

AUTOCALIBRATION OF A CAMERA PAIR

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ABSTRACT. Given a pair of cameras with identical intrinsic parameters and known pixel shape, there exists a uniparametric set of possible 3D Euclidean reconstructions. We provide for this set a closed-form explicit parameterization. Therefore, given a single piece of data from the scene, the set of solutions can be efficiently searched for the best-fit reconstruction. An experiment with real images, showing the applicability of the method, is included.

1. INTRODUCTION

The problem of autocalibration of a set of cameras has been extensively studied [3]. The case of three uncalibrated cameras with the same intrinsic parameters is of particular importance, since three is the minimal number of cameras generically providing a unique solution of the problem of recovering generic constant intrinsic parameters, and was historically the first studied [6]. Considering only two cameras, the first paper to show the possibility of obtaining the focal lengths assuming all other internal camera parameters to be known was [2]. In [7] an alternative algorithm was proposed, along with a study of degenerate configurations, and in [1] a closed-form formula was given along with a degeneration analysis. In [4] a formula valid for the case in which one camera rotates around either one or two of the coordinate axes defined by the other one was provided and, more recently, in [5] a linear formula for computing the focal length was obtained. A deeper study of the particular case of constant focal length is provided in [11], deriving closed-form solutions and studying their stability.

Given a pair of cameras with identical intrinsic parameters, Kruppa equations are the basic constraint on the internal parameters of the cameras. As a counting of parameters reveals, with the only additional assumption of cameras with known pixel shape, there exists a uniparametric set of possible 3D Euclidean reconstructions. The main result of this paper is the obtainment of this set by means of an equation that leads in a straightforward way to a closed-form explicit parametrization of the set of 3D reconstructions. Therefore, given a single piece of data from the scene, the set of solutions can be efficiently searched for the best-fit reconstruction.

The key point in our analysis is the close relationship between the configuration expressed by Kruppa equations and Poncelet's Porism of classical projective geometry. This relationship has also been mentioned in [8], where necessary and sufficient conditions for the epipolar and Kruppa constraints to be satisfied, given four corresponding points in two calibrated images, were given.

In section 2 a brief review of the projective geometry of the conic is included, as well as some basic notions on projective calibration. In section 3 we establish the relationship between Kruppa equations and Poncelet's Porism. In section 4 we make use of this result to give an explicit parametrization of the set of possible projected absolute conics and of possible planes at infinity. In section 5 the variables of the parametrization are geometrically interpreted and in section 6 we introduce the additional restriction of known pixel shape to reduce the set of solutions to a

one-dimensional family which is also explicitly parametrized. Finally, in section 7 experimental results on real images are given.

2. BACKGROUND TOPICS

2.1. Projective geometry of the conic. For the sake of completeness, we briefly summarize here the basics of projective geometry of conics we need in the paper. A general reference for the topic is [10].

Any non-degenerate conic can be parameterized in the form $x = p_0(\theta)$, $y = p_1(\theta)$, $z = p_2(\theta)$ where the p_i 's are three independent second degree polynomials. This parameterization is unique up to a projective transformation of the parameter θ , i.e., a transformation of the form

$$(1) \quad \theta' = (a\theta + b)/(c\theta + d),$$

with $ad - bc \neq 0$. Therefore the cross-ratio of four points on a conic is well-defined as the cross-ratio of their respective parameters. Homographies of the conic are given by homographic transformations of the parameter, i.e., transformations $\theta' = h(\theta)$ of the form (1).

Particularly interesting is the case of four points harmonically separated on a conic. Given two unordered pairs of points $\{\mathbf{p}_0, \mathbf{p}_1\}$ and $\{\mathbf{p}_2, \mathbf{p}_3\}$ on a conic, they are *harmonically separated*, i.e., the cross ratio $\{\mathbf{p}_0, \mathbf{p}_1; \mathbf{p}_2, \mathbf{p}_3\} = -1$, if and only if the tangents to the conic at each point of one of the pairs are concurrent with the line defined by the other pair (see figure 1). Equivalently, one can say that the pole of each of the lines $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$ lies on the other line. Lines with this property are said to be *conjugate* with respect to the conic. If we denote by \mathbf{A} the matrix of the conic, it results that the lines are conjugate if and only if $(\mathbf{p}_0 \times \mathbf{p}_1)^\top \mathbf{A}^* (\mathbf{p}_2 \times \mathbf{p}_3) = 0$, where \mathbf{A}^* stands for the adjoint matrix of \mathbf{A} .

Involutions are involutive homographies, i.e., homographies $\theta' = \sigma(\theta)$ such that $\sigma \circ \sigma = \text{Id}$. Let \mathbf{m} and \mathbf{n} be the fixed points of the involution σ . Then it can be shown that $\sigma(\mathbf{p})$ is uniquely determined by the equation $\{\mathbf{m}, \mathbf{n}; \mathbf{p}, \sigma(\mathbf{p})\} = -1$. In the particular case of an involution of a conic, the previously given interpretation of harmonically separated points shows that σ can be geometrically defined by means of the point \mathbf{v} where the tangents at the fixed points \mathbf{m} and \mathbf{n} intersect. This point is called the *vertex* of the involution. For each point \mathbf{p} of the conic, its image $\mathbf{p}' = \sigma(\mathbf{p})$ is just the other intersection point of the line $\mathbf{v}\mathbf{p}$ with the conic.

Involutions are also given by symmetrical (1, 1) algebraic correspondences, i.e., by equations of the form

$$\alpha\theta\theta' + \beta(\theta + \theta') + \gamma = 0,$$

where $\alpha\gamma - \beta^2 \neq 0$.

2.2. Projective calibration. Let us consider two cameras with projection matrices $\hat{\mathbf{P}}$ and $\hat{\mathbf{P}}'$. As is well known, the two projections $\mathbf{x} = \hat{\mathbf{P}}\hat{\mathbf{X}}$, $\mathbf{x}' = \hat{\mathbf{P}}'\hat{\mathbf{X}}$ of the same 3D point $\hat{\mathbf{X}}$ meet the so-called *epipolar relationship*,

$$\mathbf{x}'^\top \mathbf{F} \mathbf{x} = 0$$

for some rank-two matrix \mathbf{F} called *fundamental matrix* that depends on the projection matrices. Conversely, given the fundamental matrix \mathbf{F} we can recover the camera matrices up to a 3D homography, i.e., we can obtain matrices \mathbf{P} and \mathbf{P}' such that $\hat{\mathbf{P}} = \mathbf{P}\mathbf{H}$, $\hat{\mathbf{P}}' = \mathbf{P}'\mathbf{H}$ for some unknown 4×4 regular matrix \mathbf{H} [3, sec. 8.5]. From now on we will assume that a projective calibration $(\mathbf{P}, \mathbf{P}')$ is given.

The epipolar relationship has the following nice geometrical interpretation as an homography between two pencils of lines: If \mathbf{e} and \mathbf{e}' are the *epipoles*, defined by the relations $\mathbf{F}\mathbf{e} = \mathbf{e}'^\top \mathbf{F} = \mathbf{0}$, the homography assigns to any line through the pencil

defined by \mathbf{e} , passing through a point $\mathbf{x} \neq \mathbf{e}$, the line l' of coordinates $\mathbf{F}\mathbf{x}$. This line belongs to the pencil defined by \mathbf{e}' since $\mathbf{e}'^\top \mathbf{F}\mathbf{x} = 0$. According to Steiner's theorem, the points of intersection of corresponding lines define a conic, which in this case is given by the matrix $\mathbf{F}_S = (\mathbf{F} + \mathbf{F}^\top)/2$ (see figure 2). Conversely, given the conic \mathbf{F}_S and the epipoles \mathbf{e} and \mathbf{e}' , we can recover the fundamental matrix \mathbf{F} as follows. Let \mathbf{F}_S^* be the adjoint matrix of \mathbf{F}_S , and let us define $\mathbf{x}_a = \mathbf{F}_S^*(\mathbf{e} \times \mathbf{e}')$. Then $\mathbf{F} = \mathbf{F}_S + \lambda[\mathbf{x}_a]_\times$, for some scalar λ that can be determined uniquely by the condition $\mathbf{F}\mathbf{e} = \mathbf{0}$ or $\mathbf{e}'^\top \mathbf{F} = \mathbf{0}$ [3, sec. 8.4].

3. COMPATIBLE CONICS, KRUPPA CONFIGURATION AND PONCELET'S PORISM

3.1. Compatible conics and Kruppa configuration. It is well-known that the projection of the absolute conic at the plane at infinity depends only on the intrinsic parameters of the camera [3, p. 200]. From now on we suppose that both cameras have the same intrinsic parameters. We will call a conic ω *compatible* with a given epipolar geometry if there exists some conic in 3D space which projects onto ω with both cameras of an associated projective calibration.

Note that the compatibility of a conic ω is equivalent to the reducibility to two conics of the quartic curve given by the intersection of the two retroprojected cones of ω . The two spatial conics thus obtained will be called *the associated 3D conics* to ω . This twofold multiplicity of the solutions corresponds to the existence of two Euclidean reconstructions compatible with a given essential matrix (cheirality [3, chap. 20]).

The Kruppa equations express a necessary condition for the compatibility of a conic ω : Let us consider a conic Ω in 3D space projecting onto ω . Then there are two planes tangent to Ω in the pencil of planes defined by the line joining the projection centers. The projections of each of these planes will be two epipolarly related lines tangent to ω [3, p. 454]. This leads to the following definition: A conic ω is in *Kruppa configuration with respect to \mathbf{F}* if the tangents from the epipoles to ω are related by the epipolar homography. Therefore, being in Kruppa configuration with respect to \mathbf{F} is a necessary condition for a conic to be compatible with \mathbf{F} , which later will be proved to be also sufficient.

By the geometrical interpretation of the epipolar relationship given in the previous section, the tangents to ω from \mathbf{e} and \mathbf{e}' intersect pairwise on two points \mathbf{p} and \mathbf{p}' of \mathbf{F}_S , so that ω has a circumscribed quadrilateral $\mathbf{e}\mathbf{p}\mathbf{e}'\mathbf{p}'$ which is in turn inscribed in \mathbf{F}_S (see figure 3). For each choice of two different points \mathbf{p}, \mathbf{p}' of \mathbf{F}_S we have a dual pencil of Kruppa conics consisting in all the dual conics including the four lines $\mathbf{e}\mathbf{p}, \mathbf{e}'\mathbf{p}, \mathbf{e}\mathbf{p}', \mathbf{e}'\mathbf{p}'$.

3.2. Kruppa configuration and Poncelet's Porism. Relevant to the Kruppa configuration is Poncelet's Porism (see figure 4 and [10, p. 181]), which assures that once there is an n -gon inscribed in one conic and circumscribed to another, then there are infinite such n -gons, one for each choice of a point in the first conic as vertex. Therefore we can state the following remark:

Remark 3.1. The conics in Kruppa configuration with respect to \mathbf{F} are those satisfying Poncelet's Porism for a quadrilateral inscribed in \mathbf{F}_S with \mathbf{e} and \mathbf{e}' as opposed vertices.

Given a $2k$ -gon inscribed in s and circumscribed to s' , we can consider the mapping which assigns to each point \mathbf{q} of s the opposed vertex \mathbf{q}' of the corresponding $2k$ -gon given by Poncelet's Porism, as shown in figure 4.

Theorem 3.1. *Given two conics s, s' verifying Poncelet's Porism for $n = 2k$, the mapping assigning to any point of s the opposite point of the corresponding $2k$ -gon is an involution of s .*

Proof. Let s and s' be any two conics. The conic s' defines a relation on s by the rule that two points \mathbf{p} , \mathbf{p}' are related whenever the line \mathbf{pp}' is tangent to s' . Let $(p_0(\theta), p_1(\theta), p_2(\theta))^\top$ be a projective parameterization of s and θ and θ' be the parameters of \mathbf{p} and \mathbf{p}' . Then \mathbf{p} and \mathbf{p}' are related if and only if $(\mathbf{p} \times \mathbf{p}')^\top \mathbf{A}'^*(\mathbf{p} \times \mathbf{p}') = 0$, where \mathbf{A}'^* is the adjoint matrix of the conic s' . This a relation given by a symmetric polynomial of degree 2 in each variable θ , θ' , which is called a *symmetrical algebraic (2, 2) correspondence* on the conic [10, p. 179]. More generally, we can also consider the relation between two points $\mathbf{p} = \mathbf{p}_0$ and $\mathbf{p}' = \mathbf{p}_k$ defined by the existence of a chain of points of s , $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k$ such that two consecutive points are related as before. Clearly, this is a symmetrical algebraic correspondence. Moreover, for each \mathbf{p} there exist at most two related points \mathbf{p}' , since there are two tangents from \mathbf{p} to s' . So this is also a symmetrical algebraic (2, 2) correspondence.

In case that s and s' are in the configuration of Poncelet's Porism for $n = 2k$, the correspondence is one-to-one, and therefore the algebraic relation degenerates to a symmetrical (1, 1) correspondence, i.e., an involution [10, p. 55]. \square

From now on we will only consider the case $n = 4$. A pair of conics (s, s') will be called a *Poncelet pair* if there exists a quadrilateral (*Poncelet quadrilateral*) inscribed in s and circumscribed to s' . Recall that any homography has, generically, two fixed points which, in our case, can be nicely geometrically identified. Let \mathbf{r} be one of the four points of intersection of s and s' , which might be complex (see figure 5), and let us consider the Poncelet quadrilateral with vertex \mathbf{r} , which is degenerate since there is only one tangent from \mathbf{r} to s' . One of its vertices will be the fixed point $\mathbf{f}_1 = \mathbf{q} = \mathbf{q}' \in s$ of the involution defined on s by s' . Therefore \mathbf{f}_1 is given by the tangents to s' at two of its points of intersection with s . Analogously, taking the tangents to s' at the other pair of intersection points we obtain the other fixed point \mathbf{f}_2 of the involution. This proves the necessary condition part of the following theorem.

Theorem 3.2. *The necessary and sufficient condition for a pair of conics (s, s') in general position to be a Poncelet pair is that the tangents to s' at the points of intersection of s and s' intersect pairwise on s .*

Proof. It remains only to prove the sufficiency condition. This will be done in a similar manner to the proof of Poncelet's Porism given in [10, p. 181]. Consider the correspondence between points \mathbf{p} , \mathbf{p}' of s given by the property that there is a polygon $\mathbf{p}\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\mathbf{p}'$ with $\mathbf{q}_i \in s$ and all its sides tangent to s' . From the theory of algebraic correspondences on a conic, it is known that this is a (2, 2) correspondence given by a symmetric quadratic equation $f(\theta, \theta') = 0$. A united point $\mathbf{p} = \mathbf{p}'$ of the correspondence is a vertex of a quadrilateral inscribed in s and circumscribed to s' , i.e., a Poncelet quadrilateral.

Since each united point \mathbf{p} of the correspondence is given by a double root θ_0 of the fourth degree equation $f(\theta, \theta) = 0$, the existence of three united points ensures that the equation is identically zero and thus every point is a united point. Now, the result follows from observing that a degenerate Poncelet quadrilateral with three different vertices provides three double points of the correspondence. \square

The vertex of the involution can be explicitly constructed as the pole \mathbf{v} with respect to s of the line $\mathbf{f}_1\mathbf{f}_2$ joining the fixed points of the involution. Consequently, two points of s will be opposed vertices of a Poncelet quadrilateral if and only if their are collinear with \mathbf{v} (see figure 6).

Next corollary sums up the previous discussion.

Corollary 3.3. *The following properties are equivalent:*

- (1) ω is in Kruppa configuration with respect to \mathbf{F} .
- (2) (\mathbf{F}_S, ω) is a Poncelet pair with \mathbf{e} and \mathbf{e}' satisfying one of the following equivalent conditions:
 - (a) \mathbf{e} and \mathbf{e}' are opposed vertices of a Poncelet quadrilateral.
 - (b) \mathbf{e} and \mathbf{e}' are related by the involution defined on \mathbf{F}_S by ω .
 - (c) \mathbf{e} and \mathbf{e}' are collinear with the vertex of the involution defined on \mathbf{F}_S by ω .
 - (d) The line joining \mathbf{e} and \mathbf{e}' is conjugate with respect to \mathbf{F}_S with the line joining the fixed points \mathbf{f}_1 and \mathbf{f}_2 of the involution defined on \mathbf{F}_S by ω .
- (3) The tangents to ω at the points of intersection of ω with \mathbf{F}_S intersect pairwise on \mathbf{F}_S on two points (\mathbf{f}_1 and \mathbf{f}_2) defining a line conjugate with respect to \mathbf{F}_S to the line \mathbf{ee}' .

Proof. (1) \Leftrightarrow (2.a) is remark 3.1.

(2.a) \Leftrightarrow (2.b) is due to theorem 3.1.

(2.b) \Leftrightarrow (2.c) is due to the properties of involutions on a conic stated in section 2.1.

(2.c) \Leftrightarrow (2.d) is due to the relation between the vertex and the fixed points of the involution, stated in section 2.1, and the definition of conjugate lines with respect to a conic.

(2.d) \Leftrightarrow (3) is due to theorem 3.2 and the geometrical construction of the fixed points of the involution. \square

4. A PARAMETERIZATION OF THE CONICS IN KRUPPA CONFIGURATION

We will make use of the properties of Poncelet's pairs to propose a parameterization of the set of conics in Kruppa configuration with respect to \mathbf{F} . Since the absolute conic has real coefficients but no real points, i.e., it has a real definite matrix, we are only interested in conics ω in Kruppa configuration with respect to \mathbf{F} with these two characteristics. We will call a real projective parameterization $c = c(\theta)$ of \mathbf{F}_S *standard* if $c(0) = \mathbf{e}$ and $c(\infty) = \mathbf{e}'$. Standard parameterizations are unique up to a reparameterization of the form $\theta' = k\theta$. With respect to a standard parameterization, the fixed points of the involution produced on \mathbf{F}_S by ω satisfy the following property.

Theorem 4.1. *Let ω be a real definite conic in Kruppa configuration with respect to \mathbf{F} and let $c = c(\theta)$ be a standard parameterization of \mathbf{F}_S . Then, the parameters of the fixed points of the involution induced on \mathbf{F}_S by ω are $\rho, -\rho \in \mathbf{R} \setminus \{0\}$. Furthermore, the other vertices of the degenerate Poncelet quadrilaterals defined by the fixed points have parameters $\rho e^{\pm i\phi}$ and $\rho e^{\pm i\psi}$ for some arguments $\phi, \psi \in (0, \pi)$ (see figure 7).*

Proof. Let $c = c(\theta)$ be a standard parameterization of \mathbf{F}_S . Since the epipoles are related by the involution, i.e., 0 and ∞ are swapped, the equation of the involution is of the form $\theta\theta' = k$ with $k \neq 0$. Let θ_0 and θ'_0 be the parameters of the other two points of the Poncelet's quadrilateral with vertices \mathbf{e} and \mathbf{e}' . The lines joining \mathbf{e} with $c(\theta_0)$ and $c(\theta'_0)$ must be either real or complex conjugate, since they are tangent lines from the real point \mathbf{e} to the conic of real coefficients ω . They cannot be real since ω has no real points. Therefore they are complex conjugate and so $\theta'_0 = \bar{\theta}_0$, so that $k = \theta_0\bar{\theta}_0 = |\theta_0|^2 > 0$. Then the fixed points of the involution, satisfying $\theta^2 = k$, have parameters $\theta = \rho = \sqrt{k}$ and $\theta = -\rho$.

To prove the second statement, first observe that the tangents from the real points $c(\pm\rho)$ to ω are non-real conjugate. Therefore the vertices must have parameters $\theta_1, \bar{\theta}_1 = \rho_1 e^{\pm i\phi}$ and $\theta_2, \bar{\theta}_2 = \rho_2 e^{\pm i\psi}$ with $\phi, \psi \in (0, \pi)$. To see that $\rho_1 = \rho_2 = \rho$, it is enough to observe that θ_i and $\bar{\theta}_i$, being opposed vertices of a Poncelet quadrilateral, are related by the involution, so that $\theta_i\bar{\theta}_i = \rho_i^2 = k = \rho^2$. \square

Now let us explicitly obtain the set of conics ω in Kruppa configuration with respect to \mathbf{F} and the set of their associated 3D-planes (see figure 9). Let us consider a projective coordinate system for which $\mathbf{e} = (0, 0, 1)^\top$, $\mathbf{e}' = (1, 0, 0)^\top$ and $\mathbf{x}_a = \mathbf{F}_S^*(\mathbf{e} \times \mathbf{e}') = (0, 1, 0)^\top$. Choosing any other real point of \mathbf{F}_S as the unit point $(1, 1, 1)$, the equation of \mathbf{F}_S turns out to be $y^2 = xz$, so that the mapping $\theta \mapsto c(\theta) = (\theta^2, \theta, 1)^\top$ is a standard parameterization of the conic. Let $\mathbf{f}_1 = c(\rho)$, $\mathbf{f}_2 = c(-\rho)$, $\rho \in \mathbf{R} \setminus \{0\}$ be the fixed points of the involution induced on \mathbf{F}_S by ω . The pole of the line through $\mathbf{f}_1 = c(\rho)$ and $\mathbf{f}_2 = c(-\rho)$ will be the vertex \mathbf{v} of the involution. Let us take a real line through \mathbf{v} , which will intersect \mathbf{F}_S in two points, \mathbf{r} and \mathbf{s} . The associated conics ω are those with tangent $\mathbf{f}_1\mathbf{r}$ at \mathbf{r} and $\mathbf{f}_1\mathbf{s}$ at \mathbf{s} , which form a pencil determined by the degenerated conics given by the product of the lines $\mathbf{f}_1\mathbf{r}$ and $\mathbf{f}_1\mathbf{s}$ and the double line \mathbf{rs} (see [10, p. 160]), i.e., the pencil $\omega = \mathbf{C} + \lambda\mathbf{C}'$, where

$$(2) \quad \begin{aligned} \mathbf{C} &= (\mathbf{r} \times \mathbf{s})(\mathbf{r} \times \mathbf{s})^\top, \\ \mathbf{C}' &= (\mathbf{f}_1 \times \mathbf{r})(\mathbf{f}_1 \times \mathbf{s})^\top + (\mathbf{f}_1 \times \mathbf{s})(\mathbf{f}_1 \times \mathbf{r})^\top. \end{aligned}$$

This is a self-dual pencil, i.e., a pencil of conics whose dual conics also form a pencil. Note that since $\mathbf{r}, \mathbf{s} = c(\rho e^{\pm i\phi})$, the conic ω depends on three real parameters: $\rho \neq 0$, $\phi \in [0, 2\pi)$ and $\lambda \in \mathbf{R}$.

A straightforward computation shows that, with our choice of coordinates,

$$(3) \quad \begin{aligned} \mathbf{F} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{C}(\rho, \phi) &= \begin{pmatrix} 1 & -\rho(\cos\phi + 1) & \rho^2 \cos\phi \\ -\rho(\cos\phi + 1) & 2\rho^2(\cos\phi + 1) & -\rho^3(\cos\phi + 1) \\ \rho^2 \cos\phi & -\rho^3(\cos\phi + 1) & \rho^4 \end{pmatrix}, \\ \mathbf{C}'(\rho, \phi) &= \begin{pmatrix} 1 & -2\rho \cos\phi & \rho^2 \\ -2\rho \cos\phi & 4\rho^2 \cos^2\phi & -2\rho^3 \cos\phi \\ \rho^2 & -2\rho^3 \cos\phi & \rho^4 \end{pmatrix}. \end{aligned}$$

Therefore, conics in Kruppa configuration with respect to \mathbf{F} are given by

$$(4) \quad \omega(\rho, \phi, \lambda) = \mathbf{C}(\rho, \phi) + \lambda\mathbf{C}'(\rho, \phi).$$

Note that if we use the points $c(-\rho)$ and $c(\rho e^{\pm i\psi})$ to define the pencil, the expression of the conics will be $\omega(-\rho, \psi + \pi, \mu)$.

Not all these conics are definite, as the projection of the absolute conic must be.

Theorem 4.2. *The conic $\omega(\rho, \phi, \lambda)$ is definite if and only if $\lambda > 0$, $\rho \neq 0$, and $\phi \neq 0, \pi$.*

Proof. Let us denote by Δ_i the principal minor of ω of order i . It is straightforward to check that

$$\begin{aligned} \Delta_1 &= \rho^4(\lambda + 1), \\ \Delta_2 &= \rho^6(1 - \cos\phi)(2\lambda + \cos\phi + 1), \\ \Delta_3 &= 4\rho^6(1 - \cos\phi)^3(\cos\phi + 1)\lambda. \end{aligned}$$

The matrix ω will be definite if and only if all the $\Delta_i > 0$ or $\Delta_1, \Delta_3 < 0$ and $\Delta_2 > 0$. The first possibility occurs if and only if $\lambda > 0$ and $\phi \neq 0, \pi$. The second never occurs. \square

We already know that compatible conics are conics in Kruppa configuration with respect to \mathbf{F} . The converse seems to be well-known folklore. We provide here a proof for the sake of completeness.

Theorem 4.3. *Conics in Kruppa configuration with respect to \mathbf{F} are compatible conics.*

Proof. Let ω be a conic in Kruppa configuration with respect to \mathbf{F} . We consider a projective coordinate system adapted to \mathbf{F}_S as described above and a standard parameterization $c = c(\theta)$ of this conic. We can therefore express $\omega = \mathbf{C} + \lambda\mathbf{C}'$ using formulas (3). Let us take the pair of projection matrices compatible with \mathbf{F} given by $\mathbf{P} = (\mathbf{I} | 0)$ and $\mathbf{P}' = (\mathbf{e}'_\times \mathbf{F} | \mathbf{e}')$ (see [3, p. 238]). The cones backprojection of ω are the quadrics $\mathbf{Q} = \mathbf{P}^\top \omega \mathbf{P}$ and $\mathbf{Q}' = \mathbf{P}'^\top \omega \mathbf{P}'$. For ω to be compatible, \mathbf{Q} and \mathbf{Q}' must intersect in a pair of 3D conics. These conics, in turn, will be contained in the product of two planes, which is a rank-two quadric of the pencil defined by \mathbf{Q} and \mathbf{Q}' and so it will correspond to a double root of the generalized eigenvalue equation $\det(\mathbf{Q} + \mu\mathbf{Q}') = 0$ [9]. An easy computation shows that we have indeed this double root $\mu = -\rho^2$. Let us denote the corresponding rank-two quadric as $\mathbf{A} = \mathbf{Q} - \rho^2\mathbf{Q}'$, which is rank-two and therefore breaks indeed into a pair of planes. Now it follows that the quartic given by the intersection of the cones \mathbf{Q} and \mathbf{Q}' is contained in the pair of conics given by the intersection of any of the cones with the pair of planes, and so ω is a compatible conic. \square

It is interesting as well to have the explicit coordinates of the planes associated to the conic ω , which are the conics in which the quadric $\mathbf{A} = \mathbf{Q} - \rho^2\mathbf{Q}'$ of the previous proof breaks up. Let π_1 and π_2 be these planes, so $\mathbf{A} \sim \pi_1\pi_2^\top + \pi_2\pi_1^\top$. This makes apparent that π_1 and π_2 lie in the pencil of planes defined by the column space of \mathbf{A} . Thus if we find a point of each plane (which can be done by intersecting \mathbf{A} with any line not incident with the base line of the pencil of planes), we easily compute:

$$(5) \quad \pi_1 = \begin{pmatrix} 1 \\ -\rho(1 + 2 \cos \phi) \\ \rho^2(1 + 2 \cos \phi) \\ \rho^3 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 \\ -\rho[\lambda(2 \cos \phi - 1) + 2]/(2 + \lambda) \\ -\rho^2[\lambda(2 \cos \phi - 1) + 2]/(2 + \lambda) \\ -\rho^3 \end{pmatrix}.$$

Note that π_1 does not depend on λ . In fact, it can be checked that the backprojections of the points of \mathbf{F}_S given by $\mathbf{f}_1 = c(\rho)$, $\mathbf{r}, \mathbf{s} = c(\rho e^{\pm i\phi})$ lie in it, so that π_1 is the plane defined by these three spatial points. We will denote by $\pi_1(\rho, \phi)$ and $\pi_2(\rho, \phi, \lambda)$ the expressions defined by the equations (5). Now consider the other two points of intersection of ω with \mathbf{F}_S , which, from theorem 4.1 are $c(\rho e^{\pm i\psi})$. Then ω also belongs to the pencil of conics given by the parameters $-\rho$ and $\psi + \pi$, i.e., $\omega = \omega(-\rho, \psi + \pi, \mu)$ for some μ . Therefore its associated planes can also be written as $\pi_1(-\rho, \psi + \pi)$ and $\pi_2(-\rho, \psi + \pi, \mu)$ and we must have

$$(6) \quad \begin{aligned} \pi_1(-\rho, \psi + \pi) &= \pi_2(\rho, \phi, \lambda), \\ \pi_2(-\rho, \psi + \pi, \mu) &= \pi_1(\rho, \phi), \end{aligned}$$

since the other possibility of coincidence between the planes would yield $\rho = -\rho$, which is impossible. Equations (6) yield

$$(7) \quad \begin{aligned} \cos \psi &= \frac{\lambda \cos \phi + 2}{\lambda + 2} \\ \mu &= \frac{(\lambda + 2)(1 + \cos \phi)}{1 - \cos \phi}. \end{aligned}$$

Therefore the expression for π_1 in equations (5) provides all the planes in which the 3D conics associated to conics in Kruppa configuration lie as ρ runs over $\mathbf{R} \setminus \{0\}$

and $\phi \in (0, \pi)$. In particular, we have obtained that $\pi_2(\rho, \phi, \lambda)$ is the plane through the retroprojection of the points $c(-\rho)$ and $c(\rho e^{\pm i\psi})$. We are in conditions to prove the following result:

Corollary 4.4. (1) *The mapping*

$$(8) \quad (0, \infty) \times (0, \pi) \times (0, \infty) \ni (\rho, \phi, \lambda) \mapsto \omega(\rho, \phi, \lambda)$$

given by (4) establishes a bijection onto the set of definite conics in Kruppa configuration with respect to \mathbf{F} .

(2) *The mapping*

$$(9) \quad (\mathbf{R} \setminus \{0\}) \times (0, \pi) \ni (\rho, \phi) \mapsto \pi_1(\rho, \phi)$$

given by (5) establishes a bijection onto the set of 3D-planes associated to definite conics in Kruppa configuration with respect to \mathbf{F} .

This provides explicitly the triparametric family of Euclidean reconstructions compatible with the data.

Proof. The mapping given by equation (8) is well defined, i.e., $\omega(\rho, \phi, \lambda)$ is a definite conic in Kruppa configuration with respect to \mathbf{F} . This is a consequence of the discussion on formula (4) and theorem 4.2. To see that it is a bijection, note that given a definite conic ω in Kruppa configuration with respect to \mathbf{F} then, as a consequence of theorem 4.1, it has associated two pairs of parameters (ρ, ϕ) and $(-\rho, \psi)$. The criterion $\rho > 0$ selects the first pair. Therefore ω is a member of the pencil of conics generated by $\mathcal{C}(\rho, \phi)$ and $\mathcal{C}'(\rho, \phi)$ and, again by theorem 4.2, its parameter λ is real and greater than 0.

Regarding the second part, we have already seen that the mapping given by equation (9) is surjective. Injectivity is immediate. \square

5. INTERPRETATION OF THE PARAMETERS ρ AND ϕ

Next we study how the parameters ρ and ϕ are related with the elements of the camera motion.

Lemma 5.1. *Let $\mathbf{r}, \mathbf{s}, \mathbf{e}$ and \mathbf{e}' be four points of the projective plane (see figure 8). Let $\mathbf{a} = \mathbf{e}\mathbf{r} \cap \mathbf{e}'\mathbf{s}$ and $\mathbf{b} = \mathbf{e}'\mathbf{r} \cap \mathbf{e}\mathbf{s}$ and let \mathbf{q} be a point on the line \mathbf{ab} . Let us suppose that no three points among $\mathbf{r}, \mathbf{s}, \mathbf{e}, \mathbf{e}'$ and \mathbf{q} are aligned. Let \mathbf{H} be the homography with fixed points \mathbf{q}, \mathbf{r} and \mathbf{s} that maps \mathbf{e} onto \mathbf{e}' . Let c be a projective parameterization of the conic through the points $\mathbf{r}, \mathbf{s}, \mathbf{e}, \mathbf{e}'$ and \mathbf{q} such that $c(0) = \mathbf{e}$ and $c(\infty) = \mathbf{e}'$. Then the ratio $\theta_{\mathbf{r}} : \theta_{\mathbf{s}} : \theta_{\mathbf{q}}$ of the parameters of the points \mathbf{r}, \mathbf{s} and \mathbf{q} coincide with the ratio $\lambda_{\mathbf{r}} : \lambda_{\mathbf{s}} : \lambda_{\mathbf{q}}$ of the eigenvalues of \mathbf{H} at the corresponding points.*

Proof. Let us take an adapted coordinate system such that $\mathbf{a} = (1, 0, 0)^{\top}$, $\mathbf{r} = (0, 1, 0)^{\top}$, $\mathbf{s} = (0, 0, 1)^{\top}$. Then \mathbf{e} , being aligned with \mathbf{a} and \mathbf{r} has coordinates $(1, \alpha, 0)^{\top}$ for some parameter $\alpha \neq 0$. Analogously, $\mathbf{e}' = (1, 0, \beta)^{\top}$ for some $\beta \neq 0$ and so $\mathbf{b} = (1, \alpha, \beta)^{\top}$. The point \mathbf{q} , being aligned with \mathbf{a} and \mathbf{b} , must have coordinates $(\gamma, \alpha, \beta)^{\top}$ for some $\gamma \neq 1$. The matrix of \mathbf{H} with respect the adapted coordinate system can be easily computed as

$$\begin{pmatrix} 1 - \gamma & 0 & 0 \\ -\alpha & 1 & 0 \\ \beta(1 - \gamma) & 0 & (1 - \gamma)^2 \end{pmatrix},$$

which has eigenvalues $\lambda_{\mathbf{r}} = 1$, $\lambda_{\mathbf{s}} = (1 - \gamma)^2$, $\lambda_{\mathbf{q}} = 1 - \gamma$.

Now we consider the conic through the points $\mathbf{r}, \mathbf{s}, \mathbf{e}, \mathbf{e}'$ and \mathbf{q} and its projective parameterization c such that $c(0) = \mathbf{e}$, $c(1) = \mathbf{q}$, $c(\infty) = \mathbf{e}'$. By Chasles Theorem

([10, p. 133]), the parameter $\theta_{\mathbf{r}}$ can be obtained as

$$\begin{aligned}\theta_{\mathbf{r}} &= \{0, 1; \theta_{\mathbf{r}}, \infty\} \\ &= \{c(0), c(1); c(\theta_{\mathbf{r}}), c(\infty)\} \\ &= \{\mathbf{e}, \mathbf{q}; \mathbf{r}, \mathbf{e}'\} \\ &= \{\mathbf{se}, \mathbf{sq}; \mathbf{sr}, \mathbf{se}'\} = 1/(1 - \gamma),\end{aligned}$$

where the last equality results from a direct computation. The parameter $\theta_{\mathbf{s}}$ can be computed analogously as $\theta_{\mathbf{s}} = 1 - \gamma$ and therefore

$$\theta_{\mathbf{r}} : \theta_{\mathbf{s}} : \theta_{\mathbf{q}} = 1/(1 - \gamma) : 1 - \gamma : 1 = \lambda_{\mathbf{r}} : \lambda_{\mathbf{s}} : \lambda_{\mathbf{q}} = 1 : (1 - \gamma)^2 : 1 - \gamma.$$

□

Let us now consider a projective calibration for two cameras with identical intrinsic parameters. Let F_S be the associated conic and let c denote a standard parameterization of this conic. Let π be the plane defined by the retroprojection of the points $\mathbf{q} = c(\rho)$, $\mathbf{r} = c(\rho e^{i\phi})$ and $\mathbf{s} = c(\rho e^{-i\phi})$. These three points are therefore the fixed points of the homography H induced by π [3, p. 358]. This homography maps \mathbf{e} onto \mathbf{e}' , so we are in the situation considered in the lemma. For the Euclidean reconstruction corresponding to the parameters (ρ, ϕ) , the matrix H is given, as is well known, by $H \sim KRK^{-1}$, where R is the rotation of the camera motion and K is the intrinsic parameters matrix. Therefore the eigenvalues of H , ρ , $\rho e^{i\phi}$ and $\rho e^{-i\phi}$, are proportional to those of R and thus ϕ is the rotation angle. This leads to the known unimodular constraint [3, p. 458]. Moreover, being $c(\rho)$ a real eigenvector of H , $K^{-1}c(\rho)$ is the real eigenvector of R , i.e., the direction of the rotation axis of the camera motion. Therefore $c(\rho)$ is the projection of this direction.

6. AUTOCALIBRATION OF TWO CAMERAS WITH KNOWN PIXEL SHAPE

In this section we apply the previous results to the obtainment of a parameterization of the set of solutions for the autocalibration problem of two cameras with identical intrinsic parameters and known pixel shape. As is well known, this is equivalent, by means of a coordinate change, to supposing that the cameras have square pixels (see [12]), which is what we assume from now on. Let us denote by $\mathbf{i} = (1, i, 0)^T$ and $\mathbf{j} = (1, -i, 0)^T$ the cyclic points at infinite of the image plane with respect to a Euclidean coordinate system. The CCD of a camera with square pixels implements an Euclidean coordinate system and therefore \mathbf{i} and \mathbf{j} belong to ω . This can also be directly checked from the expression of ω in terms of the intrinsic parameter matrix, $\omega \sim K^{-T}K^{-1}$. Changing to a coordinate system adapted to F_S as in section 4, the cyclic points will have arbitrary known coordinates $\mathbf{i} = (a, b, c)^T$ and $\mathbf{j} = (\bar{a}, \bar{b}, \bar{c})^T$.

The condition that \mathbf{i} and \mathbf{j} belong to ω imposes a restriction on the allowable pairs (ρ, ϕ) in the parameterization of the set of conics in Kruppa configuration with respect to F given by corollary 4.4. Namely, since there must be a λ such that $\omega = C(\rho, \phi) + \lambda C'(\rho, \phi)$ satisfies $\mathbf{i}^T \omega \mathbf{i} = \mathbf{j}^T \omega \mathbf{j} = 0$, the condition

$$\begin{vmatrix} \mathbf{i}^T C \mathbf{i} & \mathbf{i}^T C' \mathbf{i} \\ \mathbf{j}^T C \mathbf{j} & \mathbf{j}^T C' \mathbf{j} \end{vmatrix} = 0$$

must hold. This determinant is a real polynomial F in the variables ρ and $\cos \phi$ given by

$$(10) \quad F(\rho, \cos \phi) = 2\rho(1 - \cos \phi)[A(\rho) \cos^2 \phi + B(\rho) \cos \phi + C(\rho)]$$

where

$$\begin{aligned}
A(\rho) &= -4\rho^2 \left[(b^2\bar{b}\bar{c} - \bar{b}^2bc)\rho^2 + (-b^2\bar{a}\bar{c} + \bar{b}^2ac)\rho - \bar{b}^2ab + b^2\bar{a}\bar{b} \right], \\
B(\rho) &= -2\rho \left[(-b^2\bar{c}^2 + \bar{b}^2c^2)\rho^4 + (-2\bar{b}^2bc + 2b^2\bar{b}\bar{c} - 2\bar{b}\bar{c}ac + 2bc\bar{a}\bar{c})\rho^3 + \right. \\
&\quad \left. (2\bar{b}^2ac - 2b^2\bar{a}\bar{c})\rho^2 + (-2\bar{a}\bar{b}ac - 2\bar{b}^2ab + 2ab\bar{a}\bar{c} + 2b^2\bar{a}\bar{b})\rho - b^2\bar{a}^2 + \bar{b}^2a^2 \right], \\
C(\rho) &= (\bar{b}\bar{c}c^2 - bc\bar{c}^2)\rho^6 + (-\bar{b}^2c^2 + b^2\bar{c}^2 + c^2\bar{a}\bar{c} - \bar{c}^2ac)\rho^5 + \\
&\quad (-ab\bar{c}^2 + 2\bar{b}\bar{c}ac - 2bc\bar{a}\bar{c} + \bar{a}\bar{b}c^2)\rho^4 + (2b^2\bar{a}\bar{c} - 2\bar{b}^2ac)\rho^3 + \\
&\quad (-bc\bar{a}^2 + 2\bar{a}\bar{b}ac + \bar{b}\bar{c}a^2 - 2ab\bar{a}\bar{c})\rho^2 + (-\bar{a}^2ac - \bar{b}^2a^2 + b^2\bar{a}^2 + a^2\bar{a}\bar{c})\rho + \\
&\quad \bar{a}\bar{b}a^2 - ab\bar{a}^2.
\end{aligned}$$

Note that given ρ, ϕ satisfying (10), the value of λ giving the corresponding ω is

$$(11) \quad \lambda = -\frac{\mathbf{i}^\top \mathbf{C} \mathbf{i}}{\mathbf{i}^\top \mathbf{C}' \mathbf{i}} = -\frac{\mathbf{j}^\top \mathbf{C} \mathbf{j}}{\mathbf{j}^\top \mathbf{C}' \mathbf{j}},$$

so that

$$\omega \sim (\mathbf{i}^\top \mathbf{C}' \mathbf{i}) \mathbf{C} - (\mathbf{i}^\top \mathbf{C} \mathbf{i}) \mathbf{C}' = (\mathbf{j}^\top \mathbf{C}' \mathbf{j}) \mathbf{C} - (\mathbf{j}^\top \mathbf{C} \mathbf{j}) \mathbf{C}'.$$

Disregarding the factors of $F(\rho, \cos \phi)$ leading to the degenerate cases $\rho = 0$ and $\cos \phi = 1$, we obtain the second-degree equation in $\cos \phi$

$$(12) \quad A(\rho) \cos^2 \phi + B(\rho) \cos \phi + C(\rho) = 0.$$

Next we deal with the obtainment of the values of ρ corresponding to definite conics in Kruppa configuration. Some values of ρ will correspond to two conics while others to one or to none.

6.1. First restriction: $\cos \phi$ must be real. Since $\cos \phi$ must be real, the range of possible values of ρ are within those that make the discriminant $\text{Discrim}(\rho) = B(\rho)^2 - 4A(\rho)C(\rho) \geq 0$. It can be checked that

$$\text{Discrim}(\rho) = G(\rho)^2 H(\rho^2)$$

where $G(x)$ and $H(x)$ are real polynomials. $H(x)$ is the quadratic polynomial

$$H(x) = \alpha x^2 + 2\beta x + \gamma, \text{ where}$$

$$\alpha = (a\bar{b} - \bar{a}b)^2 \leq 0$$

$$\beta = (|b|^4 + |b|^2 - \bar{a}c|^2) \geq 0$$

$$\gamma = (b\bar{c} - \bar{b}c)^2 \leq 0.$$

As the sign of $\text{Discrim}(\rho)$ is that of $H(\rho^2)$, we are interested in the discriminant of $H(x)$, which turns out to be

$$\Delta(H) = 16|\mathbf{i}^\top \mathbf{F}_S \mathbf{j}|^2 |\mathbf{i}^\top \mathbf{F}_S \mathbf{i}| |\mathbf{j}^\top \mathbf{F}_S \mathbf{j}| > 0$$

assuming that \mathbf{i} and \mathbf{j} do not belong to \mathbf{F}_S . Hence $H(x)$ has two real roots. The signs of these roots must be positive, since $\gamma/\alpha \geq 0$ and $\beta/\gamma \leq 0$. So $H(\rho^2)$ will be non-negative in two symmetric compact intervals, I and $-I$.

6.2. Second restriction: $|\cos \phi| \leq 1$. To obtain the subintervals within $I \cup (-I)$ where $|\cos \phi| \leq 1$ we have to find the real roots of the equations

$$A(\rho) \pm B(\rho) + C(\rho) = 0,$$

that appear when substituting $\cos \phi = \pm 1$ in (12). The knowledge of these points together with that of the vertical asymptotes, which can be obtained solving

$$A(\rho) = 0,$$

allows to determine the intervals of ρ within which $|\cos \phi| \leq 1$.

6.3. Third restriction: ω must be definite. We have seen in theorem 4.2 that ω is definite if and only if $\lambda > 0$, $\rho \neq 0$ and $\phi \neq 0, \pi$. We recall that λ is given by equation (11). The endpoints of the intervals of ρ for which $\lambda > 0$ will be among the solutions of the systems of equations

$$\left. \begin{aligned} A(\rho) \cos^2 \phi + B(\rho) \cos \phi + C(\rho) &= 0 \\ \mathbf{i}^\top \mathbf{C} \mathbf{i} &= 0 \end{aligned} \right\}$$

and

$$\left. \begin{aligned} A(\rho) \cos^2 \phi + B(\rho) \cos \phi + C(\rho) &= 0 \\ \mathbf{i}^\top \mathbf{C}' \mathbf{i} &= 0 \end{aligned} \right\},$$

leading to the equations

$$\begin{aligned} (\bar{a}\bar{b} - \bar{a}b)\rho^2 - b\bar{c} + \bar{b}c &= 0, \text{ or} \\ (\bar{a}\bar{b} - \bar{a}b)\rho^2 + (\bar{a}c - a\bar{c})\rho + b\bar{c} - \bar{b}c &= 0, \text{ or} \\ a\bar{a}\rho^4 - (\bar{a}\bar{b} + \bar{a}b)\rho^3 + (b\bar{c} + \bar{b}c)\rho - c\bar{c} &= 0, \end{aligned}$$

which provides at most eight such endpoints of intervals.

Note that the intervals of values of ρ for which there is either one or two real definite conics are symmetric, as each conic associated to ρ can also be seen as associated to $-\rho$.

The previous analysis allows for an easy obtainment of a parameterization of the set of 3D Euclidean reconstructions associated to the pair of images.

7. EXPERIMENTAL RESULTS

Two images of size 1536×1024 of the same scene have been taken with a digital camera keeping constant the internal parameters. A total of 77 matched points were hand-picked, from which a projective calibration were performed using the Gold Standard algorithm for estimating the fundamental matrix, followed by projective bundle adjustment, with a final residual error of 0.38 pixels. Then the possible projections of the 3D conics, assuming square pixels, were parametrized and sampled. The solution providing an angle between points 37, 38 and 39 closest to a right angle was selected. No additional Euclidean bundle adjustment was performed.

Figure 10 shows the images with the matched points. A few of the points that later were used for ground-truth comparison in table 1 are numbered. Figure 11 shows the curve given by equation (12) relating ρ and $\cos \phi$. Valid solutions, i.e., those providing definite conics in Kruppa configuration with respect to the epipolar geometry, are highlighted. Figures 12 and 13 show two views of the reconstructed scene, including camera positions.

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Pair	Actual	Computed
(42,43)	89	89
(44,45)	89	88.4
(43,44)	121	122.2
(42,45)	121	117.3
(42,44)	150	149.0
(16,20)	12.5	13.5
(20,21)	14.2	11.8
(75,76)	44	41.7
(74,76)	115	112.8

TABLE 1. Actual and computed distances in cm between some point pairs. The first measure has been employed for normalization.

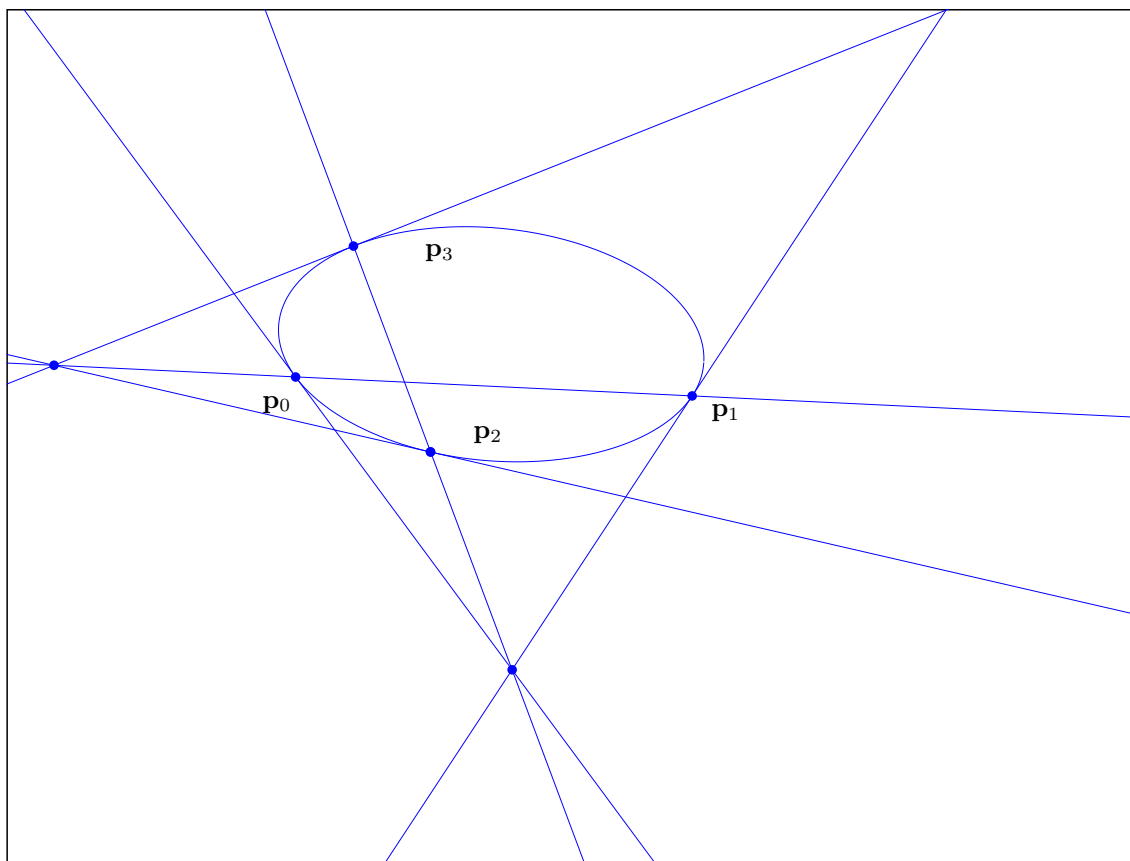


FIGURE 1. Points harmonically separated on a conic. The lines p_0p_1 and p_2p_3 are conjugate with respect to the conic.

- [6] S. Maybank and O. Faugeras, A theory of self-calibration of a moving camera. In *International Journal of Computer Vision*, 8 (2), pages 123–141. 1992
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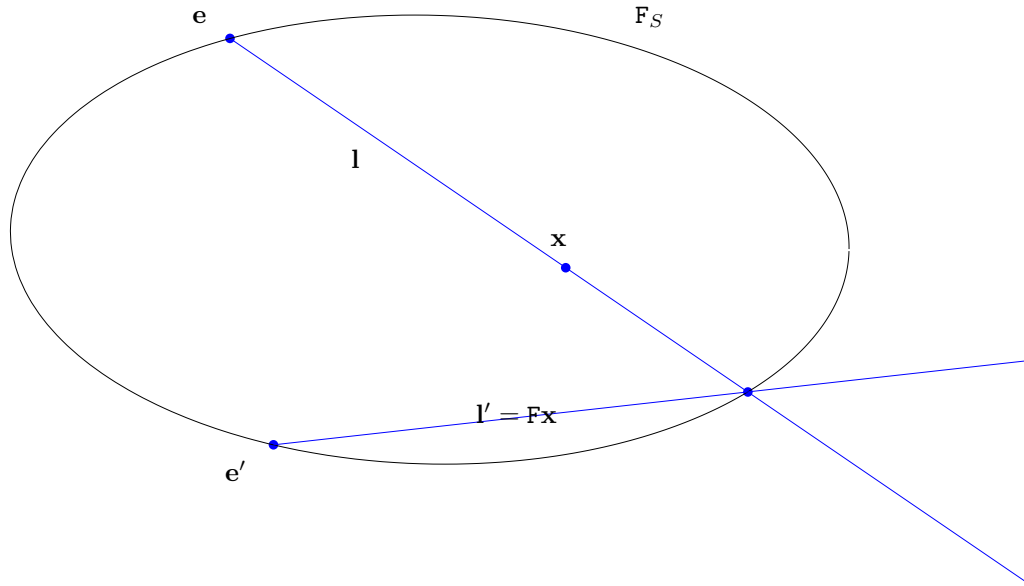


FIGURE 2. Steiner's construction of the epipolar relationship.

- [9] L. Quan, "Conic Reconstruction and Correspondence From Two Views", IEEE Trans. on Pattern Analysis and Machine Intelligence, vol 18, no. 2, pp. 151–160, Feb. 1996.
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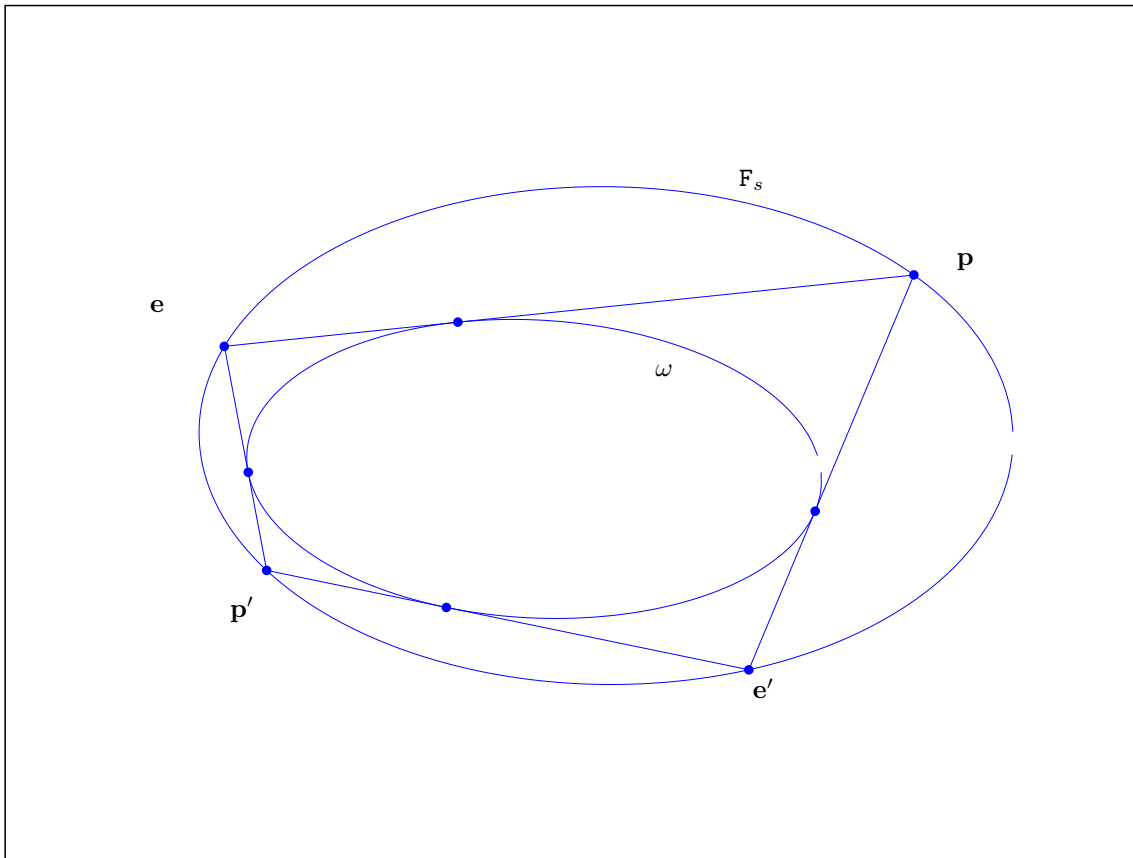


FIGURE 3. Geometric interpretation of Kruppa equations.

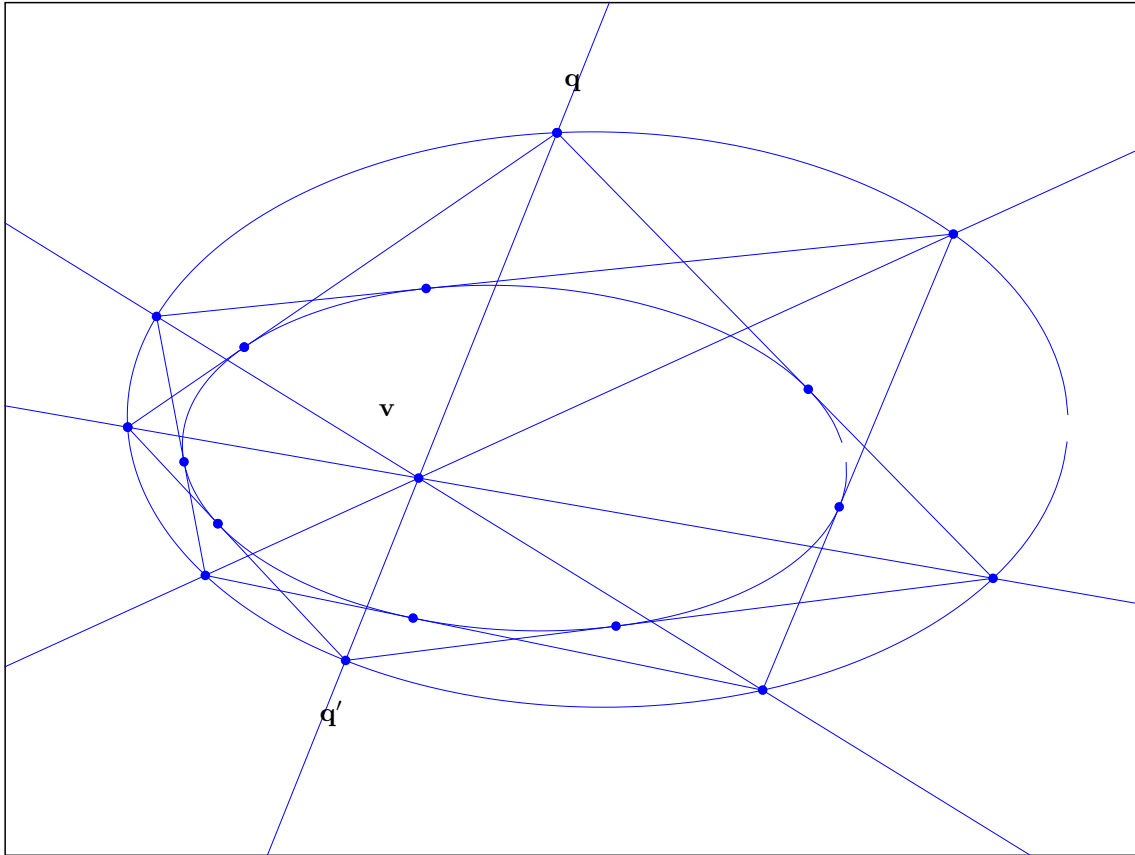


FIGURE 4. Illustration of Poncelet's porism.

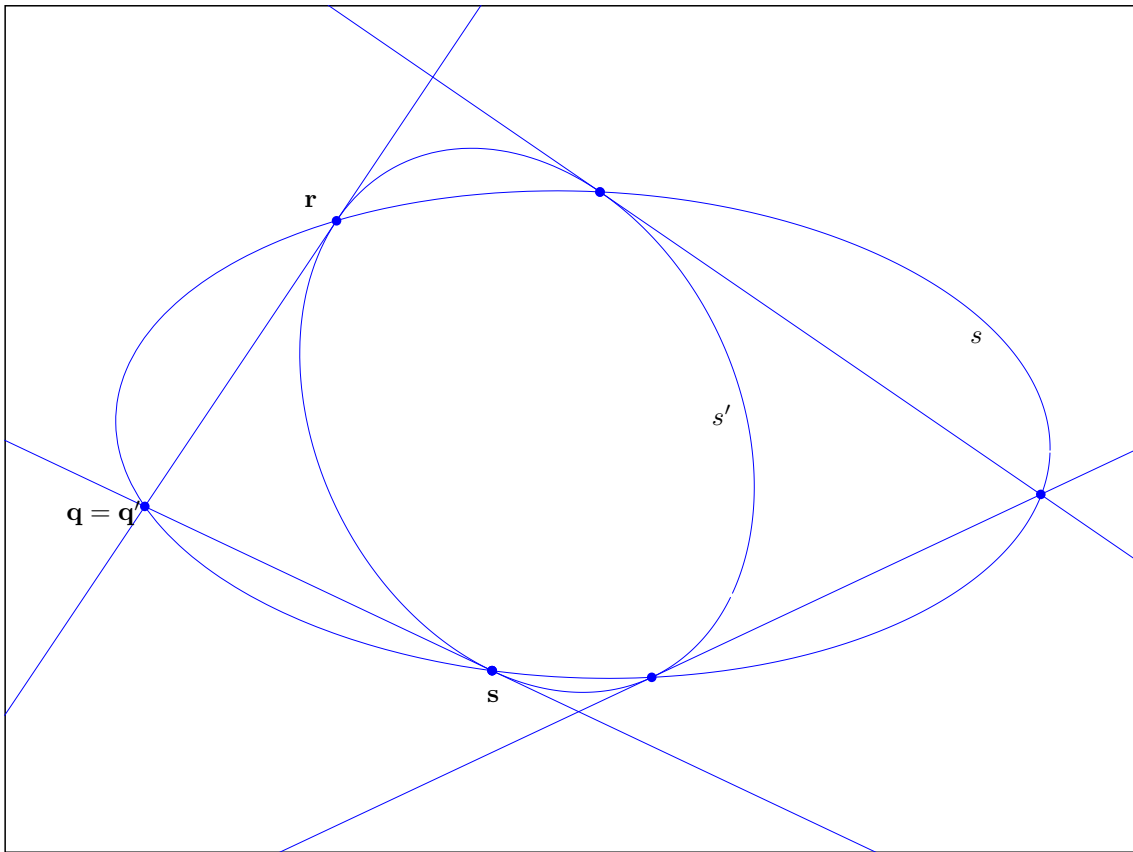


FIGURE 5. Illustration of the construction of the fixed points of the quadrangle homography as well as theorem 3.2.

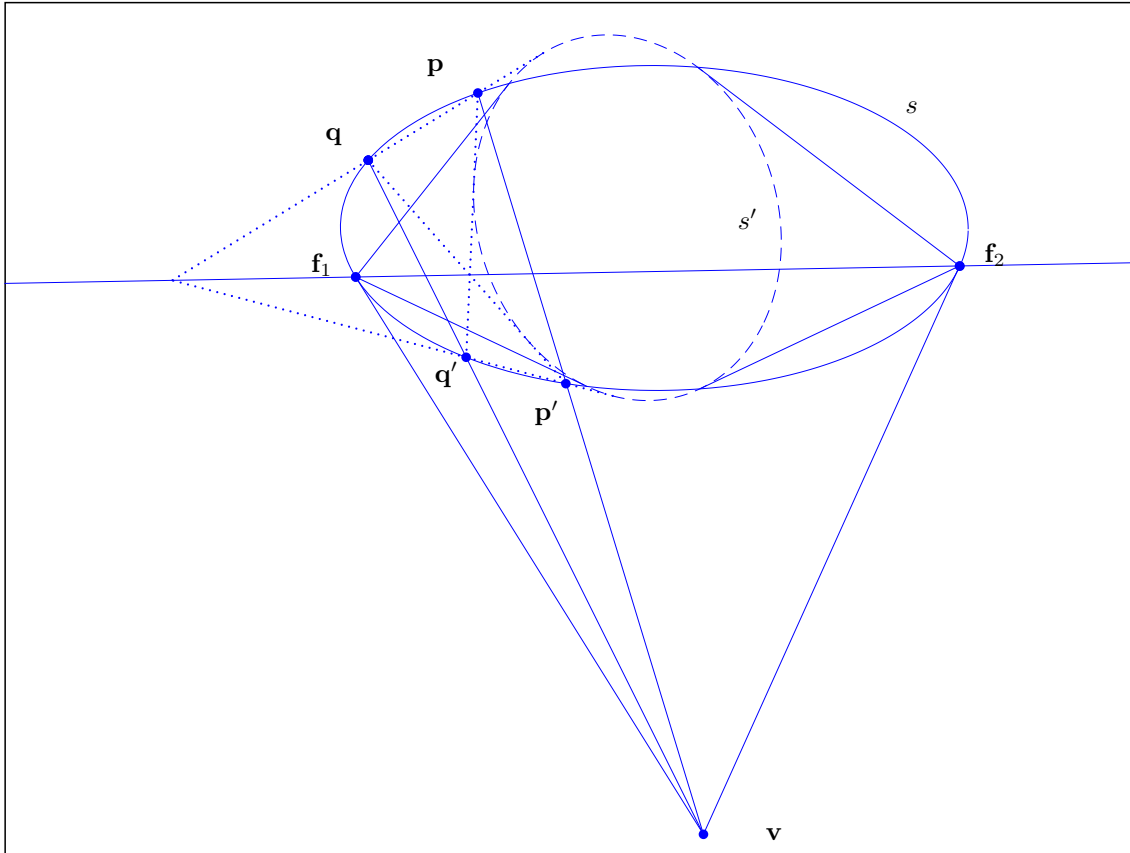


FIGURE 6. Pencil of lines joining opposed vertices of Poncelet's quadrilaterals of a pair of admissible conics.

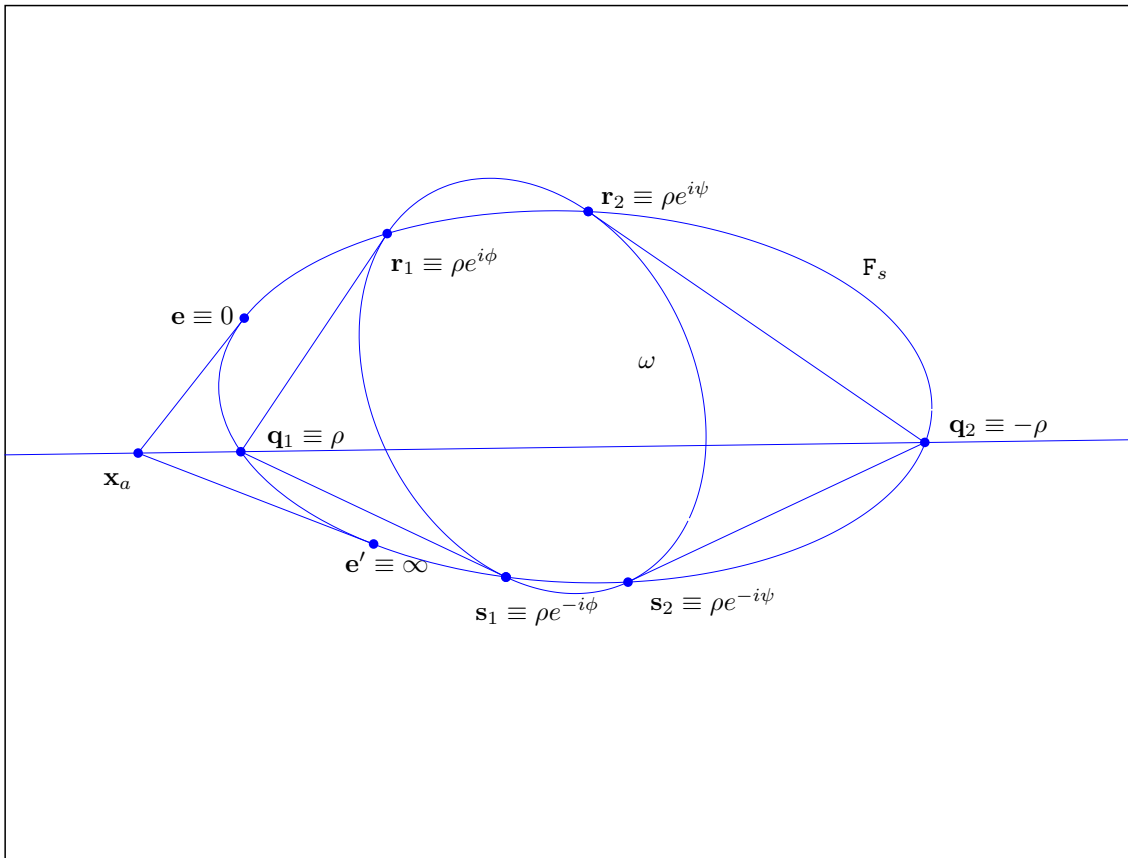


FIGURE 7. Condition of compatibility of ω with epipolar geometry.

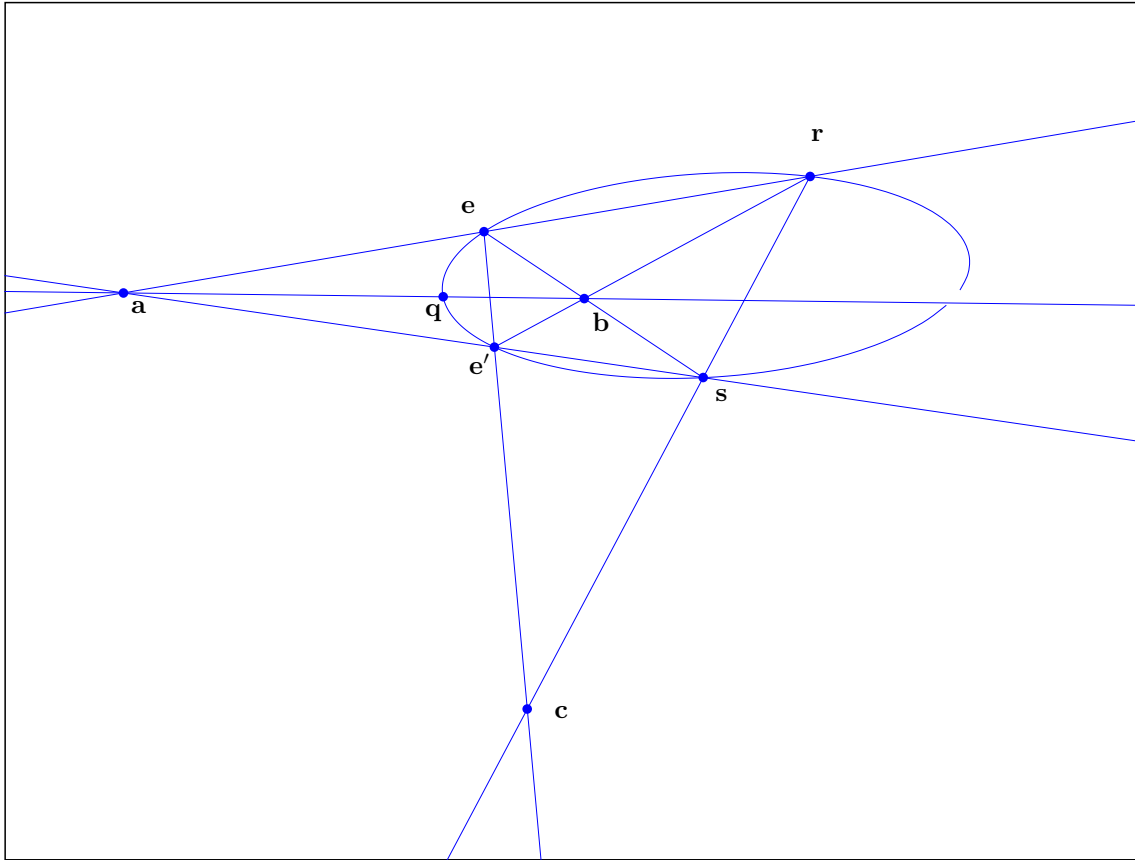


FIGURE 8. Illustration of the elements of lemma 5.1.

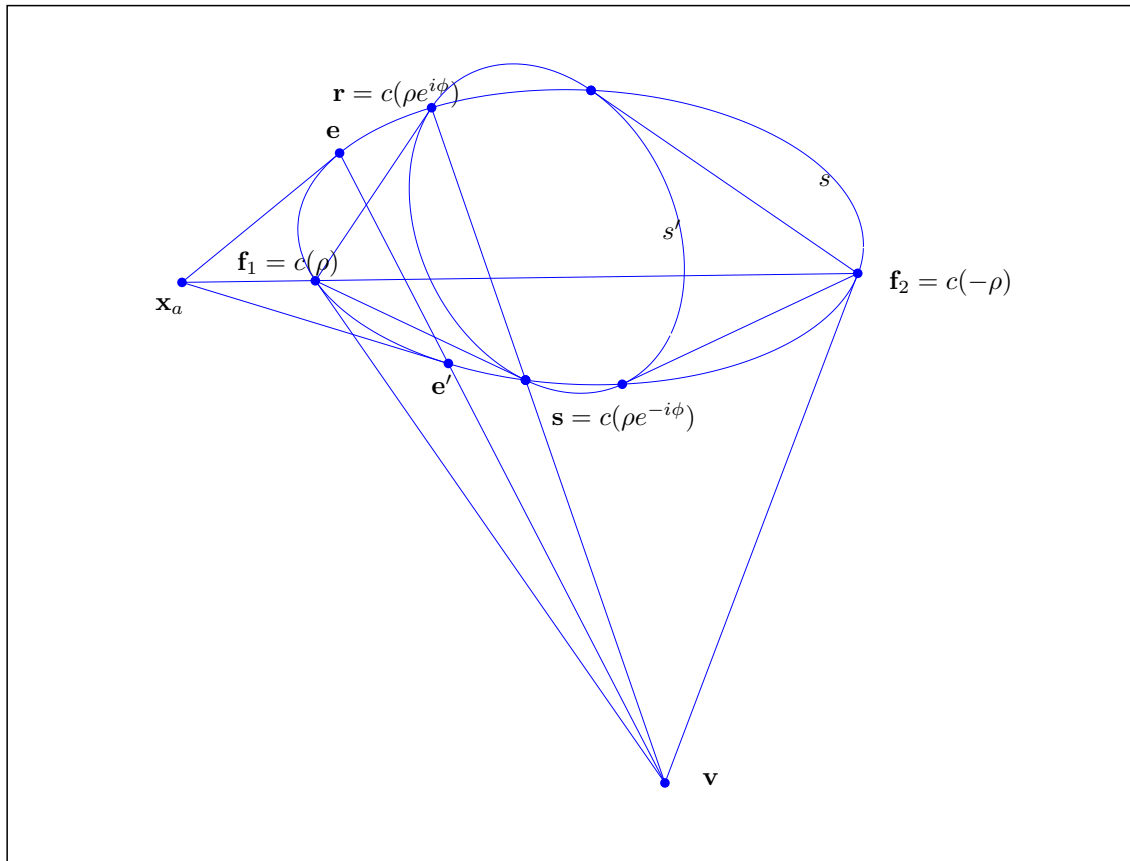


FIGURE 9. Illustration of the elements of lemma 5.1.



FIGURE 10. Images employed for the 3D reconstruction with marked points.

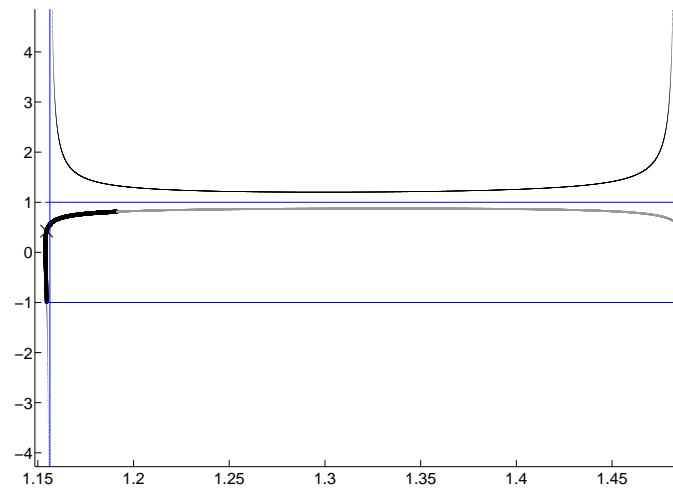
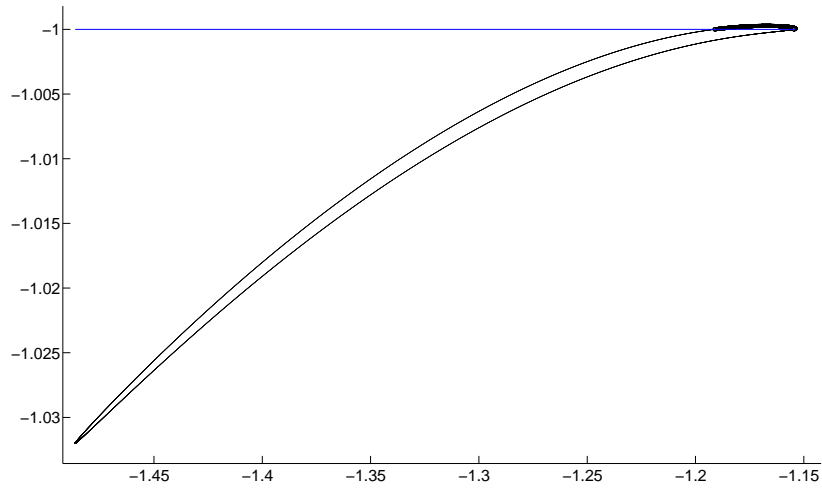


FIGURE 11. Left and right components of the curve of equation (12), with ρ in the horizontal axes and $\cos \phi$ in the vertical axes. Valid solutions (real definite conics) are shown in bold line. The solution corresponding to the final reconstruction is marked with a cross.

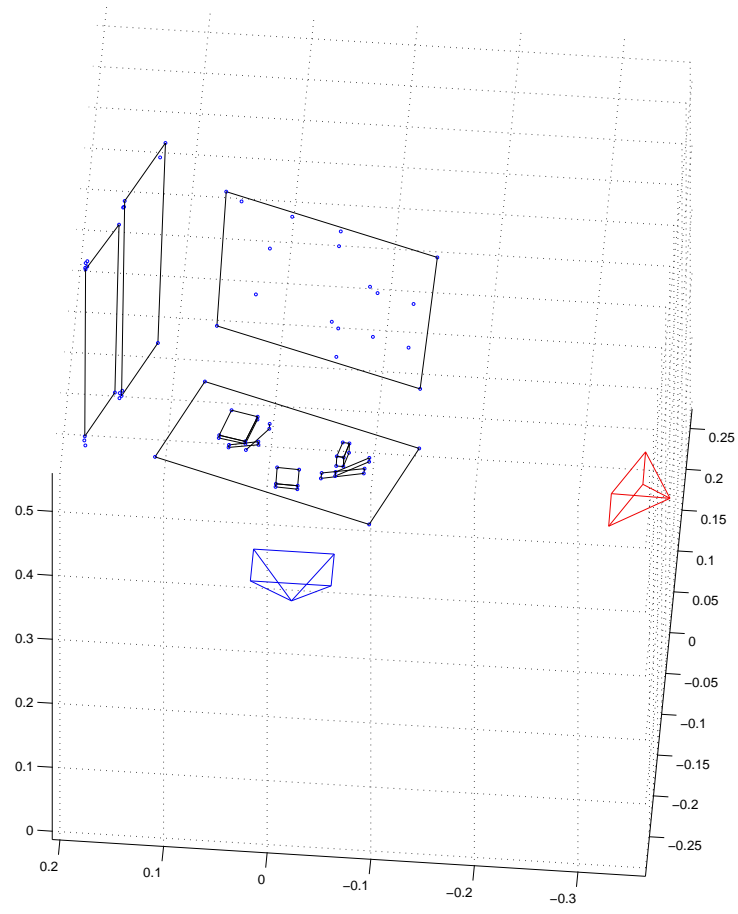


FIGURE 12. One view of the marked points of the 3D reconstruction.

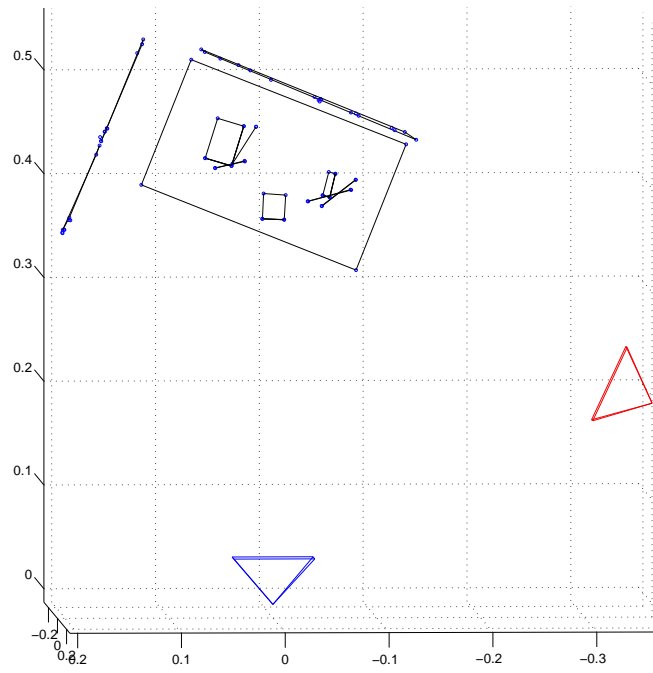


FIGURE 13. A second view of the marked points of the 3D reconstruction.