Diversity of Lorentz-Zygmund spaces of operators defined by approximation numbers

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February 13, 2023

Dedicated to Professor Oleg Vladimirovich Besov on the occasion of his 90th birthday

2020 Math. Subj. Class. Primary: 47B10, 46M35. Secondary: 47B06, 46B70.
Key words: Spaces of operators defined by approximation numbers; dependence on the parameters; logarithmic interpolation spaces; Lorentz-Zygmund spaces.

Abstract

We prove the following dichotomy result for the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ of all operators $T \in \mathscr{L}(X,Y)$ whose approximation numbers belong to the Lorentz-Zygmund sequence spaces $\ell_{p,q}(\log \ell)_\alpha$: If X and Y are *infinite-dimensional* Banach spaces, then the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ with $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$ are all different from each other, but otherwise, if X or Y are *finite-dimensional*, they are all equal (to $\mathscr{L}(X,Y)$).

Moreover we show that the scale $\{\mathscr{L}^{(a)}_{\infty,q}(X,Y)\}_{0<q<\infty}$ is strictly increasing in q, where $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ is the space of all operators in $\mathscr{L}(X,Y)$ whose approximation numbers are in the limiting Lorentz sequence space $\ell_{\infty,q}$.

1 Introduction

Diversity of spaces of functions is a classical problem that has attracted the attention of many authors (see, for example, [32, 2.3.9]). In particular, Besov [5] has also contributions to this question. We are interested here in diversity, but for certain spaces of operators. Namely, for Banach spaces X, Y and parameters $0 < p < \infty$ and $0 < q \leq \infty$, let $\mathscr{L}^{(a)}_{p,q}(X,Y)$ be the space of all bounded linear operators $T \in \mathscr{L}(X,Y)$ whose approximation numbers $(a_n(T))$ belong to the Lorentz sequence space $\ell_{p,q}$. As one can see in the monographs by Pietsch [26, 27] and König [15], these spaces of operators play an important role in Operator Theory.

Intimately related with $\mathscr{L}^{(a)}_{p,q}(X,Y)$ are the spaces $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$, formed by all $T \in \mathscr{L}(X,Y)$ such that $\sum_{n=1}^{\infty} a_n(T)^q n^{-1} < \infty$, and the Lorentz-Zygmund spaces of operators $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$, consisting of all $T \in \mathscr{L}(X,Y)$ such that $(a_n(T)) \in \ell_{p,q}(\log \ell)_\alpha$, where $\ell_{p,q}(\log \ell)_\alpha$ is the Lorentz-Zygmund sequence space and $\alpha \in \mathbb{R}$. We refer to [6] for properties of the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$, and to [10, 9] for properties of $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$. Note that for Hilbert spaces H,K and for $q = 1$, the space $\mathscr{L}^{(a)}_{\infty,1}(H,K)$ was introduced by Macaev [18]. It is very useful in a number of questions concerning operators between Hilbert spaces (see [18, 13]).

If the operator T belongs to any of these spaces of operators, then one has information on the speed of convergence to zero of its approximation numbers $a_n(T)$ and therefore on the degree of compactness of T. As we will show later, if X and/or Y are finite-dimensional, then all these spaces of operators coincide with $\mathscr{L}(X,Y)$. It is well known that there exist Banach spaces with few operators on them (see, for example, [19]). Therefore it is important to know whether, for arbitrary infinite-dimensional spaces X and Y, these spaces of operators are always different whenever their parameters are distinct.

At this point there are two options, either to base the arguments on deep results on approximation numbers (see [22]), or on results of interpolation theory and basic properties of approximation numbers. Following this second way, Cwikel and the present authors have shown in [8, Theorem 4.3] that, if X and Y are infinite-dimensional, all spaces $\mathscr{L}^{(a)}_{p,q}(X,Y)$ with $0 < p < \infty$ and $0 < q \le \infty$ are different from each other. The arguments in [8] do not apply to the spaces $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ with $0 < q < \infty$. Via a different approach we show in Section 4 that for arbitrary infinite-dimensional Banach spaces X and Y the scale $\{\mathscr{L}^{(a)}_{\infty,q}(X,Y)\}_{0<q<\infty}$ is strictly increasing in q. To prove this, we use the representation of operators in $\mathscr{L}^{(a)}_{p,q}(X,Y)$ in terms of finite rank operators, which is an outstanding consequence of the description of $\mathscr{L}^{(a)}_{p,q}(X,Y)$ as approximation space (see [25]). As one can see in the recent paper by Pietsch [28], results of this type are of interest in the study of traces on operator ideals defined over the class of all Banach spaces.

In Section 4 we also consider the corresponding problem for Lorentz-Zygmund spaces of operators $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$. For this aim we distinguish several cases. Sometimes we base the arguments on our previous achievements with Cwikel [8], some other times we rely on the representation of the operators in $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ in terms of finite rank operators, and finally we also use ideas of interpolation theory.

The organization of the paper is as follows. Section 2 is devoted to Lorentz-Zygmund sequence spaces. Properties of Lorentz-Zygmund function spaces on non-atomic measures spaces have been studied in great detail in the literature (see, for example, [2, 11, 23]), but concerning Lorentz-Zygmund sequence spaces there are much less papers. These spaces are important special cases of the more general class of weighted Lorentz sequence spaces $\ell_{p,q}(w)$ with so-called quasi-logarithmic weights. Baumbach [1] investigated these spaces, in particular he completely clarified under which conditions there are embeddings between two spaces of this type. We specify his results to the subclass of Lorentz-Zygmund sequence spaces.

In Section 3 we deal with logarithmic perturbations $(A_0, A_1)_{\varrho,q}$ of the real interpolation method in the quasi-Banach setting. These are the most natural interpolation methods in the context of Lorentz-Zygmund spaces. Among other things, we prove that $(A_0, A_1)_{\varrho,q}$ contains a subspace isomorphic to ℓ_q provided that (A_0, A_1) is a quasi-Banach Gagliardo couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. This extends a result that Lévy [16] established for the real method and for Banach couples, and it will be useful for us in Section 4, where we prove the results mentioned earlier on spaces of operators.

2 Lorentz-Zygmund sequence spaces

In what follows, given two non-negative real sequences (η_n) and (τ_n), we write $\eta_n \lesssim \tau_n$ if there is a constant $c > 0$ such that $\eta_n \leq c\tau_n$ for all $n \in \mathbb{N}$. We put $\eta_n \sim \tau_n$ if $\eta_n \lesssim \tau_n$ and $\tau_n \lesssim \eta_n$.

Let $w : [1, \infty) \to (0, \infty)$ be a function with $w(1) = 1$ and satisfying the condition

$$\lim_{t\to\infty} \frac{w(at)}{w(t)} = 1 \quad \text{for all } a > 0\,. \tag{2.1}$$

These so-called *quasi-logarithmic* weights were investigated by Baumbach [1]. He studied the corresponding *weighted Lorentz sequence spaces* $\ell_{p,q}(w)$, $0 < p < \infty$, $0 < q \leq \infty$, consisting of all bounded scalar sequences $\xi = (\xi_n)$ such that the quasi-norm

$$\|\xi\|_{\ell_{p,q}(w)} := \begin{cases} \left(\sum\limits_{n=1}^{\infty} \left(n^{1/p-1/q} w(n) \xi_n^* \right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup\limits_{n\in\mathbb{N}} n^{1/p} w(n) \xi_n^* & \text{if } q = \infty \end{cases} \tag{2.2}$$

is finite. Here (ξ_n^*) is the non-increasing rearrangement of ξ, given by

$$\xi_n^* := \inf \big\{ \delta > 0 : \operatorname{card}\{k \in \mathbb{N} : |\xi_k| \geq \delta\} < n \big\}\,.$$

Here we are only interested in the subclass of *Lorentz-Zygmund sequence spaces*, i.e. the spaces

$$\ell_{p,q}(\log \ell)_\alpha := \ell_{p,q}(w_\alpha) \quad \text{with } w_\alpha(t) = (1 + \log t)^\alpha \text{ and } \alpha \in \mathbb{R}\,.$$

Clearly all weights w_α satisfy condition (2.1). The quasi-norm (2.2) on $\ell_{p,q}(\log \ell)_\alpha$ is given by

$$\|\xi\|_{p,q,\alpha} := \begin{cases} \left(\sum\limits_{n=1}^{\infty} \left(n^{1/p-1/q} (1 + \log n)^\alpha \xi_n^* \right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup\limits_{n\in\mathbb{N}} n^{1/p} (1 + \log n)^\alpha \xi_n^* & \text{if } q = \infty\,. \end{cases}$$

We refer to [2, p. 10] and [3, p. 285] for properties of these spaces. Note that for $\alpha = 0$ we recover the usual *Lorentz sequence space* $(\ell_{p,q}, \|\cdot\|_{p,q})$ (see [15, Section 1.c] and [27, Section 2.1]). In particular, if $p = q$, then $\ell_{p,p}$ is space of p-summable sequences ℓ_p.

The limit case $p = \infty$ of Lorentz sequence spaces will also be of interest for us. Recall that (see [10, 9]) the space $\ell_{\infty,q}$, $0 < q < \infty$, consists of all bounded sequences of scalars $\xi = (\xi_n)$ having a finite quasi-norm

$$\|\xi\|_{\infty,q} := \Big(\sum_{n=1}^{\infty} (\xi_n^*)^q n^{-1} \Big)^{1/q}\,.$$

As pointed out in the monograph [15, Proposition 1.c.10], the Lorentz sequence spaces $\ell_{p,q}$ are all different from each other. The main index is p, determining that sequences in $\ell_{p,q}$ have a power decay as $n^{-1/p}$. The index q yields a further logarithmic decay as $(1+\log n)^{-1/q}$. The spaces $\ell_{p,q}$ are lexicographically ordered (by inclusion resp. continuous embeddings), that is,

$$\ell_{p_0,q_0} \hookrightarrow \ell_{p_1,q_1} \iff \begin{cases} p_0 < p_1 \text{ and } 0 < q_0, q_1 \leq \infty \quad \text{or} \\ p_0 = p_1 \text{ and } 0 < q_0 \leq q_1 \leq \infty\,. \end{cases}$$

Moreover, all these embeddings are strict, unless $(p_1, q_1) = (p_2, q_2)$.

For Lorentz-Zygmund sequence spaces the picture is more involved, because in addition to the influence of p and q we also have the action of $(1 + \log n)^\alpha$. Hence, both indices q, α are responsible for the logarithmic decay of sequences in $\ell_{p,q}(\log \ell)_\alpha$. This gives more flexibility, which is very useful as the following example illustrates. For every $\lambda > 0$, the sequence

$$(\xi_n) := \big(n^{-1/p}(1 + \log n)^\lambda\big) \text{ belongs to } \ell_{p+\varepsilon} \text{ for all } \varepsilon > 0, \text{ but not to } \ell_{p,\infty}\,.$$

That means, the classical Lorentz spaces cannot distinguish sequences having the same power-like decay and additional logarithmic factors with *different positive* exponents. However, Lorentz-Zygmund spaces can 'see' the difference, since we have, even for arbitrary $\lambda \in \mathbb{R}$ and $0 < q < \infty$,

$$(\xi_n) \in \ell_{p,q}(\log \ell)_\alpha \Longleftrightarrow \alpha < -\lambda - 1/q \qquad \text{and} \qquad (\xi_n) \in \ell_{p,\infty}(\log \ell)_\alpha \Longleftrightarrow \alpha \le -\lambda\,.$$

Baumbach [1] studied weighted Lorentz sequence spaces $\ell_{p,q}(w)$ with quasi-logarithmic weights. In particular he found necessary and sufficient conditions for the existence of embeddings between these spaces, see [1, Satz 3.1 and Satz 3.4]. Specified to the subclass of Lorentz-Zygmund spaces $\ell_{p,q}(\log \ell)_\alpha$, his results can be formulated as follows.

Theorem 2.1. *Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$ and $-\infty < \alpha_0, \alpha_1 < \infty$. Then there is an embedding $\ell_{p_0,q_0}(\log \ell)_{\alpha_0} \hookrightarrow \ell_{p_1,q_1}(\log \ell)_{\alpha_1}$ if, and only if, one of the following three conditions is satisfied.*

(i) $p_0 < p_1$

(ii) $p_0 = p_1, \alpha_1 \le \alpha_0$ *and* $\alpha_1 + 1/q_1 < \alpha_0 + 1/q_0$

(iii) $p_0 = p_1, q_0 = q_1$ *and* $\alpha_0 = \alpha_1$

Furthermore, all embeddings in (i) and (ii) are strict.

We conclude this section by a few remarks.

Remark 2.2. It follows from this theorem that any two Lorentz-Zygmund sequence spaces are always different, except when all three parameters coincide.

Remark 2.3. It is clear that the lexicographical ordering of the family of Lorentz spaces $\ell_{p,q}$ has no analogue for the family of Lorentz-Zygmund spaces. Any two Lorentz spaces are always comparable (meaning that one is contained in the other), but there are many pairs of incomparable Lorentz-Zygmund spaces, as can easily be seen from the conditions (i)-(iii) in the previous theorem.

3 Logarithmic interpolation methods

By real interpolation of the couple (ℓ_{p_0}, ℓ_{p_1}) one gets Lorentz sequence spaces (see [4, 31]). In order to obtain Lorentz-Zygmund spaces from this couple we work with logarithmic perturbations of the usual real method. In this section we first review these logarithmic interpolation methods in the quasi-Banach setting, and then we extend a result of Lévy [16] on existence of subspaces isomorphic to ℓ_q in the real interpolation spaces $(A_0, A_1)_{\theta,q}$ to logarithmic methods.

This will allow us to establish a result on strict embeddings of logarithmic interpolation spaces which we will need in the last section.

Recall that if $(A, \|\cdot\|)$ is a quasi-Banach space with constant $c_A \geq 1$ in the quasi-triangle inequality and $0 < r \leq 1$ satisfies that $2^{1/r-1} = c_A$ then, according to the Aoki-Rolewicz theorem (see [26, Section 6.2] or [15, Proposition 1.c.5]), there is an equivalent r-norm $\|\cdot\|_*$ on A, that means it satisfies $\|x+y\|_*^r \leq \|x\|_*^r + \|y\|_*^r$ for any $x, y \in A$. Then we say that A is an r-Banach space. Note that every r-norm is also an s-norm for all $0 < s < r$.

Let (A_0, A_1) be a *quasi-Banach couple*, that is to say, two quasi-Banach spaces A_0, A_1 which are continuously embedded into the same Hausdorff topological vector space. When A_0, A_1 are both r-Banach spaces, then we refer to (A_0, A_1) as an *r-Banach couple* and, if $r = 1$, as a *Banach couple*.

We endow the intersection $A_0 \cap A_1$ with the quasi-norm $\|a\| = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$ and the sum $A_0 + A_1$ with the quasi-norm $K(1, \cdot)$, where

$$K(t,a) = \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\right\} \quad , \quad t > 0, a \in A_0 + A_1\,,$$

is the Peetre's K-functional of the quasi-Banach couple (A_0, A_1). If $\|\cdot\|_{A_j}$ is an r-norm for $j = 0, 1$, then it is useful to consider the functional

$$K_r(t,a) = \inf\left\{\left(\|a_0\|_{A_0}^r + t^r\|a_1\|_{A_1}^r\right)^{1/r} : a = a_0 + a_1, a_j \in A_j\right\},$$

which is an r-norm in $A_0 + A_1$ and satisfies

$$K(t,a) \leq K_r(t,a) \leq 2^{1/r-1}K(t,a) \quad \text{for all } t > 0 \text{ and } a \in A_0 + A_1. \tag{3.1}$$

Let (A_0, A_1) be a quasi-Banach couple, $0 < q \leq \infty$ and $\varrho(t) = t^\theta(1 + |\log t|)^{-\alpha}$, where $0 < \theta < 1$ and $\alpha \in \mathbb{R}$. Then the *logarithmic interpolation space* $(A_0, A_1)_{\varrho,q}$ is formed by all elements $a \in A_0 + A_1$ for which the quasi-norm

$$\|a\|_{\varrho,q} = \begin{cases} \left(\int_0^\infty \left(\frac{K(t,a)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\ \sup\limits_{t>0} \frac{K(t,a)}{\varrho(t)} & \text{if } q = \infty \end{cases}$$

is finite (see, for example, [14, 20, 24, 12] and the references given there). If $\alpha = 0$ we recover the usual *real interpolation space* $(A_0, A_1)_{\theta,q}$ (see [4, 31, 3]).

Let (A_0, A_1) be any quasi-Banach couple. It is not hard to check that $\|\cdot\|_{\varrho,q}$ is equivalent to the quasi-norm

$$\|a\|_{\varrho,q;r} = \begin{cases} \left(\sum\limits_{m\in\mathbb{Z}} j_m(a)^q\right)^{1/q} & \text{if } q < \infty \\ \sup\limits_{m\in\mathbb{Z}} j_m(a) & \text{if } q = \infty\,, \end{cases}$$

where

$$j_m(a) = 2^{-\theta m}(1 + |m|)^\alpha K_r(2^m, a) \quad \text{for } m \in \mathbb{Z} \text{ and } a \in A_0 + A_1\,.$$

The description of $(A_0, A_1)_{\varrho,q}$ by means of $\|\cdot\|_{\varrho,q;r}$ shows that

$$(A_0, A_1)_{\varrho,p} \hookrightarrow (A_0, A_1)_{\varrho,q} \quad \text{provided that } 0 < p \leq q \leq \infty. \tag{3.2}$$

Note that $\|\cdot\|_{\varrho,q;r}$ is even an r-norm, if (A_0, A_1) is an r-Banach couple and $q \geq r$.

If $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$ and $\varrho(t) = t^\theta(1+|\log t|)^{-\alpha}$, it turns out that

$$(\ell_{p_0}, \ell_{p_1})_{\varrho,q} = \ell_{p,q}(\log \ell)_\alpha \quad \text{(with equivalent quasi-norms)}$$

(see [21, Example 3] or [20, Théorème 1]).

For $j = 0, 1$, we write $A_j^\sim$ for the *Gagliardo completion* of A_j, formed by all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{A_j^\sim} = \sup\{t^{-j}K(t,a) : t > 0\},\ j = 0,1,$$

see [3, 4]. Note that

$$\|a\|_{A_0^\sim} = \lim_{t\to\infty} K(t,a) \qquad \text{and} \qquad \|a\|_{A_1^\sim} = \lim_{t\to 0} \frac{K(t,a)}{t}\,.$$

We say that the quasi-Banach couple (A_0, A_1) is a *Gagliardo couple* if $A_j = A_j^\sim$ for $j = 0, 1$.

Lévy established in [16, 17] that if (A_0, A_1) is a Banach couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$ and $1 \le q < \infty$, then $(A_0, A_1)_{\theta,q}$ contains a subspace isomorphic to ℓ_q. Cwikel and the present authors [8] extended this result to quasi-Banach couples and to the full range of q, $0 < q < \infty$. Next we study the existence of subspaces isomorphic to ℓ_q in logarithmic interpolation spaces. For this aim we follow the approach and ideas developed in [8].

Lemma 3.1. *Let (A_0, A_1) be a couple of r-Banach spaces for some $0 < r \le 1$. Suppose that there is a constant $C \ge 1$ such that, for some fixed $\theta \in (0,1)$ and $\alpha \in \mathbb{R}$,*

$$K_r(t,a) \le C\,\varrho(t)\,\|a\|_{A_0+A_1} \quad \text{for all } t > 0 \text{ and all } a \in A_0 \cap A_1\,, \tag{3.3}$$

where $\varrho(t) = t^\theta(1+|\log t|)^{-\alpha}$. Then there is a constant $D \ge 1$ such that

$$\|a\|_{A_0^\sim \cap A_1^\sim} \le D\,\|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1\,.$$

Proof. Pick θ_1, θ_2 with $0 < \theta_1 < \theta < \theta_2 < 1$. Then there are constants $M_1, M_2 \ge 1$ such that

$$\varrho(t) = t^\theta(1+|\log t|)^{-\alpha} \le \begin{cases} M_1 t^{\theta_1} & \text{for } 0 < t \le 1\,, \\ M_2 t^{\theta_2} & \text{for } 1 \le t < \infty\,. \end{cases}$$

This together with (3.1) implies that for all $a \in A_0 \cap A_1$ the following inequalities hold

$$K_r(t,a) \le C M_1 t^{\theta_1}\|a\|_{A_0+A_1} \quad \text{for all } 0 < t \le 1\,,$$

$$K_r(t,a) \le C M_2 t^{\theta_2}\|a\|_{A_0+A_1} \quad \text{for all } 1 \le t < \infty\,.$$

Then the iteration procedure in the proof of [8, Lemma 3.1] shows that there are constants $C_k = C_k(C, M_k, \theta_k) \ge 1, k = 1, 2$, such that

$$K_r(t,a) \le C_1 t\|a\|_{A_0+A_1} \quad \text{for } 0 < t \le 1\,, \tag{3.4}$$

$$K_r(t,a) \le C_2\|a\|_{A_0+A_1} \quad \text{for } 1 \le t < \infty\,. \tag{3.5}$$

Taking the limit as $t \to 0$ in (3.4) and the limit as $t \to \infty$ in (3.5), and using the inequality $K(t,a) \le K_r(t,a)$, for all $a \in A_0 \cap A_1$ we obtain that

$$\|a\|_{A_1^\sim} = \lim_{t\to 0} \frac{K(t,a)}{t} \le \lim_{t\to 0} \frac{K_r(t,a)}{t} \le C_1 \|a\|_{A_0+A_1}$$

and

$$\|a\|_{A_0^\sim} = \lim_{t\to\infty} K(t,a) \le \lim_{t\to\infty} K_r(t,a) \le C_2 \|a\|_{A_0+A_1} .$$

Setting $D = \max\{C_1, C_2\}$, we conclude that

$$\|a\|_{A_0^\sim \cap A_1^\sim} \le D \|a\|_{A_0+A_1} \quad \text{for all } a \in A_0 \cap A_1.$$

□

We can now proceed as in [8, Theorem 3.2] to establish the following result.

Theorem 3.2. *Let (A_0, A_1) be a Gagliardo couple of quasi-Banach spaces, let $0 < q \le \infty$ and $\varrho(t) = t^\theta (1 + |\log t|)^{-\alpha}$ with $0 < \theta < 1$ and $\alpha \in \mathbb{R}$. Then the following conditions are equivalent.*

(i) $A_0 \cap A_1$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.

(ii) $(A_0, A_1)_{\varrho,q}$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.

Proof. $(i) \Longrightarrow (ii)$: The description of $(A_0, A_1)_{\varrho,q}$ in terms of the J-functional (see [4, Theorem 3.11.3] and [24, Theorem 2.1]) yields that $A_0 \cap A_1 \subseteq (A_0, A_1)_{\varrho,q} \subseteq \overline{A_0 \cap A_1}^{\|\cdot\|_{A_0+A_1}}$. Hence, if (i) holds, then $(A_0, A_1)_{\varrho,q} = A_0 \cap A_1$ is closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$.

$(ii) \Longrightarrow (i)$: Since (A_0, A_1) is a Gagliardo couple, we have $A_0 \cap A_1 = A_0^\sim \cap A_1^\sim$. Moreover, according to the Aoki-Rolewicz theorem, we can assume that A_0 and A_1 are both r-normed for some $0 < r \le 1$. Under the assumption (ii), it follows from [29, Corollary 2.12] that there is a constant $M > 0$ such that

$$\|a\|_{\varrho,q} \le M \|a\|_{A_0+A_1} \quad \text{for all } a \in (A_0, A_1)_{\varrho,q} .$$

Using that $(A_0, A_1)_{\varrho,q} \hookrightarrow (A_0, A_1)_{\varrho,\infty}$ and (3.1), we obtain that there is $C > 0$ such that

$$\sup_{t>0} \varrho(t)^{-1} K_r(t,a) \le C \|a\|_{A_0+A_1} \quad \text{for all } a \in A_0^\sim \cap A_1^\sim .$$

The constant C is bigger than or equal to 1 because

$$\|a\|_{A_0+A_1} = K(1,a) \le \varrho(1)^{-1} K_r(1,a) \le C \|a\|_{A_0+A_1} .$$

Consequently, applying Lemma 3.1, we derive that $A_0 \cap A_1 = A_0^\sim \cap A_1^\sim$ is closed in $A_0 + A_1$. □

Theorem 3.3. *Let (A_0, A_1) be a quasi-Banach couple, let $0 < q < \infty$ and $\varrho(t) = t^\theta (1 + |\log t|)^{-\alpha}$ with $0 < \theta < 1$ and $\alpha \in \mathbb{R}$. If $(A_0, A_1)_{\varrho,q}$ is not closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$, then $(A_0, A_1)_{\varrho,q}$ contains a subspace isomorphic to ℓ_q .*

Proof. Due to the Aoki-Rolewicz theorem we can assume that A_0 and A_1 are both r-normed for some $0 < r \le 1$. Consider on $A_0 + A_1$ the r-norm $K_r(1, \cdot)$ and on $(A_0, A_1)_{\varrho,q}$ the r-norm $\|a\|_{\varrho,q;r} = \|(j_m(a))_{m\in\mathbb{Z}}\|_q$, where

$$j_m(a) = 2^{-\theta m}(1 + |m|)^\alpha K_r(2^m, a)\,.$$

Given any $0 < \varepsilon < 1$, proceeding as in [8, Theorem 3.3], one can construct inductively a sequence $(x_n) \subseteq (A_0, A_1)_{\varrho,q}$ and an increasing sequence $(N_n) \subseteq \mathbb{N}$ with the following properties:

$$\text{(a)}\quad \|x_n\|_{\varrho,q;r} = 1 \qquad\qquad \text{for all } n \in \mathbb{N}\,,$$

$$\text{(b)}\quad \Big(\sum_{|m|>N_n} j_m(x_n)^q\Big)^{1/q} \le 2^{-n/r}\varepsilon \quad \text{for all } n \in \mathbb{N}\,,$$

$$\text{(c)}\quad \Big(\sum_{|m|\le N_{n-1}} j_m(x_n)^q\Big)^{1/q} \le 2^{-n/r}\varepsilon \quad \text{for all } n \ge 2\,.$$

Then, by the same arguments as in [8, Theorem 3.3], it follows that the sequence (x_n) is isomorphic to the unit vector basis of ℓ_q. In fact, for every sequence of scalars with finitely many non-zero entries we have

$$M_0\Big(\sum_{n=1}^{\infty}|\alpha_n|^q\Big)^{1/q} \le \Big\|\sum_{n=1}^{\infty}\alpha_n x_n\Big\|_{\varrho,q;r} \le M_1\Big(\sum_{n=1}^{\infty}|\alpha_n|^q\Big)^{1/q}$$

with constants

$$M_0 = M_0(\varepsilon, r) = (1 - 3\varepsilon^r)^{1/r} \quad \text{and} \quad M_1 = M_1(\varepsilon, r) = (1 + 2\varepsilon^r)^{1/r}\,. \tag{3.6}$$

□

Remark 3.4. In the special case when (A_0, A_1) is a Banach couple with $A_0 \hookrightarrow A_1$ and $1 \le q < \infty$, the fact that $(A_0, A_1)_{\varrho,q}$ contains a subspace isomorphic to ℓ_q was shown in [7, Lemma 7].

Theorem 3.3 allows us to establish the following result on strict embeddings of logarithmic interpolation spaces.

Theorem 3.5. *Let (A_0, A_1) be a quasi-Banach Gagliardo couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$ with respect to the quasi-norm of $A_0 + A_1$. Let $0 < \theta < 1$, $\alpha \in \mathbb{R}$, $\varrho(t) = t^\theta(1 + |\log t|)^{-\alpha}$ and $0 < p < q \le \infty$. Then the embedding*

$$(A_0, A_1)_{\varrho,p} \hookrightarrow (A_0, A_1)_{\varrho,q} \tag{3.7}$$

is strict.

Proof. Pick $0 < r \le 1$ such that (A_0, A_1) is an r-Banach couple. As it is pointed out in (3.2), the embedding (3.7) holds. To show that it is strict we proceed by contradiction. Assume that $X := (A_0, A_1)_{\varrho,p} = (A_0, A_1)_{\varrho,q}$. Then, for any $p \le s \le q$ we have $(A_0, A_1)_{\varrho,s} = X$. So we can assume that $q < \infty$ in (3.7).

Now let $\varepsilon > 0$ be sufficiently small so that $M_0(\varepsilon, r)$ in (3.6) satisfies that $M_0(\varepsilon, r) \geq 1/2$. We shall fix later the value of ε satisfying one more additional condition. We can construct a sequence $(x_n) \subseteq X$ satisfying properties $(a), (b), (c)$, so that (x_n) is equivalent to the unit vector basis (e_n) in ℓ_p. In particular, we get

$$\|\sum_{n=1}^{N} x_n\|_{\varrho,p;r} \geq M_0(\varepsilon, r) N^{1/p} \geq N^{1/p}/2 \quad \text{for all } N \in \mathbb{N}\,. \tag{3.8}$$

By our assumption, $\|\cdot\|_{\varrho,p;r}$ and $\|\cdot\|_{\varrho,q;r}$ are equivalent quasi-norms on X. Hence there is a constant $C \geq 1$ such that

$$\|a\|_{\varrho,q;r} \leq \|a\|_{\varrho,p;r} \leq C\|a\|_{\varrho,q;r} \qquad \text{for all } a \in X\,. \tag{3.9}$$

Now we consider the vectors $y_n := x_n/\|x_n\|_{\varrho,q;r}$. Then clearly

$$\text{(a}^*\text{)} \quad \|y_n\|_{\varrho,q;r} = 1 \quad \text{for all } n \in \mathbb{N}\,.$$

Moreover, due to $\|x_n\|_{\varrho,p;r} = 1$ and (3.9) we have $j_m(y_n) = j_m(x_n)/\|x_n\|_{\varrho,q;r} \leq C j_m(x_n)$. Since $p < q$, this implies

$$\text{(b}^*\text{)} \quad \Big(\sum_{|m|>N_n} j_m(y_n)^q\Big)^{1/q} \leq \Big(\sum_{|m|>N_n} j_m(y_n)^p\Big)^{1/p} \leq C\Big(\sum_{|m|>N_n} j_m(x_n)^p\Big)^{1/p} \leq 2^{-n/r}\, C\,\varepsilon\,.$$

Similarly

$$\text{(c}^*\text{)} \quad \Big(\sum_{|m|\leq N_{n-1}} j_m(y_n)^q\Big)^{1/q} \leq C\Big(\sum_{|m|\leq N_{n-1}} j_m(x_n)^p\Big)^{1/p} \leq 2^{-n/r}\, C\,\varepsilon\,.$$

The conditions (a*), (b*), (c*) are in complete analogy to (a), (b), (c). We have just replaced p by q and ε by $C\varepsilon$. Let $\widetilde{M}_j = M_j(C\varepsilon, r), j = 0, 1$, be the constants given by (3.6). Choose $\varepsilon > 0$ small enough so that in addition to $M_0(\varepsilon, r) \geq 1/2$, we also have that $\widetilde{M}_0 > 0$. Then, the arguments in [8, Theorem 3.3] yield that (y_n) is equivalent to the unit vector basis in ℓ_q. For any finite sequence (α_n), we have

$$\widetilde{M}_0\big(\sum_{n=1}^{\infty} |\alpha_n|^q\big)^{1/q} \leq \big\|\sum_{n=1}^{\infty} \alpha_n y_n\big\|_{\varrho,q;r} \leq \widetilde{M}_1\big(\sum_{n=1}^{\infty} |\alpha_n|^q\big)^{1/q}\,. \tag{3.10}$$

Now we take

$$\alpha_n := \|x_n\|_{\varrho,q;r} \leq \|x_n\|_{\varrho,p;r} = 1\,.$$

So $\alpha_n x_n = y_n$. By (3.8) and (3.10), for any $N \in \mathbb{N}$ we get

$$\begin{aligned}
\frac{1}{2}N^{1/p} &\leq \big\|\sum_{n=1}^{N} x_n\big\|_{\varrho,p;r} \leq C\big\|\sum_{n=1}^{N} x_n\big\|_{\varrho,q;r} \\
&= C\big\|\sum_{n=1}^{N} \alpha_n y_n\big\|_{\varrho,q;r} \leq C\widetilde{M}_1\big(\sum_{n=1}^{N} |\alpha_n|^q\big)^{1/q} \\
&\leq C\widetilde{M}_1 N^{1/q}\,.
\end{aligned}$$

Consequently,

$$N^{1/p-1/q} \leq 2C\widetilde{M}_1 \quad \text{for any } N \in \mathbb{N}$$

which is impossible. □

4 Spaces of operators

This is the central section of the paper. It contains our main results on the spaces $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ and $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$.

In what follows, X, Y are Banach spaces, and $\mathscr{L}(X,Y)$ stands for the space of all bounded linear operators from X into Y. The *approximation numbers* of an operator $T \in \mathscr{L}(X,Y)$ are defined by

$$a_n(T) = \inf\{\|T - R\| : R \in \mathscr{L}(X,Y) \text{ with } \operatorname{rank} R < n\}, \quad n \in \mathbb{N}.$$

Clearly, $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq 0$. The faster the asymptotic decay to 0 of $(a_n(T))$, the better T can be approximated by finite rank operators, and therefore the better is the 'degree of compactness' of T. We measure the asymptotic decay rate of $(a_n(T))$ with the help of the Lorentz-Zygmund sequence spaces.

Subsequently, it will be useful to work with generalized approximation numbers, where the error of approximation by finite rank operators is not measured in the operator norm, but in a stronger quasi-norm $\|\cdot\|_E$. If $(E, \|.\|_E)$ is a quasi-normed space which is continuously embeddded into $\mathscr{L}(X,Y)$, then we set

$$a_n(T; E) = \inf\{\|T - R\|_E : R \in \mathscr{L}(X,Y) \text{ with } \operatorname{rank} R < n\}, \quad n \in \mathbb{N}.$$

For $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$, the Lorentz-Zygmund space of operators $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ consists of all $T \in \mathscr{L}(X,Y)$ having a finite quasi-norm $\|T\|_{p,q,\alpha} = \|(a_n(T))\|_{p,q,\alpha}$ (see [6]). When $\alpha = 0$, we recover the Lorentz spaces of operators $(\mathscr{L}^{(a)}_{p,q}(X,Y), \|\cdot\|_{p,q})$ (see [30, 27, 15]), and if in addition $p = q$, then we obtain the space $(\mathscr{L}^{(a)}_{p}(X,Y), \|\cdot\|_{p})$ (see [26]). The limit case $p = \infty, \alpha = 0, 0 < q < \infty$ is also of interest for us. The space $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ is formed by all $T \in \mathscr{L}(X,Y)$ for which the quasi-norm $\|T\|_{\infty,q} = \|(a_n(T))\|_{\infty,q}$ is finite (see [10, 9]). When H, K are Hilbert spaces and $q = 1$, the space $\mathscr{L}^{(a)}_{\infty,1}(H,K)$ was introduced by Macaev (see [18] and [13]).

For arbitrary infinite-dimensional Banach spaces X and Y, Cwikel and the present authors have shown in [8, Theorem 4.3] that all spaces $\mathscr{L}^{(a)}_{p,q}(X,Y)$ of Lorentz type with $0 < p < \infty$ and $0 < q \leq \infty$ are different from each other. Next we study this problem for the spaces $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ and the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$.

Our first main result concerns the Macaev-type spaces $\mathscr{L}^{(a)}_{\infty,q}(X,Y)$ of operators.

Theorem 4.1. *Let X and Y be arbitrary infinite-dimensional Banach spaces. Then the scale $\{\mathscr{L}^{(a)}_{\infty,q}(X,Y)\}_{0<q<\infty}$ is strictly increasing.*

Proof. We proceed by contradiction. Assume that the scale is not strictly increasing. Then there exist $0 < p < q < \infty$ such that $\mathscr{L}^{(a)}_{\infty,p}(X,Y) = \mathscr{L}^{(a)}_{\infty,q}(X,Y)$. Hence the quasi-norms of these spaces are equivalent, that means there is a constant $C_1 > 0$ such that for all finite-rank operators $T \in \mathscr{L}(X,Y)$ the inequality

$$\|T\|_{\infty,p} = \left(\sum_{k=1}^{\infty} \frac{a_k(T)^p}{k}\right)^{1/p} \leq C_1 \left(\sum_{k=1}^{\infty} \frac{a_k(T)^q}{k}\right)^{1/q} = C_1 \|T\|_{\infty,q} \tag{4.1}$$

holds. We will show that (4.1) implies that

$$\mathscr{L}^{(a)}_{1,\infty}(X,Y) = \mathscr{L}^{(a)}_{1,r}(X,Y) \quad \text{for some } 0 < r < \infty,$$

which is a contradiction in view of [8, Theorem 4.3].

Let $T \in \mathscr{L}^{(a)}_{1,\infty}(X,Y)$ with $\|T\|_{1,\infty} = 1$. By the representation theorem for the space $\mathscr{L}^{(a)}_{1,\infty}(X,Y)$ (see [27, 2.3.8]) there are a constant $C_2 > 0$ and operators $T_n \in \mathscr{L}(X,Y)$ such that rank$(T_n) \le 2^n$, $T = \sum_{n=0}^{\infty} T_n$ (convergence in the operator norm) and

$$\sup_{n\in\mathbb{N}\cup\{0\}} 2^n \|T_n\| \le C_2 \|T\|_{1,\infty} < \infty\,.$$

For every $n \in \mathbb{N}\cup\{0\}$ we have, with some constant $C_3 > 0$,

$$\|T_n\|_{\infty,q} = \left(\sum_{k=1}^{2^n} \frac{a_k(T_n)^q}{k}\right)^{1/q} \le \left(\sum_{k=1}^{2^n} \frac{1}{k}\right)^{1/q} \|T_n\| \le C_3\, n^{1/q} \|T_n\|\,.$$

By (4.1) this implies, with $C = C_1C_2C_3$,

$$\|T_n\|_{\infty,p} \le C\, n^{1/q} 2^{-n} \|T\|_{1,\infty} = C\, n^{1/q} 2^{-n}\,.$$

Now we use generalized approximation numbers $a_n(\,.\,;E)$ with $E = \mathscr{L}^{(a)}_{\infty,p}(X,Y)$. The quasi-norm $\|.\|_{\infty,p}$ on E is equivalent to an s-norm for some $0 < s < 1$, say, with some constant $M > 0$. Since

$$T = \sum_{n=0}^{\infty} T_n \quad \text{and} \quad \operatorname{rank}\Big(\sum_{n=0}^{N-1} T_n\Big) \le \sum_{n=0}^{N-1} 2^n < 2^N\,,$$

we conclude that, for some constant $C_4 > 0$ and all $N \in \mathbb{N}$,

$$a_{2^N}(T;E) \le \Big\| \sum_{n=N}^{\infty} T_n \Big\|_{\infty,p} \le M \left(\sum_{n=N}^{\infty} \|T_n\|^s_{\infty,p}\right)^{1/s} \le M\,C \left(\sum_{n=N}^{\infty} \frac{n^{s/q}}{2^{sn}}\right)^{1/s} \le C_4 N^{1/q} 2^{-N}\,.$$

Consequently we can find an operator $R \in \mathscr{L}(X,Y)$ of rank less than 2^N such that

$$\|T - R\|_{\infty,p} \le 2 a_{2^N}(T;E) \le 2\,C_4 N^{1/q} 2^{-N}\,.$$

In order to get a lower bound for $\|T-R\|_{\infty,p}$, we use that $a_{k+2^N}(T) \le a_k(T-R) + a_{2^N}(R) = a_k(T-R)$. This gives

$$\|T-R\|_{\infty,p} = \left(\sum_{k=1}^{\infty} \frac{a_k(T-R)^p}{k}\right)^{1/p} \ge \left(\sum_{k=1}^{2^N} \frac{a_{k+2^N}(T)^p}{k}\right)^{1/p}$$

$$\ge \left(\sum_{k=1}^{2^N} \frac{1}{k}\right)^{1/p} \cdot a_{2^{N+1}}(T) \ge C_5 N^{1/p} a_{2^{N+1}}(T)\,.$$

Together with the previous estimate this implies that, for some constant $C_6 > 0$ and all $k \in \mathbb{N}$,

$$a_k(T) \le \frac{C_6}{k\,(1+\log k)^{\nu}}\,, \qquad \text{where} \qquad \nu = 1/p - 1/q > 0\,.$$

Now take any r such that $1/\nu < r < \infty$. Then $\nu r > 1$, and so we arrive at

$$\|T\|_{1,r} = \left(\sum_{k=1}^{\infty} \left(k^{1-1/r} a_k(T)\right)^r\right)^{1/r} \le C_6 \left(\sum_{k=1}^{\infty} \frac{1}{k\,(1+\log k)^{\nu r}}\right)^{1/r} < \infty\,.$$

Hence we have shown that assumption (4.1) implies $\mathscr{L}^{(a)}_{1,\infty}(X,Y) = \mathscr{L}^{(a)}_{1,r}(X,Y)$ for all $r \in (1/\nu, \infty)$. But this is impossible by [8, Theorem 4.3]. The proof is finished. □

Our second main result concerns the family of Lorentz-Zygmund spaces of operators.

Theorem 4.2. *For arbitrary infinite-dimensional Banach spaces X and Y the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ with $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$ are all different from each other, that means*

$$\mathscr{L}^{(a)}_{p_0,q_0,\alpha_0}(X,Y) \neq \mathscr{L}^{(a)}_{p_1,q_1,\alpha_1}(X,Y) \ \textit{unless} \ (p_0,q_0,\alpha_0) = (p_1,q_1,\alpha_1)\,.$$

Proof. We distinguish several cases. An essential tool in Case 2 are generalized approximation numbers, while the proof of Case 3 is based on logarithmic interpolation.
<u>Case 1</u>. First we suppose that $p_0 < p_1$.
Choose r_0, r_1 such that $p_0 < r_0 < r_1 < p_1$. It follows from Theorem 2.1 that

$$\mathscr{L}^{(a)}_{p_0,q_0,\alpha_0}(X,Y) \hookrightarrow \mathscr{L}^{(a)}_{r_0}(X,Y) \hookrightarrow \mathscr{L}^{(a)}_{r_1}(X,Y) \hookrightarrow \mathscr{L}^{(a)}_{p_1,q_1,\alpha_1}(X,Y)\,.$$

Hence, if $\mathscr{L}^{(a)}_{p_0,q_0,\alpha_0}(X,Y) = \mathscr{L}^{(a)}_{p_1,q_1,\alpha_1}(X,Y)$, then we get $\mathscr{L}^{(a)}_{r_0}(X,Y) = \mathscr{L}^{(a)}_{r_1}(X,Y)$, a contradiction to [8, Theorem 4.3].
<u>Case 2</u>. Now let $p_0 = p_1 = p$ and $\alpha_0 > \alpha_1$.
We treat this case with similar ideas to those in the proof of Theorem 4.1. We again proceed by contradiction. Assume that

$$\mathscr{L}^{(a)}_{p,q_0,\alpha_0}(X,Y) = \mathscr{L}^{(a)}_{p,q_1,\alpha_1}(X,Y)\,. \tag{4.2}$$

We are going to show that for any $0 < r < p$ there exists $0 < s < \infty$ such that

$$\mathscr{L}^{(a)}_{r,s}(X,Y) = \mathscr{L}^{(a)}_{r,\infty}(X,Y)$$

which is impossible by [8, Theorem 4.3]. Let $T \in \mathscr{L}^{(a)}_{r,\infty}(X,Y)$. According to the representation theorem of $\mathscr{L}^{(a)}_{r,\infty}(X,Y)$ in terms of finite rank operators (see [27, 2.3.8]), there exist $C_0 > 0$ and $(T_n) \subseteq \mathscr{L}(X,Y)$ such that rank $(T_n) \leq 2^n$, $T = \sum_{n=0}^{\infty} T_n$ with convergence in the operator norm and

$$\sup_{n\in\mathbb{N}\cup\{0\}} 2^{n/r}\|T_n\| \leq C_0\|T\|_{r,\infty} < \infty\,.$$

Note that for any $0 < q < \infty$ and $\alpha \in \mathbb{R}$, we have with some constant $C_1 = C_1(p,q,\alpha) > 0$

$$\begin{aligned}\|T_n\|_{p,q,\alpha} &= \Big(\sum_{k=1}^{2^n}\big(k^{1/p-1/q}(1+\log k)^{\alpha}a_k(T_n)\big)^q\Big)^{1/q}\\ &\leq \Big(\sum_{k=1}^{2^n}\big(k^{1/p-1/q}(1+\log k)^{\alpha}\big)^q\Big)^{1/q}\|T_n\| \leq C_1 2^{n/p}n^{\alpha}\|T_n\|\,.\end{aligned}$$

This also holds for $q = \infty$. It follows from [6, Theorem 3.2], [26, Theorem 6.1.6] and equality (4.2) that the quasi-norms of $\mathscr{L}^{(a)}_{p,q_0,\alpha_0}(X,Y)$ and $\mathscr{L}^{(a)}_{p,q_1,\alpha_1}(X,Y)$ are equivalent. Thus there is a constant $C_2 > 0$ such that $\|S\|_{p,q_0,\alpha_0} \leq C_2\|S\|_{p,q_1,\alpha_1}$ for all finite rank operators $S \in \mathscr{L}(X,Y)$, whence

$$\|T_n\|_{p,q_0,\alpha_0} \leq C_2\|T_n\|_{p,q_1,\alpha_1} \leq C_1C_2 2^{n/p}n^{\alpha_1}\|T_n\|\,.$$

Now we work again with generalized approximation numbers, but this time with respect to $E = \mathscr{L}^{(a)}_{p,q_0,\alpha_0}(X,Y)$. Since $\operatorname{rank}\big(\sum_{n=0}^{N-1} T_n\big) \le \sum_{n=0}^{N-1} 2^n < 2^N$ and $\|\cdot\|_{p,q_0,\alpha_0}$ is equivalent to a t-norm for some $0 < t < 1$, say with a constant $C_3 > 0$, we obtain the estimate

$$\begin{aligned} a_{2^N}(T;E) &\le \Big\| \sum_{n=N}^{\infty} T_n \Big\|_{p,q_0,\alpha_0} \le C_3 \Big(\sum_{n=N}^{\infty} \|T_n\|^t_{p,q_0,\alpha_0} \Big)^{1/t} \\ &\le C_2 C_3 \Big(\sum_{n=N}^{\infty} \|T_n\|^t_{p,q_1,\alpha_1} \Big)^{1/t} \le C_1 C_2 C_3 \Big(\sum_{n=N}^{\infty} \big(2^{n/p} n^{\alpha_1} \|T_n\|\big)^t \Big)^{1/t} . \end{aligned}$$

Moreover, $\|T_n\| \le C_0 2^{-n/r} \|T\|_{r,\infty}$. Recall that $r < p$, whence $1/p - 1/r < 0$ and

$$\begin{aligned} a_{2^N}(T;E) &\le C_0 C_1 C_2 C_3 \Big(\sum_{n=N}^{\infty} \big(2^{n(1/p-1/r)} n^{\alpha_1}\big)^t \Big)^{1/t} \|T\|_{r,\infty} \\ &\le C_4 2^{N(1/p-1/r)} N^{\alpha_1} \|T\|_{r,\infty} . \end{aligned}$$

Now we estimate $a_{2^N}(T;E)$ from below. Pick $R \in \mathscr{L}(X,Y)$ with $\operatorname{rank}(R) \le 2^N$ such that

$$\begin{aligned} 2a_{2^N}(T;E) \ge \|T - R\|_{p,q_0,\alpha_0} &= \Big(\sum_{k=1}^{\infty} \big(k^{1/p-1/q} (1+\log k)^{\alpha_0} a_k(T-R)\big)^{q_0} \Big)^{1/q_0} \\ &\ge \Big(\sum_{k=1}^{2^N} \big(k^{1/p-1/q} (1+\log k)^{\alpha_0} a_{k+2^N}(T)\big)^{q_0} \Big)^{1/q_0} . \end{aligned}$$

Here we used again that $a_{k+2^N}(T) \le a_k(T-R) + a_{2^N}(R) = a_k(T-R)$. Therefore

$$2a_{2^N}(T;E) \ge \Big(\sum_{k=1}^{2^N} \big(k^{1/p-1/q} (1+\log k)^{\alpha_0}\big)^{q_0} \Big)^{1/q_0} a_{2^{N+1}}(T) \ge C_5 2^{N/p} N^{\alpha_0} a_{2^{N+1}}(T) .$$

Collecting these inequalities we obtain that

$$a_{2^{N+1}}(T) \le C_6 2^{-N/r} N^{\alpha_1 - \alpha_0} \|T\|_{r,\infty} \quad \text{for all } N \in \mathbb{N} ,$$

or, equivalently, that

$$a_k(T) \le C_7 k^{-1/r} (1+\log k)^{\alpha_1-\alpha_0} \|T\|_{r,\infty} \quad \text{for all } k \in \mathbb{N} .$$

Choosing now $0 < s < 1$ with $s(\alpha_0 - \alpha_1) > 1$ we get the inequality

$$\begin{aligned} \|T\|_{r,s} &= \Big(\sum_{k=1}^{\infty} \big(k^{1/r-1/s} a_k(T)\big)^s \Big)^{1/s} \\ &\le C_7 \Big(\sum_{k=1}^{\infty} \frac{1}{k\,(1+\log k)^{s(\alpha_0-\alpha_1)}} \Big)^{1/s} \|T\|_{r,\infty} \le C_8 \|T\|_{r,\infty} , \end{aligned}$$

since the last series converges. This implies the embedding $\mathscr{L}^{(a)}_{r,\infty}(X,Y) \hookrightarrow \mathscr{L}^{(a)}_{r,s}(X,Y)$. The reverse embedding is obvious, because $\ell_{r,s} \hookrightarrow \ell_{r,\infty}$. Hence $\mathscr{L}^{(a)}_{r,\infty}(X,Y) = \mathscr{L}^{(a)}_{r,s}(X,Y)$, which is a contradiction to [8, Theorem 4.3].

Case 3. Finally, let $p_0 = p_1 = p$, $\alpha_0 = \alpha_1 = \alpha$ and $0 < q_0 \neq q_1 \leq \infty$.
Take now any r_0, r_1 with $0 < r_0 < p < r_1 < \infty$. By [8, Lemma 4.1 and Lemma 4.2], we know that $(\mathscr{L}^{(a)}_{r_0}(X,Y), \mathscr{L}^{(a)}_{r_1}(X,Y))$ is a Gagliardo couple and that $\mathscr{L}^{(a)}_{r_0}(X,Y) \cap \mathscr{L}^{(a)}_{r_1}(X,Y) = \mathscr{L}^{(a)}_{r_0}(X,Y)$ is not closed in $\mathscr{L}^{(a)}_{r_0}(X,Y) + \mathscr{L}^{(a)}_{r_1}(X,Y) = \mathscr{L}^{(a)}_{r_1}(X,Y)$. Let $0 < \theta < 1$ such that $1/p = (1-\theta)/r_0 + \theta/r_1$ and put $\varrho(t) = t^\theta(1 + |\log t|)^{-\alpha}$. According to [6, Theorem 5.2], we have that

$$\big(\mathscr{L}^{(a)}_{r_0}(X,Y), \mathscr{L}^{(a)}_{r_1}(X,Y)\big)_{\varrho,q_j} = \mathscr{L}^{(a)}_{p,q_j,\alpha}(X,Y).$$

Consequently, Theorem 3.5 yields that

$$\mathscr{L}^{(a)}_{p,q_0,\alpha}(X,Y) \neq \mathscr{L}^{(a)}_{p,q_1,\alpha}(X,Y).$$

This completes the proof. □

Combining this theorem on diversity of Lorentz-Zygmund spaces of operators with the *strict embeddings* relations of Lorentz-Zygmnd sequence spaces, we get the following result.

Corollary 4.3. *Let X and Y be arbitrary infinite-dimensional Banach spaces, and let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ and $\alpha_0, \alpha_1 \in \mathbb{R}$. If*

$$p_0 < p_1 \qquad \text{or} \qquad p_0 = p_1,\ \alpha_1 \leq \alpha_0 \ \text{ and } \ \alpha_1 + 1/q_1 < \alpha_0 + 1/q_0\,,$$

then $\mathscr{L}^{(a)}_{p_0,q_0,\alpha_0}(X,Y)$ is strictly embedded in $\mathscr{L}^{(a)}_{p_1,q_1,\alpha_1}(X,Y)$. In particular, the scale

$$\{\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)\}_{\alpha\in\mathbb{R}} \quad \text{is strictly decreasing in } \alpha\,, \text{ for any fixed } p \text{ and } q, \text{ and}$$
$$\{\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)\}_{0<q\leq\infty} \quad \text{is strictly increasing in } q\,, \text{ for any fixed } p \text{ and } \alpha\,.$$

We conclude the paper by a comment on the case when at least one of the Banach spaces X and Y is finite-dimensional, say of dimension N. Then every operator $T \in \mathscr{L}(X,Y)$ has rank at most N, hence $a_n(T) = 0$ for all $n > N$, and therefore

$$\|T\|_{p,q,\alpha} = \Big(\sum_{k=1}^{N} \big(k^{1/p-1/q}(1+\log k)^\alpha a_k(T)\big)^q\Big)^{1/q} < \infty\,.$$

This shows that $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y) = \mathscr{L}(X,Y)$ for all parameters p, q, α.
Theorem 4.2 and this simple observation yield the dichotomy mentioned in the abstract.

Corollary 4.4. *Let X and Y be arbitrary Banach spaces. Then the spaces $\mathscr{L}^{(a)}_{p,q,\alpha}(X,Y)$ with $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$ are*

- *either all different from each other (if X and Y are infinite-dimensional)*
- *or all equal to $\mathscr{L}(X,Y)$ (if X or Y are finite-dimensional).*

Acknowledgments. The authors would like to thank Prof. Albrecht Pietsch for some stimulating mails and Leo R. Ya. Doktorski for pointing out to us the paper [1].

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