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Explicit Formulas for the 3-Jet Lift of a Matrix Group. Apllications to Conformal Geometry

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Abstract:

The 3-jet lift G^3 of a matrix group G is isomorphic, via a map that we give explicitly, to a semidirect product of G itself and a nilpotent group builded up from the first two prolongations of its Lie algebra. Using this isomorphism, we write down the formulas for the most natural representations of G^3, as well as for one additional representation of the 2-jet lift G^2 appearing when G is of finite type 2. We apply these results to the case of the (linear) conformal group and we point out the geometric implications of these representations.

1 Introduction

The r-jet lift G^r (with $r \in \mathbf{N}$, $\mathbf{r} > \mathbf{0}$) of a Lie subgroup G of the general linear group $GL(n, \mathbf{R})$ is the Lie group of r-jets at $0 \in \mathbf{R^n}$ of (local, 0-preserving) diffeomorphisms of $\mathbf{R^n}$ which preserve, in an obvious sense and up to order $(r-1)$, the canonical G-structure on $\mathbf{R^n}$.

The r-jet lift of $GL(n, \mathbf{R})$ itself is the structure group of the principal r-frame bundle $F^r(M)$ over any n-dimensional manifold M, and some of its properties have been studied in this context [**5**,**8**,**10**]. The second order frame bundle $F^2(M)$ has been extensively studied and explicit expressions concerning the 2-jet lift of $GL(n, \mathbf{R})$ are well known; see e.g. [**3**]. A proper use of such expressions leads [**1**] to a new proof of the first structure equation —first found in [**10**]— for connections in $F^2(M)$.

For a proper Lie subgroup G of $GL(n, \mathbf{R})$, the group G^r comes into play when one deals with "$(r-1)$-integrable" G-structures ("uniformly $(r-1)$-flat" in the sense of [**4**]), i.e. G-structures $P \subset LM$ having, over each point of M, contact of order $(r-1)$ with coordinate sections of the linear frame bundle LM —let us remember that 1-integrable G-structures are those admitting symmetric (linear) connections [**2**,**4**]. For each $(r-1)$-integrable G-structure P, one can define its r-jet lift P^r as the set of r-jets at $0 \in \mathbf{R^n}$ of (local) diffeomorphisms of $\mathbf{R^n}$ into M yielding such order of contact, and it turns out that $P^r(M, G^r)$ becomes a principal subbundle of $F^r(M)$.

Consider the Lie algebra g of G. It may happen that the so called "r-prolongation" g_r vanishes. Remember that when $g_r = 0$ and $g_{r-1} \neq 0$, one says that G is of *finite type* r. In the case g_r vanishes, G^{r+1} becomes isomorphic to G^r, and G^r *inherits some additional properties from the* $(r+1)$ *order*, mainly an *additional natural representation* that it is in the background of the appearance, in some cases [**4**], of Cartan connections on the r-jet lifts P^r of $(r-1)$-integrable G-structures P.

The (pseudo) orthogonal conformal group $CO(n, \mathbf{R})$ is (for $n \geq 3$) of finite type 2 and every $CO(n, \mathbf{R})$-structure is 1-integrable. Both $CO(n, \mathbf{R})$ and its 2-jet lift $CO(n, \mathbf{R})^\mathbf{2}$ can be considered subgroups of the *full* conformal group $O(n+2, \mathbf{R})/\mathbf{Z_2}$. A lot of work has been done in the study of 2-jet lifts of conformal structures in dimension ≥ 3, see e.g. [**6**,**7**], mainly by specific methods of the conformal case. We are interested in a more systematic treatment of connections on 2-jet lifts $P^2(M, G^2)$ of 1-integrable G-structures P when G is of finite type 2, and their relationships with Cartan connections when they exist. Such treatment needs an appropriate handling of the 3-jet lift of a general matrix group. To our knowledge, no explicit formulas are up to now available.

In this paper we give (Lemma) a map which shows the isomorphism between G^3 and a semidirect product of G itself and a nilpotent group builded up from the first two prolongations of the Lie algebra g of G. This map allows us to write down (Theorem) explicit formulas for the most natural representations of G^3 and for the additional representation of G^2 appearing when G is of finite type 2. When G is the (linear) conformal group $CO(n, \mathbf{R})$, one can check that this additional representation is the restriction to $CO(n, \mathbf{R})^\mathbf{2}$ of the adjoint representation of the full conformal group $O(n+2, \mathbf{R})/\mathbf{Z_2}$. This last representation of $CO(n, \mathbf{R})^\mathbf{2}$ controls the behaviour of conformal Cartan connections on the 2-jet lift of a $CO(n, \mathbf{R})$-structure.

2 Basic definitions

For each $v \in \mathbf{R^n}$, let $\mathcal{D}_{0,v}(\mathbf{R^n})$ be the set of diffeomorphisms φ between open subsets of $\mathbf{R^n}$, containing 0 and v respectively, such that $\varphi(0) = v$. The *r-jet* $j_0^r(\varphi)$ is the equivalence class of φ in $\mathcal{D}_{0,v}(\mathbf{R^n})$ induced by the equivalence relation

$$\varphi \overset{r}{\sim} \varphi' \iff D^s\varphi|_0 = D^s\varphi'|_0 , \quad 1 \leq s \leq r.$$

The set $G^r(n) := \mathcal{D}_{0,0}(\mathbf{R^n})/\underset{\sim}{r} = \{j_0^r(g) : g \in \mathcal{D}_{0,0}(\mathbf{R^n})\}$ is a Lie group, the *group of r-jets*, with product law

$$j_0^r(g)\cdot j_0^r(g') := j_0^r(g\circ g')$$

and differentiable structure such that the bijection

$$\begin{array}{rccc} (u_0,u_1,\ldots,u_{r-1}): & G^r(n) & \longrightarrow & GL(n,\mathbf{R})\times \mathbf{S^2(n)}\times\cdots\times \mathbf{S^r(n)} \\ & j_0^r(g) & \longmapsto & \left(Dg\big|_0\,,\, D^2g\big|_0\,,\ldots,\, D^rg\big|_0\right) \end{array}$$

becomes a global chart; we denote by $S^r(n)$ the subspace of symmetric r-linear maps in $L^r(n)\equiv L(\mathbf{R^n}\times \overset{\mathbf{r}}{\cdots}\times\mathbf{R^n},\mathbf{R^n})$. This bijection induces a product law on $GL(n,\mathbf{R})\times\mathbf{S^2(n)}\times\cdots\times\mathbf{S^r(n)}$ which can be explicitly computed by applying the chain rule to the derivatives of the composition of functions. The canonical Lie group isomorphism $G^1(n)\to GL(n,\mathbf{R})$, $\mathbf{j_0^1(g)}\mapsto\mathbf{Dg}|_\mathbf{0}$, will be understood in what follows.

The set $F^r(\mathbf{R^n}) := \{\mathbf{j_0^r}(\varphi) : \varphi\in\mathcal{D}_{\mathbf{0},\mathbf{v}}(\mathbf{R^n}),\ \mathbf{v}\in\mathbf{R^n}\}$ is a (trivial) $G^r(n)$-principal bundle over $\mathbf{R^n}$ [**5**], the *bundle of r-frames over* $\mathbf{R^n}$, with projection $\pi^r : F^r(\mathbf{R^n})\to\mathbf{R^n}$, $\mathbf{j_0^r}(\varphi)\mapsto\varphi(\mathbf{0})$, $G^r(n)$-right action $R^r_{j_0^r(g)}j_0^r(\varphi)\equiv j_0^r(\varphi)\cdot j_0^r(g) := j_0^r(\varphi\circ g)$, and (global) trivialization

$\Psi^r : F^r(\mathbf{R^n})\to\mathbf{R^n}\times\mathbf{G^r(n)}$, $\mathbf{j_0^r}(\varphi)\mapsto(\varphi(\mathbf{0})\,,\,\mathbf{j_0^r}(\mathbf{T}_{-\varphi(\mathbf{0})}\circ\varphi))$, T_v being the translation by $v\in\mathbf{R^n}$. The canonical principal fibre bundle isomorphism $F^1(\mathbf{R^n})\to\mathbf{LR^n}$, $\mathbf{j_0^1}(\varphi)\mapsto(\varphi_*|_\mathbf{0} : \mathbf{R^n}\to\mathbf{T}_{\varphi(\mathbf{0})}\mathbf{R^n})$, will be understood in what follows.

3 The r-jet lift of a matrix group

Let G be a Lie subgroup of $GL(n,\mathbf{R})$. The *canonical G-structure* $G\mathbf{R^n}$ is the subset of the linear frame bundle $L\mathbf{R^n}$ obtained by moving with G the global section σ_I of $\pi^1 : L\mathbf{R^n}\to\mathbf{R^n}$ induced by the identity chart $I\equiv(u^1,u^2,\ldots,u^n)$ of $\mathbf{R^n}$

$$G\mathbf{R^n} := (\mathbf{Im}\,\sigma_\mathbf{I})\cdot\mathbf{G}.$$

Every diffeomorphism $\varphi : U\subset\mathbf{R^n}\to\mathbf{V}\subset\mathbf{R^n}$ raises a principal bundle isomorphism $\tilde\varphi : L\mathbf{R^n}|_\mathbf{U}\to\mathbf{LR^n}|_\mathbf{V}$, $\mathbf{j_0^1}(\psi)\equiv\mathbf{l}\mapsto\mathbf{j_0^1}(\varphi\circ\psi)\equiv\varphi_*|_{\psi(\mathbf{0})}\circ\mathbf{l}$.

The *r-jet lift* G^r *of* G is defined as the subset of $G^r(n)$ induced by those diffeomorphisms $g\in\mathcal{D}_{0,0}(\mathbf{R^n})$ whose associated $\tilde g$ takes σ_I into a (local) section of $L\mathbf{R^n}$ having $(r-1)$-order contact over $0\in\mathbf{R^n}$ with some section of $G\mathbf{R^n}$

$$G^r := \{j_0^r(g)\in G^r(n) : \exists\,\sigma\in Sec(G\mathbf{R^n})\text{ with }\mathbf{j_0^{r-1}}(\tilde{\mathbf{g}}\circ\sigma_\mathbf{I}) = \mathbf{j_0^{r-1}}(\sigma\circ\mathbf{g})\}.$$

It can be described equivalently by

$$\begin{aligned} G^r = \{j_0^r(g)\in G^r(n) : \exists\,\kappa\in\mathcal{F}(\mathbf{R^n},\mathbf{G})\text{ with }\ &\mathbf{Dg}\big|_\mathbf{0} = \kappa(\mathbf{0}) \\ \text{and }\ & D^{s+1}g\big|_0 = D^s\kappa|_0\ (1\le s\le r-1)\}; \end{aligned}$$

we have identified $D^s\kappa|_0 \in L(\mathbf{R^n}\times \overset{\mathbf{s}}{\cdots} \times\mathbf{R^n}, \mathbf{gl(n,R)})$ with the element of $L^{s+1}(n)$ given by $D^s\kappa|_0(v_1,\dots,v_{s+1}) \equiv D^s\kappa|_0(v_1,\dots,v_s)(v_{s+1})$. We have that $(GL(n,\mathbf{R}))^{\mathbf{r}} = \mathbf{G^r(n)}$ and G^1 is canonically isomorphic to G.

Let us show up some algebraic properties of G^r:

(i) For every $0 < r' < r$, the projection $\beta^{r,r'} : G^r \longrightarrow G^{r'}$, $j_0^r(g) \longmapsto j_0^{r'}(g)$, is a Lie group homomorphism and the invariant Lie subgroup $G^{r,r'} \equiv Ker\ \beta^{r,r'}$ is nilpotent. In particular, $G^{r,r-1}$ $(r > 1)$ is abelian.

(ii) For $r > 1$, one sees very easily the existence of the following splitting exact sequence

$$1 \longrightarrow G^{r,1} \hookrightarrow G^r \underset{\gamma}{\overset{\beta}{\rightleftarrows}} G \longrightarrow 1,$$

with $\hookrightarrow$ the inclusion $(G^{r,1} := \{j_0^r(g) \in G^r : Dg|_0 = I\} = \{j_0^r(Dg|_0^{-1} \circ g) : j_0^r(g) \in G^r\})$, $\beta(j_0^r(g)) := Dg|_0$ and $\gamma(a) := j_0^r(a)$.

Thus G^r is canonically *isomorphic* with the semidirect product $G \otimes G^{r,1}$, associated to the homomorphism of G into $Aut(G^{r,1})$ given by

$$a \longmapsto (j_0^r(g) \mapsto j_0^r(a \circ g \circ a^{-1})),$$

and this isomorphism is defined by the map

$$\begin{array}{rccc} \chi : & G^r & \longrightarrow & G \otimes G^{r,1} \\ & j_0^r(g) & \longmapsto & (Dg|_0\ ,\ j_0^r(Dg|_0^{-1} \circ g)). \end{array}$$

Remark The following sequence is also exact

$$1 \longrightarrow G^{r,r-1} \hookrightarrow G^r \longrightarrow G^{r-1} \longrightarrow 1;$$

however it does not split, unless $r = 2$ (in which case both sequences are the same) or $G^{r,r-1} = \{0\}$ (trivial case); thus, in general, one *cannot* write $G^r \simeq G^{r-1} \otimes G^{r,r-1}$ (the group G^{r-1} has no natural structure as a subgroup of G^r).

By looking at curves in G^r through the identity $j_0^r(I)$, it is easy to see [**4**] that the Lie algebra g^r of G^r must be isomorphic —as a linear space— to the direct sum $\sum_{s=0}^{r-1} g_s$, g_s being the *s-prolongation of* g, defined as $g_0 := g$ and, for $s > 0$, as the vector space

$$g_s := \{t \in S^{s+1}(n)\ :\ t(v_1,\dots,v_s,\cdot) \in g,\ \forall\, v_1,\dots,v_s \in \mathbf{R^n}\} = \mathbf{L(R^n, g_{s-1})} \cap \mathbf{S^{s+1}(n)}$$

(obviously $(gl(n,\mathbf{R}))_{\mathbf{s}} = \mathbf{S^{s+1}(n)}$; moreover: $g_s = 0 \Rightarrow g_{s+1} = 0$).

One can also prove [**4**,**6**,**8**] that the Lie algebra g^r of G^r is isomorphic to the

Lie algebra $(\sum_{s=0}^{r-1} g_s, [\ ,\ ]_0^r)$, with the Lie bracket $[\ ,\ ]_0^r$ defined as follows

$$[t_p, t'_q]_0^r := \frac{(p+q+1)!}{p!(q+1)!}\mathbf{S}(t_p(t'_q(\cdot, \overset{q+1}{\ldots}, \cdot), \overset{p}{\ldots}, \cdot)) - \frac{(p+q+1)!}{(p+1)!q!}\mathbf{S}(t'_q(t_p(\cdot, \overset{p+1}{\ldots}, \cdot), \overset{q}{\ldots}, \cdot)),$$

if $p+q \leq r-1$

$[t_p, t'_q]_0^r := 0$, if $p+q > r-1$,

where $t_p \in g_p,\ t'_q \in g_q,\ p,q \geq 0$ (thus $[t_p, t'_q]_0^r \in g_{p+q}$), and the *symmetrizer* $\mathbf{S}$ is given by ($\forall\, t \in L^s(n)$ and $\forall\, v_1, \ldots, v_s \in \mathbf{R^n}$):

$$\mathbf{S}(t)(v_1, \ldots, v_s) := \tfrac{1}{s!}\textstyle\sum_{\sigma \in P_s} t(v_{\sigma(1)}, \ldots, v_{\sigma(s)}).$$

It follows that the abelian group $G^{r,r-1}$ $(r > 1)$ is isomorphic to the vector group g_{r-1}. If $g_{r-1} = 0$, the homomorphism $\beta^{r,r-1} : G^r \to G^{r-1}$ is a Lie group isomorphism.

REMARKS.

(i) For $r > 1$, consider the Lie algebra $(\sum_{s=1}^{r-1} g_s, [\ ,\ ]_1^r)$, with $[\ ,\ ]_1^r$ being simply the restriction of $[\ ,\ ]_0^r$. One always has

$$(\sum_{s=0}^{r-1} g_s, [\ ,\ ]_0^r) = g \niplus (\sum_{s=1}^{r-1} g_s, [\ ,\ ]_1^r)\ (semidirectsum).$$

Moreover —and corresponding to the above Remark on groups— $(\sum_{s=0}^{r-2} g_s, [\ ,\ ]_0^{r-1})$ closes as a Lie subalgebra of $(\sum_{s=0}^{r-1} g_s, [\ ,\ ]_0^r)$ if and only if either $r = 2$ or $g_{r-1} = 0$ (trivial case).

(ii) The Lie bracket $[\ ,\ ]_0^r$ on g^r can be extended in a natural way to some kind of "bracket" $[\ ,\ ]_{-1}^r$ on $\mathbf{R^n} + \mathbf{g^r}$, defined as follows

$$[\ ,]_{-1}^r\big|_{g^r \times g^r} := [\ ,\ ]_0^r$$
$$[v, v']_{-1}^r := 0$$
$$[t_p, v]_{-1}^r := t_p(v) =: -[v, t_p]_{-1}^r,$$

where $v, v' \in \mathbf{R^n},\ \mathbf{t_p} \in \mathbf{g_p},\ \mathbf{p} \geq \mathbf{0}$. Certainly $\mathbf{R^n} + \mathbf{g^r}$ always closes under $[\ ,\ ]_{-1}^r$; however, the Jacobi-identity is fulfilled —only then it becomes possible to define the *Lie algebra* $\mathbf{R^n} + \mathbf{g^r} \equiv (\mathbf{R^n} + \mathbf{g^r}, [\ ,\ ]_{-\mathbf{1}}^{\mathbf{r}})$— *if and only if* either $r = 1$ (we get the affine algebra $\mathbf{R^n} \oplus \mathbf{g}$) or the prolongation g_r vanishes.

Besides the adjoint representation, G^r has another natural representation; let us show how this one comes out.

The *r-jet lift* $G^r(\mathbf{R^n})$ *of* $G\mathbf{R^n}$ is defined as the subset of $F^r(\mathbf{R^n})$ induced by those diffeomorphisms $\varphi \in \mathcal{D}_{0,v}(\mathbf{R^n}),\ \mathbf{v} \in \mathbf{R^n}$, whose associated $\tilde{\varphi}$ takes σ_I into a (local) section of $L\mathbf{R^n}$ having $(r-1)$-order contact over $\varphi(0) \in \mathbf{R^n}$ with some section of $G\mathbf{R^n}$

$$G^r(\mathbf{R^n}) := \{\mathbf{j_0^r}(\varphi) \in \mathbf{F^r}(\mathbf{R^n}) : \exists\, \sigma \in \mathbf{Sec(GR^n)} \text{ with } \mathbf{j_0^{r-1}}(\tilde{\varphi} \circ \sigma_{\mathbf{I}}) = \mathbf{j_0^{r-1}}(\sigma \circ \varphi)\}.$$

One sees very easily that $G^r(\mathbf{R^n})$ is a (trivial) G^r-principal bundle over $\mathbf{R^n}$, in fact a reduction of $F^r(\mathbf{R^n})$ (see section 2).

The canonical isomorphism $G^1 \simeq G$ yields a natural representation of G^1 in $\mathbf{R^n}$. For $r > 1$, the above construction of $G^r(\mathbf{R^n})$ leads to a *natural representation* $\mathcal{A}^{G^r}$ *of* G^r *in the vector space* $\mathbf{R^n} + \mathbf{g^{r-1}}$, as follows:

(i) Every diffeomorphism $g : U \to V$ in $\mathcal{D}_{0,0}(\mathbf{R^n})$ raises a diffeomorphism $\mathcal{I}_g^{r-1} : F^{r-1}(\mathbf{R^n})|_{\mathbf{U}} \to \mathbf{F^{r-1}}(\mathbf{R^n})|_{\mathbf{V}},\ \mathbf{j_0^{r-1}}(\psi) \mapsto \mathbf{j_0^{r-1}}(\mathbf{g} \circ \psi \circ \mathbf{g^{-1}})$ (it is *not* a principal bundle isomorphism!). One can prove [**4**] that the tangent map $\mathcal{I}_{g\,*}^{r-1}\big|_{j_0^{r-1}(I)}$, which depends in fact on $j_0^r(g)$, preserves the subspace $T_{j_0^{r-1}(I)}(G^{r-1}(\mathbf{R^n}))$ of $T_{j_0^{r-1}(I)}(F^{r-1}(\mathbf{R^n}))$; thus we can write

$$\mathcal{I}_{g\,*}^{r-1}\big|_{j_0^{r-1}(I)} : T_{j_0^{r-1}(I)}(G^{r-1}(\mathbf{R^n})) \longrightarrow \mathbf{T_{j_0^{r-1}(I)}}(\mathbf{G^{r-1}}(\mathbf{R^n})).$$

(ii) The global trivialization (see section 2) $\Psi^{r-1} : G^{r-1}(\mathbf{R^n}) \to \mathbf{R^n} \times \mathbf{G^{r-1}}$ and the associated canonical identification $T_{j_0^{r-1}(I)}G^{r-1}(\mathbf{R^n}) \simeq \mathbf{R^n} + \mathbf{g^{r-1}}$ lead to the (injective) representation

$$\begin{array}{rccc} \mathcal{A}^{G^r}: & G^r & \longrightarrow & GL(\mathbf{R^n} + \mathbf{g^{r-1}}) \\ & j_0^r(g) & \longmapsto & (\mathcal{I}_{g\,*}^{r-1}\big|_{j_0^{r-1}(I)} : \mathbf{R^n} + \mathbf{g^{r-1}} \to \mathbf{R^n} + \mathbf{g^{r-1}}). \end{array}$$

This representation of G^r induces a representation a^{g^r} *of the Lie algebra* g^r, given by

$$a^{g^r} \equiv \mathcal{A}_*^{G^r}\big|_{j_0^r(I)} : g^r \longrightarrow gl(\mathbf{R^n} + \mathbf{g^{r-1}}).$$

Let us consider what happens if $g_{r-1} = 0$. In that case, the above cited Lie group isomorphism $\beta^{r,r-1} : G^r \to G^{r-1}$ gives an obvious representation

$$\overline{\mathcal{A}}^{G^{r-1}} := \mathcal{A}^{G^r} \circ (\beta^{r,r-1})^{-1} : G^{r-1} \to GL(\mathbf{R^n} + \mathbf{g^{r-1}}).$$

4 The 2-jet lift of G

We now give a short review of the 2-jet lift G^2 of a matrix group G. The material of this section is fairly well known, although our description shows up the semidirect product structure of $G^2(n)$, so the outcoming formulas are not the usual ones [**3**].

As the Abelian group $G^{2,1}(n) = \{j_0^2(Dg|_0^{-1} \circ g) : j_0^2(g) \in G^2(n)\}$ is isomorphic —via the map u_1— to the vector group $S^2(n)$, we get the isomorphism

$$\begin{array}{rccc} (u_0, \bar{u}_1) := (I, u_1) \circ \chi : & G^2(n) & \xrightarrow{\simeq} & GL(n, \mathbf{R}) \otimes \mathbf{S^2(n)} \\ & j_0^2(g) & \longmapsto & (Dg|_0 \, , \, Dg|_0^{-1} \, D^2 g|_0), \end{array}$$

where the product law in $GL(n, \mathbf{R}) \otimes \mathbf{S^2(n)}$ is given by

$$(a, t) \cdot (a', t') := (aa', a'^{-1} t(a', a') + t').$$

Restricting $(u_0, \bar{u}_1)$ to

$$G^2 = \{j_0^2(g) \in G^2(n) : \exists\, \kappa \in \mathcal{F}(\mathbf{R^n}, \mathbf{G}) \text{ with } \mathbf{Dg}|_\mathbf{0} = \kappa(\mathbf{0}) \text{ and } \mathbf{D^2 g}\big|_\mathbf{0} = \mathbf{D}\kappa|_\mathbf{0}\}$$

we get, just because $Dg|_0^{-1} \, D^2 g|_0 = (\kappa^{-1} D\kappa)|_0 \in S^2(n) \cap L(\mathbf{R^n}, \mathbf{g})$, the *isomorphism* (which will be understood in what follows)

$$\begin{array}{rccc} (u_0, \bar{u}_1) : & G^2 & \longrightarrow & G \otimes g_1 \\ & j_0^2(g) & \longmapsto & (Dg|_0 \, , Dg|_0^{-1} \, D^2 g|_0). \end{array}$$

Before looking into the representation Ad^{G^2} and $\mathcal{A}^{G^2}$ of G^2, we want to introduce the following useful notations: $\forall\, t \in L^r(n) \simeq L(\mathbf{R^n}, \mathbf{L^{r-1}(n)})$, $\forall\, a \in GL(n, \mathbf{R})$ and $\forall\, \alpha \in gl(n, \mathbf{R})$, we write

$$t_a \equiv at(a^{-1}, \ldots, a^{-1}) \in L^r(n) t^\alpha$$

$$L^r(n) t^\alpha \equiv \alpha t - t(\alpha\, \cdot, \cdot, \ldots, \cdot) - \ldots - t(\cdot, \ldots, \cdot, \alpha\, \cdot) \in L^r(n).$$

Note that if we call $Q \in Hom(GL(n, \mathbf{R}), \mathbf{GL}(\mathbf{L^r(n)}))$, $Q(a) : t \to t_a$, and $q \in Hom(gl(n, \mathbf{R}), \mathbf{gl}(\mathbf{L^r(n)}))$, $q(\alpha) : t \to t^\alpha$, it is very easy to see that $q = Q_*|_I$.

In the case $t \in S^2(n) \subset L^2(n) \simeq L(\mathbf{R^n}, \mathbf{gl}(\mathbf{n}, \mathbf{R}))$, it is easy to check the following properties:

(i) Both maps $t \mapsto t_a$ and $t \mapsto t^\alpha$ belong to $gl(S^2(n))$,

(ii) $t_{ab} = (t_b)_a$ and $t^{[\alpha, \beta]} = (t^\beta)^\alpha - (t^\alpha)^\beta$,

(iii) $t_a(v) = at(a^{-1} v) a^{-1}$ and $t^\alpha(v) = [\alpha, t(v)] - t(\alpha v)$, $\forall\, v \in \mathbf{R^n}$,

(iv) $t^{t(v)}(w) - t^{t(w)}(v) = 2[t(v), t(w)]$, $\forall\, v, w \in \mathbf{R^n}$,

(v) $(t^\alpha)_a = (t_a)^{a\alpha a^{-1}}$.

Remark. We can rewrite the product law in $GL(n, \mathbf{R}) \otimes \mathbf{S^2(n)}$ as follows:

$$(a, t) \cdot (a', t') = (aa', t_{a'^{-1}} + t'), \quad \textit{thus} \quad (a, t)^{-1} = (a^{-1}, -t_a).$$

With this notations, the representations Ad^{G^2} (adjoint) and $\mathcal{A}^{G^2}$ —and the corresponding representations ad^{g^2} and a^{g^2}— have the form

$$\begin{array}{rccc} Ad^{G^2}: & G^2 \simeq G \ltimes g_1 & \longrightarrow & Aut(g \oplus g_1) \\ & (a,t) & \longmapsto & \begin{pmatrix} Ad^G_a & 0 \\ -(t^{(\cdot)})_a & (\cdot)_a \end{pmatrix}, \end{array}$$

or $Ad^{G^2}_{(a,t)}(\beta, s) = (a\beta a^{-1}, s_a - (t^\beta)_a)$;

$$\begin{array}{rccc} ad^{g^2}: & g^2 \simeq g \oplus g_1 & \longrightarrow & Der(g \oplus g_1) \\ & (\alpha,t) & \longmapsto & \begin{pmatrix} ad^g_\alpha & 0 \\ -t^{(\cdot)} & (\cdot)^\alpha \end{pmatrix}, \end{array}$$

or $ad^{g^2}_{(\alpha,t)}(\beta, s) = ([\alpha,\beta], s^\alpha - t^\beta) = [(\alpha,t),(\beta,s)]^2_0$;

$$\begin{array}{rccc} \mathcal{A}^{G^2}: & G^2 \simeq G \ltimes g_1 & \longrightarrow & GL(\mathbf{R^n} + \mathbf{g}) \\ & (a,t) & \longmapsto & \begin{pmatrix} a & 0 \\ Ad^G_a t(\cdot) & Ad^G_a \end{pmatrix}, \end{array}$$

or $\mathcal{A}^{G^2}_{(a,t)}(v, \beta) = (av, a(t(v)+\beta)a^{-1})$;

$$\begin{array}{rccc} a^{g^2}: & g^2 \simeq g \oplus g_1 & \longrightarrow & gl(\mathbf{R^n} + \mathbf{g}) \\ & (\alpha,t) & \longmapsto & \begin{pmatrix} \alpha & 0 \\ t(\cdot) & ad^g_\alpha \end{pmatrix}, \end{array}$$

or $a^{g^2}_{(\alpha,t)}(v, \beta) = (\alpha v, t(v) + [\alpha, \beta])$.

The case $g_1 = 0$. In that case, one has the isomorphism

$$\begin{array}{ccc} G & \longrightarrow & G^2 \simeq G \ltimes \{0\} \\ a & \longmapsto & (a, 0) \end{array}$$

and

$$\begin{array}{rccc} \overline{\mathcal{A}}^G: & G & \longrightarrow & Aut(\mathbf{R^n} \oplus \mathbf{g}) \\ & a & \longmapsto & \begin{pmatrix} a & 0 \\ 0 & Ad^G_a \end{pmatrix}, \end{array}$$

or $\overline{\mathcal{A}}^G_a(v, \beta) = (av, a\beta a^{-1})$ (we write $\overline{\mathcal{A}}^G_a \in Aut(\mathbf{R^n} \oplus \mathbf{g})$ because $\overline{\mathcal{A}}^G$ preserves the bracket in the Lie algebra $\mathbf{R^n} \oplus \mathbf{g}$).

5 The 3-jet lift of G

In analogous way as we did for G^2, we first describe G^3 as a semidirect product of G itself and a (now *non-abelian*) nilpotent group builded up on the first two prolongations g_1 and g_2 of its Lie algebra g.

The nilpotent group $G^{3,1}(n) = \{j_0^3(g) \in G^3(n) : Dg|_0 = I\}$ is isomorphic —via (u_1, u_2)— to the group $(S^2(n) \times S^3(n), \star)$, $\star$ being the product law induced by applying the chain rule to the derivatives of the composition of functions

$$(t, T) \star (t', T') := (t + t', T + 3\mathbf{S}(t(\cdot)t'(\cdot, \cdot)) + T').$$

Now, instead of using the chart (u_1, u_2), it will be more convenient (see the next lemma) to use the chart $(u_1, u_2 - \frac{3}{2}\mathbf{S}(u_1(\cdot)u_1(\cdot, \cdot)))$, which establishes an isomorphism between $G^{3,1}(n)$ and the group $(S^2(n) \times S^3(n), \diamond)$, where $\diamond$ stands for the product law

$$(t, T) \diamond (t', T') := (t + t', T + \tfrac{1}{2}[t, t']_0^3 + T'),$$

and the bracket $[t, t']_0^3 := 3\mathbf{S}(t(\cdot)t'(\cdot, \cdot)) - 3\mathbf{S}(t'(\cdot)t(\cdot, \cdot))$ was given in section 3. In that way we get the isomorphism

$$\begin{aligned}(u_0, \bar{u}_1, \bar{u}_2) := (I, u_1, u_2 - \tfrac{3}{2}\mathbf{S}(u_1(\cdot)u_1(\cdot, \cdot))) \circ \chi : G^3(n) \longrightarrow \\ \longrightarrow GL(n, \mathbf{R}) \otimes (\mathbf{S^2(n)} \times \mathbf{S^3(n)}, \diamond)\end{aligned}$$

$$\begin{aligned}j_0^3(g) \longmapsto (Dg|_0\,,\ Dg|_0^{-1}\ D^2g\big|_0\,,\ Dg|_0^{-1}\ D^3g\big|_0 - \\ -\frac{3}{2}\mathbf{S}(Dg|_0^{-1}\ D^2g\big|_0(\cdot)\ Dg|_0^{-1}\ D^2g\big|_0(\cdot, \cdot))),\end{aligned}$$

where the product law in $GL(n, \mathbf{R}) \otimes (\mathbf{S^2(n)} \times \mathbf{S^3(n)}, \diamond)$ is given by

$$\begin{aligned}(a, t, T) \cdot (a', t', T') &:= (aa'\,,\ (t_{a'^{-1}}\,,\ T_{a'^{-1}}) \diamond (t'\,,\ T')) = \\ &= (aa'\,,\ t_{a'^{-1}} + t'\,,\ T_{a'^{-1}} + \tfrac{1}{2}[t_{a'^{-1}}, t']_0^3 + T')\end{aligned}$$

(what implies $(a, t, T)^{-1} = (a^{-1}, -t_a, -T_a)$).

Remark. Note that the term $\frac{1}{2}[t_{a'^{-1}}, t']_0^3$ *prevents* this product to be a semidirect product of the form $G^2(n) \otimes S^3(n)$; the group $G^2(n)$ has no natural structure as a subgroup of $G^3(n)$ (see the first remark in section 3).

As the following lemma establishes, the above chart $(u_0, \bar{u}_1, \bar{u}_2)$ provides, by restriction, an *isomorphic* image of G^3 in terms of G and the first two prolongations of g, and this is a very interesting fact for computations.

Lemma. The group G^3 is isomorphic —via $(u_0, \bar{u}_1, \bar{u}_2)$— to the group $G \otimes (g_1 \times g_2, \diamond)$.

Proof. As we said above, it holds: $j_0^3(g) \in G^3 \quad \Leftrightarrow \quad \exists\, \kappa \in \mathcal{F}(\mathbf{R^n}, \mathbf{G})$ such that
(1) $Dg|_0 = \kappa(0) \in G$,
(2) $D^2g|_0 = D\kappa|_0(\cdot) \in S^2(n)$,
(3) $D^3g|_0 = D^2\kappa|_0(\cdot, \cdot) \in S^3(n)$.

As for G^2, conditions (1) and (2) lead to

$$\bar{u}_1(j_0^3(g)) \in g_1.$$

Now we consider condition (3). Let us call $\kappa_1(\cdot) \equiv \kappa^{-1}D\kappa \in \mathcal{F}(\mathbf{R^n}, \mathbf{L}(\mathbf{R^n}, \mathbf{g}))$ and $\kappa_2(\cdot,\cdot) \equiv \kappa^{-1}D^2\kappa \in \mathcal{F}(\mathbf{R^n}, \mathbf{L}_{\text{sim}}(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{gl(n,R)}))$. Note that although $\kappa_1|_0(\cdot) \in L(\mathbf{R^n}, \mathbf{g})$, in general $\kappa_2|_0(\cdot,\cdot)$ *does not* belong to $L(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{g})$; so conditions (1) and (3) *do not* lead to $Dg|_0^{-1}\, D^3g|_0 \in g_2$.

From $D(\kappa^{-1})(\cdot) = -\kappa^{-1}D\kappa(\cdot)\kappa^{-1}$, it immediately follows

$$\kappa_2(\cdot,\cdot) = D\kappa_1(\cdot,\cdot) + \kappa_1(\cdot)\kappa_1(\cdot);$$

then, at the origin of $\mathbf{R^n}$,

$$\kappa_2|_0(\cdot,\cdot) \in \kappa_1|_0(\cdot)\kappa_1|_0(\cdot) + L(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{g}).$$

Thus conditions (1) and (3) lead to

$$(3') \qquad \kappa_2|_0(\cdot,\cdot) \in (\kappa_1|_0(\cdot)\kappa_1|_0(\cdot) + L(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{g})) \cap \mathbf{S^3(n)}.$$

Now observe that, given any $s \in g_1 := L(\mathbf{R^n}, \mathbf{g}) \cap \mathbf{S^2(n)}$, the identity

$$s(\cdot)s(\cdot,\cdot) \equiv \tfrac{3}{2}\mathbf{S}(s(\cdot)s(\cdot,\cdot)) + \tfrac{1}{2}s^{s(\cdot)}(\cdot,\cdot)$$

holds *with* $s^{s(\cdot)} \in L(\mathbf{R^n}, \mathbf{g_1}) \subset \mathbf{L}(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{g})$; applying this identity to $s = \kappa_1|_0$, we get from (3')

$$\kappa_2|_0(\cdot,\cdot) - \ \tfrac{3}{2}\ \mathbf{S}(\kappa_1|_0(\cdot)\kappa_1|_0(\cdot,\cdot)) \in L(\mathbf{R^n} \times \mathbf{R^n}, \mathbf{g}) \cap \mathbf{S^3(n)} = \mathbf{g_2};$$

thus again (1) and (3) lead to

$$Dg|_0^{-1}\, D^3g|_0 - \tfrac{3}{2}\mathbf{S}(\,Dg|_0^{-1}\, D^2g|_0\,(\cdot)\ Dg|_0^{-1}\, D^2g|_0\,(\cdot,\cdot)) \in g_2,$$

or, in other words, $\bar{u}_2(j_0^3(g)) \in g_2$.

Once we know that $(u_0, \bar{u}_1, \bar{u}_2)(G^3) \subset G \times g_1 \times g_2$, the assert of the lemma follows immediately.

□

Corollary. If $g_2 = 0$, every $j_0^3(g) \in G^3$ satisfies

$$Dg|_0^{-1}\, D^3g|_0 - \tfrac{3}{2}\mathbf{S}(\,Dg|_0^{-1}\, D^2g|_0\,(\cdot)\ Dg|_0^{-1}\, D^2g|_0\,(\cdot,\cdot)) = 0.$$

Remark.

Thus the *algebraic* condition for a matrix group G of being of finite type 2 *analytic* property: every $G\mathbf{R^n}$-preserving diffeomorphism $\varphi \in \mathcal{D}_{0,v}(\mathbf{R^n}),\ \mathbf{v} \in \mathbf{R^n}$, must satisfy

$$D\varphi|_0^{-1}\, D^3\varphi|_0 - \tfrac{3}{2}\mathbf{S}(\,D\varphi|_0^{-1}\, D^2\varphi|_0\,(\cdot)\ D\varphi|_0^{-1}\, D^2\varphi|_0\,(\cdot,\cdot)) = 0.$$

In particular, let $CO(n,\mathbf{R})$ (with $n \geq 3$) be the linear conformal group with respect to some (pseudo) euclidean scalar product η in $\mathbf{R^n}$; as it is well known, $CO(n,\mathbf{R})$ is of finite type 2. It thus follows that every conformal (local) diffeomorphism in $(\mathbf{R^n},\eta)$ satisfies this condition, as one immediately checks.

Using the above isomorphism $(u_0,\bar{u}_1,\bar{u}_2)$, we get the following

Theorem. The representations Ad^{G^3} (adjoint) and $\mathcal{A}^{G^3}$ —and the corresponding representations ad^{g^3} and a^{g^3}— are given by

$$\begin{array}{llll} Ad^{G^3}: & G^3 \simeq G \otimes (g_1 \times g_2, \diamond) & \to & Aut(g+g_1+g_2,[\cdot,\cdot]_0^3) \\ & (a,t,T) & \mapsto & \begin{pmatrix} Ad^G_a & 0 & 0 \\ -(t^{(\cdot)})_a & (\cdot)_a & 0 \\ -(T^{(\cdot)}-\frac{1}{2}[t^{(\cdot)},t]_0^3)_a & ([t,\cdot]_0^3)_a & (\cdot)_a \end{pmatrix}, \end{array}$$

or $\quad Ad^{G^3}_{(a,t,T)}(\beta,s,S)=(a\beta a^{-1}, s_a-(t^\beta)_a, (S-T^\beta+[t,s]_0^3+\frac{1}{2}[t^\beta,t]_0^3)_a)$;

$$\begin{array}{llll} ad^{g^3}: & g^3 \simeq G \otimes (g_1 \times g_2, \diamond) & \to & Der(g+g_1+g_2,[\cdot,\cdot]_0^3) \\ & (\alpha,t,T) & \mapsto & \begin{pmatrix} ad^g_\alpha & 0 & 0 \\ -t^{(\cdot)} & (\cdot)^\alpha & 0 \\ -T^{(\cdot)} & [t,\cdot]_0^3 & (\cdot)^\alpha \end{pmatrix}, \end{array}$$

or $\quad ad^{g^3}_{(\alpha,t,T)}(\beta,s,S)=([\alpha,\beta], s^\alpha-t^\beta, S^\alpha-T^\beta+[t,s]_0^3)\equiv[(\alpha,t,T),(\beta,s,S)]_0^3)$;

$$\begin{array}{llll} \mathcal{A}^{G^3}: & G^3 \simeq G \otimes (g_1 \times g_2, \diamond) & \to & GL(\mathbf{R^n}+\mathbf{g}+\mathbf{g_1}) \\ & (a,t,T) & \mapsto & \begin{pmatrix} a & 0 & 0 \\ Ad^G_a t(\cdot) & Ad^G_a & 0 \\ (T(\cdot)-\frac{1}{2}t^{t(\cdot)})_a & -(t^{(\cdot)})_a & (\cdot)_a \end{pmatrix}, \end{array}$$

or $\quad \mathcal{A}^{G^3}_{(a,t,T)}(v,\beta,s)=(av\ ,\ a(t(v)+\beta)a^{-1}\ ,\ (T(v)-\frac{1}{2}t^{t(v)}-t^\beta+s)_a)$;

$$\begin{array}{llll} a^{g^3}: & g^3 \simeq G \otimes (g_1 \times g_2, \diamond) & \to & gl(\mathbf{R^n}+\mathbf{g}+\mathbf{g_1}) \\ & (\alpha,t,T) & \mapsto & \begin{pmatrix} \alpha & 0 & 0 \\ t(\cdot) & ad^g_\alpha & 0 \\ T(\cdot) & -t^{(\cdot)} & (\cdot)^\alpha \end{pmatrix}, \end{array}$$

or $\quad a^{g^3}_{(\alpha,t,T)}(v,\beta,s)=(\alpha v, t(v)+[\alpha,\beta], T(v)+s^\alpha-t^\beta)$.

Proof. A careful computation is needed; for more details see [**9**]. The expressions for $Ad^{G^3}_{(a,t,T)} := Int^{G^3}_{(a,t,T)\,*}\big|_{(I,0,0)}$ and $\mathcal{A}^{G^3}_{j_0^3(g)\simeq(a,t,T)} := \mathcal{I}^2_{g\,*}\big|_{j_0^2(I)\simeq(0,I,0)}$ follow from the following expressions of the inner automorphisms of G^3

$$\begin{aligned} Int^{G^3}_{(a,t,T)} : G\otimes (g_1 \times g_2, \diamond) &\to G\otimes (g_1 \times g_2, \diamond) \\ (u_0, \bar{u}_1, \bar{u}_2) &\mapsto (a,t,T)(u_0,\bar{u}_1,\bar{u}_2)(a,t,T)^{-1} = \end{aligned}$$
$$= (au_0a^{-1}, (t_{u_0^{-1}} + \bar{u}_1 - t)_a, (T_{u_0^{-1}} + \bar{u}_2 + \tfrac{1}{2}([t_{u_0^{-1}}, \bar{u}_1]^3_0 + [t,\bar{u}_1]^3_0 + [t, t_{u_0^{-1}}]^3_0))_a),$$

and of the diffeomorphism

$$\begin{aligned} \mathcal{I}^2_g : \ F^2(\mathbf{R^n})|_{\mathbf{U}} \simeq \mathbf{U} \times \mathbf{GL(n,R)} \times \mathbf{S^2(n)} &\to F^2(\mathbf{R^n})|_{\mathbf{V}} \\ (u, u_0, u_1) &\mapsto \end{aligned}$$
$$\mapsto (g(u),\ Dg\big|_u u_0\, Dg^{-1}\big|_0,\ D^2g\big|_u (u_0\, Dg^{-1}\big|_0, u_0\, Dg^{-1}\big|_0) +$$
$$+ Dg\big|_u u_1 (Dg^{-1}\big|_0, Dg^{-1}\big|_0) + Dg\big|_u u_0\, D^2g^{-1}\big|_0),$$

respectively.

□

The case $g_2 = 0$. In that case, the map

$$\begin{aligned} G^2 \cong G\otimes g_1 &\longrightarrow G^3 \cong G\otimes (g_1 \times \{0\}, \diamond) \\ (a,t) &\longmapsto (a,t,0) \end{aligned}$$

gives an isomorphism $G^2 \simeq G^3$ and a representation $\overline{\mathcal{A}}^{G^2}$ which preserves the bracket in the Lie algebra $(\mathbf{R^n} + \mathbf{g} + \mathbf{g_1}, [\cdot,\cdot]^{\mathbf{2}}_{\mathbf{-1}})$

$$\begin{aligned} \overline{\mathcal{A}}^{G^2} :\ G^2 \simeq G\otimes g_1 &\longrightarrow Aut(\mathbf{R^n} + \mathbf{g} + \mathbf{g_1}, [\cdot,\cdot]^{\mathbf{2}}_{\mathbf{-1}}) \\ (a,t) &\longmapsto \begin{pmatrix} a & 0 & 0 \\ Ad^G_a t(\cdot) & Ad^G_a & 0 \\ -\frac{1}{2}(t^{t(\cdot)})_a & -(t^{(\cdot)})_a & (\cdot)_a \end{pmatrix}, \end{aligned}$$

or $\overline{\mathcal{A}}^{G^2}_{(a,t)}(v,\beta,s) = (av\, ,\ a(t(v)+\beta)a^{-1}\, ,\ (-\frac{1}{2}t^{t(v)} - t^{\beta} + s)_a)$.

Remarks.

(i) In the case $g_2 = 0$, observe that: $\overline{\mathcal{A}}^{G^2}_{(a,t)}|_{g+g_1} = Ad^{G^2}_{(a,t)}$ *and* $\overline{a}^{g^2}_{(\alpha,t)} = ad^{\mathbf{R^n}+\mathbf{g^2}}_{(0,\alpha,t)}$.

(ii) When $G = CO(n,\mathbf{R})$, the linear conformal group, it is well known [**6**] that $CO(n,\mathbf{R})^{\mathbf{2}}$ is isomorphic to a Lie subgroup $\mathcal{L}_0$ of the full conformal group $\mathcal{L} \equiv O(n+2)/\mathbf{Z}_2$, as follows:

$$CO(n,\mathbf{R})^{\mathbf{2}} \simeq \mathbf{CO(n,R)} \otimes \mathbf{co(n,R)_1} \xrightarrow{\ \simeq\ } \mathcal{L}_\mathbf{0}$$

$$(kb, -\tilde{\mu}) \longmapsto A_{(k,b,\mu)} \equiv \begin{pmatrix} k & 0 & 0 \\ b\mu^\sharp & b & 0 \\ (2k)^{-1}\eta(\mu^\sharp,\mu^\sharp) & k^{-1}\mu & k^{-1} \end{pmatrix},$$

with $k \in \mathbf{R}^+$, $\mathbf{b} \in \mathbf{O(n,R)}$, $\mu \in \mathbf{R^{n*}}$, $\tilde{\mu} \in \mathbf{co(n,R)_1}$, $\mu^\sharp \in \mathbf{R^n}$ (we have used the isomorphisms: $\mathbf{R^{n*}} \longrightarrow \mathbf{R^n}$, $\mu \longmapsto \mu^\sharp$, determined by $\eta(\mu^\sharp, \cdot) = \mu$, with η a standard scalar product of arbitrary signature, and $\mathbf{R^{n*}} \longrightarrow \mathbf{co(n,R)_1}$, $\mu \longmapsto \tilde{\mu}$, where

$$\tilde{\mu}(v,w) := \mu(v)w + \mu(w)v - \eta(v,w)\mu^\sharp).$$

This induces the following isomorphism between the Lie algebras

$$co(n,\mathbf{R})^{\mathbf{2}} \simeq \mathbf{co(n,R)} \ni \mathbf{co(n,R)_1} \xrightarrow{\simeq} \mathbf{l_0}$$

$$(\kappa I + \beta, -\tilde{\mu}) \longmapsto a_{(\kappa,b,\mu)} \equiv \begin{pmatrix} \kappa & 0 & 0 \\ \mu^\sharp & \beta & 0 \\ 0 & \mu & -\kappa \end{pmatrix}$$

with $\kappa \in \mathbf{R}$, $\beta \in \mathbf{o(n,R)}$, $\mu \in \mathbf{R^{n*}}$.

Moreover one sees very easily the isomorphism

$$(\mathbf{R^n} + \mathbf{co(n,R)} + \mathbf{co(n,R)_1}, [\ ,\ ]^{\mathbf{2}}_{-\mathbf{1}}) \xrightarrow{\simeq} \mathbf{l}$$

$$(w, \kappa I + \beta, -\tilde{\mu}) \longmapsto \begin{pmatrix} \kappa & w^\natural & 0 \\ \mu^\sharp & \beta & w \\ 0 & \mu & -\kappa \end{pmatrix}$$

(being $w^\natural = \eta(w, \cdot)$) with

$$\begin{array}{llllll} [w,w']^2_{-1} & = 0, & & & & \\ [\beta,w']^2_{-1} & = \beta w', & [\beta,\beta']^2_{-1} & = \beta\beta' - \beta'\beta, & & \\ [\tilde{\mu},w']^2_{-1} & = \tilde{\mu}(w'), & [\tilde{\mu},\beta']^2_{-1} & = -\tilde{\mu}^{\beta'} = \widetilde{\mu\beta'}, & [\tilde{\mu},\tilde{\mu}']^2_{-1} & = 0. \end{array}$$

The last isomorphism yields (after a straightforward computation) the correspondence

$$\overline{\mathcal{A}}^2_{(kb,-\tilde{\mu})}(w, \kappa I + \beta, -\tilde{\nu}) \longmapsto Ad^{\mathcal{L}}_{A_{(k,b,\mu)}} a_{(\kappa,\beta,\nu)}.$$

Thus we conclude that

$$\overline{\mathcal{A}}^2_{j^2_0(g)} = Ad^{\mathcal{L}}_{j^2_0(g)}, \quad \forall\, j^2_0(g) \in CO(n,\mathbf{R})^{\mathbf{2}};$$

for details see [**9**].

6 Final comment

As we mentioned in the introduction we are interested in the relatonships between true connections and Cartan connections on 2-jet lifts $P^2(M,G^2)$ of 1-integrable

G-structures when P is of finite type 2. The behaviour of such geometrical objects under the principal action is controlled by the above representations Ad^{G^2} and $\overline{\mathcal{A}}^{G^2}$, respectively. Using such representations we have been able to prove that, given a Cartan connection on P^2 and a symmetric connection on P, a distinguished connection on P^2 arises which is not the trivial prolongation of the latter one; in the conformal case, the geodesics of that second order connection are closely related to the so called "conformal circles".

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