sigma_1 \longrightarrow \text{bvpm}(\Sigma_2, \dots, \Sigma_n; (X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*)$$

be the measure associated to Γ given by $\varphi_1(A_1)(A_2, \dots, A_n) = \Gamma(A_1, \dots, A_n)$. By Lemma 1.5 we get that φ_1 is well defined and with bounded variation. Similarly to the proof of Claim 1 above it can be proved that, for every $A_1 \in \Sigma_1$, the polymeasure $\varphi_1(A_1)$ is separately regular. Call Z the space of separately regular polymeasures with bounded variation defined on $\Sigma_2 \times \dots \times \Sigma_n$ and with values in $(X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^*$. Note that the induction hypothesis tells us that

$$\begin{aligned} Z &= C(K_2 \times \dots \times K_n, X_1 \hat{\otimes}_\varepsilon X_2 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n)^* \\ &= (X_1 \hat{\otimes}_\varepsilon C(K_2, X_2) \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon C(K_n, X_n))^* \\ &= I(X_1; (C(K_2, X_2) \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon C(K_n, X_n))^*). \end{aligned}$$

Now we can continue similarly to the proof of the case $n = 2$ to prove that φ_1 is regular, and the proof finishes similarly to the case $n = 2$. \square

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