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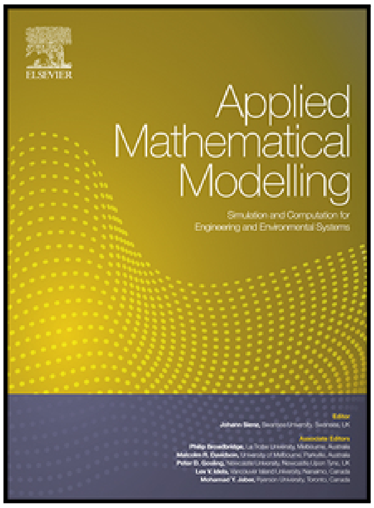

Well posedness of fluid-solid mixture models for biofilm spread

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- Study of fluid-solid mixture models combining quasi-stationary and time dependent equations set in moving domains

- We characterize time derivatives of quasi-stationary mechanical magnitudes as solutions of additional boundary value problems

- Mechanical submodels are well posed when complemented by such characterizations of time derivatives

- We give conditions for well posedness of time dependent boundary value problems set in moving domains for specific geometries

- We establish well posedness of stationary transport submodels under sign conditions motivated by simulations and experiments

Well posedness of fluid-solid mixture models for biofilm spread

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Abstract

Two phase solid-fluid mixture models are ubiquitous in biological applications. For instance, models for growth of tissues and biofilms combine time dependent and quasi-stationary boundary value problems set in domains whose boundary moves in response to variations in the mechano-chemical variables. For a model of biofilm spread, we show how to improve its stability properties by characterizing the time derivatives of relevant quasi-stationary magnitudes in terms of additional boundary value problems. We also give conditions for well posedness of time dependent submodels set in moving domains depending on the motion of the boundary. After constructing solutions for transport, diffusion and elliptic submodels for volume fractions, displacements, velocities, pressures and concentrations with the required regularity, we are able to handle the full model of biofilm spread in moving domains assuming we know the dynamics of the boundary. These techniques are general and can be applied in models with a similar structure arising in biological applications.

Keywords: Fluid-solid mixture models, thin film approximations, evolution equations in moving domains, quasi-stationary approximations, stationary transport equations

1. Introduction

Biofilms are bacterial aggregates that adhere to moist surfaces. Bacteria are encased in a self-produced polymeric matrix [1] which shelters them from chemical and mechanical aggressions. Biofilms formed on medical equipment,

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such as implants and catheters, are responsible for hospital-acquired infections [2]. In industrial environments, they cause substantial economical and technical problems, associated to food poisoning, biofouling, biocorrosion, contaminated ventilation systems, and so on [3, 4, 5]. Modeling biofilm spread is important to be able to eradicate them.

A wide variety of models have been proposed to understand different patterns and behaviors, see [6] for continuous descriptions of basic culture systems and [7, 8, 9, 10] for individual based, cellular automata and immersed boundary approaches to microscopic effects on biofilm structure, for instance. Biofilm behavior is strongly affected by environmental conditions. While biofilms grown in flows are formed by scattered bacteria immersed in large matrix chunks [11], in biofilms spreading on surfaces bacteria are glued together by very small volume fractions of extracellular matrix [12]. Fingers and streamers develop in the first case, wrinkled patterns are observed in the latter [13]. Two phase fluid flow models have been successful explaining quantitative observations of fluid-like behavior, more precisely, accelerated spread on surfaces due to swelling [12]. However, describing the formation of wrinkles requires models able to account also for solid-like behavior, such as viscoelastic, poroelastic or solid-fluid mixture descriptions [14, 15, 16]. Both fluid-like and solid like-behavior are also observed in tissues [17].

Solid-fluid mixture models provide a somewhat unifying point of view since they behave like monophasic elastic, monophasic viscoelastic, or truly biphasic mixture/poroelastic [18, 19] depending on the volume fractions of fluid and solid biomass, their densities, the Lam constants of the solid, the viscosity of the fluid, the hydraulic permeability of the solid/fluid system, the characteristic times for solid displacement and the characteristic matrix network length. The situation we have in mind is the study of *Bacillus Subtilis* biofilm spread on air/agar interfaces, for which fluid-like accelerated spread and solid-like wrinkling are experimentally documented [12, 13]. A solid-fluid mixture model in the line of those established in [20] for swelling tissues captures both features [14]. We focus on this type of models, under additional assumptions on the parameters taken from experimental studies.

We describe here biofilms in terms of solid-fluid mixtures, see Figure 1. At each point $\mathbf{x}$ of the biofilm we have a solid fraction of biomass $\phi_{\mathrm{s}}(\mathbf{x},t)$ (cell biomass, polymeric threads) and a volume fraction of water $\phi_{\mathrm{f}}(\mathbf{x},t)$ containing dissolved substances (nutrients, autoinducers and so on), in such a way that $\phi_{\mathrm{s}}(\mathbf{x},t)+\phi_{\mathrm{f}}(\mathbf{x},t)=1$. The solid and fluid volume fractions move with velocities $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{f}}$, respectively.

We consider a biofilm occupying a time dependent region Ω^t. Figure 2 represents schematic views of two dimensional slices. The upper boundary Γ^t_+ separates the biofilm from an outer fluid. A lower boundary Γ^t_- separates the biofilm from the substratum it attaches to. In our case, the fluid is air and the substratum is agar, from which nutrients are taken. The main variables satisfy

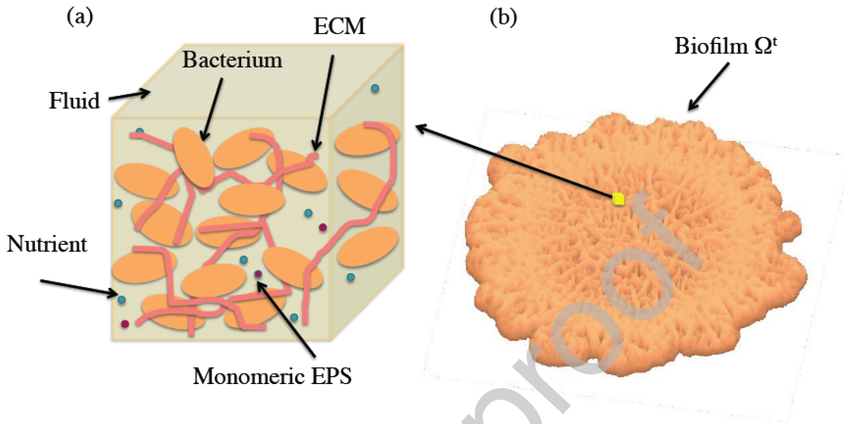

Figure 1: (a) Schematic view of biofilm microscopic structure. Cells are embedded in a network of polymeric threads forming the extracellular matrix (ECM), while a liquid solution containing nutrients and chemicals flows through the network. (b) Schematic view of a biofilm spreading on a surface.

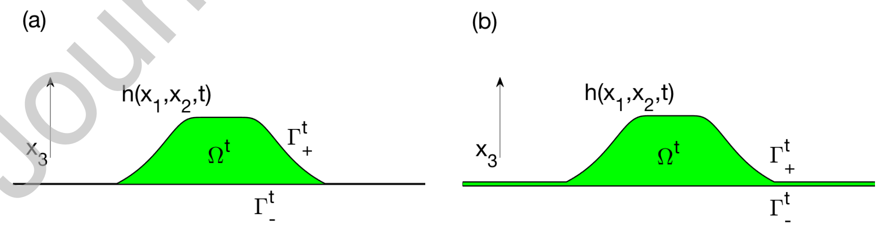

Figure 2: Schematic representation of a biofilm slice Ω^t spreading on a surface (a) occupying a finite region and ending at triple contact points, (b) spreading over precursor layers. The upper boundary Γ^t_+ represents the biofilm/air interface. The lower boundary Γ^t_- represents the biofilm/agar interface, which provides nutrients and resources necessary for biofilm growth in our framework.

a set of quasi-stationary equations

$$\begin{array}{lcl}
\operatorname{div}(\mathbf{v}_{\mathrm{f}}\phi_{\mathrm{f}}) & = & -k_{\mathrm{s}}\frac{c}{c+K_{\mathrm{s}}}\phi_{\mathrm{s}}, \\
\operatorname{div}(k_{\mathrm{h}}(\phi_{\mathrm{s}})\nabla(p-\pi(\phi_{\mathrm{s}})) & = & \operatorname{div}(\mathbf{v}_{\mathrm{s}}), \\
\mu\Delta\mathbf{u}_{\mathrm{s}}+(\mu+\lambda)\nabla(\operatorname{div}(\mathbf{u}_{\mathrm{s}})) & = & \nabla p, \\
-d\Delta c+\operatorname{div}(\mathbf{v}_{\mathrm{f}}c) & = & -k_{\mathrm{c}}\frac{c}{c+K_{\mathrm{c}}}\phi_{\mathrm{s}},
\end{array} \tag{1}$$

constrained by the additional conditions

$$\phi_{\mathrm{f}}\mathbf{v}_{\mathrm{f}}=-k_{\mathrm{h}}(\phi_{\mathrm{s}})\nabla(p-\pi(\phi_{\mathrm{s}}))+\phi_{\mathrm{f}}\mathbf{v}_{\mathrm{s}}, \quad \mathbf{v}_{\mathrm{s}}=\frac{\partial\mathbf{u}_{\mathrm{s}}}{\partial t}, \quad \phi_{\mathrm{f}}+\phi_{\mathrm{s}}=1, \tag{2}$$

in the region occupied by the biofilm Ω^t, which varies with time. In this quasi-static framework, the displacement vector $\mathbf{u}_{\mathrm{s}}(\mathbf{x},t)$ and the scalar pressure $p(\mathbf{x},t)$, volume fraction $\phi_{\mathrm{s}}(\mathbf{x},t)$ and concentration $c(\mathbf{x},t)$ fields depend on time through variations of the boundary Γ^t, which expands due to cell division and swelling. The positive functions $k_{\mathrm{h}}(\phi_{\mathrm{s}})$ and $\pi(\phi_{\mathrm{s}})$ represent the permeability and the osmotic pressure. Typically, $\pi(\phi_{\mathrm{s}})=\Pi\phi_{\mathrm{s}}$, $\Pi>0$, and $k_{\mathrm{h}}(\phi_{\mathrm{s}})=(1-\phi_{\mathrm{s}})^2/\zeta$, $\zeta>0$. This system is subject to a set of boundary conditions:

$$\begin{array}{ll}
p-\pi=p_{\mathrm{ext}}-\pi_{\mathrm{ext}}, & \text{on } \Gamma^t=\Gamma^t_+\cup\Gamma^t_-, \\
(\hat{\boldsymbol{\sigma}}(\mathbf{u}_{\mathrm{s}})-p\mathbf{I})\mathbf{n}=\mathbf{t}_{\mathrm{ext}}, \ \dfrac{\partial c}{\partial\mathbf{n}}=0, & \text{on } \Gamma^t_+, \\
\mathbf{u}_{\mathrm{s}}=0, \ c=c_0, & \text{on } \Gamma^t_-,
\end{array} \tag{3}$$

where $\mathbf{n}$ is the outer unit normal and

$$\hat{\boldsymbol{\sigma}}(\mathbf{u}_{\mathrm{s}})=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{s}}))\,\mathbf{I}+2\mu\,\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{s}}), \quad \varepsilon_{ij}(\mathbf{u})=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big), \ i,j=1,\dots,n,$$

$n=2,3$, represent elastic stress and strain tensors. Boundary conditions for ϕ_{f} would be required only in regions of the border where the characteristic curves of the associated transport problem enter the domain [21], if any. The displacement and velocity vectors have components $\mathbf{u}=(u_1,\dots,u_n)$ and $\mathbf{v}=(v_1,\dots,v_n)$, $n=2,3$, respectively. All the parameters appearing in the model, k_{s}, K_{s}, k_{c}, K_{c}, μ, λ, d are positive constants. External pressures, tractions and concentrations are denoted by p_{ext}, π_{ext}, $\mathbf{t}_{\mathrm{ext}}$ and c_0. For ease of the reader, we have summarized the modeling in Section 2. The equation for the solid displacements suggests some kind of poroelastic constitutive law for the mixture [19].

The model is complemented with an equation for the dynamics of Γ^t, $t>0$. If we consider biofilms represented by the scheme in Figure 2(a), the contact points between biofilm, air and agar require specific additional information to avoid singularities. Working with the geometry represented in Figure 2(b), we avoid this difficulty by introducing precursor layers [12, 22]. Then, Γ^t_- is fixed. The upper boundary Γ^t_+ is parametrized by a height function $h(x_1,x_2,t)$, which satisfies the equation [14]

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x_1}\left[\int_0^h(\mathbf{v}\cdot\hat{\mathbf{x}}_1)\,dx_3\right]+\frac{\partial}{\partial x_2}\left[\int_0^h(\mathbf{v}\cdot\hat{\mathbf{x}}_2)\,dx_3\right]=\mathbf{v}\cdot\hat{\mathbf{x}}_3\big|_0, \tag{4}$$

where the composite velocity of the mixture $\mathbf{v} = \phi_\mathrm{f}\mathbf{v}_\mathrm{f} + \phi_\mathrm{s}\mathbf{v}_\mathrm{s}$ has components $\mathbf{v}\cdot\hat{\mathbf{x}}_i = v_{\mathrm{s},i} - k_\mathrm{h}(\phi_\mathrm{s})\frac{\partial(p-\pi)}{\partial x_i}$, $i = 1, 2, 3$.

At present, only perturbation analysis and numerical studies are available for this type of models [12, 14] in simple geometries. Asymptotic studies yield thin film type approximations for (1)-(4) assuming circular geometries and radial symmetry. Non standard lubrication equations for the height h are obtained, which admit families of self-similar solutions in radial geometries. However, the construction of reliable numerical solutions of the model in general experimental configurations faces difficulties due to the lack of well-posedness results.

In this paper, we assume we know the dynamics of the upper boundary Γ^t_+, given by a smooth curve $x_3 = h(x_1, x_2, t)$, and develop an existence and stability theory for the model equations. To simplify the analysis, take constant $k_\mathrm{h}(\phi_\mathrm{s}) = k_\mathrm{h} > 0$ and $k_\mathrm{h}(\phi_\mathrm{s})/\phi_\mathrm{f} = \xi_\infty > 0$. In this quasi-stationary framework, the displacements $\mathbf{u}_\mathrm{s}$ depend on time through the motion of the boundary. However, we lack equations for the velocities, other than the relation $\frac{\partial \mathbf{u}_\mathrm{s}}{\partial t} = \mathbf{v}_\mathrm{s}$. In Section 3 we obtain a system of equations characterizing the velocity:

$$\begin{aligned} \mathrm{div}(\hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s})) = \mu\Delta\mathbf{v}_\mathrm{s} + (\mu+\lambda)\nabla(\mathrm{div}(\mathbf{v}_\mathrm{s})) = \nabla p_t, &\quad \text{in } \Omega^t, \\ \mathbf{v}_\mathrm{s} = 0, &\quad \text{on } \Gamma^t_-, \\ \hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s})\mathbf{n} = \frac{\partial \mathbf{g}}{\partial t} + \mathbf{r}(\mathbf{g}, \mathbf{u}_\mathrm{s}), &\quad \text{on } \Gamma^t_+, \end{aligned} \tag{5}$$

with $\mathbf{g} = -p\mathbf{n} = -(p_\mathrm{ext} - \pi_\mathrm{ext})\mathbf{n}$ and $\mathbf{r}$ to be defined later. A similar equation is obtained for p_t from the equation for p. Taking the divergence of the equations for $\mathbf{u}_\mathrm{s}$ and $\mathbf{v}_\mathrm{s}$ we find additional equations to close the system

$$\frac{de}{dt} = k_\mathrm{h}(2\mu+\lambda)\Delta e - k_\mathrm{h}\Pi\Delta\phi_\mathrm{s}, \quad \text{in } \Omega^t, \tag{6}$$

$$\frac{de_t}{dt} = k_\mathrm{h}(2\mu+\lambda)\Delta e_t - k_\mathrm{h}\Pi\Delta\phi_{s,t}, \quad \text{in } \Omega^t, \tag{7}$$

where $e = \mathrm{div}(\mathbf{u}_\mathrm{s})$ and $e_t = \mathrm{div}(\mathbf{v}_\mathrm{s})$. We will neglect $\Pi\Delta\phi_{s,t}$ in (7) because Π and $\Delta\phi_\mathrm{s}$ are small compared to other terms. Notice that (6) and (7) are time dependent problems set in time dependent domains, while most results in the literature refer to fixed domains.

The construction of solutions for such systems combines a number of difficulties that we will address in stages. For ease of the reader, we summarize the derivation of the model in Section 2. Section 3 characterizes the time derivatives of $\mathbf{u}_\mathrm{s}$ and p, solutions of elliptic problems in time dependent domains, by means of additional boundary value problems. In this way we improve the stability of the model, since solving additional partial differential equations in each spatial domain is more effective than approximating time derivatives by quotients of differences of solutions calculated in variable spatial domains. Section 4 establishes well posedness results for linear parabolic problems (7) set in domains with moving boundaries for specific types of parametrizations. Section 5 considers the elliptic and stationary transport problems involved in the quasi-stationary submodels, separately and in fixed domains, under hypotheses motivated by asymptotic studies and numerical solutions. Finally, Section 6 considers the full coupled time dependent problem and Section 7 discusses the validity of the results under weaker assumptions. Section 8 summarizes our conclusions and open issues.

2. The model equations

In this section, we summarize the derivation of the model equations (1)-(3). As said before, we have in mind applications to experiments of biofilm spread on agar/air interfaces. We model biofilms as solid-fluid mixtures, composed of a solid biomass phase and a liquid phase formed by water carrying dissolved chemicals (nutrients, autoinducers, waste), as often done with tissues and cellular media [19, 20]. Additional details and further references can be found in [12, 14, 19, 20]. Under the equipresence hypothesis of mixtures, the volume fraction of solid $\phi_{\rm s}(\mathbf{x},t)$ and the volume fraction of fluid $\phi_{\rm f}(\mathbf{x},t)$ satisfy

$$\phi_{\rm s} + \phi_{\rm f} = 1. \tag{8}$$

We work in isothermal conditions, both in space and time.

2.1. Mass balance

At each location $\mathbf{x}$, the volume fraction of biomass increases as dictated by its growth rate while the volume fraction of fluid decreases accordingly [12, 20]

$$\frac{\partial(\rho_{\rm s}\phi_{\rm s})}{\partial t} + {\rm div}(\rho_{\rm s}\phi_{\rm s}\mathbf{v}_{\rm s}) = r_{\rm s}(\rho_{\rm s}\phi_{\rm s}, c), \quad r_{\rm s}(\rho_{\rm s}\phi_{\rm s}, c) = k_{\rm s}\tfrac{c}{c+K_{\rm s}}\rho_{\rm s}\phi_{\rm s},$$
$$\frac{\partial(\rho_{\rm f}\phi_{\rm f})}{\partial t} + {\rm div}(\rho_{\rm f}\phi_{\rm f}\mathbf{v}_{\rm f}) = -r_{\rm s}(\rho_{\rm s}\phi_{\rm s}, c),$$

where $\mathbf{v}_{\rm s}$ and $\mathbf{v}_{\rm f}$ denote the velocities of the solid and fluid components, respectively, c is the substrate concentration and $r_{\rm s}(\rho_{\rm s}\phi_{\rm s}, c)$ stands for the production of biomass due to nutrient consumption. We describe it by a Monod type law. The parameters $K_{\rm s}$ (starvation threshold) and $k_{\rm s}$ (intake rate) are positive constants.

When the densities of the constituents are constant, the mass balance laws for $\phi_{\rm s}$ and $\phi_{\rm f}$ become

$$\frac{\partial\phi_{\rm s}}{\partial t} + {\rm div}(\phi_{\rm s}\mathbf{v}_{\rm s}) = r_{\rm s}(\phi_{\rm s}, c), \quad \frac{\partial\phi_{\rm f}}{\partial t} + {\rm div}(\phi_{\rm f}\mathbf{v}_{\rm f}) = -\frac{\rho_{\rm s}}{\rho_{\rm f}} r_{\rm s}(\phi_{\rm s}, c).$$

The composite density of the mixture is then $\rho(\mathbf{x}) = \rho_{\rm s}\phi_{\rm s}(\mathbf{x}) + \rho_{\rm f}\phi_{\rm f}(\mathbf{x})$.

Measured values for densities of phases in cellular systems are very similar (relative differences of order 10^{-2}). The assumption $\rho_{\rm f} = \rho_{\rm s} = \rho = \rho_{\rm w}$, $\rho_{\rm w}$ being the density of water, leads to [12]

$$\frac{\partial\phi_{\rm s}}{\partial t} + {\rm div}(\phi_{\rm s}\mathbf{v}_{\rm s}) = r_{\rm s}(\phi_{\rm s}, c), \quad r_{\rm s}(\phi_{\rm s}, c) = k_{\rm s}\tfrac{c}{c+K_{\rm s}}\phi_{\rm s}, \tag{9}$$
$$\frac{\partial\phi_{\rm f}}{\partial t} + {\rm div}(\phi_{\rm f}\mathbf{v}_{\rm f}) = -r_{\rm s}(\phi_{\rm s}, c). \tag{10}$$

The substrate concentration c is then governed by:

$$\frac{\partial c}{\partial t} + \operatorname{div}(\mathbf{v}_\mathrm{f} c) - \operatorname{div}(d\nabla c) = -r_n(\phi_\mathrm{s}, c), \quad r_n(\phi_\mathrm{s}, c) = \phi_\mathrm{s} k_\mathrm{c} \frac{c}{c + K_\mathrm{c}}, \tag{11}$$

where $r_n(\phi_\mathrm{s}, c)$ represents consumption by the biofilm. The parameters d (diffusivity), k_c (uptake rate) and K_c (half-saturation) are positive constants. We impose zero-flux boundary conditions on the air–biofilm interface and constant Dirichlet boundary condition on the agar–biofilm interface. In equations (9), (10), (11), typical parameter values are such that the time derivatives can be neglected. In this way we obtain the equations for ϕ_f and c listed in (1). The solutions depend on time though the motion of the biofilm boundary.

Adding up equations (9) and (10), we obtain a conservation law for the growing mixture:

$$0 = \operatorname{div}(\phi_\mathrm{s}\mathbf{v}_\mathrm{s} + \phi_\mathrm{f}\mathbf{v}_\mathrm{f}) = \operatorname{div}(\mathbf{v}) = \operatorname{div}(\mathbf{v}_\mathrm{s} + \mathbf{q}), \tag{12}$$

where $\mathbf{v} = \phi_\mathrm{s}\mathbf{v}_\mathrm{s} + \phi_\mathrm{f}\mathbf{v}_\mathrm{f}$ is the averaged velocity of the mixture and

$$\mathbf{q} = \phi_\mathrm{f}(\mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{s}) \tag{13}$$

is the filtration flux. The mixture is thus assumed to be incompressible. Equation (4) follows integrating (12), see [12, 14] for details, though we will not use it here.

2.2. Driving forces

We model next the nature of the forces present in the mixture [20].

Stress in the fluid. Under isothermal and isotropic conditions, the fluid partial stress of an inviscid fluid in the case of infinitesimal deformations is

$$\boldsymbol{\sigma}_\mathrm{f} = -\phi_\mathrm{f}\, p\mathbf{I}, \tag{14}$$

p being the pore hydrostatic pressure. This assumes large numbers of small pores in the mixture. In case large regions filled with fluid were present, the standard stress law for viscous fluids should be considered.

Stress in the solid biomass. We hypothesize that the partial solid stress has two components

$$\boldsymbol{\sigma}_\mathrm{s} = \hat{\boldsymbol{\sigma}}_\mathrm{s} - \phi_\mathrm{s}\, p\mathbf{I}. \tag{15}$$

The second one describes the mechanical interaction between fluid and solid. The first one represents stresses arising from the strain of the wet solid and is related to the solid fibrous structure. Under small infinitesimal deformations, we can assume a linear isotropic elastic strain-stress relation

$$\hat{\boldsymbol{\sigma}}_\mathrm{s} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}(\mathbf{u}_\mathrm{s}))\,\mathbf{I} + 2\mu\,\boldsymbol{\varepsilon}(\mathbf{u}_\mathrm{s}), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{16}$$

where $\mathbf{u}_\mathrm{s}$ is the displacement vector of the solid, $\boldsymbol{\varepsilon}(\mathbf{u})$ the deformation tensor, and λ, μ, the Lamé constants. Depending on the fibrous microstructure, other choices, such as viscoelastic strain-stress relations, are possible.

Inertial forces. The inertial forces in the fluid and solid phases are $\rho_s \mathbf{a}_s$ and $\rho_f \mathbf{a}_f$, where $\mathbf{a}_s$ and $\mathbf{a}_f$ are the corresponding accelerations. In our framework, fluid and solid velocities $\mathbf{v}_f$ and $\mathbf{v}_s$ are very small and inertial forces are negligible.

Interaction forces. Mechanical considerations imply that interaction forces acting on the two phases are equal in magnitude and opposite in direction, so that they cancel each other at the mixture level.
Filtration forces are due to the interaction between solid and fluid particles, which is a function of their relative flows, that is, the filtration flux. The forces per unit volume $\rho_s \mathbf{f}_s$ and $\rho_f \mathbf{f}_f$ satisfy $\rho_s \mathbf{f}_s + \rho_f \mathbf{f}_f = 0$. Ignoring inertial effects and assuming that there is no coupling between concentration and viscous effects, the filtration force for isotropic solids with isotropic permeability is taken to be

$$\mathbf{f}_f = -\frac{1}{k_h}\mathbf{q}, \tag{17}$$

where k_h (hydraulic permeability) is a positive function of ϕ_f and $\mathbf{q}$ is the filtration flux. In our framework, $k_h(\phi_f) = \phi_f^2/\zeta$, where ζ is a friction parameter often set equal to $\zeta = \mu_f/\xi(\phi_f)^2 > 0$ and ξ is the "mesh size" of the underlying biomass network.
Interaction forces derived from chemical potentials satisfy $\phi_s \nabla \pi_s + \phi_f \nabla \pi_f = 0$. Here, we deal with osmotic forces. The concentration of polymeric substances present in the biofilm causes swelling, that is, water absorption, and leads to the introduction of an osmotic pressure. In our framework, we assume the osmotic pressure is a function of the biomass fraction $\pi_f = \Pi(\phi_s)$ [12].
Buoyancy forces are interactive forces caused by density gradients of the solid and the fluid. We assume $\rho_f = \rho_s = \rho = \rho_w$, thus, we neglect them here.

2.3. Equations of motion

The theory of mixtures hypothesizes that the motion of each phase obeys the usual momentum balance equations [20]. In the absence of external body forces, and using the expressions for the stress tensors (14) and (15)-(16), the equations become

$$\operatorname{div} \hat{\boldsymbol{\sigma}}_s + \phi_s(-\nabla p + \nabla \pi_s) + \phi_s \mathbf{f}_s = 0, \quad \phi_f(-\nabla p + \nabla \pi_f) + \phi_f \mathbf{f}_f = 0. \tag{18}$$

Combining (18), (17), and (13) we obtain

$$\mathbf{q} = -k_h \nabla(p - \pi) = \phi_f(\mathbf{v}_f - \mathbf{v}_s). \tag{19}$$

This is Darcy's law in the presence of concentration gradients.

Adding up the equations in (18), we find an expression relating solid displacements and pressure

$$\operatorname{div} \hat{\boldsymbol{\sigma}}_s(\mathbf{u}_s) - \nabla p = 0. \tag{20}$$

Equation (20) suggests a constitutive law for the mixture similar to poroelastic constitutive laws already proposed, see [15] and references therein. At the biofilm boundary, the jumps in the total stress vector and the chemical potential vanish:

$$(\hat{\boldsymbol{\sigma}}_{\mathrm{s}} - p\mathbf{I})\mathbf{n} = \mathbf{t}_{\mathrm{ext}}, \quad p - \pi = p_{\mathrm{ext}} - \pi_{f,\mathrm{ext}},$$

when applicable.

The solid velocity is then $\mathbf{v}_{\mathrm{s}} = \frac{\partial \mathbf{u}_{\mathrm{s}}}{\partial t}$. These equations are complemented by (9) and (12), which now becomes

$$\operatorname{div}(\mathbf{v}_{\mathrm{s}}) = -\operatorname{div}(\mathbf{q}) = \operatorname{div}(k_{\mathrm{h}} \nabla (p - \pi)). \tag{21}$$

In this way, we find the equations and boundary conditions for $\mathbf{u}_{\mathrm{s}}$, $\mathbf{v}_{\mathrm{s}}$ and p listed in (1) and (3). Combining with (13) and (8), we obtain (2).

3. Differentiation of quasi-stationary problems

In the previous section, we have defined the velocity $\mathbf{v}_{\mathrm{s}}$ as the time derivative of the displacement $\mathbf{u}_{\mathrm{s}}$. The change in time of $\mathbf{u}_{\mathrm{s}}$ is due to the motion of the upper boundary Γ_+^t, that is, time variations in h. In this section we seek an equation characterizing $\mathbf{v}_{\mathrm{s}}$. We expect $\mathbf{v}_{\mathrm{s}}$ to solve the same boundary value problem as $\mathbf{u}_{\mathrm{s}}$, but differentiating all sources with respect to time. However, since the boundary Γ^t of Ω^t moves with time, we need to calculate the adequate boundary conditions too. We explain here how to obtain boundary value problems for time derivatives of relevant magnitudes in moving domains.

In the region Ω^t occupied by the moving biofilm, the displacements $\mathbf{u}_{\mathrm{s}}$ of the solid phase satisfy equations (1) with boundary conditions (3). To simplify later computations, it is convenient to recast these equations in the general linear elasticity framework. We use the Einstein summation convention that implies summation over a set of indexed terms in a formula when repeated in it. In the next equations, summation over α, β, m is implied, but not over j. The components of the displacement $u_j(t), j = 1, \dots, n$, n being the dimension, fulfill

$$\begin{aligned}
-\frac{\partial}{\partial x_\alpha}\left(c_{j\alpha m\beta}\frac{\partial u_m(t)}{\partial x_\beta}\right) &= f_j(t), \quad j = 1, \dots, n, \qquad && \text{in } \Omega^t, \\
u_j(t) &= 0, \quad j = 1, \dots, n, && \text{on } \Gamma_d^t, \\
c_{j\alpha m\beta}\frac{\partial u_m(t)}{\partial x_\beta} n_\alpha(t) &= g_j(t), \quad j = 1, \dots, n, && \text{on } \Gamma_n^t,
\end{aligned} \tag{22}$$

where $\mathbf{n}(t)$ is the outer unit normal vector and $c_{j\alpha m\beta}$ the elastic constants. Γ_n^t and Γ_d^t are parts of the boundary Γ^t where we enforce conditions on the stresses of the displacements, respectively.

The elastic constants $c_{j\alpha m\beta}$ for isotropic solids like the ones we consider are

$$c_{j\alpha m\beta} = \lambda \delta_{j\alpha}\delta_{m\beta} + \mu(\delta_{jm}\delta_{\alpha\beta} + \delta_{j\beta}\delta_{\alpha m})$$

where δ_{jm} stands for the Kronecker delta, whereas λ and μ represent the Lamé constants. The stress tensor is

$$\sigma_{j\alpha} = c_{j\alpha m\beta}\varepsilon_{m\beta} = \lambda\delta_{j\alpha}\varepsilon_{pp} + 2\mu\varepsilon_{j\alpha}.$$

In this framework, the velocity $\mathbf{v}$ is the 'Frèchet derivative' or 'domain derivative' of $\mathbf{u}$ with respect to t [23], which is characterized by the solution of a boundary value problem, as we show next.

Theorem 3.1. *We assume that the body* $\mathbf{f}$ *and boundary* $\mathbf{g}$ *forces are differentiable in time, with values in* $[L^2(\Omega^t)]^n$ *and* $[L^2(\Gamma^t)]^n$*, respectively, with* $t > 0$*,* $n = 2, 3$ *being the dimension. Moreover, the* C^2 *boundaries* Γ^t *are obtained deforming* Γ^0 *along a smooth vector field* $\boldsymbol{\nu}$*. Then, the time derivative* $\mathbf{v}(t) = \frac{\partial \mathbf{u}(t)}{\partial t}$*,* $t > 0$*, of the displacement given by (22) satisfies*

$$\begin{aligned}
-\frac{\partial}{\partial x_\alpha}\left(c_{j\alpha m\beta}\frac{\partial v_m(t)}{\partial x_\beta}\right) &= \frac{\partial f_j(t)}{\partial t}, \quad j = 1,..,n, \qquad \mathbf{x} \in \Omega^t,\\
v_j(t) &= 0, \quad j = 1,..,n, \qquad \mathbf{x} \in \Gamma^t_d,\\
c_{j\alpha m\beta}\frac{\partial v_m(t)}{\partial x_\beta}n_\alpha(t) &= \frac{\partial g_j(t)}{\partial t} + r_j(g_j(t), \mathbf{u}(t)), \quad j = 1,..,n, \qquad \mathbf{x} \in \Gamma^t_n,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
r_j = c_{j\alpha m\beta}\frac{\partial u_m(t)}{\partial x_\beta}\frac{\partial \nu_q}{\partial x_\alpha}n_q(t) + c_{j\alpha m\beta}\frac{\partial u_m(t)}{\partial x_\beta}\frac{\partial(\nu_p n_\alpha(t))}{\partial x_p}\\
+c_{j\alpha m\beta}\frac{\partial u_m(t)}{\partial x_\beta}\frac{\partial \nu_p}{\partial x_p}n_\alpha(t) - g_j(t)\mathbf{n}(t)\cdot\nabla\boldsymbol{\nu}\,\mathbf{n}(t), \quad j = 1, \ldots, n.
\end{aligned} \tag{24}$$

As a corollary, we get the expressions of interest for our model.

Corollary 3.2. *Under the previous hypotheses, the time derivative* $\mathbf{v}_\mathrm{s}(t)$*,* $t > 0$*, of the solution* $\mathbf{u}_\mathrm{s}$ *of (1) with boundary conditions (3) satisfies*

$$\begin{aligned}
\mathrm{div}(\hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s})) = \mu\Delta\mathbf{v}_\mathrm{s} + (\mu+\lambda)\nabla(\mathrm{div}(\mathbf{v}_\mathrm{s})) &= \nabla p_t, \quad \mathbf{x} \in \Omega^t,\\
\mathbf{v}_\mathrm{s} &= 0, \quad \mathbf{x} \in \Gamma^t_-,\\
(\hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s})\mathbf{n})_j &= \frac{\partial g_j}{\partial t} + r_j(g_j, \mathbf{u}_\mathrm{s}), \; j = 1, 2 \quad \mathbf{x} \in \Gamma^t_+,
\end{aligned} \tag{25}$$

with $\mathbf{g} = -p\mathbf{n} = -(p_\mathrm{ext} - \pi_\mathrm{ext})\mathbf{n}$ *and* $\mathbf{r}$ *is defined by (24) with* $c_{j\alpha m\beta} = \lambda\delta_{j\alpha}\delta_{m\beta} + \mu(\delta_{jm}\delta_{\alpha\beta} + \delta_{j\beta}\delta_{\alpha m})$.

Corollary 3.3 *Under the previous hypotheses, assuming* $k_\mathrm{h}(\phi_\mathrm{s}) = k_\mathrm{h}$ *constant and* $\pi(\phi_\mathrm{s}) = \Pi\phi_\mathrm{s}$*, the derivative* $p_t(t) = \frac{\partial p(t)}{\partial t}$*,* $t > 0$*, of the solution* p *of (1) with Dirichlet boundary conditions* $p = p_\mathrm{ext}(t)$ *satisfies,*

$$\begin{aligned}
&k_\mathrm{h}\Delta p_t = \mathrm{div}(\mathbf{v}_{s,t}) + k_\mathrm{h}\Pi\Delta\phi_{s,t}, && \mathbf{x} \in \Omega^t,\\
&p_t = p'_\mathrm{ext}(t), && \mathbf{x} \in \Gamma^t.
\end{aligned}$$

Proof of Theorem 3.1. We will follow a similar variational approach to that employed in [23] for 2D exterior elasticity problems with zero Dirichlet boundary conditions on a moving boundary. We are going to calculate the derivative at $t = 0$. Similar arguments hold for any $t > 0$.

Step 1: Variational formulation. First, we write the boundary value problem for $\mathbf{u}$ in variational form [24]. The boundary value problem (22) becomes: Find $\mathbf{u}^t \in [H^1_{\Gamma^t_d}(\Omega^t)]^n$ such that

$$b^t(\Omega^t; \mathbf{u}^t, \mathbf{w}^t) = \ell^t(\Omega^t; \mathbf{w}^t), \qquad \forall\, \mathbf{w}^t \in [H^1_{\Gamma^t_d}(\Omega^t)]^n, \tag{26}$$

where

$$b^t(\Omega^t; \mathbf{u}^t, \mathbf{w}^t) = \int_{\Omega^t} c_{j\alpha m\beta} \frac{\partial u^t_m}{\partial x^t_\beta} \frac{\partial \overline{w}^t_j}{\partial x^t_\alpha}\, d\mathbf{x}^t, \qquad \forall\, \mathbf{u}^t, \mathbf{w}^t \in [H^1_{\Gamma^t_d}(\Omega^t)]^n, \tag{27}$$

$$\ell^t(\Omega^t; \mathbf{w}^t) = \int_{\Omega^t} f_j(t) w^t_j\, d\mathbf{x}^t + \int_{\Gamma^t_n} g_j(t) w^t_j\, d\mathbf{S}_{\mathbf{x}^t}, \qquad \forall\, \mathbf{w}^t \in [H^1_{\Gamma^t_d}(\Omega^t)]^n. \tag{28}$$

Here, $H^1_{\Gamma^t_d}(\Omega^t)$ denotes the usual Sobolev space of $H^1(\Omega^t)$ functions vanishing on $\Gamma^t_d \subset \partial\Omega^t$. $H^1(\Omega^t)$ is formed by all functions whose square, and the squares of their derivatives, are integrable in Ω^t, that is, belong to $L^2(\Omega^t)$. When $\mathbf{f}(t) \in [L^2(\Omega^t)]^n$, $\mathbf{g} \in [L^2(\Gamma^t)]^n$ and $\mathrm{meas}(\Gamma^t_d) \neq 0$, this problem admits a unique solution $\mathbf{u}^t \in [H^1_{\Gamma^t_d}(\Omega^t)]^n$ [24], which in fact belongs to $[H^2(\Omega^t)]^n$, vanishes on Γ^t_d and satisfies $\boldsymbol{\sigma}(\mathbf{u}^t)\,\mathbf{n} = \mathbf{g}$ on $\Gamma^t_n = \partial\Omega^t \setminus \Gamma^t_d$. For $t = 0$, we have u^0. Here, $\sigma_{\alpha j}(\mathbf{u}^t) = c_{j\alpha m\beta} \frac{\partial u^t_m}{\partial x_\beta}$.

Step 2: Change of variables. We now transform all the quantities appearing in (27)-(28) back to the initial configuration Ω^0. The process is similar to transforming deformed configurations back to a reference configuration in continuum mechanics [25]. We are assumig that the evolution of the moving part of the boundary $\Gamma^t = \{\mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x}) \,|\, \mathbf{x} \in \Gamma^0\}$ is given by a family of deformations $\mathbf{x}^t = \phi^t(\mathbf{x}) = \mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x})$ starting from a smooth surface $\Gamma^0 \in C^2$ (twice differentiable) and following a smooth vector field $\boldsymbol{\nu} \in C^2(\Omega)$, $\Omega^t \subset \Omega$, $t > 0$. The deformation gradient is the jacobian of the change of variables [26]

$$\mathbf{J}^t(\mathbf{x}) = \nabla_{\mathbf{x}} \boldsymbol{\phi}^t(\mathbf{x}) = \left(\frac{\partial x^t_i}{\partial x_j}(\mathbf{x})\right) = \mathbf{I} + t\,\nabla\boldsymbol{\nu}(\mathbf{x}), \tag{29}$$

and its inverse $(\mathbf{J}^t)^{-1} = \left(\frac{\partial x_i}{\partial x^t_j}\right)$ is the jacobian of the inverse change of variables. Then, volume and surface elements are related by

$$d\mathbf{x}^t = \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x}, \quad dS_{\mathbf{x}^t} = \det \mathbf{J}^t(\mathbf{x}) \|\, (\mathbf{J}^t(\mathbf{x}))^{-T}\mathbf{n}\| dS_{\mathbf{x}} \tag{30}$$

and the chain rule for derivatives reads $\nabla_{\mathbf{x}} u_m(\mathbf{x}^t(\mathbf{x})) = (J^t(\mathbf{x}))^T \nabla_{\mathbf{x}^t} u_m(\mathbf{x}^t(\mathbf{x}))$, that is, $\nabla_{\mathbf{x}^t} u_m = (\mathbf{J}^t)^{-T} \nabla_{\mathbf{x}} u_m$. For each component we have

$$\frac{\partial u_m}{\partial x^t_\beta}(\mathbf{x}^t(\mathbf{x})) = \frac{\partial u_m}{\partial x_k}(\mathbf{x}^t(\mathbf{x}))(J^t)^{-1}_{k\beta}(\mathbf{x}). \tag{31}$$

We define $\tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{u}^t \circ \phi^t(\mathbf{x}) = \mathbf{u}^t(\mathbf{x}^t(\mathbf{x}))$, definition that extends to $\tilde{\mathbf{w}}$ and other functions. Changing variables and using (30)-(31) we have:

$$b^t(\Omega^t; \mathbf{u}^t, \mathbf{w}^t) = \int_{\Omega^t} c_{j\alpha m\beta} \frac{\partial u_m^t}{\partial x_\beta^t}(\mathbf{x}^t) \frac{\partial w_j^t}{\partial x_\alpha^t}(\mathbf{x}^t)\, d\mathbf{x}^t =$$

$$\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_p}(\mathbf{x})(J^t)^{-1}_{p\beta}(\mathbf{x}) \frac{\partial \tilde{w}_j}{\partial x_q}(\mathbf{x})(J^t)^{-1}_{q\alpha}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x} = \tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \quad (32)$$

$$\ell^t(\Omega^t; \mathbf{w}^t) = \int_{\Omega^t} f_j(\mathbf{x}^t, t) w_j^t(\mathbf{x}^t)\, d\mathbf{x}^t + \int_{\Gamma_n^t} g_j(\mathbf{x}^t, t) w_j^t(\mathbf{x}^t)\, dS_{\mathbf{x}^t} =$$

$$\int_{\Omega^0} \tilde{f}_j(\mathbf{x}, t)\tilde{w}_j(\mathbf{x}) \det \mathbf{J}^t\, d\mathbf{x} + \int_{\Gamma_n^0} \tilde{g}_j(\mathbf{x}, t)\tilde{w}_j(\mathbf{x}) \det \mathbf{J}^t \| (\mathbf{J}^t)^{-T}\mathbf{n}\| dS_{\mathbf{x}} = \tilde{\ell}^t(\Omega^0; \tilde{\mathbf{w}}). \quad (33)$$

For arbitrary test functions $\mathbf{w}^t \in [H^1_{\Gamma_d^t}(\Omega^t)]^n$, $\tilde{\mathbf{w}} \in [H^1_{\Gamma_d^t}(\Omega^0)]^n$ is a test function in Ω^0. Therefore, we obtain the equivalent variational formulation: Find $\tilde{\mathbf{u}} \in [H^1_{\Gamma_d^t}(\Omega^0)]^n$ such that

$$\tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) = \tilde{\ell}^t(\Omega^0; \mathbf{w}), \qquad \forall\, \mathbf{w} \in [H^1_{\Gamma_d^t}(\Omega^0)]^n, \quad (34)$$

with $\tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w})$ and $\tilde{\ell}^t(\Omega^0; \mathbf{w})$ defined in (32)-(33) replacing $\tilde{\mathbf{w}}$ by $\mathbf{w}$.

Let us analyze the dependence on t of the terms appearing in the expression for $\tilde{b}^t$ and $\tilde{\ell}^t$. From the definitions of the Jacobian matrices (29) we obtain [23, 26]

$$\det \mathbf{J}^t(\mathbf{x}) = 1 + t\,\mathrm{div}(\boldsymbol{\nu}(\mathbf{x})) + O(t^2), \quad (35)$$

$$(\mathbf{J}^t)^{-1}(\mathbf{x}) = \mathbf{I} - t\,\nabla\boldsymbol{\nu}(\mathbf{x}) + O(t^2), \quad (36)$$

$$\det \mathbf{J}^t(\mathbf{x})\, \|(\mathbf{J}^t(\mathbf{x}))^{-T}\mathbf{n}\| = 1 + t\,\mathrm{div}_\Gamma(\boldsymbol{\nu}(\mathbf{x})) + O(t^2), \quad (37)$$

where $\mathrm{div}_\Gamma(\boldsymbol{\nu}(\mathbf{x})) = \mathrm{div}(\boldsymbol{\nu}(\mathbf{x})) - \mathbf{n}\cdot\nabla\boldsymbol{\nu}(\mathbf{x})\mathbf{n}$. Inserting (35)-(36) in (32) we find the following expansions. When $p = \beta$ and $q = \alpha$ we get

$$\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha}\, d\mathbf{x} + t\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha} \mathrm{div}(\boldsymbol{\nu})\, d\mathbf{x} \quad (38)$$
$$-t\int_{\Omega^0} c_{j\alpha m\beta} \left[\frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial \nu_\beta}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha} + \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha} \frac{\partial \nu_\alpha}{\partial x_\alpha} \right] d\mathbf{x} + O(t^2),$$

whose leading term is $b^0(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w})$. When $p \neq \beta$ and $q \neq \alpha$ the summands are $O(t^2)$. The remaining terms provide the contribution

$$-t\int_{\Omega^0} c_{j\alpha m\beta} \left[\frac{\partial \tilde{u}_m}{\partial x_p} \frac{\partial \nu_p}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha} + \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_q} \frac{\partial \nu_q}{\partial x_\alpha} \right] d\mathbf{x} + O(t^2),$$

with $p \neq \beta$, $q = \alpha$ in the first one and $q \neq \alpha$, $p = \beta$ in the second one. Adding up the contributions we get

$$\tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) = b^0(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) + t[I_1(\tilde{\mathbf{u}}) + I_2(\tilde{\mathbf{u}}) + I_3(\tilde{\mathbf{u}})] + O(t^2), \quad (39)$$

where

$$
\begin{array}{rl}
I_1(\tilde{\mathbf{u}}) = & \int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha} \operatorname{div}(\boldsymbol{\nu})\, d\mathbf{x}, \\
I_2(\tilde{\mathbf{u}}) = & -\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_p} \frac{\partial \nu_p}{\partial x_\beta} \frac{\partial w_j}{\partial x_\alpha}\, d\mathbf{x}, \\
I_3(\tilde{\mathbf{u}}) = & -\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial w_j}{\partial x_q} \frac{\partial \nu_q}{\partial x_\alpha}\, d\mathbf{x} = \int_{\Omega^0} \frac{\partial}{\partial x_\alpha} \left(c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \right) \frac{\partial w_j}{\partial x_q} \nu_q\, d\mathbf{x} \\
& + \int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} \frac{\partial^2 w_j}{\partial x_\alpha \partial x_q} \nu_q\, d\mathbf{x} - \int_{\Gamma^0} c_{j\alpha m\beta} \frac{\partial \tilde{u}_m}{\partial x_\beta} n_\alpha \frac{\partial w_j}{\partial x_q} \nu_q\, dS_{\mathbf{x}}.
\end{array} \tag{40}
$$

Similarly, from the definition (33) of the linear form $\tilde{\ell}^t$ and the definition of 'material derivative' $\dot{\mathbf{f}}$

$$
\tilde{\mathbf{f}}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}^t(\mathbf{x}), t) = \mathbf{f}(\mathbf{x}, 0) + t\, \dot{\mathbf{f}}(\mathbf{x}, 0) + O(t^2), \tag{41}
$$

we find the expansion

$$
\begin{array}{rl}
\tilde{\ell}^t(\Omega^0; \mathbf{w}) & = \int_{\Omega^0} f_j(0) w_j\, d\mathbf{x} + t \int_{\Omega^0} [f_j(0)\operatorname{div}(\boldsymbol{\nu}) + \dot{f}_j(0)] w_j\, d\mathbf{x} \\
& + \int_{\Gamma_n^0} g_j(0) w_j\, dS_{\mathbf{x}} + t \int_{\Gamma_n^0} [g_j(0)\operatorname{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0)] w_j\, dS_{\mathbf{x}} + O(t^2)
\end{array} \tag{42}
$$

whose leading term is $\ell^0(\Omega^0; \mathbf{w})$.

Step 3. Variational problem for the domain derivative $\mathbf{u}'$. Let us compare the transformed function $\tilde{\mathbf{u}}$ and the solution $\mathbf{u}^0$ of $b^0(\Omega^0; \mathbf{u}^0, \mathbf{w}) = \ell^0(\Omega^0; \mathbf{w})$. For any $\mathbf{w} \in [H^1_{\Gamma_d^t}(\Omega^0)]^n$ we have

$$
\begin{array}{c}
b^0(\Omega^0; \tilde{\mathbf{u}} - \mathbf{u}^0, \mathbf{w}) = b^0(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) - \ell^0(\Omega^0; \mathbf{w}) = \\
b^0(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) - \tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) + \tilde{\ell}^t(\Omega^0; \mathbf{w}) - \ell^0(\Omega^0; \mathbf{w}).
\end{array} \tag{43}
$$

Well posedness of the variational problems (26) with respect to changes in domains Ω^t and sources $\mathbf{f}(t), \mathbf{g}(t)$, implies uniform bounds on the solutions for $t \in [0, T]$: $\|\mathbf{u}^t\|_{[H^1(\Omega^t)]^n} \leq C(T)$, $\|\tilde{\mathbf{u}}\|_{[H^1(\Omega^0)]^n} \leq C(T)$. Expansions (39)-(42) show that the right hand side in (43) tends to zero as $t \to 0$. Well posedness of the variational problem again implies $\tilde{\mathbf{u}} \to \mathbf{u}^0$ in $[H^1_{\Gamma_d^t}(\Omega^0)]^n$ as $t \to 0$.

Dividing by t equation (43) and using (39)-(42), we find

$$
\begin{array}{rcl}
b^0(\Omega^0; \frac{\tilde{\mathbf{u}} - \mathbf{u}^0}{t}, \mathbf{w}) & = & \frac{1}{t}[b^0(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w}) - \tilde{b}^t(\Omega^0; \tilde{\mathbf{u}}, \mathbf{w})] + \frac{1}{t}[\tilde{\ell}^t(\Omega^0; \mathbf{w}) - \ell^0(\Omega^0; \mathbf{w})] \\
& = & -[I_1(\tilde{\mathbf{u}}) + I_2(\tilde{\mathbf{u}}) + I_3(\tilde{\mathbf{u}})] + \int_{\Omega^0} [f_j(0)\operatorname{div}(\boldsymbol{\nu}) + \dot{f}_j(0)] w_j\, d\mathbf{x} \\
& + & \int_{\Gamma_n^0} [g_j(0)\operatorname{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0)] w_j\, dS_{\mathbf{x}} + O(t).
\end{array}
$$

Then, the limit $\dot{\mathbf{u}} = \lim_{t \to 0} \frac{\tilde{\mathbf{u}} - \mathbf{u}^0}{t}$ satisfies

$$
\begin{array}{rl}
b^0(\Omega^0; \dot{\mathbf{u}}, \mathbf{w}) = & \int_{\Omega^0} [f_j(0)\operatorname{div}(\boldsymbol{\nu}) + \dot{f}_j(0)] w_j d\mathbf{x} - [I_1(\mathbf{u}^0) + I_2(\mathbf{u}^0) + I_3(\mathbf{u}^0)] \\
& + \int_{\Gamma_n^0} [g_j(0)\operatorname{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0)] w_j\, dS_{\mathbf{x}}.
\end{array} \tag{44}
$$

As before, the function $\dot{\mathbf{u}}$ is the so called 'material derivative', that is, $\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u}^0 \boldsymbol{\nu}$. The domain derivative becomes $\mathbf{u}' = \dot{\mathbf{u}} - \nabla \mathbf{u}^0 \boldsymbol{\nu}$. Then,

$$
b^0(\Omega^0; \mathbf{u}', \mathbf{w}) = b^0(\Omega^0; \dot{\mathbf{u}}, \mathbf{w}) - b^0(\Omega^0; \nabla \mathbf{u}^0 \boldsymbol{\nu}, \mathbf{w}), \tag{45}
$$

where

$$b^0(\Omega^0; \nabla \mathbf{u}^0 \boldsymbol{\nu}, \mathbf{w}) = \int_{\Omega^0} \frac{\partial}{\partial x_\beta} \left(c_{j\alpha m\beta} \frac{\partial u_m^0}{\partial x_p} \nu_p \right) \frac{\partial w_j}{\partial x_\alpha} \, d\mathbf{x}.$$

Notice that $\mathbf{u}'$ vanishes on Γ_d whenever $\dot{\mathbf{u}}$ and $\boldsymbol{\nu}$ do so.

Step 4. Differential equation for the domain derivative $\mathbf{u}'$. We evaluate the different terms in the right hand side of (44) to calculate the right hand side in (45). First, notice that $-\frac{\partial}{\partial x_\alpha}\left(c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\right) = f_j(0)$ in Ω^0 and $c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta} n_\alpha = g_j(0)$ on Γ_n^0, $u_j^0 = 0$ on Γ_d^0, $j = 1, ..., n$, imply:

$$\begin{aligned} I_3(\mathbf{u}^0) \; &= \int_{\Omega^0} \left(\tfrac{\partial f_j(0)}{\partial x_q} \nu_q + f_j(0) \tfrac{\partial \nu_q}{\partial x_q} \right) w_j \, d\mathbf{x} - \int_{\Gamma^0} f_j(0) w_j n_q \nu_q \, d\mathbf{x} \\ &- \int_{\Gamma_n^0} g_j(0) \tfrac{\partial w_j}{\partial x_q} \nu_q \, dS_{\mathbf{x}} + \int_{\Omega^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \tfrac{\partial^2 w_j}{\partial x_q \partial x_\alpha} \nu_q \, d\mathbf{x}. \end{aligned}$$

Using $\frac{\partial u_m^0}{\partial x_p}\frac{\partial \nu_p}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\left(\frac{\partial u_m^0}{\partial x_p}\nu_p\right) - \frac{\partial^2 u_m^0}{\partial x_p \partial x_\beta}\nu_p$, we get

$$\begin{aligned} I_2(\mathbf{u}^0) \; &= -b^0(\Omega^0; \nabla \mathbf{u}^0 \boldsymbol{\nu}, \mathbf{w}) - \int_{\Omega^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \tfrac{\partial \nu_p}{\partial x_p} \tfrac{\partial w_j}{\partial x_\alpha} \, d\mathbf{x} \\ &- \int_{\Omega^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \nu_p \tfrac{\partial^2 w_j}{\partial x_\alpha \partial x_p} \, d\mathbf{x} + \int_{\partial\Omega^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \nu_p n_p \tfrac{\partial w_j}{\partial x_\alpha} \, d\mathbf{x}. \end{aligned}$$

As a result of the two previous identities

$$\begin{aligned} &-[I_1(\mathbf{u}^0) + I_2(\mathbf{u}^0) + I_3(\mathbf{u}^0)] = b^0(\Omega^0; \nabla \mathbf{u}^0 \boldsymbol{\nu}, \mathbf{w}) - \int_{\Gamma^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \nu_p n_p \tfrac{\partial w_j}{\partial x_\alpha} \, d\mathbf{x} \\ &- \int_{\Omega^0} \left(\tfrac{\partial f_j(0)}{\partial x_q} \nu_q + f_j(0) \tfrac{\partial \nu_q}{\partial x_q} \right) w_j \, d\mathbf{x} + \int_{\Gamma^0} f_j(0) w_j n_q \nu_q \, d\mathbf{x} + \int_{\Gamma_n^0} g_j(0) \tfrac{\partial w_j}{\partial x_q} \nu_q \, dS_{\mathbf{x}} \end{aligned}$$

and (45) becomes

$$\begin{aligned} &b^0(\Omega^0; \mathbf{u}', \mathbf{w}) = -\int_{\Gamma^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \nu_p n_p \tfrac{\partial w_j}{\partial x_\alpha} \, d\mathbf{x} + \int_{\Gamma^0} f_j(0) w_j n_q \nu_q \, d\mathbf{x} + \\ &\int_{\Omega^0} f_j'(0) w_j \, d\mathbf{x} + \int_{\Gamma_n^0} [g_j(0) \mathrm{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0)] w_j \, dS_{\mathbf{x}} + \int_{\Gamma_n^0} g_j(0) \tfrac{\partial w_j}{\partial x_q} \nu_q \, dS_{\mathbf{x}}. \end{aligned}$$

Integrating by parts in $b^0(\Omega^0; \mathbf{u}', \mathbf{w})$ and choosing $\mathbf{w}$ with compact support inside Ω^0, this identity yields the following equation for $\mathbf{u}'$ in Ω^0

$$-\frac{\partial}{\partial x_\alpha} \left(c_{j\alpha m\beta} \frac{\partial u_m'}{\partial x_\beta}(\mathbf{x}) \right) = f_j'(\mathbf{x}, 0), \quad j = 1, ..., n. \tag{46}$$

However, to obtain a pointwise boundary condition for $\mathbf{u}'$ we need to rewrite the integral on Γ^0 in such a way that no derivatives of the test function $\mathbf{w}$ are involved.

Step 5: Boundary condition for the domain derivative $\mathbf{u}'$. We integrate by parts the original expressions of $I_i(\mathbf{u}^0)$, $i = 1, 2, 3$ to get

$$I_1 \;\; = \;\; -\int_{\Omega^0} \tfrac{\partial}{\partial x_\alpha} \left(c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \mathrm{div}(\boldsymbol{\nu}) \right) w_j \, d\mathbf{x} + \int_{\Gamma^0} c_{j\alpha m\beta} \tfrac{\partial u_m^0}{\partial x_\beta} \mathrm{div}(\boldsymbol{\nu}) n_\alpha w_j \, dS_{\mathbf{x}},$$

$$
\begin{aligned}
I_2 &= -\int_{\Omega^0} c_{j\alpha m\beta} \frac{\partial}{\partial x_\beta}\left(\frac{\partial u_m^0}{\partial x_p}\nu_p\right)\frac{\partial w_j}{\partial x_\alpha} d\mathbf{x} - \int_{\Omega^0} c_{j\alpha m\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_p}\left(\frac{\partial u_m^0}{\partial x_\beta}\nu_p\right) w_j d\mathbf{x} \\
&+ \int_{\Omega^0} c_{j\alpha m\beta}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial \nu_p}{\partial x_p}\right) w_j d\mathbf{x} + \int_{\Gamma^0} c_{j\alpha m\beta}\frac{\partial^2 u_m^0}{\partial x_p \partial x_\beta}\nu_p w_j n_\alpha d\mathbf{x} \\
I_3 &= \int_{\Omega^0} \frac{\partial}{\partial x_q}\frac{\partial}{\partial x_\alpha}\left(c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\nu_q\right) w_j d\mathbf{x} + \int_{\Omega^0}\frac{\partial}{\partial x_q}\left(f_j(0)\nu_q\right) w_j d\mathbf{x} \\
&- \int_{\Gamma^0} c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial \nu_q}{\partial x_\alpha} n_q w_j dS_{\mathbf{x}}
\end{aligned}
$$

Adding up to compute $-[I_1 + I_2 + I_3]$, integrating by parts $b^0(\Omega^0; \mathbf{u}', \mathbf{w})$, inserting (46) in (45) and setting $\boldsymbol{\nu} = 0$ on Γ_d we find

$$
\begin{aligned}
\int_{\Gamma_n^0} c_{j\alpha m\beta}\frac{\partial u_m'}{\partial x_\beta} n_\alpha w_j \, dS_{\mathbf{x}} &= \int_{\Gamma^0} c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial \nu_q}{\partial x_\alpha} n_q w_j \, dS_{\mathbf{x}} \\
&- \int_{\Gamma^0} c_{j\alpha m\beta}\left[\frac{\partial^2 u_m^0}{\partial x_p \partial x_\beta}\nu_p n_\alpha + \frac{\partial u_m^0}{\partial x_\beta}\operatorname{div}(\boldsymbol{\nu}) n_\alpha\right] w_j \, dS_{\mathbf{x}} \\
&+ \int_{\Gamma_n^0}[g_j(0)\operatorname{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0)] w_j \, dS_{\mathbf{x}}.
\end{aligned}
$$

Now, using identifies $-c_{j\alpha m\beta}\frac{\partial^2 u_m^0}{\partial x_p \partial x_\beta}\nu_p n_\alpha = -\frac{\partial}{\partial x_p}(g_j(0)\nu_p) + c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial(\nu_p n_\alpha)}{\partial x_p}$, and $g_j(0)\operatorname{div}_\Gamma(\boldsymbol{\nu}) + \dot{g}_j(0) = \frac{\partial}{\partial x_p}(g_j(0)\nu_p) - g_j(0)\mathbf{n}\cdot\nabla\boldsymbol{\nu}\,\mathbf{n} + g_j'(0)$, we obtain

$$
\begin{aligned}
c_{j\alpha m\beta}\frac{\partial u_m'}{\partial x_\beta} n_\alpha &= c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial \nu_q}{\partial x_\alpha} n_q + c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial(\nu_p n_\alpha)}{\partial x_p} \\
&+ c_{j\alpha m\beta}\frac{\partial u_m^0}{\partial x_\beta}\frac{\partial \nu_p}{\partial x_p} n_\alpha - g_j(0)\mathbf{n}\cdot\nabla\boldsymbol{\nu}\,\mathbf{n} + g_j'(0)
\end{aligned}
\tag{47}
$$

on Γ_n^0. □

4. Study of diffusion problems in time dependent domains

As discussed in the introduction, the practical use of the equations for time derivatives of velocities and pressures obtained in Section 3 leads to solving diffusion problems set in moving domains. We study here parabolic problems of the form

$$
\begin{array}{lll}
e_t - \kappa\Delta e = f(\mathbf{x}, t), & \mathbf{x} \in \Omega^t, & t > 0, \\
e(\mathbf{x}, t) = g(t), & \mathbf{x} \in \Gamma^t, & t > 0, \\
e(\mathbf{x}, 0) = e_0(\mathbf{x}), & \mathbf{x} \in \Omega^t. &
\end{array}
\tag{48}
$$

As in Section 3 we assume that the evolution of the moving part of the boundary is given by a family of deformations [26]

$$
\Gamma^t = \left\{\mathbf{x} + t\,\boldsymbol{\nu}(\mathbf{x}) \,|\, \mathbf{x} \in \Gamma^0\right\},
$$

starting from a smooth surface $\Gamma^0 \in C^2$ (twice differentiable) and following a smooth vector field $\boldsymbol{\nu} \in C(\overline{\Omega^0}) \cup C^2(\Omega^0)$. We can assume $e(t) = 0$ by making the change $e = \hat{e} + g$. Then $\hat{e}$ solves (48) with zero Dirichlet boundary condition, initial datum $e_0(\mathbf{x}) - g(0)$ and right hand side $f(\mathbf{x}, t) - g'(t)$. Therefore, we will work with zero Dirichlet boundary conditions in the sequel. To solve (48) we will first refer it to a fixed domain and then construct converging Faedo-Galerkin approximations.

4.1. Variational formulation in the undeformed configuration

As usual, we denote as $H_0^1(\Omega^t)$ the subspace of $H^1(\Omega^t)$ formed by functions whose trace vanishes on Γ^t with the induced norm. Multiplying (48) by $w^t \in H_0^1(\Omega^t)$ and integrating, we find

$$\int_{\Omega^t} e_t(\mathbf{x}^t, t) w^t(\mathbf{x}^t)\, d\mathbf{x}^t + \kappa \int_{\Omega^t} \nabla_{\mathbf{x}^t} e(\mathbf{x}^t, t) \cdot \nabla_{\mathbf{x}^t} w^t(\mathbf{x}^t)\, d\mathbf{x}^t = \int_{\Omega^t} f(\mathbf{x}^t, t) w^t(\mathbf{x}^t)\, d\mathbf{x}^t$$

for each t. We use (29), (30), (31) to refer these integrals to a fixed domain.

We define $\tilde{w}(\mathbf{x}) = w^t \circ \phi^t(\mathbf{x}) = w^t(\mathbf{x}^t(\mathbf{x}))$, ϕ^t as in (29). Notice that

$$\begin{aligned} e_t(\mathbf{x}^t(\mathbf{x}), t) &= \frac{d}{dt}[e(\mathbf{x}^t(\mathbf{x}), t)] - \nabla_{\mathbf{x}^t} e(\mathbf{x}^t(\mathbf{x}), t) \cdot \frac{d\mathbf{x}^t}{dt} \\ &= \frac{d}{dt}\tilde{e}(\mathbf{x}, t) - [(\mathbf{J}^t)^{-T} \nabla_{\mathbf{x}} \tilde{e}(\mathbf{x}, t)]^T \tilde{\boldsymbol{\nu}}(\mathbf{x}), \end{aligned} \tag{49}$$

where ^{T} denotes transpose. After changing variables, problem (48) reads: Find $e \in C([0,T], L^2(\Omega^0)) \cap L^2(0,T; H_0^1(\Omega^0))$ such that $e(\mathbf{x}, 0) = e_0(\mathbf{x})$ and

$$\begin{aligned} &\int_{\Omega^0} \tilde{e}_t(\mathbf{x}, t)\tilde{w}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x} - \int_{\Omega^0} \nabla_{\mathbf{x}} \tilde{e}(\mathbf{x}, t)^T (\mathbf{J}^t(\mathbf{x}))^{-1} \tilde{\boldsymbol{\nu}}(\mathbf{x}) \tilde{w}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x} \\ &+\kappa \int_{\Omega^0} \nabla_{\mathbf{x}} e(\mathbf{x}, t)^T ((\mathbf{J}^t(\mathbf{x}))^T \mathbf{J}^t(\mathbf{x}))^{-1} \nabla_{\mathbf{x}} \tilde{w}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x} \\ &= \int_{\Omega^0} \tilde{f}(\mathbf{x}, t)\tilde{w}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{50}$$

Since $w^t \in H_0^1(\Omega^t)$, we have $\tilde{w} \in H_0^1(\Omega^0)$. In fact, we can take the same arbitrary function $w \in H_0^1(\Omega^0)$ for all t.

Let us set $\mathbf{A}(x,t) = \kappa((\mathbf{J}^t(\mathbf{x}))^T \mathbf{J}^t(\mathbf{x}))^{-1} \det \mathbf{J}^t(\mathbf{x})$, $c(\mathbf{x}, t) = \det \mathbf{J}^t(\mathbf{x})$ and $\mathbf{b}(\mathbf{x}, t) = -(\mathbf{J}^t(\mathbf{x}))^{-1} \tilde{\boldsymbol{\nu}}(\mathbf{x}) \det \mathbf{J}^t(\mathbf{x})$. The matrix $\mathbf{J}^t(\mathbf{x})$ is defined in (29). It admits two non zero derivatives with respect to t, higher order derivatives vanish. It has C^k regularity in space for a C^{k+1} field $\boldsymbol{\nu}$. These coefficients are smooth and bounded up to the boundary, provided $\kappa > 0$ remains constant. Approximate expressions for $(\mathbf{J}^t(\mathbf{x}))^{-1}$ and $\det \mathbf{J}^t(\mathbf{x})$ are given in (35) and (36). They imply that $c(\mathbf{x}, t) > 0$ and $\mathbf{A}(\mathbf{x}, t)$ is definite positive for $t \leq \tau$ small enough depending on $\boldsymbol{\nu}$. Existence of solutions for (50) in $[0,T]$ is a consequence of the results in the next section, applied in a sequence of small intervals covering $[0,T]$.

4.2. Construction of stable solutions

Consider a basis $\{\phi_1, \phi_2, \ldots, \phi_M \ldots\}$ of the Hilbert space $L^2(\Omega)$. We choose the normalized eigenfunctions $\phi_j \in H^2(\Omega) \cap H_0^1(\Omega)$, $j \in \mathbb{N}$, of $-\Delta$ in $H_0^1(\Omega)$, see [27].

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded C^2 domain. Given a function $f \in C^1([0,T]; L^2(\Omega))$ there exists a unique solution $u \in C([0,T]; H^2(\Omega)) \cap$*

$H^1(0,T;H_0^1(\Omega))$ of

$$\begin{aligned}&\int_\Omega u_t(\mathbf{x},t)w(\mathbf{x})\,c(\mathbf{x},t)\,d\mathbf{x}+\int_\Omega \nabla u(\mathbf{x},t)\cdot\mathbf{b}(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x}+\\&\int_\Omega \nabla u(\mathbf{x},t)^T\mathbf{A}(\mathbf{x},t)\nabla w(\mathbf{x})\,d\mathbf{x}=\int_\Omega f(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x},\end{aligned}\tag{51}$$

for all $w\in H_0^1(\Omega)$, $t\in[0,T]$, provided

- $\mathbf{A}(\mathbf{x},t)\in C^1(\overline{\Omega}\times[0,T])$, $\mathbf{b}(\mathbf{x},t)\in C^1(\overline{\Omega}\times[0,T])$ *and* $c(\mathbf{x},t)\in C^2(\overline{\Omega}\times[0,T])$,
- *the matrices $\mathbf{C}^M(t)$ with elements $\int_\Omega c(t)\phi_m\phi_k d\mathbf{x}$, $m,k=1,\ldots,M$, are invertible for $t\in[0,T]$,*
- *the matrices $\mathbf{A}(\mathbf{x},t)$ are uniformly coercive, that is, $\boldsymbol{\xi}^T\mathbf{A}(\mathbf{x},t)\boldsymbol{\xi}\geq a_0|\boldsymbol{\xi}|^2$, $a_0>0$, for all $\boldsymbol{\xi}\in\mathbb{R}^n$, and the scalar field $c(\mathbf{x},t)$ is bounded from below, $c(\mathbf{x},t)\geq c_0>0$, for all $\mathbf{x}\in\mathbb{R}^n$ and $t>0$,*
- $u_0\in L^2(\Omega)$ *and* $w_0=\mathrm{div}(\mathbf{A}(\mathbf{x},0)\nabla u_0(\mathbf{x}))+\mathbf{b}(\mathbf{x},0)\cdot\nabla u_0(\mathbf{x})\in L^2(\Omega)$.

Moreover, the solution depends continuously on parameters and data.
We obtain a solution for the original time dependent problem set in a moving domain undoing the change of variables.

Proof. *Existence.* We use the Faedo-Galerkin method [28, 29]. First, we change variables $u(\mathbf{x},t)=e^{\lambda t}v(\mathbf{x},t)$, $u_t(\mathbf{x},t)=e^{\lambda t}[v_t(\mathbf{x},t)+\lambda v(\mathbf{x},t)]$, with $\lambda>0$ to be selected large enough. We obtain similar variational equations for v with an additional term λcv and g and f multiplied by $e^{-\lambda t}$. Then we seek approximate solutions $v^M(\mathbf{x},t)=\sum_{m=1}^M\alpha_m(t)\phi_m(\mathbf{x})$ such that

$$\begin{aligned}&\int_\Omega c(\mathbf{x},t)\,v_t^M(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x}+\int_\Omega\sum_{p,q=1}^n a_{pq}(\mathbf{x},t)\frac{\partial v^M}{\partial x_p}(\mathbf{x},t)\frac{\partial w}{\partial x_q}(\mathbf{x})\,d\mathbf{x}\\&+\int_\Omega\lambda c(\mathbf{x},t)v^M(\mathbf{x},t)w(\mathbf{x},t)\,d\mathbf{x}+\int_\Omega\sum_{p=1}^n b_p(\mathbf{x},t)\frac{\partial v^M}{\partial x_p}(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x}\\&=\int_\Omega e^{-\lambda t}f(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x},\\&v^M(\mathbf{x},0)=\sum_{m=1}^M\alpha_m(0)\phi_m(\mathbf{x}),\quad\alpha_m(0)=\int_\Omega u_0(\mathbf{x})\phi_m(\mathbf{x})d\mathbf{x},\end{aligned}\tag{52}$$

for all $w\in V^M=\mathrm{span}\{\phi_1,\phi_2,\ldots,\phi_M\}$. We find a system of M differential equations for the coefficient functions $\alpha_m(t)$ setting $w=\phi_k$, $k=1,\ldots,M$,

$$\begin{aligned}&\sum_{m=1}^M\alpha_m'(t)\int_\Omega c(t)\phi_m\phi_k\,d\mathbf{x}=-\sum_{m=1}^M\alpha_m(t)\int_\Omega\sum_{p=1}^n b_p(t)\frac{\phi_m}{\partial x_p}\phi_k\,d\mathbf{x}-\\&\sum_{m=1}^M\alpha_m(t)\int_\Omega\left[\sum_{p,q=1}^n a_{pq}(t)\frac{\partial\phi_m}{\partial x_p}\frac{\partial\phi_k}{\partial x_q}+\lambda c(t)\phi_m\phi_k\right]d\mathbf{x}+\int_\Omega e^{-\lambda t}f(t)\phi_k\,d\mathbf{x}.\end{aligned}\tag{53}$$

This can be written as a linear system with continuous and bounded coefficients in $[0,T]$

$$\frac{d}{dt}\boldsymbol{\alpha}^M = \mathbf{C}^M(t)^{-1}\mathbf{A}^M(t)\boldsymbol{\alpha}^M + \mathbf{C}^M(t)^{-1}\mathbf{g}^M(t) + \mathbf{C}^M(t)^{-1}\mathbf{f}^M(t)$$

with initial datum $\boldsymbol{\alpha}^M(0)$, which admits a unique solution $\boldsymbol{\alpha}^M(t)$, $t \in [0,T]$ [30]. Multiplying identity (53) by α_k and adding over k, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega c(\mathbf{x},t)|v^M(\mathbf{x},t)|^2 d\mathbf{x} + \int_\Omega \sum_{p,q=1}^n a_{pq}(\mathbf{x},t)\frac{\partial v^M}{\partial x_p}(\mathbf{x},t)\frac{\partial v^M}{\partial x_q}(\mathbf{x},t)\, d\mathbf{x} \\
&+\int_\Omega (\lambda c - \frac{1}{2}c_t)(\mathbf{x},t)|v^M(\mathbf{x},t)|^2\, d\mathbf{x} + \int_\Omega \sum_{p=1}^n b_p(\mathbf{x},t)\frac{\partial v^M}{\partial x_p}(\mathbf{x},t)v^M(\mathbf{x},t)\, d\mathbf{x} \\
&= \int_\Omega e^{-\lambda t} f(\mathbf{x},t)v^M(\mathbf{x},t)\, d\mathbf{x}.
\end{aligned} \tag{54}$$

Integrating in $[0,T]$ and using coercivity, lower bounds for A and c, L^∞ bounds, as well as Young's inequality [27], we find

$$\begin{aligned}
\frac{c_0}{2}\int_\Omega |v^M(\mathbf{x},t)|^2 d\mathbf{x} + \frac{a_0}{2}\int_0^t\int_\Omega |\nabla v^M(\mathbf{x},t)|^2 d\mathbf{x}ds + \frac{\lambda c_0}{2}\int_0^t\int_\Omega |v^M(\mathbf{x},t)|^2 d\mathbf{x}ds \\
\leq \frac{\|c\|_{L^\infty_{xt}}}{2}\ \|v^M(0)\|_{L^2(\Omega)} + \frac{1}{2}\ \|f\|_{L^2(0,T,L^2(\Omega))}
\end{aligned}$$

for λ large enough depending on a_0, $\|c_t\|_{L^\infty_{xt}}$, c_0, $\|\mathbf{b}\|_{L^\infty_{xt}}$, n. Gronwall inequality, and the fact that $v^M(0) \to u_0$ in L^2, imply that v^M is bounded in $L^\infty(0,T,L^2(\Omega))$ and $L^2(0,T,H^1_0(\Omega))$. We extract a subsequence $v^{M'}$ converging a limit v weakly star in $L^\infty(0,T,L^2(\Omega))$ and weakly in $L^2(0,T,H^1_0(\Omega))$. Moreover, $\frac{d}{dt}\int_\Omega c(t)v^{M'}(t)\phi_k\, d\mathbf{x}$ tends to $\frac{d}{dt}\int_\Omega c(t)v(t)\phi_k\, d\mathbf{x}$ in the sense of distributions in $\mathcal{D}'(0,T)$ for any k. Similar convegences hold for $u^{M'}$ and $u = e^{\lambda t}v$. We undo the change in (52), multiply by a function $\psi \in C^\infty_c([0,T))$, integrate over t and pass to the limit as $M' \to \infty$ to find

$$\begin{aligned}
-\int_\Omega c(\mathbf{x},0)\, u(\mathbf{x},0)w(\mathbf{x})\psi(0)\, d\mathbf{x} - \int_0^t\int_\Omega c_t(\mathbf{x},t)u(\mathbf{x},t)w(\mathbf{x})\psi(t)\, d\mathbf{x}ds+ \\
\int_0^t\int_\Omega \left[\sum_{p,q=1}^n a_{pq}(\mathbf{x},t)\frac{\partial v}{\partial x_p}(\mathbf{x},t)\frac{\partial w}{\partial x_q}(\mathbf{x}) + \sum_{p=1}^n b_p(\mathbf{x},t)\frac{\partial v^M}{\partial x_p}(\mathbf{x},t)w(\mathbf{x})\right]\psi(t)\, d\mathbf{x}ds \\
= \int_0^t\int_\Omega e^{-\lambda t} f(\mathbf{x},t)w(\mathbf{x})\psi(t)\, d\mathbf{x}ds,
\end{aligned}$$

for any $w \in H^1_0(\Omega)$, so that the limiting solution satisfies the condition on the initial data and the equation

$$cu_t - \mathrm{div}(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u = f \tag{55}$$

in the sense of distributions [24, 29].

Uniqueness. To prove uniqueness, we assume there are two solutions u_1 and u_2, and set $u = u_1 - u_2$. We subtract the equations satisfied by both, multiply by u, set $u = e^{\lambda t} v$ and integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c(\mathbf{x},t)|v(\mathbf{x},t)|^2 d\mathbf{x} + \int_\Omega \sum_{p,q=1}^n a_{pq}(\mathbf{x},t)\frac{\partial v}{\partial x_p}(\mathbf{x},t)\frac{\partial v}{\partial x_q}(\mathbf{x},t)\,d\mathbf{x}$$
$$+\int_\Omega \sum_{p=1}^n b_p(\mathbf{x},t)\frac{\partial v}{\partial x_p}(\mathbf{x},t)v(\mathbf{x},t)\,d\mathbf{x} + \int_\Omega (\lambda c - \frac{1}{2}c_t)(\mathbf{x},t)|v(\mathbf{x},t)|^2\,d\mathbf{x} = 0.$$

Using uniform coercivity, the L^∞ bounds, and taking λ large enough, we see that $\int_\Omega c(\mathbf{x},t)|v(\mathbf{x},t)|^2 \leq \int_\Omega c(\mathbf{x},0)|v(\mathbf{x},0)|^2 = 0$. Therefore, the solution is unique.

Regularity. Next, we differentiate with respect to t to get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t(\mathbf{x},t)w(\mathbf{x})\,c(\mathbf{x},t)\,d\mathbf{x} + \int_\Omega \nabla u_t(\mathbf{x},t)\cdot\mathbf{b}(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x}\\
&+\int_\Omega \nabla u_t(\mathbf{x},t)^T\mathbf{A}(\mathbf{x},t)\nabla w(\mathbf{x})\,d\mathbf{x} + \int_\Omega u_t(\mathbf{x},t)w(\mathbf{x})\,c_t(\mathbf{x},t)\,d\mathbf{x}\\
&=\int_\Omega f_t(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x} - \int_\Omega u(\mathbf{x},t)w(\mathbf{x})\,c_{tt}(\mathbf{x},t)\,d\mathbf{x}\\
&+\int_\Omega \nabla u(\mathbf{x},t)\cdot\mathbf{b}_t(\mathbf{x},t)w(\mathbf{x})\,d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x},t)^T\mathbf{A}_t(\mathbf{x},t)\nabla w(\mathbf{x})\,d\mathbf{x},
\end{aligned} \tag{56}$$

with $u_t(\mathbf{x},0) = w_0(\mathbf{x})$. The functions $\nabla u\cdot\mathbf{b}_t$, $\nabla u^T\mathbf{A}_t$, uc_{tt} define linear forms in $H^1(\Omega)$. Arguing as in Theorem 4.1, we see that the function u_t is the unique solution in $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ of this problem. Then, (55) implies that $-\mathrm{div}(\mathbf{A}\nabla u)+\mathbf{b}\cdot\nabla u = -cu_t + f \in C([0,T];L^2(\Omega))$ zero Dirichlet boundary condition. Elliptic regularity theory ensures that $u \in C([0,T];H^2(\Omega))$.

Stability. The limiting solution inherits all the bounds established on the approximating sequence. Therefore its $L^\infty([0,T];H^2(\Omega))$ and $H^1(0,T,H_0^1(\Omega))$ norms are bounded from above in terms of constants depending on the parameters of the problem and the norms of the data. □

Theorem 4.2 *Under the hypotheses of Theorem 4.1, if $f \in L^q(\Omega\times[0,T])$ and $u_0 \in L^q(\Omega)$, then u, its first and second order spatial derivatives, and u_t belong to $L^q(\Omega\times[0,T])$, $1 < q < \infty$.*

Proof. We set $v/c = u$. Then $\nabla u = \nabla v/c - v/c^2\nabla c$ and $cu_t = v_t - c_t/cv$. Therefore, v is a solution of

$$v_t - \mathrm{div}\left(\frac{\mathbf{A}}{c}\nabla v\right) + \mathbf{A}\frac{\nabla c}{c^2}\nabla v + \frac{\mathbf{b}}{c}\cdot\nabla v + \left[\mathrm{div}\left(\mathbf{A}\frac{\nabla c}{c^2}\right) + \mathbf{b}\cdot\frac{\nabla c}{c^2} - \frac{c_t}{c}\right]v = f.$$

The result is a consequence of the regularity result stated in Theorem 9.1 in [31]. □

5. Well posedness results for the quasi-stationary submodels

In this section we establish the pertinent existence and regularity results for the elliptic submodels and the stationary transport problem in fixed domains.

Constructing solutions for the stationary transport problems considered here is a non trivial issue. We are able to obtain them by a regularization procedure under sign hypotheses on the velocity fields motivated by asymptotic studies in specific experimental set-ups [12, 14], which will have to be preserved by any implemented scheme. Besides the standard constitutive law $\pi(\phi_\mathrm{s}) = \Pi\phi_\mathrm{s}$, $\Pi > 0$, we will use the simplifying assumptions $k_\mathrm{h}(\phi_\mathrm{s}) = k_\mathrm{h} > 0$, $k_\mathrm{h}(\phi_\mathrm{s})/\phi_\mathrm{f} = \xi_\infty > 0$, $c/(c + K_\mathrm{s}) = g_\mathrm{s} > 0$ and $c/(c + K_\mathrm{c}) = g_\mathrm{c} > 0$. Setting these functions equal to a constant is motivated by the fact that ϕ_s and c are expected to depart slightly from background values ϕ_∞ and c_∞. We will comment in Section 7 on how to relieve these assumptions.

5.1. Elliptic problems for displacements, velocities and concentrations

For ease of calculations, we make a change of notation in the model: $p - \pi$ becomes p while p becomes $p + \pi$. We will keep this notation in the sequel. Consider the first the submodel for mechanical fields:

$$\begin{aligned}
&\mu\Delta\mathbf{u}_\mathrm{s} + (\mu + \lambda)\nabla\mathrm{div}(\mathbf{u}_\mathrm{s}) - \nabla p = \Pi\nabla\phi_\mathrm{s}, && \text{on } \Omega,\\
&\mu\Delta\mathbf{v}_\mathrm{s} + (\mu + \lambda)\nabla\mathrm{div}(\mathbf{v}_\mathrm{s}) = \nabla p', && \text{on } \Omega,\\
&k_\mathrm{h}\Delta p - \mathrm{div}(\mathbf{v}_\mathrm{s}) = 0, && \text{on } \Omega,\\
&\Delta p' = (2\mu + \lambda)\Delta e', && \text{on } \Omega,\\
&p = p_\mathrm{ext}, \quad p' = p'_\mathrm{ext} && \text{on } \Gamma,\\
&\mathbf{u} = 0, \quad \mathbf{v} = 0, && \text{on } \Gamma_-,\\
&(\hat{\boldsymbol{\sigma}}(\mathbf{u}_\mathrm{s}) - (p + \Pi\phi_\mathrm{s})\mathbf{I})\mathbf{n} = \mathbf{g}, \quad (\hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s}) - p'\mathbf{I})\mathbf{n} = \mathbf{g}', && \text{on } \Gamma_+.
\end{aligned} \tag{57}$$

We denote by $H^1_{0,-}(\Omega)$ the Sobolev space of $H^1(\Omega)$ functions vanishing on Γ_-.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open bounded domain with C^4 boundary $\partial\Omega$. Let us assume that $\phi_\mathrm{s} \in H^1(\Omega)$ and $e' \in H^2(\Omega)$. Given positive constants μ, λ, k_h, Π, there exists a unique solution $\mathbf{u}_\mathrm{s} \in [H^2(\Omega)]^n \times [H^1_{0,-}(\Omega)]^n$, $\mathbf{v}_\mathrm{s} \in [H^3(\Omega)]^n \times [H^1_{0,-}(\Omega)]^n$, $p \in H^4(\Omega)$, $p' \in H^2(\Omega)$ of (57) for any $p_\mathrm{ext}, p'_\mathrm{ext} \in \mathbb{R}$ and $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^n$.*
Moreover, if $\phi_\mathrm{s} \in W^{1,q}(\Omega)$ and $e' \in W^{1,q}(\Omega)$, $n < q < \infty$, then $p' \in W^{1,q}(\Omega)$, $\mathbf{v}_\mathrm{s} \in W^{2,q}(\Omega)$, $p \in W^{3,q}(\Omega)$ and $\mathbf{u}_\mathrm{s} \in W^{2,q}(\Omega)$.

Proof. The equation for p' uncouples from the rest and provides a solution $p' \in H^2(\Omega)$ by classical theory for Laplace equations [27]. Next, the equation for $\mathbf{v}$ is a classical Navier elasticity system which admits a unique solution $\mathbf{v}_\mathrm{s} \in [H^2(\Omega)]^n \times [H^1_{0,-}(\Omega)]^n$ [24]. Since the source $\nabla p' \in [H^1(\Omega)]^n$, elliptic regularity theory implies $\mathbf{v}_\mathrm{s} \in [H^3(\Omega)]^n$. Now, $\mathrm{div}(\mathbf{v}_\mathrm{s}) \in H^2(\Omega)$ implies that the unique solution p of the corresponding Poisson problem has $H^4(\Omega)$ regularity thanks to the C^4 regularity of $\partial\Omega$ [27]. Finally, the equation for $\mathbf{u}_\mathrm{s}$ is again a classical Navier elasticity system with L^2 right hand side which admits a unique solution $\mathbf{u}_\mathrm{s} \in [H^2(\Omega)]^n \cap [H^1_{0,-}(\Omega)]^n$.
When $\phi_\mathrm{s} \in W^{1,q}(\Omega)$ and $e' \in W^{1,q}(\Omega)$, we obtain the increased regularity [32]. Notice that since the boundary values are constant, we can construct extensions to $H^k(\Omega)$ and $W^{k,q}$ for the necessary k, q [24, 27]. $\square$

Now, the equation for the concentrations is:

$$\begin{aligned} -d\Delta c + \mathrm{div}(\mathbf{v}_\mathrm{f} c) &= -k_\mathrm{c} g_\mathrm{c} \phi_\mathrm{s}, && \mathbf{x} \in \Omega, \\ c &= c_0 && \mathbf{x} \in \Gamma_-, \\ \frac{\partial c}{\partial \mathbf{n}} &= 0 && \mathbf{x} \in \Gamma_+, \end{aligned} \tag{58}$$

given positive constants $d, c_0, k_\mathrm{c}, g_\mathrm{c}$ and known functions $\mathbf{v}_\mathrm{f}$ and ϕ_s.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open bounded domain with C^2 boundary $\partial\Omega$. Given positive constants $k_\mathrm{c}, g_\mathrm{c}, d, c_0$, a vector function $\mathbf{v}_\mathrm{f} \in [H^1(\Omega)]^n \cap C(\overline{\Omega})$, and a positive function $\phi_\mathrm{s} \in L^2(\Omega)$ there exists a unique nonnegative solution $c \in H^1(\Omega)$ of (58) provided d is sufficiently large.*

Proof. Set $c = \tilde{c} + c_0$. The resulting problem admits the variational formulation: Find $\tilde{c} \in H^1_{0,-}(\Omega)$ such that

$$\begin{aligned} d \int_\Omega \nabla \tilde{c} \cdot \nabla w \, d\mathbf{x} - \int_\Omega \mathbf{v}_\mathrm{f} \cdot \tilde{c} \nabla w \, d\mathbf{x} + \int_{\Gamma_+} \tilde{c} w \mathbf{v}_\mathrm{f} \cdot \mathbf{n} \, dS_\mathbf{x} \\ = -k_\mathrm{c} g_\mathrm{c} \int_\Omega \phi_\mathrm{s} w \, d\mathbf{x} + c_0 \int_\Omega \mathbf{v}_\mathrm{f} \cdot \nabla w \, d\mathbf{x}, \end{aligned}$$

for all $w \in H^1_{0,-}(\Omega)$. The continuous bilinear form is coercive provided d is large enough compared to $\|\mathbf{v}_\mathrm{f}\|_\infty$. Thus, we have a unique solution $\tilde{c} \in H^1_{0,-}(\Omega)$ with $H^2(\Omega)$ regularity.

Let is denote by $c^- \geq 0$ the negative part of c, that is, $c^- = 0$ when $c \geq 0$ and $c^- = -c$ when $c < 0$. The function $c^- \in H^1_{0,-}(\Omega)$ satisfies

$$d \int_\Omega |\nabla c^-|^2 \, d\mathbf{x} - \int_\Omega \mathbf{v}_\mathrm{f} \cdot c^- \nabla c^- \, d\mathbf{x} + \int_{\Gamma_+} |c^-|^2 \mathbf{v}_\mathrm{f} \cdot \mathbf{n} \, dS_\mathbf{x} = -k_\mathrm{c} g_\mathrm{c} \int_\Omega \phi_\mathrm{s} c^- \, d\mathbf{x} \leq 0.$$

Coercivity implies $c^- = 0$ and $c \geq 0$ provided d is large enough compared to $\|\mathbf{v}_\mathrm{f}\|_\infty$.

For uniqueness, assume we have two positive solutions c_1 and c_2 in $H^1(\Omega)$ and set $c = c_1 - c_2 \in H^1_{0,-}(\Omega)$. Then u is a solution of

$$\begin{aligned} -d\Delta c + \mathrm{div}(\mathbf{v}_\mathrm{f} c) &= 0, && \mathbf{x} \in \Omega, \\ c &= 0, && \mathbf{x} \in \Gamma_-, \\ \frac{\partial c}{\partial \mathbf{n}} &= 0, && \mathbf{x} \in \Gamma_+. \end{aligned}$$

The variational equation with test function c and coercivity imply $c = 0$, that is, $c_1 = c_2$. $\square$

5.2. Conservation law for volume fractions

Consider the equation

$$\mathrm{div}(-\mathbf{v}_\mathrm{f} \phi_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s} \phi_\mathrm{f} = k_\mathrm{s} g_\mathrm{s}, \quad \mathbf{x} \in \Omega, \tag{59}$$

where k_s and g_s are positive constants and $\mathbf{v}_\mathrm{f}$ a known function.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a thin open, bounded subset, with C^3 boundary $\partial\Omega$. Let $\mathbf{v}_\mathrm{f} \in [H^2(\Omega) \cap C(\overline{\Omega})]^n$ such that $\mathrm{div}(\mathbf{v}_\mathrm{f}) \leq 0$ in Ω, $\mathrm{div}(\mathbf{v}_\mathrm{f}) \in L^\infty(\Omega)$ and $\mathbf{v}_\mathrm{f} \cdot \mathbf{n} \leq 0$ a.e. on $\partial\Omega$. We assume that $\nabla \mathbf{v}_\mathrm{f} \in [L^\infty(\Omega)]^{n^2}$ with $\|\nabla \mathbf{v}_\mathrm{f}\|_{[L^\infty]^{n^2}}$ small enough compared to $k_\mathrm{s} g_\mathrm{s}$. Then, given positive constants k_s and g_s, there exists a solution $\phi_\mathrm{f} \in L^2(\Omega)$ of (59) in the sense of distributions. Moreover,*

- *$0 \leq \phi_\mathrm{f} \leq 1$ on Ω and ϕ does not vanish in sets of positive measure.*
- *$\phi_\mathrm{f} \in H^1(\Omega)$ is the unique solution of the variational formulation in $H^1(\Omega)$ and*
$$\frac{1}{2} k_\mathrm{s} g_\mathrm{s} \|\nabla \phi\|_{L^2} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^2]^n}.$$
- *If we assume that Ω is a thin domain for which $\mathbf{n} \sim \mathbf{e}_n$ and $\mathrm{div}(\mathbf{v}_\mathrm{f}) \in W^{1,q}(\Omega)$, $n < q < \infty$, then $\nabla \phi_\mathrm{f} \in L^q(\Omega)$ and*
$$\frac{1}{2} k_\mathrm{s} g_\mathrm{s} \|\nabla \phi\|_{L^q} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^q]^n}.$$

Proof. *Existence.* For each $\varepsilon > 0$, we follow [33] and let $\phi_\varepsilon \in H^1(\Omega)$ be the solution of the variational formulation
$$\begin{aligned} b(\phi_\epsilon, w) = \varepsilon \int_\Omega \nabla \phi_\varepsilon \cdot \nabla w \, d\mathbf{x} + \int_\Omega \mathbf{v}_\mathrm{f} \cdot \phi_\varepsilon \nabla w \, d\mathbf{x} - \int_{\partial\Omega} \phi_\varepsilon w \, \mathbf{v}_\mathrm{f} \cdot \mathbf{n} dS_\mathbf{x} \\ + \int_\Omega k_\mathrm{s} g_\mathrm{s} \, \phi_\varepsilon w \, d\mathbf{x} = \int_\Omega k_\mathrm{s} g_\mathrm{s} \, w \, d\mathbf{x} = L(w), \quad \forall w \in H^1(\Omega) \end{aligned}$$
of
$$-\varepsilon \Delta \phi_\varepsilon - \mathrm{div}(\mathbf{v}_\mathrm{f} \phi_\varepsilon) + k_\mathrm{s} g_\mathrm{s} \phi_\varepsilon = k_\mathrm{s} g_\mathrm{s} \text{ in } \Omega, \quad \frac{\partial \phi_\varepsilon}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega. \tag{60}$$
The bilinear form $b(\varphi, w)$ is continuous on $H^1(\Omega)$ [24], while the linear form L is continuous on $L^2(\Omega)$. Since $\mathrm{div}(\mathbf{v}_\mathrm{f}) \leq 0$ and $\mathbf{v}_\mathrm{f} \cdot \mathbf{n} \leq 0$, the bilinear form b is also coercive in $H^1(\Omega)$. Indeed,
$$\int_\Omega \mathbf{v}_\mathrm{f} \cdot \phi_\varepsilon \nabla \phi_\varepsilon d\mathbf{x} = \frac{1}{2} \int_\Omega \mathbf{v}_\mathrm{f} \cdot \nabla |\phi_\varepsilon|^2 d\mathbf{x} = \frac{1}{2} \int_{\partial\Omega} |\phi_\varepsilon|^2 \mathbf{v}_\mathrm{f} \cdot \mathbf{n} d\mathbf{x} - \frac{1}{2} \int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f}) |\phi_\varepsilon|^2 d\mathbf{x}.$$
The positive term $-\int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f}) |\phi_\varepsilon|^2 d\mathbf{x}$ is finite because $|\phi_\varepsilon|^2 \in L^2(\Omega)$ thanks to Sobolev embeddings [27, 34]. Since the bilinear form $\varepsilon \int_\Omega \nabla \phi \cdot \nabla w \, d\mathbf{x} + \int_\Omega k_\mathrm{s} g_\mathrm{s} \phi w \, d\mathbf{x}$ is coercive in $H^1(\Omega)$, we have a unique solution $\phi_\varepsilon \in H^1(\Omega)$ by Lax Milgram's theorem [27]. We set $w = \phi_\varepsilon$ and apply Young's inequality [27] to obtain the uniform bound $\|\phi_\varepsilon\|_{L^2} \leq \mathrm{meas}(\Omega)^{1/2}$ from
$$\begin{aligned} 0 \leq \varepsilon \int_\Omega |\nabla \phi_\varepsilon|^2 \, d\mathbf{x} - \frac{1}{2} \int_{\partial\Omega} |\phi_\varepsilon|^2 \mathbf{v}_\mathrm{f} \cdot \mathbf{n} \, dS_\mathbf{x} + \int_\Omega \left[-\frac{1}{2} \mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s} \right] |\phi_\varepsilon|^2 \, d\mathbf{x} \\ = \int_\Omega k_\mathrm{s} g_\mathrm{s} \phi_\varepsilon \, d\mathbf{x} \leq \|k_\mathrm{s} g_\mathrm{s}\|_{L^2} \left(\int_\Omega |\phi_\varepsilon|^2 \right)^{1/2}. \end{aligned}$$

Each of the positive terms in the left hand side of the above inequality are uniformly bounded too. Thus, we can extract a subsequence $\phi_{\varepsilon'}$ such that $\phi_{\varepsilon'}$ tends weakly in $L^2(\Omega)$ to a limit ϕ, and $\varepsilon \nabla \phi_\varepsilon$ tends strongly to zero. Setting $w \in C_c^\infty(\Omega)$ in the variational formulation, and taking limits [29, 35], ϕ is a solution of (59) in the sense of distributions. The variational equation holds with $\epsilon = 0$, replacing the boundary integral by the duality ${}_{H^{-1/2}(\partial\Omega)} < \phi\, \mathbf{v}_\mathrm{f} \cdot \mathbf{n}, w >_{H^{1/2}(\partial\Omega)}$ for $w \in H^1(\Omega)$ [36].

L^∞ *estimates.* Setting $\psi_\varepsilon = \phi_\varepsilon - 1$ and $w = \psi_\varepsilon^+$ we get

$$\varepsilon \int_\Omega |\nabla \psi_\varepsilon^+|^2 \, d\mathbf{x} - \frac{1}{2} \int_{\partial\Omega} |\psi_\varepsilon^+|^2 \, \mathbf{v}_\mathrm{f} \cdot \mathbf{n} dS_\mathbf{x} + \int_\Omega \left[-\frac{1}{2} \mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s} \right] |\psi_\varepsilon^+|^2 \, d\mathbf{x}$$
$$= \int_\Omega \mathrm{div}(\mathrm{v}_\mathrm{f}) \psi_\varepsilon^+ d\mathbf{x} \leq 0.$$

Thus, $\psi_\varepsilon^+ = 0$ and $\phi_\varepsilon \leq 1$. Similarly, we set $\psi_\varepsilon = -\phi_\varepsilon$ to find

$$\varepsilon \int_\Omega |\nabla \psi_\varepsilon^+|^2 \, d\mathbf{x} - \frac{1}{2} \int_{\partial\Omega} |\psi_\varepsilon^+|^2 \mathbf{v}_\mathrm{f} \cdot \mathbf{n} \, dS_\mathbf{x} + \int_\Omega \left[-\frac{1}{2} \mathrm{div}(\mathrm{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s} \right] |\psi_\varepsilon^+|^2 \, d\mathbf{x}$$
$$= - \int_\Omega k_\mathrm{s} g_\mathrm{s} \psi_\varepsilon^+ d\mathbf{x} \leq 0.$$

Thus, $\psi_\varepsilon^+ = 0$ and $\phi_\varepsilon \geq 0$. Weak limits ϕ in L^2 inherit these properties. Moreover, (59) implies that ϕ cannot vanish in sets of positive measure.

H^1 *Regularity.* Elliptic regularity for system (60) implies that $\phi_\varepsilon \in H^2(\Omega)$ [27, 37]. We multiply (60) by $\Delta\phi_\varepsilon$ and integrate over Ω to get

$$-\varepsilon \int_\Omega |\Delta \phi_\varepsilon|^2 d\mathbf{x} - \int_\Omega \mathbf{v}_\mathrm{f} \cdot \nabla \phi_\varepsilon \Delta \phi_\varepsilon d\mathbf{x} + \int_\Omega [-\mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s}] \, \phi_\varepsilon \Delta \phi_\varepsilon d\mathbf{x}$$
$$= \int_\Omega k_\mathrm{s} g_\mathrm{s} \Delta \phi_\varepsilon d\mathbf{x}.$$

Integrating by parts, and using the boundary condition, we find

$$-\varepsilon \int_\Omega |\Delta \phi_\varepsilon|^2 d\mathbf{x} + \int_\Omega \left[\frac{1}{2} \mathrm{div}(\mathbf{v}_\mathrm{f}) - k_\mathrm{s} g_\mathrm{s} \right] |\nabla \phi_\varepsilon|^2 d\mathbf{x} + \frac{1}{2} \int_{\partial\Omega} |\nabla \phi_\varepsilon|^2 \mathbf{v}_\mathrm{f} \cdot \mathbf{n} dS_\mathbf{x} =$$
$$\int_\Omega \nabla[-\mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s}] \cdot \phi_\varepsilon \nabla \phi_\varepsilon d\mathbf{x} - \int_\Omega v_{\mathrm{f},j,x_k} \phi_{\varepsilon,x_j} \phi_{\varepsilon,x_k} d\mathbf{x}.$$

We know that $0 \leq \phi_\varepsilon \leq 1$. Therefore,

$$\int_\Omega \left[-\frac{1}{2} \mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s} \right] |\nabla \phi_\varepsilon|^2 d\mathbf{x} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^2]^n} \|\nabla \phi_\varepsilon\|_{L^2}$$
$$+ \int_\Omega |v_{\mathrm{f},j,x_k} \phi_{\varepsilon,x_j} \phi_{\varepsilon,x_k}| d\mathbf{x}.$$

If $\|\nabla \mathbf{v}_\mathrm{f}\|_{[L^\infty]^{n^2}}$ is small enough compared to $k_\mathrm{s} g_\mathrm{s}$

$$\frac{1}{2} k_\mathrm{s} g_\mathrm{s} \|\nabla \phi_\varepsilon\|_{L^2} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^2]^n}.$$

We extract a subsequence $\phi_{\varepsilon'}$ converging weakly in $H^1(\Omega)$ to a limit ϕ, strongly in $L^2(\Omega)$, and pointwise in Ω. The traces of ϕ on $\partial\Omega$ belong to $L^2(\partial\Omega)$, and are weak limits of traces of $\phi_{\varepsilon'}$. Passing to the limit in the variational formulation for (60), $\phi \in H^1(\Omega)$ is a solution with $\epsilon = 0$ which inherits these bounds.

Uniqueness. Given two solutions $\phi_1, \phi_2 \in H^1(\Omega)$, we set $\psi = \phi_1 - \phi_2$. Subtracting the variational equations we get for the test function $\psi \in H^1(\Omega)$

$$-\frac{1}{2}\int_{\partial\Omega} (\mathbf{v}_\mathrm{f} \cdot \mathbf{n})|\psi|^2 \, dS_\mathbf{x} + \int_\Omega \left[-\frac{1}{2}\mathrm{div}(\mathbf{v}_\mathrm{f}) + k_\mathrm{s} g_\mathrm{s}\right] |\psi|^2 \, d\mathbf{x} = 0,$$

that is, $\phi_1 = \phi_2$ in view of the signs. □

$W^{1,q}$ *regularity.* By elliptic regularity, $\phi_\varepsilon \in W^{3,q}(\Omega)$, since the source in (60) belongs to $W^{1,q}(\Omega)$ and Ω is a C^3 domain. Following [33], we differentiate (60) with respect to x_k, multiply by $h(\phi_\varepsilon)\phi_{x_k}$ for $h(\phi_\varepsilon) = (|\nabla\phi_\varepsilon|^2 + \delta)^{(q-2)/2}$, add k and integrate over Ω to get

$$\begin{aligned} -\varepsilon \int_\Omega \Delta(\nabla\phi_\varepsilon) \cdot h(\phi_\varepsilon)\nabla\phi_\varepsilon \, d\mathbf{x} + \int_\Omega k_\mathrm{s} g_\mathrm{s} h(\phi_\varepsilon)|\nabla\phi_\varepsilon|^2 \, d\mathbf{x} \\ - \int_\Omega v_{\mathrm{f},i}\phi_{\varepsilon,x_i x_k} h(\phi_\varepsilon)\phi_{\varepsilon,x_k} d\mathbf{x} - \int_\Omega v_{\mathrm{f},i,x_k}\phi_{\varepsilon,x_i} h(\phi_\varepsilon)\phi_{\varepsilon,x_k} d\mathbf{x} \\ - \int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f}) h(\phi_\varepsilon)|\nabla\phi_\varepsilon|^2 d\mathbf{x} - \int_\Omega \nabla(\mathrm{div}(\mathbf{v}_\mathrm{f})) \cdot h(\phi_\varepsilon)\phi_\varepsilon \nabla\phi_\varepsilon \, d\mathbf{x} = 0. \end{aligned}$$

Sum over repeated indices is intended. Notice that Lemma 3.1 from [33] holds in our framework for our thin domains, so that the first term is nonnegative. Notice that $((|\nabla\phi_\varepsilon|^2 + \delta)^{q/2})_{x_i} = q h(\phi_\varepsilon)\phi_{\varepsilon,x_k}\phi_{\varepsilon,x_i,x_k}$. Integrating by parts, the third term becomes

$$\frac{1}{q}\int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f})(|\nabla\phi_\varepsilon|^2 + \delta)^{q/2} d\mathbf{x} - \frac{1}{q}\int_{\partial\Omega} (|\nabla\phi_\varepsilon|^2 + \delta)^{q/2} \mathbf{v}_\mathrm{f} \cdot \mathbf{n} \, dS_\mathbf{x}.$$

Putting all together we get

$$\begin{aligned} \int_\Omega k_\mathrm{s} g_\mathrm{s} h(\phi_\varepsilon)|\nabla\phi_\varepsilon|^2 \, d\mathbf{x} \le -\frac{1}{q}\int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f})(|\nabla\phi_\varepsilon|^2 + \delta)^{q/2} d\mathbf{x} \\ + \int_\Omega v_{\mathrm{f},i,x_k}\phi_{\varepsilon,x_i}\phi_{\varepsilon,x_k} h(\phi_\varepsilon) \, d\mathbf{x} + \int_\Omega \mathrm{div}(\mathbf{v}_\mathrm{f}) h(\phi_\varepsilon)|\nabla\phi_\varepsilon|^2 \, d\mathbf{x} \\ + \int_\Omega \nabla(\mathrm{div}(\mathbf{v}_\mathrm{f})) \cdot h(\phi_\varepsilon)\phi_\varepsilon\nabla\phi_\varepsilon \, d\mathbf{x}. \end{aligned}$$

We let $\delta \to 0$ and use that $\|\nabla\mathbf{v}_\mathrm{f}\|_{[L^\infty]^{n^2}}$ is small enough to find

$$\frac{1}{2} k_\mathrm{s} g_\mathrm{s} \int_\Omega |\nabla\phi_\varepsilon|^q \, d\mathbf{x} \le \|\nabla(\mathrm{div}(\mathbf{v}_\mathrm{f}))\|_{L^q} \||\nabla\phi_\varepsilon|\|_{L^q}^{q-1},$$

which yields the bound we seek letting $\varepsilon \to 0$. □

6. Well posedness results for the full model with a known boundary dynamics

Once we have analyzed the different submodels, we consider the whole system when the boundary of the domains Ω^t moves with time according to a given dynamics.

$$\begin{aligned}
&\mu\Delta\mathbf{u}_\mathrm{s} + (\mu+\lambda)\nabla\mathrm{div}(\mathbf{u}_\mathrm{s}) - \nabla p = \Pi\nabla\phi_\mathrm{s}, && \text{in } \Omega^t,\\
&\mu\Delta\mathbf{v}_\mathrm{s} + (\mu+\lambda)\nabla\mathrm{div}(\mathbf{v}_\mathrm{s}) = \nabla p', && \text{in } \Omega^t,\\
&k_\mathrm{h}\Delta p = \mathrm{div}(\mathbf{v}_\mathrm{s}), && \text{in } \Omega^t,\\
&\Delta p' = (2\mu+\lambda)\Delta e', && \text{in } \Omega^t,\\
&p = p_\mathrm{ext}, \quad p = p'_\mathrm{ext} && \text{on } \Gamma^t,\\
&\mathbf{u}_\mathrm{s} = 0, \quad \mathbf{v}_\mathrm{s} = 0, && \text{on } \Gamma^t_-,\\
&(\hat{\boldsymbol{\sigma}}(\mathbf{u}_\mathrm{s}) - (p + \Pi\phi_\mathrm{s})\mathbf{I})\mathbf{n} = \mathbf{g}, && \text{on } \Gamma^t_+,\\
&(\hat{\boldsymbol{\sigma}}(\mathbf{v}_\mathrm{s}) - p'\mathbf{I})\mathbf{n} = \mathbf{g}'(\nabla\mathbf{u}_\mathrm{s}), && \text{on } \Gamma^t_+,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\mathrm{div}(-\mathbf{v}_\mathrm{f}\phi_\mathrm{f}) + k_\mathrm{s}g_\mathrm{s}\phi_\mathrm{f} = k_\mathrm{s}g_\mathrm{s}, && \text{in } \Omega^t,\\
&\mathbf{v}_\mathrm{f} = -\xi_\infty\nabla p + \mathbf{v}_\mathrm{s}, \quad \phi_\mathrm{f} + \phi_\mathrm{s} = 1, && \text{in } \Omega^t,
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\frac{de'}{dt} = k_\mathrm{h}(2\mu+\lambda)\Delta e', && \text{in } \Omega^t,\\
&e' = e_\mathrm{ext}, && \text{on } \Gamma^t,\\
&e'(0) = e_0, && \text{on } \Omega^0,
\end{aligned} \tag{63}$$

$$\begin{aligned}
&-d\Delta c + \mathrm{div}(\mathbf{v}_\mathrm{f}c) = -k_\mathrm{c}g_\mathrm{c}\phi_\mathrm{s}, && \text{in } \Omega^t,\\
&c = c_0 && \text{on } \Gamma^t_-,\\
&\frac{\partial c}{\partial \mathbf{n}} = 0 && \text{on } \Gamma^t_+.
\end{aligned} \tag{64}$$

We keep the simplifying assumptions made in Section 5, $k_\mathrm{h}(\phi_\mathrm{s}) = k_\mathrm{h} > 0$, $k_\mathrm{h}(\phi_\mathrm{s})/\phi_\mathrm{f} = \xi_\infty > 0$, $c/(c+K_\mathrm{s}) = g_\mathrm{s} > 0$ and$c/(c+K_\mathrm{c}) = g_\mathrm{c} > 0$. As in Section 5.1, we change notation in system (1) and conditions (3): $p-\pi$ becomes p while p becomes $p+\pi$ in the equations for $\mathbf{u}_\mathrm{s}$ and p. The equations for $\mathbf{v}_\mathrm{s}$ and p' follow from Section 3. Equations for ϕ_f, c, $\mathbf{v}_\mathrm{f}$, ϕ_s correspond to (1), (2) and (3). Equation (63) comes from (7) neglecting a small order term, as explained before. We have the following result.

Theorem 6.1. *Let $\Omega^t \subset \Omega \subset \mathbb{R}^n$, $n = 2,3$, $t \in [0,T]$, be a family of open bounded C^4 thin domains, for which $\mathbf{n} \sim \mathbf{e}_n$. The lower boundary Γ_- is fixed, while the upper boundary Γ^t_+ is obtained deforming Γ^0_+ along a vector field $\nu(\mathbf{x}) \in C(\overline{\Omega}) \cap C^4(\Omega)$. Assume that*

- $e_\mathrm{ext}(t)$, $\mathbf{g}(t)$, $\mathbf{g}'(t)$, $p_\mathrm{ext}(t)$, $p'_\mathrm{ext}(t)$, $c_0(t) \in C([0,T])$, $e_0 \in L^2(\Omega^0) \cap L^q(\Omega^0)$, *for $q > n$,*

- e_{ext}, $\mathbf{g}'$, Π *and* p_{ext} *are small enough.*

Given positive constants μ, λ, Π, k_{h}, k_{s}, k_{c}, g_{s}, g_{c}, ξ_∞, *and* d *large enough, system (61)-(64) admits a unique solution* $e' \in H^2(\Omega^t) \cap W^{2,q}(\Omega^t)$, $\mathbf{u}_{\mathrm{s}} \in [H^2(\Omega^t)]^n \cap W^{2,q}(\Omega^t)$, $\mathbf{v}_{\mathrm{s}}, \mathbf{v}_{\mathrm{f}} \in [H^3(\Omega^t)]^n$, $p \in H^4(\Omega^t)$, $p' \in H^2(\Omega^t)$, $\phi_{\mathrm{f}}, \phi_{\mathrm{s}} \in H^1(\Omega^t) \cap W^{1,q}(\Omega^t)$, $c \in H^2(\Omega^t)$, *for* $q > n$, *satisfying* $c \geq 0$ *and* $0 \leq \phi_{\mathrm{f}}, \phi_{\mathrm{s}} \leq 1$, $t \in [0,T]$. *Moreover, the norms of the solutions are bounded in terms of the parameters and data of the problem.*

Proof. Assume first that $\mathbf{g}'(\nabla \mathbf{u}_{\mathrm{s}})$ does not depend on $\mathbf{u}_{\mathrm{s}}$. Then, the result is a consequence of Corollary 4.3, Theorems 5.1-5.3 and Sobolev embeddings [34] (neither L^q regularity nor conditions on the domain geometry nor smallness assumptions are needed). We calculate the unknowns according to the sequence e', p', $\mathbf{v}_{\mathrm{s}}$, p, $\mathbf{v}_{\mathrm{f}}$, ϕ_{f}, ϕ_{s}, $\mathbf{u}_{\mathrm{s}}$, and c.

When $\mathbf{g}'(\nabla \mathbf{u}_{\mathrm{s}})$ does depend on $\mathbf{u}_{\mathrm{s}}$, we construct e' thanks to Corollary 4.3. For each fixed $t > 0$, $e' \in H^2(\Omega^t) \cap W^{2,q}(\Omega^t)$ and we can construct $p' \in H^2(\Omega^t) \cap W^{2,q}(\Omega^t)$. Next, we solve the quasi-stationary system by means of an iterative scheme. At each step ℓ, we freeze $\Pi \nabla \phi_{\mathrm{s}}^{(\ell-1)}$ in the equation for $\mathbf{u}_{\mathrm{s}}^{(\ell)}$ and $\mathbf{g}'(\nabla \mathbf{u}_{\mathrm{s}}^{\ell-1})$ in the boundary condition for $\mathbf{v}_{\mathrm{s}}^{(\ell)}$. Initially, we set $\phi_{\mathrm{s}}^{(0)} = \phi_\infty \in (0,1)$ constant and $\phi_{\mathrm{f}}^{(0)} = 1 - \phi_\infty$. We set $\mathbf{u}^{(0)} = 0$. Theorem 5.1, Theorem 5.2, Theorem 5.3 guarantee the existence of $\mathbf{v}_{\mathrm{s}}^{(1)}$, $p^{(1)}$, $\mathbf{u}_{\mathrm{s}}^{(1)}$, $\mathbf{v}_{\mathrm{f}}^{(1)}$, $\phi_{\mathrm{f}}^{(1)}$, $\phi_{\mathrm{s}}^{(1)}$, and $c^{(1)}$, with the stated regularity.

In a similar way, given all the fields at step $\ell - 1$, we can construct the solutions for step ℓ. Notice that $\mathbf{v}_{\mathrm{f}}^{(\ell-1)} \in W^{2,q}$ implies $\mathbf{v}_{\mathrm{f}}^{(\ell-1)} \in W^{1,\infty}(\Omega)$ and $\mathbf{v}_{\mathrm{f}}^{(\ell-1)} \in C(\overline{\Omega})$. To apply Theorem 5.3 we also need to satisfy smallness and sign assumptions that we will consider later. Assuming they hold, we get for the elliptic system involving $\mathbf{v}_{\mathrm{s}}^{(\ell)}$, $\mathbf{u}_{\mathrm{s}}^{(\ell)}$, $p^{(\ell)}$ and for the transport equation for $\phi_{\mathrm{s}}^{(\ell)}$

$$
\begin{aligned}
\|p^{(\ell)}\|_{H^2(\Omega^t)} + \|\mathbf{v}_{\mathrm{s}}^{(\ell)}\|_{H^2(\Omega^t)} + \|\mathbf{u}_{\mathrm{s}}^{(\ell)}\|_{H^2(\Omega^t)} \leq C_1^t [\Pi \|\nabla \phi_{\mathrm{s}}^{(\ell-1)}\|_{L^2(\Omega^t)} \\
+ \|\nabla p'\|_{L^2(\Omega^t)} + \|p_{\mathrm{ext}}\|_{H^{3/2}(\Gamma_+^t)} + \|\mathbf{g}'(\nabla \mathbf{u}_{\mathrm{s}}^{(\ell-1)})\|_{H^{1/2}(\Gamma_+^t)} + \|\mathbf{g}\|_{H^{1/2}(\Gamma_+^t)}],
\end{aligned}
$$

$$
\begin{aligned}
\|p^{(\ell)}\|_{W^{2,q}(\Omega^t)} + \|\mathbf{v}_{\mathrm{s}}^{(\ell)}\|_{W^{2,q}(\Omega^t)} + \|\mathbf{u}_{\mathrm{s}}^{(\ell)}\|_{W^{2,q}(\Omega^t)} \leq C_2^t [\Pi \|\nabla \phi_{\mathrm{s}}^{(\ell-1)}\|_{L^q(\Omega^t)} + \\
\|\nabla p'\|_{L^q(\Omega^t)} + \|p_{\mathrm{ext}}\|_{W^{1-\frac{1}{q},q}(\Gamma_+^t)} + \|\mathbf{g}'(\nabla \mathbf{u}_{\mathrm{s}}^{(\ell-1)})\|_{W^{1-\frac{1}{q},q}(\Gamma_+^t)} + \|\mathbf{g}\|_{W^{1-\frac{1}{q},q}(\Gamma_+^t)}],
\end{aligned}
$$

$$
\|p^{(\ell)}\|_{W^{3,q}(\Omega^t)} \leq C_3^t [\|\mathbf{v}_{\mathrm{s}}^{(\ell)}\|_{W^{1,q}(\Omega^t)} + \|p_{\mathrm{ext}}\|_{W^{3-1/q,q}(\Gamma^t)}]
$$

$$
\|\mathbf{v}_{\mathrm{f}}^{(\ell)}\|_{W^{2,q}(\Omega^t)} \leq \xi_\infty \|p^{(\ell)}\|_{W^{3,q}(\Omega^t)} + \|\mathbf{v}_{\mathrm{s}}^{(\ell)}\|_{W^{2,q}(\Omega^t)}
$$

$$
\frac{1}{2} k_{\mathrm{s}} g_{\mathrm{s}} \|\nabla \phi_{\mathrm{f}}^{(\ell)}\|_{L^q} \leq \|\nabla \mathrm{div}(\mathbf{v}_{\mathrm{f}}^{(\ell)})\|_{[L^q]^n}.
$$

Notice that $\nabla \phi_{\mathrm{f}}^{(\ell)} = -\nabla \phi_{\mathrm{s}}^{(\ell)}$. Combining the above inequalities, and provided Π and $\mathbf{g}'$ are small enough, we obtain an upper bound for $\|\mathbf{v}_{\mathrm{f}}^{(\ell)}\|_{W^{2,q}(\Omega^t)}$, $\|\mathbf{v}_{\mathrm{s}}^{(\ell)}\|_{W^{2,q}(\Omega^t)}$, $\|p^{(\ell)}\|_{W^{2,q}(\Omega^t)}$, $\|\phi_{\mathrm{s}}\|_{W^{1,q}(\Omega^t)}$, in terms of constants depending on the problem data and parameters, and also on time, but remain bounded in time for $t \in [0, T]$.

We guarantee by induction the smallness of $\|\mathbf{v}_\mathrm{f}^{(\ell)}\|_{[W^{1,\infty}]}$ and $\mathrm{div}(\mathbf{v}_\mathrm{f}^{(\ell)}) \leq 0$, $\mathbf{v}_\mathrm{f}^{(\ell)} \cdot \mathbf{n} \leq 0$. Initially, $\phi_\mathrm{s}^{(0)}$ is constant and $\nabla\phi_\mathrm{s}^{(0)} = 0$. We construct $\mathbf{v}_\mathrm{s}^{(1)}$ and $p^{(1)}$ in such a way that $\|\mathbf{v}_\mathrm{s}^{(1)}\|_{[W^{2,q}]^n}$, $\|p^{(1)}\|_{[W^{3,q}]^n}$ and $\|\mathbf{v}_\mathrm{f}^{(1)}\|_{[W^{2,q}]^n}$ are bounded in terms of the problem parameters and data. By Sobolev injections for $n < q < \infty$, $\|\mathbf{v}_\mathrm{s}^{(1)}\|_{[W^{1,\infty}]^n}$ satisfies a similar bound, and can be made as small as required by making $\mathbf{g}'$ and p_ext small. Then, $\|\nabla\phi_\mathrm{f}^{(1)}\|_{L^q}$ is bounded by $\|\mathbf{v}_\mathrm{f}^{(1)}\|_{[W^{2,q}]^n}$ and is equally small. Furthermore, $\mathrm{div}(\mathbf{v}_\mathrm{f}^{(1)})\phi_\mathrm{f}^{(1)} + \mathbf{v}_\mathrm{f}^{(1)}\nabla\phi_\mathrm{f}^{(1)} = -k_\mathrm{s} g_\mathrm{s} \phi_\mathrm{f}^{(1)} \leq 0$. Since $\mathbf{v}_\mathrm{f}^{(1)}$ and $\nabla\phi_\mathrm{f}^{(1)}$ are small compared to $-k_\mathrm{s} g_\mathrm{s} \phi_\mathrm{f}^{(1)} \leq 0$ which is almost constant. Thus, $\mathrm{div}(\mathbf{v}_\mathrm{f}^{(1)}) \leq 0$. Finally, $\int_A \mathrm{div}(\mathbf{v}_\mathrm{f}^{(1)})\, d\mathbf{x} = \int_{\partial A} \mathbf{v}_\mathrm{f}^{(1)} \cdot \mathbf{n}\, dS_\mathbf{x} \leq 0$ for all $A \subset \Omega$ so that $\mathbf{v}_\mathrm{f}^{(1)} \cdot \mathbf{n} \leq 0$ on $\partial\Omega$.

By induction, if $\|\mathbf{v}_\mathrm{f}^{(\ell-1)}\|_{[W^{1,\infty}]^n}$ is small and $\mathbf{v}_\mathrm{f}^{(\ell-1)}$ satisfies the sign conditions, we can repeat the argument to show that this holds for $\mathbf{v}_\mathrm{f}^{(\ell)}$ too and that it also satisfies the sign conditions. We need to estimate $\|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f}^{(\ell-1)})\|_{[L^q]^n}$, which is possible since Π is small.

These estimates allow us to extract subsequences converging weakly to limits $\mathbf{v}_\mathrm{s}$, $\mathbf{u}_\mathrm{s}$, p, ϕ_s satisfying variational formulations of the equations. Problem (64) is already studied in Theorem 5.2. $\square$

7. Extensions

The results obtained in sections 5 and 6 are derived under a few assumptions. We discuss here modifications to deal with the original functions $g_\mathrm{s}(c)$, $g_\mathrm{c}(c)$ and $k_\mathrm{h}(\phi_\mathrm{f})$. For completeness, we also discuss the implications of dropping the modeling assumption $\rho_\mathrm{f} = \rho_\mathrm{s} = \rho$ and removing the technical sign assumptions on $\mathbf{v}_\mathrm{f}$.

7.1. Variable sources $g_\mathrm{c}(c)$ and $g_\mathrm{s}(c)$

When $g_\mathrm{c}(c) = \frac{c}{c+K_\mathrm{c}}$, we need to ensure that c has a lower positive bound to proceed. A simple argument works for thin enough films, the case under study here.

Theorem 7.1. *Under the hypotheses of Theorem 5.2. there exists a unique nonnegative solution $c \in H^1(\Omega)$ of*

$$\begin{array}{rcll} -d\Delta c + \mathrm{div}(\mathbf{v}_\mathrm{f} c) & = & -k_\mathrm{c} \frac{c}{c+K_\mathrm{c}} \phi_\mathrm{s}, & \mathbf{x} \in \Omega, \\ c & = & c_0 & \mathbf{x} \in \Gamma_-, \\ \dfrac{\partial c}{\partial \mathbf{n}} & = & 0 & \mathbf{x} \in \Gamma_+, \end{array} \tag{65}$$

provided d is sufficiently large, for $K_\mathrm{c} > 0$. Moreover, $\|c\|_{H^2}$ and $\|c\|_{L^\infty}$ are bounded from above by constants depending on d, k_c, K_c, Ω, c_0, $\|\mathbf{v}_\mathrm{f}\|_{[L^2]^n}$ and $\|\phi_\mathrm{s}\|_{L^2}$. If $\mathrm{div}(\mathbf{v}_\mathrm{f}) \leq 0$, then c is stricly positive in Ω and, provided that $0 \leq$

$\phi_s \le 1$ *and* $M = \max\{x_n \,|\, (x_1, \dots, x_{n-1}, x_n) \in \Omega\} < \frac{2dK_c}{k_c c_{\max}} c_0$*, we also have* $c \ge -\frac{k_c c_{\max}}{2dK_c} M + c_0 > 0$ *in* $\overline{\Omega}$.

Proof. We consider the iterative scheme: Find $\tilde{c}^{(m+1)} = c^{(m+1)} - c_0 \in H^1_{0,-}(\Omega)$ such that

$$b_m(\tilde{c}^{(m+1)}, w) := d\int_\Omega \nabla\tilde{c}^{(m+1)} \cdot \nabla w \, d\mathbf{x} - \int_\Omega \mathbf{v}_f \tilde{c}^{(m+1)} \cdot \nabla w \, d\mathbf{x} + \int_{\Gamma_+} \tilde{c}^{(m+1)} w \mathbf{v}_f \cdot \mathbf{n} \, dS_\mathbf{x} + \int_\Omega a_m \tilde{c}^{(m+1)} w \, d\mathbf{x} = c_0 \int_\Omega \mathbf{v}_f \cdot \nabla w \, d\mathbf{x} - c_0 \int_\Omega a_m w \, d\mathbf{x} := L_m(w),$$

for all $w \in H^1_{0,-}(\Omega)$, where $a_m = k_c \frac{\phi_s}{c^{(m)} + K_c}$. Setting $c^{(0)} = c_0 > 0$, we argue as in Theorem 5.2 and construct $c^{(1)} = \tilde{c}^{(1)} + c_0 \ge 0$. Then $0 \le a_1 \le k_c \|\phi_s\|_\infty / K_c$. Let us assume we have $\tilde{c}^{(m)} \in H^1_{0,-}(\Omega)$, $c^{(m)} = \tilde{c}^{(m)} + c_0 \ge 0$ and $0 \le a_m \le k_c \|\phi_s\|_\infty / K_c$. We argue as in Theorem 5.2 and construct a unique solution $\tilde{c}^{(m+1)} \in H^1_{0,-}(\Omega)$. Setting $w = \tilde{c}^{(m+1)}$ we get, thanks to Sobolev and trace inequalities, [24, 27],

$$\gamma(d, \Omega) \|\tilde{c}^{(m+1)}\|_{H^1} \le c_0 \left[\|\mathbf{v}_f\|_{[L^2]^n} + \frac{k_c}{K_c} \|\phi_s\|_{L^2} \right],$$

for d large enough and $\mathbf{v}_f$ small enough compared to d, $\gamma(d, \Omega) > 0$ not depending on m. Morover, the function $c^{(m+1)} = \tilde{c}^{(m+1)} + c_0$ satisfies

$$d \int_\Omega |\nabla c^{(m+1)-}|^2 \, d\mathbf{x} - \int_\Omega \mathbf{v}_f \cdot c^{(m+1)-} \nabla c^{(m+1)-} \, d\mathbf{x} + \int_{\Gamma_+} |c^{(m+1)-}|^2 \mathbf{v}_f \cdot \mathbf{n} \, dS_\mathbf{x} + \int_\Omega a_m |c^{(m+1)-}|^2 \, d\mathbf{x} = 0.$$

Coercivity for d large enough implies $c^{(m+1)-} = 0$, $c^{(m+1)} \ge 0$. By induction, the estimates on a_m and $\|\tilde{c}^{(m)}\|_{H^1}$ are valid for all m. When Ω is a C^2 domain, elliptic regularity theory implies $\tilde{c}^{(m)} \in H^2(\Omega)$. Sobolev embeddings in dimension $n = 2, 3$ imply $\tilde{c}^{(m)} \in L^\infty(\Omega)$ [24]. All norms are bounded in terms of the L^2 norms of $\mathbf{v}_f$ and ϕ_s. We can extract a subsequence $\tilde{c}^{(m_j)}$ tending to a limit $\tilde{c}$ weakly in H^2, weakly star in L^2, strongly in H^1 and pointwise. This allows us to pass to the limit in the variational equations for $\tilde{c}^{(m)}$ obtaining a solution $c = \tilde{c} + c_0 \ge 0$ for (65). The limit $\tilde{c}$ inherits the H^2 and L^∞ bounds.

In case we had two such solutions c_1 and c_2, setting $c = c_1 - c_2$ we get

$$-d\Delta c + \mathrm{div}(\mathbf{v}_f c) = -k_c K_c \phi_s \frac{c}{(c_1 + K_c)(c_2 + K_c)}, \quad \mathbf{x} \in \Omega,$$

with $c = 0$ on Γ_- and $\frac{\partial c}{\partial \mathbf{n}} = 0$ on Γ_+. Writing the variational form with test function c, we find $c = 0$ for d large as before.

Let us prove now strict positivity. If $\mathrm{div}(\mathbf{v}_f) \le 0$ and $0 \le \phi_s \le 1$, then $-d\Delta c + \mathbf{v}_f \cdot \nabla c = -k_c \frac{c}{c + K_c} \phi_s - \mathrm{div}(\mathbf{v}_f) c \ge -\frac{k_c}{K_c} c \ge -\frac{k_c}{K_c} \|c\|_{L^\infty}$. By elliptic comparison principles [31], $c > \underline{c} + c_0$ where $-d\Delta \underline{c} + \mathbf{v}_f \cdot \nabla \underline{c} = -\frac{k_c}{K_c} c$ in Ω with $\underline{c} = 0$

on Γ_- and $\frac{\partial \underline{c}}{\partial \mathbf{n}} = 0$ on Γ_+. For $\mathbf{v}_\mathrm{f} \in C(\overline{\Omega})$, $\underline{c} \in C^2(\Omega) \cap C(\overline{\Omega})$ and $d\Delta \underline{c} = \frac{k_\mathrm{c}}{K_\mathrm{c}} \underline{c} > 0$ at any strict minimum in Ω. A uniform lower bound is obtained observing that the problem $-d\Delta w = -\frac{k_\mathrm{c}}{K_\mathrm{c}} \|c\|_{L^\infty}$ in Ω with boundary conditions $w = 0$ on Γ_- and $\frac{\partial w}{\partial \mathbf{n}} = 0$ on Γ_+ admits subsolutions $\underline{w}(\mathbf{x}) = \frac{k_\mathrm{c}\|c\|_{L^\infty}}{dK_\mathrm{c}} x_n(\frac{x_n}{2} - M)$, where x_n is the component in the normal direction to Γ_-. Taking $\frac{k_\mathrm{c}\|c\|_{L^\infty}}{dK_\mathrm{c}}(x_n - M) < 0$, that is, M larger than the longest vertical diameter in Ω, we have $\frac{\partial \underline{w}}{\partial \mathbf{n}} < 0$ on Γ_+. Then, $c \geq \underline{\underline{c}} + c_0 \geq w + c_0 \geq \underline{w} + c_0 \geq -\frac{k_\mathrm{c}\|c\|_{L^\infty}}{2dK_\mathrm{c}} M + c_0 > 0$ in $\overline{\Omega}$, where $\underline{\underline{c}}$ solves $-d\Delta \underline{\underline{c}} + \mathbf{v}_\mathrm{f} \cdot \nabla \underline{\underline{c}} = -\frac{k_\mathrm{c}}{K_\mathrm{c}} \|c\|_{L^\infty}$ with zero boundary conditions. $\square$

We now revisit Theorem 5.3 replacing constant g_s with $g_\mathrm{s}(c) = \frac{c}{c+K_\mathrm{s}}$.

Theorem 7.2. *Under the hypotheses of Theorem 5.3, assume that $c \in W^{1,q}(\Omega) \cap C(\overline{\Omega})$, $q > n$, is a strictly positive function, bounded from below by a positive constant $c_{\min}$, that is, $c \geq c_{\min} > 0$ in Ω and $\nabla \mathbf{v}_\mathrm{f} \in [L^\infty(\Omega)]^{n^2}$ with $\|\nabla \mathbf{v}_\mathrm{f}\|_{[L^\infty]^{n^2}}$ small enough compared to $c_{\min}$. Then, there is a unique solution $\phi_\mathrm{f} \in H^1(\Omega)$ of*

$$\mathrm{div}(-\mathbf{v}_\mathrm{f}(\mathbf{x})\phi_\mathrm{f}) + k_\mathrm{s} \frac{c}{c+K_\mathrm{s}} \phi_\mathrm{f} = k_\mathrm{s} \frac{c}{c+K_\mathrm{s}}, \quad \mathbf{x} \in \Omega, \tag{66}$$

such that

$$\frac{1}{2} \frac{c_{\min} k_\mathrm{s}}{\|c\|_{L^\infty} + K_\mathrm{s}} \|\nabla \phi\|_{L^2} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^2]^n} + 2k_\mathrm{s}K_\mathrm{s}\|\nabla c\|_{[L^2]^n},$$
$$\frac{1}{2} \frac{c_{\min} k_\mathrm{s}}{\|c\|_{L^\infty} + K_\mathrm{s}} \|\nabla \phi\|_{L^q} \leq \|\nabla \mathrm{div}(\mathbf{v}_\mathrm{f})\|_{[L^q]^n} + 2k_\mathrm{s}K_\mathrm{s}\|\nabla c\|_{[L^q]^n}.$$

Moreoever, $0 \leq \phi_\mathrm{f} \leq 1$ on Ω, and ϕ does not vanish in sets of positive measure.

Proof. We follow exactly the same steps as in the proof of Theorem 5.3 with $g_\mathrm{s} = g_\mathrm{s}(c) = \frac{c}{c+K_\mathrm{s}} = 1 - \frac{K_\mathrm{s}}{c+K_\mathrm{s}}$ using the fact that $\frac{c_{\min}}{\|c\|_{L^\infty}+K_\mathrm{s}} \leq g_\mathrm{s}(c) \leq 1$ and including the terms that involve $\nabla g_\mathrm{s} = \nabla g_\mathrm{s}(c) = \frac{K_\mathrm{s}\nabla c}{(c+K_\mathrm{s})^2}$. $\square$

Thanks to Theorems 7.1 and 7.2, Theorem 6.1 still holds replacing constant g_s and g_c by variable $g_\mathrm{s}(c)$ and $g_\mathrm{c}(c)$ under the geometrical assumptions on Ω stated in Theorem 7.1, with standard modifications in the proof.

7.2. Variable permeability

Establishing similar results when we allow for slightly varying smooth coefficients $k_\mathrm{h}(\phi_\mathrm{f}) = \frac{\phi_\mathrm{f}^2 \zeta_\infty}{\mu_\mathrm{f}}$ and $\xi(\phi_\mathrm{f}) = \frac{k_\mathrm{h}(\phi_\mathrm{f})}{\phi_\mathrm{f}}$ requires surmounting a number of difficulties.

First, in equations (6) and (7), $k_\mathrm{h} = k_\mathrm{h}(\phi_\mathrm{s}(\mathbf{x}, t))$ and new source terms appear, of the form $k_\mathrm{h}' \nabla \phi_\mathrm{s} \nabla(p - \pi(\phi_\mathrm{s}))$. These additional terms are expected to be smaller than the rest and discarded. Thus, we could keep the same set of equations except that k_h now depends on ϕ_s. Also, the term $k_\mathrm{h}\Delta p_t$ in Corollary 2.3 is replaced by $\mathrm{div}(k_\mathrm{h}(\phi_\mathrm{s})\nabla p_t)$. Other additional terms are expected to be smaller and could be discarded.

We need to revisit Section 4 with variable coefficients $\kappa(\mathbf{x},t) = k_{\rm h}(\phi_{\rm s}(\mathbf{x},t))$ in the time dependent problem (48). This coefficient is bounded, but not C^1, since the functions $\phi_{\rm s}$ we construct lack that regularity. We should regularize $k_{\rm h}(\phi_{\rm s}(\mathbf{x},t))$, apply the results in Section 4 to the regularized problems and seek the solutions we need in some weak limit, which is a non trivial issue.

In Section 5.1, Theorem 5.1 still holds if we replace $k_{\rm h}\Delta p$, $k_{\rm h}$ constant, by $\mathrm{div}(k_{\rm h}\nabla p)$, $k_{\rm h} \in L^\infty$.

To construct solutions of the full final model, one could consider a more refined iterative scheme and proceed as in the proof of Theorem 6.1 making use of the updated auxiliary results.

$$\begin{aligned}
&\mu\Delta\mathbf{u}_{\rm s}^{(\ell)} + (\mu+\lambda)\nabla\mathrm{div}(\mathbf{u}_{\rm s}^{(\ell)}) - \nabla p^{(\ell)} = \Pi\nabla\phi_{\rm s}^{(\ell-1)}, && \text{in } \Omega^t,\\
&\mu\Delta\mathbf{v}_{\rm s}^{(\ell)} + (\mu+\lambda)\nabla\mathrm{div}(\mathbf{v}_{\rm s}^{(\ell)}) = \nabla p^{'(\ell)}, && \text{in } \Omega^t,\\
&\mathrm{div}(k_{\rm h}(\phi_{\rm s}^{(\ell-1)})p^{'(\ell)}) = \mathrm{div}(\mathbf{v}_{\rm s}^{(\ell)}), && \text{in } \Omega^t,\\
&\Delta p^{'(\ell)} = (2\mu+\lambda)\Delta e^{'(\ell)}, && \text{in } \Omega^t,\\
&p^{(\ell)} = p_{\rm ext}, \quad p^{(\ell)} = p'_{\rm ext} && \text{on } \Gamma^t,\\
&\mathbf{u}_{\rm s}^{(\ell)} = 0, \quad \mathbf{v}_{\rm s}^{(\ell)} = 0, && \text{on } \Gamma^t_-,\\
&(\hat{\boldsymbol\sigma}(\mathbf{u}_{\rm s}^{(\ell)}) - (p^{(\ell)} + \Pi\phi_{\rm s}^{(\ell-1)})\mathbf{I})\mathbf{n} = \mathbf{g}, && \text{on } \Gamma^t_+,\\
&(\hat{\boldsymbol\sigma}(\mathbf{v}_{\rm s}^{(\ell)}) - p^{'(\ell)}\mathbf{I})\mathbf{n} = \mathbf{g}'(\nabla\mathbf{u}_{\rm s}^{\ell-1}), && \text{on } \Gamma^t_+,
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\mathrm{div}(-\mathbf{v}_{\rm f}^{(\ell)}\phi_{\rm f}^{(\ell)}) + k_{\rm s}g_{\rm s}(c^{(\ell-1)})\phi_{\rm f}^{(\ell)} = k_{\rm s}g_{\rm s}(c^{(\ell-1)}), && \text{in } \Omega^t,\\
&\mathbf{v}_{\rm f}^{(\ell)} = -\xi_\infty(\phi_{\rm s}^{(\ell)})\nabla p^{(\ell)} + \mathbf{v}_{\rm s}^{(\ell)}, \quad \phi_{\rm f}^{(\ell)} + \phi_{\rm s}^{(\ell)} = 1, && \text{in } \Omega^t,
\end{aligned} \tag{68}$$

$$\begin{aligned}
&\frac{de^{'(\ell)}}{dt} = k_{\rm h}(\phi^{(\ell-1)})(2\mu+\lambda)\Delta e^{'(\ell)}, && \text{in } \Omega^t,\\
&e^{'(\ell)} = e_{\rm ext}, && \text{on } \Gamma^t,\\
&e^{'(\ell)}(0) = e_0, && \text{on } \Omega^0,
\end{aligned} \tag{69}$$

$$\begin{aligned}
&-d\Delta c^{(\ell)} + \mathrm{div}(\mathbf{v}_{\rm f}^{(\ell)}c^{(\ell)}) = -k_{\rm c}g_{\rm c}(c^{(\ell)})\phi_{\rm s}^{(\ell)}, && \text{in } \Omega^t,\\
&c^{(\ell)} = c_0 && \text{on } \Gamma^t_-,\\
&\frac{\partial c^{(\ell)}}{\partial\mathbf{n}} = 0 && \text{on } \Gamma^t_+.
\end{aligned} \tag{70}$$

The main difficulty consists in constructing pressures and volume fraction fields with high enough regularity.

7.3. Sign and size assumptions on $\mathbf{v}_{\rm f}$

We have obtained our results under the technical conditions $\mathrm{div}(\mathbf{v}_{\rm f}) \leq 0$, $\mathbf{v}_{\rm f}\cdot\mathbf{n} \leq 0$ and $\mathbf{v}_{\rm f}$ small compared to other parameters. These conditions allow us to construct solutions with a minimum regularity for the stationary transport equation with velocity fields $\mathbf{v}_{\rm f}$ and source terms. Existing abstract theories for this type of problems require the assumption $\mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} = 0$, which avoids the need of boundary conditions since characteristics do not touch the boundary. Here, we have somehow relaxed these assumptions, still avoiding boundary conditions. Establishing rigorous existence results of smooth enough solutions for stationary transport problems dropping these hypotheses is an open question and a difficult issue in transport theory. In our framework, existing asymptotic constructions of approximate solutions by perturbation techniques [12], end up approximating ϕ_{f} by a constant plus the solution of an elliptic problem. On one hand, this allows to consider general boundary conditions. On the other hand, it ensures regularity of the volume fractions and pressures. Under which conditions we can justify this procedure is unclear.

7.4. Variable density

If we consider for $\rho_{\mathrm{s}} \neq \rho_{\mathrm{f}}$, then the composite density of the mixture $\rho(\mathbf{x}) = \rho_{\mathrm{s}}\phi_{\mathrm{s}}(\mathbf{x}) + \rho_{\mathrm{f}}\phi_{\mathrm{f}}(\mathbf{x})$ varies in space. In our previous analysis is we need to multiply $r_{\mathrm{s}}(\phi_{\mathrm{s}}, c)$ in equation (10) for ϕ_{f} and $r_n(\phi_{\mathrm{s}}, c)$ in equation (11) for c by a constant factor $\frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{f}}}$. In the equations of motion, all terms in the equation for the solid phase are multiplied by ρ_{s} and all terms in the equation for the fluid phase are multiplied by ρ_{f}. Being constant, both scale out. Al the results we have established persist replacing k_{c} and k_{s} by $k_{\mathrm{c}}\frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{f}}}$ and $k_{\mathrm{c}}\frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{f}}}$.

A more relevant change arises in (12). The conservation law obtained when adding the balance equations for ϕ_{s} and ϕ_{f} becomes

$$\operatorname{div}(\mathbf{v}) = (1 - \rho_{\mathrm{s}}/\rho_{\mathrm{f}}) r_{\mathrm{s}}(\phi_{\mathrm{s}}, c),$$

incompressibility of the mixture is lost. The equation for p in (57) becomes

$$\operatorname{div}(k_{\mathrm{h}} \nabla p) = \operatorname{div}(\mathbf{v}_{\mathrm{s}}) - (1 - \rho_{\mathrm{s}}/\rho_{\mathrm{f}}) r_{\mathrm{s}}(\phi_{\mathrm{s}}, c).$$

The function r_{s} is bounded, however, its derivatives involve derivatives of ϕ_{s}, solutions of the stationary conservation law, whose regularity is limited. The regularity of the volume fraction fields is a major technical problem in the analysis of these models.

8. Discussion and conclusions

The study of biological aggregates and tissues often leads to complex mixture models, combining transport equations for volume fractions of different phases, with continuum models for mechanical behavior of the mixture and chemical species [19, 38, 39]. These models are set in domains that change with time, because cells grow, die and move and because of fluid transport within the biological network. Here, we have considered a fluid-solid mixture description of the spread of cellular systems called biofilms, which could be adapted to general tissues. These models involve different time scales, so that part of the equations are considered quasi-stationary, that is, they are stationary problems solved at different times in different domains and with some time dependent coefficients.

Such equations are coupled to time dependent problems set in moving domains and to variables not directly characterized by means of equations.

In this paper, we have developed mathematical frameworks to tackle some of the difficulties involved in the construction of solutions for these multiphysics systems and the study of their behavior. First, we have shown how to improve these models by characterizing time derivatives of solutions of stationary boundary value problems with varying coefficients set in moving domains in terms of complementary boundary value problems derived for them. In this way we obtain a quasi-stationary elliptic system for the mechanical variables of the solid phase, not only displacements and pressure, but also velocity, that can be solved at each time coupled to the other submodels. This option is more stable than evaluating velocities as quotients of differences of displacements calculated in meshes of different spatial domains. On one side, the error committed is easier to control. On the other side, the computational is cost smaller, since we use a single mesh at each time. Once we know the velocity of the solid phase and the pressure, the velocity of the fluid phase follows by a Darcy type law. Next, we have devised an strategy to construct solutions of an auxiliary class of time dependent linear diffusion problems set in moving domains with parametrizations satisfying a number of conditions. We are able to refer the model to a fixed domain and then solve by Galerkin type schemes. The complete model involves a quasi-stationary transport problem. We show that we can construct smooth enough solutions by a regularization procedure, under sign hypothesis on the fluid velocity field suggested by asymptotic solutions constructed in simple geometries. Once we know how to construct stable solutions of each submodel satisfying adequate regularity properties, an iterative scheme allows us to solve the full problem when the time evolution of the boundary of the spatial region occupied by the biological film is known.

In applications one must couple these models with additional lubrication type equations for the motion of the film boundary, see equation (4). Perturbation analysis [14] provides approximate solutions with selfsimilar dynamics for h. Establishing existence and regularity results for such complex models that can guide construction of reliable numerical solutions is a completely open problem. The techniques we have developed are general and can be applied in models with a similar structure arising in other biological applications.

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