

INTEGRATION IN ČECH THEORIES AND A BOUND ON ENTROPY

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ABSTRACT. The evaluation of Alexander–Spanier cochains over formal simplices in a topological space leads to a notion of integration of Alexander–Spanier cohomology classes over Čech homology classes. The integral defines an explicit and non-degenerate pairing between the Alexander–Spanier cohomology and the Čech homology. Instead of working on the limits that define both groups, most of the discussion is carried out “at scale \mathcal{U} ”, for an open covering \mathcal{U} . As an application, we generalize a result of Manning to arbitrary compact spaces X : we prove that the topological entropy of $f: X \rightarrow X$ is bounded from below by the logarithm of the spectral radius of the map induced in the first Čech cohomology group.

1. INTRODUCTION

The topological entropy [1, 12] of a dynamical system is a measure of its complexity. Positive entropy is interpreted as the existence of chaotic dynamics. The precise computation of the entropy is very complicated except for some specific examples, but often in applications it is sufficient to obtain a positive lower bound for it.

The action of a map $f: X \rightarrow X$ on the homology or cohomology groups of X is a good indicator of how involved the dynamics is. The entropy conjecture states that for C^1 maps on compact manifolds the entropy is bounded from below by the logarithm of the spectral radius of the induced map $f_*: H_*(X) \rightarrow H_*(X)$. The conjecture has been proved in several instances, for example in the C^∞ case [27] or when X is a torus [18] or a nilmanifold [17], but remains open in general. A classical partial result is due to Manning [16]: the conjecture holds, even for continuous maps, if only the spectral radius of f_* over the first homology group is considered.

The statement in [16] holds for spaces X more general manifolds, but they must still satisfy some nice local features. Manning remarks at the end of his paper: “*Although Čech cohomology theory would seem most appropriate for relating cohomology eigenvalues to topological entropy as defined in [1] by refinements of open covers we have been unable to exploit this approach.*” Motivated by this, our Theorems 25 and 26 generalize the theorem of Manning to arbitrary compact spaces in terms of Čech cohomology. When X is locally connected we recover the sharp bound $h(f) \geq \log|\lambda|$ for every eigenvalue λ of the map f_* induced in the first Čech cohomology group while in the completely general, non–locally connected case, we

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obtain a weaker inequality: $h(f) \geq (\log|\lambda|)/d$, where $d \in \mathbb{Z}^+$ is the degree of λ as an algebraic number. This bound is still positive when $|\lambda| > 1$, as Manning’s original bound.

In dynamical systems Čech theories are sometimes more useful than singular cohomology and homology theories because they are better suited (due to the continuity axiom they satisfy) to describe pathological spaces with bad local topology such as strange attractors. A standard reference for these theories is [5]. The traditional approach to Čech theories is based on simplicial complexes that are extrinsic to the space under inspection, such as the nerve of an open covering in the original description by Čech [23] or the Vietoris complex associated to it [25]. By contrast, in this article we work with intrinsic descriptions both for cohomology and homology. On one hand we make use of Alexander-Spanier cohomology [2, 21], a theory whose q -cochains are simply functions from $(q + 1)$ -tuples of points of X that satisfy an algebraic relation that codifies the coboundary operator. This cohomology theory coincides with Čech cohomology theory in paracompact spaces [11] but does not have a simple dual construction of chains and homology groups. Regarding homology, we take the approach of Vietoris. For a fixed open covering \mathcal{U} of X we work with \mathcal{U} -small chains and define a homology group at scale \mathcal{U} that disregards the local structure of X contained in any element of \mathcal{U} and is a reinterpretation of (and, in particular, isomorphic to) the homology group of the Vietoris complex $V(\mathcal{U})$. The limit of these groups as \mathcal{U} ranges over the open coverings of X is called the Vietoris homology group of X (the original formulation considered open coverings by balls of radius ϵ in a metric space [10]). In paracompact spaces it is isomorphic to the Čech homology group [15, 11].

One of the main ingredients in the generalization of Manning’s theorem is a bilinear pairing between Čech homology and cohomology that is interesting on its own. In Section 6 we define a notion of integration of Alexander-Spanier cohomology classes over Čech homology classes reminiscent of the integration of differential forms over simplices. The definition is possible thanks to the intrinsic descriptions mentioned above and is based on the evaluation of the Alexander-Spanier cochains over sufficiently small simplices, which we call “formal” because their information is reduced to its set of vertices. Integration defines an explicit and non-degenerate bilinear pairing between Čech cohomology and homology on compact topological spaces in a similar fashion to what integration of differential forms does between de Rham’s cohomology and singular homology on differentiable manifolds. It is worth noticing that the concept of formal simplex that we use was already present in the literature from the early developments of algebraic topology under the name of abstract or symbolic simplex, see for example [2]. Also, it was known that Alexander-Spanier cohomology and Vietoris homology theories could be constructed from the same system of complexes [11]. Our contribution in this framework is to establish an elementary and explicit pairing between these theories. As a side note, there are related notions of integration in the literature, see [3] or [6, p. 128].

Alexander-Spanier cocycles can be stratified in terms of the open covering of X where the cocycle condition holds. In view of the definition of the integral, an Alexander-Spanier cocycle whose coboundary identically vanishes at each element of \mathcal{U} (which we gather in a group denoted $H_{\mathcal{U}}^*(X)$) can be integrated over \mathcal{U} -small chains. In Section 7 we prove that the integral defines a nondegenerate pairing at scale \mathcal{U} if coefficients are taken in a field. The article studies these cohomology and homology groups at scale \mathcal{U} for a fixed \mathcal{U} . They lead to straightforward proofs of the results in [7], where Čech homology was obtained as a

limit of homology groups based on ϵ -continuous simplices (see Section 3), and give a clear interpretation of a result of Keesee [13] that states that any Alexander-Spanier cohomology class has a representative that takes a finite number of different values (see Proposition 11).

Sections 2 and 3 contain background material on Čech theories, notably the definition of Alexander-Spanier cohomology and the homology groups at scale \mathcal{U} . Some results about the first homology group and the cohomology groups are presented in Sections 4 and 5. The integral is defined in Section 6, and we prove its nondegeneracy in the ensuing section. The results on entropy and their proofs build on all the preliminary work and are presented in the last part of the article. We also provide examples that show the need to use eigenvalues in cohomology instead of homology to bound the entropy. The non-locally connected case requires some technical work that is the content of Section 9.

2. BACKGROUND ON ČECH HOMOLOGY AND COHOMOLOGY

2.1. Classical definition. Let X be any topological space. Consider an open covering \mathcal{U} of X . There are two classical simplicial complexes associated to \mathcal{U} . One is its nerve (or Čech complex [23]) $N(\mathcal{U})$, which is defined as follows:

- (i) It has a vertex for each member of \mathcal{U} .
- (ii) A (finite) collection of vertices spans a simplex if and only if the corresponding members of \mathcal{U} have a nonempty intersection.

Another one, which will be more useful for us, is the Vietoris complex $V(\mathcal{U})$ of \mathcal{U} . Its original conception [25] and later and recent developments (which have shifted their denomination to Vietoris-Rips complex) have been mainly restricted to metric spaces but here we present a definition in terms of an open covering:

- (i) Its vertices are the points of X .
- (ii) A collection of vertices x_0, \dots, x_q spans a simplex in $V(\mathcal{U})$ if there exists $U \in \mathcal{U}$ that contains all of them.

These two complexes are, in the sense explained by Dowker [4], “dual” to each other and, in particular, have isomorphic homology and cohomology groups.

Refining the open covering \mathcal{U} produces complexes that approximate the space X ever better, and in the limit the homology and cohomology groups of $N(\mathcal{U})$ or $V(\mathcal{U})$ constitute algebraic topological invariants of X . Let us describe this more precisely. Given a refinement \mathcal{V} of \mathcal{U} , there are simplicial maps $N(\mathcal{V}) \rightarrow N(\mathcal{U})$ and $V(\mathcal{V}) \rightarrow V(\mathcal{U})$. The latter is an inclusion while the former is not uniquely defined, because $V \in \mathcal{V}$ may be contained in several $U \in \mathcal{U}$, but all the choices give rise to contiguous maps and, in particular, are homotopic [22, Section 3.5]. These bonding maps provide the family of all (Čech or Vietoris) complexes that arise from open coverings \mathcal{U} of X the structure of an inverse system, denoted by $\{N(\mathcal{U})\}$ and $\{V(\mathcal{U})\}$, respectively. Applying the homology functor we obtain two inverse systems of homology groups. Their inverse (or projective) limits, referred to as Čech and Vietoris homology groups, are evidently isomorphic by the result of Dowker, although this fact was already known by that time [15, VII (26.1)]. The construction for cohomology is analogous, but let us remark that there is no Vietoris cohomology theory formulated as such. Since the cohomology functor is contravariant we obtain directed systems of groups that are pro-isomorphic and, in particular, have isomorphic (direct) limits. The groups defined in this way

are called Čech homology and Čech cohomology groups and denoted by $\check{H}_*(X)$ and $\check{H}^*(X)$, respectively.

The coefficient group becomes relevant in some parts of the article. Unless explicitly stated, we assume it is an R -module G and usually omit it from the notation.

2.2. Alexander–Spanier cohomology. As mentioned in the Introduction, it is also possible to describe the cohomology groups $\check{H}^q(X)$ in an intrinsic manner using the language of Alexander–Spanier cohomology. We first recall its definition from [22] (see also [21]). For $q \geq 0$ a q -cochain is a (not necessarily continuous) map $\xi: X^{q+1} \rightarrow G$. The set $C^q(X)$ of q -cochains forms an R -module. The coboundary of a cochain is defined in the usual fashion: $\delta\xi(x_0, \dots, x_{q+1}) := \sum_{j=0}^q (-1)^j \xi(x_0, \dots, \hat{x}_j, \dots, x_{q+1})$. A cochain is said to be locally zero if there exists an open covering \mathcal{U} of X such that $\xi(x_0, \dots, x_q) = 0$ whenever x_0, \dots, x_q all belong to the same element of \mathcal{U} . The subset of locally zero cochains is a submodule of $C^q(X)$ and we denote by $\overline{C}^q(X)$ the quotient module of all cochains modulo the locally zero ones. The coboundary of a locally zero cochain is again locally zero, and so δ descends to a coboundary operator on $\overline{C}^q(X)$ which we denote again with the same symbol. The cohomology of the complex $\{\overline{C}^q(X), \delta\}$ is, by definition, the Alexander–Spanier cohomology of X (with coefficients in G). It is isomorphic to the Čech cohomology of X when X is paracompact ([11] and [22, Corollary 8, p. 334]) or, in general, when they are defined using the same family of coverings [4].

Now suppose an open covering \mathcal{U} of X is fixed. \mathcal{U} shall be thought of as the “scale” below which the structure of X is disregarded. With minor modifications (already mentioned by Spanier [21, Appendix A]) the above definitions can be adapted to define another cohomology group as follows. Say that a q -cochain ξ is \mathcal{U} -locally zero if $\xi(x_0, \dots, x_q) = 0$ whenever x_0, \dots, x_q belong to some member of \mathcal{U} . Heuristically, these cochains ignore all the structure of X below scale \mathcal{U} . It is straightforward to check that the coboundary of a \mathcal{U} -locally zero cochain is again \mathcal{U} -locally zero. Thus we may quotient $C^q(X)$ by the submodule of \mathcal{U} -locally zero cochains to obtain a module $\overline{C}_{\mathcal{U}}^q(X)$. The coboundary operator descends to a coboundary operator on this module yielding a cochain complex $\{\overline{C}_{\mathcal{U}}^q(X), \delta\}$. We denote the cohomology of this complex by $H_{\mathcal{U}}^q(X)$. This is, therefore, very similar to the definition of the Alexander–Spanier cohomology of X but using a single covering \mathcal{U} instead of letting \mathcal{U} run over all open coverings. Notice that, by definition, the class $\bar{\xi} \in \overline{C}_{\mathcal{U}}^q(X)$ of a cochain ξ is a cocycle if and only if $\delta\xi$ vanishes over every member of \mathcal{U} ; similarly, $\bar{\xi} = \bar{\eta}$ means that ξ and η agree when evaluated on tuples that are contained in some member of \mathcal{U} .

If \mathcal{V} refines \mathcal{U} then every \mathcal{U} -locally zero cochain is also \mathcal{V} -locally zero and so there is a natural homomorphism $\overline{C}_{\mathcal{U}}^q(X) \rightarrow \overline{C}_{\mathcal{V}}^q(X)$. This map commutes with the coboundary operator and hence induces a homomorphism $\pi_{\mathcal{U}\mathcal{V}}: H_{\mathcal{U}}^q(X) \rightarrow H_{\mathcal{V}}^q(X)$. There is, then, a direct system whose terms are $H_{\mathcal{U}}^q(X)$ and whose bonding maps are the maps $\pi_{\mathcal{U}\mathcal{V}}$.

Proposition 1. *The direct limit of $\{H_{\mathcal{U}}^q(X); \pi_{\mathcal{U}\mathcal{V}}\}$ is precisely the Alexander–Spanier cohomology of X . Therefore, under general hypothesis (described above) it is also isomorphic to the Čech cohomology of X , $\check{H}^q(X)$.*

Proof. It follows directly from its definition that $\overline{C}^q(X)$ can be identified with the direct limit of $\{\overline{C}_{\mathcal{U}}^q(X)\}$. Then the result owes to the fact that the homology functor commutes with direct limits. \square

3. AN ALTERNATIVE DESCRIPTION OF ČECH HOMOLOGY

In this section we perform a quick reformulation of the Vietoris homology theory. We introduce the notion of a formal simplex small with respect to an open covering \mathcal{U} and construct the homology groups at scale \mathcal{U} , $H_*^{\mathcal{U}}(X)$. These are isomorphic to $H_*(V(\mathcal{U}))$. As an application we recover easily the results of [7] on ϵ -homology groups.

A formal q -simplex in X is just an ordered collection $\sigma = (x_0 \dots x_q)$ of $(q+1)$ points in X . We call the x_i the vertices of σ and say that σ is \mathcal{U} -small if all its vertices are contained in some element of \mathcal{U} . (A formal q -simplex is just a point in X^{q+1} , but it is best to picture it as a subset of X).

A formal q -chain in X is a finite linear combination $c = \sum_i k_i \sigma_i$ of formal q -simplices σ_i with coefficients $k_i \in G$. We shall say that c is \mathcal{U} -small if all the σ_i are \mathcal{U} -small and denote by $S_q^{\mathcal{U}}(X)$ the set of all \mathcal{U} -small formal q -chains. The boundary of a formal simplex $\sigma = (x_0 \dots x_q)$ is defined in the usual way to be the formal chain $\partial\sigma = \sum_{j=0}^q (-1)^j (x_0 \dots \widehat{x}_j \dots x_q)$, where $(x_0 \dots \widehat{x}_j \dots x_q)$ is the $(q-1)$ -formal simplex determined by the ordered collection x_0, \dots, x_q from which x_j has been removed. The boundary of a \mathcal{U} -small formal simplex is clearly \mathcal{U} -small again, and so there is a chain complex $\{S_q^{\mathcal{U}}(X), \partial\}$. We shall denote its homology by $H_q^{\mathcal{U}}(X; G)$ and refer to it as the homology of X at scale \mathcal{U} .

Note that if we disregard the ordering of the vertices that define a formal simplex and discard simplices with repeated vertices we obtain the geometric simplices from the Vietoris complex $V(\mathcal{U})$. Otherwise stated, formal simplices are just ordered simplices of $V(\mathcal{U})$. Hence there is a map from $S_q^{\mathcal{U}}(X)$ to the simplicial chains of $V(\mathcal{U})$ defined by sending a formal simplex $(x_0 \dots x_q)$ to the same oriented simplex of $V(\mathcal{U})$ if all the vertices are different and to zero otherwise. This map induces isomorphisms in homology (see for example [19, Theorem 13.6, p. 77]) and therefore $H_q(V(\mathcal{U}); G)$ can be identified with $H_q^{\mathcal{U}}(X; G)$. Furthermore, if \mathcal{V} refines \mathcal{U} , the previous isomorphism conjugates the bonding maps $H_q(V(\mathcal{V})) \rightarrow H_q(V(\mathcal{U}))$ and $H_q^{\mathcal{V}}(X) \rightarrow H_q^{\mathcal{U}}(X)$. Therefore, the limit of the inverse system of groups $\{H_q^{\mathcal{U}}(X)\}$ is isomorphic to the Vietoris q -th homology group, which is in turn isomorphic to the Čech homology group $\check{H}_q(X)$. Throughout this article we directly refer to the limit of the homology groups at scale \mathcal{U} as the Čech homology group.

An element $\gamma \in \check{H}_q(X)$ consists of a family $(\gamma^{\mathcal{U}})_{\mathcal{U}}$ of homology classes, where each $\gamma^{\mathcal{U}} \in H_q^{\mathcal{U}}(X)$. In turn each $\gamma^{\mathcal{U}}$ is represented by a (nonunique) chain c that is \mathcal{U} -small. Somewhat abusing language, we shall call such a chain c a \mathcal{U} -small representative of γ .

Intuitively $H_q^{\mathcal{U}}(X)$ ignores all the structure of X at scales smaller than \mathcal{U} . This idea is expressed in the following mathematical statement:

Proposition 2. *Let $c = \sum_{i=0}^n k_i \sigma_i$ be a q -cycle with $q \geq 1$. Assume that the vertices of all the σ_i are contained in a single element $U \in \mathcal{U}$. Then c is nullhomologous in $H_q^{\mathcal{U}}(X)$.*

Proof. Pick an arbitrary point $a \in U$. For every formal simplex $\sigma = (x_0 \ x_1 \ \dots \ x_p)$ whose vertices lie in U consider $C_a(\sigma) = (a \ x_0 \ x_1 \ \dots \ x_p)$, a sort of formal cone of σ over a which is still contained in U . This operation on formal simplices satisfies

$$\partial C_a(\sigma) + C_a(\partial\sigma) = \sigma$$

and the identity holds for chains as well if we extend C_a linearly. If we apply the identity to a cycle c , we obtain that $c = \partial C_a(c)$ is a boundary and the conclusion follows. \square

For later reference we record here the following remark which is not obvious from the definition of $H_*^{\mathcal{U}}(X)$:

Remark 3. If the covering \mathcal{U} is finite, then $H_*^{\mathcal{U}}(X)$ is finitely generated.

Proof. Since \mathcal{U} is finite its nerve $N(\mathcal{U})$ is a finite simplicial complex and therefore $H_*(N(\mathcal{U}))$ is finitely generated too. By the result of Dowker [4] mentioned earlier this is isomorphic to $H_*(V(\mathcal{U}))$, which in turn is isomorphic to $H_*^{\mathcal{U}}(X)$. \square

Compared with singular simplices, formal simplices carry very little information: just the vertices of the simplex. For this reason it is very easy to define functors from other homology theories to the homology groups $H_*^{\mathcal{U}}(X)$. To illustrate this idea we consider the description of Čech homology in terms of ϵ -continuous simplices given in [7]. First, we need some definitions. The space X will be assumed to be compact and metric. A map f between metric spaces is called ϵ -continuous if there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ for every x, x' at a distance at most δ . Roughly speaking, f does not exhibit discontinuities of size greater than ϵ .

Let Δ_q denote the standard q -simplex. An ϵ -continuous q -simplex is an ϵ -continuous map $\sigma : \Delta_q \rightarrow X$. The free group generated by these is denoted by $S_q^\epsilon(X)$. There is a boundary operator $\partial : S_q^\epsilon(X) \rightarrow S_{q-1}^\epsilon(X)$ defined in the exact same way as in the singular theory and satisfies $\partial^2 = 0$. The homology of the chain complex $(S_*^\epsilon(X), \partial)$ is denoted by $H_*^\epsilon(X)$. Given $0 < \epsilon' < \epsilon$, any ϵ' -continuous simplex is automatically an ϵ -continuous simplex, so there exist natural inclusion maps $S_q^{\epsilon'}(X) \rightarrow S_q^\epsilon(X)$ which carry over to the homology groups and yield an inverse system $\{H_*^\epsilon(X)\}$. Then:

Theorem 4 ([7, Corollary 1, p. 88]). *The inverse limit of the system is isomorphic to $\check{H}_*(X)$.*

The proof in [7] proceeds by establishing that the homology theory just defined has the continuity property and agrees with the singular theory over simplicial complexes, and therefore must coincide with Čech homology. We shall give an alternative proof by establishing an isomorphism between the limits of $\{H_*^\epsilon(X)\}$ and $\{H_*^{\mathcal{U}}(X)\}$.

The main ideas are very natural: given a (small in a sense later explained) ϵ -continuous simplex σ we may produce a formal simplex $D(\sigma)$ by disregarding all the information in σ except for its vertices; conversely, given a formal simplex τ we may “interpolate” to produce an ϵ -continuous simplex $I(\tau)$ whose vertices are those of the formal simplex.

We begin by replacing the chain complex $S_q^\epsilon(X)$ with its subcomplex $E_q^\epsilon(X)$ generated by the (ϵ -continuous) maps $\tau : \Delta_q \rightarrow X$ whose image has a diameter less than ϵ . Note that if $\tau \in S_q^\epsilon(X)$ is an arbitrary ϵ -continuous simplex there exists δ such that τ carries any set of diameter less than δ to a set of diameter less than ϵ and so subdividing Δ_q barycentrically

enough times we see that the restriction of τ to each member of the subdivision belongs to $E_q^\epsilon(X)$. Then a standard argument (see for example [19, §31, pp. 175ff.] or [24, Appendix I, p. 211ff.]) shows that the inclusion $i: E_q^\epsilon(X) \subseteq S_q^\epsilon(X)$ induces isomorphisms between the homology of both chain complexes.

Discretization. We now define a “discretization operator” $D: E_q^\epsilon(X) \rightarrow S_q^\mathcal{U}(X)$. Fix an open covering \mathcal{U} of X and let $\epsilon > 0$ be a Lebesgue number for \mathcal{U} . Recall that every chain in $E_q^\epsilon(X)$ is of the form $\sum k_i \tau_i$ where each $\tau_i: \Delta_q \rightarrow X$ is a map whose image has a diameter less than ϵ and is therefore contained in some member of \mathcal{U} . Hence we can define $D(\tau_i)$ to be the \mathcal{U} -small formal simplex $(\tau(v_0) \dots \tau(v_q))$ and then extend linearly to chains by $D(c) = \sum k_i D(\tau_i)$. Geometrically, D simply forgets all the information that a chain carries except for the vertices of its constituent summands. It is straightforward to check that D commutes with the boundary operator so it induces a homomorphism in homology.

Interpolation. Now we want to define an “interpolation operator” $I: S_q^\mathcal{U}(X) \rightarrow E_q^\epsilon(X)$ which essentially “fills in” formal simplices to turn them into maps defined on all of Δ_q . Although this can be done in several ways, it should be compatible with the boundary operator. Fix $\epsilon > 0$ and choose a covering \mathcal{U} of X whose members all have a diameter less than ϵ .

We introduce the auxiliary map $J_q: \Delta_q \rightarrow \Delta_q$ defined as follows. Given a point $p \in \Delta_q$ with barycentric coordinates $(\lambda_0, \dots, \lambda_q)$, find the smallest i such that $\lambda_i > 0$ and set $J_q(p) := v_i$, the i -th vertex of Δ_q . Given a \mathcal{U} -small formal simplex $(x_0 \dots x_q)$ in X , we define a piecewise constant map $\tau: \Delta_q \rightarrow X$ at the vertices by $\tau(v_i) = x_i$ and then set $\tau(p) = \tau(J_q(p))$ for any other $p \in \Delta_q$.

The image of τ is precisely $\{x_0, \dots, x_q\}$, which is contained in some member of \mathcal{U} . In particular the diameter of the image of τ is less than ϵ , and so τ is indeed an element of $E_q^\epsilon(X)$. We can therefore define a homomorphism $I: S_q^\mathcal{U}(X) \rightarrow E_q^\epsilon(X)$ by linearly extending the definition above to chains. It is straightforward to check that I commutes with the boundary operators so it induces a homomorphism in homology.

Let us examine the composition $D \circ I$. First we need to find what its source and target spaces are: given an open covering \mathcal{U} we find a Lebesgue number ϵ for \mathcal{U} and then another covering \mathcal{U}' whose members all have a diameter less than ϵ . Then $D \circ I: S_q^{\mathcal{U}'}(X) \rightarrow S_q^\mathcal{U}(X)$. It is straightforward to check that the map $D \circ I$ is just the inclusion $S_q^{\mathcal{U}'}(X) \subseteq S_q^\mathcal{U}(X)$.

The analysis of the reverse composition $I \circ D$ is only slightly more complicated. Given ϵ , we find a covering \mathcal{U} whose members have a diameter less than ϵ and then in turn some $\epsilon' < \epsilon$ which is a Lebesgue number for \mathcal{U} . Then the $I \circ D$ is a map $E_q^{\epsilon'}(X) \rightarrow E_q^\epsilon(X)$ which works in the following way: given a simplex $\tau': \Delta_q \rightarrow X$, it discretizes it by retaining only its values on the vertices of Δ_q and then interpolates it to produce a new simplex $\tau: \Delta_q \rightarrow X$ using the prescription given above for I .

The simplices τ and τ' can be easily connected through an ϵ -continuous homotopy $\{\tau^t\}: \Delta_q \times [0, 1] \rightarrow X$ defined by $\tau^0 = \tau'$ and $\tau^t = \tau$ for $0 < t \leq 1$. This construction allows to define a prism operator $P: E_q^{\epsilon'}(X) \rightarrow E_{q+1}^\epsilon(X)$ (see for instance [8, Theorem 2.10, p. 112]) that satisfies

$$P(\partial\tau') + \partial P(\tau') = \tau - \tau'$$

Then, P provides a chain equivalence between $I \circ D$ and the inclusion $E_q^{\epsilon'}(X) \rightarrow E_q^\epsilon(X)$.

To sum up, the composite maps $D_* \circ I_*$ and $I_* \circ D_*$ coincide with the bonding maps of the inverse systems $\{H_*^{\mathcal{U}}(X)\}$ and $\{H_*^\epsilon(X)\}$ respectively.

Now we can easily show that the limits of the systems $\{H_*^\epsilon(X)\}$ and $\{H_*^{\mathcal{U}}(X)\}$ are isomorphic. For if $\mathcal{U}(\epsilon)$ is the covering of X given by all the open balls of radius ϵ and ϵ' is very small, the bonding map $H_*^{\epsilon'}(X) \rightarrow H_*^\epsilon(X)$ is equal to $I_* \circ D_*$ and so it factors through $H_*^{\mathcal{U}(\epsilon)}(X)$. Similarly, if ϵ is a Lebesgue number for an open covering \mathcal{U} of X and all the elements of a refinement \mathcal{U}' of \mathcal{U} have diameter smaller than ϵ , the bonding map $H_*^{\mathcal{U}'}(X) \rightarrow H_*^{\mathcal{U}}(X)$ factors through $H_*^\epsilon(X)$. Therefore, inserting alternatively homology groups at scale \mathcal{U} and ϵ -homology groups in an appropriate fashion, we build an inverse sequence that can be interpreted as a cofinal subsystem both of $\{H_*^\epsilon(X)\}$ and $\{H_*^{\mathcal{U}}(X)\}$. Since the inverse limit of an inverse system and any cofinal system are isomorphic, it follows that $\varprojlim_\epsilon H_*^\epsilon(X)$ is isomorphic to $\varprojlim_{\mathcal{U}} H_*^{\mathcal{U}}(X)$, which is in turn isomorphic to $\check{H}_*(X)$.

4. 1-DIMENSIONAL HOMOLOGY

We devote this section to an analysis of the groups $H_1^{\mathcal{U}}(X; G)$ and $\check{H}_1(X; G)$. We will not be systematic at all, but only pursue the subject to the extent that is needed for the proofs of the results about entropy in Sections 8 and 9.

Terminology. In the sequel we will mostly work with the homology groups $H_*^{\mathcal{U}}(X)$ and therefore with \mathcal{U} -small formal simplices and chains. For the sake of brevity we will drop the expression “formal” to refer to the simplices, chains, etc.

First we show that $H_1^{\mathcal{U}}(X; \mathbb{Z})$ is generated by special cycles, which we call *simple* and *elementary*, of the form

$$(x_1 \ x_2) + (x_2 \ x_3) + \dots + (x_{n-1} \ x_n)$$

where $x_n = x_1$ (cycle condition), all the x_1, \dots, x_{n-1} are different points in X (*elementary*) and it is possible to choose pairwise distinct $U_i \in \mathcal{U}$ so that each \mathcal{U} -small simplex $(x_i \ x_{i+1})$ is contained in U_i (*simple*). Intuitively, a cycle is simple if it has no “self-intersections” at scale \mathcal{U} . A simple elementary chain has an analogous definition dropping the condition that $x_1 = x_n$ and requiring that all the x_i be different. Notice that an elementary chain, not necessarily simple, corresponds to an edge path in $V(\mathcal{U})$ and if the chain is closed (i.e. it is a cycle) so is the path. If the chain or cycle is also simple then the edge path can be projected to an equivalent edge path $N(\mathcal{U})$.

Although the ensuing discussion is very elementary in nature and the details could have been left to the reader, we include it to illustrate how to manipulate \mathcal{U} -small cycles. The first step towards the proposed description is the following lemma:

Lemma 5. *Let c be a \mathcal{U} -small cycle. Then $c \sim \sum_i k_i e_i$ where all the e_i are \mathcal{U} -small elementary cycles and $k_i \in G$. Moreover, every simplex that appears in some e_i was already part of c .*

There are several alternative ways to prove the assertion. Let $\{x_i\}$ be the (finite) collection of vertices among the \mathcal{U} -small 1-simplices that constitute c . Then, c can be seen as a (homology) cycle in the complete graph K with vertices $\{x_i\}$. Since the first homology group of a finite graph is generated by closed paths and closed paths in K correspond to elementary cycles in $H_1^{\mathcal{U}}(X; \mathbb{Z})$, c is an integer combination of elementary cycles as desired.

It only remains to write an elementary cycle as a combination of simple elementary cycles. With a view towards later applications we are going to describe in detail the process when applied to an arbitrary elementary \mathcal{U} -small chain $e = \sum_{i=1}^{n-1} (x_i x_{i+1})$, not necessarily a cycle.

Lemma 6. *Let e be a \mathcal{U} -small elementary chain. Then $e \sim e_0 + s_1 + \dots + s_r$ where e_0 is a simple elementary chain and the s_i are simple elementary cycles.*

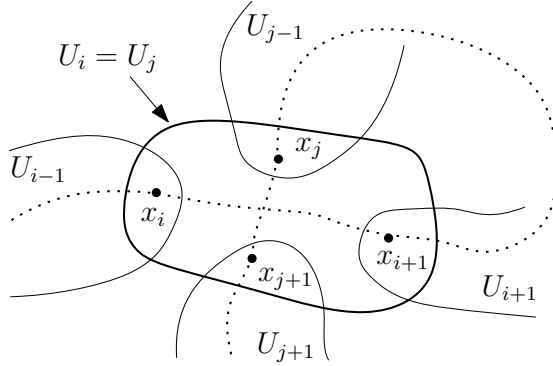
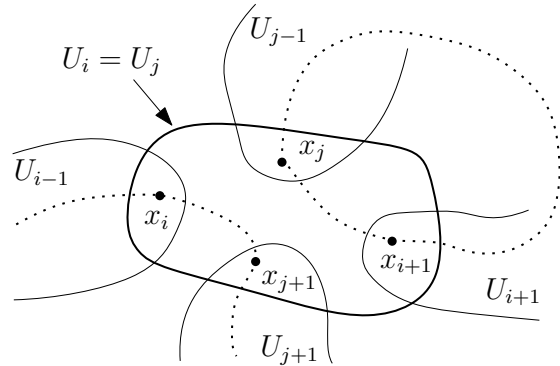
Proof. As usual, take $e = \sum_{i=1}^{n-1} (x_i x_{i+1})$. For each i choose an arbitrary $U_i \in \mathcal{U}$ that contains $(x_i x_{i+1})$, and suppose that $U_i = U_j$ for some $j > i$ (if this does not happen, then e is already simple). Fix any such i and choose $j > i$ to be the smallest with the property $U_i = U_j$. Consider the portion (*) of the chain between indices i and j :

$$e = \dots + (x_{i-1} x_i) + \underbrace{(x_i x_{i+1}) + \dots + (x_j x_{j+1})}_{(*)} + (x_{j+1} x_{j+2}) + \dots$$

We remove the simplices $(x_i x_{i+1})$ and $(x_j x_{j+1})$ from e and replace them with $(x_i x_{j+1}) + (x_j x_{i+1})$ to obtain a new chain e' , still \mathcal{U} -small. Moreover,

$$e - e' = (x_i x_{i+1}) + (x_j x_{j+1}) - (x_i x_{j+1}) - (x_j x_{i+1})$$

is easily checked to be a cycle and it is entirely contained in $U_i = U_j$, so by Proposition 2 it is nullhomologous. Hence $e \sim e'$. Figures 1 and 2 provide a schematic drawing of the situation. The dotted segments connecting the vertices x_k are just pictorial aids to suggest the simplices of the chains, but of course they are not true subsets of X .


 FIGURE 1. Original chain e

 FIGURE 2. Modified chain e'

Now we can reorder the simplices in e' in the following fashion:

$$e \sim e' = \left((x_1 x_2) + \dots + (x_{i-1} x_i) + (x_i x_{j+1}) + (x_{j+1} x_{j+2}) + \dots + (x_{n-1} x_n) \right) + \left((x_j x_{i+1}) + (x_{i+1} x_{i+2}) + \dots + (x_{j-1} x_j) \right)$$

The second term is evidently a simple elementary cycle. Now we repeat this process on the first summand of e' , and so on until we reach an expression having the form stated in the lemma. \square

Notice that if the starting chain e is in fact a cycle then $0 = \partial e = \partial e_0 + \sum_i \partial s_i = \partial e_0$, and so the e_0 summand is also a cycle. Thus, a combination of Lemmas 5 and 6 yields the following:

Proposition 7. *The \mathcal{U} -small elementary simple cycles generate $H_1^{\mathcal{U}}(X; \mathbb{Z})$ and, consequently, $H_1^{\mathcal{U}}(X; G)$ for general coefficients G .*

Now we discuss the relation between $H_1^{\mathcal{U}}(X; G)$ and $\check{H}_1(X; G)$ when X is locally connected. It was mentioned earlier on that $H_q^{\mathcal{U}}(X; G)$ essentially ignores all the structure of X below scale \mathcal{U} . Suppose that each member U of the covering \mathcal{U} is connected, so that at this scale X has no relevant 0-dimensional features (these being related to connectedness). It is then natural to expect that $H_1^{\mathcal{U}}(X; G)$ is “smaller” than $\check{H}_1(X; G)$. Indeed, we shall prove that the projection $\check{H}_1(X; G) \rightarrow H_1^{\mathcal{U}}(X; G)$ is surjective. In other words, as soon as a 1-cycle is \mathcal{U} -small, it can be refined to be \mathcal{V} -small for an arbitrary open covering \mathcal{V} in a coherent manner so as to define an element in $\check{H}_1(X; G)$.

Consider a \mathcal{U} -small 1-chain $c = \sum_i k_i \sigma_i$. Each $\sigma_i = (x_i y_i)$ is contained in some $U_i \in \mathcal{U}$. Given any open covering \mathcal{V} of X , $\mathcal{V}|_{U_i} = \{V \cap U_i : V \in \mathcal{V}\}$ is an open covering of U_i and since U_i is connected, a standard argument produces an elementary \mathcal{V} -small chain e_i in U_i such that $\partial e_i = y_i - x_i$. Consider the \mathcal{V} -small chain $c' := \sum_i k_i e_i$ obtained from c by replacing each 1-simplex σ_i with the chain e_i that connects x_i to y_i . We say that c' is a *refinement* of c . Observe that $c \sim c'$ in $H_1^{\mathcal{U}}(X; G)$. Indeed, since $\partial e_i = y_i - x_i = \partial \sigma_i$ we have that $e_i - \sigma_i$ is a cycle contained in U_i and so by Proposition 2 it is nullhomologous. Hence $c' - c = \sum_i k_i (e_i - \sigma_i) \sim 0$ in $H_1^{\mathcal{U}}(X; G)$.

When X is locally connected, open coverings that consist of connected sets constitute a cofinal subfamily of the family of all open coverings. Together with the construction from the previous paragraph, this entails the following:

Proposition 8. *Suppose X is locally connected and there is a cofinal sequence of open coverings (this is the case if X is compact and metric, for example). Assume \mathcal{U} is an open covering of X that consists of connected sets. Then any element of $H_1^{\mathcal{U}}(X; G)$ comes from an element of $\check{H}_1(X; G)$ or, in other words, the projection $\check{H}_1(X; G) \rightarrow H_1^{\mathcal{U}}(X; G)$ is surjective.*

Suppose for the next definition and lemma that $G = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$. The norm of a \mathcal{U} -small 1-cycle $c = \sum k_i \sigma_i$ is defined as $\|c\|_1 = \sum |k_i|$. For later purposes we show how to modify the refinement construction just described to control the norm of the refinements:

Lemma 9. *Let \mathcal{U} be an open covering that consists of connected sets and consider a \mathcal{U} -small chain c . Then for any open covering \mathcal{V} there exists a \mathcal{V} -small chain c' such that: (i) $c \sim c'$ in $H_1^{\mathcal{U}}(X; G)$ and (ii) $\|c'\|_1 \leq \|c\|_1 |\mathcal{V}|$, where $|\mathcal{V}|$ denotes the cardinality of \mathcal{V} .*

Proof. Start by constructing $c' = \sum_i k_i e_i$ as before. By Lemma 6 applied in U_i at scale \mathcal{V} we may write $e_i \sim e_{i0} + s_{i1} + \dots + s_{ir_i}$ in $H_1^{\mathcal{V}}(U_i; G)$ where each e_{i0} is an elementary simple chain and each s_{ij} is a cycle, and all these are \mathcal{V} -small and contained in U_i . By Proposition 2 all the s_{ij} are homologous to zero in $H_1^{\mathcal{U}}(X; G)$, so we have $e_i \sim e_{i0}$ at scale \mathcal{U} for each i . Moreover, $\|e_{i0}\|_1 \leq |\mathcal{V}|$ because e_{i0} is an elementary simple chain. Therefore in $H_1^{\mathcal{U}}(X; G)$ we have $c \sim c' := \sum k_i e_{i0}$ and $\|c'\|_1 \leq \sum |k_i| \|e_{i0}\|_1 \leq \sum |k_i| |\mathcal{V}| = \|c\|_1 |\mathcal{V}|$. \square

5. ALEXANDER-SPANIER COCYCLES WITH A FINITE IMAGE

Let $f: X \rightarrow Y$ be a map, not necessarily continuous, and suppose that \mathcal{V} is an open covering of Y and \mathcal{U} is an open covering of X that refines $f^{-1}(\mathcal{V})$. The usual formula $f_{\sharp}(\xi) := \xi \circ f$ correctly defines a homomorphism $\overline{C}_{\mathcal{V}}^q(Y) \rightarrow \overline{C}_{\mathcal{U}}^q(X)$ and, similarly, the prescription

$$f_{\sharp}(x_0 \dots x_q) := (f(x_0) \dots f(x_q))$$

for any \mathcal{U} -small q -simplex extends to a homomorphism $f_{\sharp}: S_q^{\mathcal{U}}(X) \rightarrow S_q^{\mathcal{V}}(Y)$. These commute with the coboundary and boundary operators, respectively, and therefore induce homomorphisms $f_*: H_q^{\mathcal{U}}(X) \rightarrow H_q^{\mathcal{V}}(Y)$ and $f^*: H_{\mathcal{V}}^q(Y) \rightarrow H_{\mathcal{U}}^q(X)$.

The following proposition estimates how close must two maps $f, g: X \rightarrow Y$ be in order to induce the same map in homology and cohomology for a fixed choice of open coverings. The condition is a direct translation of the notion of contiguity in the Vietoris complex:

Proposition 10. *Suppose that for every \mathcal{U} -small simplex $(x_0 \dots x_q)$ in X there exists $V \in \mathcal{V}$ which contains both simplices $(f(x_0) \dots f(x_q))$ and $(g(x_0) \dots g(x_q))$. Then $f_* = g_*: H_q^{\mathcal{U}}(X) \rightarrow H_q^{\mathcal{V}}(Y)$ and similarly for cohomology.*

Proof. Observe that the assumption implies that f and g carry \mathcal{U} -small simplices to \mathcal{V} -small simplices, so f_* and g_* are well defined. For every \mathcal{U} -small simplex $(x_0 \dots x_q)$ in X define

$$D(x_0 \dots x_q) := \sum_{i=0}^q (-1)^i (f(x_0) \dots f(x_i) g(x_i) \dots g(x_q)).$$

This is a $(q+1)$ -chain in Y which is \mathcal{V} -small because of the hypothesis of the proposition. Extending D linearly to $D: S_q^{\mathcal{U}}(X) \rightarrow S_{q+1}^{\mathcal{V}}(Y)$ we obtain a prism operator which is easily checked to satisfy $\partial D + D\partial = g_{\sharp} - f_{\sharp}$. Thus D is a chain homotopy between g_{\sharp} and f_{\sharp} and so the two induce the same homomorphism $f_* = g_*: H_q^{\mathcal{U}}(X) \rightarrow H_q^{\mathcal{V}}(Y)$. The argument for cohomology is analogous. \square

Now we describe a construction which provides some sort of “sampling” of a space X . It will also justify why it is interesting to consider maps that are not necessarily continuous. We first assume that X is paracompact, although we will subsequently specialize to the case when X is compact. Let \mathcal{V} be an open covering of X and let \mathcal{U} be a star refinement of \mathcal{V} . This means that (*) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ that contains every element of \mathcal{U} that intersects V , including U itself. The existence of such a refinement \mathcal{U} can be easily established when X is compact metric using a Lebesgue number for \mathcal{V} , but in general it is equivalent to the assumption that X be paracompact. Assume that every member of \mathcal{U} is nonempty.

Write $\mathcal{U} = \{U_i : i \in I\}$ and endow the index set I with a well-ordering. For each i pick a point $u_i \in U_i$ and consider the map

$$\begin{aligned} p: X &\rightarrow X \\ x &\longmapsto u_i \quad i \in I \text{ smallest index with } x \in U_i. \end{aligned}$$

The identity map on X , id_X , and p verify the hypothesis of Proposition 10. We conclude that they define the same map $H_q^{\mathcal{U}}(X) \rightarrow H_q^{\mathcal{V}}(X)$ (and similarly in cohomology). That is, we

have

$$p_* = j_{\mathcal{U}, \mathcal{V}}: H_q^{\mathcal{U}}(X) \rightarrow H_q^{\mathcal{V}}(X) \quad \text{and} \quad p^* = \pi_{\mathcal{V}, \mathcal{U}}: H_{\mathcal{V}}^q(X) \rightarrow H_{\mathcal{U}}^q(X).$$

From now on we concentrate on the case when X is compact. Then one can always replace \mathcal{U} with a finite subcover and this is still a star refinement of \mathcal{V} . Let $X_0 = \{u_i : i \in I\}$. This is a “finite sample” of X . Evidently the image of p is contained in X_0 , so we can write $p = i \circ p_0$ where p_0 is the map p whose target space is restricted to X_0 and $i: X_0 \subseteq X$ denotes the inclusion. Hence we can factor p_* and p^* through $H_*^{\mathcal{V}}(X_0)$ and $H_{\mathcal{V}}^*(X_0)$, respectively, to obtain the following commutative diagrams:

$$\begin{array}{ccc} H_q^{\mathcal{U}}(X) & \xrightarrow{j_{\mathcal{U}, \mathcal{V}}} & H_q^{\mathcal{V}}(X) \\ & \searrow (p_0)_* & \uparrow i_* \\ & & H_q^{\mathcal{V}}(X_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} H_{\mathcal{U}}^q(X) & \xleftarrow{\pi_{\mathcal{V}, \mathcal{U}}} & H_{\mathcal{V}}^q(X) \\ & \swarrow (p_0)^* & \downarrow i^* \\ & & H_{\mathcal{V}}^q(X_0) \end{array}$$

The useful feature of these diagrams is that the homology and cohomology groups of X_0 are entirely finitistic in nature since X_0 is finite. The number of q -simplices in X_0 is finite, with a very crude bound being $|X_0|^{q+1}$. This places an upper bound on the rank of $H_q^{\mathcal{V}}(X_0)$. Similarly, any Alexander–Spanier cochain defined on X_0 necessarily has a finite image with no more than $|X_0|^{q+1}$ elements. Thus we have the following proposition, originally due to Keesee [13].

Proposition 11. *Let X be compact. Then, every element of $\check{H}^*(X)$ has a representative cocycle whose image is finite.*

Proof. Let $z \in \check{H}^q(X)$. Recall from Proposition 1 that the Čech cohomology group is the direct limit of the \mathcal{V} -cohomology groups, $H_{\mathcal{V}}^q(X)$, where \mathcal{V} ranges over all open coverings of X . By compactness, we can restrict the limit to finite open coverings. Thus, there exists a finite open covering \mathcal{V} of X and $z_{\mathcal{V}} \in H_{\mathcal{V}}^q(X)$ such that

$$\begin{array}{ccc} H_{\mathcal{V}}^q(X) & \xrightarrow{\pi_{\mathcal{V}}} & \check{H}^q(X) \\ z_{\mathcal{V}} & \longmapsto & z \end{array}$$

Let $z_{\mathcal{U}} := \pi_{\mathcal{V}, \mathcal{U}}(z_{\mathcal{V}}) \in H_{\mathcal{U}}^q(X)$, which in the limit also represents the element z since $\pi_{\mathcal{U}} \circ \pi_{\mathcal{V}, \mathcal{U}} = \pi_{\mathcal{V}}$. By the commutative diagram above we then have $z_{\mathcal{U}} = (p_0)^* i^*(z_{\mathcal{V}})$, and evidently $i^*(z_{\mathcal{V}})$ is represented by some cocycle ξ with a finite image. Thus $z_{\mathcal{U}}$ and also z are represented by the cocycle $(p_0)_* \xi$, which also has a finite image. \square

Remark 12. The previous arguments trivially show that $H_{\mathcal{U}}^*(X)$ is finitely generated because every open covering \mathcal{U} is a star refinement of the trivial open covering $\{X\}$.

6. DEFINITION OF THE INTEGRAL

In the usual definition of cohomology as the homology of a dual chain complex there is an obvious manner in which cohomology acts on homology: by evaluation of cochains on chains. The descriptions of Čech homology and cohomology considered in this paper are not so obviously dual to each other; however, it is possible to define a pairing between cohomology and homology classes in terms of an integral reminiscent of de Rham’s theory of integration

of differential forms over simplices. We first devote a few lines to a heuristic motivation of the integral.

6.1. Motivation. Consider a space X . Borrowing the language of differential geometry just for heuristic purposes, we can think of a q -cochain ξ as a q -differential form and of a (formal) q -simplex $\sigma = (x_0 \dots x_q)$ as a base point x_0 together with q “vectors” $\overrightarrow{x_0x_i}$ that are “approximately tangent” to X at x_0 . Then the evaluation $\xi(\sigma) = \xi(x_0, \dots, x_q)$ can be thought of as the evaluation of the form ξ_{x_0} over the tuple of q tangent vectors $(\overrightarrow{x_0x_1}, \dots, \overrightarrow{x_0x_q})$. This analogy suggests how to define the integral of a cohomology class $z \in \check{H}^q(X)$ over a homology class $\gamma \in \check{H}_q(X)$: one takes a cocycle ξ that represents z and an approximation $c = \sum k_i \sigma_i$ of γ , computes the “Riemann sum” $\sum k_i \xi(\sigma_i)$ and then lets the size of the simplices of the approximation go to zero. There is no *a priori* reason why the limit should exist; however, the cocycle condition $\delta\xi = 0$ ensures that it does.

To explain this in more detail let us consider a particularly simple situation. Suppose that $\gamma: [0, 1] \rightarrow X$ is a continuous closed path in X and ξ is a 1-cocycle. To integrate ξ over γ as suggested above we would consider a partition $\{t_i\}$ of $[0, 1]$ and then evaluate ξ over the “approximate tangent vectors” $\overrightarrow{\gamma(t_i)\gamma(t_{i+1})}$ to obtain the Riemann sum $\sum \xi(\gamma(t_i), \gamma(t_{i+1}))$. Then we would take progressively finer partitions $\{t_i\}$ of $[0, 1]$ letting their diameters go to zero. What happens if we refine a partition by inserting $t_* \in (t_i, t_{i+1})$? The Riemann sum is modified by

$$\xi(\gamma(t_i), \gamma(t_*)) + \xi(\gamma(t_*), \gamma(t_{i+1})) - \xi(\gamma(t_i), \gamma(t_{i+1})) = \delta\xi(\gamma(t_i), \gamma(t_*), \gamma(t_{i+1})).$$

By the cocycle condition, this expression vanishes when the points are close enough. Thus, when the partition is fine enough further refinement do not change the value of the Riemann sum and, as a consequence, the limit of the sums as the diameter of the partition tends to zero trivially exists.

The same ideas also translate directly to an integration over Čech 1-homology classes γ : in that case the progressive refinement of the partition $\{t_i\}$ is actually part of the definition of γ , afforded by taking representatives $\gamma^{\mathcal{U}}$ of γ that are \mathcal{U} -small for progressively fine coverings \mathcal{U} .

6.2. Formal definition. Now we define formally the integral of an Alexander-Spanier cohomology class $z \in \check{H}^q(X)$ over a Čech homology class $\gamma \in \check{H}_q(X)$. For the definition to make algebraic sense we must be able to multiply coefficients in homology with coefficients in cohomology. This can be ensured in many ways; for instance by taking homology with \mathbb{Z} -coefficients and cohomology with coefficients in an arbitrary group. We will not pursue the maximum generality in this regard, so we just require that the group of coefficients G be a ring itself, and in fact will soon specialize to the case when G is a field \mathbb{K} .

We need some preliminaries. Let $\xi: X^{q+1} \rightarrow G$ be a q -cochain and let $\sigma = (x_0 \dots x_q)$ be a q -simplex. We can evaluate ξ over σ to obtain an element in G in the obvious way, setting $\xi(\sigma) := \xi(x_0, \dots, x_q)$. If $c = \sum k_i \sigma_i$ is a q -chain we define the evaluation of ξ over c by extending linearly the definition above: $\xi(c) := \sum k_i \xi(\sigma_i)$. Denoting by $\xi(\cdot)$ the standard evaluation of ξ on a tuple of points and the evaluation of ξ on a simplex or a chain is a slight abuse of notation but should not cause any confusion. The following algebraic property is key to what follows and can be easily deduced from the definitions.

Lemma 13 (Stokes' lemma). *Let ξ be a $(q-1)$ -cochain and $c = \sum k_i \sigma_i$ a q -chain. Then $(\delta\xi)(c) = \xi(\partial c)$.*

Now let $z \in \check{H}^q(X)$ and $\gamma \in \check{H}_q(X)$. Let ξ be a q -cocycle that represents z and let \mathcal{U} be an open covering of X such that $\delta\xi$ is zero over each member of \mathcal{U} . Let c be a \mathcal{U} -small representative of γ . We define the integral of z over γ as

$$(1) \quad \int_{\gamma} z := \xi(c).$$

There are several choices involved in this definition and to justify that the definition is correct we need to prove the following:

Theorem 14. *The right hand side of (1) is independent of the choices of c , \mathcal{U} , and ξ .*

Proof. (i) Let c_1 and c_2 be two \mathcal{U} -small chains that represent γ . Then $c_1 - c_2 = \partial d$ for some \mathcal{U} -small chain d . By the linearity of evaluation and Stokes' lemma we have

$$\xi(c_1) - \xi(c_2) = \xi(c_1 - c_2) = \xi(\partial d) = (\delta\xi)(d)$$

and the latter term vanishes because $\delta\xi$ is zero over every member of \mathcal{U} and d is \mathcal{U} -small. Thus the right hand side of Equation (1) does not depend on the particular \mathcal{U} -small chain c chosen to represent γ .

(ii) Let \mathcal{V} be another open covering such that $\delta\xi$ is locally zero over \mathcal{V} and suppose first that \mathcal{V} refines \mathcal{U} . Let $c^{\mathcal{V}}$ and $c^{\mathcal{U}}$ be representatives of γ that are \mathcal{V} - and \mathcal{U} -small respectively. Observe that $\gamma^{\mathcal{V}} = [c^{\mathcal{V}}]$ gets mapped to $\gamma^{\mathcal{U}} = [c^{\mathcal{U}}]$ in $H_*^{\mathcal{U}}(X)$ by the bonding morphism $H_q^{\mathcal{V}}(X) \rightarrow H_q^{\mathcal{U}}(X)$. Thus there exists a \mathcal{U} -small chain d such that $c^{\mathcal{V}} - c^{\mathcal{U}} = \partial d$ in the group of \mathcal{U} -chains. Again by linearity of evaluation and Stokes' lemma we have

$$\xi(c^{\mathcal{U}}) - \xi(c^{\mathcal{V}}) = \xi(c^{\mathcal{U}} - c^{\mathcal{V}}) = \xi(\partial d) = (\delta\xi)(d) = 0,$$

so $\xi(c^{\mathcal{U}}) = \xi(c^{\mathcal{V}})$. In the general case when \mathcal{V} does not refine \mathcal{U} , one simply goes through the common refinement $\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

(iii) Finally, to check that the right hand side of Equation (1) is independent of ξ , let ξ_1 and ξ_2 be two representatives of z . Then there exist η and an open covering \mathcal{U} such that $\xi_1 - \xi_2 = \delta\eta$ over each member of \mathcal{U} and also $\delta\xi_1$ and $\delta\xi_2$ vanish over each member of \mathcal{U} . Let c be a \mathcal{U} -small representative of γ . Then

$$\xi_1(c) - \xi_2(c) = (\xi_1 - \xi_2)(c) = (\delta\eta)(c) = \eta(\partial c) = \eta(0) = 0,$$

and $\xi_1(c) = \xi_2(c)$ as desired. \square

One would expect that the integral be linear in both z and γ , and this is certainly the case. For later convenience we reformulate this as follows. The integral can be considered as a map

$$\int : \check{H}^q(X) \times \check{H}_q(X) \rightarrow G$$

and then we have:

Proposition 15. *The integral is a bilinear form.*

Proof. This is straightforward using the freedom in c , \mathcal{U} and ξ afforded by the previous theorem. Just to illustrate this let us show that $\int_{\gamma}(z + z') = \int_{\gamma} z + \int_{\gamma} z'$ for any $\gamma \in \check{H}_q(X)$ and $z, z' \in \check{H}^q(X)$. Let ξ be a cocycle that represents z and let \mathcal{U} be an open covering such that $\delta\xi$ is zero over each member of \mathcal{U} . Let ξ' and \mathcal{U}' be analogous objects for z' . Then $\xi + \xi'$ represents $z + z'$ and its coboundary is zero over each member of the common refinement $\mathcal{U} \vee \mathcal{U}'$. Observe that $\delta\xi$ and $\delta\xi'$ are still zero over each member of $\mathcal{U} \vee \mathcal{U}'$, and so we may use the latter covering to compute the three integrals $\int_{\gamma} z$, $\int_{\gamma} z'$ and $\int_{\gamma}(z + z')$ by choosing a representative c of γ which is $(\mathcal{U} \vee \mathcal{U}')$ -small. Then clearly

$$\int_{\gamma}(z + z') = (\xi + \xi')(c) = \xi(c) + \xi'(c) = \int_{\gamma} z + \int_{\gamma} z'.$$

□

It is convenient to observe that the definition of the integral makes sense not only in the limit of Čech homology and cohomology, but already at scale \mathcal{U} for any open covering \mathcal{U} of X . Indeed, given $z \in H_{\mathcal{U}}^q(X)$ and $\gamma \in H_q^{\mathcal{U}}(X)$, one can pick a cocycle ξ which represents z and satisfies $\partial\xi = 0$ over every \mathcal{U} -small simplex and a \mathcal{U} -small chain c which represents γ and define

$$\int_{\gamma} z := \xi(c).$$

The proof that this is independent of ξ and c is completely analogous to steps (i) and (iii) in the proof of Theorem 14. In particular, the integral in this generalized sense is still bilinear.

Remark 16. There is no need to record \mathcal{U} in the notation or even to distinguish this notationally from the integral defined at the beginning of this section, and the reason is the following. A cohomology class $z_{\mathcal{U}} \in H_{\mathcal{U}}^*(X)$ represented by a cocycle ξ can be viewed as a cohomology class $z_{\mathcal{V}}$ at scale \mathcal{V} for any refinement \mathcal{V} of \mathcal{U} or as an element z of $\check{H}^*(X)$ represented by the same cocycle ξ . If we want to integrate z over a Čech homology class $\gamma = (\gamma^{\mathcal{U}})$ all we have to do is take a \mathcal{U} -small cycle c that represents γ and return $\xi(c)$. Exactly the same computation can be carried out to compute the integral of $z_{\mathcal{U}}$ along $\gamma^{\mathcal{U}}$ because c is a representative of $\gamma^{\mathcal{U}}$. Note that we could have chosen also a representative c' of $\gamma^{\mathcal{V}}$ for some refinement \mathcal{V} to compute $\int_{\gamma^{\mathcal{U}}} z_{\mathcal{U}}$ because c and c' are homologous as \mathcal{U} -small cycles.

The integral also behaves well under the action of continuous maps:

Lemma 17. *Let $f: X \rightarrow Y$ be a map. Then*

$$\xi(f_{\#}c) = f^{\#}\xi(c)$$

where ξ is a q -cochain in Y and c is a q -chain in X .

Lemma 18. *Let $f: X \rightarrow Y$ be a map and \mathcal{U} and \mathcal{V} open coverings of X and Y , respectively, such that \mathcal{U} refines $f^{-1}\mathcal{V}$. Then*

$$\int_{\gamma} f^*z = \int_{f_*\gamma} z$$

where $z \in H_{\mathcal{V}}^q(Y)$ and $\gamma \in H_q^{\mathcal{U}}(X)$.

Proof. Let $z \in H_{\mathcal{V}}^q(Y)$ and $\gamma \in H_q^{\mathcal{U}}(X)$. Let ξ be a q -cocycle that represents z . Since $\delta\xi$ is zero over each member of \mathcal{V} and \mathcal{U} refines $f^{-1}\mathcal{V}$, we conclude that $\delta f^{\sharp}\xi$ vanishes over each member of \mathcal{U} . Letting c be a \mathcal{U} -small representative of γ , since $f_{\sharp}c$ is a \mathcal{V} -small representative of $f_*\gamma$ we have that

$$\int_{\gamma} f^*z = f^{\sharp}\xi(c) = \xi(f_{\sharp}c) = \int_{f_*\gamma} z.$$

□

7. NONDEGENERATION OF THE INTEGRAL

Throughout all this section the *coefficients are taken in a field \mathbb{K} which we shall omit from the notation.*

To motivate the forthcoming discussion let us recall a well known result in multivariable calculus. Let a vectorfield V be defined on some open subset X of \mathbb{R}^3 and assume that the integral of V along any closed curve contained in X vanishes. Then (under suitable smoothness assumptions) V is the gradient of some function f ; a so-called potential for the vectorfield. The standard proof constructs explicitly the potential f as follows. One fixes some reference point $p_0 \in X$ and then for each $p \in X$ chooses a path γ_p contained in X that joins p_0 with p (if X is not connected this has to be done on each connected component). Then $f(p)$ is defined as the integral of the vectorfield V along the path γ_p . The assumption that the integral of V along any closed curve is zero ensures that $f(p)$ is independent of the path γ_p , and checking that the gradient of f is precisely V is then just a matter of simple differential calculus.

The same construction can be used, with appropriate modifications, in our setting. Suppose X is an arbitrary space. Let \mathcal{U} be an open covering of X and let $\bar{\xi} \in \bar{C}_{\mathcal{U}}^1(X)$, i.e. ξ is a 1-cochain such that $\delta\xi$ vanishes over every member of \mathcal{U} and so $\bar{\xi}$ is a representative of a cohomology class $z \in H_{\mathcal{U}}^1(X)$. Assume further that ξ has the property that for every \mathcal{U} -small cycle c , the evaluation $\xi(c)$ vanishes. Then there exists a “potential” for ξ ; that is, a 0-cochain V such that $\delta V = \xi$ on each element of \mathcal{U} . To define V we can fix a reference point $p_0 \in X$ and for any other point p that can be joined to p_0 by a \mathcal{U} -small 1-chain σ ($\partial\sigma = p - p_0$) set $V(p) = \xi(\sigma)$, which is in spirit $\int_{\sigma} \xi$ although we prefer not to use this notation because σ is not a cycle. Repeat the procedure until the definition reaches every point of X to account for multiple “ \mathcal{U} -components”.

Note that the existence of V that solves $\delta V = \xi$ implies that $z = 0$. Also, the assumption that $\xi(c)$ vanishes for each \mathcal{U} -small cycle is equivalent to the condition that the integral of z over any $\gamma \in H_1^{\mathcal{U}}(X)$ vanishes. In other words, we have the following result for $q = 1$:

$$(2) \quad \int_{\gamma} z = 0 \quad \forall \gamma \in H_1^{\mathcal{U}}(X) \quad \Rightarrow \quad z = 0 \in H_1^{\mathcal{U}}(X).$$

Proposition 19. *The statement in (2) is true for every $q \geq 0$.*

Proof. Let us construct the higher dimensional analogue of the “potential”. Let $\bar{\xi} \in \bar{C}_{\mathcal{U}}^q(X)$ be a cocycle which represents z . The goal is to find a cochain $\bar{\eta} \in \bar{C}_{\mathcal{U}}^{q-1}(X)$ that satisfies $\delta\bar{\eta} = \bar{\xi}$; that is, $\delta\eta(c) = \xi(c)$ for every \mathcal{U} -small q -chain c .

Let $B_{q-1}^{\mathcal{U}}(X) \subseteq S_{q-1}^{\mathcal{U}}(X)$ be the subspace of boundaries; that is, the set of \mathcal{U} -small $(q-1)$ -chains that bound a \mathcal{U} -small q -chain. In turn, $S_{q-1}^{\mathcal{U}}(X)$ is a subspace of $S_{q-1}^{\{X\}}(X)$. Notice that this last group contains all formal $(q-1)$ -chains in X regardless of their size.

Take a basis $\mathcal{B} = \{d_i\}$ of $B_{q-1}^{\mathcal{U}}(X)$. For each d_i let c_i be a \mathcal{U} -small q -chain such that $d_i = \partial c_i$. There exists a unique linear map η defined on $B_{q-1}^{\mathcal{U}}(X)$ such that $\eta(d_i) = \xi(c_i)$. Extend arbitrarily this to another linear map (again denoted by η) defined on all of $S_{q-1}^{\mathcal{U}}(X)$, and then again to a linear map defined on all of $S_{q-1}^{\{X\}}(X)$. In particular $\eta(x_0, \dots, x_{q-1}) = \eta(x_0 \dots x_{q-1})$ is well defined for every tuple of points in X .

We claim that $\delta\eta(c) = \xi(c)$ for every \mathcal{U} -small q -chain c . Let c be such a chain. Its boundary is an element of $B_{q-1}^{\mathcal{U}}(X)$ and can therefore be written as a finite linear combination of elements of \mathcal{B} , say $\partial c = \sum k_i d_i$. Then

$$\delta\eta(c) = \eta(\partial c) = \sum k_i \eta(d_i) = \sum k_i \xi(c_i) = \xi\left(\sum k_i c_i\right) = \xi(c')$$

where $c' := \sum k_i c_i$ satisfies $\partial c' = \partial c$. Therefore $\gamma = [c' - c]$ is an element of $H_q^{\mathcal{U}}(X)$. By assumption $0 = \int_{\gamma} z = \xi(c' - c)$, so $\xi(c') = \xi(c)$ and the equality above leads to $\delta\eta(c) = \xi(c)$, as was to be shown. \square

It is also natural to ask if a dual result holds, that is, whether an element $\gamma \in H_q^{\mathcal{U}}(X)$ which satisfies $\int_{\gamma} z = 0$ for every $z \in H_{\mathcal{U}}^q(X)$ must necessarily be zero.

Proposition 20. *The statement dual to (2) is true for every $q \geq 0$.*

Proof. By contradiction, at the level of chains and cochains, given a \mathcal{U} -small cycle $c = \sum_{i=0}^n k_i \sigma_i$ that is not a boundary we need to find a cocycle $\xi \in C_{\mathcal{U}}^q(X)$ such that $\xi(c) \neq 0$. Assume without loss of generality that no linear combination of the simplices σ_i yields a nontrivial boundary.

Define $\nu: \text{span}\{\sigma_i\} \oplus B_q^{\mathcal{U}}(X) \rightarrow \mathbb{K}$ by $\nu(\sigma_0) = 1$, $\nu(\sigma_i) = 0$ if $i \neq 0$ and $\nu \equiv 0$ on $B_q^{\mathcal{U}}(X)$. This map extends linearly to the vector space $S_q^{\mathcal{U}}(X)$ of \mathcal{U} -small chains. The extension, which we denote by ν as well, satisfies $\delta\nu(d) = \nu(\partial d) = 0$ for every \mathcal{U} -small $(q+1)$ -simplex d so it may be seen as a cocycle that represents a cohomology class in $H_{\mathcal{U}}^q(X)$. From the definition of ν we obtain that $\nu(c) = \sum_i k_i \nu(\sigma_i) = k_0 \neq 0$, as desired. \square

Let us formulate the previous results in algebraic terms. Regard again the integral as a bilinear pairing

$$\int : H_{\mathcal{U}}^q(X) \times H_q^{\mathcal{U}}(X) \rightarrow \mathbb{K}.$$

This then defines linear maps from one of its source groups to the dual of the other in a canonical fashion:

$$\begin{array}{ccc} H_{\mathcal{U}}^q(X) & \rightarrow & \text{Hom}(H_q^{\mathcal{U}}(X), \mathbb{K}) \\ z & \mapsto & I(z, \cdot): \gamma \mapsto \int_{\gamma} z \end{array} \qquad \begin{array}{ccc} H_q^{\mathcal{U}}(X) & \rightarrow & \text{Hom}(H_{\mathcal{U}}^q(X), \mathbb{K}) \\ \gamma & \mapsto & I(\cdot, \gamma): z \mapsto \int_{\gamma} z \end{array}$$

We shall call these the canonical maps associated to the bilinear form \int at scale \mathcal{U} .

Proposition 21. *Let \mathcal{U} be a finite open covering of X . Then, the canonical maps at scale \mathcal{U} are isomorphisms.*

Proof. The conclusion follows from Propositions 19 and 20 and the fact that homology and cohomology groups $H_q^{\mathcal{U}}(X; \mathbb{K})$ and $H_q^{\mathcal{U}}(X; \mathbb{K})$ are finite dimensional by Remarks 3 and 12 (in fact, if one group is finite dimensional then so is the other). \square

One may wonder whether the proposition above holds true not only at a specific scale \mathcal{U} but also in the limit. That is, if we now consider the integral as a bilinear pairing $\int : \check{H}^q(X) \times \check{H}_q(X) \rightarrow \mathbb{K}$, are the canonical maps isomorphisms? The following theorem provides an answer for compact spaces. The assumption on compactness is needed to have a cofinal system of finite open coverings on X .

Theorem 22. *Let X be compact. Then:*

(1) *The canonical map*

$$\begin{array}{ccc} \check{H}_q(X) & \rightarrow & \text{Hom}(\check{H}^q(X), \mathbb{K}) \\ \gamma & \mapsto & I(\cdot, \gamma) \end{array}$$

is an isomorphism.

(2) *The canonical map*

$$\begin{array}{ccc} \check{H}^q(X) & \rightarrow & \text{Hom}(\check{H}_q(X), \mathbb{K}) \\ z & \mapsto & I(z, \cdot) \end{array}$$

is injective. It is an isomorphism if either one of $\check{H}_q(X)$ or $\check{H}^q(X)$ are finite dimensional (in which case both are).

The theorem is an algebraic consequence of Proposition 21 and its proof is perhaps less cumbersome when formulated as such. The construction of the integral, in the abstract, can be described as follows. We have a direct system of vector spaces $\{V_i\}$ (the cohomology groups $H_{\mathcal{U}}^q(X)$) and an inverse system of vector spaces $\{W_i\}$ (the homology groups $H_q^{\mathcal{U}}(X)$) both indexed over the same set F . We shall denote by $\alpha_{ij}: V_i \rightarrow V_j$ and by $\beta_{ij}: W_j \rightarrow W_i$ the bonding maps of the systems and write $\alpha_i: V_i \rightarrow \varinjlim V_i$ and $\beta_i: \varprojlim W_i \rightarrow W_i$ for the canonical maps that relate each term of the system with the limit. For each index i there is a bilinear form $B_i: V_i \times W_i \rightarrow \mathbb{K}$ which is compatible with the bonding maps α_{ij} and β_{ij} in the sense that whenever $j \geq i$ the following holds:

$$(3) \quad B_j(\alpha_{ij}(v_i), w_j) = B_i(v_i, \beta_{ij}(w_j)) \quad \text{for every } v_i \in V_i, w_j \in W_j.$$

This is the abstract formalization of Remark 16. Finally, the integral is a bilinear map $B: (\varinjlim V_i) \times (\varprojlim W_i) \rightarrow \mathbb{K}$ defined as follows: to compute $B(v, w)$ we find an index i big enough so that there exists $v_i \in V_i$ such that $\alpha_i(v_i) = v$ and then set

$$(4) \quad B(v, w) := B_i(v_i, \beta_i(w))$$

or equivalently

$$B(\alpha_i(v_i), w) = B_i(v_i, \beta_i(w)).$$

In general this does not yield a well defined map B because the right hand side depends on the index i , but in the case of the integral it does, see Remark 16.

Proposition 23. *In the setting just described, assume that the canonical maps associated to each B_i are isomorphisms (and that Equation (4) correctly defines B). Then the following hold:*

(1) *The canonical map*

$$\begin{array}{ccc} \varprojlim W_i & \rightarrow & \text{Hom}(\varinjlim V_i, \mathbb{K}) \\ w & \mapsto & B(\cdot, w) \end{array}$$

is an isomorphism.

(2) *The canonical map*

$$\begin{array}{ccc} \varinjlim V_i & \rightarrow & \text{Hom}(\varprojlim W_i, \mathbb{K}) \\ v & \mapsto & B(v, \cdot) \end{array}$$

is injective. It is an isomorphism if either one of $\varinjlim V_i$ or $\varprojlim W_i$ are finite dimensional (in which case both are).

Proof. For the sake of brevity we shall write $V = \varinjlim V_i$ and $W = \varprojlim W_i$.

(1) We only prove that $w \mapsto B(\cdot, w)$ is surjective, leave injectivity to the reader. Let $f: V \rightarrow \mathbb{K}$ be a homomorphism. For each index i define $f_i := f \circ \alpha_i$, which is a homomorphism from V_i to \mathbb{K} . Since the canonical map $w_i \mapsto B_i(\cdot, w_i)$ is an isomorphism by assumption, there exists a unique $w_i \in W_i$ such that $f_i(v_i) = B_i(v_i, w_i)$ for each $v_i \in V_i$. Now, for any other index $j \geq i$ we also have a homomorphism f_j and an element $w_j \in W_j$ constructed in exactly the same manner and such that $f_j(v_j) = B_j(v_j, w_j)$ for every $v_j \in V_j$. We claim that $\beta_{ij}(w_j) = w_i$. Indeed, for any $v_i \in V_i$ we have

$$B_i(v_i, \beta_{ij}(w_j)) = B_j(\alpha_{ij}(v_i), w_j) = f_j(\alpha_{ij}(v_i)) = f_i(v_i)$$

Thus $\beta_{ij}(w_j)$ also satisfies the defining property of w_i and, by the uniqueness of w_i , we conclude that $w_i = \beta_{ij}(w_j)$ as claimed. This implies that $(w_i)_{i \in F}$ is an element of the inverse limit $W = \varprojlim W_i$ or, in other words, there exists $w \in W$ such that $\beta_i(w) = w_i$ for every $i \in F$. But then for any element $v \in V$ we may choose an index i and an element $v_i \in V_i$ such that $\alpha_i(v_i) = v$, so that

$$f(v) = f_i(v_i) = B_i(v_i, w_i) = B_i(v_i, \beta_i(w)) = B(v, w)$$

where in the last step we have made use of Equation (4).

(2) This follows from part (1). For if $v \neq 0$, there is a linear map $f: V \rightarrow \mathbb{K}$ such that $f(v) \neq 0$ and by (1) $f = B(\cdot, w)$ for some $w \in W$. Since $B(v, w) = f(v) \neq 0$, $B(v, \cdot)$ is not identically zero. □

Proof of Theorem 22. Apply Proposition 23 to the bilinear pairings determined by the integral at scales defined by finite open coverings of X . This can be done because Proposition 21 ensures that at each level \mathcal{U} the integral $\int : H_{\mathcal{U}}^q(X) \times H_{\mathcal{U}}^q(X) \rightarrow \mathbb{K}$ is nondegenerate; that is, both canonical maps are isomorphisms. The theorem follows from the fact that finite open coverings are cofinal among all open coverings of X . □

8. APPLICATION: GENERALIZATION OF A THEOREM OF MANNING ON ENTROPY

In the following we assume X is a compact space and $f: X \rightarrow X$ a continuous map. Let us recall first the definition of entropy of a map. This notion roughly computes the base of the exponential growth rate of the number of pieces of orbits that are distinguishable, in the sense that they lie at least ϵ apart. Since our discussion is purely topological, we use the definition of topological entropy by Adler, Konheim and McAndrew [1], where proximity is interpreted in terms of the elements of an open covering. Given an open covering \mathcal{U} of X we let $s(\mathcal{U})$ denote the minimum number of elements of \mathcal{U} that covers X , that is, the smallest size of a subcover of \mathcal{U} (which is finite by compactness). If we set $\mathcal{U}^n = \mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-n}\mathcal{U}$, the limit

$$\lim_{n \rightarrow +\infty} \frac{s(\mathcal{U}^n)}{n}$$

exists and is denoted by $h(f, \mathcal{U})$. It satisfies the relation: \mathcal{V} refines $\mathcal{U} \rightarrow h(f, \mathcal{V}) \geq h(f, \mathcal{U})$.

Definition 24. *The topological entropy of f is $h(f) = \sup h(f, \mathcal{U})$, where the supremum ranges among all open coverings \mathcal{U} of X .*

Of course, this definition of topological entropy is equivalent in a metric space to the definitions that use (n, ϵ) -separated or spanning sets which are more common in dynamical systems [12].

In [16] Manning proved the following theorem: if $f: X \rightarrow X$ is a continuous map of a compact manifold X , then its entropy $h(f)$ is bounded below by the logarithm of the spectral radius of $f_*: H_1(X; \mathbb{C}) \rightarrow H_1(X; \mathbb{C})$. That is, $h(f) \geq \log |\lambda|$ for any eigenvalue λ of f_* . In his paper Manning actually showed that his arguments work in more general compact metric spaces. Theorem 2 in [16] states that the inequality holds as long as “two local niceness properties” are satisfied: the first one is slightly weaker than local path-connectedness and the second property asks for any small loop to be homotopically trivial. As pointed out in the Introduction, Manning believed that Čech cohomology was more appropriate to relate entropy to eigenvalues. This is the language employed in the generalizations of the result of Manning to arbitrary compact spaces X presented below.

The original proof of Manning roughly kept track of the length of the iterates $f^n(\gamma)$ of a path γ . In our work, while there are several technical issues that arise from the possible complicated local topology of X , the philosophy of the proof is somehow similar. The integral plays a capital role in our argument as the measure of length is replaced by the integral of some cohomology class.

Let us state the theorems.

Theorem 25. *Let X be compact and locally connected and $f: X \rightarrow X$ be continuous. Assume that $f^*: \check{H}^1(X; \mathbb{C}) \rightarrow \check{H}^1(X; \mathbb{C})$ has an eigenvalue $\lambda \in \mathbb{C}$ with modulus $|\lambda| > 1$. Then the topological entropy of f satisfies $h(f) \geq \log |\lambda|$.*

Notice that this generalizes the classical Manning’s inequality, since for a manifold one has $\check{H}^1(X; \mathbb{C}) = H^1(X; \mathbb{C})$ and for a compact manifold the latter is finite dimensional and isomorphic to the dual of $H_1(X; \mathbb{C})$.

When X is not locally connected the lower bound we obtain is smaller than the expected $\log |\lambda|$ but is still positive, so it ensures that $h(f) > 0$:

Theorem 26. *Let X be a compact space and $f: X \rightarrow X$ a continuous map. Assume that $f^*: \check{H}^1(X; \mathbb{C}) \rightarrow \check{H}^1(X; \mathbb{C})$ has an eigenvalue $\lambda \in \mathbb{C}$ with modulus $|\lambda| > 1$. Then $h(f) \geq (\log|\lambda|)/d$, where $d \in \mathbb{Z}^+$ is the degree of the algebraic number λ . In particular, $h(f) > 0$.*

Since the degree of rational numbers is 1, the theorem yields the standard bound of $\log|\lambda|$ for eigenvalues in \mathbb{Q} .

The proofs of the two theorems have a common initial part given in Subsection 8.2 below. After that their proofs diverge. That of Theorem 25 (the locally connected case) is quite straightforward using the machinery already developed and is given in Subsection 8.3. The proof of Theorem 26 is more involved and so we devote a full section (Section 9) to it. Before embarking on this, however, we explain why the theorems involve eigenvalues in cohomology instead of homology.

8.1. Why are the statements formulated in cohomological terms? Another aspect in which Theorems 25 and 26 differ from the original result of Manning is that they assume λ to be an eigenvalue in (Čech) cohomology rather than homology. When $\check{H}^1(X; \mathbb{C})$ or $\check{H}_1(X; \mathbb{C})$ are finite dimensional these two spaces are dual to each other (see Theorem 22) and so one can equivalently assume λ to be an eigenvalue in homology. However, as proved earlier in general Čech homology is isomorphic to the dual of Čech cohomology, so there might exist eigenvalues in homology which are not present in cohomology and violate Manning’s inequality. We now describe an example of compact, locally connected metric space X where this phenomenon occurs.

The space X is shown in Figure 3. It consists of a biinfinite sequence of circumferences $\{C_i\}_{i \in \mathbb{Z}}$ labeled from left to right (with C_0 being the biggest one in the middle) and two limiting points L and R at both ends of the sequence.

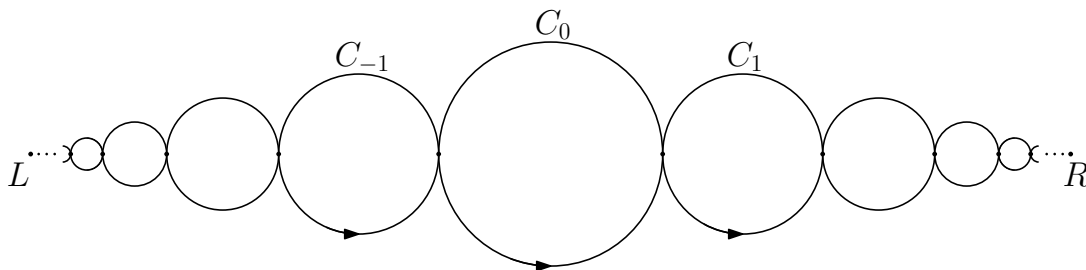


FIGURE 3.

The map $f: X \rightarrow X$ is defined as follows:

- (i) It fixes the endpoints L and R . Other than that, f sends each point $C_{i+1} \cap C_i$ to $C_i \cap C_{i-1}$.
- (ii) f sends each oriented (as in the figure) C_i onto a curve that goes around C_{i-1} twice.

Notice that conditions (i) and (ii) are compatible but force f to behave differently in the upper and lower arcs. Now if $\gamma \in \check{H}_1(X; \mathbb{C})$ is the “sum” $\sum_i C_i$, we see that $f_*(\gamma) = 2\gamma$, so that $\lambda = 2$ is an eigenvalue of f_* .

The computation of the entropy is straightforward as it can be circumscribed to the non-wandering set, which in this case is just $\{L, R\}$, an attractor-repeller decomposition of X . (For

a discussion and proof of this fact see [12, p. 130ff.]). This implies that $h(f) = h(f|_{\{L,R\}}) = 0$, so Manning's inequality fails to be true. In cohomology one sees easily that f^* has no eigenvalues and so Theorem 25 holds vacuously.

Another way of manufacturing counterexamples to the homological version of the theorem, albeit non locally connected ones, is to exploit the fact that Manning's inequality may be strongly violated in dimension zero. As an illustration let us take [9, Corollary 3]: if Z is a compact, metric, totally disconnected space and $g: Z \rightarrow Z$ is a continuous map having a nonzero topological entropy, then every $\lambda \in \mathbb{C}$ with $|\lambda| \neq 0, 1$ is an eigenvalue of $g_*: \check{H}_0(Z; \mathbb{C}) \rightarrow \check{H}_0(Z; \mathbb{C})$. In particular if g has a finite entropy then the inequality $h(g) \geq \log |\lambda|$ for every eigenvalue λ is manifestly false. This is the case, for example, of Smale's horseshoe with its usual dynamics which has an entropy $h(g) = \log 2$ (see [12, 2.5c and p. 121]).

Let SZ be the suspension of Z . From the fact that the Mayer-Vietoris sequence for Čech homology with coefficients in a field is exact [5], it can be easily deduced that $\check{H}_1(SZ; \mathbb{C})$ is isomorphic to $\check{H}_0(Z; \mathbb{C})$, where reduced 0-dimensional homology is intended. Then, we can plug the dynamics of g in the base level $Z \times \{0\}$ of SZ and extend it to a dynamics on $f: SZ \rightarrow SZ$ that fixes the poles N, S , preserves the upper and lower cones and makes $(Z \times \{0\}, \{N, S\})$ an attractor-repeller decomposition. As above, the entropy of f is readily seen to be equal to $h(g)$. However, the maps f_* and g_* induced in $\check{H}_1(SZ; \mathbb{C})$ and $\check{H}_0(Z; \mathbb{C})$ are conjugate by Δ , the connecting homomorphism of the Mayer-Vietoris sequence associated to the decomposition of SZ in upper and lower cones. In particular both maps have the same eigenvalues, which shows that Manning's inequality fails for f since it fails (by assumption) in dimension zero for g .

8.2. Proof of Theorems 25 and 26. The first steps are common and serve as an outline of both proofs. The argument starts with a nonzero eigenvector z in $\check{H}^*(X; \mathbb{C})$ of eigenvalue λ that is arbitrary in the proof of Theorem 25 but needs to be carefully selected (in a manner to be described in the next section) for the proof of Theorem 26.

Step 1. Let $z \in \check{H}^1(X; \mathbb{C})$ be an eigenvector of eigenvalue λ , i.e. a solution of $f^*z = \lambda z$ (*), and \mathcal{V} an open covering of X such that $z = \pi_{\mathcal{V}}(z_{\mathcal{V}})$ for some $z_{\mathcal{V}} \in H_{\mathcal{V}}^1(X; \mathbb{C})$. Since the cohomology classes $\lambda z_{\mathcal{V}}$ and $f^*z_{\mathcal{V}} \in H_{f^{-1}\mathcal{V}}^*(X; \mathbb{C})$ define the same element in the direct limit $\check{H}^1(X; \mathbb{C})$, there exists some open covering \mathcal{U} , finer than $\mathcal{V} \vee f^{-1}\mathcal{V}$, for which the projections of $\lambda z_{\mathcal{V}}$ and $f^*z_{\mathcal{V}}$ to $H_{\mathcal{U}}^1(X; \mathbb{C})$ are equal. In other words, $z_{\mathcal{V}}$ (or, formally, $\pi_{\mathcal{V}\mathcal{U}}(z_{\mathcal{V}})$) is a solution of (*) at scale \mathcal{U} . Note that \mathcal{U} can be chosen arbitrarily fine and will henceforth be fixed.

Step 2. Fix a natural number n and denote \mathcal{V}_n a finite subcover of $\mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-n}\mathcal{U}$. For any given \mathcal{V}_n -small 1-cycle s_n , the cycles $s_n, f_{\sharp} s_n, \dots, f_{\sharp}^n s_n$ are \mathcal{U} -small and represent homology classes $\gamma_n, f_* \gamma_n, \dots, f_*^n \gamma_n \in H_1^{\mathcal{U}}(X)$, respectively. For every $0 \leq j < n$

$$\int_{f_*^{j+1} \gamma_n} z = \int_{f_*^j \gamma_n} f^* z = \lambda \int_{f^j \gamma_n} z,$$

where the middle term shall be interpreted as an integral at scale $f^{-1}\mathcal{U}$, whereas the first and last belong to scale \mathcal{U} and we use the properties of the integral formulated in Remark 16 and

Lemma 18. By induction,

$$(5) \quad \int_{f_*^n \gamma_n} z = \lambda^n \int_{\gamma_n} z$$

Step 3. By Corollary 11, there exists a representative η of z whose image is finite. Then,

$$(6) \quad \left| \int_{f_*^n \gamma_n} z \right| = \left| \int_{[f_*^n s_n]} [\eta] \right| \leq \|f_*^n s_n\|_1 \|\eta\|_\infty \leq \|s_n\|_1 \|\eta\|_\infty.$$

where the first inequality is a consequence of the definition of the integral as a sum of evaluations over simplices and the second inequality accounts for the possible alteration of the norm of the chains due to cancellation of simplices after applying f_*^n to the chain s_n .

At this point the arguments needed to address Theorems 25 and 26 diverge. The idea is to bound $\|s_n\|_1$ from below in terms of $|\mathcal{V}_n|$ in (6) and control the integral of z along γ_n in (5) at the same time. This is easy in the locally connected case using the refinements constructed in page 10 because the value of the integral does not depend on n . However, in the general case there is no easy and coherent choice of γ_n and s_n and the argument is more delicate. The idea is to use some arithmetical properties of z to deduce a lower bound for the absolute value of the integral in terms of $|\mathcal{V}_n|$.

8.3. Final part of the proof of Theorem 25. Using the local connectedness of X , after passing to a refinement we can assume that the open covering \mathcal{U} is composed of connected sets. Then, we can use the arguments in page 10 to choose wisely the cycles s_n . Take a \mathcal{U} -small 1-cycle s_0 that defines a class $\gamma = [s_0] \in H_1^{\mathcal{U}}(X)$ such that $\int_\gamma z \neq 0$. This exists by Proposition 21. For every n , let s_n be the \mathcal{V}_n -small refinement of s_0 produced by Lemma 9, which satisfies $\|s_n\|_1 \leq \|s_0\|_1 |\mathcal{V}_n|$. Since s_n and s_0 are homologous as \mathcal{U} -small cycles,

$$\int_{\gamma_n} z = \int_\gamma z$$

and then Equation (6) yields

$$|\mathcal{V}_n| \geq C |\lambda|^n \quad \text{where} \quad C = \frac{1}{\|\eta\|_\infty \|s_0\|_1} \left| \int_\gamma z \right|$$

It is then clear that $\lim_{n \rightarrow +\infty} \frac{1}{n} \log |\mathcal{V}_n| \geq \log |\lambda|$ and, since \mathcal{V}_n was a subcover of $\mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-n}\mathcal{U}$ it follows that $h(f, \mathcal{U}) \geq \log |\lambda|$. Thus, we conclude that $h(f) \geq \log |\lambda|$. \square

9. PROOF OF THEOREM 26

The case in which X is not locally connected is more difficult as it is not clear whether there is a choice of cycles s_n that allows to control the right hand side of (5) and the norm of s_n at the same time. The argument we provide is based on a careful inspection of z and s_n . Although the eigenvector z is a cohomology class in $H_{\mathcal{U}}^1(X; \mathbb{C})$, we will prove that we may assume z to be a linear combination of rational cohomology classes with algebraic coefficients. Then, we benefit from the nice arithmetic properties of algebraic numbers to obtain a Diophantine lemma that bounds from below the absolute value of the integral of z along $\gamma_n = [s_n]$ (provided it is non-zero) in terms of the norm of s_n . Finally, we have to

choose s_n in a way that its norm is controlled by the size of \mathcal{V}_n and that guarantees that the value of the integral of z does not vanish.

The preliminary work for the choice of s_n has already been done in Section 4, as it is enough to pick a suitable simple elementary cycle. However, we still have to go through some technical lemmas to justify the arithmetic properties of z .

9.1. Remarks about coefficients. Selection of eigenvector. From now on we are going to make use simultaneously of coefficients in \mathbb{Z} , \mathbb{Q} and \mathbb{C} , so these will always be reflected in the notation. Observe first that a homology (or cohomology, for which the remark also applies verbatim) class γ with coefficients in G can be also regarded as a class with coefficients in any larger group G' , for example the case of \mathbb{Z} or \mathbb{Q} and \mathbb{C} . In view of the definition of the integral this subtlety does not affect the computation as a representative c of γ with coefficients in G is also a representative for the class with coefficients in G' .

We also need a brief discussion on the relationship between $H_{\mathcal{U}}^1(X; \mathbb{Q})$ and $H_{\mathcal{U}}^1(X; \mathbb{C})$. By the universal coefficient theorem [22] (and using that \mathcal{U} is finite and so all modules are of finite type), $H_{\mathcal{U}}^q(X; \mathbb{Q}) \otimes \mathbb{C} \cong H_{\mathcal{U}}^q(X; \mathbb{C})$ as \mathbb{Q} -vector spaces, where the isomorphism is given on the generators of the tensor product by $[\xi] \otimes \lambda \mapsto \lambda[\xi]$.

Suppose $f: X \rightarrow X$ is a continuous map. Let us denote by $f_{\mathbb{Q}}^*$ and $f_{\mathbb{C}}^*$ the endomorphisms induced by f in $\check{H}^1(X; \mathbb{Q})$ and $\check{H}^1(X; \mathbb{C})$, respectively, and denote by $P_{\lambda}(t) \in \mathbb{Q}[t]$ the minimal polynomial of λ over \mathbb{Q} in the sense of field extensions.

Lemma 27. *If $f_{\mathbb{C}}^*$ has an eigenvalue $\lambda \in \mathbb{C}$, then there exists a nonzero $w \in \check{H}^1(X; \mathbb{Q})$ whose minimal polynomial for $f_{\mathbb{Q}}^*$ is well defined and equals P_{λ} .*

Proof. Since the lemma is purely algebraic we shall give a proof that works in a more general setting. Let E be an abstract \mathbb{Q} -vector space and $g: E \rightarrow E$ an endomorphism of E . The \mathbb{C} -vector space $E \otimes \mathbb{C}$ is generated by the elements of the form $u \otimes 1$, $u \in E$ and the map g induces an endomorphism $\hat{g}: E \otimes \mathbb{C} \rightarrow E \otimes \mathbb{C}$ which acts on the generators as $\hat{g}(u \otimes 1) = g(u) \otimes 1$. We view $E \otimes \mathbb{C}$ as a \mathbb{C} -vector space with the complex multiplication absorbed by the second factor: $\nu \cdot u \otimes \mu = u \otimes \nu\mu$. In our setting g corresponds to $f_{\mathbb{Q}}^*$ and \hat{g} to the conjugation of $f_{\mathbb{C}}^*$ by the isomorphism of the universal coefficient theorem.

Any subspace $F \subset E$ spanned by a finite collection $\{u_j\}$ of linearly independent vectors has an associated (\mathbb{C} -linear) subspace \hat{F} of $E \otimes \mathbb{C}$ spanned by $\{u_j \otimes 1\}$. Incidentally, note that $\{u_j \otimes 1\}$ are linearly independent in $E \otimes \mathbb{C}$ because they are clearly independent as elements of the complexification of E , $E^{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$. One can check that:

$$\widehat{F_1 \cap F_2} = \hat{F}_1 \cap \hat{F}_2 \quad \text{and} \quad \widehat{g(F)} = \hat{g}(\hat{F}).$$

Assume that \hat{g} has an eigenvector $v \in E \otimes \mathbb{C}$ of eigenvalue λ . Taking $\{u_j\}$ to be a basis of E we have that $\{u_j \otimes 1\}$ is a basis of $E \otimes \mathbb{C}$, and so v is a linear combination of finitely many of them. Thus there exists an $F \subset E$ spanned by finitely many of the u_j such that $v \in \hat{F}$. Among all such finite dimensional \mathbb{Q} -linear subspaces F of E , choose one with the smallest dimension. Then we must have $g(F) = F$ since, otherwise, we could replace F with $F \cap g(F)$ because $v \in \hat{F} \cap \hat{g}(\hat{F}) = \widehat{F \cap g(F)}$.

The minimal polynomials of \hat{g} in \hat{F} and g in F coincide. Denote the latter by P_F . Then, $P_F(\lambda) = 0$, so $P_F = Q \cdot P_{\lambda}$ for some $Q(t) \in \mathbb{Q}[t]$. The conclusion follows from the fact that any nonzero vector in $\text{Im}(Q(g)|_F)$ has minimal polynomial equal to P_{λ} . \square

Remark 28. In the sequel we will make no notational distinction between $f_{\mathbb{Q}}^*$ and $f_{\mathbb{C}}^*$ and d will denote the degree of λ over \mathbb{Q} , $d = \deg(\lambda) = \deg(P_\lambda)$.

Lemma 29. *If f^* has an eigenvalue $\lambda \in \mathbb{C}$, there exists an eigenvector $z \in \check{H}^1(X; \mathbb{C})$ for λ of the form $z = \sum_{j=0}^{d-1} \mu_j w_j$, where $\mu_j \in \mathbb{C}$ are algebraic numbers and $w_j \in \check{H}^1(X; \mathbb{Q})$.*

Proof. By Lemma 27, there exists $w \neq 0 \in \check{H}^1(X; \mathbb{Q})$ whose minimal polynomial is $P_\lambda(t)$. Let $\lambda = \lambda_1, \dots, \lambda_d \in \mathbb{C}$ be its complex roots. Then, $P_\lambda(t) = (t - \lambda) \cdot (t - \lambda_2) \dots (t - \lambda_d)$.

Observe that $\{w, f^*w, \dots, (f^*)^{d-1}w\}$ are linearly independent in $\check{H}^1(X; \mathbb{Q})$ and therefore also in $\check{H}^1(X; \mathbb{C})$. Define an element $z \in \check{H}^1(X; \mathbb{C})$ by

$$z := (f^* - \lambda_2) \dots (f^* - \lambda_d)w.$$

(If $\deg(P_\lambda) = 1$ then $z = w$). Evidently $(f^* - \lambda)z = P_\lambda(f^*)w = 0$, and so $f^*z = \lambda z$. Also, z is nonzero. To check this expand its definition to get

$$\begin{aligned} z &= (f^*)^{d-1}w - (\lambda_2 + \dots + \lambda_d)(f^*)^{d-2}w + \dots + (-1)^{d-1}\lambda_2 \dots \lambda_d w \\ &= \mu_0(f^*)^{d-1}w + \mu_1(f^*)^{d-2}w + \dots + \mu_{d-1}w \end{aligned}$$

which is a linear combination of $\{w, f^*w, \dots, (f^*)^{d-1}w\}$ with complex coefficients μ_j at least one of which is nonzero ($\mu_0 = 1$). In particular, $z \neq 0$ and z is an eigenvector of eigenvalue λ .

Finally, observe that the μ_j are all algebraic. Indeed, all the μ_j (which are sums and products of the λ_j) are algebraic numbers since the set of algebraic numbers is a field (see for instance [14, Corollary 2.6, p. 232]). \square

Remark 30. In a very similar way to Step 1 in Subsection 8.2, we can assume that the description of z in the lemma is valid at scale \mathcal{U} . Indeed, we can find an open covering \mathcal{V} of X and classes $z' \in H_{\mathcal{U}}^1(X; \mathbb{C})$, $w'_j \in H_{\mathcal{U}}^1(X; \mathbb{Q})$ such that $\pi_{\mathcal{V}}(z') = z$, $\pi_{\mathcal{V}}(w'_j) = w_j$. Then, for some \mathcal{U} finer than \mathcal{V} , $\pi_{\mathcal{V}\mathcal{U}}(\sum \mu_j w'_j) = \pi_{\mathcal{V}\mathcal{U}}(z')$. So, with an already familiar abuse of notation

$$z = \sum \mu_j w_j \text{ in } H_{\mathcal{U}}^1(X; \mathbb{C}),$$

where $w_j \in H_{\mathcal{U}}^1(X; \mathbb{Q})$. As in Step 1, by refining \mathcal{U} even further we can also guarantee that the eigenvector equation $f^*z = \lambda z$ holds in $H_{\mathcal{U}}^1(X; \mathbb{C})$.

9.2. A Diophantine approximation lemma. Let us start with a theorem due to Schmidt that generalizes several celebrated classical results that roughly state that irrational algebraic numbers are badly approximated by rational numbers. The precise result is Theorem 2 from [20] (see also [26]):

Theorem 31. *Let ν_1, \dots, ν_m be real algebraic numbers such that $\{1, \nu_1, \dots, \nu_m\}$ are linearly independent over \mathbb{Q} . For every $\epsilon > 0$ there are only finitely many m -tuples of nonzero integers a_1, \dots, a_m with*

$$\text{dist}(a_0 + \nu_1 a_1 + \dots + \nu_m a_m, \mathbb{Z}) \geq \frac{1}{|a_1 \dots a_m|^{1+\epsilon}}$$

The inequality immediately implies

$$(7) \quad |a_0 + \nu_1 a_1 + \dots + \nu_m a_m| \geq A \cdot (\max |a_i|)^{-(m+\epsilon)}$$

for all $(m+1)$ -tuples of integers a_j , with $a_j \neq 0$ if $j > 0$, and some constant A that only depends on ϵ . Now, if we assume that the left hand side of the inequality is not zero, at the

expense of replacing A by a larger constant we can suppose that (a_0, a_1, \dots, a_m) ranges over all \mathbb{Z}^{m+1} and the numbers $1, \nu_1, \dots, \nu_m$ are not necessarily rationally independent (use the linear relations to simplify in (7) until it only depends on a maximal subset of independent ν_j , the factors are absorbed by A). Then, it is clear that the inequality also applies to complex ν_j (note that rational linear independence of complex numbers is weaker than independence of its real or imaginary parts).

Plug $\mu_0 = 1, \mu_1, \dots, \mu_{d-1}$ from Lemma 29 in (7) to deduce that for every $\epsilon > 0$ there exists a constant $A > 0$ such that

$$(8) \quad \left| \sum_{j=0}^{d-1} \mu_j C_j \right| \geq \frac{A}{(\max |C_j|)^{d-1+\epsilon}}$$

for every $(C_0, C_1, \dots, C_d) \in \mathbb{Z}^{d+1}$ unless the sum on the left vanishes.

In view of the decomposition of z proved in Lemma 29, the possible nonzero values that the integral of z along an arbitrary homology class γ may take can be bounded from below in terms of the number of simplices that compose a \mathcal{U} -small representative of γ .

Lemma 32. *Let \mathcal{U} be a finite open covering of X . Suppose $z \in H_{\mathcal{U}}^1(X; \mathbb{C})$ has the form*

$$z = \sum_{j=0}^{d-1} \mu_j w_j$$

where each μ_j is an algebraic complex number and all the $w_j \in H_{\mathcal{U}}^1(X; \mathbb{Q})$. Then, for every $\epsilon > 0$ there exist a constant $B > 0$ such that, for any \mathcal{U} -small cycle $[c] \in H_1^{\mathcal{U}}(X; \mathbb{Z})$, the integral $\int_{[c]} z$ is either zero or its absolute value is bounded below as

$$\left| \int_{[c]} z \right| \geq \frac{B}{\|c\|_1^{d-1+\epsilon}}.$$

Proof. Each w_j is represented by some cocycle ξ_j (with values in \mathbb{Q}) whose coboundary vanishes at scale \mathcal{U} , and we may assume without loss of generality that $\text{im } \xi_j$ is finite by Corollary 11. Thus, there exists an integer $D > 0$ such that the image of each $D\xi_j$ consists of integer numbers.

Write $c = \sum_i k_i \sigma_i$ with $k_i \in \mathbb{Z}$. By definition the integral $\int_{[c]} z = \sum_j \mu_j \int_{[c]} w_j$ is just the sum

$$S = \sum_j \mu_j \sum_i k_i \xi_j(\sigma_i) \quad \text{so} \quad DS = \sum_j \mu_j C_j, \quad \text{where we define } C_j := D \sum_i k_i \xi_j(\sigma_i) \in \mathbb{Z}.$$

We can estimate C_j by $|C_j| \leq D \|\xi_j\|_{\infty} \sum_i |k_i| = D \|\xi_j\|_{\infty} \|c\|_1$, so in particular

$$\max |C_j| \leq D \|c\|_1 M, \quad \text{where } M := \max \|\xi_j\|_{\infty}.$$

Now inequality (8) shows that if S is nonzero, then it is bounded below by

$$|S| \geq \frac{1}{|D|} \frac{A}{(\max |C_j|)^{d-1+\epsilon}} \geq \frac{A}{D^{d+\epsilon} M^{d-1+\epsilon} \|c\|_1^{d-1+\epsilon}}.$$

It only remains to group everything other than $\|c\|_1^{d-1+\epsilon}$ into a constant B . This constant depends on D and M (which are fixed once the choice of the cochains ξ_j is done), A , which is determined by the μ_j and depends on ϵ , and ϵ itself. The result follows. \square

9.3. Proof of Theorem 26. Consider the eigenvector z of eigenvalue λ from Lemma 29 and the open covering \mathcal{U} found in Remark 30. Recall that from Steps 1-3 in Subsection 8.2 we have

$$(9) \quad |\lambda^n| \left| \int_{[s_n]} z \right| \leq \|s_n\|_1 \|\eta\|_\infty$$

Since z can be thought of as a cohomology class at scale \mathcal{U} and $z \neq 0 \in \check{H}^1(X; \mathbb{C})$, it defines a non-trivial element of $H_{\mathcal{V}_n}^1(X; \mathbb{C})$ which by the non degeneration of the integral at a specific scale (Proposition 21) has a nonzero integral over some element in $H_1^{\mathcal{V}_n}(X; \mathbb{C})$. By Proposition 7 the simple elementary cycles generate $H_1^{\mathcal{V}_n}(X; \mathbb{Z})$ and therefore also $H_1^{\mathcal{V}_n}(X; \mathbb{C})$, so in fact there exists a \mathcal{V}_n -small simple elementary cycle s_n such that $\int_{[s_n]} z \neq 0$.

For the rest of the proof consider $\epsilon > 0$ fixed. Now, by Lemma 32 applied to $z = \sum_{j=0}^{d-1} \mu_j w_j$ there exist a constant $B > 0$ such that

$$\left| \int_{[s_n]} z \right| \geq \frac{B}{\|s_n\|_1^{d-1+\epsilon}}$$

Combine this equation with (9) and recall that, by definition, a \mathcal{V}_n -small simple cycle satisfies $\|s_n\|_1 \leq |\mathcal{V}_n|$. We deduce

$$B|\lambda|^n \leq \|\eta\|_\infty |\mathcal{V}_n|^{d+\epsilon}.$$

Notice that in this inequality none of $\|\eta\|_\infty, d, \epsilon$ depends on n . A straightforward computation then shows that

$$\frac{\log |\lambda|}{d + \epsilon} \leq \lim_{n \rightarrow +\infty} \frac{1}{n} \log |\mathcal{V}_n|,$$

and since \mathcal{V}_n was an arbitrary finite subcover of $\mathcal{U} \vee \cdots \vee f^{-n}\mathcal{U}$ for every n (and $\epsilon > 0$ is arbitrary) this bounds $h(f, \mathcal{U})$ from below. It follows that $h(f) \geq (\log |\lambda|)/d > 0$. This finishes the proof of Theorem 26.

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