

# The method of virtual copies and contractions of simple Lie algebras

**Rutwig Campoamor-Stursberg**

Instituto de Matemática Interdisciplinar IMI-UCM,  
Plaza de Ciencias 3, E-28040 Madrid, Spain

E-mail: [rutwig@ucm.es](mailto:rutwig@ucm.es)

**Abstract.** Contractions of simple Lie algebras onto semidirect sums are analyzed in the context of the method of virtual copies, and various existence conditions are obtained in terms of the branching rules of representations and the properties of the characteristic representation associated to an embedding of semisimple algebras. Contractions of the Lie algebra  $\mathfrak{sp}(2N, \mathbb{R})$  with respect to nonmaximal symplectic subalgebras are studied in some detail. In particular, a realization of the subalgebra in terms of cubic polynomial operators is constructed.

## 1. Introduction

Invariants functions by the coadjoint representation of a Lie algebra not only constitute a valuable tool in representation theory, where their eigenvalues can be used to characterize representations, but also provide a powerful tool for physical applications, where invariant operators can be related to various symmetry properties of a given system [1, 2, 3]. For the case of semisimple Lie algebras  $\mathfrak{s}$ , the construction of the Casimir operators follows from the particular properties of these algebras, and can be reduced to the computation of the centre of the universal enveloping algebra  $\mathcal{U}(\mathfrak{s})$  [4, 5, 6]. However, this approach is not universally valid, as nonsemisimple algebras can admit rational and even nonrational invariants, a fact that partially invalidates the purely algebraic ansatz [7]. In order to circumvent this obstruction, alternative methods based on the theory of partial differential equations or differential forms have been developed, which are moreover applicable to arbitrary Lie algebras [8, 9, 10, 11, 12]. For certain types of Lie algebras, however, there exists the possibility of computing the invariants in terms of the Casimir operators of a semisimple subalgebra  $\mathfrak{s}$ , using properly defined operators in the enveloping algebra that transform like the generators of  $\mathfrak{s}$ . This idea has been fruitful in various contexts, such as the boson realizations of Lie algebras [13] or the study of certain kinematical algebras [14].

The purpose of this work is to review the main properties of the so-called method of virtual copies, originally formulated in [15] in relation to the Hamilton algebras in relativistic theories [16]. The procedure is valid for certain types of semidirect sums  $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus} \Gamma \mathfrak{r}$  having a solvable nonnilpotent radical  $\mathfrak{r}$ , extending naturally a well-known method valid for the semidirect sums of simple Lie algebras with the Heisenberg algebra  $\mathfrak{h}_N$  [17]. Virtual copies are constructed by means of higher-order operators in the generators that transform like the generators of the semisimple



subalgebra  $\mathfrak{s}$ , up to a functional factor that can be identified with an invariant of the radical. As a consequence, these operators do not belong to the enveloping algebra of the semidirect sum, but to its field of fractions [18], justifying that such copies are called virtual. The construction is compatible with generalized Inönü-Wigner contractions of Lie algebras, a fact that allows us to determine necessary conditions for the existence of virtual copies of simple Lie algebras. It is shown how the classification of (maximal) semisimple Lie subalgebras of simple Lie algebras and the branching rules of representations can be used to study the possible contractions of simple algebras onto semidirect sums that allow the construction of virtual copies. A hierarchy of semidirect sums with a symplectic Levi factor for which a virtual copy generated by cubic operators exists is obtained by contraction of the simple Lie algebra  $\mathfrak{sp}(2N, \mathbb{R})$  for any  $N \geq 2$ .

Unless otherwise stated, any Lie algebra  $\mathfrak{g}$  and any representation is considered over the field  $\mathbb{R}$  of real numbers. The Einstein summation convention is used.

## 2. Maurer-Cartan equations. Invariants of Lie algebras

Given a Lie algebra  $\mathfrak{g} = \{X_1, \dots, X_n \mid [X_i, X_j] = C_{ij}^k X_k\}$  in terms of generators and commutation relations, a polynomial operator  $C_p = \alpha^{i_1 \dots i_p} X_{i_1} \dots X_{i_p}$  in the generators of  $\mathfrak{g}$  is called invariant whenever it commutes with all elements in  $\mathfrak{g}$ , that is, the condition  $[X_i, C_p] = 0$  is satisfied for  $i = 1, \dots, n$ . Such an operator can be shown to belong to the centre of the enveloping algebra  $\mathcal{U}(\mathfrak{g})$ , and is usually called Casimir operator. For semisimple Lie algebras, Casimir operators can be constructed using structural properties [6]. If the Lie algebra is not semisimple, polynomial invariant functions do not necessarily exist, and additional methods to compute the invariants are required [8, 11]. A method that works independently on the particular structure of the Lie algebra is given by the analytical realization [9]. Here the generators of the Lie algebra  $\mathfrak{s}$  are realized in the space  $C^\infty(\mathfrak{g}^*)$  by means of the differential operators:

$$\hat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}, \quad (1)$$

where  $\{x_1, \dots, x_n\}$  denote the coordinates in a dual basis of  $\{X_1, \dots, X_n\}$ . The invariants of  $\mathfrak{g}$  therefore correspond to the solutions of the following system of partial differential equations:

$$\hat{X}_i F = 0, \quad 1 \leq i \leq n. \quad (2)$$

This system has exactly  $\mathcal{N}(\mathfrak{g})$  functionally independent solutions, determined by the formula

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \sup_{x_1, \dots, x_n} \text{rank} \left( C_{ij}^k x_k \right), \quad (3)$$

where  $A(\mathfrak{g}) := (C_{ij}^k x_k)$  is the matrix associated to the commutator table of  $\mathfrak{g}$  over the given basis [19]. If a solution of (2) is of polynomial type, the symmetrization map defined on monomials by<sup>1</sup>

$$\text{Sym}(x_{i_1}^{a_1} \dots x_{i_p}^{a_p}) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} x_{\sigma(i_1)}^{a_1} \dots x_{\sigma(i_p)}^{a_p} \quad (4)$$

allows to recover the Casimir operators in their usual form as elements in the enveloping algebra of  $\mathfrak{g}$ , where the variables  $x_i$  are replaced by the corresponding generator  $X_i$  [18]. A maximal set of functionally independent invariants is usually called a fundamental basis.

The formalism of differential forms (see e.g. [20]) allows us not only to reformulate formula (3), but also to find an alternative procedure to compute Casimir operators [12]. Given the

<sup>1</sup> Here  $\Sigma_p$  denotes the symmetric group in  $p$  letters.

structure tensor  $\{C_{ij}^k\}$  of a Lie algebra  $\mathfrak{g}$  over the basis  $\{X_1, \dots, X_n\}$ , we identify the dual space  $\mathfrak{g}^*$  with the left-invariant 1-forms  $\{\omega^1, \dots, \omega^n\}$  on the simply connected Lie group  $G$  whose algebra is isomorphic to  $\mathfrak{g}$ . Then the coboundary operator  $d$  on  $\mathfrak{g}^*$  given by

$$d\omega(X_i, X_j) = -\omega([X_i, X_j]) = -C_{ij}^k \omega(X_k), \quad \omega \in \mathfrak{g}^*.$$

is well defined, and the structure tensor is codified by the so-called Maurer-Cartan equations

$$d\omega^k = -\frac{1}{2} C_{ij}^k \omega^i \wedge \omega^j, \quad 1 \leq i < j \leq \dim(\mathfrak{g}), \quad (5)$$

The condition  $d^2\omega^i = 0$  for all  $i$  can be shown to be equivalent to the Jacobi condition [21]. Now define the linear subspace  $\mathcal{L}(\mathfrak{g}) = \mathbb{R}\{d\omega^i\}_{1 \leq i \leq \dim \mathfrak{g}}$  of  $\wedge^2 \mathfrak{g}^*$  spanned by the 2-forms  $d\omega^i$ . Given a generic element  $\omega = a_i d\omega^i$  of  $\mathcal{L}(\mathfrak{g})$  with real coefficients  $a_i$ , there always exists a positive integer  $j_0(\omega) \in \mathbb{N}$  such that  $\wedge^{j_0(\omega)} \omega \neq 0$  and  $\wedge^{j_0(\omega)+1} \omega \equiv 0$ . Taking into account all elements, we define the quantity  $j_0(\mathfrak{g})$  as the maximal possible rank:

$$j_0(\mathfrak{g}) = \max \{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{g})\}.$$

This number is a numerical invariant of the Lie algebra  $\mathfrak{g}$  that can be used to rewrite equation (3) as<sup>2</sup>

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - 2j_0(\mathfrak{g}). \quad (6)$$

In this context, we see that if  $F$  is a solution to the system (2), then its total differential  $dF$  annihilates the differential operators  $\hat{X}_i$  in (1):

$$dF(X_i) = \hat{X}_i(F) = 0.$$

The main advantage of this procedure is that it works for any representation of the Lie algebra, not merely for the adjoint representation [8].

### 3. Semidirect sums of simple and Heisenberg-Weyl Lie algebras

Within the class of Lie algebras that are not semisimple, this type possibly constitutes the most important example for which a unified method to compute the Casimir operators can be proposed. The algebraic approach of [17] was extended to physical applications by Ch. Quesne in [13], with later generalizations to other classes of Lie algebras (see [22, 23] and references therein). Given a semidirect sum  $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{w}(n)$  of a simple Lie algebra  $\mathfrak{s}$  with basis  $\{E_{ij}\}$  and the Heisenberg-Weyl algebra  $\mathfrak{w}(n)$  spanned by  $n$  boson pairs  $b_i, b_j^\dagger$  and the unit operator  $\mathbb{I}$ , the quadratic operators

$$E'_{ij} = E_{ij}\mathbb{I} - b_i^\dagger b_j \quad (7)$$

were shown to verify the conditions

$$[E'_{ij}, b_k^\dagger] = [E'_{ij}, b_k] = 0, \quad [E_{ij}, E'_{kl}] = [E_{ij}, E_{kl}]'. \quad (8)$$

These identities ensure that the operators  $E'_{ij}$  in the enveloping algebra of  $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{w}(n)$  transform like the original generators of  $\mathfrak{s}$ , up to the identity operator, so that they span a copy  $\mathfrak{s}'$  of  $\mathfrak{s}$ . It is straightforward to verify that the commutation relation

$$[E'_{ij}, E'_{kl}] = \mathbb{I} [E_{ij}, E_{kl}]',$$

<sup>2</sup> In addition, the scalar  $j_0(\mathfrak{g})$  also corresponds to the number of internal labels required to describe irreducible representations of  $\mathfrak{g}$  [9].

is satisfied. It follows from the construction that for a Casimir operator

$$C_p = \sum_{i_1 \dots i_p} E_{i_1 i_2} \dots E_{i_p i_1},$$

of the simple Lie algebra  $\mathfrak{s}$ , the replacement of the generators  $E_{ij}$  by the operators  $E'_{ij}$  provides an invariant  $C'_p$  of the copy  $\mathfrak{s}'$  of  $\mathfrak{s}$  in the enveloping algebra  $\mathcal{U}(\mathfrak{s} \vec{\oplus}_R \mathfrak{w}(n))$ . It follows in particular that

$$[C'_p, b_k^\dagger] = [C'_p, b_k] = [C'_p, E_{ij}] = 0,$$

showing that  $C'_p$  is a Casimir operator of the Lie algebra  $\mathfrak{s} \vec{\oplus}_R \mathfrak{w}(n)$ .

The procedure works for any real semisimple Lie algebra, allowing us to compute easily the invariants of semidirect sums. A particularly important case is given by the centrally extended Schrödinger algebra  $\widehat{S}(N)$ , as well as the generalized conformal Galilean algebras [22, 24, 25], although it should be observed that, for the latter type, not all Casimir operators of the Lie algebra can be constructed using this method [26]. Once the validity of the method has been established, we can revert the problem and ask what multiplets  $\widehat{\Gamma}$  of a semisimple Lie algebra  $\mathfrak{s}$  are compatible with a Heisenberg-Weyl algebra, such that the semidirect sum

$$\mathfrak{g} = \mathfrak{s} \vec{\oplus}_{\widehat{\Gamma}} \mathfrak{h}_N$$

is well defined. Obvious constraints are given by the dimension  $\dim \widehat{\Gamma} = 2N + 1$ , as well as the fact that the factor space  $V = \mathfrak{h}_N / Z(\mathfrak{h}_N)$  must also be a representation of  $\mathfrak{s}$ . Not excluding the possibility that the inner multiplicity of the trivial representation  $\Gamma_0$  is greater than 1, we have the complete decomposition

$$\widehat{\Gamma} \simeq \bigoplus_{i=1}^s \Gamma_i \oplus \Gamma_0.$$

An irreducible representation  $\Lambda$  of  $\mathfrak{s}$  is called a constituent of  $\Gamma$  if for some  $j$  we have  $\Lambda = \Gamma_j$ . Now, as the bracket of the Heisenberg-Weyl algebra can be identified with a skew-symmetric nondegenerate bilinear form  $\varphi : V \wedge V \rightarrow Z(\mathfrak{h}_N)$  (see e.g. [22]), if  $\Lambda$  is a constituent of  $V = \mathfrak{h}_N / Z(\mathfrak{h}_N)$ , then it must have the same multiplicity as its dual:

$$\text{mult}_V(\Lambda) = \text{mult}_V(\Lambda^*).$$

In particular, the multiplicity of a self-conjugate irreducible representation  $\Lambda$  depends on the fact whether the wedge product of  $\Lambda$  with itself contains a copy of the trivial representation:

- (i) If  $\Lambda \wedge \Lambda \supset \Gamma_0$ , then  $\text{mult}_V(\Gamma)$  can adopt arbitrary positive integer values.
- (ii) If  $\Gamma \wedge \Gamma \not\supset \Gamma_0$ , then  $\text{mult}_V(\Gamma)$  is necessarily a positive even integer.

For complex simple Lie algebras the preceding result provides the classification of  $\mathfrak{h}_N$ -multiplets. In Table 1, the self-conjugate fundamental irreducible representations that are  $\mathfrak{h}_N$ -compatible (these actually correspond to the so-called symplectic representations, see e.g. [27]) are given. In order to obtain the analogue for the real simple Lie algebras, we observe that for a simple real Lie algebra  $\mathfrak{s}$  with complexification  $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C} \simeq \widehat{\mathfrak{s}}$  and a  $\mathfrak{h}_N$ -compatible representation  $\Gamma$  of  $\mathfrak{s}$  we have that  $\widehat{\Gamma} = \Gamma \otimes_{\mathbb{R}} \mathbb{C}$  is a  $\mathfrak{h}_N$ -compatible representation of  $\widehat{\mathfrak{s}}$ .

**Table 1.** Compatible fundamental representations.

$\mathfrak{s}$	$\Gamma$	$\dim \Gamma$
$A_{4q+1} \ (q \geq 0)$	$\Gamma_{2q+1}$	$\binom{4q+2}{2q+1}$
$B_{4q+1} \ (q \geq 2)$	$\Gamma_{4q+1}$	$2^{4q+1}$
$B_{4q+2} \ (q \geq 2)$	$\Gamma_{4q+2}$	$2^{4q+2}$
$C_n$	$\Gamma_{2q+1} \ (q \geq 0)$	$\binom{2n}{2q+1} - \binom{2n}{2q-1}$
$D_{2q+2} \ (q \geq 2)$	$\Gamma_{4q+1}$	$2^{4q+1}$
	$\Gamma_{4q+2}$	$2^{4q+1}$
$E_7$	$\Gamma_1$	56
	$\Gamma_3$	27664
	$\Gamma_7$	912

#### 4. Virtual copies in enveloping algebras

The main idea behind the so-called method of virtual copies developed in [15] is to construct, up to a polynomial factor, a copy of a semisimple Lie algebra  $\mathfrak{s}$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g} = \mathfrak{s} \oplus_{\Gamma} \mathfrak{t}$  by means of operators having degree  $d \geq 3$  in the generators and such that analogous identities to those in (8) are satisfied. In contrast to the case of Heisenberg radicals, the general procedure is not restricted to Lie algebras possessing a nilpotent radical. In order to construct invariants of the semidirect sum  $\mathfrak{g}$  using the Casimir operators of the Levi subalgebra  $\mathfrak{s}$ , we require that the inequality  $\mathcal{N}(\mathfrak{g}) \geq \mathcal{N}(\mathfrak{s})$  is satisfied. To this extent, we use general functions of the generators instead of the quadratic operators in (7) and determine the constraints that must be satisfied to guarantee that these operators transform in the same way as in the identities (8). Starting from a basis  $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$  of  $\mathfrak{g}$  such that  $\{X_1, \dots, X_n\}$  spans the Levi subalgebra  $\mathfrak{s}$  and

$$[X_i, X_j] = C_{ij}^k X_k. \quad (9)$$

denotes the structure tensor in  $\mathfrak{s}$ ,<sup>3</sup> we define operators  $X'_i$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  by

$$X'_i = X_i f(Y_1, \dots, Y_m) + P_i(Y_1, \dots, Y_m). \quad (10)$$

$P_i$  is required to be a homogeneous polynomial of degree  $k$ , while  $f$  must be homogeneous of degree  $k-1$ . The generalization of the constraints (8) is thus given by the identities

$$[X'_i, Y_k] = 0, \quad 1 \leq i \leq n, \quad 1 \leq k \leq m, \quad (11)$$

$$[X'_i, X_j] = [X_i, X_j]' := C_{ij}^k (X_k f + P_k). \quad (12)$$

As these relations must be satisfied for all generators of  $\mathfrak{g}$  and all operators in (10), the functions  $f$  and  $P_i$  are subjected to certain structural conditions. Expanding equation (11) we are led to the expression

$$[X'_i, Y_j] = [X_i f, Y_j] + [P_i, Y_j] = X_i [f, Y_j] + [X_i, Y_j] f + [P_i, Y_j].$$

As we are dealing with homogeneous functions in the variables of  $\mathfrak{s}$  and  $\mathfrak{t}$ , we can reorder the terms taking into account their degree. Now  $X_i [f, Y_j]$  is homogeneous of degree  $k-1$  in the variables  $\{Y_1, \dots, Y_m\}$ , while  $[X_i, Y_j] f + [P_i, Y_j]$  is of degree  $k$ . This allows us to separate the preceding expression into two parts:

$$\begin{aligned} [f, Y_j] &= 0, \\ [X_i, Y_j] f + [P_i, Y_j] &= 0. \end{aligned} \quad (13)$$

<sup>3</sup> Hence  $\{Y_1, \dots, Y_m\}$  spans the radical  $\mathfrak{t}$ .

The first equation implies that  $f$  must be a Casimir operator of the radical  $\mathfrak{r}$ . The second equality of (13) tells us, to some extent, how far the polynomial  $P_i$  is from being also an invariant of  $\mathfrak{r}$ . Indeed, if  $[X_i, Y_j] = 0$ , then  $[P_i, Y_j] = 0$ . Developing equation (12) leads to the identity

$$[X'_i, X_j] = [X_i, X_j] f - X_i [X_j, f] + [P_i, X_j].$$

Here  $[X_i, X_j] f - X_i [X_j, f]$  is homogeneous of degree  $k-1$  in the variables of  $\mathfrak{r}$  and degree one in the variables of  $\mathfrak{s}$ , with  $[P_i, X_j]$  being of degree zero in the variables of  $\mathfrak{s}$ . By homogeneity, the system

$$[X_i, X_j] f - X_i [X_j, f] = C_{ij}^k X_k f, \quad [P_i, X_j] = C_{ij}^k P_k$$

must hold for all indices  $i, j$ . Using equation (9), the first of these equations can be reduced to

$$X_i [X_j, f] = 0,$$

from which we conclude that  $f$  must be a Casimir operator of  $\mathfrak{g}$  that depends only on the variables of the radical  $\mathfrak{r}$ . The remaining equation implies that the operators  $P_i$  transform under the generators  $X_j$  in analogous form to the generators of  $\mathfrak{s}$ . It follows from all these conditions that the operators  $X'_i$  transform under the commutator like follows:

$$\begin{aligned} [X'_i, X'_j] &= [X_i f + P_i, X_j f + P_j] = [X_i f + P_i, X_j f] + [X_i f + P_i, P_j] \\ &= C_{ij}^k X_k f^2 + C_{ij}^k P_k f + [X'_i, P_j] = [X_i, X_j]' f. \end{aligned} \quad (14)$$

It should be observed that the case of radicals  $\mathfrak{r}$  satisfying  $\mathcal{N}(\mathfrak{r}) = 0$ , i.e., having no invariants, is excluded from our analysis. In such a case  $f$  reduces to a constant and thus the operators (10) correspond to a change of basis in the subalgebra  $\mathfrak{s}$ . Summarizing, up to the functional (nonconstant) factor  $f$ , the commutators behave like those of the simple Lie algebra  $\mathfrak{s}$ . As  $f$  does not correspond in general to the identity operator, we have that  $[X''_i, X''_j] \neq [X_i, X_j]''$ , and thus the operators  $X'_i$  generate a copy of  $\mathfrak{s}$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  up to this factor  $f$ , hence justifying the terminology of virtual copies [15]. We remark that the operators  $X'_i$  cannot be replaced by  $X''_i = X_i + P_i/f$ , because the second term usually belongs to the field of fractions of  $\mathcal{U}(\mathfrak{g})$  [18].

In spite of this factor  $f$ , we can still use the invariants of  $\mathfrak{s}$  to construct Casimir operators of the semidirect sum  $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{\Gamma} \mathfrak{r}$ . If  $C = \sum \alpha^{i_1 \dots i_p} X_{i_1} \dots X_{i_p}$  is a Casimir operator of degree  $p$  of  $\mathfrak{s}$ , then the operator  $C' = \sum \alpha^{i_1 \dots i_p} X'_{i_1} \dots X'_{i_p}$  naturally commutes with all the generators of the radical  $\mathfrak{r}$  by equation (11). For commutators with generators of  $\mathfrak{s}$ , the iteration of equation (14) shows that

$$[\alpha^{i_1 \dots i_p} X'_{i_1} \dots X'_{i_p}, X_\alpha] = [\alpha^{i_1 \dots i_p} X_{i_1} \dots X_{i_p}, X_\alpha]' f^p = 0,$$

and thus  $C'_p$  is a Casimir operator of  $\mathfrak{g}$  of degree  $p(\deg f + 1)$ . It can be shown that the invariants constructed by this method are functionally independent (see e.g. [15]), from which we deduce the inequality  $\mathcal{N}(\mathfrak{s}) + 1 \leq \mathcal{N}(\mathfrak{g})$ . This establishes a first restriction on the applicability of the procedure. If for a given semidirect sum  $\mathfrak{g}$  with Levi subalgebra  $\mathfrak{s}$  we have that  $\mathcal{N}(\mathfrak{g}) \leq \mathcal{N}(\mathfrak{s})$ , then no virtual copy of  $\mathfrak{s}$  in the enveloping algebra of  $\mathfrak{g}$  can be constructed by this method.

Although a classification of semidirect sums  $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{\Gamma} \mathfrak{r}$  for which a virtual copy of  $\mathfrak{s}$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  does not currently exist, we can at least enumerate certain types that certainly do not admit such a construction, based on properties of the radical  $\mathfrak{r}$ . In addition to the cases already commented that do not allow a virtual copy, further types for which either the construction is not possible or the Lie algebra  $\mathfrak{g}$  cannot exist occur whenever one of the following constraints is given:

- (i) The radical  $\mathfrak{r}$  is an Abelian algebra.
- (ii)  $\mathfrak{r}$  is a solvable Lie algebra all whose derivations are inner ( $\dim \text{Der}(\mathfrak{r}) \leq \dim \mathfrak{r}$ ).
- (iii) For the Chevalley cohomology  $H^2(\mathfrak{r}, \mathfrak{r})$  of  $\mathfrak{r}$ , the inequality  $\dim H^2(\mathfrak{r}, \mathfrak{r}) < \dim \mathfrak{s}$  holds.

The first case is a direct consequence of the constraints (11) and (12), that imply that the representation  $\Gamma$  of  $\mathfrak{s}$  is trivial, hence the semidirect sum is actually a direct sum of  $\mathfrak{s}$  and  $\mathfrak{r}$ . For the remaining two cases, the proof follows from a well-known identity valid for any Lie algebra  $\mathfrak{r}$  such that  $[\mathfrak{r}, \mathfrak{r}] \neq \mathfrak{r}$ :

$$\dim \text{Der}(\mathfrak{r}) \leq \dim \mathfrak{r} + \dim H^2(\mathfrak{r}, \mathfrak{r}).$$

The Levi factor  $\mathfrak{s}$  of the semidirect sum  $\mathfrak{s} \oplus_{\Gamma} \mathfrak{r}$  must always be a subalgebra of the algebra of derivations  $\text{Der}(\mathfrak{r})$ , as the representation  $\Gamma$  of  $\mathfrak{s}$  acts on the radical by derivations [28]. In both cases (ii) and (iii), the existence of a semisimple Lie algebra contained in  $\text{Der}(\mathfrak{r})$  is excluded by dimension.

#### 4.1. Contractions of semidirect sums

An interesting problem arises when virtual copies of simple Lie algebras in the enveloping algebras of semidirect sums are combined with contractions of Lie algebras. It is natural to ask under what conditions the existence of a virtual copy in the enveloping algebra of the contraction can be guaranteed. We recall that a contraction of a Lie algebra  $\mathfrak{g}$  is defined in terms of a family of nonsingular endomorphisms  $\Phi_\varepsilon$  of  $\mathfrak{g}$  with  $\varepsilon \in (0, 1]$  and such that the limit

$$[X, Y]_\infty := \lim_{\varepsilon \rightarrow 0} \Phi_\varepsilon^{-1} [\Phi_\varepsilon(X), \Phi_\varepsilon(Y)] \quad (15)$$

exists for any pair of generators  $X, Y \in \mathfrak{g}$ . In these conditions, equation (15) defines a Lie algebra  $\mathfrak{g}'$  called the contraction (with respect to  $\Phi_\varepsilon$ ) of  $\mathfrak{g}$  [29, 30]. If there exists a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}$  such that the matrix  $A_\Psi$  associated to  $\Psi_\varepsilon$  is diagonal, i.e.,

$$(A_\Psi)_{ij} = \delta_{ij} \varepsilon^{n_j}, \quad n_j \in \mathbb{R},$$

then we speak of a generalized Inönü-Wigner contraction [30]. Although not any contraction of Lie algebras is equivalent to one of this type, they are without discussion the most important case within physical applications, so that we restrict to this type.

Consider a semidirect sum  $\mathfrak{g} = \mathfrak{s} \oplus_R \mathfrak{r}$  so that a virtual copy of  $\mathfrak{s}$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  exists. If such a copy is generated by the operators

$$X'_i = \alpha^{i_1 \dots i_{k-1}} X_i Y_{i_1} \dots Y_{i_{k-1}} + \beta_i^{j_1 \dots j_k} Y_{j_1} \dots Y_{j_k}, \quad 1 \leq i \leq \dim \mathfrak{s} \quad (16)$$

satisfying equation (10), we can analyze whether there exist nontrivial contractions  $\mathfrak{g}' = \mathfrak{s} \oplus_R \mathfrak{r}'$  of  $\mathfrak{g}$  leaving the Levi subalgebra  $\mathfrak{s}$  invariant and such that we can find a virtual copy of the latter in  $\mathcal{U}(\mathfrak{g}')$ . One possible approach was outlined in [15], based on automorphisms of the type

$$\Phi_\varepsilon(X_i) = X_i, \quad \Phi_\varepsilon(Y_j) = \varepsilon^{n_j} Y_j, \quad n_i \in \mathbb{Z}. \quad (17)$$

On the transformed basis, the operators (16) have the form

$$X''_i = \varepsilon^{-(n_{i_1} + \dots + n_{i_{k-1}})} \alpha^{i_1 \dots i_{k-1}} X_i Y_{i_1} \dots Y_{i_{k-1}} + \varepsilon^{-(n_{j_1} + \dots + n_{j_k})} \beta_i^{j_1 \dots j_k} Y_{j_1} \dots Y_{j_k}.$$

We further define the maximal exponents of the contraction parameter  $\varepsilon$  as

$$M_0 = \max \{n_{i_1} + \dots + n_{i_{k-1}} \mid \alpha^{i_1 \dots i_{k-1}} \neq 0\}, \quad M_i = \max \{n_{i_1} + \dots + n_{i_k} \mid \beta_i^{i_1 \dots i_{k-1}} \neq 0\}.$$

It can be shown that a virtual copy of the Levi factor in the enveloping algebra of the contraction exists whenever the exponents all have the same value:

$$M_0 = M_1 = \cdots = M_n. \quad (18)$$

The existence of a solution is thus equivalent to the combinatorial problem of finding integers  $n_\ell$  ( $1 \leq \ell \leq \dim \mathfrak{r}$ ) in (17) such that the exponents in (18) have the same constant value. Assuming that the latter holds, then the operators

$$\widehat{X}_i'' = \lim_{\varepsilon \rightarrow 0} \varepsilon^{M_0} X_i''$$

define a virtual copy of  $\mathfrak{s}$  in the enveloping  $\mathcal{U}(\mathfrak{g}')$ . Although the contraction provides a nonequivalent realization in terms of operators, their order is always preserved.

### 5. Semidirect sums as contractions of simple Lie algebras

The analysis in [15] considered only semidirect sums and their contractions, leaving aside a second type of Lie algebras for which the existence problem of virtual copies can be analyzed, namely the contractions of semisimple Lie algebras. In this section we briefly outline how this analysis can be systematized using the branching rules of representations [31]. As an example of the procedure, we consider contractions of the symplectic Lie algebras  $\mathfrak{sp}(2N, \mathbb{R})$  to obtain a family of semidirect sums with symplectic Levi factor and a solvable nonnilpotent radical.

Suppose that  $\mathfrak{s}'$  is a semisimple subalgebra of a simple Lie algebra  $\mathfrak{s}$ . If  $\text{ad}(\mathfrak{s})$  denotes the adjoint representation of  $\mathfrak{s}$ , then its restriction to the subalgebra  $\mathfrak{s}'$  leads to the decomposition

$$\text{ad}(\mathfrak{s}) \downarrow \text{ad}(\mathfrak{s}') \oplus \Lambda,$$

where  $\Lambda$  is called the characteristic representation [31, 32]. In terms of commutators, this means that, given a basis  $\{X_1, \dots, X_m\}$  of  $\mathfrak{s}'$ , we can extend it to a basis  $\{X_1, \dots, X_m, Y_1, \dots, Y_{n-m}\}$  of  $\mathfrak{s}$  such that  $\{Y_1, \dots, Y_{n-m}\}$  spans the representation space of  $\Lambda$ . The commutators are generically given by

$$[\mathfrak{s}', \mathfrak{s}'] \subset \mathfrak{s}', \quad [\mathfrak{s}', \Lambda] \subset \Lambda, \quad [\Lambda, \Lambda] \subset \mathfrak{s}' + \Lambda = \mathfrak{s}.$$

If the semidirect sum  $\mathfrak{g} = \mathfrak{s}' \overrightarrow{\oplus}_{\Lambda} \mathfrak{r}$  arises as a contraction of  $\mathfrak{s}$ , where  $\mathfrak{r}$  is a solvable nonabelian Lie algebra, then there exist integers  $n_j$  with  $1 \leq j \leq \dim \Lambda$  such that family of automorphisms is given by

$$\Psi_\varepsilon(X_i) = X_i, \quad \Psi_\varepsilon(Y_j) = \varepsilon^{n_j} Y_j, \quad (19)$$

where at least for two indices  $i_0 \neq j_0$  we have  $n_{i_0} \neq n_{j_0}$ . Otherwise, if all the integers  $n_j$  are equal, then the contraction of  $\mathfrak{s}$  determined by the automorphisms  $\Psi_\varepsilon$  in (19) would lead to the inhomogeneous Lie algebra  $\mathfrak{s}' \overrightarrow{\oplus}_{\Lambda} \mathbb{R}^{\dim \Lambda}$  (see e.g. [33]), which has already been seen to not admit a virtual copy of  $\mathfrak{s}$  in the enveloping algebra. The structure of the radical  $\mathfrak{r}$  is thus heavily dependent on the characteristic representation  $\Lambda$ , providing an effective test to see whether a contraction of a semisimple algebra gives rise to a semidirect sum with the expected properties. Two of these restrictions, first proved in [34], are of special relevance in this context:

- (i) If the characteristic representation  $\Lambda$  is irreducible, then  $\mathfrak{r}$  is an Abelian Lie algebra.
- (ii) If the characteristic representation  $\Lambda$  does not contain the trivial representation  $\Gamma_0$ , then  $\mathfrak{r}$  is a nilpotent Lie algebra.



It follows that if the radical of  $\mathfrak{s}' \oplus_{\Lambda} \mathfrak{r}$  is solvable nonnilpotent, then  $\Lambda$  must be reducible and necessarily contain the trivial representation.<sup>4</sup> Another constraint is given by the multiplicity  $\text{mult}_{\Lambda} \Gamma_0$  of the trivial representation  $\Gamma_0$  in  $\Lambda$ . Indeed, if the inequality  $\text{mult}_{\Lambda} \Gamma_0 > \dim \mathfrak{s}'$  holds, then a virtual copy cannot be found, because in such cases the invariants of the Lie algebra  $\mathfrak{s}' \oplus_{\Lambda} \mathfrak{r}$  will not depend on the variables of  $\mathfrak{s}'$  [23, 35]. The analysis should therefore be restricted to the case where the constraints  $\text{mult}_{\Lambda} \Gamma_0 < \dim \mathfrak{s}'$  and  $\mathcal{N}(\mathfrak{r}) > 0$  are satisfied. It must be remarked, however, that these are necessary, but usually not sufficient conditions to guarantee that a virtual copy of the Levi factor  $\mathfrak{s}'$  in the enveloping algebra of  $\mathfrak{g} = \mathfrak{s}' \oplus_{\Lambda} \mathfrak{r}$  can be found, as the same characteristic representation  $\Lambda$  can be associated to nonequivalent radicals [33, 34]. This can be easily seen if we suppose that the decomposition of  $\Lambda$  into irreducible representations of  $\mathfrak{s}'$  is given by

$$\Lambda = \Lambda_1 + \cdots + \Lambda_s, \quad s > 1.$$

If we now define block-diagonal endomorphisms  $\Psi_{\varepsilon}$  of  $\mathfrak{s}$  by

$$\Psi_{\varepsilon}|_{\mathfrak{s}'} = \text{Id}_{\mathfrak{s}'}, \quad \Psi_{\varepsilon}|_{\Lambda_k} = \varepsilon^{n_k} \text{Id}_{\Lambda_k}, \quad 1 \leq k \leq s,$$

it follows at once that the action of  $\mathfrak{s}'$  on  $\Lambda$  is always the same, independently on the values of the vector  $(\varepsilon_1, \dots, \varepsilon_s)$ . However, whether the radical is Abelian, nilpotent or solvable essentially depends on the relations between the exponents  $\varepsilon_k$ . Indeed, if  $X \in \Lambda_i, Y \in \Lambda_j$  are two elements such that  $0 \neq [X, Y] \in \Lambda_k$ , then the commutator is preserved by contraction only if the equality  $\varepsilon_i + \varepsilon_j - \varepsilon_k = 0$  is satisfied.

In spite of this limitation of the branching rules of representations to characterize the possible radicals, the classification of (maximal) semisimple subalgebras (see e.g. [31, 36] and references therein) can be used to obtain a systematic overview of the contractions of a simple Lie algebra  $\mathfrak{s}$  onto semidirect sums. If the chosen subalgebra  $\mathfrak{s}'$  turns out to be not maximal in  $\mathfrak{s}$ , then we can always find a maximal subalgebra  $\mathfrak{s}''$  of  $\mathfrak{s}$  such that  $\mathfrak{s}' \subset \mathfrak{s}'' \subset \mathfrak{s}$  and consider the branching rule

$$\text{ad}(\mathfrak{s}) \downarrow \text{ad}(\mathfrak{s}') \oplus \Lambda \downarrow \text{ad}(\mathfrak{s}'') \oplus \Lambda'.$$

A suitable strategy is therefore to begin the analysis with the maximal subalgebras and then to proceed recursively, using the transitivity of branching rules [32].

Without entering into the details, we briefly illustrate how to carry out this process, using the exceptional (complex) Lie algebra  $F_4$ . The notations for representations and branching rules are adapted from [32] and [37]. The Lie algebra  $F_4$  admits the following maximal semisimple subalgebras:<sup>5</sup>

$$\mathfrak{sl}(2), \mathfrak{sl}(2) \times G_2, \mathfrak{so}(9), \mathfrak{sl}(2) \times \mathfrak{sp}(6), \mathfrak{sl}(3) \times \mathfrak{sl}(3),$$

where the last three are also maximal rank subalgebras. We inspect the corresponding branching rule for the adjoint representation  $[1, 0^3]$  of  $F_4$  for each of these subalgebras:

(i)  $\mathfrak{sl}(2)$ :

$$[1, 0^3] \downarrow (22) + (14) + (10) + (2).$$

As the characteristic representation  $\Lambda = (22) + (14) + (10)$  does not contain the trivial representation, any contraction preserving the subalgebra has a nilpotent radical. Such Lie algebras will generally have invariants that only depend on the variables of the radical [12].

<sup>4</sup> It is important to realize that the converse does not generally hold, as the semidirect sums of simple Lie algebras with the Heisenberg-Weyl algebra  $\mathfrak{h}_N$  show.

<sup>5</sup> The algebra  $A_1 \times A_3$  listed in [32] is not maximal, as it is contained in  $B_4$  (see [38]).

(ii)  $\mathfrak{sl}(2) \times G_2$ :

$$[1, 0^3] \downarrow (4) \otimes [0, 1] + (0) \otimes [1, 0] + (2) \otimes [0, 0].$$

The only contraction having a Levi factor isomorphic to  $\mathfrak{sl}(2) \times G_2$  is necessarily an inhomogeneous Lie algebra. Reducing further with respect to the subalgebra  $G_2 \subset \mathfrak{sl}(2) \times G_2$  we obtain the branching rule

$$[1, 0^3] \downarrow 5[0, 1] + [1, 0] + 3[0, 0].$$

In this case, there exists at least one contraction onto a semidirect sum  $\mathfrak{g}$  with a solvable nonnilpotent radical  $\mathfrak{r}$ , and a virtual copy of  $G_2$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  can be found.

(iii)  $\mathfrak{so}(9)$ :

$$[1, 0^3] \downarrow [0, 1, 0^2] + [0^3, 1].$$

Here  $\Lambda$  is a spinor representation, and thus the contraction is an inhomogeneous Lie algebra. In order to get a contraction onto a semidirect sum with a nonabelian radical, we must further reduce with respect to a semisimple subalgebra  $\mathfrak{s}' \subset \mathfrak{so}(9)$ , from which  $\mathfrak{so}(9)$  possesses five, three of them also having maximal rank [37].

(iv)  $\mathfrak{sl}(2) \times \mathfrak{sp}(6)$ :

$$[1, 0^3] \downarrow (1) \otimes [0^2, 1] + (0) \otimes [2, 0^2] + (2) \otimes [0^3].$$

As the subalgebra has maximal rank, the contraction is inhomogeneous. Considering the further embedding  $\mathfrak{sp}(6) \subset \mathfrak{sl}(2) \times \mathfrak{sp}(6)$  leads to the branching rule

$$[1, 0^3] \downarrow 2[0^2, 1] + [2, 0^2] + 3 \otimes [0^3].$$

Here the trivial representation has multiplicity three, and a semidirect sum with a solvable nonnilpotent radical can be found by contraction. It further admits a virtual copy of  $\mathfrak{sp}(6)$  in its enveloping algebra.

(v)  $\mathfrak{sl}(3) \times \mathfrak{sl}(3)$ :

$$[1, 0^3] \downarrow (0, 2) \otimes (1, 0) + (2, 0) \otimes (0, 1) + (0^2) \otimes (1, 1) + (1, 1) \otimes (0^2). \quad (20)$$

Again the semidirect sum with Levi factor  $\mathfrak{sl}(3) \times \mathfrak{sl}(3)$  is necessarily inhomogeneous. From this decomposition we observe that both copies of  $\mathfrak{sl}(3)$  are not equivalent, so that we have to consider some further reduction separately for each copy. The Lie algebra  $A_2$  possesses two inequivalent  $A_1$  subalgebras, one being regular with embedding index  $j_f = 1$ , while the other is an  $S$ -subalgebra with index  $j_f = 4$ , that will be denoted by  $A_1$  and  $A_1^{(4)}$ , respectively. We must thus inspect four subcases that follow from equation (20):

(a) First reduction with respect to the  $A_1$ :

$$[1, 0^3] \downarrow \{(2) + (1) + (0)\} \otimes \{(1, 0) + (0, 1)\} + (0) \otimes (1, 1) + \{(2) + (1)^2 + (0)\} \otimes (0^2).$$

(b) First reduction with respect to the  $A_1^{(4)}$ :

$$[1, 0^3] \downarrow \{(4) + (0)\} \otimes \{(1, 0) + (0, 1)\} + (0) \otimes (1, 1) + \{(4) + (2)\} \otimes (0^2).$$

(c) Second reduction with respect to the  $A_1$ :

$$[1, 0^3] \downarrow \{(2, 0) + (0, 2)\} \otimes \{(1) + (0)\} + (0^2) \otimes \{(2) + (1)^2 + (0)\} + (1, 1) \otimes (0).$$

(d) Second reduction with respect to the  $A_1^{(4)}$ :

$$[1, 0^3] \downarrow \{(2, 0) + (0, 2)\} \otimes (2) + (0^2) \otimes \{(4) + (2)\} + (1, 1) \otimes (0).$$

The four cases have in common that the characteristic representation of the subalgebra does not contain the trivial representation, so that any contraction preserving the subalgebra would have a nilpotent radical. A second reduction with respect to the subalgebras

$$A_2 \subset A_1 \times A_2, \quad A_2 \subset A_1^{(4)} \times A_2$$

can thus be considered. For the embedding  $A_2 \subset A_1 \times A_2$ , the cases (a) and (b) lead to the branching rule

$$[1, 0^3] \downarrow 6 \{(1, 0) + (0, 1)\} + (1, 1) + 8(0^2),$$

while for the subalgebra  $A_1^{(4)} \times A_2$ , the cases (c) and (d) lead to the branching rule

$$[1, 0^3] \downarrow 3 \{(2, 0) + (0, 2)\} + 8(0^2) + (1, 1).$$

In both cases, the trivial representation has multiplicity eight, and contractions of  $F_4$  onto semidirect sums with a solvable (nonnilpotent) radical and a Levi factor isomorphic to either  $A_1 \times A_2$  and  $A_1^{(4)} \times A_2$ , respectively, can be constructed.

### 5.1. Some contractions of $\mathfrak{sp}(2N, \mathbb{R})$

In this paragraph we illustrate the explicit construction of a semidirect sum and the virtual copy of its Levi factor, as well as the resulting Casimir operators. To this extent, we consider the case of the symplectic Lie algebras  $\mathfrak{sp}(2N, \mathbb{R})$ . These Lie algebras can be conveniently realized in terms of creation and annihilation operators [14]: Let  $a_i, a_j^\dagger$  ( $i, j = 1 \dots, N$ ) be the linear operators satisfying the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.$$

The operators  $\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\}$  span the real symplectic Lie algebra  $\mathfrak{sp}(2N, \mathbb{R})$ . For computational purposes, it is convenient to consider the basis

$$X_{i,j} = a_i^\dagger a_j, \quad X_{-i,j} = a_i^\dagger a_j^\dagger, \quad X_{i,-j} = a_i a_j, \quad (1 \leq i, j \leq N). \quad (21)$$

As these generators satisfy the condition  $X_{i,j} + \eta_i \eta_j X_{-j,-i} = 0$ , where  $\eta_i = \text{sign}(i)$ , the commutators can be comprised in a unique expression:

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} + \eta_i \eta_j \delta_{j,-l} X_{k,-i} - \eta_i \eta_j \delta_{i,-k} X_{-j,l}. \quad (22)$$

Over this basis, the Casimir operators  $C_{2k}$  of  $\mathfrak{sp}(2N, \mathbb{R})$  are obtained in analytic form from the coefficients of the characteristic polynomial

$$\det(M - T \text{Id}_{2N}) = T^{2N} + C_{2k} T^{2N-2k}, \quad (23)$$

with the matrix  $M$  defined by

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,N} & -x_{-1,1} & \dots & -x_{-1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{N,1} & \dots & x_{N,N} & -x_{-1,N} & \dots & -x_{-N,N} \\ x_{1,-1} & \dots & x_{1,-N} & -x_{1,1} & \dots & -x_{N,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{1,-N} & \dots & x_{N,-N} & -x_{1,N} & \dots & -x_{N,N} \end{pmatrix}. \quad (24)$$

We now analyze whether there exists contractions of  $\mathfrak{sp}(2N, \mathbb{R})$  onto a semidirect sum the Levi subalgebra of which is isomorphic to  $\mathfrak{sp}(2N-2, \mathbb{R})$ . As can be easily verified, this subalgebra is regular but not maximal [37]. It is contained in the maximal rank subalgebra  $\mathfrak{sp}(2(N-1), \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ , and the adjoint representation  $[2, 0^{N-1}]$  of  $\mathfrak{sp}(2N, \mathbb{R})$  decomposes as follows:

$$[2, 0^{N-1}] \downarrow [2, 0^{N-2}] \otimes (0) + [0^{N-1}] \otimes (2) + [1, 0^{N-2}] \otimes (1), \quad (25)$$

where  $(\ell)$  denotes the  $\ell+1$ -dimensional irreducible representation of  $\mathfrak{sl}(2, \mathbb{R})$  with highest weight  $\ell$ . While the two first terms correspond to the adjoint representation of  $\mathfrak{sp}(2(N-1), \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ , the last term is an irreducible representation, and thus a contraction preserving the subalgebra would lead to an inhomogeneous Lie algebra. In order to obtain a semidirect sum with Levi factor isomorphic to  $\mathfrak{sp}(2(N-1), \mathbb{R})$ , we consider the following family of automorphisms defined with respect to the basis (21):

$$\begin{aligned} \Psi_\varepsilon(X_{i,j}) &= X_{i,j}, & \Psi_\varepsilon(X_{N,N}) &= X_{N,N}, & \Psi_\varepsilon(X_{-N,N}) &= \varepsilon^2 X_{-N,N}, & \Psi_\varepsilon(X_{N,-N}) &= \varepsilon^2 X_{N,-N}, \\ \Psi_\varepsilon(X_{i,N}) &= \varepsilon X_{i,N}, & \Psi_\varepsilon(X_{N,i}) &= \varepsilon X_{N,i}, & \Psi_\varepsilon(X_{-i,N}) &= \varepsilon X_{-i,N}, & \Psi_\varepsilon(X_{i,-N}) &= \varepsilon X_{i,-N}. \end{aligned} \quad (26)$$

It is immediate to see that, in the contraction, the symplectic Lie algebra of rank  $N-1$  is preserved,<sup>6</sup> and the characteristic representation  $\Lambda$  for the embedding into  $\mathfrak{sp}(2N, \mathbb{R})$  is easily obtained from (25):

$$[2, 0^{N-1}] \downarrow [2, 0^{N-2}] \otimes + 3[0^{N-1}] + 2[1, 0^{N-2}] = \text{ad}(\mathfrak{sp}(2(N-1), \mathbb{R})) \oplus \Lambda.$$

The two copies of the  $2(N-1)$ -dimensional representation  $[1, 0^{N-2}]$  are spanned by the sets  $\{X_{i,N}, X_{-i,N}\}$  and  $\{X_{N,i}, X_{i,-N}\}$ , respectively, while the trivial representation is spanned by the elements  $X_{N,N}, X_{-N,N}$  and  $X_{N,-N}$ . Using equation (22), it is routine to verify that the only nontrivial brackets of the radical are given for the indices  $1 \leq i \leq N-1$  by

$$\begin{aligned} [X_{i,N}, X_{-i,N}] &= -X_{-N,N}, & [X_{N,i}, X_{i,-N}] &= X_{N,-N}, & [X_{N,N}, X_{\pm i,N}] &= -X_{\pm i,N}, \\ [X_{N,N}, X_{N,i}] &= X_{N,i}, & [X_{N,N}, X_{i,-N}] &= X_{i,-N}, & [X_{N,N}, X_{N,-N}] &= 2X_{N,-N}, \\ [X_{N,N}, X_{-N,N}] &= -2X_{-N,N}. \end{aligned}$$

It follows that, in the limit  $\varepsilon \rightarrow 0$ , the automorphisms (26) define a contraction of  $\mathfrak{sp}(2N, \mathbb{R})$  onto the semidirect sum  $\mathfrak{G}_N = \mathfrak{sp}(2(N-1), \mathbb{R}) \overrightarrow{\oplus}_{\Lambda} \mathfrak{r}_N$  possessing a solvable nonnilpotent radical  $\mathfrak{r}_N$ . A short computation using the Maurer-Cartan equations (5) of  $\mathfrak{G}_N$  and formula (6) shows that the identities  $\mathcal{N}(\mathfrak{G}_N) = N$  and  $\mathcal{N}(\mathfrak{r}_N) = 1$  are satisfied. The only Casimir operator of the radical is quadratic in the generators and given explicitly by  $f = X_{-N,N}X_{N,-N}$ . The remaining  $N-1$  invariants of  $\mathfrak{G}_N$  are constructed via the virtual copy method, using the matrix (24) for appropriate operators. As  $f$  is quadratic, the polynomials  $P_i$  in (10) must be cubic in the generators of  $\mathfrak{r}_N$ . For indices  $1 \leq i, j \leq N-1$  we define the operators

$$\begin{aligned} X'_{i,j} &= fX_{i,j} - \frac{1}{2}(X_{N,-N}(X_{i,N}X_{-j,N} + X_{-j,N}X_{i,N}) + X_{-N,N}(X_{i,-N}X_{N,j} + X_{N,j}X_{i,-N})), \\ X'_{-i,j} &= fX_{-i,j} - X_{N,-N}X_{-j,N}X_{-i,N} + X_{-N,N}X_{N,i}X_{N,j}, \\ X'_{i,-j} &= fX_{i,-j} + X_{N,-N}X_{j,N}X_{i,N} - X_{-N,N}X_{i,-N}X_{j,-N}. \end{aligned} \quad (27)$$

A quite long but routine computation shows that the conditions (11) and (12) are fulfilled, so that the operators in (27) generate a virtual copy of the symplectic algebra  $\mathfrak{sp}(2(N-1), \mathbb{R})$  in the enveloping algebra of  $\mathfrak{G}_N$ . Replacing now the variables  $x_{i,j}$  in (24) by the cubic polynomials  $x'_{i,j}$ , we obtain  $N-1$  independent invariant functions  $C'_{2k}$  of  $\mathfrak{G}_N$ . Observe that these functions

<sup>6</sup> It actually preserves the reductive subalgebra  $\mathfrak{sp}(2N, \mathbb{R}) \oplus \mathbb{R}$ .

have degree  $6k$  for any  $1 \leq k \leq N - 1$ . In this case, as can be seen from the operators in (27), the term  $f$  always arises as a common factor, so that the functions can be factorized as

$$C'_{2k} = f^{2k-1} J_{2k+2}, \quad 1 \leq k \leq N - 1.$$

The polynomials  $J_{2k+2}$  have degree  $2k + 2$  in the generators and do not have a further common factor, so that they can be chosen as primitive invariants. Adding  $f$  to the list and symmetrizing the functions using the map (4), we obtain a complete set  $\{f, J_4, J_6, \dots, J_{2N}\}$  of Casimir operators for the semidirect sum  $\mathfrak{G}_N$ . We further remark that, as  $\mathfrak{G}_N$  is obtained as a contraction of  $\mathfrak{sp}(2N, \mathbb{R})$ , the invariants could have also been obtained by a limiting process, applying the method described in [30].<sup>7</sup> We further observe that, starting from the same maximal subalgebra, we could have considered the contraction of  $\mathfrak{sp}(2N, \mathbb{R})$  onto the semidirect sum with Levi factor  $\mathfrak{sl}(2, \mathbb{R})$ . In this case, however, the multiplicity of the trivial representation exceeds the dimension of the latter Lie algebra, implying that the invariants depend only on generators of the radical, and hence no virtual copy of  $\mathfrak{sl}(2, \mathbb{R})$  can be constructed.

Further examples of this type, using the realization (21), can be obtained considering the maximal subalgebras

$$\mathfrak{sp}(2k, \mathbb{R}) \oplus \mathfrak{sp}(2N - 2k, \mathbb{R}) \subset \mathfrak{sp}(2N, \mathbb{R}), \quad 1 \leq k \leq N.$$

According to this (regular) embedding, the adjoint representation decomposes as:

$$[2, 0^{N-1}] \downarrow [2, 0^{k-1}] \otimes [0^{2N-2k}] + [0^{2k}] \otimes [2, 0^{2N-2k-1}] + [1, 0^{2k-1}] \otimes [1, 0^{2N-2k-1}].$$

As the subalgebra is of maximal rank for any value of  $k$ , the contraction would lead to an inhomogeneous Lie algebra. Using that  $k \leq N$ , we can further restrict to the subalgebra  $\mathfrak{sp}(2N - 2k, \mathbb{R}) \subset \mathfrak{sp}(2k, \mathbb{R}) \oplus \mathfrak{sp}(2N - 2k, \mathbb{R})$ , where we can always ensure that the inequality  $\text{mult}_\Lambda \Gamma_0 < \dim \mathfrak{sp}(2N - 2k, \mathbb{R})$  holds. In these conditions, for each value of  $k$  we can find contractions isomorphic to a semidirect sum  $\mathfrak{g}$  with Levi factor  $\mathfrak{sp}(2N - 2k, \mathbb{R})$  and such that a virtual copy of the latter in the enveloping algebra of  $\mathfrak{g}$  exists.

## 6. Concluding remarks

We have briefly reviewed the main properties of the method of virtual copies originally proposed in [15], a procedure that allows the explicit construction of the invariants of semidirect sums  $\mathfrak{s} \oplus_{\Lambda} \mathfrak{t}$  in terms of the Casimir operators of the Levi subalgebra  $\mathfrak{s}$ . This method enlarges naturally the procedure valid for the semidirect sums of simple Lie algebras with the Heisenberg algebras  $\mathfrak{h}_N$ , a type of algebras that has turned out to be of interest in physical applications [13, 17, 25]. Combining the contractions of Lie algebras with the classification problem of (maximal) semisimple subalgebras [36], we have presented a possible approach to obtain all the contractions of simple Lie algebras that are isomorphic to a semidirect sum and possess the property of having a virtual copy. In this context, the multiplicity of the trivial representation in the characteristic representation of the embedding is shown to play a central role. Although many important types of semisimple sums  $\mathfrak{g}$  that admit a virtual copy of their Levi factor in the enveloping algebra do not arise as a contraction of a semisimple Lie algebra (as a matter of fact, the Schrödinger or Hamilton algebras), the systematic analysis of the contractions of simple Lie algebras allows us to deduce some necessary conditions for the existence of such virtual copies, further showing a relevant connection between the branching rules of representations and the

<sup>7</sup> This “contraction” of invariants would also be valid for the inhomogeneous algebras, even in the absence of a virtual copy of the Levi factor [23].

structural properties of the radicals in semidirect sums of Lie algebras. Such a connection may imply that the method of virtual copies, besides its application to the invariant problem of Lie algebras that are not semisimple, could also be of potential interest in the realization of Lie algebras in terms of higher-order nonlinear differential operators. How to adequately formulate such a correspondence between virtual copies and operators will hopefully be reported in a future paper.

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