UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Invariantes analíticos de singularidades aisladas de hipersuperficie e invariantes combinatorios de semigrupos numéricos.
Analytic invariants of isolated hypersurface singularities and combinatorial invariants of numerical semigroups

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Patricio Almirón Cuadros

Directores

Maria Alberich Carramiñana
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Abstract

This work is about analytic invariants of isolated hypersurface singularities and combinatorial invariants of numerical semigroups.

The first part deals with analytic and topological invariants of an isolated hypersurface singularity. Our main contributions are the following: first we provide a closed formula for the minimal Tjurina number in an equisingularity class of a plane branch in terms of topological invariants of the branch, secondly we address a question of Dimca and Greuel about the quotient of the Milnor and Tjurina numbers of an isolated plane curve singularity; we extend this question to isolated surface singularities in $\mathbb{C}^3$ which gives the clue to provide a complete answer to Dimca and Greuel's question. Moreover, we show the connection of the extended question with an old standing conjecture posed by Durfee. Finally, we establish K. Saito's continuous limit distribution for the spectrum of Newton non-degenerate isolated hypersurface singularities and link this problem with our generalization of Dimca and Greuel's question. As a consequence, this provides a new way of understanding the important role of Durfee's conjecture in the context of isolated hypersurface singularities.

The second part deals with numerical semigroups and their combinatorics. First, we address Wilf's conjecture on numerical semigroups, which asks for a lower bound of its conductor in terms of the genus and the embedding dimension of the numerical semigroup. In this direction, we present two necessary conditions for a numerical semigroup to have negative Eliahou number, which is a number whose positivity implies Wilf's conjecture. As a consequence, we show 36 new examples of numerical semigroups with negative Eliahou number satisfying Wilf's conjecture. One of our main contributions to Wilf's conjecture is to propose its extension to modules over a numerical semigroup, which provides a new insight in some related problems to Wilf's conjecture. We provide a formula for the conductor of a semimodule over a numerical semigroup with two generators and as a consequence we prove the generalization of Wilf's conjecture in this particular case. Secondly, we also study the value set of modules over the local ring of an irreducible plane curve singularity with one Puiseux pair providing a partial generalization of a Theorem of Bresinsky and Teissier about the value semigroup of an irreducible plane curve singularity. As a consequence, we deduce some new features about the value set of Kähler differentials of an irreducible plane curve singularity with one Puiseux pair.

Resumen

Este trabajo trata sobre invariantes analíticos de singularidades aisladas de hipersuperficie e invariantes combinatorios de semigrupos numéricos.

La primera parte trata sobre invariantes analíticos y topológicos de una singularidad aislada de hipersuperficie. Nuestras principales contribuciones son las siguientes: primero proporcionamos una fórmula cerrada para el número mínimo de Tjurina en una clase de equisingularidad de una rama plana en términos de invariantes topológicos de la rama, segundo abordamos una pregunta de Dimca y Greuel sobre el cociente del número de Milnor y del número de Tjurina de una singularidad aislada de curva plana; extendemos esta pregunta a singularidades aisladas de superficie en $\mathbb{C}^3$, lo cual es clave para proporcionar una respuesta completa a la pregunta de Dimca y Greuel. Además, mostramos la conexión de esta extensión con una conjetura planteada por Durfee. Finalmente, establecemos la distribución continua límite de K. Saito para el espectro de singularidades aisladas de hipersuperficie Newton no degeneradas y vinculamos este problema con nuestra generalización de la pregunta de Dimca y Greuel. Como consecuencia, esto proporciona una nueva forma de entender la importancia de la conjetura de Durfee en el contexto de las singularidades aisladas de hipersuperficie.

La segunda parte trata sobre semigrupos numéricos y su combinatoria. Primero, abordamos la conjetura de Wilf sobre semigrupos numéricos, que propone un límite inferior de su conductor en términos del género y la dimensión de embebimiento del semigrupo numérico. En esta dirección, presentamos dos condiciones necesarias para que un semigrupo numérico tenga número de Eliahou negativo, que es un número cuya positividad implica la conjetura de Wilf. Como consecuencia, mostramos 36 nuevos ejemplos de semigrupos numéricos con número de Eliahou negativo que satisfacen la conjetura de Wilf. Una de nuestras principales contribuciones a la conjetura de Wilf es proponer su extensión a módulos sobre un semigrupo numérico, lo que proporciona una nueva perspectiva en algunos problemas relacionados con la conjetura de Wilf. Proporcionamos una fórmula para el conductor de un semimódulo sobre un semigrupo numérico con dos generadores y como consecuencia demostramos la generalización de la conjetura de Wilf en este caso particular. En segundo lugar, también estudiamos el conjunto de valores de módulos sobre el anillo local de una singularidad aislada de curva plana irreducible con un par de Puiseux que proporciona una generalización parcial de un Teorema de Bresinsky y Teissier sobre el semigrupo de valores de una singularidad aislada de curva plana irreducible. Como consecuencia, deducimos algunas características nuevas sobre el conjunto de valores de diferenciales de Kähler de una singularidad aislada de curva plana irreducible con un par de Puiseux.

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Since I am one of the more than 450 million people who have Spanish as their mother tongue, which makes it the second most widely spoken native language in the world, it is my wish to write most of these acknowledgments in Spanish. I hope the non-Spanish-speaking reader will understand this decision.

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Contents

Introduction

This work deals with analytic invariants of isolated hypersurface singularities and combinatorial invariants of numerical semigroups. Before to start with the introduction of each of them, let us briefly summarize their main contents.

The first part deals with analytic and topological invariants of an isolated hypersurface singularity, more specifically with the study of the relations between the Milnor number, of topological nature, the Tjurina number and the geometric genus, of analytic nature, and the spectral values, of Hodge-theoretical nature. Our main contributions are the following: first we provide a closed formula for the minimal Tjurina number in an equisingularity class of a plane branch in terms of topological invariants of the branch, secondly we address a question of Dimca and Greuel about the quotient of the Milnor and Tjurina numbers of an isolated plane curve singularity; we extend this question to isolated surface singularities in $\mathbb{C}^3$ which gives the clue to provide a complete answer to Dimca and Greuel's question. Moreover, we show the connection of the extended question with an old standing conjecture posed by Durfee. Finally, we establish K. Saito's continuous limit distribution for the spectrum of Newton non-degenerate isolated hypersurface singularities and link this problem with our generalization of Dimca and Greuel's question. As a consequence, this provides a new way of understanding the important role of Durfee's conjecture in the context of isolated hypersurface singularities.

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Part I. Analytic invariants of isolated hypersurface singularities

The first part of this memoir is completely devoted to study the interconnections between analytic and topological invariants of a germ of isolated hypersurface singularity. Let us consider an holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. The set of zeroes of f defines a hypersurface X in $\mathbb{C}^{n+1}$. Smooth points of a hypersurface are precisely those points where at least one partial derivative does not vanish; which in particular allows to regularly define a tangent space on it. Singular points are then those points where all partial derivatives vanish. Moreover, a singular point $p \in X$ is said to be isolated if there exist a sufficiently small neighborhood U of p such that any $q \in U \setminus \{p\}$ is a smooth point of X. Working with isolated singularities lead us to work with germs of functions. A germ of function at a point p is an equivalence class defined by the following equivalence relation: given two functions f and g, we say that f defines the same germ at p as g if there exists an open neighborhood V of p such that $f = g$ in V.

The study of an isolated hypersurface singularity $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by a germ of holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ can be done from two different perspectives. From the topological point of view, two germs (X, x) and (Y, y) of hypersurface singularity are topologically equivalent if there exists a homeomorphism $(\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ mapping (X, x) to (Y, y). In this setting, the work of Milnor [118] presents a fibration whose fibers, $F_\delta = \{z \in B_\epsilon \mid f(z) = \delta\}$ with $B_\epsilon \subset \mathbb{C}^{n+1}$ a small open ball of radii ϵ small enough and $\delta \in \mathbb{C} \setminus \{0\}$ small enough, are smooth and they encode the topology of the germ. Milnor proved [118] that F_δ has the homotopy type of a bouquet of μ spheres, i.e. $H_n(F_\delta) = \mathbb{Z}^\mu$. Nowadays, $\mu = \operatorname{rank} H_n(F_\delta)$ is called the *Milnor number* and it constitutes one of the main topological invariants of an isolated hypersurface singularity.

From the analytic point of view, two germs (X, x) and (Y, y) of hypersurface singularity are analytically equivalent if there exists a local analytic isomorphism $(\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ mapping (X, x) to (Y, y). Obviously, this implies that the notion of analytic equivalence is stronger than the topological one. In this setting, Tjurina in 1969 [166] proved that one can explicitly compute the miniversal deformation of a germ of isolated hypersurface singularity, which is a complex analytic germ of flat morphism $\pi : (\mathcal{X}, \mathbf{0}) \to (S, \mathbf{0})$ such that (X, x) is analytically isomorphic to $(\pi^{-1}(\mathbf{0}), \mathbf{0})$, any other deformation of (X, x) can be obtained by base change from it and S has minimal dimension between all deformations with this property [1]. More concretely, Tjurina [166] showed that if $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a defining equation for (X, x) then the base space S of its miniversal deformation is isomorphic to the $\mathbb{C}$–algebra $T_f := \mathbb{C}\{x_0, \dots, x_n\}/(f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$. In 1982, Mather and Yau [114] showed that the algebra T_f is a complete analytic invariant; its dimension is nowadays called the *Tjurina number*, τ, and it is thus one of the main analytic invariants of an isolated hypersurface singularity.

We will focus on the relationship between the Milnor number and the Tjurina number. If $(X, \mathbf{0}) \subset (\mathbb{C}^{n+1}, 0)$ is a germ of isolated hypersurface singularity defined by an equation f, the Milnor number, μ, and the Tjurina number, τ can be computed as follows

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \dots, x_n\}}{(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})}, \qquad \tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \dots, x_n\}}{(f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})}.$$

From these equalities, the inequality $\mu - \tau \geq 0$ becomes obvious. It was a trending topic during the seventies and the eighties up to what generality could this inequality be extended to more general singularities; as the the works of Greuel [68, 67], Looijenga-Steenbrink [109] or Wahl [179] show.

One of the motivations to study this topic is the following question posed in 2017 by Dimca and Greuel [46, Question 4.2]

[1] In fact Tjurina's result works for any isolated complete intersection singularity, see Appendix B. Also, generic fibers of the miniversal deformation are homeomorphic to the Milnor fiber of (X, x)

Question. *3.13 Is it true that $\mu/\tau < 4/3$ for any isolated plane curve singularity?*

One of the strategies to attack this problem is the computation of the minimal Tjurina number in a fixed family of singularities. In our case, it is natural to consider families of deformations with constant Milnor number and then to use the upper semicontinuity of τ (see Definition 1.42) to compute the minimal Tjurina number. As Milnor number of a plane curve can be computed from discrete invariants of a resolution of singularities (see Section 1.3.1), the main difficulty of the question relies on the computation of the Tjurina number.

In 1997, Peraire [131] provided an algorithm (see Section 2.3.2) which computes the minimal Tjurina number in the equisingularity class of any plane branch from its semigroup of values (see Definition 1.1.2). Using this algorithm we run several examples with a computer validating Dimca and Greuel guess. Unfortunately, we were technically unable to provide an estimate as the one showed in Dimca and Greuel Question 3.13. Therefore, we focused first on some particular families of plane curves. The main goal of Chapter 2 is to provide a closed formula for the computation of the minimal Tjurina number in any topological class of irreducible plane curve singularities. As a consequence, Chapter 2 contains our first attacks to Dimca and Greuel Question 3.13.

In 1988, Briançon, Granger and Maisonobe [25] provided a recursive formula for computing the minimal Tjurina number of a semi-quasihomogeneous plane curve singularity, i.e. a plane curve singularity defined by an equation with initial term $y^n - x^m$. Recall that those curves are irreducible if and only if $\gcd(n, m) = 1$. So the class of semi-quasihomogeneous plane curves includes some non-irreducible plane curves. Using this formula, in [6] we provided a positive answer to Dimca and Greuel Question 3.13 in this particular case.

Proposition. *2.3 Let C be a plane curve singularity (not necessarily irreducible) defined by an equation $f = y^n - x^m + h.o.t.$, then*

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

In [6], we also showed that the particular case of $\gcd(n, m) = 1$ could be treated alternatively from the point of view of studying the deformations of the quasihomogeneous initial term $y^n - x^m$ (see subsection 2.3.1). This approach is the one used by Zariski [197, Chap. VI] to study in detail the moduli space of a plane branch with one Puiseux pair. In particular, the main result in this direction is the identification of the dimension of the generic component of the moduli space with the minimal dimension of a miniversal equisingular deformation of a plane branch with one Puiseux pair. Those results of Zariski were extended by Teissier [165] to the case of any plane branch with any number of Puiseux pairs (see also Section 2.2). More specifically, Teissier showed that in order to compute the minimal Tjurina number of a plane branch in a fixed equisingularity class, there is no need to explicitly compute the moduli space of plane branches but just to understand the deformations of the monomial curve associated to the value semigroup of a plane branch (see Section 2.2.1).

In 2016, Genzmer [62] (see also Theorem 2.23) provided a formula for the dimension of the generic component of the moduli space of plane branches in terms of the sequence of multiplicities resulting from a resolution of singularities of a general element in this component. Using this formula together with Teissier's results (see Subsection 2.2.2) we provided [3] a closed formula for the minimal Tjurina number in a fixed equisingularity class of a plane branch in terms of topological invariants.

Theorem. *2.25 For any equisingular class of a germ of plane branch with multiplicity n and sequence of multiplicities $\{e_p\}_{p \in \mathcal{N}_O}$, it holds*

$$\tau_{min} = \sigma(n) + \frac{n^2+3n-6}{2} + \sum_{p\,free} \frac{(e_p-1)(e_p+2)+2\sigma(e_p+1)}{2} + \sum_{p\,sat.} \frac{e_p(e_p-1)+2\sigma(e_p+2)}{2},$$

where the summation runs on all points p equal or infinitely near to the origin and σ is defined as $\sigma(k) = (k-2)(k-4)/4$ if k is even, and $\sigma(k) = (k-3)^2/4$ if k is odd.

From this formula, we were able to provide a positive answer to Dimca and Greuel Question 3.13 for any irreducible plane curve singularity:

Corollary. *2.30 For any plane branch singularity,*

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Chapter 3 is focused in the study of Dimca and Greuel's question 3.13 from a more general perspective. To do that, we move to study isolated surface singularities in $\mathbb{C}^3$. In the case of isolated surface singularities in $\mathbb{C}^3$, there are several results concerning formulas for the difference $\mu-\tau$ provided by Wahl [179], Looijenga and Steenbrink [109] and Yau [190, 187, 191, 189]; moreover some of them work in a more general setting than just surface singularities (see Section 3.1). In [179], Wahl proved that for an isolated surface singularity the difference $\mu - \tau$ can be sharply bounded by twice another important invariant of a singularity, the geometric genus p_g. We can shortly define the geometric genus by means of resolution of singularities. If $(X, 0) \subset (\mathbb{C}^N, 0)$ is an isolated hypersurface singularity and $\widetilde{X} \to X$ is a resolution of singularities of X, then the geometric genus can be defined as $p_g := \dim_{\mathbb{C}} H^{N-2}(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. In this context, there is the following long standing conjecture by Durfee [44, Conjecture 5.3].

Conjecture (Durfee's Conjecture 1978). *3.19 For any isolated surface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$*

$$6p_g \leq \mu.$$

Still open, lots of particular cases are known where Durfee's conjecture 3.19 holds (see Section 3.3); in particular, this is the case for a surface singularity in $\mathbb{C}^3$ defined by an equation of the form $F(x, y, z) = f(x, y) + z^2$. This particular case was proven by Tomari in 1993 [168]; moreover he proved that $8p_g(F) < \mu(F)$ (Theorem 3.14). Thus, Tomari's inequality, combined with Wahl's Theorem 3.10 which provides $\mu(F) - \tau(F) \leq 2p_g(F)$, are the key of our complete solution [5] to Dimca and Greuel question 3.13:

Theorem. *3.15 For any germ C of plane curve singularity*

$$\frac{\mu(C)}{\tau(C)} < \frac{4}{3}.$$

From this general point of view, it is natural to extend Dimca and Greuel's question to more general singularities. For isolated complete intersection singularities (ICIS) in $\mathbb{C}^N$ of dimension $n = N - r$ defined by an ideal $\mathcal{I} = (f_1, \ldots, f_r)$, Hamm [73, Satz 1.7] showed that the Milnor fiber of $(X, 0)$ is homotopy equivalent to a bouquet of spheres, extending previous results of Milnor [118].

On the other hand, Tjurina's work [166] identified $\mathrm{Ext}^1_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)})$ as the base space of the miniversal deformation of a normal isolated singularity with $\mathrm{Ext}^2_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)}) = 0$, e.g. ICIS (see also Appendix B for a survey on Tjurina's result). Thus, following Hamm and Tjurina's results, the Milnor and Tjurina numbers of an ICIS can be defined as

$$\mu := \mathrm{rk}\, H_n(F), \quad \tau := \dim_{\mathbb{C}}(\mathrm{Ext}^1_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)})),$$

where F is the Milnor fiber of the smoothing of $(X, 0)$, $\mathcal{O}_{(X,0)} = \mathbb{C}\{x_1, \ldots, x_N\}/\mathcal{I}$ and $\Omega^1_{(X,0)}$ is the corresponding module of differential 1–forms at $(X, 0)$. Thus, we propose the following general problem which extends Dimca and Greuel's question:

Problem. *3.1 Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity of dimension n and codimension $k = N - n$. Is there an optimal $\frac{b}{a} \in \mathbb{Q}$ with $b < a$ such that*

$$\mu - \tau < \frac{b}{a}\mu \text{ ?}$$

Here optimal means that there exists a family of singularities such that μ/τ tends to $\frac{a}{a-b}$ when the multiplicity at the origin tends to infinity.

Chapter 3 also contains two partial solutions to Problem 3.1 in the case of space curves with the semigroup of a plane branch and in the case of surface singularities satisfying Durfee's conjecture. The first one uses the results of Teissier about the deformations of the monomial curve associated to a plane branch explained in Section 2.2. This monomial curve is defined from the semigroup of values of an irreducible plane curve singularity. If this semigroup is generated by more than two elements, then the monomial curve is no longer a plane curve. Moreover, there exist deformations of it with the same semigroup of values which are not a plane curve. In this direction subsection 3.2.1 is devoted to provide a partial answer to Problem 3.1 in this particular case:

Corollary. *3.17 Let $(C, \mathbf{0}) \subset (\mathbb{C}^N, \mathbf{0})$ be an irreducible space curve with semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ of an irreducible plane curve singularity. Then,*

$$\mu(C) - \tau(C) < \frac{\mu(C)}{4}.$$

On the other hand, we can also use the results where Durfee's conjecture holds in order to provide the following partial result in the hypersurface case of dimension 2 of the Problem 3.1.

Proposition. *3.31 Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity of one of the following types:*

(1) Quasi-homogeneous singularity,

(2) $(X, 0)$ of multiplicity 3,

(3) absolutely isolated singularity,

(4) suspension of the type $\{f(x, y) + z^N = 0\}$,

(5) the link of the singularity is an integral homology sphere,

(6) the topological Euler characteristic of the exceptional divisor of the minimal resolution is positive.

Then

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

Moreover, we show in Example 3.28 that this bound is sharp. Therefore, this motivates us to propose the following conjecture.

Conjecture. *3.32 For any $(X,0) \subset (\mathbb{C}^3,0)$ isolated surface singularity:*

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

Durfee's conjecture implies our conjecture about the quotient between the Milnor number and the Tjurina number in the case of surface singularities in $\mathbb{C}^3$. Thus, conjecture 3.32 provides a necessary condition for Durfee's conjecture to be true.

Chapter 4 is devoted to study a generalization of Durfee's conjecture to the case of isolated hypersurface singularities provided by K. Saito in [146], and some related problems. In 1983, K. Saito [146] initiated the study of the distribution of another set of invariants which are called the exponents (see subsection 1.3.3). They are analytic invariants of Hodge-theoretical nature and for $n \geq 2$ they are generally not topological invariants. Briefly speaking, the set of exponents of a germ of isolated hypersurface singularity $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ is a set of μ, μ being the Milnor number, rational numbers which are certain logarithms of the monodromy eigenvalues of the Milnor fiber $\{\alpha_1,\ldots,\alpha_\mu\} \subset \mathbb{Q} \cap (0,n+1)$ (see subsection 1.3.3). It is also usual to consider them into a normalized generating function:

$$\chi_f(T) := \frac{1}{\mu}\sum_{j=1}^{\mu} T^{\alpha_j}.$$

Observe that with the change of variable $T = \exp(2\pi t)$, $\chi_f(t)$ can be seen as the Fourier representation of the discrete probability density of the set of exponents in the real interval $(0,n+1)$ defined by

$$\frac{1}{\mu}\sum_{i=1}^{\mu} \delta(s-\alpha_i)ds,$$

where $\delta(s)$ is Dirac's delta function. The general philosophy behind K. Saito's distribution problem is the following one: we start with a finite set of rational numbers in a real interval; then we should define a limit in order to push this finite set to be an infinite set of rational numbers. This infinite set should be distributed in the real interval in a uniform way which is provided by the continuous probability distribution $N_{n+1}ds$ defined as

$$N_{n+1}(s)ds := \int_{x_0+\cdots+x_n=s} \varphi(x_0)\cdots\varphi(x_n)dx_0\cdots dx_n,$$

where φ is the indicator function of the unit interval $[0,1]$, i.e. $\varphi(x) := \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{if } x \notin [0,1]. \end{cases}$

From this point of view, K. Saito [146, (2.5)] proposed the following

Problem. *4.1 Let $\alpha_1,\ldots,\alpha_\mu$ be the exponents of an isolated hypersurface singularity. To show whether the continuous distribution $N_{n+1}ds$ is the "limit" of the distribution of the exponents as f "moves".*

K. Saito provided a solution to Problem 4.1 in the case of quasi-homogeneous singularities and in the case of irreducible plane curve singularities [146, (3.7),(3.9)]. In the case of quasi-homogeneous singularities of degree 1 with respect to the weights $(w_0,\ldots,w_n)$, K. Saito showed [146, (3.7)] that

$$\lim_{w_0,\dots,w_n\to 0}\chi_f(t)=\Big(\frac{\sin(\pi t)}{\pi t}\exp(i\pi t)\Big)^{n+1},$$

where he observed that the Fourier transform $\mathcal{F}$ of the continuous distribution $N_{n+1}ds$ is

$$\mathcal{F}(N_{n+1})=\Big(\frac{\sin(\pi t)}{\pi t}\exp(i\pi t)\Big)^{n+1}.$$

Therefore, "convergence "in Problem 4.1 means that the Fourier transformation of the distribution converges uniformly on compact sets. On the other hand, for an irreducible plane curve with Puiseux characteristic pairs $(n_1,q_1),\dots(n_g,q_g)$, K. Saito [146, (3.7)] showed that $\lim_{n_g\to\infty}\chi_f(t)=\mathcal{F}(N_2)$. Therefore, the general choice of a limit is unclear.

Following some ideas used by K. Saito for the quasi-homogeneous case, the main result of Chapter 4 is to establish K. Saito's continuous limit distribution for the spectrum of Newton non-degenerate hypersurface singularities.

Theorem. *4.20 For a fixed Newton diagram Γ, consider the Newton diagrams $\varpi\Gamma$ obtained from Γ by scaling with the factor ϖ. Then we have*

$$\lim_{\varpi\to\infty}\chi_{f_\varpi}=\mathcal{F}(N_{n+1}),$$

where f_ϖ is any Newton non-degenerate function germ of $n+1$ variables with Newton diagram $\varpi\Gamma$.

K. Saito [146, (2.5) ii), (2.8) i)] further suggested to describe up to what extent the spectral distribution is bounded by N_{n+1} and introduced the notion of (weakly) dominating values. Consider the function

$$\Phi\colon[0,1]\to\mathbb{R},\quad r\mapsto\int_0^r N_{n+1}(s)-\frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds$$

defined by the difference of the continuous and discrete exponent distributions. By definition, $0<r<(n+1)/2$ is a *dominating value* if $\Phi(r)>0$ for all f in $n+1$ variables. In the case of irreducible plane curve singularities, we have found the following previously unknown dominating value, thus providing a particular answer to K. Saito's problem.

Theorem. *4.18 For any irreducible plane curve singularity $C=f^{-1}(0)$ with value semigroup $\Gamma:=\langle\overline{\beta}_0,\overline{\beta}_1,\dots,\overline{\beta}_g\rangle$ different from $\langle 2,3\rangle$ and $\langle 2,5\rangle$, we have $\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1})>0$. In other words,*

$$\Big(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\Big)^2>\frac{2}{\mu},$$

where μ is the Milnor number of C. Moreover, we have $\lim_{n_g\to\infty}\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1})=0$, where $n_g=\gcd(\overline{\beta}_0,\dots,\overline{\beta}_{g-1})$.

Besides that, our interest on K. Saito's notion of dominating values lies in the fact that it provides an explanation and generalization of Durfee's Conjecture 3.19. To conclude, let us summarize our idea of a possible strategy to attack Problem 3.1 in the case of isolated hypersurface singularities. As showed by M. Saito [149, 152] the Brieskorn lattice, i.e. the $\mathbb{C}\{t\}$–module

$$H_0'':=\frac{\Omega^{n+1}_{(\mathbb{C}^{n+1},0)}}{df\wedge d\Omega^{n-1}_{(\mathbb{C}^{n+1},0)}},$$

plays a key role for the understanding of the analytic structure of an isolated hypersurface singularity. An important part of the structure of the Brieskorn lattice is that it carries a V–filtration in the sense of Kashiwara and Malgrange [84, 113]. After a result of M. Saito [147], the geometric genus equals the number of spectral values less or equal than 1; moreover this result can be read in terms of the V–filtration as

$$p_g = \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}.$$

On the other hand, H_0'' possess natural actions of the operators t and ∂_t^{-1} defined by $t[\omega] := [f\omega]$ and $\partial_t^{-1}[\omega] := [df \wedge \alpha]$, where $\omega \in \Omega^{n+1}_{(\mathbb{C}^{n+1},0)}$, $\alpha \in \Omega^{n}_{(\mathbb{C}^{n+1},0)}$ and $[\bullet]$ denotes the class of a $(n+1)$–form in H_0'' (see Section 1.3.3 for a more detailed treatment). Therefore, we have the following equality

$$\mu - \tau = \dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} tH_0''}{\partial_t^{-1} tH_0''}.$$

After Wahl's Theorem 3.10, i.e. $\mu - \tau \leq 2p_g$ for isolated surface singularities in $\mathbb{C}^3$, we think that it is natural to ask for the following generalization of Wahl's Theorem 3.10.

Question. *4.11 Let $(X,0) \in (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity with $n \geq 2$. Does exist $C(n) \in \mathbb{Q}$ such that*

$$\dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''} \leq C(n) \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}?$$

Observe that for $n = 2$, Question 4.11 can be answered by means of Wahl's Theorem 3.10 with $C(2) = 2$. On the other hand, for $n = 1$ our solution to Dimca and Greuel's Question 3.13, Theorem 3.15, implies that $C(1) = 1/2$. As far as the author's knowledge, Question 4.11 is open for $n \geq 3$. Moreover, it would provide a powerful tool combined with the extension of Durfee's conjecture provided by K. Saito in [146]. K. Saito [146] asked for 1 to be a dominating value, i.e. $\Phi_f(1) > 0$, for any isolated hypersurface singularity in $n+1$ variables with $n \geq 2$. In fact, K. Saito [146, §2] showed that $\Phi_f(1)$ can be explicitly computed as

$$\int_0^1 (N_{n+1}(s) - \frac{1}{\mu} \sum_{i=1}^{\mu} \delta(s - \alpha_i))ds = \frac{1}{(n+1)!} - \frac{p_g}{\mu}.$$

Therefore, $\Phi_f(1) > 0$ translates into the inequality $p_g < \mu/((n+1)!)$; thus providing a generalization of Durfee's conjecture as pointed out by K. Saito [146, pg. 203]. That generalization together with a solution to Question 4.11 implies the following sequence of inequalities

$$\mu - \tau = \dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''} \leq C(n) \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}} = C(n) p_g \leq \frac{C(n)}{(n+1)!}\mu. \quad (0.1)$$

Equation (0.1) provides in a single line of inequalities a relation between the main analytic and topological invariants of an isolated hypersurface singularity. Moreover, it would show how the topology, i.e. the Milnor number and the embedding dimension, constrain the Tjurina number, the distribution of spectral values and the geometric genus. From this point of view, we think that the first part of this memoir provides a strong motivation to investigate Equation (0.1) in order to gain understanding of the interplay between those important invariants of a hypersurface singularity.

Part II. Combinatorial invariants of numerical semigroups and modules over them

The second part of this memoir is devoted to the study of combinatorial invariants of numerical semigroups and modules over them. A semigroup $(\Gamma, +)$ over $\mathbb{N}$ is an additive submonoid of the natural numbers[2]; usually we will drop the $+$ symbol if there is no confusion. Observe that in particular $k\mathbb{N}$ is a semigroup over $\mathbb{N}$ with infinite complement over $\mathbb{N}$. However, we will be interested in the following class of semigroups over the natural numbers:

Definition 0.1. A *numerical semigroup* Γ is an additive submonoid of the natural numbers with finite complement.

A direct consequence of the finitude of the complement is that any numerical semigroup Γ over the natural numbers is finitely generated, this means that there exist $a_1, \ldots, a_e \in \Gamma$ such that

$$\Gamma := \{n \in \mathbb{N} : \ n = \alpha_1 a_1 + \cdots + \alpha_e a_e, \ \alpha_i \in \mathbb{N}\} =: \langle a_1, \ldots, a_e \rangle.$$

Two of the main combinatorial invariants of a numerical semigroup are its genus and its embedding dimension. The elements in $\mathbb{N} \setminus \Gamma$ are said to be the *gaps* of the semigroup, and the cardinal of the set of gaps, which we denote by $g(\Gamma)$, is called the *genus* of Γ. On the other hand, the *embedding dimension*, $e(\Gamma)$, of a semigroup is defined as the cardinal of a minimal set of generators of Γ. Since $\mathbb{N} \setminus \Gamma$ is a finite set, there exists $c \in \Gamma$ such that $c + \mathbb{N} \subset \Gamma$ and then the structure of that part of the semigroup is isomorphic to $\mathbb{N}$. The minimal $c := c(\Gamma)$ such that $c + \mathbb{N} \subset \Gamma$ is called *conductor* of the semigroup. The *Frobenius number* is defined as the greatest gap of a semigroup, that is $c(\Gamma) - 1$.

Numerical semigroups became notorious in Mathematics due to the Frobenius problem, which asks for a closed formula for the Frobenius number. A measure of the difficulty of such problem is the well known fact that the computation of $c(\Gamma) - 1$ is a NP–hard problem under Turing reductions, which means that it cannot be solved with a polynomial time algorithm (see Ramírez-Alfonsín book [138] for a good account on this).

In 1978, H. Wilf [185] asked a question relating the genus, the embedding dimension and the conductor of a numerical semigroup. Nowadays, this question is known as Wilf's conjecture for numerical semigroups.

Conjecture. 5.1***(Wilf's conjecture 1978)*** *Let Γ be a numerical semigroup Γ, assume that Γ is minimally generated by $e(\Gamma)$ elements, $g(\Gamma) = |\mathbb{N} \setminus \Gamma|$ and that $c(\Gamma)$ is the smallest natural number such that $c(\Gamma) + \mathbb{N} \subset \Gamma$. Then,*

$$\frac{g(\Gamma)}{c(\Gamma)} \leq 1 - \frac{1}{e(\Gamma)}.$$

Chapter 5 and Chapter 6 are devoted to deepen Wilf's conjecture and extend it to more general structures over a semigroup. As we will see in Chapter 5, the state of the art about the solution of the conjecture shows that it is far from being solved either far from being well understood. Our main contributions to this topic try to shed some light into this problem.

An equivalent formulation of Wilf's conjecture can be done in terms of the elements in the semigroup which are strictly smaller than the conductor. The *δ–invariant* of a semigroup is defined

[2] Being a controversial subject in mathematics, it is necessary to clarify that in this work we consider 0 to be a natural number.

as $\delta(\Gamma) := |\{s \in \Gamma : s < c(\Gamma)\}|$. Therefore, one has the obvious equality $c(\Gamma) = g(\Gamma) + \delta(\Gamma)$ which allows to read Wilf's conjecture as

$$e(\Gamma)\delta(\Gamma) \geq c(\Gamma).$$

Our approach for understanding Wilf's conjecture has been to define a linear function, which we call *Wilf function of a semigroup*, as

$$W_\Gamma(k) := k\delta(\Gamma) - c(\Gamma).$$

From this point of view, we ask for

$$\mu_\Gamma := \min\{k \in \mathbb{N} : W_\Gamma(k) \geq 0\}$$

and then Wilf's conjecture can be translated as: does $e(\Gamma) \geq \mu_\Gamma$ hold for any numerical semigroup Γ?

In Chapter 5, we show how this new approach to Wilf's conjecture is useful in order to gain some understanding of this conjecture and its related problems. In particular, some related questions have emerged from partial solutions to Wilf's conjecture. The first one is a question proposed by Moscariello and Sammartano in 2015 [119]:

Conjecture (Frögohämosa-conjecture). 5.2 *Let $\Gamma \neq \mathbb{N}$ be a numerical semigroup. The equality $c(\Gamma) = e(\Gamma) \cdot \delta(\Gamma)$ holds if and only if Γ has embedding dimension 2 or there exist $m, q \in \mathbb{N} \setminus \{0\}$ with $m > 1$ such that*

$$\Gamma = W_{m,q} := \{0, m, 2m, 3m, \ldots, (q-1)m, qm, qm+1, qm+2, \ldots\}.$$

It is an intriguing question in numerical semigroup theory to understand why those and only those semigroups satisfy the equality in Wilf's conjecture. One of our contributions has been to show that the semigroups appearing in Frögohämosa-conjecture are precisely those with an extreme behavior of its Wilf function, which in particular offers a possible explanation to understand the semigroups satisfying equality in Wilf's conjecture

Theorem. 5.30 *Let Γ be a numerical semigroup. Then,*

1. *$\Gamma = \mathbb{N}$ if and only if $W_\Gamma(k) \geq 0$ for $1 \leq k \leq m$.*
2. *$\Gamma = W_{m,q}$ for $m, q \in \mathbb{N} \setminus \{0\}$ if and only if $W_\Gamma(k) \leq 0$ for all $1 \leq k \leq m$.*
3. *$\Gamma = \langle a, b\rangle$ with $\gcd(a, b) = 1$ is the numerical semigroup with minimal embedding dimension between those satisfying $W_\Gamma(k) \geq 0$ for all $2 \leq k \leq m$ and $W_\Gamma(2) = 0$.*

The second related problem with Wilf's conjecture that we are going to study is in connection with a new invariant introduced by Eliahou in 2018 [51] (see Section 5.4 for a precise definition), called *Eliahou number* of a numerical semigroup, $E(\Gamma)$. Its main property is the following one:

$$W_\Gamma(e(\Gamma)) \geq E(\Gamma).$$

Therefore, any numerical semigroup with positive Eliahou number satisfies the Wilf conjecture. Unfortunately, there exist numerical semigroups with negative Eliahou number [36, 51, 52] and thus the Wilf conjecture is reduced to study those semigroups.

If we denote by e_s the number of minimal generators less than $c(\Gamma)$, our approach will reduce the study of the semigroups with negative Eliahou number to the investigation of semigroups with $\mu_\Gamma \geq e_s$. We will show that the Eliahou number is bounded below by $W_\Gamma(e_s)$. Thus, the negativity of the Eliahou number implies $\mu_\Gamma > e_s$. Therefore providing the following necessary condition for a semigroup to have negative Eliahou number.

Theorem. *5.36 Let Γ be a numerical semigroup with Eliahou number $E(\Gamma) < 0$. Let us denote by e_s the number of minimal generators which are strictly less than $c(\Gamma)$ and by e_c the ones which are bigger or equal than $c(\Gamma)$. Then,*

$$W_\Gamma(e(\Gamma)) < e_c\delta(\Gamma).$$

In particular, $\mu_\Gamma > e_s$.

The characterization of numerical semigroups with negative Eliahou number is a huge challenge, as observed in [36, 38, 52], and very few general properties about them are known. One of the main contributions of Chapter 5 is to establish necessary conditions for a numerical semigroup in order to have negative Eliahou number, see Section 5.4. We present two of them: the first one is the already mentioned strict inequality $\mu_\Gamma > e_s$, which has the advantage of making it possible to find semigroups satisfying it in a much easier way than fulfilling the straightforward inequality $E(\Gamma) < 0$, as Example 5.35 shows.

The second necessary condition for a semigroup to have negative Eliahou number is based on the recent concept introduced in 2021 by Rosales et al. [141] of concentration of a numerical semigroup: set $\text{next}_\Gamma(s) := \min\{x \in \Gamma : s < x\}$; the *concentration of a numerical semigroup* is defined as

$$\mathbf{C}(\Gamma) = \max\{\text{next}_\Gamma(s) - s : s \in \Gamma \setminus \{0\}\}.$$

If we call $m(\Gamma) := \min(\Gamma \setminus \{0\})$ the *multiplicity* of the semigroup, it is clear that a numerical semigroup with concentration 1 is of the form $\{0, m(\Gamma), \rightarrow\}$, where the arrow $\rightarrow$ means that from $m(\Gamma)$ on all natural numbers belong to the set. We will show that, if a semigroup has negative Eliahou number, then the square of its concentration should be lower bounded by certain constant depending on its multiplicity and its conductor.

Corollary. *5.38 Let Γ be a numerical semigroup with multiplicity m and concentration $\mathbf{C}(\Gamma) = k$, and write the conductor of Γ as $c(\Gamma) = Lm + \rho$ with $2 \leq \rho \leq m$. If $E(\Gamma) < 0$, then $m/k^2 < (L+1)/(L-1)$.*

Summarizing, it is certainly well-known that finding numerical semigroups with negative Eliahou number is an acute trouble (see for example [36, 38, 52]). While the requirement of necessary conditions appears to enlarge the set of candidates to study the Wilf conjecture, they may lead to a more simple characterization of the interesting families of semigroups at issue in order to prove Wilf's conjecture. For instance, we have been able to find 36 new examples of numerical semigroups with negative Eliahou number, all of them satisfying Wilf's conjecture, see Example 5.42.

In Chapter 6 we propose and motivate an extension of Wilf's conjecture to certain additive structures over a numerical semigroup Γ. In analogy with an ideal over a ring, a similar structure can be defined in the context of numerical semigroups. A set $\Delta \subset \mathbb{N}$ is called *Γ–semimodule* if it is closed under addition with respect to elements of the semigroup, i.e. $\Delta + \Gamma \subset \Delta$. Usually, when referring to Γ–semimodules we will be thinking about *normalized Γ–semimodules,* i.e. those Γ–semimodules satisfying $\Gamma \subset \Delta$. There is no loss of generality in this assumption since given a Γ–semimodule, Δ, there always exists another one, $\Delta^\circ = \Delta - \min(\Delta)$, containing Γ and isomorphic to Δ. Observe that, Γ is itself a normalized Γ–semimodule.

Analogously to the numerical semigroup case, the conductor, genus, δ–invariant and embedding dimension can be defined for a general Γ–semimodule Δ. Hence it makes sense to study the Wilf function associated to a Γ–semimodule Δ :

$$W_\Delta(k) := k\delta(\Delta) - c(\Delta).$$

It is then reasonable to ask whether the natural generalization of Wilf's conjecture to this setting, i.e. $W_\Delta(e(\Delta)) \geq 0$, may hold. Our starting point has been to consider numerical semigroups of the form $\Gamma = \langle \alpha, \beta \rangle$ and its Γ–semimodules of the form $\Delta = \Gamma \cup (\Gamma + g)$ with $g \in \mathbb{N} \setminus \Gamma$. In this setting our first result has been to provide the following formula for the conductor of such Γ–semimodules.

Theorem. *6.11 Let $\Gamma = \langle \alpha, \beta \rangle$. Let Δ be a Γ-semimodule. Assume that Δ is minimally generated by $\{g_0 = 0, g_1, \ldots, g_s\}$ and define its Γ–semimodule of syzygies as*

$$\mathrm{Syz}(\Delta) := \bigcup_{i,j \in I, i \neq j} \Big((\Gamma + g_i) \cap (\Gamma + g_j) \Big) = \bigcup_{i=0}^{s} (\Gamma + h_i).$$

Let M denote the biggest (with respect to the order of the natural numbers) minimal generator of $\mathrm{Syz}(\Delta)$*. Then*

$$c(\Delta) = M - \alpha - \beta + 1.$$

From Theorem 6.11 we have been able to provide an explicit expression for the Wilf function in the following particular case.

Proposition. *6.30 Let $\Gamma = \langle \alpha, \beta \rangle$. Let $g = \alpha\beta - a\alpha - b\beta$ be a gap of Γ. Let Δ be the Γ–semimodule generated by $\{0, g\}$. Then*

$$-W_\Delta(e(\Delta)) = \begin{cases} a\alpha - 2ab \text{ if } & \min\{\alpha\beta - b\beta, \alpha\beta - a\alpha\} = \alpha\beta - b\beta, \\ b\beta - 2ab \text{ if } & \min\{\alpha\beta - b\beta, \alpha\beta - a\alpha\} = \alpha\beta - a\alpha. \end{cases}$$

Proposition 6.30 implies that the natural generalization of Wilf's conjecture does not hold in general. This is because, in the particular case of the hypothesis of Proposition 6.30, $W_\Delta(e(\Delta))$ attains positive and negative values (see also Example 6.46). Therefore, we propose the following extension of Wilf's conjecture for Γ–semimodules:

Question. *6.28 Let Γ be a numerical semigroup with any number of minimal generators and Δ be a Γ–semimodule with any number of minimal generators.*

(1) Find a characterization of

$$\widetilde{\mu}_\Gamma := \min\{k \in \mathbb{N} : W_\Delta(k) \geq 0 \text{ for all } \Gamma\text{-semimodules } \Delta\}.$$

(2) Is $\widetilde{\mu}_\Gamma$ related to any invariant of Γ?

(3) For all Γ-semimodules Δ with e *minimal generators, characterize*

$$\widetilde{\mu}_{\Gamma,\mathrm{e}} := \min\{k \in \mathbb{N} : W_\Delta(k) \geq 0 \text{ for all } \Delta \text{ with } \mathrm{e} \text{ minimal generators}\}.$$

(4) Can $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)}$ be computed from $\widetilde{\mu}_\Gamma$?

(5) For a fixed Γ, describe those Γ–semimodules Δ such that $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)} \leq \mathrm{e}(\Delta)$.

(6) Find a characterization of those numerical semigroups Γ such that for any Γ–semimodule Δ one has $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)} \leq \mathrm{e}(\Delta)$, i.e. $W_\Delta(\mathrm{e}(\Delta)) \geq 0$.

Chapter 6 also contains several results concerning the particular case of $\Gamma = \langle \alpha, \beta \rangle$ and its Γ–semimodules minimally generated by two elements. Proposition 6.30 allows us to detect a new symmetry property between the gaps of a numerical semigroup $\Gamma = \langle \alpha, \beta \rangle$, as we see in Section 6.3. Those stronger symmetries enlarge the well known symmetry between gaps and elements of semigroups of this type. Those new symmetries have motivated us to introduce the new concepts of supersymmetric and self-symmetric gaps.

Definition. *6.36 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let us denote by $\mathcal{T}_r$ the set of lattice points of $\mathcal{L} := \mathbb{N}^2$ inside (and not in the border of) the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ and $\mathcal{T}_u$ the set of points of $\mathcal{L}$ inside (and not in the border of) the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$. The set of* supersymmetric gaps *is defined to be*

$$\mathsf{SG} := \begin{cases} \mathcal{T}_u & \text{if} \quad |\mathcal{T}_u| < |\mathcal{T}_r| \\ \mathcal{T}_r & \text{if} \quad |\mathcal{T}_r| < |\mathcal{T}_u|. \end{cases}$$

We also define the set of self-symmetric gaps

$$\mathsf{SSG} := \{g \in \mathbb{N} \setminus \Gamma : \ W(g) = 0\}.$$

Also, we have detected that this set of gaps together with its symmetries codify the inner structure of the gaps of a numerical semigroup with two generators.

Theorem. *6.38 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Then the set $\mathsf{SG} \cup \mathsf{SSG}$ of supersymmetric and self-symmetric gaps completely determines the set of gaps of Γ. In particular, it determines Γ itself.*

In Section 6.3.3, we have also proposed a possible generalization of those new concepts to general numerical semigroups. In this generalization, the main underlying idea is that the Wilf function associated to the semimodules generated by $\{0, g\}$ with $g \in \mathbb{N} \setminus \Gamma$ reveals some new properties of the inner structure of the set of gaps of a numerical semigroup. In this direction we have been able to prove the following particular case of Question 6.28.

Theorem. *6.54 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with two generators. Let $g \in \mathbb{N} \setminus \Gamma$ be a gap of the semigroup. Let Δ be the Γ–semimodule minimally generated by $\{0, g\}$. Then,*

$$W_{\Delta}(3) \geq 0.$$

In particular, $\widetilde{\mu}_{\Gamma,2} = 3$.

In a more general setting of a numerical semigroup Γ with any number of generators, we have related the Wilf function of such Γ–semimodules with the Wilf function of the numerical semigroup as follows.

Theorem. *6.19 Let Γ be a numerical semigroup. Then*

$$\max(W_{\Delta}(2)) \leq W_{\Gamma}(4),$$

where max *runs over all possible Γ–semimodules with $e(\Delta) = 2$.*

Chapter 7 deals with value set of modules over the local ring of an irreducible plane curve singularity with one Puiseux pair. In the case of an irreducible plane curve singularity C with any number of Puiseux pairs defined by a germ of function f, one has naturally a discrete valuation $v : R := \mathbb{C}\{x, y\}/(f) \to \mathbb{N} \cup \{\infty\}$. Moreover, the value set of the local ring of the curve $\Gamma(C) := v(R) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ has a natural structure of numerical semigroup and it constitutes a topological invariant of the singularity (see Section 1.1.2). Also, if we denote by $n_i := \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1})/gcd(\overline{\beta}_0, \ldots, \overline{\beta}_i)$ we know that $\Gamma(C)$ satisfy the following properties

(1) $n_i\overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$,

(2) $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$ for all $i = 1, \ldots, g$.

In 1972, Bresinsky [23, Theorem 2] and Teissier [165, Chap. I. 3.2] independently proved that for any numerical semigroup Γ satisfying conditions (1) and (2) there exist a plane branch $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ such that $\Gamma = \Gamma(C)$ (see also Theorem 3.16). The analytic counterpart of Γ is the value set of Kähler differentials. The value set of Kähler differentials of an irreducible plane curve singularity is defined as $\Delta' = v(Rdx + Rdy)$ and it was proved by Delorme in 1978 [42] that it is an analytic invariant. Moreover, one can easily check that Δ' has a natural structure of Γ–semimodule, its normalization is given by $\Delta = \Delta' - (n-1)$, where n is the multiplicity of C at the singular point, and $\Delta = v(R + Rdy/dx)$.

In Chapter 7 we focus on the study of the set of values of modules over the local ring of an irreducible plane curve singularity with one Puiseux pair, i.e. irreducible plane curves with $\Gamma(C) = \langle \overline{\beta}_0, \overline{\beta}_1 \rangle$. Our main goal is to extend to Γ–semimodules Bresinsky-Teissier Theorem 3.16. For this purpose we define the following class of Γ–semimodules of a numerical semigroup minimally generated by two elements.

Definition. *7.3 Let Γ be a numerical semigroup of the form $\langle \alpha, \beta \rangle$. A Γ-semimodule L of a numerical semigroup Γ is called an increasing semimodule if it satisfies the following property:*

If L has minimal set of generators $\{g_0 = 0, g_1, \ldots, g_s\}$ and we put $g_{s+1} = \infty$, $u_0 = 0$, then for all $0 \leq i \leq s$ we have $g_{i+1} > u_i$, where $u_i = \min\{(\Gamma + g_i) \cap E_{i-1}\}$ for $1 \leq i \leq s$ and $E_i = \bigcup_{0 \leq j \leq i} (\Gamma + g_j)$ for $0 \leq i \leq s$.

We will see that the class of increasing Γ–semimodules contains the class of value sets of Kähler differentials of any plane branch of semigroup Γ. We will provide a geometric method to compute all possible increasing Γ–semimodules of a given Γ and as a consequence we will see that this class has a natural tree structure. After that, we will show that any increasing Γ–semimodule can be realized as the value set of a certain module over the local ring of an irreducible plane curve with one Puiseux pair.

Theorem. *7.16 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with $\alpha < \beta$. Let L be an increasing Γ-semimodule, and set $b := c(\Gamma) - \beta - 1$. Then there exist a tuple $(a_1, \ldots, a_b) \in \mathbb{C}^b$ and $z \in \mathbb{C}\{t\}$ such that $L = v(R + zR)$, where R is the local ring of the germ of plane curve singularity defined by the Puiseux parameterization*

$$C : \begin{cases} x(t) := t^{\alpha} \\ y(t) := t^{\beta} + \sum_{i=1}^{b} a_i t^{i+\beta}. \end{cases}$$

Theorem 7.16 provides an extension of Bresinsky-Teissier Theorem in this particular case. The converse statement of Theorem 7.16 in the case of value sets of Kähler differentials was proved in 1978 by Delorme [42, Lemma 12]; who showed that any value set of Kähler differentials of an irreducible plane curve singularity with one Puiseux pair satisfies the conditions of being an increasing semimodule. Delorme's result combined with our Theorem 7.16 provide the following corollary.

Corollary. *7.22 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Then, L is an increasing Γ–semimodule with first non-zero minimal generator equal to $\beta - \alpha$ if and only if L can be realized as the set of values of Kähler differentials of an irreducible plane curve singularity with value semigroup Γ.*

In particular, this implies that the value sets of Kähler differentials can be computed from the combinatorics of the increasing semimodules over the semigroup of the curve. Finally, since our proof of Theorem 7.16 is constructive then we have been able to implement it in the case of value set of Kähler differentials, as one can see in Appendix A.

List of Publications

The main results of this memoir have given rise to the following research articles:

Chapter 2

1. P.Almirón, G. Blanco: **A note on a question of Dimca and Greuel**, C.R. Acad. Sci. Paris, Ser. I (2019). DOI: 10.1016/j.crma.2019.01.002.

2. M.Alberich-Carramiñana, P. Almirón, G.Blanco, A. Melle-Hernández: **The minimal Tjurina number of irreducible germs of plane curve singularities**. Indiana Univ. Math. J. 70 No. 4 (2021), 1211–1220. DOI: 10.1512/iumj.2021.70.8583.

3. P. Almirón: **The 4/3 problem for germs of isolated plane curve singularities**. In Extended Abstracts GEOMVAP 2019. Birkhäuser. Trends in Mathematics. 2021, 15, 139-143.DOI: 10.1007/978 − 3 − 030 − 84800 − 2_23

Chapter 3

4. P. Almirón: **On the quotient of Milnor and Tjurina numbers for two-dimensional isolated hypersurface singularities**. To appear in Mathematische Nachrichten. Preprint in arxiv:1910.12843.

Chapter 4

5. P. Almirón, M. Schulze: **Limit spectral distribution for non-degenerate hypersurface singularities**. To appear in Comptes Rendus Mathématique. arxiv: 2012.06360.

Chapter 5

6. P. Almirón, J.J. Moyano-Fernández: **Eliahou number, Wilf function and concentration of a numerical semigroup**. To appear in Quaestiones Mathematicae. DOI: 10.2989/16073606.2022.2041126.

Chapter 6

7. P. Almirón, J.J. Moyano-Fernández: **A formula for the conductor of a semimodule of a numerical semigroup with two generators**. Semigroup Forum **103**, 278–285 (2021). DOI:10.1007/s00233-021-10182-1.

8. P. Almirón, J.J. Moyano-Fernández: **Supersymmetric gaps of a numerical semigroup with two generators**. To appear in Communications in Algebra. DOI: 10.1080/00927872.2022.2058521.

9. P. Almirón, J.J. Moyano-Fernández: **An Extension of the Wilf conjecture to semimodules over a numerical semigroup**. Preprint in arxiv:2012.01358.

Chapter 7

10. M.Alberich-Carramiñana, P. Almirón, J.J. Moyano-Fernández: **Curve singularities with one Puiseux pair and value sets of modules over their local rings**.Arxiv:2105.07943.

Part I

Tjurina number of hypersurface singularities and the 4/3 problem for plane curves

Chapter 1

Hypersurface singularities

On this chapter we are going to review some basic concepts about hypersurface singularities. We start with plane curve singularities, by reviewing the notion of equisingularity and its natural relationship with topological equivalence. We will also review the construction of embedded resolution of plane curve singularities and we will introduce the semigroup of values of an irreducible plane curve singularity as an equisingularity invariant. After that, we will shift to general hypersurface singularities and we will introduce the main numerical invariants that will be needed along this memoir which are the Milnor number, the Tjurina number and the exponents and spectrum of an hypersurface singularity. To finish, we will recall the notion of Newton non-degenerate singularities.

1.1 Brief introduction to plane curve singularities

The local study of isolated plane curve singularities is a vast research topic which has been developed during centuries. Probably, when starting to talk about plane curve singularities from a basic and an historical point of view, one should mention the names of I. Newton and V. Puiseux. After that, M. Noether, F. Enriques, O. Chisini and O. Zariski compound a significant representation of the group of mathematicians that developed the basic tools of the theory on its early stages. A basic introduction to plane curve singularities must mention at least the notions of blow ups, infinitely near points, Newton-Puiseux series, equisingularity, etc. We will mainly follow the books of E. Casas-Alvero [31] and C.T.C Wall [182] to briefly overview some basic tools to work with topological aspects about plane curve singularities.

Let us consider S a smooth complex algebraic or analytic surface. We take local coordinates (x, y) at a distinguished point $O \in S$. Let $f \in \mathbb{C}\{x, y\}$ be a holomorphic function defining a germ $(C, O) := \{f(x, y) = 0\} \subset (S, O)$ of isolated plane curve singularity, or simply isolated plane curve singularity if the local context is clear. Frequently in geometry, working with local coordinates leads to look for some kind of parameterization of our object of study. To obtain such a parameterization, one of the main results is the Newton-Puiseux Theorem (see [31, Chapter 1] or [182, Chapter 2]). Briefly, Newton-Puiseux Theorem says that $f(x, y)$ can be expressed as product of simple factors of the type $y - g_j(x^{1/m_j})$ with $g_j \in \mathbb{C}\{x^{1/m_j}\}$. More concretely, one can factorize $f(x, y) = f_1(x, y) \cdots f_r(x, y)$ with each f_i being an irreducible element of $\mathbb{C}\{x, y\}$ and $f_i \neq f_j$ if $i \neq j$; after that, if we denote by $\epsilon^{m_i} = 1$ a m_i–th root of the unit different from 1 then we can see C as a union of curves C_i defined by the equations $f_i(x, y) = \prod_{\epsilon^{m_i}=1}(y - g_i(\epsilon x^{1/m_i})) = 0$. Moreover, a local parameterization of each C_i can be expressed as $(x(t), y(t)) = (t^{m_i}, g_i(t^{m_i}))$. We usually call *branches* to the irreducible components of C, *Puiseux series* to the elements g_i and *Puiseux parameterization* to a parameterization as before.

Let us assume for a while that $f(x,y)$ is irreducible, i.e. $C : f(x,y) = 0$ is a single branch. In this setting, it is natural to compare whether two branches are topologically equivalent or not. A reasonable definition for topological equivalence of germs of plane curves is the following.

Definition 1.1. Two plane curve singularities C_1, C_2 have the same topological type if and only if C_1 and C_2 are topologically equivalent as embedded surfaces in $\mathbb{C}^2$, that is, if there exists $U_1, U_2 \subset \mathbb{C}^2$ open neighborhoods of the origin and a homeomorphism $T : U_1 \to U_2$ such that C_1 is defined in U_1, C_2 is defined in U_2 and $T(C_1 \cap U_1) = C_2 \cap U_2$.

Since this definition is of local nature, Puiseux series should play an important role in the description of the topological type of a branch. Despite the Puiseux series are infinite, if one wants to study topological properties of the branch there are only a finite number of terms for which one needs to take care of. Those exponents are called *characteristic exponents.*

Definition 1.2. Let $s = \sum_{j>0} a_j x^{j/n}$ be a Puiseux series, we say that it has *polydromy order* n if n and $\gcd\{j|\ a_j \neq 0\}$ have no common factors. We define the *characteristic exponents of* s as a finite set of rational numbers $\{\beta_1/n, \ldots, \beta_k/n\}$ defined as follows: if we denote by (n) the set of integers which are multiples of n then $\beta_1 := \min\{j|\ a_j \neq 0 \quad j \notin (n)\}$, and inductively if $e_{i-1} := \gcd(n, \beta_1, \ldots, \beta_{i-1}) \neq 1$, $\beta_i := \min\{j|\ a_j \neq 0 \quad j \notin (e_{i-1})\}$. Since n is the polydromy order of s, there exists k for which $e_k = 1$.

At the beginning of the twentieth century, the works of K. Brauner [22], W. Burau [29] and O. Zariski [193] provide a new framework for the study of topological equivalence of plane curve singularities. More concretely, those works provide a detailed description of the geometry of the knot associated to a plane branch through the combinatorics of the Puiseux exponents. Let us briefly recall the main idea.

Let us denote $D_\epsilon := \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 \leq \epsilon^2\}$ the disc with center O and by S_ϵ its boundary sphere. As we have said, if $C : f(x,y) = 0$ is a branch then we have a Puiseux parameterization of C in a neighborhood of O given by $(x(t), y(t)) = (t^n, s(t) = \sum_{j>n} a_j t^j)$. Therefore, the knot defined by $C \cap S_\epsilon$ can be parametrized by setting $t = \epsilon^{1/n} e^{2\pi i\theta}$. Moreover, if ϵ is small enough the knot $K = C \cap S_\epsilon$ is a 1–manifold smoothly embedded in S_ϵ and there is an isomorphism of the pair $(D_\epsilon, C \cap D_\epsilon)$ to the cone on $(S_\epsilon, C \cap S_\epsilon)$ (see [182, Lemma 5.2.1]). By considering successive truncations of the Puiseux series one can examine the possible values of y for each value of x. Let C_k be the curve given by $(x(t), y_k(t)) = (t^n, s_k(t) = \sum_{l>n}^{k} a_l t^l)$ and consider the knot $K_k = C_k \cap S_\epsilon$. Since the terms with $k < \beta_1$ in $s_k(t)$ are all multiples of n then we have a unique value of y_k for each value of x. When $k = \beta_1$ then the values of y_{k-1} splits into n/e_1 values for y_k. After that, we are seeing the same picture until we reach $k = \beta_2$ where now each value splits into e_1/e_2 points. Finally the general rule is as follows: each time we reach a characteristic exponent β_k the previous values split in e_{k-1}/e_k new ones. Since the sequence of e_i's is decreasing, this process comes to an end after a finite number of step. If we denote $e_0 := n$, the pairs $(\beta_i/e_i, e_{i-1}/e_i)$ emerge as natural invariant of the singularity, those pairs are called *Puiseux pairs* of the branch.

One can look at the Puiseux pairs in the Newton process. To do that, let us define the numbers $p_i := e_{i-1}/e_i$, $q'_1 := \beta_1/e_1$ and $q'_i := (\beta_i - \beta_{i-1})/e_i$ and observe (see [50, Appendix to Chapter I]) that the Puiseux series can be written in a multiplicative way as

$$y = x^{q'_1/p_1}(a_1 + x^{q'_2/p_1p_2}(a_2 + \cdots (a_{g-1} + x^{q'_g/p_1\cdots p_g}(a_g + \cdots)))).$$

The idea given by Brauner [22] is that, thanks to the pairs (p_i, q'_i), one can give a description of the link as iterated torus knots (see [50, Appendix to Chapter I] or [45, Chapter 2]). More precisely, let us define the numbers $q_1 := q'_1$ and $q_{i+1} := p_i p_{i+1} q_i + q'_{i+1}$. At the level $k = \beta_1$, we have a (p_1, q_1)–torus knot, i.e the linear flow parametrized by $(e^{in\theta}, \epsilon_1 e^{i\beta_1\theta})$ in the solid torus

$|x| = 1, |y| \leq 1$ cross the meridian n/e_1 times and the parallel β_1/e_1 times. After that, we consider a tubular neighborhood of the invariant curve defined by $(e^{in\theta}, \epsilon_1 e^{i\beta_1\theta})$, which is again a torus and we repeat the process obtaining a sequence of iterated torus knots of type (p_i, q_i). We will call *Newton pairs* to the pairs (p_i, q_i).

Example 1.3. Let C be the branch with equation $f = (y^2 - x^3)^2 - x^5y$. With the help of SINGULAR [35], we can obtain a Puiseux parameterization of this curve:

$$\begin{cases} x(t) = t^4 \cdot (1 - 3t + 6t^2 + h.o.t.) \\ y(t) = \quad t^6 - 5t^7 + 15t^8 + h.o.t. \end{cases}$$

Since $1 - 3t + 6t^2 + h.o.t.$ is a unit in the ring $\mathbb{C}\{t\}$ of convergent power series in t, we can perform an analytic change of coordinates $x'(t) = x(t)/(1 - 3t + 6t^2 + h.o.t.)$ in order to obtain a Puiseux parameterization of the form $(x'(t), y(t)) = (t^4, t^6 - 5t^7 + h.o.t.)$.

According to the previous discussion, the Puiseux characteristic exponents are $\{6/4, 7/4\}$. Now, from the formulas introduced before we can compute the Puiseux pairs $\{(3, 2), (7, 2)\}$ and the Newton pairs $\{(2, 3), (2, 13)\}$. Moreover, observe that from the Puiseux parameterization we have the multiplicative form:

$$y = (x')^{6/4}(1 - 5(x')^{1/4} + \cdots).$$

Finally, the association of Puiseux pairs, Newton pairs and characteristic exponents with the description of the cone $(S_\epsilon, C \cap S_\epsilon)$ lead to the following theorem.

Theorem 1.4 (Burau, Brauner, Zariski). *Let C_1, C_2 be branches. Then the following statements are equivalent:*

1. *C_1 and C_2 have the same Puiseux pairs/ Newton pairs/characteristic exponents.*
2. *The cones $(S_\epsilon, C_1 \cap S_\epsilon)$ and $(S_\epsilon, C_2 \cap S_\epsilon)$ are homeomorphic.*
3. *C_1 and C_2 are topologically equivalent.*

Proof. See for example [182, Thm 5.5.8] □

Obviously, if we start with a plane curve singularity C with more than one branch, there are analogous constructions. In this case, $C \cap S_\epsilon$ have structure of link where each component is the knot associated to a branch of C. We refer to the book of Wall [182, Chapter 5] and to the book of Dimca [45, Chapter 2] for more details.

As can be seen in [45] or [182] one can continue the study of local topological properties of plane curve singularities via studying the associated link. However, the techniques we are going to use in the main chapters of this memoir are more proximate to an alternative but equivalent approach to topological equivalence of plane curves singularities. This is an algebro-geometric approach pivoting through the central concepts of equisingularity and resolution of singularities.

1.1.1 Embedded resolution of plane curve singularities

We continue with the study of local topological properties of plane curves singularities via resolution of singularities. As can be seen in the book of Casas-Alvero [31, Chapters 3 and 5], a good approach is the notion of infinitely near points that was introduced by M. Noether at the end of nineteenth century. Infinitely near points appear when we perform successive quadratic transformations in the complex plane. This kind of transformation is called blow up. To blow up a point of a plane curve singularity is a way to analyze the different tangent directions to the curve at this point. Moreover, the process of performing successive blow up at points in the curve gives rise to a non-singular curve together with a union of projective lines.

Definition 1.5. Let S be a smooth complex surface and $O \in S$ a fixed point. Let $\overline{S}$ be the set of all pairs (x,l) where $x \in S$ and $l \in \mathbb{P}^1$ a line through O containing x, i.e. $\overline{S} := \{(x,l) \in S \times \mathbb{P}^1 \mid x \in l, O \in l\} \subset S \times \mathbb{P}^1$. Consider the projection $\pi : \overline{S} \ni (x,l) \mapsto x \in S$. The variety $\overline{S}$ together with the natural projection π is called the *blow up of the point* O. The line $E_O := \pi^{-1}(O) \cong \mathbb{P}^1$ is called *exceptional divisor* of the blowing up.

Remark 1.6. The blow up surface $\overline{S}$ can be explicitly described with the help of the usual charts associated to $\mathbb{P}^1$. More concretely if we denote $U_1 = \{(\alpha : \beta) \in \mathbb{P}^1 \mid \alpha \neq 0\}$ and $U_2 = \{(\alpha : \beta) \in \mathbb{P}^1 \mid \beta \neq 0\}$, then

$$\overline{S} = \{((x,y);(\alpha:\beta)) \in S \times U_1 \mid x(\beta/\alpha) = y\} \cup \{((x,y);(\alpha:\beta)) \in S \times U_2 \mid x = y(\alpha/\beta)\}.$$

Observe that in the charts $S \times U_1$ and $S \times U_2$ we have local coordinates $(x, y, z_1 := \beta/\alpha)$ and $(x, y, z_2 := \alpha/\beta)$. In this way, thanks to the relations $xz_1 = y$ and $yz_2 = x$, we can take local coordinates $(u_1, v_1) = (x, z_1)$ and $(u_2, v_2) = (y, z_2)$ of $\overline{S}$ with respect to those charts such that the projection to S is described in the first chart as $(u_1, v_1) \mapsto (u_1, u_1v_1)$ and in the second chart as $(u_2, v_2) \mapsto (u_2v_2, v_2)$.

Observe that the exceptional divisor can be naturally identified with the tangent directions of S at O. This is the reason why E_O is also called *first infinitesimal neighborhood* of O. Also, $\overline{S} \setminus E_O$ is dense with respect to the Zariski topology in $\overline{S}$ and hence π is a birrational equivalence between $\overline{S}$ and S. Let us choose local coordinates (x, y) at the distinguished point $O \in S$. We denote by $\mathcal{O}_{(S,O)} \simeq \mathbb{C}\{x,y\}$ the local ring at O which is isomorphic to the ring of convergent power series in the variables (x, y). Before to proceed with the study of blowing up points at a curve, we shall define the notions of multiplicity and intersection multiplicity.

Definition 1.7. Let $(C,O) := \{f(x,y) = 0\} \subset (S,O)$, $f \in \mathbb{C}\{x,y\}$ be a germ of plane curve. The *multiplicity* of C at O, that we denote by $e_O(C)$, is the smallest degree of the terms of a defining function f. If $(C',O) := \{g(x,y) = 0\} \subset (S,O)$ is another germ of plane curve then the *intersection multiplicity at* O of C with C' is defined as:

$$[C \cdot C']_O := \dim_{\mathbb{C}} \frac{\mathcal{O}_{(S,O)}}{(f,g)} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f,g)}.$$

Now, we want to blow up an isolated singular point of a plane curve and to study the intersection behavior of the curve with lines passing through the origin. The key fact is that after blowing up a point we will obtain another curve in the surface $\overline{S}$ which intersects the exceptional divisor at a finite number of points. On those points, the curve has less or equal multiplicity than the multiplicity of the original curve at the singular point. This provides a finite number of points where the new curve can be blown up again decreasing the multiplicity; thus, providing a process which eventually reach a non-singular curve. This process is called a resolution of the singularity.

Definition 1.8. The pull-back of C by the projection morphism associated to the blow up at O, $\overline{C} = \pi^*(C)$, is called the *total transform* of C by π. We observe that $\overline{C} = \widetilde{C} + e_O(C)E_O$, where we denote by $\widetilde{C}$ (or C_O) the curve on $\overline{S}$ not containing E_O. We call $\widetilde{C}$ the *strict transform* of C by π.

Remark 1.9. We can identify $\widetilde{C} = \overline{\pi^{-1}(C \setminus \{O\})}$, where the bar denotes here the Zariski closure in $\overline{S}$.

It can be seen that $\widetilde{C}$ intersects E_O at a finite number of points. Hence, one can consider $\widetilde{C} \cap E_O$ as a divisor on $E_O \subset \mathbb{P}^1$. Moreover, the multiplicity at O can be expressed as a finite sum of intersection multiplicities of the curve with the points in the exceptional divisor. In this way one can write $e_O(C) = \sum_{p \in E_O} [\widetilde{C} \cdot E_O]_p$ where $[\widetilde{C} \cdot E_O]_p$ denotes the multiplicity of intersection at a point $p \in E_O$. In particular, $e_p(\widetilde{C}) \leq e_O(C)\ \forall p \in E_O$.

Example 1.10. Let C be the branch with equation $f = (y^2 - x^3)^2 - x^5 y$ considered in Example 1.3. Let S^1 be the blowing up of the origin. Let us take the local coordinates $(u_1, v_1) \in S^1$ in S^1 at the intersection point p_1 of the exceptional divisor E_1 and the strict transform $C^{(1)}$ of the curve. Thus, the blowing up map is given by $\pi_1 : S^1 \to \mathbb{C}^2$ with $(u_1, v_1) \mapsto (u_1, u_1 v_1)$. The total transform of the branch in this chart reads as

$$\overline{C} :\ u_1^4(v_1^4 + u_1^2 - 2u_1 v_1^2 - u_1^2 v_1) = 0.$$

Observe that the strict transform, which we denoted by $C^{(1)}$, is defined by the equation $v_1^4 + u_1^2 - 2u_1 v_1^2 - u_1^2 v_1 = 0$ so it is singular at $(u_1, v_1) = (0, 0)$. The exceptional divisor E_1 has equation $u_1 = 0$ and it is tangent to $C^{(1)}$ at p_1 (see Figure 1.2).

Moreover, it is easy to check that in the other chart, we cannot see any intersection point of the strict transform with the exceptional divisor. From now on we will only consider the charts were such an intersection point is appearing.

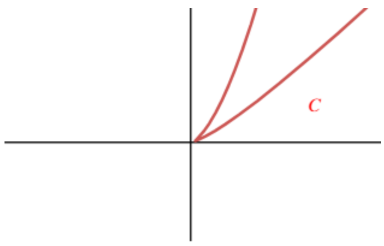


Figure 1.1: Branch $C : (y^2 - x^3)^2 - x^5 y = 0$

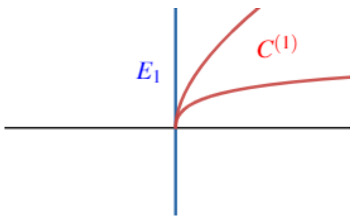


Figure 1.2: Representation of the strict transform $C^{(1)}$ and the exceptional divisor E_1 after blowing up the origin O

We now blow up the unique intersection point p_1 of E_1 with $C^{(1)}$. Take local coordinates (u_2, v_2) in S^2 at the unique intersection point, p_2, of the strict transform $C^{(2)}$ and the exceptional divisor E_2. The blowing up map is $\pi_2 : S^2 \to S^1$ with $(u_2, v_2) \mapsto (u_2 v_2, v_2)$. The total transform of $C^{(1)}$ is

$$\overline{C^1} :\ v_2^2((v_2 - u_2)^2 - u_2^2 v_2) = 0.$$

Observe that the strict transform, $C^{(2)}$ has equation $(v_2 - u_2)^2 - u_2^2 v_2$, it is again singular, it intersects $E_2 : v_2 = 0$ and it also intersects the strict transform of $\widetilde{E_1} : u_2 = 0$ at p_2 (see Figure 1.3).

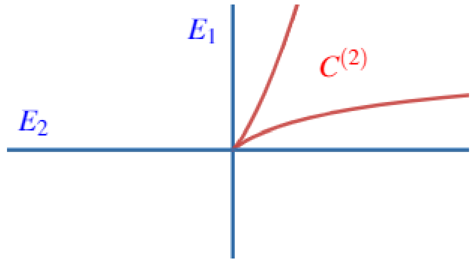


Figure 1.3: Representation of the strict transform $C^{(2)}$, the strict transform E_1 and the exceptional divisor E_2 after blowing up p_1

Let us blow up p_2. Take local coordinates $(u_3, v_3) \in S^3$ in S^3 at the unique intersection point, q, of the strict transform of the first exceptional divisor $\widetilde{E_1}$ and the exceptional divisor E_3. The blowing up map is $\pi_3 : S^3 \to S^2$ with $(u_3, v_3) \mapsto (u_3v_3, v_3)$. The total transform of C^2 is

$$\overline{C^2} : \ v_3^2((1-u_3)^2 - u_3^2 v_3) = 0.$$

The strict transform $C^{(3)}$ with equation $1 + u_3^2 - 2u_3 - u_3^2 v_3 = 0$ is smooth and it is tangent to the exceptional divisor $E_3 : \ v_3 = 0$ at the point $(u_3, v_3) = (1, 0)$ (see Figure 1.4).

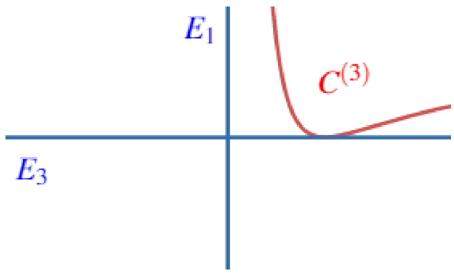


Figure 1.4: Representation of the strict transform $C^{(3)}$, the strict transform E_1 and the exceptional divisor E_3 after blowing up p_2

Example 1.10 shows a special situation since the exceptional divisor is tangent to the strict transform. This motivates to introduce the notion of transversality. Two curves are said to be *transversal* at a common point if and only if they have different tangents. Moreover, we can translate the notion of being transversal in terms of multiplicity. Observe that, in general if one has two curves C, C' passing through O, their intersection multiplicity satisfies $[C \cdot C']_O \geq e_O(C)e_O(C')$. Therefore, it can be checked that two curves are transversal at O if and only if $[C \cdot C']_O = e_O(C)e_O(C')$.

Roughly speaking, a resolution of singularities of a plane curve is to obtain via blow ups a non-singular strict transform. However, Example 1.10 shows that one can obtain a smooth strict transform which is not transversal to the exceptional divisor. This forces to define the stronger notion of embedded resolution of a plane curve singularity.

Definition 1.11. Let C be a germ of isolated plane curve singularity on the smooth surface S. We say that $\pi : \overline{S} \to S$ is an *embedded resolution* of C if $\overline{S}$ is smooth, the reduced total transform $(\overline{C})_{red} = (\pi^*(C))_{red}$ is a normal crossing divisor (the union of transversal smooth curves with no three of them intersecting at a point) and the strict transform $\widetilde{C}$ is smooth.

Remark 1.12. Following Definition 1.8, if $\pi : \overline{S} \to S$ is a sequence of blow ups from a resolution of singularities of C, then $\overline{C} = \pi^*(C)$ can be expressed as sum of divisors $\overline{C} = \widetilde{C} + \sum \eta_i E_i$ where the η_i can be computed from the multiplicities at the points where the blow ups has been done. In this way, $\eta_i \geq 1$. The reduced total transform is then obtained by dropping out the multiplicities of the exceptional divisors, i.e. $(\overline{C})_{red} := \widetilde{C} + \sum E_i$. We need this notion in the previous definition in order to be consistent with the definition of normal crossing divisor. Observe that strictly speaking $\eta_i E_i$ with $\eta_i \geq 2$ is not a smooth curve.

As we have seen in Example 1.10, we may blow up the points on the intersection $\widetilde{C} \cap E_O$ obtaining a new strict transform and a new exceptional divisor. Iterating this process of blow up points we construct a sequence of blow up centered at some distinguished points.

Definition 1.13. We call E_O the first neighborhood of O and hence its points are called points on the first neighborhood of O. Inductively, for any $i > 1$, define the points on the i–th neighborhood of O to be the points on the first neighborhood of any point on the $(i-1)$–th neighborhood of O. The points in any neighborhood of O are called *points infinitely near* to O. We denote the set of them by $\mathcal{N}_O$.

Since infinitely near points are defined inductively, we have a natural partial order in $\mathcal{N}_O$; for any $p, q \in \mathcal{N}_O$ we say p *precedes* q, $p \leq q$, if and only if q is infinitely near to p.

A point $p \in \mathcal{N}_O$ is lying on a surface S_p which may be obtained by successively blow up the points preceding p. Denote by $\pi_p : S_p \to S$ this composition of blow ups. In the same way, we denote by $\overline{C}_p, \widetilde{C}_p$ the total and the strict transform of a germ of curve C by π_p. Moreover, the definition of infinitely near points replace the study of the singular point O by the study of the whole set of infinitely near points $\mathcal{N}_O$ and its proximity relations (see Definition 1.18).

To continue, we need to extend the definition of multiplicity at a point to the set of infinitely near points.

Definition 1.14. Let C be a germ of curve at $O \in S$ and $p \in \mathcal{N}_O$. We define the *multiplicity of C at p*, $e_p(C)$, to be the multiplicity $e_p(\widetilde{C}_p)$ of the strict transform at p. If we consider the total transform, we define the *value of C at p* as $v_p(C) := e_p(\overline{C}_p)$.

Therefore, we will say that an infinitely near point p to O is *simple* if $e_p(C) = 1$ and it is *multiple* if $e_p(C) > 1$. Since the process of blowing up points leads to an embedded resolution of a plane curve singularity, the following results are naturally needed.

Theorem 1.15. *A reduced germ of curve contains at most finitely many multiple infinitely near points.*

Proof. See for example [31, Thm 3.7.1] □

Theorem 1.16. *Given a curve C on a non-singular surface S, there exists a finite sequence of blow ups points $S^{(r)} \to \cdots \to S^{(1)} \to S$ such that if we denote by $\pi : S^{(r)} = S' \to S$ their composition, then the inverse image of the singular points of C is a union of nonsingular curves (each of them isomorphic to $\mathbb{P}^1$) meeting transversally on the nonsingular surface S' and the strict transform $\widetilde{C}$ by π is a nonsingular curve that meets transversally these curves.*

Proof. See for example [182, Thm 3.4.4] □

In particular, Theorem 1.16 shows that any curve has an embedded resolution which is a composition of blow up points.

Example 1.17. Let us continue with the example 1.10. In order to move the origin from q to the intersection point $p_3 = (u_3, v_3) = (1, 0)$ of the exceptional divisor E_3 and $C^{(3)}$, we consider a translation of the origin $\lambda : S^3 \to S^3$ given by $(u_4, v_4) = (1 - u_3, v_3)$. In this new coordinates, the strict transform C^3 becomes $u_4^2 + 2u_4v_4 - v_4 - u_4^2v_4 = 0$.

Let us blow up $p_3 = (u_4, v_4) = (0, 0)$. Take local coordinates $(u_5, v_5) \in S^4$ in S^4 at the unique intersection point, p_4, of the strict transform, C^4, and the exceptional divisor E_5. The blowing up map is $\pi_4 : S^4 \to S^3$ with $(u_5, v_5) \mapsto (u_5, u_5v_5)$. The total transform of C^3 is

$$\overline{C^3} : \; u_5(u_5 - v_5 - u_5^2v_5 + 2v_5^2) = 0.$$

The strict transform C^4 with equation $u_5 - v_5 - u_5^2v_5 + 2v_5^2$ intersects the exceptional divisor $E_5 : u_5 = 0$ and the strict transform of the exceptional divisor $\widetilde{E_3} : v_5 = 0$ at p_4 (see Figure 1.5). So again the situation is not transversal.

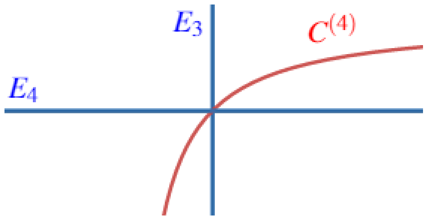


Figure 1.5: fourth blow up

To finish, the blow up of p_4 finally provides an embedded resolution of the singularity since now the situation is transversal (see Figure 1.6).

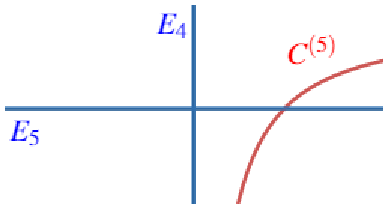


Figure 1.6: Last blow up

The embedded resolution results as a composition of blow ups together with a translation of the origin after the third blow up $\pi = \pi_1 \circ \pi_2 \circ \pi_3 \circ \lambda \circ \pi_4 \circ \pi_5$. Also, in this process we have obtained the sequence of multiplicities of the strict transforms: $e_O = 4, e_{p_1} = 2, e_{p_2} = 2, e_{p_3} = 1, e_{p_4} = 1$. Observe also, that since the strict transform C^3 (Figure 1.4) does not pass through the intersection point q of E_1 and E_3 then $e_q = 0$.

In this composition it is important to know the nature of the point we are blowing up. For example, if the point is the intersection point of two exceptional divisors or not. The relative position of the infinitely near points in a sequence of blown ups are encoded in the proximity relations.

Definition 1.18. Let $p, q \in \mathcal{N}_O$. We say that q is *proximate to* p, denoted by $q \to p$, if and only if q belongs (as an ordinary or infinitely near point) to the exceptional divisor E_p of blowing up p.

Furthermore, proximity relations allow to compute inductively the multiplicity of the curve at an infinitely near point.

Proposition 1.19. *For any (ordinary or infinitely near) point p on a germ of curve C,*

$$e_p(C) = \sum_{q \to p} e_q(C).$$

Proof. See for example [31, Thm 3.5.3] □

Definition 1.20. Let p be any point infinitely near to O, consider $\pi_p : S_p \to S$ the composition of blow ups giving rise to p. Then p belongs to either one or two components of the exceptional divisor $F = \pi_p^{-1}(O)$. We say that p is *free* if it belongs to one component. We say that p is *satellite* if it belongs to two components.

Example 1.21. We can see that the point p_1 appearing in Example 1.10 only belongs to the exceptional divisor E_1. Also, the point p_3 appearing in Example 1.17 only belongs to the exceptional divisor E_3. Thus, p_1, p_3 are free points.

On the other hand, the points p_2, p_4 in those examples are satellite points since p_2 belongs to E_1 and E_2 and p_4 belongs to E_3 and E_4.

Remark 1.22. Observe that the intersection point q of E_1 and E_3 is a satellite point, since it is a infinitely near point to O. However, this point does not lie on C since $e_q(C) = 0$. We will mainly focus on the infinitely near points lying on C, i.e. those infinitely near points with $e_p(C) \geq 1$.

Definition 1.23. Let C be a germ of curve. A (proper or infinitely near) point p on C will be called *singular point* of C if and only if either

(a) p is multiple on C, or

(b) p is a satellite point, or

(c) p precedes a satellite point on C.

The previous definition means that $p \in \mathcal{N}_O$ is singular if and only if $[\widetilde{C}_p \cdot E_p] > 1$ where E_p is the germ at p of the exceptional divisor of the composition of blowing-ups giving rise to p and $\widetilde{C}_p$ the strict transform at p [31, Lemma 3.8.1]. Moreover, from Theorem 1.16 and the definition of singular points, there are finitely many infinitely near singular points on a germ of curve.

Example 1.24. Continuing with the example 1.17 we can sketch the intersections of the divisors appeared during the resolution process as we can see in Figure 1.7. Usually, this sketch is encoded in different graphs structures such as the dual graph or the Enriques diagram (see [180, Chap. 3, Sec. 3.6] and [31, Chap. 3, Sec. 3.9]).

1.1.2 Equisingularity class of a branch

The notion of equisingularity of hypersurface germs was presented by Zariski in 1965 on his founding papers [195]. In those papers, Zariski gives three equivalent definitions of equisingularity of germs of plane curve singularities. As it is pointed out by Casas-Alvero in [31, Chapter 3, Section 3.8, pg. 93], those three definitions of equisingularity in the case of plane curves can be summarized in the following definition.

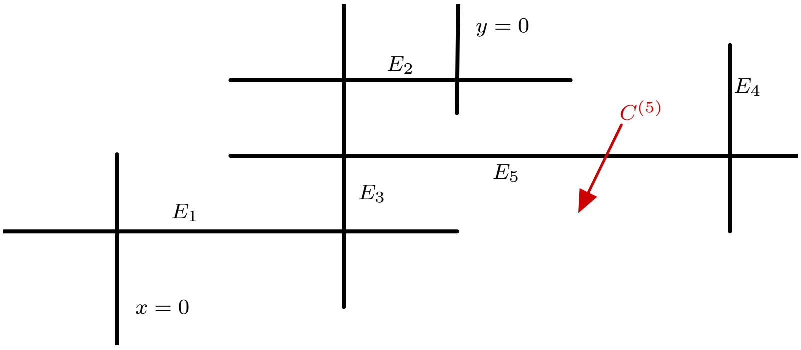


Figure 1.7: Structure of the resolution process of $(y^2 - x^3)^2 - x^5y = 0$

Definition 1.25. Two germs of curve ξ, ζ are said to be *equisingular* if and only if both are reduced and non-empty and there exists a bijection $\varphi : \mathcal{S}(\xi) \to \mathcal{S}(\zeta)$ such that both φ, φ^{-1} preserve natural ordering and proximity of infinitely near points: for any $p, q \in \mathcal{S}(\xi)$, p is infinitely near (resp. proximate) to q if and only if $\varphi(p)$ is infinitely near (resp. proximate) to $\varphi(q)$.

Thus, equisingularity constitute an equivalence relation in the set of germs of plane curves. We will call *equisingularity class* or *equisingularity type* to the corresponding classes of this equivalence relation. As introduced by Zariski [195], equisingularity is an abstract notion inherent to the algebraic structure of the local ring of the germ. As pointed out by Zariski [195], equisingularity for plane curves allows to classify singularities by induction in the numbers of blow ups needed to resolve the singularity without using its Puiseux series. Moreover, this algebraic classification fits well for classifying singularities in higher dimension.

One can thought that this purely algebraic definition of equisingularity is rather involved. However, thanks to the work of Brauner [22], Burau [29] and Zariski [193] for plane curves, the notion of equisingularity is equivalent to the notion of topological equivalence. In particular, by Theorem 1.4, two irreducible plane curves are equisingular if and only if they are topologically equivalent.

For plane branches, equisingularity can be encoded with the help of a numerical semigroup. We conclude this brief introduction to plane curve singularities with the definition of this important equisingularity invariant. Let us assume for a while that ξ is an irreducible germ of plane curve singularity. Then, there exists a Puiseux parameterization, $(t^n, s(t^n) = \sum_{j \geq n} a_j t^j)$ in such a way that the morphism

$$R := \mathbb{C}\{x, y\}/(f) \hookrightarrow \mathbb{C}\{t\}$$

induces a discrete valuation $v : R \to \mathbb{N} \cup \infty$ of the field of fractions of R:

1. $v(g) := \mathrm{ord}_t(g(x(t), y(t))) = [\xi, \{g = 0\}]$,
2. $v(g) = \infty \Leftrightarrow g \in (f)$,
3. $v(u) = 0$ if $u \in R$ is a unit,
4. $v(gh) = v(g) + v(h)$,
5. $v(g + h) \geq \min\{v(g), v(h)\}$.

We can naturally define the set of values of the discrete valuation as the set of intersection multiplicities of germs with the fixed germ ξ:

Definition 1.26. We define the *semigroup of values of* ξ as

$$\Gamma(\xi) := \Gamma = \{[\xi,\eta] \,|\, \eta \text{ a germ not containing } \xi\} = \{v(g) |\, g \in R \setminus \{0\}\}.$$

Recall that a numerical semigroup is an additive submonoid of the natural numbers of finite complement (Definition 0.1). Thus, it is finitely generated. We are going to show that Γ is a numerical semigroup. Moreover, the set of minimal generators of Γ is obtained from the Puiseux pairs/ Puiseux exponents. Hence, we will obtain that Γ is a topological invariant of the plane branch.

Let us assume that ξ has Puiseux exponents $\{\beta_1/\beta_0, \ldots, \beta_g/\beta_0\}$. We define the germ ξ_i of equation:

$$\sigma_i(x) = \sum_{\substack{j\,\in\,\beta_0\mathbb{Z} \\ 1\,\le\,j\,<\,\beta_1}} a_j x^{j/\beta_0} + \cdots + \sum_{\substack{j\,\in\,e_{i-1}\mathbb{Z} \\ \beta_{i-1}\,\le\,j\,<\,\beta_i}} a_j x^{j/\beta_0}.$$

The germs ξ_i are special truncations of the Puiseux series of ξ. They are called *maximal contact elements*. Following Zariski's notation [197], we denote by $\overline{\beta}_i := [\xi_i, \xi]$ the maximal contact values. It can be proved that the semigroup of values $\Gamma(\xi)$ is finitely generated and the minimal set of generators is precisely the set of maximal contact values.

Theorem 1.27. *The semigroup of values of an irreducible plane cuve singularity ξ is finitely generated by*

$$\overline{\beta}_{i+1} = \frac{\beta_0 - e_1}{e_i}\beta_1 + \frac{e_1 - e_2}{e_i}\beta_2 + \cdots + \frac{e_{i-1} - e_i}{e_i}\beta_i + \beta_{i+1},$$

where $\{\beta_1/\beta_0, \ldots, \beta_g/\beta_0\}$ is the set of characteristic exponents and $\overline{\beta}_0 = \beta_0$ and $\overline{\beta_1} = \beta_1$.

Proof. See for example [182, Thm. 4.3.5] □

As we announced, an easy consequence of Theorem 1.27 is that Γ is a numerical semigroup and it is a numerical invariant of the equisingularity class.

Corollary 1.28. *Two germs ξ, ξ' are equisingular if and only if $\Gamma(\xi) = \Gamma(\xi')$.*

Let us continue with a more detailed description of the semigroup of values. First of all, we notice that the expression of the minimal generators can be simplified. Recall that we denote $e_i = \gcd(\beta_0, \ldots, \beta_i)$ and $n_i = e_{i-1}/e_i$. Then,

$$\overline{\beta}_i = n_{i-1}\overline{\beta}_{i-1} - \beta_{i-1} + \beta_i \quad i = 2, \ldots, g, \tag{1.1}$$

and $\overline{\beta}_0 = \beta_0$, $\overline{\beta}_1 = \beta_1$.

From this expression, it can be seen that (see [197, Chapter II, §3])

$$c(\Gamma) := n_g\overline{\beta}_g - \beta_g - (\beta_0 - 1) \tag{1.2}$$

is the minimal natural number such that for all $j \ge c(\Gamma)$ we have $j \in \Gamma$. We call $c(\Gamma)$ the *conductor* of Γ. Also, one can easily check that for each $\overline{\beta}_i$ we have

$$n_i\overline{\beta}_i \in \langle \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} \rangle := \{n \in \mathbb{N} |\, n = a_0\overline{\beta}_0 + \cdots + a_{i-1}\overline{\beta}_{i-1}\} \quad \text{and} \quad \overline{\beta}_{i+1} > n_i\overline{\beta}_i.$$

To finish, we come back to the description of the equisingularity class as an "iterated union of torus knots ". From this point of view, since the semigroup is also a equisingularity invariant, one expect that the toric structure should be reflected on the semigroup structure. Let us check that. First, recall that we have defined the Newton pairs (p_i, q'_i) of the irreducible plane curve singularity from the characteristic exponents of its Puiseux series as $p_i := e_{i-1}/e_i$, $q'_1 := \beta_1/e_1$ and $q'_i := (\beta_i - \beta_{i-1})/e_i$. Therefore, making use of the Equation (1.1), we can describe the generators of the torus knot as $p_i = n_i$ and $q_i = \overline{\beta}_i/(n_{i+1} \cdots n_g)$.

Since, $\gcd(p_i, q_i) = 1$ then

$$\langle p_i, q_i \rangle = \{n \in \mathbb{N} |\ n = ap_i + bq_i\}$$

is a numerical semigroup (see Definition 0.1). Let us show how to construct Γ from the semigroups of the Newton Pairs:

We start with $\Gamma_1 = \langle p_1, q_1 \rangle$ the semigroup associated to the first torus knot. Now we want "to glue"Γ_1 with the semigroup associated to the second Newton pair $\langle p_2, q_2 \rangle$. To do so, one defines

$$\Gamma_2 := \langle p_1 p_2, q_1 p_2, q_2 \rangle,$$

so that Γ_2 is the semigroup associated to the truncation of the Puiseux series up to the second Puiseux exponent. Recursively, we consider the semigroup up to the pair (p_i, q_i)

$$\Gamma_i := \langle p_1 p_2 \cdots p_i, q_1 p_2 \cdots p_i, \ldots, q_{i-1} p_i, q_i \rangle$$

and we "glue it"with the semigroup of the next pair (p_{i+1}, q_{i+1}) to obtain

$$\Gamma_{i+1} := \langle p_1 p_2 \cdots p_{i+1}, q_1 p_2 \cdots p_{i+1}, \ldots, q_i p_{i+1}, q_{i+1} \rangle.$$

The process ends with $\Gamma = \Gamma_g$.

Remark 1.29. In general, the notion of gluing semigroups can be explicitly defined and we refer to [144, Chapter 8] for a detailed treatment of that topic. It is closely related to the characterization of complete intersection numerical semigroups proved by Delorme [41]. Let us briefly explain this assertion.

A numerical semigroup Γ is said to be a complete intersection if the associated algebra $A := \bigoplus_{\nu \in \Gamma} K t^{\nu} \subset K[t]$ is a complete intersection algebra; this definition is independent of the field K. Thus, if G is a set of generators of Γ, Delorme proved that Γ is a complete intersection numerical semigroup if and only if either $1 \in G$ or there exists a disjoint partition $G = a_1 G_1 \sqcup a_2 G_2$ such that $\gcd(a_1, a_2) = 1$, $< G_1 >$ and $< G_2 >$ are complete intersection numerical semigroups and $a_1 \in < G_2 >$ and $a_2 \in < G_1 >$.

Observe that in the case of the value semigroup of the plane branch the partition can be obtained with the help of the Newton pairs.

Example 1.30. Following with the example 1.24, let us compute the semigroup of the branch C with equation $(y^2 - x^3)^2 - x^5 y$. To do so, recall that in example 1.3 we showed that the Newton characteristic exponents are $\{6/4, 7/4\}$. Thus, we can use Theorem 1.27 to compute the minimal generators of the semigroup and we obtain $\Gamma = \langle 4, 6, 13 \rangle$.

Also, we can obtain the same result with the "gluing method". In example 1.3 we also computed the Newton pairs of C which are $(2, 3), (2, 13)$. In this way, $\Gamma_1 = \langle 2, 3 \rangle$. According with the previous discussion we have

$$\Gamma = \Gamma_2 = \langle 2 \cdot 2, 3 \cdot 2, 13 \rangle = \langle 4, 6, 13 \rangle.$$

1.2 Normal two-dimensional singularities

Increasing one dimension from the plane curve case, we move to study germs of surface singularity in $\mathbb{C}^3$. As usual, we will deal with surface singularities having only isolated 0–dimensional singularities, i.e. isolated singular points. Thus, if we have a germ $(X, 0) \subset (\mathbb{C}^3, 0)$ of isolated surface singularity at 0 we will say that it is a germ of *normal surface singularity.*[1]

As in the case of plane curve singularities, there exist a resolution of singularities of a germ of a normal surface. A resolution of singularities of $(X, 0) \subset (\mathbb{C}^3, 0)$ consists of a manifold M and a proper analytic map $\pi : M \to X$ such that π is biholomorphic on the inverse image of on the singular locus $\mathrm{Sing}(X)$, which in our case is just the origin, of $(X, 0)$ and such that $\pi^{-1}(X \setminus \mathrm{Sing}(X))$ is dense in M. As usual, resolutions of normal singularities are not unique. However, in 1939, Zariski proved in [194, Theorem of Reduction of Singularities] that any normal two–dimensional singularity can be canonically resolved by a finite sequence of blow ups plus normalizations, such a resolution is usually called Zariski's canonical resolution or just canonical resolution. Before to continue, let us provide the following definition.

Definition 1.31. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a germ of normal surface singularity. A resolution $\pi : M \to X$ of singularities of $(X, 0)$ is a minimal resolution if for any other resolution $\pi' : M' \to X$ there is a unique holomorphic map $\rho : M' \to M$ such that $\pi' = \pi \circ \rho$.

Unfortunately, one can show that Zariski's canonical resolution is not necessarily a minimal resolution. To see that let us recall a characterization of a minimal resolution of a germ of normal surface singularity. To do that, we need first to introduce the notion of exceptional curve of the first kind.

Definition 1.32. A 1–dimensional analytic subset A in a 2–dimensional complex manifold is exceptional of the first kind if there is a proper holomorphic map $\varphi : M \to Y$ with Y a manifold such that $\varphi(A)$ is a point p and $\varphi : M \setminus A \to Y \setminus \{p\}$ is biholomorphic. If A is irreducible, it is called an *exceptional curve of the first kind.*

Moreover, exceptional curves of the first kind can be characterized by means of the following lemma.

Lemma 1.33. *A* 1*–dimensional analytic subset* A *in a* 2*–dimensional complex manifold* M *is exceptional of the first kind if and only if it is compact, connected and has a weighted graph which upon successively collapsing vertices with genus* 0 *and weight* -1 *becomes the empty graph.*

Proof. See [98, Chapter V, Corollary 5.8]. □

Remark 1.34. If A is a 1–dimensional analytic subset in a 2–dimensional complex manifold M its weighted graph is defined as follows: let $I \subset \mathbb{N}$ and $\{A_i\}_{i \in I}$ be the irreducible components of A, the vertices of the graph are in one to one with the set of irreducible components $\{A_i\}_{i \in I}$. The weight of a vertex $i \in I$ is defined as the self intersection of A_i. Two vertices $i \neq j$ are joined by a segment if and only if A_i intersects A_j.

Now we are ready to recall the following important theorem which allows to compute a minimal resolution of a germ of normal surface singularity.

[1] If $(X, 0)$ is a germ of analytic 2–dimensional singularity in $\mathbb{C}^3$, the natural definition of normal comes from the properties of its local ring $\mathcal{O}_X$. Thus, X is normal if $\mathrm{Spec}(\mathcal{O}_X)$ is isomorphic to $\mathrm{Spec}(\widetilde{\mathcal{O}}_X)$, where $\widetilde{\mathcal{O}}_X$ is the integral closure of $\mathcal{O}_X$. We refer to Chapter III of Laufer's book [98] for a justification of our definition, more concretely [98, Thm. 3.1 and Thm. 3.12].

Theorem 1.35. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a germ of normal surface singularity. Let $\pi : \widetilde{M} \to X$ be any resolution of X. The minimal resolution of X may be obtained from $\widetilde{M}$ by successively collapsing all exceptional curves of the first kind which lie above* 0.

Proof. See [98, Chapter V, Theorem 5.9] □

Because of Theorem 1.35, it is also common to define a minimal resolution of a germ of normal surface singularity as a resolution with no exceptional curve of the first kind. Moreover, Theorem 1.35 allows to show that Zariski's canonical resolution is not necessarily minimal since it may have exceptional curves of the first kind as we will see in Example 1.38. Before to proceed with Example 1.38 let us recall a few important results about germs of normal surface singularities in $\mathbb{C}^3$ of multiplicity 2.

It is well known that any germ of normal surface singularity in $\mathbb{C}^3$ of multiplicity 2 can be expressed in a suitable system of coordinates as

$$\{z^2 = f(x, y)\} \quad \text{where } f \text{ defines a germ of isolated plane curve singularity.}$$

It is also common to call such singularities *normal two–dimensional double point singularity* by obvious reasons. In 1978, Laufer [101] present a detailed study of the canonical resolution of a normal two–dimensional double point singularity. Morever, he proved that the canonical resolution can be obtained from a resolution of the plane curve singularity. To do that, the following lemma is a key ingredient.

Lemma 1.36. *Let $X = \{(x, y, z) \in \mathbb{C}^3 | \ z^2 = f(x, y)\}$ be an analytic subvariety of a polydisc in $\mathbb{C}^3$ with $p = (0, 0, 0) \in X$ with p isolated singular point. Let $\phi : X \to \mathbb{C}^2$ be given by $\phi(x, y, z) = (x, y)$. Let $\pi : M \to \mathbb{C}^2$ be the blow–up of $\mathbb{C}^2$ at $(0, 0)$. Let $\pi^* : X^* \to X$ be the blow–up at p induced by π via ϕ. Let $\omega : X' \to X^*$ be the normalization of X^*. There is a map ϕ_1 such that the following diagram is commutative:*

$$\begin{array}{ccc} X' & & \\ \downarrow{\scriptstyle \omega} & & \\ X^* & \xrightarrow{\pi^*} & X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} \\ M & \xrightarrow{\pi} & \mathbb{C}^2 \end{array}$$

The singularities of X^* and X' are all double points with ϕ_1 and $\phi' = \phi_1 \circ \omega$ respectively locally representing X^* and X' as two-fold branched covering spaces of M.

Proof. See [101, Lemma 2.2]. □

The canonical resolution of X is obtained by iterated application of Lemma 1.36. Let us briefly explain how to use Lemma 1.36 on the different iterations of the embedded resolution of the associated plane curve singularity $C : \{f(x, y) = 0\}$.

Let $\pi : (S, E) \longrightarrow (\mathbb{C}^2, \mathbf{0})$ be a resolution of $(C, \mathbf{0})$. As we have seen in the previous section, the morphism π can be seen as composition of point blow ups $\pi = \pi_{\mathbf{0}} \circ \cdots \circ \pi_{p_k}$. Denote by $\pi_{p_i} = \pi_{\mathbf{0}} \circ \cdots \circ \pi_{p_{i-1}}$ the blowing up of the point p_i. We can now apply Lemma 1.36 for each π_{p_i} to obtain the following diagram:

$$
\begin{array}{ccccccccc}
\widetilde{X}^{(k)} & & & & \widetilde{X}^{(2)} & & \widetilde{X}^{(1)} & & \\
\downarrow{\scriptstyle \omega_j} & & & & \downarrow{\scriptstyle \omega_2} & & \downarrow{\scriptstyle \omega} & & \\
X^{(k)} & \xrightarrow{\pi^*_{p_k}} & \cdots & \longrightarrow & X^{(2)} & \xrightarrow{\pi^*_{p_2}} & X^{(1)} & \xrightarrow{\pi^*_{\mathbf{0}}} & X \\
\downarrow{\scriptstyle \phi_j} & & & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} \\
M^{(k)} & \xrightarrow{\pi_{p_k}} & \cdots & \longrightarrow & M^{(2)} & \xrightarrow{\pi_{p_2}} & M^{(1)} & \xrightarrow{\pi_{\mathbf{0}}} & \mathbb{C}^2.
\end{array}
\tag{1.3}
$$

We denote $\phi'_J = \phi_j \circ \omega_j$ the two-fold branched covering of $M^{(j)}$ with branch locus $B^{(j)}$. Finally, Laufer shows with the following Theorem that the canonical resolution of X can be obtained from the previous construction together with some possible additional blow ups.

Theorem 1.37. *Let $C = \{f(x,y) = 0\}$ be a germ of plane curve singularity. Let $p = (0,0,0) \in X = \{z^2 = f(x,y)\}$ be the associated normal two-dimensional double point. Perform blow–ups on $\mathbb{C}^2$ until the branch locus $B^{(l)}$ of $\phi'_L = \phi_l \circ \omega_l$ is non-singular. Then $\widetilde{X}^{(l)}$ is isomorphic to the canonical resolution.*

Proof. See [101, Theorem 3.1]. □

Example 1.38. Let us consider the germ of normal surface singularity $(X,0)$ defined by the equation $z^2 = (x+y^2)(x^2+y^7)$. In order to keep the exposition clear we refer to [98, pg. 30–33] for the detailed computation of Zariski's canonical resolution of X. Let us briefly sketch the idea of that computation. As we have said, one could first consider the minimal embedded resolution of of the plane curve $C : (x+y^2)(x^2+y^7) = 0$. As one can see $C = C_1 \cup C_2$ has two branches: $C_1 : x+y^2 = 0$ and $C_2 : x^2+y^7 = 0$. Following the procedures of the previous section, the structure of the resolution process of C can be encoded in the following diagram

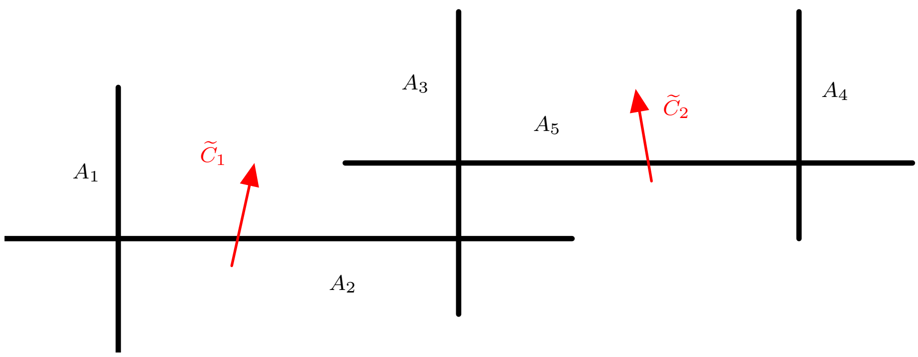


Figure 1.8: Structure of the resolution process of C. The arrows $\widetilde{C}_1$ and $\widetilde{C}_2$ denote the strict transforms of the branches of C at the end of the resolution.

Therefore, we have a sequence of $k = 5$ blow ups. Now, we can lift this process to the case $z^2 = (x+y^2)(x^2+y^7)$ by using the diagram in Equation (1.3). At step 3, in a local suitable coordinate system (u_3, v_3, z) of $X^{(2)}$, the equation at the intersection $A_3 \cap A_2$ is $z^2 = u_3^6 v_3^8$. Thus, it has two irreducible components, each of which is non-singular. A similar situation happens at step 5 where in a local suitable coordinate system (u_5, v_5, z) of $X^{(5)}$, the equation at the intersection $A_3 \cap A_2$ is $z^2 = u_5^8 v_5^{18}$. Since there is no other intersections where the divisor has more than one component, the structure of the resolution process of X can be read from Figure 1.8 by splitting the divisor A_3 in two divisors $A_{3,1}, A_{3,2}$ as Figure 1.9 shows.

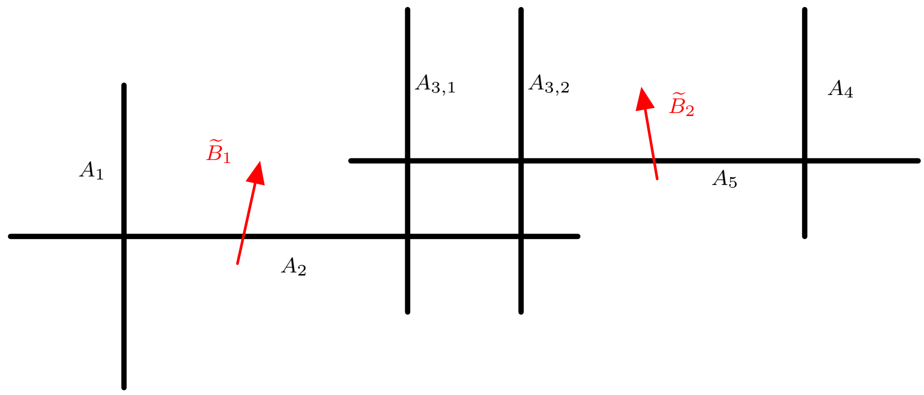


Figure 1.9: Structure of the resolution process of X. The arrows $\widetilde{B}_1$ and $\widetilde{B}_2$ denote the strict transforms of the irreducible components of the branch locus.

If we denote by $\pi : \widetilde{X} \to X$ the Zariski's canonical resolution computed in [98, pg. 30–33] then the exceptional set $\pi{-}1(0) = A_1 \cup A_2 \cup A_{3,1} \cup A_{3,2} \cup A_4 \cup A_5$ is composed by 6 exceptional curves all of them of genus 0 since they are complex projective lines. Moreover, their self–intersections are $-1, -4, -3, -3, -1, -2$, thus A_1 and A_4 are exceptional curves of the first kind as we can also see in the resolution graph of Figure 1.10.

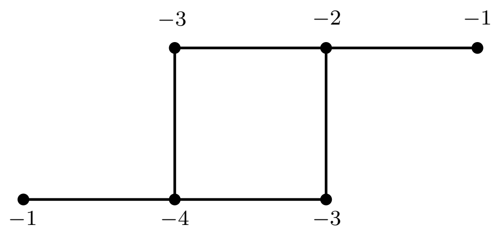


Figure 1.10: Graph of Zariski's canonical resolution of $\{z^2 = (x + y^2)(x^2 + y^7)\}$.

Following Theorem 1.35 one can now contract the exceptional curves of first kind A_1, A_4. This will produce an exceptional curve of first kind which after being contracted provides the resolution graph associated to the minimal resolution of X showed in Figure 1.11.

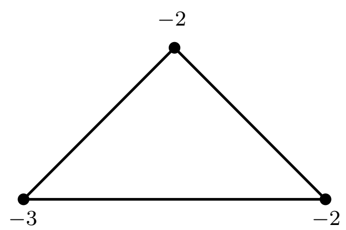


Figure 1.11: Graph of the minimal resolution of $\{z^2 = (x + y^2)(x^2 + y^7)\}$.

Finally, it can be shown that one can always obtain a minimal resolution with the following properties.

Theorem 1.39. *Let $(x, p) \subset (\mathbb{C}^3, 0)$ be a germ at p of a normal two–dimensional singularity. Then there is a unique minimal resolution $\pi : M \to V$ among all resolutions satisfying the following conditions: if $\pi^{-1}(p) = A = \cup_i A_i$ is the decomposition of $\pi^{-1}(p)$ into irreducible components then*

(a) Each A_i is non-singular,

(b) A_i and A_j, $i \neq j$, intersect transversaly wherever they intersect and

(c) no three distinct A_i meet.

Proof. See [98, Chapter V, Theorem 5.12] □

Definition 1.40. A minimal good resolution is a resolution as the one of Theorem 1.39.

Remark 1.41. The minimal resolution of Figure 1.11 is an example of minimal good resolution.

1.3 Invariants of hypersurface singularities

Let us now move to a more general context. We consider $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ being a germ of isolated hypersurface singularity. There are different ways to classify hypersurface singularities, however we are mainly interested in two equivalence relations: analytic equivalence and topological equivalence. Two germs of hypersurface singularities $(X, x), (Y, y)$ are said to be *topologically equivalent* if there exists a homeomorphism $(\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ mapping (X, x) to (Y, y). In contrast, $(X, x), (Y, y)$ are said to be *analytically equivalent* if there exists a local analytic isomorphism $(\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ mapping (X, x) to (Y, y). To each hypersurface singularity we can associate different objects, for example numbers, sets, modules, groups...; if those objects are invariant under analytic or topological equivalence then they are said to be *analytic or topological invariant.*

An easy example of topological invariants of a branch are, by Theorem 1.4, the Puiseux pairs and Puiseux characteristic exponentes defined in Section 1.1. On the other hand, an analytic invariant may vary in the topological class so, in general, they are more difficult to compute (see for example [45, Chapter 6], [71, Chapter 2], [108, Chapter 8, 9] or [190]).

Sometimes, to compute an analytic invariant from the topological data is easier if this invariant is upper semicontinuous. Since the main invariants we are going to introduce are upper semicontinuous, let us define this notion. A power series $F \in \mathbb{C}\{\mathbf{x}, \mathbf{t}\} = \mathbb{C}\{x_1, \ldots, x_{n+1}, t_1, \ldots, t_k\}$ is called *unfolding* of $f \in \mathbb{C}\{\mathbf{x}\}$ if $F(\mathbf{x}, \mathbf{0}) = f(\mathbf{x})$.

Definition 1.42. Let $F \in \mathbb{C}\{\mathbf{x}, \mathbf{t}\}$ be an unfolding of $f \in \mathbb{C}\{\mathbf{x}\}$ and assume that $\mathbf{0}$ is an isolated critical point. Assume that there are neighborhoods $U(\mathbf{0}) \subset \mathbb{C}^{n+1}$, $V(0) \subset \mathbb{C}$ and $T(\mathbf{0}) \subset \mathbb{C}^k$, such that F converges on $U \times T$ and $\mathbf{0} \in U$ is the only critical point of $f = F_{\mathbf{0}} : U \to V$ and $F_{\mathbf{t}}$ has only isolated critical points in U.

Let $I(f)$ be a numerical invariant of f, we say that $I(f, \mathbf{0})$ is *upper semicontinuous* if

$$I(f, \mathbf{0}) \geq \sum_{x \in \mathrm{Crit}\, F_{\mathbf{t}}^{-1}(y)} I(F_{\mathbf{t}}, x),$$

where $\mathrm{Crit}\, F_{\mathbf{t}}^{-1}(y)$ means the set of critical points of the function $F_{\mathbf{t}}^{-1}(y)$.

Upper semicontinuity is very useful to compute minimal values of analytic invariants under deformations with the same topological type (see for example [3, 24, 125]). On this section, we will introduce the main numerical invariants that are going to appear along this work: the Milnor number, the Tjurina number and the exponents and spectrum of an hypersurface singularity. We remark that all of them are upper semicontinous invariants. For the upper semicontinuity of Milnor and Tjurina numbers see for example [71, Thm 2.6]. For the semicontinuity of spectral numbers we refer to Steenbrink [161].

1.3.1 Milnor number

As we have seen in Section 1.1, the associated link of a plane curve singularity contains the information to give a topological classification of the singularity. This idea can be generalized to higher dimensions. In 1968, inspired by previous works of Brauner and Brieskorn, Milnor [118] provides a systematic study of the topology of an isolated hypersurface singularity defined by $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$.

More concretely, Milnor presents a fibration which allows to study the topological properties of $K = V \cap S_\epsilon \subset S_\epsilon$ where $V = f^{-1}(0)$ and $S_\epsilon := \{x \in \mathbb{C}^{n+1} | \, ||x|| = \epsilon\}$. Let us consider the map from the complement of K to the unit circle $\phi : S_\epsilon \setminus K \to S^1$ defined by $\phi(z) := \frac{f(z)}{|f(z)|}$. Then the *fibration theorem* states:

Theorem 1.43. *[118, §4] The map ϕ is the projection map of a smooth fiber bundle. Each fiber*

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus K$$

is a smooth parallelizable $2n$–dimensional manifold.

In honor to Milnor, such a fibration is called *Milnor fibration* and its fibers *Milnor fibers*. As presented in [118], the fibration theorem works for any point of a complex hypersurface. However, as it is pointed out by Milnor in [118, §6 and §7], the fibration theorem provides good topological consequences when it is applied to an isolated singular point. In particular, when V has an isolated singular point and ϵ is small enough, the closure of each fiber F_θ in S_ϵ is a smooth $2n$–dimensional manifold with boundary and the boundary is precisely K. Moreover, the closure $\overline{F}_\theta$ is embedded in S_ϵ in such a way that has the same homotopy type as its complement $S_\epsilon \setminus \overline{F}_\theta$ [118, Corollary 6.2].

Finally, the good properties of Milnor fiber allowed Milnor to compute the homotopy type of each fiber which, after that, have become one of the main topological invariants for hypersurface singularities.

Theorem 1.44. *[118, Thm 6.5 and Thm 7.2] Let us define by μ the multiplicity of 0 as an isolated solution of the collection of equations $\{\partial f/\partial z_1 = \ldots \partial f/\partial z_{n+1} = 0\}$. Then, each fiber F_θ has the homotopy type of a bouquet $\bigvee S^n$ of μ spheres.*

Moreover, the middle homology group $H_n(F_\theta)$ of the fiber is free abelian of rank μ.

Definition 1.45. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of isolated hypersurface singularity. The *Milnor number* is defined as

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \ldots, x_{n+1}\}}{(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}})}.$$

Observe that by the previous Theorem 1.44, if F is the Milnor fiber associated to f then

$$\operatorname{rank} H_n(F) = \mu \quad \text{and} \quad \chi(F) = 1 + (-1)^n \mu.$$

It is a well known fact in singularity theory that Milnor number is a topological invariant of a hypersurface singularity. A proof of this fact was pointed out by Teissier in [163, Remark 5.9]. Another proof of this fact can be found in the works of L. D. Tráng, [102, Proposition] and [103, Theoreme (3.3)].

It is also important to recall that Milnor fibration allows the definition of a *local Picard-Lefschetz monodromy transformation* on $H_n(F)$ and $H^n(F)$, from now on *monodromy transformation* on $H^n(F)$. Let us consider the disk $S := \{\theta \in \mathbb{C} | \, |\theta| < \delta\}$ and the punctured disk $S^{'} := S \setminus \{0\}$ for a sufficiently small δ and let $\gamma : [0,1] \to S^{'}$ be a loop representing an element of the fundamental group $\pi_1(S^{'}, \theta)$. Let $B = \{|x| < \epsilon\} \subset \mathbb{C}^{n+1}$ be the ball of radius ϵ small enough and fix $\delta \ll \epsilon$. Put $X = B \cap f^{-1}(S)$ and $X' = X \setminus f^{-1}(0)$. Observe that Milnor fibration theorem provides the locally trivial fibration $X' \to S'$ whose fibers are precisely the Milnor fibers. Thus, $\gamma^{-1}X' \xrightarrow{\varphi} [0,1]$ is a trivial fibration and we have a diffeomorphism $h_\gamma : F_\theta = \varphi^{-1}(0) \to F_\theta = \varphi^{-1}(1)$. Hence, we have a monodromy representation

$$\begin{array}{ccc} \pi_1(S^{'}, \theta) & \to & \operatorname{Aut} H^n(F) \\ [\gamma] & \mapsto & (h_\gamma^*)^{-1}. \end{array}$$

From the monodromy representation, we obtain the *monodromy transformation* on the cohomology of the Milnor fiber as the $\mathbb{C}$–linear transformation

$$M := (h_\gamma^*)^{-1} : H^n(F) \to H^n(F).$$

The monodromy transformation has the following important property.

Theorem 1.46 (Monodromy Theorem). *Let $f \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with isolated singularity at 0 and let $X = f^{-1}(0)$ be the germ of hypersurface singularity. Let F be the Milnor fiber of X and M the monodromy transformation on $H^n(F; \mathbb{C})$. Then,*

1. *All eigenvalues of M are roots of unity.*
2. *The size of the Jordan blocks of the the Jordan normal form of M is at most $n+1$.*

Because of its importance in singularity theory, proofs of the monodromy theorem can be found in lots of places. A very beautiful proof of the first item can be found in [26, Satz 4, pg. 11], in German. A translation to English of [26, Satz 4, pg. 11] is given in [93, Chapter I section 9]. Also it is very interesting to see the proof given by Fried [58] due to the very different techniques used there. As we will see in subsection 1.3.3, the eigenvalues of monodromy are closely related to some other analytic invariants of a singularity. Also they are involved in a famous open conjecture in singularity theory called the Monodromy Conjecture.

To finish, let us compute the Milnor number of a plane curve singularity. As we have seen in Section 1.1, the Puiseux characteristic exponents are topological invariants of a plane curve so it is natural to ask if one can compute the Milnor number from the Puiseux exponents. To do that, we need first to introduce the following invariant (see [158, §1, Chapter IV]).

Definition 1.47. Let us denote by $\mathcal{O} := \frac{\mathbb{C}\{x,y\}}{(f)}$ the local ring at the isolated singular point of the germ of plane curve $C := \{f(x,y) = 0\}$ and let $\overline{\mathcal{O}}$ the integral closure of the local ring $\mathcal{O}$. The *delta invariant of a plane curve* is defined as

$$\delta(C) := \dim_{\mathbb{C}} \frac{\overline{\mathcal{O}}}{\mathcal{O}}.$$

Moreover, the following Theorem 1.47 provides an expression of δ in terms of the sequence of multiplicities of the set of infinitely near points.

Theorem 1.48. *If (C, O) is a (reduced) germ of isolated plane curve singularity, $\mathcal{N}_O$ is the set of infinitely near points to the origin and e_p denotes the multiplicity at $p \in \mathcal{N}_O$ then*

$$\delta(C) = \sum_{p \in \mathcal{N}_O} \frac{e_p(e_p - 1)}{2}.$$

Proof. See for example [31, Thm. 3.11.12] □

Finally, the next theorem allows to compute the Milnor number for plane curve singularities in terms of the resolution.

Theorem 1.49. *[118, §10] Let C be a germ of isolated plane curve singularity and r be its number of branches. Then,*

$$\mu = 2\delta - r + 1.$$

Example 1.50. The previous formula together with the formula of the δ–invariant allow us to easily compute the Milnor number of the curve of example 1.3. The sequence of multiplicities $4, 2, 2, 1, 1$ was computed in example 1.24. Thus, we have

$$\mu = 4 \cdot 3 + 2 \cdot 1 + 2 \cdot 1 - 1 + 1 = 16.$$

1.3.2 Tjurina number

The Tjurina number is a very rich numerical invariant, in the sense that there are several techniques to compute it which have provided a vast topic of research. Mainly, there are three lines to work with it: analytic classification [114], sheaves of relative differentials (see for example [190]) and deformation theory [166]. As one can see in a quick view of the index of this work, the Tjurina number is the leitmotiv of most of the research in this memoir. For this reason, this subsection will only present a brief framework about the main aspects related with the Tjurina number.

The Tjurina number takes this name in honor to the Russian mathematician G.N. Tjurina since she gave the explicit description of the miniversal deformation of a normal isolated singularity in [166]. We will recall this result in the Appendix B. She tragically died on 21 July 1970 with the short age of 31. However, her work constitutes amazing advances on singularity theory and algebraic geometry as one can see in her Obituary [15] written by Arnol'd et al.

Despite Tjurina's result [166] appeared in 1969, the Tjurina number name for this invariant does not become popular in the bibliography until the late 1980s. As it is pointed out by Greuel in [69, footnote 8, pg. 389], the name of Tjurina number was introduced by himself in [68] in 1980. Finally, after the key result of Looijenga and Steenbrink [109] one can see that the name of Tjurina number established as the common nomenclature.

Let us introduce the definition of Tjurina's number for the case of hypersurface singularities.

Definition 1.51. Let $(X,0) \in (\mathbb{C}^n,0)$ be an isolated hypersurface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^n,0}$. The *Tjurina algebra* is defined as

$$T_f := \frac{\mathbb{C}\{x_1,\dots,x_n\}}{(f,\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})}.$$

The dimension as complex vector space of the Tjurina algebra $\tau := \dim_{\mathbb{C}} T_f$ is called *Tjurina number* of X.

It is easy to see that Tjurina number is not a topological invariant, for example take the curves $C_1 := \{y^7 - x^9 = 0\}$ and $C_1 := \{y^7 - x^9 + x^5y^5 = 0\}$ which are equisingular plane curve singularities with $\tau(C_1) = 48 \neq 45 = \tau(C_2)$. The following Theorem of Mather and Yau [114] shows that the Tjurina number is an analytic invariant.

Theorem 1.52 ([114]). *Let $f, g \in \mathbb{C}\{x_1,\dots,x_n\}$ be non units. Then the isolated hypersurface singularities defined by f and g are analytically equivalent if and only if their Tjurina algebras are isomorphic as $\mathbb{C}$–algebras.*

Remark 1.53. Linking with the introduction of this subsection, one can see in the original paper of 1982 by Mather and Yau [114] they call moduli algebra to the Tjurina algebra. The name moduli algebra is motivated by the Theorem 1.52 since after Theorem 1.52 the $\mathbb{C}$–algebra structure of the Tjurina algebra encodes the analytic type of the hypersurface singularity.

To finish, let us move to the case of irreducible plane curves. In the case of plane branches, the Tjurina number can be identified with the length of a certain torsion module. Let $f(x,y) = 0$ be an irreducible germ of plane curve singularity with any number of Puiseux pairs and let us consider $R = \mathbb{C}\{x,y\}/(f(x,y))$ the local ring of the curve and $Rdx + Rdy$ the module of Kähler differentials. This module has a torsion part which we denote by $\mathcal{T}$. In 1966, Zariski [196] gives the following result:

Theorem 1.54. *[196] Let $f(x,y) = 0$ be an irreducible germ of plane curve singularity with any number of Puiseux pairs. Then, $l(\mathcal{T}) = \tau$, where $l(\mathcal{T})$ denotes the length of $\mathcal{T}$ as R–module and τ denotes the Tjurina number.*

In [196], Zariski also proved [196, Theorem 2] that $l(\mathcal{T}) \leq c(\Gamma)$. However, from his proof we can also provide an effective way to compute $l(\mathcal{T})$ in terms of certain value set. Following the proof of [196, Theorem 2]. Let us denote $\overline{R} := \mathbb{C}\{t\}$ and by $\overline{R}D\overline{R}$ the module of differentials of $\overline{R}$ as R module. Observe that $\overline{R}D\overline{R} = \overline{R}dt$. On the other hand let us denote by RDR the submodule of $\overline{R}D\overline{R}$ generated over R by the differentials $D\xi$ with $\xi \in R$, i.e.

$$R \ni \xi \mapsto D\xi := \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy \in RDR.$$

Recall that we have a Puiseux parameterization $(t^n, s(t^n) = \sum_{j\geq n} a_j t^j)$ in such a way that the morphism

$$R = \mathbb{C}\{x,y\}/(f) \overset{\varphi}{\hookrightarrow} \overline{R} = \mathbb{C}\{t\}$$

induces a discrete valuation $v : R \to \mathbb{N} \cup \infty$ of the field of fractions of R. Thus, we have a well defined homomorphism

$$\begin{array}{rccc} \varphi' : & RDR & \longrightarrow & \overline{R} \\ & \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy & \mapsto & \varphi\left(\frac{\partial \xi}{\partial x}\right)\frac{d\varphi(x)}{dt} + \varphi\left(\frac{\partial \xi}{\partial y}\right)\frac{d\varphi(y)}{dt}. \end{array}$$

Observe that φ' allow us to extend the discrete valuation v to the R–module RDR in such a way that for $\xi \in RDR$ we have

$$v(\xi) := \mathrm{ord}_t(\varphi'(\xi)).$$

Following Zariski's proof of [196, Theorem 2] let us denote by $\mathfrak{M}$ the maximal ideal of R. Then, for any integer $\nu = v(\xi)$ in the set $v(\mathfrak{M})$ we have $\nu - 1 = v(D\xi)$ and so $\nu - 1 \in v(RDR)$. Following [197, Chap. III, Sec. 3], any differential form $\omega \in RDR$ such that $\omega = d\xi$ with $\xi \in R$ is called *exact differential form*. From a result of Berger [18, §3, Korollar 2], one can compute the length of $\mathcal{T}$ as

$$l(\mathcal{T}) = l(\overline{R}D\overline{R}/RDR) + l(\overline{R}/R). \tag{1.4}$$

Following [197, Chap. III, Sec. 3] (see also [196, Note pp. 782]), the number of linearly independent *non-exact differential forms* is precisely $r = c - l(\mathcal{T})$. In order to compute τ, it remains to compute $l(\overline{R}D\overline{R}/RDR)$. This can be done by characterizing the value set of the non-exact differential forms.

The set of gaps of Γ, $\mathbb{N} \setminus \Gamma = \{\alpha_1, \dots, \alpha_{c/2}\}$, is precisely $\mathbb{N} \setminus v(\mathfrak{M}) = \mathbb{N} \setminus \Gamma$. Moreover, $c/2 = |\mathbb{N} \setminus \Gamma| = l(\overline{R}/R)$ is the delta invariant of the curve. Here, Zariski [196, Theorem 2] deduces the inequality $l(\mathcal{T}) \leq c$ from the following inclusion of sets

$$v(\overline{R}D\overline{R}) \setminus v(RDR) \subset \{\alpha_1 - 1, \dots, \alpha_{c/2} - 1\}. \tag{1.5}$$

Equation (1.5) provides the desired bound. However, one could say a bit more since $l(\overline{R}D\overline{R}/RDR) = |v(\overline{R}D\overline{R}) \setminus v(RDR)|$. Observe that a differential form $\omega \in RDR$ is exact if and only if $v(\omega) + 1 \in \Gamma$. Thus, we can consider the set of values

$$\Lambda := \{v(t\omega) |\, \omega \in Rdx + Rdy,\ \omega \neq 0\} \cup \{0\}.$$

Thus, since $\Gamma \subset \Lambda$, Berger's identity (1.4) together with the equality

$$|\mathbb{N} \setminus \Lambda| = |v(\overline{R}D\overline{R}) \setminus v(RDR)| = \frac{c}{2} - r = |\mathbb{N} \setminus \Gamma| - |\Lambda \setminus \Gamma|$$

provides the following way to compute the Tjurina number of an irreducible plane curve singularity.

Proposition 1.55. *[30, Prop. 3] Let $f(x, y) = 0$ be a germ of irreducible plane curve singularity with any number of Puiseux pairs. Let $c(\Gamma)$ be the conductor of the semigroup. Then,*

$$\tau = c(\Gamma) - |\Lambda \setminus \Gamma|$$

Remark 1.56. Proposition 1.55 was proven by Carbonne in [30, Prop. 3]. However, we felt that his definition of the set Λ at the end of pp. 378 seems to be a bit artificial in order to conclude Proposition 1.55 [30, Prop. 3]. With this discussion we hope we have clarified the definition of Λ.

Remark 1.57. Quite recently, Abreu and Hernandes [1] have used this characterization of the Tjurina number in order to show that the Tjurina number of a plane curve singularity with semigroup $\Gamma = \langle \overline{\beta}_0, \dots, \overline{\beta}_g \rangle$ with $\gcd(\overline{\beta}_0, \dots, \overline{\beta}_{g-1}) = 2$ is constant along the equisingularity class defined by Γ. In particular, they show [1, Theorem 5.2] that for this family of plane branches $|\Lambda \setminus \Gamma|$ equals the conductor of $\Gamma_{g-1} = \langle \overline{\beta}_0/2, \ldots, \overline{\beta}_{g-1}/2 \rangle$. This formula generalizes the case $g = 2$ previously proven by Luengo and Pfister [112].

Observe that this particular family is very special since in general the Tjurina number varies in the equisingularity class (see for example Theorem 2.21).

Example 1.58. We can use Luengo-Pfister [112] and Abreu-Hernandes [1] results in order to compute the Tjurina number of the plane branch of Example 1.24. We have seen in Example 1.30 that the semigroup of this curve is $\Gamma = \langle 4, 6, 13 \rangle$ and in Example 1.50 that the Milnor number is $\mu = 16$. Thus we are in the condition of the Theorems of [1, 112] and we have:

$$\tau = \mu - c(\langle 2, 3 \rangle) = 16 - 2 = 14.$$

1.3.3 Exponents of an isolated hypersurface singularity

The set of exponents, or equivalently the singularity spectrum, is one of the most known set of discrete analytic invariants of an hypersurface singularity. The name *exponents* comes from the definition given by M. Saito in 1983 [147] in order to distinguish this set from the singularity spectrum previously defined by Steenbrink in 1976 [160]. Both, the spectrum and the exponents, are sets of μ rational numbers, where μ is the Milnor number of the hypersurface singularity. Moreover, α is a spectral number if and only if $\alpha + 1$ is an exponent (see [149, Theorem] or [93, II, Sec 8.1, Remark 8.17, pg. 117]). This is the main reason why today it is possible to speak simply of spectrum when referring to both the spectrum and the exponents, specifying if it is according to the definition of M. Saito or that of Steenbrink.

We recall that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a germ of isolated hypersurface singularity. Let us denote by t the coordinate in the target space $(\mathbb{C}, 0)$. Let us denote by $\Omega^k_{(\mathbb{C}^{n+1},0)}$ the sheaf of differential k–forms on $(\mathbb{C}^k, 0)$. After Brieskorn's work [26], the $\mathbb{C}\{t\}$–module

$$H_0'' := \frac{\Omega^{n+1}_{(\mathbb{C}^{n+1},0)}}{df \wedge d\Omega^{n-1}_{(\mathbb{C}^{n+1},0)}}$$

becomes a central object in the study of the invariants of a singularity. The module H_0'' was hence been called *Brieskorn lattice*. Brieskorn lattice is not only a $\mathbb{C}\{t\}$–module but also has a module structure over the ring $\mathbb{C}\{\{\partial_t^{-1}\}\} := \{\sum_{i\geq 0} a_i \partial_t^{-i} \mid \sum_i a_i t^i/i! \in \mathbb{C}\{t\}\}$ of microdifferential operators with constant coefficients (see [135, Microlocalisation §1]). Observe that we can naturally define a differential operator ∂_t^{-1} acting on the Brieskorn lattice as follows:

$$\partial_t^{-1}[\omega] := [df \wedge \alpha],$$

where $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$, $\alpha \in \Omega^{n}_{\mathbb{C}^{n+1},0}$ such that $d\alpha = \omega$ and $[\omega]$ denotes the class of an element of $\Omega^{n+1}_{\mathbb{C}^{n+1},0}$ in H_0''. Hence, it is easy to check that

$$\partial_t^{-1} H_0'' = \frac{df \wedge \Omega^{n}_{(\mathbb{C}^{n+1},0)}}{df \wedge d\Omega^{n-1}_{(\mathbb{C}^{n+1},0)}} \subset H_0''.$$

Thus, H_0'' has module structure over the ring $\mathbb{C}\{\{\partial_t^{-1}\}\}$. Moreover, by Pham [135, Proposition, pg. 280] (see also [135, pgs. 155–162]) H_0'' is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$–module of rank μ, where μ is the Milnor number of the singularity.

Since the action of t on H_0'' is $t\omega := f\omega$, it is natural to ask for the existence of a "good basis" of H_0'' as $\mathbb{C}\{\{\partial_t^{-1}\}\}$–module which behaves well with respect to the action of t. This was proven by M. Saito [149].

Theorem 1.59. *[149, Theorem] There exists a basis $\{v_i\}$ of H_0'' as $\mathbb{C}\{\{\partial_t^{-1}\}\}$–module and matrices with complex coefficients A_0, A_1 such that*

$$tv = A_0 v + A_1 \partial_t^{-1} v$$

where $v = (v_1, \dots, v_\mu)^t$. Moreover, A_0 is nilpotent and A_1 is semisimple.

Now, we are ready to introduce the exponents of a hypersurface singularity.

Definition 1.60. The *exponents* of f are defined as the set of eigenvalues of the matrix A_1.

Remark 1.61. The previous definition strongly depends on the special basis provided by Theorem 1.59. According to M. Saito [149], the basis $\{v_i\}$ is not intrinsic and it can be replaced by the corresponding $\mathbb{C}$–linear section v of the natural projection:

$$pr : H_0'' \to \frac{H_0''}{\partial_t^{-1} H_0''} \cong \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0}} =: \Omega_f$$

such that $\operatorname{Im} v = \sum \mathbb{C} v_i$.

In a recent paper M. Saito [152] calls such a section a *very good section* if the eigenvalues of A_1 coincide with the usual spectrum. Moreover, he shows the existence of good sections which are not very good, that is, good sections lead to a matrix A_1 whose eigenvalues do not coincide with the usual spectrum [152, Example 4.1]. In any case, the existence of a very good section is always guaranteed by [149, Theorem] and thus our definition is consistent.

The main properties of the exponents/spectrum has been developed during the eighties by several mathematicians, here we should mention at least the works of M. Saito (e.g [147, 148, 151]), K. Saito [146], Steenbrink [160, 161, 154] and Varchenko [171]. In the following proposition we mention two important properties of the spectrum that will be needed in Chapter 4.

Proposition 1.62. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of isolated hypersurface singularity. Let $\{\alpha_1, \dots, \alpha_{\mu_f}\}$ be the set of exponents of f. Then,*

(a) (Symmetry) $\alpha_i = n + 1 - \alpha_{\mu_f+1-i}$.

In particular, the exponents are symmetric with respect to $(n+1)/2$.

(b) (Thom-Sebastiani) Assume $f(x_1, \dots, x_{n+1})$ and $g(y_1, \dots, y_{m+1})$ are series in separate sets of variables. Let $\{\beta_1, \dots, \beta_{\mu_g}\}$ be the exponents of g. Then, the set of exponents of $f \oplus g$ is

$$\{\alpha_i + \beta_j : \ 1 \le i \le \mu_f \ 1 \le j \le \mu_g\}.$$

Proof. (a)[147, Proposition 3.3], (b) [146, (1.6)] and [171, Theorem 7.3].

We cannot give a complete overview about the exponents without at least briefly mentioning their relationship with the eigenvalues of the monodromy of the Milnor fiber. Observe that the Monodromy Theorem 1.46 can be shortly stated as follows: there exists $N \in \mathbb{N}$ such that $(M^N - Id)^{n+1} = 0$. Moreover, the monodromy theorem allows to decompose the monodromy operator $M = M_s M_u = M_u M_s$ into a semisimple part M_s and the unipotent part M_u.

The Monodromy Theorem 1.46 tells us that all the eigenvalues of M are of the form $\lambda = e^{2\pi i\alpha}$. Moreover, since $\dim_{\mathbb{C}} H^n(F;\mathbb{C}) = \mu$ there are exactly μ eigenvalues counted with multiplicity.

With the notation of the Remark 1.61, let us take a very good section v (see also [152])

$$pr : H_0'' \to \frac{H_0''}{\partial_t^{-1} H_0''} \cong \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0}} =: \Omega_f$$

such that $\operatorname{Im} v = \sum \mathbb{C} v_i$.

Observe that thanks to the existence of this very good section, Theorem 1.59 can be translated to Ω_f and A_0, A_1 can be considered as $\mathbb{C}$–linear endomorphisms of Ω_f [149, Theorem ']. According to M. Saito [149], it is possible to define a V–filtration on Ω_f. It can be done by considering Ω_f^α the α–eigenspace with respect to the action of A_1. If we denote by $V^\alpha \Omega_f = \bigoplus_{\beta \geq \alpha} \Omega_f^\alpha$ and $V^{>\alpha}\Omega_f = \bigoplus_{\beta > \alpha} \Omega_f^\alpha$, we can consider the corresponding graded pieces $\operatorname{Gr}_V^\alpha \Omega_f = V^\alpha \Omega_f / V^{>\alpha}\Omega_f$. Thus, we can define $\operatorname{Gr}_V A_0$ and $\operatorname{Gr}_V A_1$ as the induced maps in $\operatorname{Gr}_V \Omega_f = \bigoplus \operatorname{Gr}_V^\alpha \Omega_f$. Moreover, Saito proves the following

Proposition 1.63. *[149, Proposition, pg.29] The matrices M_u, M_s of the unipotent and semimsimple parts of the monodromy transformation can be identified with* $\exp(-2\pi i \operatorname{Gr}_V A_0)$ *and* $\exp(-2\pi i \operatorname{Gr}_V A_1)$ *respectively. But A_0, A_1 do not commute in general.*

This proposition together with the next definition provide the well known relations between the exponents, the spectrum and the eigenvalues of the monodromy. We refer to [93, II, section 8.] for a more detailed description of these relations.

Definition 1.64. Let f be a hypersurface isolated singularity. Let F be its Milnor fiber and let M be the monodromy transformation on $H^n(F;\mathbb{C})$. Let μ be the Milnor number of f. Let us denote by $\lambda_1, \ldots, \lambda_\mu$ be the eigenvalues of the monodromy M. The *spectrum* of f is a discrete invariant formed by μ rational numbers $\alpha_1, \ldots, \alpha_\mu \in \mathbb{Q} \cap (-1, n)$, called the *spectral numbers*, which are defined as

$$\alpha_j = -\frac{1}{2\pi i} \log \lambda_j.$$

Remark 1.65. Observe that the previous definition is not complete since one should justify the choice of the complex logarithm branch in order to obtain $\alpha_i \in (-1, n)$. This choice depends on the Hodge filtration on the vanishing cohomology of the Milnor fiber. Since we do not want to enter into detail we refer to [93, II, section 8] for that justification.

Remark 1.66. Exponents are also related to another important invariant called the jumping numbers. See [174, pg. 1191] [107, pg. 171] and more specifically [49, Remark 2.2]

Remark 1.67. The spectrum of a hypersurface singularity can be explicitly computed in a computer with the help of SINGULAR [35] thanks to the work of M. Schulze [155, 156].

To finish, let us mention the case of irreducible plane curve singularities. In that case, M. Saito [151, version 2000: Theorem 1.5] provides an explicit expression of the exponents in terms of the Puiseux pairs.

Theorem 1.68. *[151, Theorem 1.5] Let C be a germ of an irreducible isolated plane curve singularity with semigroup $\Gamma = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$. Let us denote by $e_i := \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_i)$ and $n_i := e_{i-1}/e_i$. Let $\{(n_1, q_1), \ldots, (n_g, q_g)\}$ be the Newton pairs of f. Then, the exponents of f in $(0, 1]$ are*

$$\bigcup_{s=1}^{g}\left\{\frac{q_s i + n_s j + n_s q_s r}{n_s \overline{\beta}_s} \,:\, 0 < i < n_s,\ 0 < j < q_s,\ 0 \le r < e_s,\ \frac{i}{n_s} + \frac{j}{q_s} < 1\right\}$$

Since the exponents of a plane curve are symmetric with respect to 1, then M. Saito's Theorem 1.68 completely determines the full set of exponents of an irreducible plane curve with a given semigroup. Also, the spectrum is determined since it is obtained by subtracting 1 to the exponents.

Example 1.69. From Theorem 1.68 we can compute the spectrum of the curve $(y^2 - x^3)^2 - x^5 y = 0$ of the Example 1.3. Recall that we computed in Example 1.30 its semigroup $\Gamma = \langle 4, 6, 13\rangle$. Also the Newton pairs are $(n_1, q_1) = (2, 3)$ and $(n_2, q_2) = (2, 13)$. Thus, the exponents in $(0, 1]$ are

$$\left\{\frac{5}{12}, \frac{11}{12}\right\} \bigcup \left\{\frac{15}{26}, \frac{17}{26}, \frac{19}{26}, \frac{21}{26}, \frac{23}{26}, \frac{25}{26}\right\}.$$

Applying the symmetry with respect to 1, one can easily obtain the exponents in $(1, 2]$. Following the above discussion we can directly compute the spectrum by subtracting 1 to the set of exponents:

$$\mathrm{Sp}(f) = \left\{\pm\frac{7}{12}, \pm\frac{11}{26}, \pm\frac{9}{26}, \pm\frac{7}{26}, \pm\frac{5}{26}, \pm\frac{3}{26}, \pm\frac{1}{12}, \pm\frac{1}{26}\right\}.$$

1.4 Newton non-degenerate hypersurface singularities

Newton non-degenerate singularities were first defined by Kouchnirenko [92] in 1976. As pointed out by Wall [181], it can be said that such singularities emerged as the need for "piecewise filtrations"related to the Newton polygon of a function expressed in Arnol'd paper [14, English translation: Section 9].

The notion of non-degeneracy of a function with respect to the Newton polygon have been widely study in the literature. Just to cite some of them, Wall [181, pg. 4] provides a weaker notion of non-degeneracy which is wide enough to include all weighted homogeneous functions with isolated singularity. As one can see in Oka's book [127], the notion of non-degeneracy can be naturally extended to complete intersection singularites. Bivià-Ausina, Fukui and Saia [19] generalize Kouchnirenko's method in order to compute colengths of some ideals they called Newton-non degenerate ideals.

Newton non-degenerate singularities constitute a family of singularities for which is possible to explicitly compute topological and analytic invariants: for instance, Kouchnirenko's famous theorem [92, Théorème I, II, II] (see also 1.86) which allows to compute the Milnor number for non-degenerate singularities; Varchenko's formula for the zeta-function of the monodromy of a Newton non-degenerate singularity [172, Theorem 4.1]. From Varchenko's result the relationship between Toric geometry and the resolution of singularities of a Newton non-degenerate singularity becomes apparent (see also [91, 128]). Finally, we should also mention that Steenbrink [160, Sec. 5.6] showed a formula for the generating series of the spectrum of a Newton non-degenerate hypersurface singularity. This formula will play a central role in Chapter 4.

1.4.1 A bit of convex geometry

As we will see in a moment, Newton non-degenerate singularities are a special family of singularities that can be studied through the analysis of the combinatorial properties of certain convex polytopes. For this reason, before to talk about Newton non-degenerate singularities, let us recall some basic definitions in convex geometry which will be also useful in Chapter 4. We refer to [56] for a general study of convex geometry.

We say that x is a *convex combination* of $x_1, \dots, x_r \in \mathbb{R}^n$ if there exist $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that

$$x = \lambda_1 x_1 + \cdots \lambda_r x_r \quad \text{with} \quad \sum_{i=1}^{r} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for all } i = 1, \dots, r.$$

Given a set $M \subset \mathbb{R}^n$, the set of all convex combinations of elements of M is called *convex hull of M*. We will denote it by $\mathrm{ConvH}(M)$. Obviously, the convex hull is the smallest convex set containing M. In the particular case where M is finite set, $\mathrm{ConvH}(M)$ is called *polytope*.

The objects in convex geometry can be thought as a reunion of smallest pieces. In this way, the description of a convex object is made through its faces which are again made by faces of small dimension and so on. Let us properly introduce this notion.

To simplify notation, let us denote by $u \cdot v := \sum u_i v_i$ the usual scalar product for $u, v \in \mathbb{R}^n$.

Definition 1.70. Let $K \subset \mathbb{R}^n$ be a closed convex set. We will say that a hyperplane $H := \{x \ : \ x \cdot u = \alpha\}$ is a *supporting hyperplane of* K if $K \cap H \neq \emptyset$ and $K \subset H^+$ or $K \subset H^-$. We call $H^+ = \{x \ : \ x \cdot u \geq \alpha\}$ (resp. $H^- = \{x \ : \ x \cdot u \leq \alpha\}$) a supporting half-space of K.

Theorem 1.71. *Any closed convex set possesses a supporting hyperplane at each of its boundary points.*

Proof. See for example [56, Chap.1, Theorem 3.9]. □

Definition 1.72. If H is a supporting hyperplane of the closed convex set K, we call $F := K \cap H$ a *face* of K. We will say k–face for a face of dimension k. We call a face F a *vertex* if its dimension is 0, an *edge* if its dimension is 1 and *facet* if its dimension is $\dim K - 1$. We write $\sigma \in K$ to indicate that σ is a face of K. For $\sigma, \tau \in K$, write $\tau \leq \sigma$ if τ is a face of σ.

Another important notion to describe a convex object are cones. Let us introduce their definition.

Definition 1.73. Let $M \subset \mathbb{R}^n$. We call the *cone of M* to the set of all nonnegative linear combinations of elements of M:

$$\sigma_M := \{x = \lambda_1 y_1 + \cdots \lambda_r y_r \ : \ y_1 \dots, y_r \in M \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for all } i = 1, \dots, r\}.$$

If $M = \{y_1, \dots, y_k\}$ is a finite set, we will say that the cone σ is generated by $\{y_1, \dots, y_k\}$

A lattice N of rank d is a free abelian group isomorphic to $\mathbb{Z}^d$. A set of d linearly independent vectors $v_1, \dots, v_d$ is a basis of N if every $x \in N$ can be expressed as $x = \sum \alpha_i v_i$ with $\alpha_i \in \mathbb{Z}$. A cone σ is *rational* in a lattice N if it is generated by a finite set $S \subset N$. The cone σ is *regular* if the elements of a minimal system of generators belong to a basis of the lattice N. A useful characterization of a regular cone is the following

Lemma 1.74. *Let σ be a k–dimensional cone such that there exists a supporting hyperplane H with $H \cap \sigma = \{0\}$. Assume that σ is generated by $\{x_1, \dots, x_k\} \subset \mathbb{R}^n$ such that for each $i = 1, \dots, k$ the coordinates of $x_i = (x_{i,1}, \dots, x_{i,n})$ are relatively prime, i.e. they satisfy* $\gcd(x_{i,1}, \dots, x_{i,n}) = 1$. *Then the following conditions are equivalent.*

1. *σ is regular.*
2. *There exists vectors $x_{k+1}, \dots, x_n$ with its coordinates relatively prime such that*

$$\det(x_1, \dots, x_n) = \pm 1.$$

3. *There exists a unimodular linear transformation that maps the canonical basis vectors $e_1, \dots, e_k$ onto $x_1, \dots, x_k$ respectively.*

Proof. See [56, Chap. V, Sec. 1, Definition 1.10 and Lemma 1.11]. □

Remark 1.75. We will usually work with cones over $\mathbb{Q}^d$ instead of $\mathbb{Z}$. In this case, given a cone $\sigma \subset \mathbb{R}^d$ we can always consider $\sigma_{\mathbb{Q}} := \sigma \cap \mathbb{Q}$. If σ is generated by $w_1, \dots, w_k$ then $\sigma_{\mathbb{Q}} = \sum \mathbb{Q}_{\geq 0} w_i$.

Definition 1.76. A fan Σ is a finite collection of cones such that:

1. If σ is a cone of Σ and $\tau \leq \sigma$, then $\tau \in \Sigma$.
2. If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both cones.

The set $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma$ is called the *support of the fan Σ*. A rational fan is called *regular* if all its cones are regular.

Sometimes, it is necessary to work with regular fans. This can always be done by subdividing our original fan. A fan Σ' is said to be a *subdivision of a fan Σ* if $|\Sigma| = |\Sigma'|$ and for every cone $\sigma' \in \Sigma'$ there exists a cone $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$. Following Khovanskii, Varchenko [91, Thm. 1, Remark] pointed out that a rational fan has always a regular subdivision.

Theorem 1.77. *Any finite rational fan has a regular subdivision.*

Proof. See Theorem [91, On English translation: Section 1.2, Theorem 1]

To finish, let us introduce the algebra associated to a rational cone. This algebra will play an important role in Sections 4.2 and 4.4. Before to introduce it, let us mention the following Lemma, also known as Gordan's Lemma

Lemma 1.78. *If σ is a rational cone in $\mathbb{R}^n$, then the monoid $\sigma \cap \mathbb{Z}^n$ is finitely generated.*

Proof. See for example [56, Chap. V, Sec. 3, Lemma 3.4]. □

Definition 1.79. Let σ be a rational cone, the algebra associated to σ is

$$A_\sigma := \{f \in \mathbb{C}[z, z^{-1}] \mid \operatorname{supp}(f) \subset \sigma\}.$$

An immediate consequence of Gordan's lemma is that A_σ is a finitely generated graded monomial algebra. Take $(a_1, \dots, a_k)$ a system of generators of the monoid $\sigma \cap \mathbb{Z}^n$, then $A_\sigma = \mathbb{C}[u_1, \dots, u_k]$ where

$$u_i := z^{a_i} \in \mathbb{C}[z, z^{-1}] \quad \text{for } i = 1, \dots, k.$$

Thus, A_σ is a finitely generated algebra. Consider now the indeterminates $X_1, \dots, X_k$. Then, we can consider the algebra homomorphism

$$\begin{array}{rccc} \varphi_A : \mathbb{C}[X_1, \dots, X_k] & \to & \mathbb{C}[u_1, \dots, u_k] \\ X_i & \mapsto & u_i. \end{array}$$

Observe that $A_\sigma \simeq \mathbb{C}[X_1, \dots, X_k]/\operatorname{Ker}\varphi_A$. Moreover, one can explicitly compute $\operatorname{Ker}\varphi_A$. We say that $(\nu, \lambda) = (\nu_1, \dots, \nu_k; \lambda_1, \dots, \lambda_k) \in \mathbb{N}^{2k}$ is a linear integral relation for $(a_1, \dots, a_k)$ if

$$\nu_1 a_1 + \cdots \nu_k a_k = \lambda_1 a_1 + \cdots \lambda_k a_k.$$

Thus, each integral relation (ν, λ) provides a binomial relationship of the form

$$u_1^{\nu_1} \cdots u_k^{\nu_k} - u_1^{\lambda_1} \cdots u_k^{\lambda_k} = 0.$$

Therefore $\operatorname{Ker}\varphi_A$ is generated by all binomial relationships of the form

$$X_1^{\nu_1} \cdots X_k^{\nu_k} - X_1^{\lambda_1} \cdots X_k^{\lambda_k} = 0$$

for (ν, λ) an integral relation for $(a_1, \dots, a_k)$. Then A_σ is a finitely generated graded monomial algebra described by the previous set of generators and relations.

Finally, observe that if σ is a rational regular cone then there is no integral relations for $(a_1, \dots, a_k)$ since the generators $(a_1, \dots, a_k)$ are linearly independent over $\mathbb{Z}$. Thus, if σ is a rational regular cone $A_\sigma \simeq \mathbb{C}[u_1, \dots, u_k]$ with $v(u_i) = a_k$ since $\operatorname{Ker}\varphi_A = 0$.

1.4.2 Newton non-degenerate singularities

Let us consider a germ of holomorphic function

$$f = \sum_{\mathbf{m} \in \mathbb{N}^{n+1}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}\{\mathbf{x}\} = \mathbb{C}\{x_0, \dots, x_n\}.$$

We denote by $\operatorname{supp} f = \{\mathbf{m} \in \mathbb{N}^{n+1} |\, c_{\mathbf{m}} \neq 0\}$ the support of f. Observe that the support is the set of monomials effectively appearing in the equation of f. One can represent these monomials as points in $\mathbb{R}_+^{n+1}$ just by assigning to each monomial $\mathbf{x}^{\mathbf{m}} = x_0^{m_0} \cdots x_n^{m_n}$ the point $\mathbf{m} = (m_0, \dots, m_n) \in \mathbb{R}^{n+1}$.

Following Kouchnirenko [92], we will say that f is "commode "(in French) if for each $i = 0, \dots, n$ and for some $m_i \in \mathbb{N}$ we have $m_i \mathbf{e}_i \in \operatorname{supp} f$, where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector with all its coordinates equal to 0 except the i–th coordinate which is equal to 1.

Definition 1.80. Given a germ of holomorphic function $f \in \mathbb{C}\{\mathbf{x}\} = \mathbb{C}\{x_0, \dots, x_n\}$, we can associate to f three different sets in $\mathbb{R}^{n+1}$:

$$\begin{aligned} \Gamma_+(f) &:= \operatorname{ConvH}\Big(\bigcup_{m \in \operatorname{supp} f \setminus \{0\}} (m + \mathbb{R}_+^{n+1})\Big), \\ N(f) &:= \text{Boundary of } \Gamma_+(f) = \text{union of compact faces of } \Gamma_+(f), \\ \Gamma_-(f) &:= \text{union of all segments joining the origin with } N(f). \end{aligned}$$

We call *Newton polyhedron of* f to $\Gamma_+(f)$. According to Kouchnirenko [92, Définition 1.6 I] we call $N(f)$ the *Newton boundary*. However, it is also well established in the literature to call $N(f)$ the *Newton polytope or Newton diagram* of f. Also, we call *Newton fan* to $\Gamma_-(f)$. Finally, we call *Newton principal part of* f to

$$f_0 := \sum_{\mathbf{m}\in N(f)} c_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}.$$

Remark 1.81. Observe that by definition, $N(f)$ is a polytope satisfying the following conditions: (a) each ray through the origin of $\mathbb{R}_+^{n+1}$ meet $N(f)$ in just one point and (b) the region lying above $N(f)$, which is $\Gamma_+(f)$ is convex.

Let F be a face and σ_F be its cone. If the context is clear we will put $\sigma := \sigma_F$ and we will refer to σ as face. For a face F, we denote by $f_{\sigma_F} := f_\sigma = \sum_{m\in\sigma_F} c_m\mathbf{x}^m$ the restriction to a cone of the support of a function. Now, we are able to introduce the notion of non-degeneracy according to Kouchnirenko.

Definition 1.82. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of holomorphic function and f_0 be its Newton principal part. We say that f_0 is *Newton non-degenerate* (or just non-degenerate) in the sense of Kouchnirenko if for each face σ of $N(f)$, the polynomials

$$\left(x_0\frac{\partial f}{\partial x_0}\right)_\sigma, \ldots, \left(x_n\frac{\partial f}{\partial x_n}\right)_\sigma$$

have no common zeroes on $(\mathbb{C}\setminus\{0\})^{n+1}$. Obviously, we will say that f is Newton non-degenerate if f_0 is Newton non-degenerate.

The following example shows that Newton non-degeneracy is a condition which strongly depends of the coefficients of f.

Example 1.83. Let us consider two curve singularities defined by $f_1 = (y^2 - x^3)^2 - x^5y$ and $f_2 = y^4 - 7x^3y^2 + x^6 - x^5y$. The Newton polyhedron of both functions coincide with the one represented in Figure 1.12. Observe that $N(f_1) = N(f_2)$ have a unique compact face determined by the line $2\alpha + 3\beta = 12$.

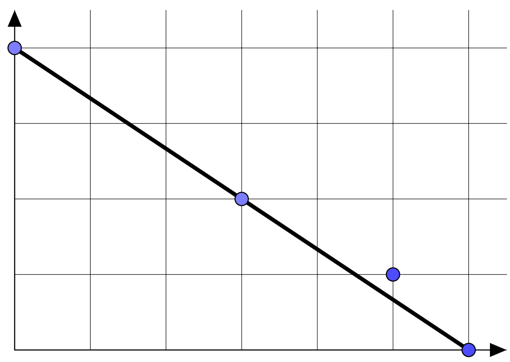


Figure 1.12: Newton polyhedron of f_1 and f_2

However, f_1 defines a Newton degenerate plane branch since

$$\Big\{(x,y)\in\mathbb{C}^2:\ 6x^3(x^3-y^2)=0,\ -4y^2(x^3-y^2)=0\Big\}=\Big\{(x,y)=(x,\pm x^{3/2})\ x\in\mathbb{C}\Big\},$$

while f_2 defines a Newton non-degenerate plane curve singularity since

$$\{(x,y)\in\mathbb{C}^2:\ 3x^3(2x^3-7y^2)=0,\ 2y^2(2y^2-7x^3)=0\}=\{(0,0)\}.$$

From example 1.83 one can thought that in order to be a degenerate function, the coefficients must satisfy certain restriction. This is a general fact proven by Kouchnirenko.

Theorem 1.84. *[92, Thm. 6.1] Let $S\subset\mathbb{N}^{n+1}$ be a finite set, the function $f=\sum_{m\in S}a_m\mathbf{x}^m$ has Newton non-degenerate principal part for a generic choice of coefficients $a_m\in\mathbb{C}^*$.*

Remark 1.85. Kouchnirenko's formulation of this theorem is a bit more sophisticated. However, in the case of a finite $S\subset\mathbb{N}^{n+1}$, $\Gamma_+(f)$ is a polytope. Therefore, $\Gamma_+(f)$ fulfils the hypothesis of [92, Thm. 6.1] and the rest of the formulation is a mere transcription in this setting.

One of the main properties of Newton non-degenerate hypersurface singularities is that we can compute its Milnor number in terms of certain volumes that can be computed from $\Gamma_-(f)$. This important result is due to Kouchnirenko [92].

Theorem 1.86. *[92, Thm. I](Kouchnirenko) The Milnor number of any Newton non-degenerate f in $n+1$ variables and commode can be written as*

$$\mu=(n+1)!V_{n+1}-n!V_n+\cdots+1!(-1)^nV_1+(-1)^{n+1},$$

where V_{n+1} is the $n+1$-dimensional Euclidean volume of $\Gamma_-(f)$ and for $1\leq j\leq n$, V_j is the j–dimensional Euclidean volume of $\Gamma_-(f)\cap H_j$ with H_j being the union of all j–dimensional coordinate planes.

Example 1.87. Let us consider the germ of hypersurface singularity in $\mathbb{C}^3$ defined by the equation $f(x,y,z)=x^4+y^5+z^6+xyz$. Let $A=(4,0,0)$, $B=(0,5,0)$, $C=(0,0,6)$, $D=(1,1,1)$ be the points corresponding to the support of f. It is easy to check that $N(f)$ consists of three triangles defined as $\sigma_1=ABD$, $\sigma_2=BCD$ and $\sigma_3=CDA$. Observe that the $(n+1)$-volume of the cone σ_1 is just $\det((4,0,0),(5,0,0),(1,1,1))/(3!)$. The same happens to σ_2 and σ_3 so $3!V_3=4\cdot5+5\cdot6+6\cdot4$. In a similar way we can compute V_2 as a sum of the volumes of the faces of each σ_i and the same for V_1. Then, $\mu=(4\cdot5+5\cdot6+6\cdot4)-(4\cdot5+5\cdot6+6\cdot4)+4+5+6-1=14$.

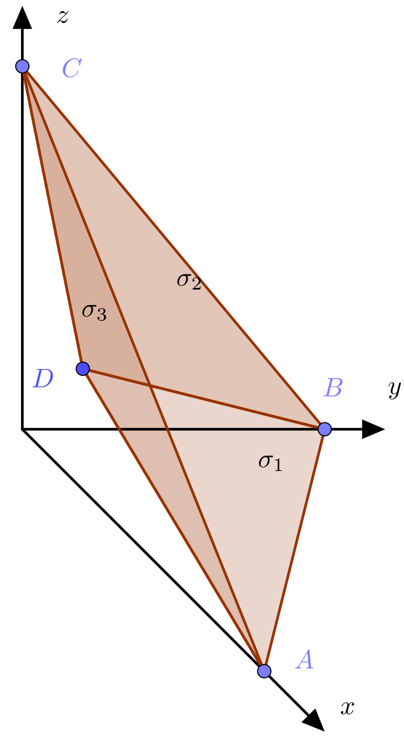


Figure 1.13: Newton polyhedron of $f(x, y, z)$.

Chapter 2

The minimal Tjurina number for plane curve singularities

Let $(C, \mathbf{0})$ be a germ of an isolated plane curve singularity with equation $f \in \mathbb{C}\{x,y\}$. The present chapter is going to be devoted to them. After Mather-Yau Theorem 1.52, the Tjurina number defined as

$$\tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f, \partial f/\partial x, \partial f/\partial y)}$$

constitutes one of the main analytic invariants of an isolated plane curve singularity. Many questions in Singularity Theory turn around knowing to what extent the topology of a germ of a singularity constrains its analytical properties. As we said in Section 1.3 the Tjurina number is an upper semicontinuous invariant (Definition 1.42). Thus, it is reasonable to consider a family of plane curve singularities with fixed Milnor number and to try to compute the minimal value of the Tjurina number in this family, which we denote by $\tau_{\min}$. The computation of $\tau_{\min}$ in terms of other invariants of the singularity is precisely the main topic of this chapter.

Looking at the main references about the Tjurina number for plane curves (for example Zariski [197], Teissier [165], Delorme [42], Briançon, Granger and Maisonobe [24], Laudal and Pfister [97, 96] and Peraire [131]) and comparing them with the main references about the Tjurina number of hypersurface singularities (for example Greuel [68, 67], Looijenga and Steenbrink [109] and Wahl [179, 178]), it is quite easy to realize that the Tjurina number of plane curve singularities constitutes itself a research topic with its own techniques. More concretely, the fact that the plane curve case allows a relatively good control of the defining equation of the curve has lead to consider the effective computation of the minimal Tjurina number for a fixed class of plane curve singularities. The general procedure is to fix the Milnor number; however, in the irreducible case we will usually fix the equisingularity class, which implies the constancy of Milnor number.

There have been several attempts to give a formula for τ_{min}. Most of them give a recursive expression of τ_{min} in terms of the semigroup of the branch, or any other equisingularity invariant. Moreover, they are mainly based on a detailed study of the set of values of Kähler differentials (see also Proposition 1.55), that we will denote by $\Delta(C) := v(\mathcal{O}_{C,\mathbf{0}} + \mathcal{O}_{C,\mathbf{0}}(dy/dx))$, or equivalently of the standard basis of the Tjurina algebra. In addition, it was proven by Delorme in [42] that $\Delta(C)$ is an analytical invariant of the singularity.

For the case of one Puiseux pair (n, m) with $\gcd(n, m) = 1$, Zariski proves in [197, §VI, pg. 107–114] that the dimension of the generic component of the moduli space (See Section 2.2.1 for a precise definition) is

$$q_{min} = \frac{(n-3)(m-3)}{2} + \left\lfloor \frac{m}{n} \right\rfloor - 1 - \mu + \tau_{min}.$$

On the other hand, Delorme in [42] gives a recursive formula for the dimension of the generic component in terms of the continued fraction of m/n (see Theorem 2.19). Delorme and Zariski's results together give a recursive formula for τ_{min} in the case of a branch with one Puiseux pair. Moreover, with the use of completely different techniques, Laudal and Pfister [97, 96] provide a similar expression for the minimal Tjurina number of an irreducible plane curve singularity with one Puiseux pair.

In 1988, Briançon, Granger and Maisonobe in [24] (see also Section 2.1) give a recursive formula for the minimal Tjurina number for the specific family of curves $f = f_0 + g$ which are a deformation of the initial term $f_0 = y^n - x^m$ such that $\deg_w(f_0) < \deg_w(g)$. Such curves are known as *semi-quasi-homogeneous singularities* with weights $w = (n, m), n, m \geq 2$. In particular, as far as the author knowledge, this is, up to the moment of writing this memoir, the only family of non-irreducible plane curve singularities for which the minimal Tjurina number can be explicitly computed from the equisingularity invariants.

Finally, in 1997 Peraire in [131] provides an algorithm which allows to compute the minimal Tjurina number of an irreducible plane curve with any number of Puiseux pairs from the equisingularity invariants (see subsection 2.3.2). Peraire's procedure to compute τ_{min} is closely related to the invariant $\Delta(C)$. More concretely, Peraire's algorithm [131, Section 3] computes, from the semigroup of values Γ, a set of generators for the value set of Kähler differentials Δ_{gen} of a plane curve singularity whose coefficients of a Puiseux parameterization belong to a certain Zariski open set. Thus, such a curve has minimal Tjurina number which can be computed by means of Proposition 1.55. Observe that Peraire's main contribution is that her algorithm provides an explicit way to compute a value set of Kähler differentials with maximal $|\mathbb{N} \setminus \Delta(C)|$, or equivalently with maximal $|\Delta(C) \setminus \Gamma|$. Moreover, since her algorithm only depends on the semigroup of values of the branch (see subsection 2.3.2 and also [131, Section 3]), her result recovers the fact that the minimal Tjurina number of an irreducible plane curve singularity is a topological invariant in the sense that two curves that are topologically equivalent has the same minimal Tjurina number.

The computation of a minimal Tjurina number in a family with fixed Milnor number lead us to study the following question posed by Dimca and Greuel in 2017 [46, Question 4.2],

Question. *3.13 Is it true that $\mu/\tau < 4/3$ for any isolated plane curve singularity?*

As Milnor number of a plane curve can be computed from a resolution of singularities (see Section 1.3.1), the main difficulty of the question relies on the computation of the Tjurina number. Moreover, since the Tjurina number is an analytic invariant, it is impossible to provide a formula from a resolution of singularities. However, we can focus ourselves on the study of the minimal Tjurina number in a fixed equisingularity class, which by the very definition is a topological invariant.

In our first approach to this question, we looked at the results of Delorme [42] and Briançon, Granger and Maisonobe in [24] in order to check the validity of the question. We provided a first positive answer for semiquasihomogeneous singularities in a joint work with Blanco [6]. Those results are exposed on Section 2.1 and Section 2.3.1.

After that, we used Peraire's algorithm [131, Section 3] to run several examples with a computer which lead us to think that Dimca and Greuel Question 3.13 were going to be true, at least for irreducible plane curve singularities. However, we were unable to provide a positive answer to Dimca and Greuel question from Peraire's result. This fact, lead us to use Genzmer's formula for the dimension of the generic component of the moduli space [62] (see also Theorem 2.23). Thanks to Genzmer formula and a previous result by Wall [182], in a joint work with Alberich-Carramiñana, Blanco and Melle-Hernández [3], we were able to provide a positive answer to Dimca and Greuel Question 3.13 in the case of irreducible plane curves through a closed formula for the minimal Tjurina number of an equisingularity class in terms of the sequence of multiplicities of the strict transform along a resolution. Sections 2.2 and Section 2.3 are devoted to explain those results.

2.1 The semiquasihomogeneous case

In 1988, Briançon, Granger and Maisonobe [24] provide a formula for the minimal Tjurina number of a semi-quasi-homogeneous singularity. We recall that f is a semi-quasi-homogeneous singularity with weights $w = (n, m)$ with $n, m \geq 2$ if $f = f_0 + g$ is a deformation of the initial term $f_0 = y^n - x^m$ such that $\deg_w(f_0) < \deg_w(g)$. In particular, if $\gcd(n, m) = 1$ then it is a branch with one Puiseux pair and if $\gcd(n, m) \geq 2$ then it is a plane curve singularity with $r = \gcd(n, m)$ irreducible components each of one with one Puiseux pair.

To do so, they gave recursive formulas to compute the τ_{min} of this type of singularities. Their main result is the following:

Theorem 2.1 ([24, §I.6]). *For semi-quasi-homogeneous singularities with initial term $y^n - x^m$,*

$$\tau_{min} = (m-1)(n-1) - \sigma(m, n).$$

The number $\sigma(a, b)$ is defined recursively for any non-negative integers a, b as follows. If $a, b \leq 2$ then $\sigma(a, b) := 0$. Otherwise, we can express $a = bq + r, 0 \leq r < b, q \geq 1$. For the cases $r = 0, 1, b-1, b/2$ there are closed formulas for $\sigma(a, b)$ denoted by $\Sigma_0, \Sigma_1, \Sigma_{b-1}, \Sigma_{b/2}$ in [24, Table 1]. Let us recall them:

(Σ_0)

a. If $a = bq$ with $q > 1$ and b odd then

$$\sigma(a, b) = \frac{(a-2)(b-2)}{4} - \frac{b}{2} + \frac{q}{4}.$$

b. If $a = bq$ with $q = 1$ and b odd then

$$\sigma(a, b) = \frac{(a-2)(b-2)}{4} - \frac{b}{2} + \frac{5}{4}.$$

c. If $a = bq$ with $q \geq 1$ and b even then

$$\sigma(a, b) = \frac{(a-2)(b-2)}{4} - \frac{b}{2} + 1.$$

(Σ_1)

a. If $a = bq + 1$ with $q \geq 1$ and b odd then

$$\sigma(a, b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} + \frac{q}{4}.$$

b. If $a = bq + 1$ with $q \geq 1$ and b even then

$$\sigma(a, b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} + \frac{1}{2}.$$

(Σ_{b-1})

a. If $a = bq + b - 1$ with $q \geq 1$ and b odd then

$$\sigma(a,b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} + \frac{q}{4} - \frac{1}{4}.$$

b. If $a = bq + b - 1$ with $q \geq 1$ and b even then

$$\sigma(a,b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} + \frac{1}{2}.$$

$(\Sigma_{b/2})$

a. If $a = bq + \frac{b}{2}$ with $q \geq 1$ and $b/2 \geq 3$ odd then

$$\sigma(a,b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} + \frac{5}{4}.$$

b. If $a = bq + \frac{b}{2}$ with $q \geq 1$ and $b/2 \geq 2$ even then

$$\sigma(a,b) = \frac{(a-2)(b-2)}{4} - \frac{b}{4} - 1.$$

If none of the above cases hold, define recursively, see [24, Tables 2 and 3], a finite sequence $(a_0,b_0),(a_1,b_1),\ldots,(a_k,b_k)$ with $(a_0,b_0) = (m,n)$ and $\sigma(a_k,b_k)$ is in one of the previous cases or $\sigma(a_{k-1},b_{k-1})$ is of the form (BP) that will be defined next and we set $(a_k,b_k) = (2,2)$ or $\sigma(a_k,b_k)$ is of the form (BP). For $i = 0,\ldots,k-1$:

(A) If $\gcd(a_i,b_i) = 1$, we can find $ub_i - va_i = 1$ with $2 \leq u < a_i$. Letting $\gamma := \lfloor \frac{a_i}{u} \rfloor$, There are two subcases:

(AE) If γ is even, define $a_{i+1} = a_i - \gamma u, b_{i+1} = b_i - \gamma v$, then

$$\sigma(a_i,b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma}{4} + \sigma(a_{i+1},b_{i+1}).$$

(AO) If γ is odd, define $a_{i+1} = (\gamma+1)u - a_i, b_{i+1} = (\gamma+1)v - b_i$, and

$$\sigma(a_i,b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma+1}{4} + \sigma(a_{i+1},b_{i+1}).$$

(B) Otherwise, $a_i = \alpha a', b_i = \alpha b'$ with $\alpha \geq 2, \gcd(a',b') = 1$, and we can find a Bezout's identity $ub' - va' = 1$ with $1 \leq u < a'$. There are again two subcases:

(BP) If α is even,

$$\sigma(a_i,b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2}.$$

(BO) If α is odd, define $a_{i+1} = |a' - 2u|$ and $b_{i+1} = |b' - 2v|$, and

$$\sigma(a_i,b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \sigma(a_{i+1},b_{i+1}).$$

Remark 2.2. A very interesting alternative proof of Briançon, Granger and Maisonobe Theorem 2.1 was given by Hao in [74].

We are going to use Theorem 2.1 to give an answer to Dimca and Greuel Question 3.13 in the case of semi-quasihomogeneous singularities.

Proposition 2.3. *For semi-quasi-homogeneous singularities with initial term $y^n - x^m$,*

$$\mu/\tau < 4/3.$$

Proof. Observe that in the recursive cases (A) and (BO) we have for $i = 0, \ldots, k-1$

$$\sigma(a_i, b_i) \leq \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \sigma(a_{i+1}, b_{i+1}). \tag{2.1}$$

Let us check this assertion by considering the different cases:

- Assume (a_i, b_i) is in the case (AE), then

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma}{4} + \sigma(a_{i+1}, b_{i+1}).$$

  Thus,

$$\sigma(a_{i+1}, b_{i+1}) - \sigma(a_i, b_i) = -\frac{(a_i-2)(b_i-2)}{4} + \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \frac{\gamma}{4}.$$

- Assume (a_i, b_i) is in the case (AO), then

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma+1}{4} + \sigma(a_{i+1}, b_{i+1}).$$

  Thus,

$$\sigma(a_{i+1}, b_{i+1}) - \sigma(a_i, b_i) = -\frac{(a_i-2)(b_i-2)}{4} + \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \frac{\gamma+1}{4}.$$

- Assume (a_i, b_i) is in the case (BP), then $i = k-1$ and $(a_k, b_k) = (2, 2)$. Thus,

$$\sigma(a_{k-1}, b_{k-1}) = \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2} \leq \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_k-2)(b_k-2)}{4} + \sigma(a_k, b_k),$$

  since $(a_k, b_k) = (2, 2)$.

- Assume (a_i, b_i) is in the case (BO), then

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \sigma(a_{i+1}, b_{i+1}).$$

  Thus,

$$\sigma(a_{i+1}, b_{i+1}) - \sigma(a_i, b_i) = -\frac{(a_i-2)(b_i-2)}{4} + \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \frac{\alpha}{2}.$$

Therefore, Equation (2.1) follows from the previous computations together with the fact that $\gamma, \alpha \geq 0$.

From Equation (2.1) together with the closed formulas appearing in $\Sigma_0, \Sigma_1, \Sigma_{b-1}, \Sigma_{b/2}$, one can deduce that, in general,

$$(n-1)(m-1) - \frac{(m-2)(n-2)}{4} - \kappa(n,m) \leq \tau_{min},$$

where $\kappa(n,m) = a_k/4b_k$ if $\sigma(a_k,b_k)$ is $\Sigma_0, \Sigma_1, \Sigma_{b-1}$ with b odd, $\kappa(n,m) = 5/4$ if $\sigma(a_k,b_k)$ is $\Sigma_0, \Sigma_1, \Sigma_{b-1}$ with b even or $\Sigma_{b/2}$ with $b/2$ odd, and $\kappa(n,m) = 0$ if $\sigma(a_k,b_k)$ is $\Sigma_{b/2}$ with $b/2$ even or in the case (BP). In any case,

$$\frac{\mu}{\tau} \leq \frac{\mu}{\tau_{min}} \leq \frac{4(n-1)(m-1)}{3nm - 2n - 2m - 4\kappa(n,m)},$$

which is bounded by $4/3$ if and only if $n + m - 8\kappa(n,m) > 3$. Observe that if $\kappa(n,m) = 0$ then there is nothing to check. If, $\kappa(n,m) = 5/4$ then one need to check those possible combinations of $n + m < 13$ but those cases can be easily computed by hand to check the inequality. Finally for $\kappa(n,m) = a_k/4b_k$ observe that $a_k/4b_k \leq a_1/4$ and that

$$n + m - 3 - 2a_1 \geq n + m - 3 - 2m + 2\frac{m-1}{u}u = n + m - 5 > 0,$$

so we are in one of the previous cases. □

2.2 The monomial curve and its deformations

Let us consider a numerical semigroup Γ minimally generated by $\{\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g\}$. Thus,

$$\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle = \{z \in \mathbb{N} |\ z = l_0\overline{\beta}_0 + l_1\overline{\beta}_1 + \cdots + l_g\overline{\beta}_g \text{ and } l_i \in \mathbb{N} \text{ for } i = 0, \ldots, g\}.$$

Assume that Γ satisfies the following conditions:

1. $n_i\overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$,
2. $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$ for all $i = 1, \ldots, g$,

where $n_i := \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1}) / \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_i)$.

A semigroup satisfying those conditions is called *semigroup of a plane branch* (see Section 3.2.1 for more details).

Let $t \in \mathbb{C}$ be a local coordinate of the germ $(\mathbb{C}, 0)$ and let $(u_0, \ldots, u_g) \in \mathbb{C}^{g+1}$ be local coordinates of the germ $(\mathbb{C}^{g+1}, \mathbf{0})$. Following Teissier [165, Chap I. Sec. 1], let $(C^\Gamma, \mathbf{0}) \subset (\mathbb{C}^{g+1}, \mathbf{0})$ be the curve defined via the parameterization

$$C^\Gamma : u_i = t^{\overline{\beta}_i}, \qquad 0 \leq i \leq g.$$

From the defining conditions of Γ, there exist for all i, numbers $l_0^{(i)}, \ldots, l_{i-1}^{(i)} \in \mathbb{N}$ such that

$$n_i\overline{\beta}_i = l_0^{(i)}\overline{\beta}_0 + \cdots + l_{i-1}^{(i)}\overline{\beta}_{i-1}.$$

Now, Teissier [165, 1.22, 2.2.3 and 2.5] shows that the germ $(C^{\Gamma}, \mathbf{0})$ is irreducible, quasi-homogeneous and it satisfies the following equations:

$$f_i = u_i^{n_i} - u_0^{l_0^{(i)}} u_1^{l_1^{(i)}} \cdots u_{i-1}^{l_{i-1}^{(i)}} = 0 \quad \text{for } 1 \leq i \leq g.$$

Morever, the following theorem attributed to Lejeune-Jalabert shows that these are the defining equations of the monomial curve:

Theorem 2.4. *[165, Chap. I Prop 2.2] If Γ is the semigroup of a plane branch, then the affine curve $C^{\Gamma} \subset \mathbb{C}^{g+1}$ is a complete intersection, and therefore, so too is the branch $(C^{\Gamma}, \mathbf{0})$. Moreover, the defining equations of $(C^{\Gamma}, \mathbf{0})$ are the $f_1, \ldots, f_g$ previously defined.*

An immediate consequence of Theorem 2.4 is that by Tjurina's theorem B.16 there is a miniversal deformation of the monomial curve. Even more, Tjurina's theorem B.16 shows how to compute explicitly the miniversal deformation of $(C^{\Gamma}, \mathbf{0})$ (See Appendix B). For commodity to the reader let us describe the deformation of $(C^{\Gamma}, \mathbf{0})$.

We have the following exact sequence

$$0 \to \mathbb{C}[u_0, \ldots, u_g]^g \to \mathbb{C}[u_0, \ldots, u_g] \xrightarrow{\varphi} \mathbb{C}[C^{\Gamma}] := \mathbb{C}[t^{\nu} \mid \nu \in \Gamma] \to 0, \tag{2.2}$$

where $\operatorname{Ker} \varphi = (f_1, \ldots, f_g)$. Let us denote by $\mathbf{f} := (f_1, \ldots, f_g)$ and let us consider the Jacobian map

$$D(\mathbf{f}) := \left(\frac{\partial f_j}{\partial x_i} \right)_{\substack{1 \leq i \leq g \\ 0 \leq j \leq g}} : \mathbb{C}[u_0, \ldots, u_g]^{g+1} \to \mathbb{C}[u_0, \ldots, u_g]^g.$$

Observe that the image of $D(\mathbf{f})$ is the submodule of $\mathbb{C}[u_0, \ldots, u_g]$ generated by

$$\overline{y}_0 := \left(\frac{\partial f_1}{\partial u_0}, \ldots, \frac{\partial f_g}{\partial u_0} \right), \ldots, \overline{y}_g := \left(\frac{\partial f_1}{\partial u_g}, \ldots, \frac{\partial f_g}{\partial u_g} \right).$$

Remark 2.5. With the notation of Appendix B, $\mathbb{C}[u_0, \ldots, u_g] \simeq \mathcal{O}_{(\mathbb{C}^{g+1}, 0)}$. Thus in Equation (B.2), the image of $D(\mathbf{f})$, which we denote by $D(\mathbf{f})\mathbb{C}[u_0, \ldots, u_g]^{g+1}$, is precisely the module $(\frac{\partial f_j}{\partial x_i})\mathcal{O}_U^N$ with $N = g + 1$ and $U = (\mathbb{C}^{g+1}, 0)$.

Let us denote by $\mathcal{N} := \pi(D(\mathbf{f}))$ the submodule of $(\mathbb{C}[u_0, \ldots, u_g]/(f_1, \ldots, f_g))^g \simeq \mathbb{C}[C^{\Gamma}]^g$ where $\pi : \mathbb{C}[u_0, \ldots, u_g]^g \to \mathbb{C}[C^{\Gamma}]^g$ is the canonical surjection.

Remark 2.6. Thus in Equation (B.2) the module $\mathcal{N}$ is precisely the module $(\frac{\partial f_j}{\partial x_i})\mathcal{O}_U^N + (f_1, \ldots, f_k)\mathcal{O}_U^k$ with $k = g$ and $U = (\mathbb{C}^{g+1}, 0)$.

Then, Theorem B.16 reads as follows. Take $s_1 \ldots, s_t \in \mathbb{C}[u_0, \ldots, u_g]^g$, with $s_i = (s_i^1, \ldots, s_i^g)$ such that their image on $\mathbb{C}[C^{\Gamma}]^g / \mathcal{N}$ form a $\mathbb{C}$–basis. Denote by $\mathbf{u} := (u_0, \ldots, u_g)$. Define $\mathbf{F} = (F_1, \ldots, F_k)$ such that

$$\begin{aligned} F_1(\mathbf{u}, \mathbf{w}) &= f_1(\mathbf{u}) + \textstyle\sum_{j=1}^t w_j s_j^1(\mathbf{u}), \\ &\vdots \qquad\qquad \vdots \\ F_k(\mathbf{u}, \mathbf{w}) &= f_k(\mathbf{u}) + \textstyle\sum_{j=1}^t w_j s_j^k(\mathbf{u}). \end{aligned}$$

If we define $(\mathcal{X},\mathbf{0}) := V(F_1,\ldots,F_k) \subset (\mathbb{C}^n \times \mathbb{C}^t,\mathbf{0})$ the zero set of $\mathbf{F}$, then the deformation defined by $(C^\Gamma,\mathbf{0}) \xrightarrow{i} (\mathcal{X},\mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^t,\mathbf{0})$ is the miniversal deformation of $(C^\Gamma,\mathbf{0})$, with i being induced by the inclusion and ϕ by the natural projection. Let us denote by G the miniversal deformation of C^Γ. Recall that according to Definition B.1, a deformation is a germ of flat morphism such that the special fiber is precisely our original object; in particular, it can be decomposed as a pair of maps (i,ϕ) where i is an inclusion and ϕ is a projection. In this way, $G:(\mathcal{X},\mathbf{0}) \to (\mathbb{C}^t,\mathbf{0})$ is a germ of flat morphism such that $(G^{-1}(\mathbf{0}),\mathbf{0})$ is isomorphic to $(C^\Gamma,\mathbf{0})$.

Therefore, $t=\tau(C^\Gamma)$ is the Tjurina number of the monomial curve (See Introduction of Chapter 3 or Appendix B for a definition of the Tjurina number of an Isolated Complete Intersection Singularity).

Proposition 2.7. *[165, I. Prop. 2.7] In the previous setting, if we denote by $\delta(\Gamma)=|\mathbb{N}\setminus\Gamma|$ then the dimension of the miniversal deformation of $(C^\Gamma,\mathbf{0})$ satisfies $\tau(C^\Gamma)=2\delta(\Gamma)$.*

Remark 2.8. As pointed out in [165, I. Prop. 2.7], the previous Proposition can be proven in a more general context. This general case was proven by Pinkham in [136, Chap. IV 14.9] for the case $(C^\Gamma,\mathbf{0})$ being Gorenstein.

Remark 2.9. The following easy example shows that there are fibers of the miniversal deformation whose semigroup is different from the defining semigroup of the monomial curve: Take $\Gamma=\langle 5,7\rangle$ and the fiber defined by $C_t: u_0^7+u_1^5+tu_0=0$ with $t\neq 0$. Then, since C is smooth, its semigroup is $\Gamma(C)=\mathbb{N}$.

At this point, one would like to be able to distinguish which of the fibers of the miniversal deformation generates a deformation with semigroup Γ. To do so, Teissier shows that we can choose $s_j=(s_{i,j})$ to be homogeneous since the vectors $\pi(\overline{y}_i)$ have negative degree $\deg(\pi(\overline{y}_i))=-\overline{\beta}_i$ because $\deg \partial f_i/\partial u_k = n_i\overline{\beta}_i-\overline{\beta}_k$ by Euler's identity. In particular, this means that if one chooses $\deg(w_j)=-\deg(s_j)$ then we endow the algebra $\mathbb{C}[u_0,\ldots,u_g,w_1,\ldots,w_\tau]$ with the unique grading for which $\deg(u_i)=\overline{\beta}_i$ and F_i are homogeneous with $\deg(F_i)=n_i\overline{\beta}_i$. This is just a consequence of the fact that Equation 2.2 provides a graded free resolution of the algebra associated to the monomial curve.

Define the following sets

$$P_- := \{j\in\{1,\ldots,\tau\}|\ \deg(w_j)<0\} \quad P_+ := \{j\in\{1,\ldots,\tau\}|\ \deg(w_j)>0\}.$$

Denote by $\tau_-(\Gamma):=\tau_-:=|P_-|$ and $\tau_+=|P_+|$. Since, there are no w_j of degree zero then $\tau=\tau_++\tau_-$. Therefore,

Theorem 2.10. *[165, I 2.10] The deformation of C^Γ, obtained by applying the base change $(D_\Gamma,\mathbf{0}) := (\mathbb{C}^{\tau_-}\times\{\mathbf{0}\},\mathbf{0}) \hookrightarrow (\mathbb{C}^{\tau(C^\Gamma)},\mathbf{0})$ to the miniversal deformation of C^Γ, is a miniversal constant semigroup deformation of C^Γ. If we denote that deformation by G_Γ and we denote by $(\mathcal{X}_\Gamma,\mathbf{0}) := (\mathcal{X},\mathbf{0})\times_{(\mathbb{C}^{\tau(C^\Gamma)},\mathbf{0})}(D_\Gamma,\mathbf{0})$, we have the following commutative diagram*

$$\begin{array}{ccc} (\mathcal{X},\mathbf{0}) & \xrightarrow{\quad G \quad} & (\mathbb{C}^{\tau(C^\Gamma)},\mathbf{0}) \\ \uparrow & & \uparrow \\ (\mathcal{X}_\Gamma,\mathbf{0}) & \xrightarrow{\quad G_\Gamma \quad} & (D_\Gamma,\mathbf{0}). \end{array}$$

Moreover, there exists a section σ_Γ of G_Γ that picks out the unique singular point of each fiber (which is a singular point with semigroup Γ).

We will call to $(\mathcal{X}_\Gamma, \mathbf{0}) \xrightarrow{G_\Gamma} (D_\Gamma, \mathbf{0})$ the *miniversal constant semigroup deformation* of $(C^\Gamma, \mathbf{0})$. Moreover, the base space $(D_\Gamma, \mathbf{0})$ has dimension τ_-.

Even more, to analyze the different analytic types of curves with fixed semigroup Γ it is enough to study constant semigroup deformations of C^Γ. This is possible thanks to the following theorem.

Theorem 2.11 ([165, I.1]). *Every branch $(C, \mathbf{0})$ with semigroup Γ is isomorphic to the generic fiber of a one parameter complex analytic deformation of $(C^\Gamma, \mathbf{0})$.*

Remark 2.12. As remarked by Teissier [165], the above statement is a short-hand way of stating the following: for every branch $(C, \mathbf{0})$ with semigroup Γ there exists a deformation $\rho : (X, \mathbf{0}) \longrightarrow (D, \mathbf{0})$ of $\mathbb{C}^\Gamma$, with a section σ, such that for any sufficiently small representative $\widetilde{\rho}$ of the germ of ρ, $(\widetilde{\rho}^{-1}(v), \sigma(v))$ is analytically isomorphic to $(C, \mathbf{0})$ for all $v \neq 0$ in the image of $\widetilde{\rho}$.

2.2.1 The generic component of the moduli space

In 1965, S. Ebey [48] proved that the moduli space of curves having a given semigroup is in bijection with a constructible algebraic subset of some affine space. To do so, Ebey need to fix the semigroup of values Γ of an irreducible plane curve singularity and then look at the class of all complete local subrings of $\mathbb{C}[[t]]$ with the same Γ. From the discussion of Section 1.1.2, we call to such a class the equisingularity class of a plane branch. Once this class is fixed, Ebey considered two groups of $\mathbb{C}$–automorphisms of $\mathbb{C}[[t]]$:

$$\mathcal{G}_1 := \{\sigma \in \mathrm{Aut}_\mathbb{C} \mid \sigma(t) := a_0 t + a_1 t^2 + \dots,\ a_0 \neq 0\},$$
$$\mathcal{G}_c := \{\sigma \in \mathrm{Aut}_\mathbb{C} \mid \sigma(t) := t + a_c t^{c+1} + \dots\},$$

where c is the conductor of the semigroup Γ. Now, the group $\mathcal{G}_1/\mathcal{G}_c$ is a connected solvable algebraic group which acts regularly on the parameter space of branches with fixed semigroup Γ (see [48, Thm. 4]). After that, Ebey can deduce the existence of such a constructible set in bijection with the moduli space from the theory of algebraic groups (see [48, Thms. 5 and 6]).

Natural questions arise from Ebey's paper: How does the constructible set which represent the moduli variety looks like ? How is its topology and structure? What's about the generic orbit of the group action? Those questions were exposed by O. Zariski on a series of lectures about this topic in the fall of 1973 at the École Polytechnique in Paris. The notes of those lectures were collected in the book "Le probleme des modules pour les branches planes"with an appendix by B. Teissier [197]. Motivated by those lectures, several mathematicians have tried to tackle the problem initiating an outbreak of a deep study of the moduli space of irreducible plane curve singularities.

As pointed out by Zariski [197, Chap. VI 1], Ebey's model of the moduli space allow to speak of the variety which represent generic orbits of the group action. This variety is what Zariski called *generic component of the moduli space*, which we are going to denote by M_{gen}, and he asked about its dimension.

From Zariski's definition the generic component may look as a very weird object to manage. However, Teissier monomial curve allows to see the dimension of the generic component as the minimal dimension of the base space of a miniversal constant semigroup deformation. Let us denote by $(C_{\boldsymbol{v}}, \mathbf{0}), \boldsymbol{v} \in S$ any fiber of the miniversal deformation of $(C^\Gamma, \mathbf{0})$. We will denote by $\tau(C_{\boldsymbol{v}})$ the dimension of the base space of the miniversal deformation of $(C_{\boldsymbol{v}}, \mathbf{0})$ and by $q(C_{\boldsymbol{v}})$ the dimension of the base space of the miniversal constant semigroup deformation of the fiber $(C_{\boldsymbol{v}}, \mathbf{0})$.

First of all, Teissier begins by defining the moduli space of plane branches with semigroup Γ from the point of view of deformations of the monomial curve. Following Teissier [165, Chap. II, Sec. 2], analytic equivalence of germs induces an equivalence relation $\sim$ on D_Γ as follows: $w \sim w'$ if and only if the germs $(G_\Gamma^{-1}(w), \sigma_\Gamma(w))$ and $(G_\Gamma^{-1}(w'), \sigma_\Gamma(w'))$ are analytically isomorphic. Thus, Teissier call $\widetilde{M}_\Gamma := D_\Gamma / \sim$ the *moduli space* associated to the semigroup Γ. Moreover, $\widetilde{M}_\Gamma$ is quasi-compact and connected [165, Chap. II, Theorem 5 (2.3)].

Let $m : D_\Gamma \longrightarrow \widetilde{M}_\Gamma$ be the natural projection and let $D_\Gamma^{(2)}$ be the following subset of D_Γ

$$D_\Gamma^{(2)} := \{\boldsymbol{v} \in D_\Gamma \mid (G^{-1}(\boldsymbol{v}), \mathbf{0}) \text{ is a plane branch}\}.$$

Then, Teissier proves in [165, Chap. II, 2.3 (2)] that $D_\Gamma^{(2)}$ is an analytic open dense subset of D_Γ and that $m(D_\Gamma^{(2)})$ is the moduli space M_Γ of plane branches with semigroup Γ in the sense of Ebey [48] and Zariski [197]. Moreover, $\widetilde{M}_\Gamma = M_\Gamma$ if and only if Γ is generated by two elements.

Following [165, Chap. II 3.2], consider $V_0 \subset D_\Gamma$ the set of points such that if $v \in V_0$ then $\tau(C_v)$ takes its minimal value. The set V_0 is an open analytic set by [165, Addendum, 2.5]. Denote $V := V_0 \cap D_\Gamma^{(2)}$. After that, Teissier calls a branch C_v *generic or general*, i.e. a fiber is generic or general, if $v \in V$. In the sequel, we will refer to this notion of genericity as Teissier genericity if there is a possibility of confusion. Here, Teissier pointed out the following question:

Question 2.13. [165, Chap II 3.2] Is $V = V_0$?

According to [165, Addendum 2.1], the germ of D_Γ at any $\boldsymbol{v}$ has a decomposition as

$$(D_\Gamma, \boldsymbol{v}) \cong (\mathbb{C}^{\mu - \tau(C_{\boldsymbol{v}})} \times D_{\Gamma,\boldsymbol{v}}, \mathbf{0}),$$

where $D_{\Gamma,\boldsymbol{v}}$ is the base space of the miniversal constant semigroup deformation of $(C_{\boldsymbol{v}}, \mathbf{0})$. Thus, one has the following relation, see [165, §II.3.4],

$$\tau(C_{\boldsymbol{v}}) - q(C_{\boldsymbol{v}}) = \mu - \tau_-. \tag{2.3}$$

In the particular case where $v \in V$ recall that by definition of V we have $\tau(C_v) = \tau_{\min}$. Then, in case $v \in V$, $\tau(C_v)$ is independent of v and Equation (2.3) implies $q := q_{\min} := q(C_v)$ is also independent of $v \in V$; since $\mu - \tau_-$ does not vary. After that, the next Theorem of Teissier allows to identify $q_{\min}$ with the dimension of the generic component of the moduli space.

Theorem 2.14. *[165, Chap. II Thm. 6] The dimension of the generic component M_1 of the moduli space of branches $\widetilde{M}_\Gamma$ with semigroup Γ is*

$$\dim M_1 = q.$$

Moreover, $M_1 \cap M_\Gamma \subset \widetilde{M}_\Gamma$ is Zariski open and dense.

Teissier's Theorem 2.14 has the following meaning: the dimension of the generic component of the moduli space equals the minimal possible dimension of a miniversal equisingular deformation of a plane branch. In particular, Teissier's Theorem 2.14 shows that the dimension of the generic component of the moduli space of an irreducible plane branch can be computed even if one doesn't know the structure of the moduli itself. In fact, this has been one of the key results that have allowed to Genzmer to provide a formula for q in terms of a resolution of singularities of the branch (See [62, Lemma 3 and Main Theorem] and Theorem 2.23).

Theorem 2.15. *[165, Chap. II Thm. 7] The set V of general points of M_Γ is contained in the generic component M_1 of M_Γ and contains a Zariski open set, Z, of M_Γ.*

It is important to notice that Theorem 2.15 shows the following inclusions $Z \subset V \subset M_1$. Moreover, we can be more precise and to set $Z = M_1 \cap M_\Gamma$ since by Equation (2.3) any germ with $v \in M_1 \cap M_\Gamma$ has minimal Tjurina number. This fact lead us to discuss for a moment the concept of "genericity "of a branch in this context.

In the case of irreducible plane curve singularities with one Puiseux pair, the computation of the minimal Tjurina number by Briançon, Granger and Maisonobe [24] is based on the computation of the Tjurina number of a curve where the coefficients of the defining equation belong to a certain Zariski open set Z'. Moreover, since in this case the monomial curve is a plane curve, then Z' is precisely the set Z provided by Theorem 2.15. However, the computation of the minimal Tjurina number of an irreducible plane curve singularity with any number of Puiseux pair provided by Peraire [131] is based on the computation of the Tjurina number of a curve C_{gen} where the coefficients of its Puiseux series belong to a certain Zariski open set Z''. Observe, that if C_{gen} has more than one Puiseux pair, the associated monomial curve is no longer a plane curve. It would be certainly an interesting problem to understand Peraire's conditions [131, (5.5)] in terms of the deformations of the associated monomial curve. Moreover, it would be interesting to see the conditions associated to the defining equation of the branch.

After this discussion, it is clear that talking about "a generic curve "could be a bit confusing. Observe that in all cases "a generic curve "has minimal Tjurina number because of Equation (2.3). However, one must be careful about the number $q(C_v)$ appearing in Equation (2.3). Observe that from Equation (2.3) one can also compute $\tau_{\min}$ and $\dim M_{gen} = q$ from the computation of $\tau(C_v)$ and $q(C_v)$ with $v \in V \setminus Z$. A particular example of this fact is the following:

Example 2.16. Let us consider the following two Puiseux parametrizations of a curve with semigroup $\Gamma = \langle 7, 8\rangle$:

$$C_1 : \begin{cases} x(t) := t^7 \\ y(t) := t^8 + t^{10} + a_{11}t^{11} + a_{12}t^{12} + a_{13}t^{13} + a_{20}t^{20}, \end{cases}$$

with $a_{12} \neq (13 + 9a_{11}^2)/8$.

$$C_2 : \begin{cases} x(t) := t^7 \\ y(t) := t^8 + t^{12} + a_{13}t^{13} + a_{18}t^{18}, \end{cases}$$

with no conditions for the coefficients.
By Example 7.2 of the ArXiv version of [76] (see ArXiv:0707.4502) C_1 and C_2 has both mininmal Tjurina number $\tau_{\min} = 36$ and hence, by Equation (2.3) has the same $q = 4$. Since they belong to different strata of the stratification provided by Hefez-Hernandes of the moduli space then C_1 is not analytically equivalent to C_2. Moreover, as it is pointed out in Example 7.2 of the ArXiv version of [76] (see ArXiv:0707.4502) the dimensions of both strata are different. By our previous discussion, $C_1, C_2 \in V$ and $C_1 \in Z$ and $C_2 \in V \setminus Z$, thus $q(C_1) = q(C_2)$.

In general, this example shows that $q(C_v)$, which is the dimension of the base space of a miniversal constant semigroup deformation of C_v, is not necessarily equal to the dimension of the corresponding strata of C_v in the moduli space of the analytic classification of plane branches.

2.2.2 Remark on the dimension of the μ–constant stratum

To finish the discussion about the dimension of the generic component of the moduli space, we would like to examine in detail the equality (2.3). As we have seen, the definition of genericity has lead us to a subtle discussion about the arguments which allow to compute the invariants $q_{\min}$, $\tau_{\min}$ and $\dim M_{gen}$ from any general point. There, Equation (2.3) plays a central role that must be understood. To do so, let us first forget for a moment the context of irreducible plane curve singularity.

Let us consider now $C : f(x,y) = 0$ to be a germ of isolated plane curve singularity (not necessarily irreducible). We can consider the miniversal deformation of the germ. Inside the base space of a miniversal deformation of a germ of plane curve singularity, C, there is a closed analytic subspace Δ^μ called the μ–constant stratum, where μ is the Milnor number of C. This stratum can be defined as follows: take the miniversal deformation $\varphi : (\mathcal{Y}, 0) \to (S, 0)$ of the plane curve singularity C. Denote by μ the Milnor number of C and by $\mathcal{Y}_s := \varphi^{-1}(s)$ a fiber of the deformation, then

$$\Delta^\mu := \{s \in S | \ \mu(\mathcal{Y}_s) = \mu\}.$$

In 1974, Wahl proved that this stratum is smooth [177, Section 4] (see also Theorem 2.61 in [71]). Moreover, its codimension can be computed from the embedded resolution of the plane curve by the following formula given by Wall in [182, Section 8].

Theorem 2.17. *[182, Thm. 8.1] Let $f(x,y) = 0$ be a germ of isolated plane curve singularity. Let s be the number of satellite points in a resolution of singularities of f and b the number of irreducible components of the germ. Then,*

$$\dim_{\mathbb{C}}(\Delta^\mu) = \sum_p \frac{(e_p - 1)(e_p - 2)}{2} - b - s + 2,$$

where the summation is running over all equal or infinitely near points to the origin and e_p denotes the multiplicity of the strict transform of the curve at the point p.

Remark 2.18. The previous result was also obtained by Mattei [115, Thm. 4.2.1] from completely different techniques using foliation theory. Also, a formula for this dimension in terms of the Newton pairs was given by Cassou-Noguès in [32, Thm. 8].

Now let us consider an irreducible plane curve singularity C. Since the codimension of the μ–constant stratum is equal in any versal deformation (see [71, Thm. 2.61, Prop. 2.63, Thm. 2.64]) we have

$$\tau_{\min} - q_{\min} = \mu - \dim_{\mathbb{C}}(\Delta^\mu) \tag{2.4}$$

Observe that the left part of Equation (2.4) can be interpreted in terms of deformations of the monomial curve C^Γ. As showed in the previous subsection, if $\tau_-(\Gamma)$ denotes the dimension of the base space of the miniversal constant semigroup deformation of C^Γ then Equation (2.3) provides the equality $\mu - \tau_-(\Gamma) = \tau_{\min} - q_{\min}$. Therefore, Equation (2.3) combined with Equation (2.4) and Theorem 2.17 gives

$$\tau_-(\Gamma) = \dim_{\mathbb{C}}(\Delta^\mu) = \sum_p \frac{(e_p - 1)(e_p - 2)}{2} - s + 1. \tag{2.5}$$

Recall that identity (2.3) is a comparison between dimensions of certain base spaces of deformations of the fibers of the miniversal deformation of the monomial curve. Moreover, there are fibers in that deformation which are not plane curves. On the other hand, identity (2.4) is comparing plane deformations of a plane curve singularity with semigroup Γ, i.e. identity (2.4) is comparing codimension of two base spaces of versal deformations of the plane curve singularity. Thus observe that the nature of Equation (2.3) and Equation (2.4) is different. However, as pointed out by Teissier [165, Addendum, Remark 2.2], the product decomposition which provides Equation (2.3) means that in a neighborhood of a point $v \in D_{\Gamma}$, the miniversal deformation G of the monomial curve C^{Γ} works as a versal deformation of the fiber C_v; thus providing some natural insight to Equation (2.5).

2.3 A closed formula in terms of the sequence of multiplicities

As we will see in subsection 2.3.2, in 1997 Peraire [131, Section 3] provides an algorithm to compute the minimal Tjurina number in an equisingularity class from the semigroup of values of the branch. Her algorithm allowed us to run several examples with a computer. This lead us to be convinced that Dimca and Greuel Question 3.13 was going to be true, at least for irreducible plane curve singularities. However, we were unable to provide a positive answer to Dimca and Greuel Question from Peraire's result.

In this section, we start with a proof of Dimca and Greuel Question in the case of one Puiseux pair through Delorme's results [42]. After that, we will recall Peraire's algorithm [131] to compute the minimal Tjurina number of an irreducible plane curve singularity with any number of Puiseux pairs. Finnaly, we will use the results of Section 2.2 together with a formula for the dimension of the generic component of the moduli space provided by Genzmer [62] to present a positive answer to Dimca and Greuel Question 3.13 for irreducible plane curve singularities. This answer will be given through a closed formula for the minimal Tjurina number of an equisingularity class in terms of the sequence of multiplicities of the strict transform along a resolution.

2.3.1 The case of one Puiseux pair revisited

Along this subsection we will assume that C is an irreducible plane curve singularity with a single Puiseux pair (a, b). We will denote by $\Gamma = \langle a, b\rangle, a < b$ with $\gcd(a, b) = 1$ the semigroup of C. Despite this case can be treated as a particular case of a semiquasihomogeneous singularity (Section 2.1), the following approach shows a first approximation to the general strategy we will use in the case of any number of Puiseux pairs.

For irreducible plane curve singularities with a single Puiseux pair, the associated monomial curve C^{Γ} is in fact a plane curve. Therefore, by choosing a suitable coordinate system, any plane curve singularity with semigroup $\Gamma =< a, b >$ can be written as (See [197, Chap. VI 2.])

$$C_v := y^a - x^b + \sum_{\substack{a\alpha + b\beta > ab \\ 1 \leq \alpha \leq b-2 \\ 1 \leq \beta \leq a-2}} v_{\alpha,\beta} x^{\alpha} y^{\beta}. \tag{2.6}$$

In 1978, following the ideas of Zariski [197], Delorme [42] computed the dimension of the generic component $q_{a,b}$ of the moduli space of plane branches with a single Puiseux pair (a, b). The formula is in terms of some inductive procedure using the continued fraction representation of b/a.

Theorem 2.19 ([42, Thm. 32]). *Consider the continued fraction representation of the Puiseux pair $b/a = [h_1, h_2, \dots, h_k]$, with $k \geq 2, h_1 > 0$ and $h_2 > 0$. Then, the dimension $q_{a,b}$ of the generic component of the moduli space is given by*

$$q_{a,b} = \frac{(a-4)(b-4)}{4} + \frac{r_0}{4} + \frac{(2-t_1)(h_1-2)}{2} - \frac{t_1 t_2}{2},$$

where the numbers r_0, t_1, t_2 are defined inductively by the following procedure

$$r_k := 0, \quad t_k := 1, \quad r_{i-1} := r_i + t_i h_i, \quad t_{i-1} := \begin{cases} 0, & \text{if } t_i = 1 \text{ and } r_{i-1} \text{ even}, \\ 1, & \text{otherwise.} \end{cases}$$

Observe that, except for the case $(a, b) = (2, 3)$, Delorme's formula for $q_{a,b}$ give the following lower bound:

$$\frac{(a-4)(b-4)}{4} \leq q_{a,b}. \tag{2.7}$$

On the other hand, for one characteristic exponent Zariski [197, section 2 Chapter VI]) already proved that $\tau_-(\Gamma)$ coincides with the number of points of the standard lattice of $\mathbb{R}^2$ that are in the interior of the triangle defined by the lines $\alpha = b-1, \ \ \beta = a-1, \ \ \alpha a + \beta b = ab$. Therefore, it is easy to see that

$$\tau_- = \frac{(a-3)(b-3)}{2} + \left\lfloor \frac{b}{a} \right\rfloor - 1,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. In this case, the Milnor number is $\mu = (a-1)(b-1)$. Combining the lower bound in Equation (2.7) and Equation (2.3) one obtains the following lower bound for τ_{min}

$$\frac{(b-4)(a-4)}{4} + (b-1)(a-1) - \frac{(b-3)(a-3)}{2} - \frac{b}{a} + 1 \leq \tau_{min}. \tag{2.8}$$

except for the case $(n, m) = (2, 3)$.

Thus, we can give an alternative prove to the one of Proposition 2.3 of Dimca and Greuel Question 3.13 in the following particular case.

Proposition 2.20. *For any plane branch C with one characteristic exponent,*

$$\frac{\mu(C)}{\tau(C)} < \frac{4}{3}.$$

Proof. It is sufficient to proof the inequality for the τ_{min} of each characteristic pair (a, b).

Dividing μ by the expression in Equation (2.8) and rewriting

$$\frac{\mu}{\tau} \leq \frac{\mu}{\tau_{min}} \leq \frac{4a(a-1)(b-1)}{3a^2 b - 2a^2 - 2ab + 6a - 4b}, \tag{2.9}$$

assuming always that $(a, b) \neq (2, 3), a < b$. The upper bound in Equation (2.9) is strictly smaller than $4/3$ if and only if $0 < b(a-4) + a(a+3)$. Therefore, the result holds if $a \geq 4$. The cases $a = 2$ and $a = 3$ follow from computing the τ_{min} using Theorem 2.19.

Indeed, let $a = 2$ and $b = 2h_1 + 1, h_1 > 1$ so the continued fraction representation is $b/a = [h_1, 2]$. Then, $r_0 = 2, t_1 = 0, t_2 = 1$ and $q_{2,b} = h_1 - b/2 - 1/2 = 0$. Analogously, if $a = 3$, then $b = 3h_1 + 1$

or $b = 3h_1 + 2$; the continued fractions are either $b/a = [h_1, 3]$ or $b/a = [h_1, 1, 2]$. Then, $r_0 = 3 + h$ or $r_0 = 2 + h$, $t_2 = 1$ or $t_2 = 0$, respectively, and $t_1 = 1$ in either case. Consequently, in both cases, $q_{3,3h_1+1} = -b/4 + 3h_1/4 + 1/4 = 0$ and $q_{3,3h_1+1} = -b/4 + 3h_1/4 + 1/2 = 0$. Finally, since $\tau_- = 0$ if $a = 2$ and $\tau_- = h_1 - 1$ if $a = 3$,

$$\frac{\mu}{\tau_{min}} = 1 < \frac{4}{3}, \qquad \frac{\mu}{\tau_{min}} < \frac{6b-6}{5b-3} < \frac{6}{5} < \frac{4}{3},$$

for $a = 2, b \geq 3$ and $a = 3, b \geq 4$, respectively. □

To finish this subsection, let us remark an alternative way to compute the dimension of the generic component of the moduli space for branches with one Puiseux pair. In 1988, Laudal, Martin and Pfister [96] provided an alternative description of the moduli space for irreducible plane curve singularities with semigroup $\Gamma =< a, b >$. Also, a more detailed explanation of this description is given in the Lecture Notes of Laudal Pfister [97, §5,§6,§7]. They study and effectively compute the kernel of the Kodaira-Spencer map (See Section B) of the μ-constant versal family defined in Equation 2.6.

Recall that a deformation of a plane curve singularity is versal if and only the Kodaira-Spencer map is surjective (See [108, Thm. 6.5]). The key point in [96] and [97] is that the kernel of the Kodaira-Spencer map has a natural structure of graded Lie algebra. From this, they are able to give a complete description of the moduli space for irreducible plane curve singularities with semigroup $\Gamma =< a, b >$ in terms of the rank of the kernel of the Kodaira-Spencer map. This allows them to recover Delorme's formula (Thm. 2.19) for the generic component of the moduli. Even more, they show the following interesting result which cannot be a priori deduced from Delorme's results:

Theorem 2.21. *[96, Thm. 2.7] Let C_v be the μ–constant versal family with semigroup $\Gamma =< a, b >$. Then, $\tau(C_v)$ takes every possible value between $\tau_{\min}$ and μ.*

2.3.2 The minimal Tjurina number from the semigroup of values

In 1997, Peraire provides an algorithm to compute the generic value set for the extended jacobian of an irreducible plane curve singularity from the semigroup of values of the curve. However, this extended method does not allow to give a formula such the one on Theorem 2.19. Let us recall her results.

Let ξ be an irreducible germ of curve defined by an equation $f = 0$, $f \in \mathbb{C}\{x, y\}$, and let $\Gamma =< \overline{\beta}_0, \ldots, \overline{\beta}_g >$ be its semigroup of values and let $c(\Gamma)$ be its conductor . Let us denote by $\mathbf{J} = (f, \partial f/\partial y, \partial f/\partial x)$ the extended jacobian ideal of ξ. Let $(t^{\beta_0}, \sigma(t^{\beta_0}))$ be the Puiseux parameterization of ξ. Recall that this parameterization induces a discrete valuation in terms of the order function in t (see Section 1.1.2). Define the ideal $L_i \subset \mathbb{C}\{x, y\}$ such that:

$$L_i := \{g \in \mathbb{C}\{x, y\} | \operatorname{ord}_t(g(t^n, \sigma(t^n))) \geq i\}$$

Thus, there is a filtration $\mathbb{C}\{x, y\} = L_0 \supset L_1 \supseteq \cdots \supseteq L_i \supseteq \cdots$ of complete ideals. Since $\mathbf{J}$ is a fractional ideal, there exist R, N such that $L_R \subset \mathbf{J} \subset L_N$. By Plucker's formula $N = c(\Gamma) + \beta_0 - 1$. If we denote $H_i := L_{c(\Gamma)+\beta_0-1+i}$ and $\mathbf{J}_i = \mathbf{J} \bigcap H_i$. Therefore, one has the following sequence of ideals

$$\mathbf{J} = \mathbf{J}_0 \supseteq \mathbf{J}_1 \supseteq \cdots \supseteq \mathbf{J}_R$$

with $\dim_{\mathbb{C}}(\mathbf{J}_i/\mathbf{J}_{i+1}) \leq 1$. We say that H_i is a gap of the ideal $\mathbf{J}$ if $\mathbf{J}_i = \mathbf{J}_{i+1}$. In such a way, if we call $\alpha(\mathbf{J})$ to the number of gaps for $\mathbf{J}$, we have:

$$\mathrm{codim}(\mathbf{J}) = c(\Gamma)/2 + \beta_0 - 1 + \alpha(\mathbf{J}).$$

Peraire's algorithm computes the non-gaps for $\mathbf{J}$ from the semigroup Γ in case that the Puiseux series of ξ has general coefficients, where general means the existence of an open Zariski set on the space of parameters. Even more, Peraire's method also provide a precise description of this Zariski open set where the general coefficients should live. From the results of Section 2.2 any curve with general coefficients in this open set has minimal Tjurina number. The algorithm works as follows:

First, let us denote by $\mathcal{M} := \{\beta_0, \ldots, \beta_g\}$ the set of characteristic exponents of ξ (see Thm.1.27) and by $\mathcal{L}(\mathcal{M}) := \{i \in \mathbb{N} | i \geq \beta_1,\ i \in (e_{k-1})\ \text{ for } i < \beta_k,\ k = 2, \ldots, g\}$. For an element $i \in \mathcal{L}(\mathcal{M})$ we denote $(i)_{+1}$ the next order in the Puiseux series, i.e. $(i)_{+1} = i + e_i$ if $\beta_i \leq i < \beta_{i+1}$ and $i + e_i < \beta_{i+1}$ and $(i)_{+1} = \beta_{i+1}$ otherwise. Peraire's algorithm starts at step $K = \beta_1 - \beta_0$ and by defining

$$\Lambda_{-1} := (0,0,0), \quad \Lambda_0 := (\beta_1 - \beta_0, \beta_1 - \beta_0, \beta_1), \quad \Omega_{\beta_1 - \beta_0} := \{\Lambda_{-1}, \Lambda_0\},$$
$$\Upsilon_{\beta_1 - \beta_0} := \emptyset, \quad \Sigma_{\beta_1 - \beta_0} := \Gamma \cup (\beta_1 - \beta_0 + \Gamma).$$

At step K if $\{j \in \mathbb{N} | j \geq K \text{ and } j \notin \Sigma_{K-1}\} = \emptyset$ then the algorithm finishes at step K. Otherwise, let us denote π_i for $i = 1,2,3$ the projection to the i–th coordinate of a vector and define:

$\Xi_K := \{\Lambda \in \Omega_{K-1} |\ K - \pi_2(\Lambda) \in \Gamma\}$.

If $\Xi_K = \emptyset$ then $\Upsilon_K = \emptyset$.

If $\Xi_K \neq \emptyset$ then

$$\Upsilon_K := \{\Lambda \in \Xi_K | \pi_3(\Lambda) > \min(\pi_3(\Xi_K)) \ \&\ \Lambda \notin \Upsilon_{K'} \text{ if } K - K' \in \Gamma \setminus \{0\}\},$$
$$\Omega_K := \Omega_{K-1} \cup \{(K, K + (i)_{+1} - i, (i)_{+1}) | i \in \pi_3(\Upsilon_K)\}.$$
$$\Sigma_K := \pi_2(\Omega_K) + \Gamma.$$

From this algorithm, Peraire provides the following way to compute the minimal Tjurina number in the equisingularity class of a plane branch.

Theorem 2.22. *[131, Thm. 7.2] Let ξ be a generic irreducible germ of curve, then the generic value set of the extended Jacobian ideal is given by $c(\Gamma) + \beta_0 - 1 + \pi_2(\Omega) + \Gamma$. Let us denote $\Delta_{gen} = \pi_2(\Omega) + \Gamma$, the Γ–semimodule generated by the second coordinates of the elements of Ω. Then, the minimal Tjurina number in the equsingularity class is*

$$\tau_{\min} = \delta(\xi) + \beta_0 - 1 + |\mathbb{N} \setminus \Delta_{gen}|.$$

Observe that the main importance about Peraire's algorithm is the computation of Δ_{gen} and hence of the cardinal $|\mathbb{N} \setminus \Delta_{gen}|$. Observe that once one can compute Δ_{gen} then the computation of the minimal Tjurina number follows from the formula given in Proposition 1.55 or just from the argument about the adapted filtration $\mathbf{J}_{\bullet}$. Thus, Peraire's results [131] provide for the first time an explicit way to compute the minimal Tjurina number of an equisingularity class from the equisingularity invariants. However, we haven't succeed into providing enough control of this algorithm in order to give a positive answer to Dimca and Greuel Question 3.13. This leads us to try to find a formula which allowed us to provide such a positive answer.

2.3.3 Formula for the minimal Tjurina number of irreducible plane curves

Recently in 2016, Genzmer in [62] has provided a formula to compute the dimension of the generic component of the moduli space of an irreducible plane curve singularity through the sequence of multiplicities of the strict transform along a resolution. He uses a previous result by Mattei [115] which uses the theory of holomorphic foliations to interpret the dimension of the generic component of the moduli space as the dimension of a certain cohomology group [62, Lemma 3]. To compute the dimension of this cohomology group Genzmer is able to decompose this dimension, by using a Mayer-Vietoris argument, as a sum of dimensions of other cohomology groups related with the resolution of the singularity. To state his formula let us first introduce the following notation.

Let $\{e_p\}_p$ be the sequence of multiplicities of the resolution of the branch (see Definition 1.14). We define

(a) $e'_p := e_p$ if p is the origin,

(b) $e'_p := e_p + 1$ if p is free and $e_p > 0$,

(c) $e'_p := e_p + 2$ if p is satellite and $e_p > 0$,

(d) $e'_p := 2$ otherwise.

Observe, that $\{e'_p\}_p$ is the sequence of multiplicities of the iterated reduced total transforms appearing in the resolution of the curve. Here, reduced total is the total transform of the curve dropping out the multiplicities of the exceptional divisors, i.e. $(\overline{C}_p)_{red} = \widetilde{C}_p + E_{p_1} + E_{p_2}$ if p is satellite or $(\overline{C}_p)_{red} = \widetilde{C}_p + E_{p_1}$ if p is free. From this sequence, the dimension of the generic component of the moduli space $q_{\min}$ for any plane branch $(C, \mathbf{0})$ in terms of the sequence of multiplicities of the reduced total transform along the minimal embedded resolution of the germ $(C, \mathbf{0})$ is

Theorem 2.23 ([62]). *The dimension of the generic component q_{min} of the moduli space of plane branches with semigroup Γ equals*

$$q_{min} = \sum_p \sigma(e'_p),$$

where the summation runs on all points p equal or infinitely near to the origin and

$$\sigma(k) := \begin{cases} \dfrac{(k-2)(k-4)}{4}, & \text{if } k \text{ is even}, \\[2ex] \dfrac{(k-3)^2}{4}, & \text{if } k \text{ is odd}. \end{cases}$$

Remark 2.24. For the case of $\Gamma = < a, a+1 >$, Genzmer formula coincides with the one proved by Zariski in [197, Chap. VI Thm. 4.16].

From Genzmer's formula together with the results exposed in Section 2.2 we are able to provide a formula for the minimal Tjurina number in an equisingularity class for irreducible plane curve singularities.

Theorem 2.25. *For any equisingular class of germs of irreducible plane curve singularity, with multiplicity n and sequence of multiplicities $\{e_p\}_{p \in \mathcal{N}_O}$, we have*

$$\tau_{min} = \sigma(n) + \frac{n^2+3n-6}{2} + \sum_{p\,free} \frac{(e_p-1)(e_p+2)+2\sigma(e_p+1)}{2} + \sum_{p\,sat.} \frac{e_p(e_p-1)+2\sigma(e_p+2)}{2},$$

where the summation runs on all points p equal or infinitely near to the origin and σ is defined as in Theorem 2.23.

Remark 2.26. A few days after our paper [3] was published in arXiv, Genzmer and Hernandes in [63] obtained an alternative formula for $\tau_{\min}$. Although the methods are rather different, the key ingredient is still the formula for the generic dimension of the moduli space obtained in [62].

A fortiori, we can see from Theorem 2.25 that the τ_{min} of an equisingularity class depends only on the minimal resolution of $(C, \mathbf{0})$ and not on the minimal embedded resolution. Furthermore, the formula works for any resolution of $(C, \mathbf{0})$, minimal or not. This is completely trivial if one looks at the formula. Observe that the minimal resolution obtains a strict transform with $e_p = 1$ in the minimal number of blow ups. Thus any other resolution, for example the minimal embedded resolution, will make the sequence of multiplicities longer but all this extra multiplicities will be $e_p = 1$ or $e_p = 0$; in both cases their contribution to the formula of Theorem 2.25 is equal to 0.

Example 2.27. Continuing with Example 1.58, let us compute with the help of our formula the minimal Tjurina number for plane branches with semigroup $\Gamma = \langle 4, 6, 13\rangle$. To do so it is enough to consider the minimal embedded resolution computed in the Examples 1.10 and 1.17. Recall that the multiplicity at the origin is $n = 4$, we have two free points p_1, p_3 with $e_{p_1} = 2$ and $e_{p_3} = 1$ and we have two satellite points p_2, p_4 with $e_{p_2} = 2$ and $e_{p_4} = 1$. Thus,

$$\tau_{\min} = 0 + \frac{16+12-6}{2} + \frac{1\cdot 4+2\cdot 0}{2} + \frac{0\cdot 3+2\cdot 0}{2} + \frac{2\cdot 1+2\cdot 0}{2} + \frac{1\cdot 0+2\cdot 0}{2} = 14.$$

As expected, the minimal Tjurina number coincides with the computation of the Tjurina number in Example 1.58, thanks to the results of Abreu-Hernandes [1] and Luengo-Pfister [112] the Tjurina number for this equisingularity class of singularities is constant.

A first corollary of our closed formula for the minimal Tjurina number in an equisingularity class is that combined with Peraire's formula 2.22 we have.

Corollary 2.28. *With the notation of Theorem 2.22. For any equisingularity class of plane branch singularities the cardinal of the gaps of Δ_{gen} is*

$$|\mathbb{N} \setminus \Delta_{gen}| = \sigma(n) + n - 2 + \sum_{p\,free} ((e_p-1) + \sigma(e_p+1)) + \sum_{p\,sat.} \sigma(e_p+2),$$

where the summation runs on all points p equal or infinitely near to the origin.

Finally, our formula for τ_{min} (Theorem 2.25) enables us to give a positive answer to Dimca and Greuel's Question 3.13 in the case of any plane branch. Before proving this, we need the following property of the sequence of multiplicities.

Lemma 2.29. *For any plane branch singularity of multiplicity n,*

$$\sum_{p\,sat.} e_p = n - 1,$$

where the summation runs on all satellite infinitely near points p.

Proof. Consider the finite sequence of positive multiplicities $\{e_p\}_p$ along the minimal embedded resolution of the plane branch. From Enriques' theorem [31, Thm. 5.5.1] one has that $n+\sum_{p\,\text{free}} e_p = \beta_g$, where β_g/n is the last characteristic exponent. On the other hand, from [31, Ex. 5.6], $\sum_p e_p = \beta_g + n - 1$, where this summation runs on all points p equal or infinitely near to the origin. Since all the satellite points for which e_p is positive are included in the sequence of points blown-up in the minimal embedded resolution, the result follows. □

Finally, we get the announced positive answer to Dimca and Greuel's question as a Corollary of Theorem 2.25.

Corollary 2.30. *For any plane branch singularity,*

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Proof. It is enough to prove the inequality for the τ_{min} of each equisingularity class of plane branches. We will show that $4\tau_{min} - 3\mu > 0$. From Theorem 2.25 we have that

$$4\tau_{min} - 3\mu = 4\sigma(n) - n^2 + 9n - 12 + \sum_{p\,\text{free}} \left(4\sigma(e_p+1) - (e_p-1)(e_p-4)\right) + \sum_{p\,\text{sat.}} \left(4\sigma(e_p+2) - e_p(e_p-1)\right).$$

Now, since $\sigma(k) \geq (k-2)(k-4)/4$, we have

$$4\tau_{min} - 3\mu \geq 3n - 4 + \sum_{\substack{p\,\text{free}\\ e_p>0}} (e_p - 1) - \sum_{p\,\text{sat.}} e_p.$$

Finally, using Lemma 2.29

$$4\tau_{min} - 3\mu \geq 2n - 3 + \sum_{\substack{p\,\text{free}\\ e_p>0}} (e_p - 1) > 0,$$

and the result follows, since $n \geq 2$. □

As a direct consequence of Theorem 2.25 we also obtain the following lower bound for τ:

Corollary 2.31. *For any plane branch,*

$$\tau \geq \frac{3n^2}{4} - 1 \quad \text{if} \quad n \text{ is even}, \qquad \tau \geq \frac{3}{4}(n^2-1) \quad \text{if} \quad n \text{ is odd}.$$

The bound in Corollary 2.31 is sharp, as one can easily check for generic curves in the equisingularity class of the singularities $y^n - x^{n+1} = 0$, i.e. the minimal Tjurina number in the equisingularity class of the singularities $y^n - x^{n+1} = 0$ coincides with the bound of Corollary 2.31. Let us make the computations. To do so, we are going to use the formula provided by Theorem 2.1 and we will compare it with the formula we have provided in Theorem 2.25. To use Theorem 2.1 observe that $\sigma(n, n+1)$ is in the cases (Σ_1) so

1. If n is odd we use (Σ_1) a. and

$$\tau_{\min} = n(n-1) - \frac{(n-2)(n-1)}{4} + \frac{n}{4} - \frac{1}{4} = \frac{3}{4}(n-1)(n+1).$$

2. If n is even we use (Σ_1) b. and

$$\tau_{\min} = n(n-1) - \frac{(n-2)(n-1)}{4} + \frac{n}{4} - \frac{1}{2} = \frac{3}{4}n^2 - 1.$$

On the other hand, to use Theorem 2.25 we observe that the sequence of multiplicities of the resolution is $n, 1, 1, \ldots, 1$ so

$$\tau_{\min} = \sigma(n) + \frac{n^2 + 3n - 6}{2}.$$

Then from the definition of $\sigma(n)$ it is easy to check that both computations coincide.

Finally, it is worth noting to say that from Theorem 2.25 one can see that these are the only topological types of singular plane branches for which the bound is reached. Moreover, this bound improves the bound previously provided by Liu in [106, Corollary 2.1].

Chapter 3

The $\mu - \tau$ problem

Let $(X, \mathbf{0}) \subset (\mathbb{C}^N, 0)$ be a germ of an isolated hypersurface singularity defined by an equation $f \in \mathcal{O}_{(\mathbb{C}^N,0)}$. For such singularities there are two important invariants: the Milnor number μ, and the Tjurina number τ. Those numbers can be expressed as:

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \dots, x_N\}}{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right)}, \quad \tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \dots, x_N\}}{\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right)}.$$

By definition, it is trivial that $\mu - \tau \geq 0$. In fact, it is a well known result by K. Saito [145] that $\mu - \tau = 0$ if and only if the hypersurface singularity is a quasihomogeneous isolated hypersurface singularity, i.e. in a suitable coordinates there exists $(w_1, \dots, w_N)$ such that the defining equation has the property $f(\lambda^{w_1} x_1, \dots, \lambda^{w_N} x_N) = \lambda^d f(x_1, \dots, x_N)$ with $d = \sum w_i$ and $\lambda \in \mathbb{C}^*$.

For isolated complete intersection singularities (ICIS) of dimension $n = N - r$ defined by an ideal $\mathcal{I} = (f_1, \dots, f_r)$, Hamm [73, Satz 1.7] was the first to show that the Milnor fiber of $(X, 0)$ is homotopy equivalent to a bouquet of spheres, extending the previous results of Milnor [118]. In contrast, the algebraic definition of the Milnor number of an ICIS through the partial derivatives of the defining equations is due to the works of Greuel [68, 67] and Lê [104] independently. On the other hand, Tjurina's work [166] identifies $\mathrm{Ext}^1_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)})$ as the base space of the miniversal deformation of a normal isolated singularity with $\mathrm{Ext}^2_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)}) = 0$, as we will review in Appendix B. Thus, thanks to the results of Hamm and Tjurina, the Milnor and Tjurina numbers of an ICIS can be defined as

$$\mu := \operatorname{rk} H_n(F), \quad \tau := \dim_{\mathbb{C}}(\mathrm{Ext}^1_{\mathcal{O}_{(X,0)}}(\Omega^1_{(X,0)}, \mathcal{O}_{(X,0)})),$$

where F is the Milnor fiber of the smoothing of $(X, 0)$, $\mathcal{O}_{(X,0)} = \mathbb{C}\{x_1, \dots, x_N\}/\mathcal{I}$ and $\Omega^1_{(X,0)}$ is the corresponding module of differential 1–forms at $(X, 0)$.

During the seventies and the eighties, it was an important topic in singularity theory to what extend the difference $\mu - \tau$ is a good measure of the quasihomogeneity of more general singularities. In the ICIS case, observe that the notion of quasi-homogeneity is just the condition that each of its defining equations $f_1, \dots, f_r$ must be quasi-homogeneous. For general isolated complete intersection singularities of dimension $n = 1$, Greuel [68] proved the inequality $\mu - \tau \geq 0$ and that equality holds in any dimension if it is quasihomogeneous. For dimension $n \geq 2$ Looijenga and Steenbrink [109] provided the general inequality $\mu - \tau \geq 0$ and Vosegaard [175] showed that if equality holds then the ICIS is quasihomogeneous.

Despite the results concerning the inequality $\mu - \tau \geq 0$, very few is known in the literature concerning sharp upper bounds for $\mu - \tau$ of the form $C\mu$ with $C \in \mathbb{Q}$. In the hypersurface case, as far as the author knowledge, Liu [106] is the first who provided some bounds of this type (see for example Proposition 3.11). One of the goals of our paper [5] is to motivate the study of the following problem by showing its connection with other problems in singularity theory:

Problem 3.1. [5, Problem 1] Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity of dimension n and codimension $k = N - n$. Is there an optimal $\frac{b}{a} \in \mathbb{Q}$ with $b < a$ such that

$$\mu - \tau < \frac{b}{a}\mu\,?$$

Here optimal means that there exist a family of singularities such that μ/τ tends to $\frac{a}{a-b}$ when the multiplicity at the origin tends to infinity.

The main results of this chapter give a complete answer to Problem 3.1 in the case $N = 2$, $r = 1$ (Theorem 3.15) and partial answers in the cases $N = 3$, $r = 1$ (Proposition 3.31) and $r = N - 1$ with arbitrary N (Corollary 3.17).

In the case of plane curve singularities, i.e. $N = 2$, $r = 1$, Problem 3.1 is precisely Dimca and Greuel Question 3.13. In contrast with the previous Chapter 2, in this chapter we will provide a positive answer to Question 3.13 (Theorem 3.15) for any plane curve (not necessarily irreducible) singularity $f(x, y) = 0$, thus providing a complete answer to Question 3.13. This general solution is based on the study of μ/τ for the normal two-dimensional double point $\{z^2 = f(x, y)\}$. As one can see, our point of view is completely different from the techniques used in [6, 3, 63, 183], to solve Question 3.13 for some special cases that we have recall in Chapter 2. Moreover, our solution provide an intrinsic reason for the bound 4/3.

As a consequence of this new approach, we can use the results of Teissier [165] explained in Section 2.2 to show that if $(C, \mathbf{0})$ is an irreducible germ of spacial curve (not necessarily plane) satisfying certain extra condition (see Sec. 2.2 for a precise statement) then $\mu - \tau < \mu/4$ (Corollary 3.17). This fact, constitute a partial answer to Problem 3.1 in the case $r = N - 1$ with arbitrary N.

To finish, we will deal with the case $N = 3$ and $r = 1$. In this case, we connect Problem 3.1 with a long standing and widely studied conjecture posed by Durfee in 1978 (see Conjecture 3.19). The cases where Durfee's conjecture holds motivate us to propose $\mu/3$ as the optimal bound for the case $N = 3$ and $r = 1$ of Problem 3.1(Proposition 3.31). We conjecture that this bound is true for any isolated hypersurface singularity in $\mathbb{C}^3$ (Conjecture 3.32). Moreover, Durfee's conjecture 3.19 implies Conjecture 3.32. In this way, Conjecture 3.32 provides an easy criterion to check the validity of Durfee's conjecture 3.19.

3.1 The difference $\mu - \tau$ revisited

Let $(X, \mathbf{0})$ be a germ of isolated hypersurface singularity. An usual tool in the study of Milnor and Tjurina number is the Poincaré-De Rham complex, see for example the works of Greuel [68, 67] and Yau [192, 188, 190]. Following [192], let

$$0 \to \mathbb{C} \to \mathcal{O}_{(X,\mathbf{0})} \xrightarrow{d^0} \Omega^1_{(X,\mathbf{0})} \xrightarrow{d^1} \Omega^2_{(X,\mathbf{0})} \to \cdots$$

be the Poincaré complex at $\mathbf{0}$, where $\Omega^p_{(X,\mathbf{0})}$ is the sheaf of differential p–forms and d^p the usual differential operator. The Poincaré numbers of X at $\mathbf{0}$ are defined as

$$p^{(i)} := \dim_{\mathbb{C}} \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}} \quad \text{for all } i \geq 0.$$

If $(X, \mathbf{0})$ is a hypersurface singularity of dimension n then Brieskorn [26] proved $p^{(i)} = 0$ if $i \leq n-2$ and Sebastiani [157] proved $p^{(n-1)} = 0$. After that, Saito [145] proved that the complex is exact if and only if the singularity is quasihomogeneous.

Theorem 3.2. *[145] If $(X, \mathbf{0})$ is a germ of isolated hypersurface singularity of dimension n then $p^{(n)} = \mu - \tau$. Even more, $p^{(n)} = 0$ if and only if $(X, \mathbf{0})$ is quasihomogeneous.*

In 1975, Greuel [67] extended the results of Brieskorn and Sebastiani to the case of isolated complete intersection of any dimension and proved that if the singularity is quasihomogeneous then $\mu = \tau$ [67, Korollar 5.8].[1] The converse was proven by Wahl [179] in the case of dimension 2 and for any dimension by Vosegaard [175]. However, in this full generality the exactness of the Poincaré complex does not imply the quasihomogeneity of the singularity as Pfister and Schönemann show in [133]. In this way, the difference $\mu - \tau$ or alternatively the quotient μ/τ must be considered as a measure of the non-quasihomogeneity of the singularity and not of the exactness of the Poincaré complex.

To give formulas for the difference $\mu - \tau$ in terms of other invariants of the singularity is in general a difficult task. For example, for isolated complete intersections of dimension $n \geq 2$, in 1985 [109] Looijenga and Steenbrink give a precise formula for this difference in terms of the mixed Hodge structure of the singularity:

Theorem 3.3. *[109] If (X, x) is an isolated complete intersection singularity of dimension $n \geq 2$, then*

$$\mu - \tau = \sum_{p=0}^{n-2} h^{p,0}(X, x) + a_1 + a_2 + a_3,$$

where $h^{p,q}(X, x)$ denotes the (p, q)–Hodge number of the mixed Hodge structure which is naturally defined on the local cohomology group $H^n(X, X - \{x\}; \mathbb{C})$ and the numbers a_1, a_2, a_3 are nonnegative invariants of a resolution of (X, x).

As we will see in the forthcoming subsection, Theorem 3.3 can be rewritten for surface singularities in $\mathbb{C}^3$ without the explicit use of Mixed Hodge structures (Theorem 3.7).

3.1.1 Wahl's formula for $\mu - \tau$ in the surface case

Let $(X, 0) \in (\mathbb{C}^3, 0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. By Section 1.2, we are in the case of a germ of normal two–dimensional singularity and hence we can consider its minimal good resolution defined as Definition 1.40. Let us consider the minimal good resolution $\widetilde{\pi} : \widetilde{X} \to X$ of $(X, 0)$ and let us denote by $A = \cup A_i$ the reduced exceptional divisor, which is a union of smooth curves A_i. Let us denote by g_i the genus of A_i and by b the first Betti number of the dual graph of A, i.e. the number of loops of the dual graph. If we denote by $g = \sum_i g_i$, then $\dim H^1(A; \mathbb{C}) = 2g + b$.

Following the book of Peters and Steenbrink [132, Chapter 4], let us define $U := \widetilde{X} \setminus A$ and let us denote by $j : U \hookrightarrow \widetilde{X}$ the inclusion map. A holomorphic differential form ω on U is said to have *logarithmic poles along* A if ω and $d\omega$ have at most a pole of order one along A. Then, we define the sheaf of of logarithmic 1–forms $\Omega^1_{\widetilde{X}}(\log A)$ as the corresponding subsheaf of $j_* \Omega^1_U$ of holomorphic differential forms on U having logarithmic poles along A.

[1] Because of [67] is published in 1975, the conclusion stated in [67, Korollar 5.8] was not yet in the nowadays common form $\mu = \tau$; moreover it was first enunciated in the form $\mu = \tau$ in the subsequent paper of Greuel of 1980 [68].

Observe that since A is the divisor of a minimal good resolution of singularities, then A has simple normal crossings. In particular, if $p \in A$ we can consider local coordinates $(z_1, \ldots, z_n)$ in which A has equation $z_1 \ldots z_l = 0$. Then,

$$\Omega^1_{\widetilde{X}}(\log A)_p = \mathcal{O}_{\widetilde{X},p}\frac{dz_1}{z_1} \bigoplus \cdots \bigoplus \mathcal{O}_{\widetilde{X},p}\frac{dz_l}{z_l} \bigoplus \mathcal{O}_{\widetilde{X},p} z_{l+1} \bigoplus \cdots \bigoplus \mathcal{O}_{\widetilde{X},p} z_n.$$

For each irreducible component A_l the *Poincaré residue map* res_l *along* A_l can be defined as follows. Assume we have local coordinates $(z_1, \ldots, z_n)$ such that $z_1 = 0$ is an equation for A_l. We can write $\omega \in \Omega^1_{\widetilde{X}}(\log A)$ as $\omega = \eta \wedge \frac{dz_1}{z_1} + \eta'$ with η, η' not containing dz_1. Then,

$$\begin{aligned} \mathrm{res}_l : \Omega^1_{\widetilde{X}}(\log A) &\to \mathcal{O}_{A_l} \\ \omega &\mapsto \eta_{|A_l}. \end{aligned}$$

If we denote by $\widetilde{A} = \sqcup A_i$ the disjoint union of the components of the exceptional divisor then we have the following exact sequence:

$$0 \to \Omega^1_{\widetilde{X}} \to \Omega^1_{\widetilde{X}}(\log A) \xrightarrow{\mathrm{res}} \mathcal{O}_{\widetilde{A}} \to 0,$$

where res is the residue map defined on each A_l as res_l.

Finally, let us denote by $\Omega^1_{\widetilde{X}}(\log A)(-A)$ the kernel of the restriction map $\Omega^1_{\widetilde{X}} \twoheadrightarrow \Omega^1_{\widetilde{A}}$. Also, we denote by $\Omega^p_{\widetilde{X}} = \bigwedge^p \Omega^1_{\widetilde{X}}$ and in the same way $\Omega^p_{\widetilde{X}}(\log A) = \bigwedge^p \Omega^1_{\widetilde{X}}(\log A)$. From this, we have complexes $\Omega^\bullet_{\widetilde{X}}$ and $\Omega^\bullet_{\widetilde{X}}(\log A)$ which allow us to consider the corresponding long exact sequences in cohomology. After that, one can define the so called *Steenbrink's invariants* [179, Section 1]. [2] Before to define Steenbrink's invariants let us remark that we will denote by $H^i_A(\bullet) = H^i(\widetilde{X} \setminus A, \bullet)$; also we denote by k the number of components A_i of A. Then Steenbrink's invariants are defined as follows:

$$\gamma := \mathrm{rk}\{H^1_A(\Omega^1_{\widetilde{X}}) \to H^1(\Omega^1_{\widetilde{X}})\} - k,\ \alpha := \dim \frac{H^0(\Omega^2_{\widetilde{X}})}{dH^0(\Omega^1_{\widetilde{X}}(\log A)(-A))},\ \beta := \dim \frac{H^0(\Omega^1_{\widetilde{A}})}{\mathrm{Im}\, H^0(\Omega^1_{\widetilde{X}})}.$$

Another important invariant of a hypersurface singularity is the geometric genus, p_g.[3] Let us introduce its definition. If X is an isolated hypersurface singularity of dimension n, a holomorphic form ω on $U' = X \setminus \{0\}$ is called of *first kind* if there exists a resolution $\pi : \widetilde{X} \to X$ of the singularity X such that $\pi^*(\omega)$ extends holomorphically to $\widetilde{X}$. Sometimes the holomorphic n–forms of first kind are denoted by $L^2(X \setminus \{0\}, \Omega^n)$ or $H^0_{L^2}(X \setminus \{0\}, \Omega^n)$, where $H^0_{L^2}$ denotes the global L^2–forms; see Laufer [99, Section III and Theorems 3.1, 3.4] and Yau [187].[4] With this notation,

$$p_g := \dim_{\mathbb{C}} \frac{H^0(X \setminus \{0\}, \Omega^n_X)}{H^0_{L^2}(X \setminus \{0\}, \Omega^n_X)} = \dim_{\mathbb{C}} \frac{H^0(\widetilde{X} \setminus A, \Omega^n_{\widetilde{X}})}{H^0(\widetilde{X}, \Omega^n_{\widetilde{X}})}.$$

Observe that, in particular, the geometric genus is the number of adjunction conditions imposed by the singularity (see Merle and Teissier [117, Section 1]).

[2] It is already pointed out by Wahl in the introduction to his paper [179] that most of the results in [179, Section 1] are due to Steenbrink. These results come from a private communication between them.

[3] The geometric genus is a invariant which, in particular, allows to define the important class of rational singularities. Rational singularities are precisely those with $p_g = 0$, see for example the works of Artin [16], Du Val [43] and Laufer [99].

[4] The labeling L^2 comes from the fact they are square integrable holomorphic forms in a compact neighborhood of the origin. This means that $\|w\|_2 = (\int_C |w|^2 d\eta)^{1/2} < \infty$. Observe that L^2 is precisely the L^p Lebesgue space with $p = 2$ first defined by Riesz [139] in 1910.

While working with top forms, i.e. n–forms where n is the dimension of the hypersurface singularity, lead us to consider the geometric genus, to work with j–forms with $j < n$ also lead to other interesting invariants. This is the case of the analytic invariant first defined by Yau [192] called *the irregularity of* X. Again, if X is an isolated hypersurface singularity of dimension n, according to Yau [192, pg. 654] the irregularity can be defined as

$$q := \dim_{\mathbb{C}} \frac{H^0(\widetilde{X} \setminus A, \Omega^{n-1}_{\widetilde{X}})}{H^0(\widetilde{X}, \Omega^{n-1}_{\widetilde{X}})}.$$

In the particular case of surface singularities, i.e. $n = 2$, Wahl [179] and Yau [192, Theorem 2.1] provided the following computation of the irregularity.

Theorem 3.4. *[179, 192] Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated hypersurface singularity. Following the previous notation, the irregularity of X is given by*

$$q = p_g - g - b - \alpha - \beta - \gamma.$$

Sketch of the proof. Consider the following exact sequence

$$H^0(\Omega^1_{\widetilde{X}}) \xrightarrow{\lambda_1} H^0(\Omega^1_{\widetilde{X}\setminus A}) \xrightarrow{\lambda_2} H^1(\widetilde{X} \setminus A, \Omega^1_{\widetilde{X}}) \xrightarrow{\lambda_3} H^0(\Omega^1_{\widetilde{X}}).$$

The result follows by observing that $\mathrm{rk}(\lambda_3) = \dim_{\mathbb{C}}(H^1(\widetilde{X} \setminus A, \Omega^1_{\widetilde{X}})) - (k + \gamma)$ together with

$$\dim_{\mathbb{C}}(H^1(\widetilde{X} \setminus A, \Omega^1_{\widetilde{X}})) = \dim_{\mathbb{C}}(H^1(\widetilde{X}, \Omega^1_{\widetilde{X}})) = k + p_g - g - b - \alpha - \beta. \qquad \square$$

Theorem 3.4 shows the relation between the irregularity, the geometric genus and other invariants of the surface singularities. However, recall that we are interested in the difference between the Milnor number and the Tjurina number. In [179], Wahl provided the following formula for the Tjurina number of a surface singularity which relates the Tjurina number with the geometric genus.

Theorem 3.5. *[179, Prop. 2.2 and Thm. 2.4] Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated hypersurface singularity. Following the previous notation, the Tjurina number of X is given by*

$$\tau(X) = 12p_g(X) + \chi_{top}(A) + K \cdot K - 1 - (b + 2\alpha + 2\beta + \gamma),$$

where K denotes the canonical divisor on $\widetilde{X}$ and $\chi_{top}(A)$ is the topological Euler characteristic of the exceptional curve A of the minimal good resolution of X.

On the other hand, in analogy to Milnor's identity for plane curves (Theorem 1.49), Laufer proved an identity which relates the Milnor number with the geometric genus of a surface singularity.

Theorem 3.6. *[100, Thm. 1] Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated hypersurface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Let $\widetilde{\pi} : \widetilde{X} \to X$ be a resolution of X. Let $A := \widetilde{\pi}^{-1}(0, 0, 0)$ be the exceptional divisor and $\chi_{top}(A)$ be its topological Euler characteristic. Let us denote by f_x, f_y, f_z the corresponding partial derivatives of f. Let K be the canonical divisor on $\widetilde{X}$, i.e. the divisor of the 2-form defined by $\omega := \frac{dy\wedge dx}{f_z} = \frac{dy\wedge dz}{f_x} = \frac{dz\wedge dx}{f_y}$ which is meromorphic in $\widetilde{X}$ with pole set contained in A. Let $K \cdot K$ be the self-intersection of the divisor K. Let p_g be the geometric genus of X. Then, the Milnor number can be computed as*

$$\mu(X) = \chi_{top}(A) + K \cdot K + 12p_g(X) - 1.$$

Finally, Theorem 3.5 together with Laufer's expression of the Milnor number of a surface singularity Theorem 3.6 gives the following reformulation of Looijenga and Steenbrink Theorem 3.3.

Theorem 3.7. *[179, Theorem 2.7] Let $(X,0) \in (\mathbb{C}^3,0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$ then*

$$\mu(X) - \tau(X) = b + 2(\alpha + \beta) + \gamma.$$

Remark 3.8. The previous Theorem in its original form [179, Theorem 2.7] is stated for any Gorenstein singularity with an smoothing, where one should identify τ with the dimension of a smoothing component and μ with the Milnor number associated to the smoothing. In the case of ICIS, and more concretely of hypersurfaces, the smoothing is unique up to isomorphism so we return to the definitions we are working on. For shake of simplicity, we have preferred to keep the exposition clear and just to enunciate Wahl Theorem in this more restrictive form.

Remark 3.9. If we put $n = 2$ in Theorem 3.3 then one can recover the numbers a_1, a_2, a_3 from Steenbrink's invariant by the following relations:

$$a_1 = \alpha + \beta + \gamma, \quad a_2 + a_3 = \alpha + \beta.$$

One can check this if one looks carefully at the exact sequences [109, (6),(8) and (9)] that define the invariants a_1, a_2, a_3.

From the previous results, Wahl [179] obtains the following upper estimate for the difference $\mu - \tau$:

Theorem 3.10. *[179, Corollary 2.9] Let $(X,0) \in (\mathbb{C}^3,0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Then*

$$\mu - \tau \leq 2p_g - \dim H^1(A;\mathbb{C}),$$

where A is the exceptional divisor of a minimal good resolution of X.

In particular, $\mu - \tau \leq 2p_g$.

Proof. By Theorem 3.7 together with Theorem 3.4 $\mu - \tau = 2p_g - 2g - b - \gamma - 2q$. Thus,

$$\mu - \tau \leq 2p_g - 2g - b = 2p_g - \dim H^1(A;\mathbb{C}). \qquad \square$$

Finally, we also notice that in the form stated here, Theorems 3.10 and 3.7 are already appearing in Yau's paper [190, Section 2] and more concretely [190, Theorem 3.4]. However, we have preferred to use Wahl's approach since also it is the only one that works with weaker hypothesis on the statements.

Moreover, Wahl shows with several examples that the bound of Theorem 3.10 is sharp. For instance Wahl's examples [179, Examples 4.6 and 4.7] are positive weight deformations of $z^2 + x^{2a+1} + y^{2a+2} = 0$ or $x^a + y^a + z^a = 0$.

3.2 Dimca and Greuel problem for plane curve singularities

In 2017, Liu [106] provided the following upper bound for the quotient between the Milnor number and Tjurina number of any isolated hypersurface singularity by using purely algebraic methods:

Proposition 3.11. *[106] For any germ of isolated hypersurface singularity defined by a germ of function $f : \mathbb{C}^n \to \mathbb{C}$ we have*

$$\frac{\mu}{\tau} \leq n,$$

where μ, τ are, respectively, the Milnor number and the Tjurina number of f.

Sketch of the proof. Let us denote by M_f the Milnor algebra $\mathbb{C}\{x_1, \dots, x_n\}/(\partial f/\partial x_1, \dots, \partial f/\partial x_1)$ and by T_f the Tjurina algebra $\mathbb{C}\{x_1, \dots, x_n\}/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_1)$. Consider the multiplication by f as endomorphism of the Milnor algebra:

$$0 \to \mathrm{Ker}(f) \to M_f \xrightarrow{f} M_f \to T_f \to 0$$

Observe that $\dim_{\mathbb{C}}(\mathrm{Ker}(f)) = \tau$. By Briançon-Skoda Theorem [25], if we denote by $(f^i)M_f$ the ideal of M_f generated by (f^i), those ideals define a decreasing filtration with

$$\dim_{\mathbb{C}} \left\{ \frac{(f^i)M_f}{(f^{i+1})M_f} \right\} \leq \tau.$$

Therefore,

$$\mu = \dim_{\mathbb{C}} M_f = \dim_{\mathbb{C}} T_f + \sum_{i=1}^{n-1} \dim_{\mathbb{C}} \left\{ \frac{(f^i)M_f}{(f^{i+1})M_f} \right\} \leq n\tau. \qquad \square$$

In the particular case of plane curve singularities this means that $\mu/\tau < 2$. Motivated to know if the previous bound was optimal, Dimca and Greuel [46] showed the following example.

Example 3.12. [46, Example 4.1] Consider the families of curves

$$X_a : x^{2a+1} + x^a y^{a+1} + y^{2a} = 0, \quad X_b : x^{2b+1} + x^{b+1} y^{b+1} + y^{2b+1} = 0.$$

X_a is a family of irreducible plane curve singularities with one Puiseux pair. On the other hand, X_b is a family of non-irreducible plane curve singularities. For those families, Dimca and Greuel compute their Tjurina and Milnor numbers $\tau(X_a) = 3a^2$, $\mu(X_a) = 2a(2a-1)$, $\mu(X_b) = 4b^2$, $\tau(X_b) = 4b^2 - (b-1)^2$. Therefore, it follows that

$$\frac{\mu(X_a)}{\tau(X_a)} \xrightarrow[a \to \infty]{} 4/3, \quad \frac{\mu(X_b)}{\tau(X_b)} \xrightarrow[b \to \infty]{} 4/3$$

From the behavior of those examples, Dimca and Greuel asked the following question.

Question 3.13. [46, Question 4.2] Is it true that $\mu/\tau < 4/3$ for any isolated plane curve singularity?

In Chapter 2 we have shown some partial answers to this question. The first result about Question 3.13 is given for semi-quasi-homogeneous singularities in 2018 by Blanco and the author [6]. In April 2019 three different answers for irreducible plane curve singularities appeared in a short time. Alberich-Carramiñana, Blanco, Melle-Hernández and the author in [3] give a positive answer to Question 3.13 for irreducible plane curve singularities through a formula for the minimal Tjurina number in an equisingularity class in terms of the sequence of multiplicities. A few days later, Genzmer and Hernandes in [63] provide an alternative proof of Dimca and Greuel's inequality for the irreducible plane curve case. Despite the fact that both papers use quite different techniques, both are based on the explicit computations about the dimension of the generic component of the moduli space of an irreducible plane curve singularity given by Genzmer in [62]. Finally, Wang in [183] give another alternative proof for the irreducible case based also in Genzmer's result about the dimension of the generic component of the moduli space [62]. Moreover, Wang's approach is of different nature since he shows that $3\mu - 4\tau$ is a strictly increasing numerical invariant under blow ups for the class of irreducible plane curve singularities [183, Corollary 5.4]. Thus, Wang results show that, in the irreducible case, Dimca and Greuel Question 3.13 allows to compare singularities.

Despite those partial positive answers to Question 3.13, there is no clue in Wang's results [183] nor in Genzmer-Hernandes [63] neither in our first results [6, 3] as to whether the numbers $3, 4$ can be inferred from deeper arguments than just explicit computations, i.e. is there an intrinsic reason to consider the invariant $3\mu - 4\tau$ instead of any other combination of the type $a\mu - b\tau$ with $(a, b) \neq (3, 4)$? Moreover, after [6, 3, 63, 183] the semiquasihomogeneous case treated by Blanco and the author in [6] was still the only non-irreducible case for which Dimca and Greuel Question had a positive answer.

Here, we are going to not only give a positive answer to Dimca and Greuel's Question 3.13 in its full generality but also a non computational explanation for the 4/3 bound. To do so, let us consider $f \in \mathbb{C}\{x, y\}$ the equation of a germ of isolated plane curve singularity in $(\mathbb{C}^2, \mathbf{0})$. Now, we consider the germ of isolated hypersurface singularity $(X, \mathbf{0}) \subset (\mathbb{C}^3, \mathbf{0})$ defined by

$$F(x, y, z) = f(x, y) + z^2 = 0.$$

For such a singularity, the geometric genus has the following upper bound proved by Tomari:

Theorem 3.14. *[168, Thm. A] Let $(X, \mathbf{0}) \subset (\mathbb{C}^3, \mathbf{0})$ be a germ of isolated hypersurface singularity defined by an equation $F(x, y, z) = z^2 + f(x, y)$ with $f(x, y)$ of order at least two. Then*

$$8p_g(X) + 1 \leq \mu(X).$$

Sketch of the proof. Let $\widetilde{\pi} : \widetilde{X} \to X$ be a resolution of X. Let $A := \widetilde{\pi}^{-1}(0, 0, 0)$ be the exceptional divisor and $\chi_{top}(A)$ be its topological Euler characteristic. Let us denote by F_x, F_y, F_z the corresponding partial derivatives of F. Let K be the canonical divisor on $\widetilde{X}$, i.e. the divisor of the 2-form defined by $\omega := \frac{dy \wedge dx}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$ which is meromorphic in $\widetilde{X}$ with pole set contained in A. Then Laufer's Theorem 3.6 provides the following equality

$$\mu(X) = \chi_{top}(A) + K \cdot K + 12p_g(X) - 1.$$

Assume $\widetilde{X}$ is the canonical resolution of X as computed in Section 1.2. Let $n_0 := n$ be the order of C at the origin of $\mathbb{C}^2$. Define $\lambda_0 := \lfloor \frac{n}{2} \rfloor - 1$ where $\lfloor \cdot \rfloor$ denotes the floor function. Let us consider the sequence of branch locus $B^{(1)}, \ldots, B^{(l)}$ obtained in the canonical resolution. Let us denote n_j for $j = 1, \ldots, l - 1$ the multiplicity of each singular branch locus at its singular point and define $\lambda_j := \lfloor \frac{n_j}{2} \rfloor - 1$. Then, [167, Sec.1 (1.4.4) and (1.4.6)] state that the geometric genus and the self-intersection of the canonical divisor can be expressed as:

$$2p_g(X) = \sum_{i=0}^{l-1} \lambda_i(\lambda_i + 1) \quad \text{and} \quad K \cdot K = -2\sum_{i=0}^{l-1} \lambda_i^2.$$

From this, we have

$$\mu(X) - 8p_g(X) = 2\sum_{i=0}^{l-1} \lambda_i + \chi_{top}(A) - 1.$$

After that, Tomari shows that [168, Claim 1.3] $\lambda_1 + \chi_{top}(A) - 1 \geq 0$. The proof finish after a careful analysis of the inequality $\lambda_1 + \chi_{top}(A) - 1 \geq 0$. □

From Tomari's Theorem 3.14 together with the results of Section 3.1.1, we can provide a full answer to Dimca and Greuel Question 3.13.

Theorem 3.15. *For any germ of plane curve singularity C*

$$\frac{\mu(C)}{\tau(C)} < \frac{4}{3}.$$

Proof. Let $C : f(x, y) = 0$ be a germ of isolated plane curve singularity. Let us consider the germ of double point $(X, 0)$ defined by the equation $F(x, y, z) = f(x, y) + z^2 = 0$. It is then trivial to check that the ideal of $(F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$ can be expressed as

$$\Big(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, z\Big) \subset \mathbb{C}\{x, y, z\}.$$

Then it is obvious that the Tjurina number of the double point $\tau(X)$ is equal to the Tjurina number $\tau(C)$ of the germ of plane curve defined by $f(x, y) = 0$. The same argument works to prove that $\mu(X) = \mu(C)$.

Let $p_g(X)$ be the geometric genus of the double point X. From Tomari's Theorem 3.14 we know that $p_g(X) < \mu(X)/8$. Combining this with Wahl's Theorem 3.10 gives

$$\mu(C) - \tau(C) = \mu(X) - \tau(X) \leq 2p_g(X) < \mu(X)/4 \Rightarrow \frac{\mu(C)}{\tau(C)} < \frac{4}{3}. \qquad \square$$

In this way, we can conclude that the bound $4/3$ for the quotient μ/τ of plane curve singularities is inferred from the rich properties of the geometric genus of the corresponding normal two-dimensional double point singularity. More concretely the restriction of $4/3$ is due to the restrictions for the adjunction conditions of a normal two-dimensional double point singularity.

3.2.1 Curves with the semigroup of a plane branch

Let $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle = \{z \in \mathbb{N} |\ z = l_0\overline{\beta}_0 + l_1\overline{\beta}_1 + \cdots + l_g\overline{\beta}_g\}$ be a numerical semigroup. Let us denote by $n_i := \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1}) / \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_i)$. Assume that Γ satisfies the following conditions:

(1) $n_i\overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$,

(2) $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$ for all $i = 1, \ldots, g$.

Recall that the semigroup of a plane branch always satisfies those conditions (see section 1.1.2). Moreover, Bresinsky [23, Theorem 2] and Teissier [165, Chap I. 3.2] showed that conditions (1) and (2) completely characterize the semigroup of values of a plane branch, i.e. Γ is the semigroup of a plane branch if and only if Γ satisfies conditions (1) and (2).

Theorem 3.16. *Let $\Gamma = \langle \overline{\beta}_0, \dots, \overline{\beta}_g \rangle$ be a numerical semigroup satisfying conditions (1) and (2). Then, there exist a plane branch $(C, \boldsymbol{0}) \subset (\mathbb{C}^2, \boldsymbol{0})$ such that $\Gamma = \Gamma(C)$.*

Proof. See [165, Chap. I Prop 3.2.1] or [23, Theorem 2] □

For this reason, a semigroup satisfying conditions (1) and (2) is called *semigroup of a plane branch*.

Let us now come back to the setting of Section 2.2. We consider the semigroup Γ of a plane branch and its monomial curve C^{Γ}. Teissier [165] and Cassou-Nogués [32] results show that there exist curves that are not plane curves in the miniversal deformation of $(C^{\Gamma}, \mathbf{0})$, even more there are curves which are not plane in the miniversal semigroup constant deformation of the monomial curve. Coming back to Chapter 2 Section 2.2.1, we recall that $V_0 \subset D_{\Gamma}$ is the set of points such that if $v \in V_0$ then $\tau(C_v) = \tau_{\min}$ assumes the minimal value between all possible values of $\tau(C_v)$ with $v \in D_{\Gamma}$. As we have seen in Theorem 2.15, V_0 contains a Zariski open dense set $Z \subset V_0$ which is non-empty. Moreover, by Peraire [131, Theorem 7.2] there exists $v \in Z$ such that the germ $(G_{\Gamma}^{-1}(\boldsymbol{v}), \sigma_{\Gamma}(v))$ is an irreducible plane curve singularity with $\tau = \tau_{\min}$. From this, we have the following corollary of Theorem 3.15 which gives a partial answer to Problem 3.1 in the case $r = N - 1$ with arbitrary N.

Corollary 3.17. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^N, \mathbf{0})$ be an irreducible space curve with semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$ of an irreducible plane curve singularity. Then,*

$$\mu - \tau < \frac{\mu}{4}$$

Proof. Since $(C, \mathbf{0}) \subset (\mathbb{C}^N, \mathbf{0})$ is an irreducible germ of curve with semigroup Γ of an irreducible plane curve singularity then $(C, \mathbf{0})$ is analytically isomorphic to the generic fiber of a one parameter complex analytic deformation of $(C^{\Gamma}, \mathbf{0})$ by Theorem 2.11. Let $v \in D_{\Gamma}$ be such that $(C, \mathbf{0})$ is analytically isomorphic to $(G_{\Gamma}^{-1}(v), \sigma_{\Gamma}(v))$.

By [165, Chap.I, 2.11.2] the fibers of the miniversal constant semigroup deformation of the monomial curve C^{Γ} are also $\delta(\Gamma) = |\mathbb{N} \setminus \Gamma|$–constant. Thus, $\delta(C) = |\mathbb{N} \setminus \Gamma|$. Since C is an irreducible germ of curve singularity, by [28, Proposition 1.2.1] $\mu = 2\delta$. This means that

$$\mu - \tau \leq \mu - \tau_{\min} < \frac{\mu}{4},$$

where the last inequality is coming from Theorem 3.15 and the fact that there exists an irreducible plane curve singularity with semigroup Γ, $\tau = \tau_{\min}$ [131] and $\mu = 2\delta$. □

Remark 3.18. Observe that if Question 2.13 has a negative answer then the previous corollary do not give us a sharp upper bound.

3.3 Durfee conjecture

Following the ideas of the solution of Dimca and Greuel question 3.13, we are going to continue by exploring how far the general strategy of finding optimal upper bounds for the geometric genus is useful for providing solutions to Problem 3.1.

In the context of $(X,0) \subset (\mathbb{C}^3,0)$ being an isolated hypersurface singularity, there exist the following long standing open conjecture due to Durfee in [44]:

Conjecture 3.19 (Durfee 1978, [44]). For any isolated surface singularity $(X,0) \subset (\mathbb{C}^3,0)$

$$6p_g \leq \mu.$$

Remark 3.20. The original statement of Durfee conjecture [44, Conjecture 5.3] refers to this inequality for any germ of isolated complete intersection singularity of any dimension. A counterexample to this general statement was provided by Kerner and Némethi in [85, 86]. However, in this more restricted form it is still an open conjecture.

Durfee conjecture 3.19 was stated by Durfee in [44] as somehow a natural question regarding the intersection form of the Milnor number. Recall that as explained in Chapter 1 section 1.3.1, the Milnor fiber F of a surface singularity in $\mathbb{C}^3$ is an oriented real connected 4–manifold with boundary ∂F. For any oriented real connected 4–manifold there is a symmetric bilinear intersection form on $H_2(F;\mathbb{R})$ that can be diagonalized with diagonal entries ± 1 and 0. Hence, one can define the signature of F as $\sigma(F)$ the signature of the intersection form on $H_2(F;\mathbb{R})$. Let us denote by (μ_+,μ_0,μ_-) the Sylvester invariants of the intersection form, i.e. the number of positive, zero and negative entries of the diagonalized matrix. Then the signature $\sigma(F) = \mu_+ - \mu_-$ and $\mu = \mu_+ + \mu_0 + \mu_-$. [5]

Remark 3.21. We refer to Dimca's book [45, Chapter 3 and Appendix A] for more details about the structure of $H_n(F;\mathbb{R})$ and its intersection form.

In his paper [44], Durfee states another conjecture, sometimes named as Durfee weak conjecture, that ask for the negativity of the signature $\sigma(F)$ [44, Conjecture 5.2]. The name weak comes from the following fact:

Proposition 3.22. *[44, Prop. 3.1]* $2p_g = \mu_+ + \mu_0$.

From this result, one can see that Durfee conjecture 3.19 implies $\sigma \leq -(2p_g + \mu_0)$ and since $2p_g + \mu_0 \geq 0$ this means that Durfee conjecture 3.19 implies that the signature of the Milnor fiber is negative.[6] Also, we would like to point out that weak conjecture is not true for general normal surfaces as it was showed by Wahl [178, (5.9.3) pg. 240]. However, it is true for any hypersurface singularity as Kóllar and Némethi prove in [90, Theorem 3]. Moreover, Durfee weak conjecture has been recently proven by Enokizono [55] for isolated complete intersection singularities.

In this spirit, Durfee conjecture has been object of an extensive study and a strong and prolific research area. Before to continue let's briefly sketch the state of the art of Durfee conjecture 3.19. In the early 90s, some especial cases were proven by different mathematicians: for $(X,0)$ of multiplicity

[5] Observe that since $H_2(F;\mathbb{R})$ is a real vector space, the triple (μ_+,μ_0,μ_-), and hence the signature, is invariant under base change on the vector space $H_2(F;\mathbb{R})$. This a consequence of the classical theorem of Sylvester also known as Sylvester's inertia law [162].

[6] Here one should say that in the original statement of the weak conjecture [44, Conjecture 5.2] there is a typo on the printed version since it is stated $\sigma \geq 0$, however the discussion after the statements support $\sigma \leq 0$. This fact is already pointed out in the Mathscinet review of Durfee's paper by Laufer .

2 Tomari's Theorem 3.14 proves a stronger inequality $8p_g < \mu$, for multiplicity 3 Ashikaga [17] proves the inequality $6p_g \leq \mu - 2$, for quasi-homogeneous singularities Xu and Yau [186] prove the inequality $6p_g \leq \mu - \text{mult}(X,0) + 1$. At the end of the 90s, the inequality $6p_g \leq \mu$ is proven for the following families of surface singularities: Némethi [123],[124] for suspension type singularities $\{g(x,y) + z^k = 0\}$ and Melle-Hernández [116] for absolutely isolated singularities. In 2017, Kerner and Némethi [87] showed that Durfee conjecture is true for Newton non-degenerate singularity with large enough Newton boundary. Recently, Kóllar and Némethi prove in [90] that Durfee conjecture 3.19 is true if the link of the isolated hypersurface singularity is an integral homology sphere. Moreover, in a recent preprint [54] Enokizono show that Durfee conjecture 3.19 is true whenever the topological Euler characteristic of the exceptional divisor of the minimal resolution is positive.

However, for the isolated complete intersection non-hypersurface case Kerner and Némethi show in 2012 that the inequality $6p_g \leq \mu$ is no longer true [85]. They propose and they study a more general refined conjecture in a series of papers [85, 86, 87]:

Conjecture 3.23 (Kerner–Némethi). [85, 86, 87] Let $(X,0) \subset (\mathbb{C}^N,0)$ be an isolated complete intersection singularity of dimension n and codimension $r = N - n$. Then,

1. for $n = 2$ and $r = 1$ one has $\mu \geq 6p_g$.
2. for $n = 2$ and arbitrary r one has $\mu > 4p_g$.
3. for $n \geq 3$ and fixed r one has $\mu \geq C_{n,r}p_g$ where $C_{n,r}$ is defined by

$$C_{n,r} := \frac{\binom{n+r-1}{n}(n+r)!}{\left\{ {n+r \atop r} \right\} r!}$$

Moreover, they show that those bounds are sharp.

Remark 3.24. In fact, the bound for the hypersurface case, i.e. $(n+1)!p_g < \mu$ was already conjectured by K. Saito in 1983 [146, Section 2 (iv), pg. 203] (See also Chapter 4).

Before continuing, let us introduce the following remarkable family of surface singularities.

Definition 3.25. A surface singularity defined by the germ of function $f : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ with $f = f_d + f_{d+1} + \cdots$ (where f_j is homogeneous of degree j) is called superisolated if the projective plane curve $C_d := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities $\{p_i\}_i$ and these points are not situated on the projective curve $\{f_{d+1} = 0\}$.

Superisolated surface singularities were first introduced by Luengo in [110] to show that the μ–constant stratum in the miniversal deformation space of an isolated surface singularity, in general, is not smooth. Moreover, they are usually used to provide counterexamples to some conjectures in singularity theory. The following example show a superisolated singularity which does not fulfil the bound 4/3.

Example 3.26. Let us consider the superisolated surface singularity

$$f = x^{14} + y^6z^8 + z^{14} + x^9z^5 + (x+y+z)^{15}.$$

We can compute with SINGULAR [35] that the Milnor number is $\mu = 2288$ and the Tjurina number is $\tau = 1660$. Therefore, $\mu/\tau > 4/3$.

In this way, Theorem 3.15 is not true in general for surface singularities.

However, it is well known (see [111]) that the geometric genus of a superisolated singularity can be expressed in terms of the degree d of the projective curve C_d. Also the Milnor number depends on this degree and of the local Milnor numbers of the singularities $\{p_i\}_i$ of C_d.

Example 3.27. Let the germ of function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with $f = f_d + f_{d+1} + \cdots$ be a superisolated singularity. Let us denote by μ_i the local Milnor numbers of the singular points $\{p_i\}_i$ of the projective plane curve $C_d := \{f_d = 0\}$. Then we have (see [111]) that

$$p_g = d(d-1)(d-2)/6, \quad \mu = (d-1)^3 + \sum_i \mu_i.$$

Therefore, it is easy to check that

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

Also, in Wahl's paper [179] it is given the following example which allow us to show that asymptotically μ/τ tends to $3/2$ for surface singularities:

Example 3.28. Let us consider $F(x, y, z) = x^d + y^d + z^d + g(x, y, z) = 0$ with $deg(g) \geq d + 1$. Then Example 4.7 in [179] shows that $\tau_{min} = (2d-3)(d+1)(d-1)/3$. Here the minimal Tjurina number is defined as the minimal value among any Tjurina number of a positive weight deformation with fixed initial term $x^d + y^d + z^d$.

After that, it is easy to see that asymptotically

$$\frac{\mu}{\tau_{\min}} \xrightarrow[d \to \infty]{} \frac{3}{2}.$$

Therefore, we are under the conditions of Problem 3.1. In fact, the cases where Durfee conjecture 3.19 is known to be true allow us to proof the following result. Before to state the result, let us give first the following definitions.

Definition 3.29. An *absolutely isolated surface singularity* is a surface singularity which can be resolved after a finite number of point blowing ups.

Definition 3.30. Recall that the link K of an isolated hypersurface singularity is diffeomorphic to the boundary of the Milnor fiber. We say that the *link is an integral homology sphere* if $H_1(K; \mathbb{Z}) = 0$.

Proposition 3.31. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity of one of the following types:*

(1) Quasi-homogeneous singularities,

(2) $(X, 0)$ of multiplicity 3,

(3) absolutely isolated singularity,

(4) suspension of the type $\{f(x, y) + z^N = 0\}$,

(5) the link of the singularity is an integral homology sphere,

(6) the topological Euler characteristic of the exceptional divisor of the minimal resolution is positive.

Then

$$\frac{\mu}{\tau} < \frac{3}{2}$$

Proof. For Quasi-homogeneous singularities $\mu = \tau$ by [179]. For the cases (2), (3), (4), (5), (6) Durfee conjecture is true by [17, 116, 123, 124, 90, 54].

Therefore by Theorem 3.10 we have that for these families

$$\frac{\mu}{\tau} < \frac{\mu}{\mu - 2p_g} < \frac{3}{2}. \qquad \square$$

Finally, since Durfee inequality is believed to be true for hypersurface singularities, as one can see from the huge number of families for which this inequality holds, the previous discussion allows us to propose the following conjecture:

Conjecture 3.32. For any $(X,0) \subset (\mathbb{C}^3,0)$ isolated surface singularity:

$$\frac{\mu}{\tau} < \frac{3}{2}.$$

Proposition 3.33. *Durfee's conjecture implies Conjecture 3.32.*

Proof. Assume Durfee's conjecture is true, $6p_g < \mu$. From Wahl's Theorem 3.10, we have the following bound $\mu - \tau < 2p_g < \mu/3$. Then $\mu/\tau < 3/2$. $\square$

Despite Durfee conjecture 3.19 is believed to be true and it is strongly supported, in general it is more difficult to compute the geometric genus of a family of surface singularities than the Milnor and Tjurina numbers. For this reason, Conjecture 3.32 provides a good tool to check the validity of Durfee Conjecture 3.19 in the most complicated cases.

Chapter 4

Limit distribution of exponents

In 1983, K. Saito [146] pointed out a very beautiful problem concerning the distribution of the exponents. His problem is based on the following observation about the distribution of exponents of a Brieskorn–Pham singularity (see [146, (2.2)]). Recall that a Brieskorn-Pham singularity is defined by an equation of the form

$$f(z_0, \ldots, z_n) = z_0^{d_0} + \cdots + z_n^{d_n}.$$

Since f is a joint of monomials, the Thom-Sebastiani property allows to easily compute the set of exponents as

$$\left\{ \frac{j_0}{d_0} + \cdots + \frac{j_n}{d_n} \,:\, 1 \leq j_i \leq d_i - 1,\ i = 0, \ldots, n \right\}.$$

It is also common to express the set of exponents as a generating series, i.e. a polynomial whose non-zero terms are t^α with α being an exponent of f. We denote by $\mathrm{Sp}_f(t)$ to such polynomial. From the particular expression of the exponents of a Brieskorn-Pham singularity, it is easy to factorize $\mathrm{Sp}_f(t)$ as

$$\mathrm{Sp}_f(t) = \prod_{i=0}^{n} \left(\sum_{j=1}^{d_i - 1} t^{j/d_i} \right) = \prod_{i=0}^{n} \frac{t - t^{1/d_i}}{t^{1/d_i} - 1}$$

Following K.Saito [146, (2.2.1)], since the Milnor number of a Brieskorn-Pham singularity is $\mu = \prod_{i=0}^{n}(d_i - 1)$, the distribution of exponents is given by the following n–dimensional integral representation

$$\int_{x_0 + \cdots + x_n = s} \prod_{i=0}^{n} \left(\frac{1}{d_i - 1} \sum_{j=1}^{d_i - 1} \delta\left(x_i - \frac{j}{d_i} \right) dx_i \right), \tag{4.1}$$

where $\delta(s)$ is Dirac's delta function. Moreover, such an integral representation is precisely the Fourier transform representation of $\mathrm{Sp}_f(t)/\mu$.

In a full general context, let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with isolated critical point 0 and Milnor number μ. Let us denote by $\alpha_1, \ldots, \alpha_\mu$ the exponents of f. K. Saito considered the normalized spectrum of f,

$$\chi_f(t) := \frac{\mathrm{Sp}_f(T)}{\mu} = \frac{1}{\mu} \sum_{j=1}^{\mu} T^{\alpha_j}, \quad T = \exp(2\pi i t),$$

as the Fourier transform of the discrete probability density on the interval $(0, n+1)$,

$$\frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds.$$

In analogy to the distribution of exponents of a Brieskorn-Pham singularity (Equation (4.1)), he defines a continuous limit probability distribution N_{n+1} as

$$N_{n+1}(s)ds := \int_{x_0+\cdots+x_n=s}\varphi(x_0)\cdots\varphi(x_n)dx_0\cdots dx_n,$$

where φ is the indicator function of the unit interval $[0,1]$, i.e. $\varphi(x) := \begin{cases}1 & \text{if } x\in[0,1],\\ 0 & \text{if } x\notin[0,1].\end{cases}$

Under the Fourier transform $\mathcal{F}$, N_{n+1} corresponds to the power

$$\mathcal{F}(N_{n+1}) = \mathcal{F}(\varphi)^{n+1}. \tag{4.2}$$

From this point of view, K. Saito [146, (2.5)] proposed the following

Problem 4.1. Let $\alpha_1,\ldots,\alpha_\mu$ be the exponents of an isolated hypersurface singularity. To show whether the continuous distribution $N_{n+1}ds$ is the "limit"of the distribution of the exponents as f "moves". Equivalently, this means to find a suitable formulation of the following

$$\lim \chi_f = \mathcal{F}(N_{n+1}),\ \text{or equivalently,}\ \lim\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds = N_{n+1}ds$$

Moreover, he was able to prove the following

Theorem 4.2. *[146, (3.7), (3.9)] With the previous notations*

(a) For quasihomogeneous f of degree 1 with respect to weights $w_0,\ldots,w_n$,

$$\lim_{w_0,\ldots,w_n\to 0}\chi_f = \mathcal{F}(N_{n+1})$$

.

(b) For irreducible plane curve singularities f with Newton pairs $(n_1,l_1),\ldots,(n_g,l_g)$,

$$\lim_{n_g\to\infty}\chi_f = \mathcal{F}(N_2).$$

This Theorem together with the fact (see [146, (3.2.1)]) that the join $f+g$ of two functions in disjoint sets of variables satisfies

$$\chi_{f+g} = \chi_f\cdot\chi_g$$

provides several examples for which the limit distribution proposed in Problem 4.1 can be properly defined.

In Section 4.4 we will stablish K. Saito's limit distribution for the case of Newton non-degenerate hypersurface singularities (Theorem 4.20), this constitute quite a natural generalization of the partial results obtained by K. Saito. Moreover, thanks to the identity $\chi_{f+g} = \chi_f\cdot\chi_g$, Theorem 4.20 provides several new examples for which K. Saito's limit distribution can be established.

K. Saito [146, (2.5) ii), (2.8) i)] further suggested to describe up to what extent the spectral distribution is bounded by N_{n+1} and introduced the notion of *(weakly) dominating values*. Consider the function

$$\varPhi\colon [0,1] \to \mathbb{R}, \quad r \mapsto \int_0^r N_{n+1}(s) - \frac{1}{\mu}\sum_{i=1}^{\mu} \delta(s-\alpha_i)ds$$

defined by the difference of the continuous and discrete spectral distributions. By definition $0 < r < \frac{n+1}{2}$ is a *dominating value* if $\varPhi(r) > 0$ for all f in $n+1$ variables. A *weakly dominating value* is defined by replacing $<$ by $\leq$ and $\int_0^r$ by $\int_0^{r-\epsilon}$ for all $\epsilon > 0$. A special emphasis is made in the case of $r = 1$. More concretely, K. Saito asked

Question 4.3. [146, (2.8) iv)] Is 1 a dominating value for all $n \geq 2$?

K. Saito [146, (2.9)] showed that the values of the function

$$N(k,r) := k! \int_{r-1}^{r} N_k(s)ds$$

can be computed recursively by using that $N(k,0) = N(k,k+1) = 0$, $N(1,1) = 1$ and

$$N(k+1,r+1) = (k-r+1)N(k,r) + (r+1)N(k,r+1).$$

In particular, $N(k,r)$ behaves like the Pascal's triangle and we have $N(n+1,1) = (n+1)!$

On the other hand, the following result of M. Saito shows that the geometric genus of a germ of isolated hypersurface singularity can be computed from the set of exponents less or equal than one.

Theorem 4.4. *[147] Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of holomorphic function with isolated singularity at 0. Assume that $n \geq 2$. Then,*

$$p_g = |\{\alpha \text{ is an exponent } \leq 1\}|.$$

All this together shows that in the particular case $r = 1$ the value of $\varPhi_f(1)$ have a quite nice form

$$\varPhi_f(1) = \int_0^1 N_{n+1}(s) - \frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds = \frac{1}{(n+1)!} - \frac{p_g}{\mu}.$$

Thus, Question 4.3 provides the following generalization of Durfee conjecture.

Question 4.5. For a function f in $n+1$ variables, is the geometric genus bounded by

$$p_g < \frac{\mu}{(n+1)!}?$$

Our interest in this problem is obviously to find a good framework to attack Problem 3.1 in the hypersurface case. We will see in Section 4.1 how one can proceed in order to try to find a solution to Problem 3.1 by studying a certain filtration in the Brieskorn lattice H_0''. From this point of view, we can see $\mu - \tau$ and p_g as dimensions of certain spaces. Thus, it would be natural to compare their dimension as Question 4.11 propose. Therefore, the combination of an answer to Question 4.11 together with K. Saito's Problem 4.1 would provide a solution to our Problem 3.1 in the hypersurface case. Moreover, this strategy would link the main invariants of an isolated hypersurface singularity in a single problem.

This Chapter also contains the results obtained by the author in a joint work with M. Schulze [11]. Those results will be explained in Section 4.3 and Section 4.4. They provide some insight in K. Saito's questions about the distribution of the exponents. In Section 4.3 we will study the dominance of certain specific values for irreducible plane curve singularities. In particular we will see that the log-canonical threshold of an irreducible plane curve singularity is a dominating value (Theorem 4.18). Also, we will show that in the case of irreducible plane curve singularities with a single Puiseux pair, there is always a non-dominating value between the values of the exponents of the curve (Proposition 4.16, see also Example 4.17). In particular, this show how difficult the study of dominating value could be.

4.1 Exponents and the difference $\mu - \tau$

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of isolated hypersurface singularity. Recall that we defined the Brieskorn lattice as the $\mathbb{C}\{t\}$–module

$$H_0'' := \frac{\Omega^{n+1}_{(\mathbb{C}^{n+1},0)}}{df \wedge d\Omega^{n-1}_{(\mathbb{C}^{n+1},0)}}.$$

As we mentioned in Chapter 1, subsection 1.3.3, H_0'' is also a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$–module of rank μ, the Milnor number of the singularity. Following [148, Section 1], let us denote by $\mathcal{G}$ the localization of H_0'' by the action of ∂_t^{-1}. The localization $\mathcal{G}$ is isomorphic to the so called Gauss-Manin system as showed by M. Saito in [150, Section 2]. In particular, it is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$–module of rank μ and moreover, it is a regular holonomic $\mathcal{D}_{(\mathbb{C},0)}$–module.

If we now denote by ∂_t to the inverse of ∂_t^{-1}, then there are natural actions of t and ∂_t on $\mathcal{G}$ satisfying the relation $[\partial_t, t] = 1$. For $\alpha \in \mathbb{Q}$, we set

$$H^\alpha := \{u \in \mathcal{G} \mid (\partial_t t - \alpha)^m u = 0 \text{ for } m \gg 0\}.$$

Remark 4.6. The existence of such an $m \gg 0$ is guaranteed by the Monodromy Theorem. More concretely, we will see in a moment that there exists an isomorphism between H^α and the generalized eigenspaces $H^n(X_\infty, \mathbb{C})_\lambda$ associated to a monodromy eigenvalue λ. Thus, by Monodromy Theorem 1.46 and M. Saito's theorem 1.59, one is able to take $m = n + 1$ as a value satisfying $(\partial_t t - \alpha)^m u = 0$.

The spaces H^α allow to define a decreasing filtration $V^\bullet$ on $\mathcal{G}$ satisfying $\bigcap_\alpha V^\alpha \mathcal{G} = \{0\}$ and $\bigcup_\alpha V^\alpha \mathcal{G} = \mathcal{G}$. The filtration $V^\bullet$ is defined as

$$V^\alpha \mathcal{G} := \sum_{\beta \geq \alpha} \mathbb{C}\{\{\partial_t^{-1}\}\} H^\beta \subset \mathcal{G}.$$

Remark 4.7. The filtration V is precisely the V–filtration characterized by Kashiwara [84] and Malgrange [113] (see also Kulikov [93, Chap II. Section 6.3]). This filtration plays an important role in the understanding of the $\mathcal{D}_{(\mathbb{C},0)}$–module structure of $\mathcal{G}$.

If we denote by

$$V^{>\alpha}\mathcal{G} := \sum_{\beta>\alpha} \mathbb{C}\{\{\partial_t^{-1}\}\}H^\beta \subset \mathcal{G},$$

then, $\operatorname{Gr}_V^\alpha \mathcal{H} = V^\alpha\mathcal{H}/V^{>\alpha}\mathcal{H}$ and $H^\alpha \cong \operatorname{Gr}_V^\alpha \mathcal{H}$, since there is an isomorphism $\partial_t^p : H^{\beta+p} \to H^\beta$ for $p \in \mathbb{Z}$ and $\beta \in (0,1] \cap \mathbb{Q}$.

Remark 4.8. In particular, one can see that after Malgrange [113, Lemme 4.5] we have $H_0'' \subset V^{>-1}$ (see [93, Chap. II (8.3.7)]).

The inclusion of $H_0'' \hookrightarrow \mathcal{G}$ allows to define a decreasing $V^\bullet$ filtration on H_0'' just by setting $V^\alpha H_0'' := H_0'' \cap V^\alpha\mathcal{G}$. Thus, if we denote by

$$\operatorname{Gr}_V^\alpha H_0'' = \frac{V^\alpha H_0''}{V^{>\alpha} H_0''}$$

then it is possible to define a Hodge filtration on H^β as

$$F_p H^\beta = \partial_t^p \operatorname{Gr}_V^{\beta+p} H_0'' \quad \text{for } p \in \mathbb{Z} \quad \text{and } \beta \in (0,1] \cap \mathbb{Q}.$$

From the Hodge filtration $F_\bullet$, the set of exponents can be defined (see [148, Definition 1.3]) as those $\{\alpha_1, \ldots, \alpha_\mu\}$ with

$$|\{i : \ \alpha_i = \beta + p\}| = \dim \frac{F_p H^\beta}{F_{p-1} H^\beta}.$$

It is far from being obvious that the definition of exponents provided in Chapter 1 is equivalent to this definition. However, the main idea behind this fact is the announced identification between the spaces H^α and the eigenspaces associated to the monodromy on the Milnor fiber. We will now sketch the main arguments in this direction and we refer to [93, Chap. II Section 8] for a detailed explanation.

Following Hertiling and Stahlke [78, Sec.2], let us consider the universal covering of the punctured disk $u : D_\infty \to D^*$. For each $t \in D_\infty$, the covering map u induces an homotopy equivalence, via the inclusion map $X_{u(t)} \hookrightarrow X_\infty$, between each fiber $X_{u(t)}$ and the fiber product $X_\infty := X' \times_{D^*} D_\infty$. This means that we have an isomorphism

$$\mathbb{Z}^\mu = H^n(X_{u(t)}, \mathbb{Z}) \cong H^n(X_\infty, \mathbb{Z})$$

which give the usual name of *Milnor lattice* to $H^n(X_\infty, \mathbb{Z})$. Let us consider the generalized eigenspace with eigenvalue λ as

$$H^n(X_\infty, \mathbb{C})_\lambda = \operatorname{Ker}(M - \lambda Id)^{n+1} = \operatorname{Ker}(M_s - \lambda Id)^{n+1} \subset H^n(X_\infty, \mathbb{C}),$$

where M is the monodromy map of the Milnor fiber and M_s its semisimple part.

Then, M. Saito Theorem 1.59 translate in the fact that the decomposition associated to the monodromy eigenspace pieces is compatible with the particular choice of a basis of the Brieskorn lattice as $\mathbb{C}\{\{\partial_t^{-1}\}[\partial_t]\}$–module [149, pg. 28–29]. More concretely, let $F^\bullet$ be the Hodge filtration associated to the Mixed Hodge Structure defined in $H^n(X_\infty, \mathbb{C})_\lambda$ by Steenbrink [160] (See also [93, Chap. II] for a detailed study of those structures). Let $\alpha \in \mathbb{Q}$ be such that $\lambda = \exp(-2\pi i\alpha)$ is an eigenvalue of the monodromy. Then, the map

$$\begin{array}{rcl}\varphi_\alpha : H^n(X_\infty, \mathbb{C})_\lambda \to & & \mathcal{G}_0 \\ A & \mapsto & s(A,\alpha)_0 = t^{\alpha-1}\exp(-N\log t)A(s,\alpha)\end{array}$$

is injective and provides an isomorphism $H^n(X_\infty, \mathbb{C})_\lambda \simeq H^\alpha$ such that the Hodge filtrations satisfy $\{F^p = F_{n-p}\}$.

The previous discussion leads to the following

Theorem 4.9. *[147, 170] Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of holomorphic function with isolated singularity at 0. Assume that $n \geq 2$. Then,*

$$p_g = |\{\alpha \text{ is a spectral number } \leq 1\}| = \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}.$$

Now we are ready to state our main motivation to the study of K. Saito's limit distribution. Recall that we showed in Chapter 1 Section 1.3.3 that the Milnor number of an isolated hypersurface can be computed as

$$\dim_{\mathbb{C}} \frac{H_0''}{\partial_t^{-1} H_0''} = \mu$$

On the other hand, we also mentioned that the action of t in H_0'' is defined as $t[w] := f[w]$ for $[w] \in H_0''$, i.e. it corresponds to multiplication by f in the Milnor algebra Ω_f. Thus, the Tjurina number coincides with the dimension of the cokernel of the action of t in H_0''

$$\dim_{\mathbb{C}} \frac{H_0''}{tH_0'' + \partial_t^{-1} H_0''} = \tau.$$

Therefore, we can obtain the difference $\mu - \tau$ as the dimension of the following quotient

$$\mu - \tau = \dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''}.$$

In particular, we can rewrite Wahl's Theorem 3.10 as follows.

Theorem 4.10. *Let $(X, 0) \in (\mathbb{C}^3, 0)$ be an isolated surface singularity defined by an equation $f \in \mathcal{O}_{\mathbb{C}^3,0}$. Then,*

$$\mu - \tau = \dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''} \leq 2 \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}$$

From this point of view, the strategy is now clear. A first step in a solution of Problem 3.1 in the hypersurface case would be to provide an answer to the following

Question 4.11. Let $(X, 0) \in (\mathbb{C}^{n+1}, 0)$ be an isolated hypersurface singularity with $n \geq 2$. Is there $C(n) \in \mathbb{Q}$ such that

$$\dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''} \leq C(n) \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}?$$

Observe that, by Wahl's Theorem 3.10, the case $n = 2$ gives $C(n) = 2$ and, even more, the bound is sharp. On the other hand, for $n = 1$ our solution to Dimca and Greuel's Question 3.13, Theorem

3.15, implies that $C(1) = 1/2$. As far as the author's knowledge, Question 4.11 is open for $n \geq 3$. So, let us assume that we have found an optimal $C(n)$ in the general case. By M. Saito's Theorem 4.9 we have

$$\int_0^1 \frac{1}{\mu} \sum_{i=1}^{\mu} \delta(s - \alpha_i) ds = p_g = \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}}.$$

Therefore, $\Phi_f(1) > 0$ translates into the inequality $p_g < \mu/((n+1)!)$; thus providing a generalization of Durfee's conjecture as pointed out by K. Saito [146, pg. 203]. That generalization together with a solution to Question 4.11 implies the following sequence of inequalities

$$\mu - \tau = \dim_{\mathbb{C}} \frac{tH_0'' + \partial_t^{-1} H_0''}{\partial_t^{-1} H_0''} \leq C(n) \dim_{\mathbb{C}} \frac{H_0''}{H_0'' \cap V_{>1}} = C(n) p_g \leq \frac{C(n)}{(n+1)!} \mu. \tag{4.3}$$

Equation (4.3) provides in a single line of inequalities a relation between the main analytic and topological invariants of an isolated hypersurface singularity. Moreover, it would show how the topology, i.e. the Milnor number and the embedding dimension, constrain the Tjurina number, the distribution of spectral values and the geometric genus, which are the main analytic invariants of a hypersurface singularity.

4.2 Spectrum of non-degenerate isolated hypersurface singularities

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a *Newton non-degenerate* isolated hypersurface singularity. This means that there are local coordinates $z_0, \ldots, z_n$ such that

$$f = f(z_0, \ldots, z_n) \in \mathbb{C}\{z_0, \ldots, z_n\} = \mathcal{O}_{\mathbb{C}^{n+1},0} =: \mathcal{O}$$

is a *Newton non-degenerate commode* power series (see Chapter 1 Section 1.4). We recall that $N(f)$ denote the Newton diagram of f. We write $\sigma \in N(f)$ to indicate that σ is a face of $N(f)$. For $\sigma, \tau \in N(f)$, we write $\tau \leq \sigma$ if τ is a face of σ. By g_σ we denote the polynomial obtained from the power series $g \in \mathcal{O}$ by restricting the monomial support to the cone of σ.

Remark 4.12. From now on we will always assume that f is a commode power series. Since we are working with isolated isolated singularities, this is not a restriction at all because of finite determinacy (see for example [71, Chap. I Sec. 2.2]). In this case, finite determinacy implies that for sufficiently large $L_0, \ldots, L_n$, the function f is analytically isomorphic to $f + \sum_{i=0}^{n} z_i^{L_i}$. Moreover, an immediate consequence is that the Newton fan is rational.

As we pointed out in Chapter 1 Section 1.4, Newton non degenerate singularities emerged as singularities possessing certain good properties with respect to a filtration related to its Newton polygon. Since $N(f)$ is a polytope satisfying that each ray through the origin of $\mathbb{R}_+^{n+1}$ meet $N(f)$ in just one point and that the region lying above $N(f)$ is convex, for each facet σ there is a uniquely defined linear function λ_σ such that $\lambda_\sigma(\sigma) = 1$. Moreover, the family $\{\lambda_\sigma\}$ defines $N(f)$. Thus, associated to $N(f)$ there is a piecewise linear homogeneous function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree 1 such that $h(\alpha u) = \alpha h(u)$ and $h(N(f)) = 1$. Following Kouchnirenko [92, 2.1], the function h allows to define a decreasing filtration $\mathcal{N}$ on $\mathcal{O}$ as

$$\mathcal{N}_\alpha \mathcal{O} := \{g \in \mathcal{O} :\ h(\operatorname{supp}(g)) \geq \alpha\}.$$

Since h is a convex function (because it is defined through the facets of $N(f)$), then the filtration is well defined and we call it *Newton filtration*.

Analogously, we define $\mathcal{N}_{>\alpha}\mathcal{O} := \{g \in \mathcal{O} : \ h(\mathrm{supp}(g)) > \alpha\}$ and we define the corresponding grading as $\mathrm{Gr}_{\mathcal{N}}^{\alpha} := \mathcal{N}^{\alpha}\mathcal{O}/\mathcal{N}^{>\alpha}\mathcal{O}$. Following Steenbrink [160, (5.6)], we denote the Newton graded ring associated to $\mathcal{O}$ by

$$A := \mathrm{gr}_{\mathcal{N}}\,\mathcal{O}$$

and the principal parts of $z_0\frac{\partial f}{\partial z_0}, \ldots, z_n\frac{\partial f}{\partial z_n}$ with respect to $\mathcal{N}$ by $F_0, \ldots, F_n$. For $\sigma \in N(F)$ let A_σ be the corresponding graded subring of A and denote by

$$d(\sigma) := \dim A_\sigma = \dim \mathbb{Q}\sigma = \dim \sigma + 1$$

its dimension.

The module $\Omega_f := \Omega^{n+1}_{(\mathbb{C}^{n+1},0)}/df \wedge \Omega^{n}_{(\mathbb{C}^{n+1},0)}$ also carries a Newton filtration which is induced by the inclusion (c.f [93, Chap. II Section 8.5])

$$\begin{array}{ccc} \Omega^{n+1}_{(\mathbb{C}^{n+1},0)} = \mathcal{O}dz_0 \wedge \cdots \wedge dz_n & \xhookrightarrow{\frac{z_0\cdots z_n}{dz_0\wedge\cdots\wedge dz_n}} & \mathcal{O} \\ \downarrow & & \downarrow \\ \Omega_f & \hookrightarrow & \mathcal{O}/\langle z_0\frac{\partial f}{\partial z_0}, \ldots, z_n\frac{\partial f}{\partial z_n}\rangle. \end{array} \tag{4.4}$$

Associated to any module together with a filtration there is always a Poincaré series. The Poincaré series of Ω_f with respect to the Newton filtration is defined as follows:

$$p_{\Omega_f} := \sum g_\alpha t^\alpha \quad \text{where} \quad g_\alpha := \dim \mathrm{Gr}^{\alpha}_{\mathcal{N}}\Omega_f = \dim \frac{\mathcal{N}^{\alpha}\Omega_f}{\mathcal{N}^{>\alpha}\Omega_f}.$$

At this point, the key result is given by M. Saito [148] and Varchenko–Khovanskiĭ [173] which allows to identify the Newton filtration $\mathcal{N}$ with the V–filtration on $H_0''/\partial_t^{-1}H_0''$ induced by the V–filtration on H_0''. Recall that we also have the isomorphism $\Omega_f \cong H_0''/\partial_t^{-1}H_0''$. Thus, this identification between filtrations translate into the following theorem.

Theorem 4.13 (M. Saito, Varchenko–Khovanskiĭ). *For Newton non-degenerate f, the Poincaré series of the Newton filtered vector space Ω_f reads*

$$p_{\Omega_f}(t) = t^{\alpha_1} + \cdots + t^{\alpha_\mu} =: \mathrm{Sp}_f(t)$$

where $\alpha_1, \ldots, \alpha_\mu$ are the exponents of f.

In this way, Theorem 4.13 reduces the problem of computing the set of exponents of a Newton non-degenerate singularity to compute the Poincaré series of the module Ω_f. Based on results of Kouchnirenko [92, pp. 12–13] (and Hochster [81]) Steenbrink [160, (5.7)] gave a formula for Newton non-degenerate f decomposing $p_{H_f} = \mathrm{Sp}_f$ with respect to the faces of the Newton diagram. The inclusion (4.4) identifies

$$\Omega_f \cong z_0\cdots z_n\mathcal{O}/\langle z_0\frac{\partial f}{\partial z_0}, \ldots, z_n\frac{\partial f}{\partial z_n}\rangle \quad \text{and} \quad \mathrm{gr}_{\mathcal{N}}\,\Omega_f \cong A/\langle F_0, \ldots, F_n\rangle =: H_f. \tag{4.5}$$

For a face $\sigma \in \Gamma$ Steenbrink first writes the Poincaré series of the subspace of $A_\sigma/\langle F_{0,\sigma}, \ldots, F_{n,\sigma}\rangle$ corresponding to the interior of the cone $\mathbb{Q}_{\geq 0}\sigma$ of σ as

$$q_\sigma(t) = \sum_{\tau \leq \sigma} (-1)^{d(\sigma)-d(\tau)} (1-t)^{d(\tau)} p_{A_\sigma}(t).$$

Denote the minimal dimension of a coordinate space containing $\sigma \in \Gamma$ by

$$k(\sigma) := \min\{k \in \mathbb{Z} \mid \exists i_1, \dots, i_k \in \{0, \dots, n\} \colon \sigma \subset \mathbb{Q}e_{i_1} + \dots + \mathbb{Q}e_{i_k}\}.$$

Then Steenbrinks formula is given by

Theorem 4.14 (Steenbrink). *For Newton non-degenerate f in $n+1$ variables, the Poincaré series of H_f can be written as*

$$p_{H_f}(t) = \sum_{\sigma \in \Gamma} (-1)^{n+1-d}(\sigma)(1-t)^{k(\sigma)} p_{A_\sigma}(t) \tag{4.6}$$

$$= \sum_{\tau \leq \sigma \in \Gamma} (-1)^{n+1-d(\sigma)} (1-t)^{k(\sigma)-d(\sigma)} q_\sigma(t). \tag{4.7}$$

Example 4.15. Let us consider the family of non-degenerate hypersurface singularities of type $T_{p,q,r}$, i.e $f(x,y,z) = x^p + y^q + z^r + xyz$ with $1/p + 1/q + 1/r < 1$. Let $A = (p,0,0)$, $B = (0,q,0)$, $C = (0,0,r)$, $D = (1,1,1)$ be the points corresponding to the support of f. Then there are 13 faces in $N(f)$ which are the ones listed in the following table.

Face	$k(\sigma)$	p_{A_σ}	Face	$k(\sigma)$	p_{A_σ}
$\sigma_1 :=$ ABD	3	$\frac{1}{(1-t^{1/p})(1-t^{1/q})(1-t)}$	$\sigma_8 :=$ AB	2	$\frac{1}{(1-t^{1/p})(1-t^{1/q})}$
$\sigma_2 :=$ BCD	3	$\frac{1}{(1-t^{1/q})(1-t^{1/r})(1-t)}$	$\sigma_9 :=$ BC	2	$\frac{1}{(1-t^{1/q})(1-t^{1/r})}$
$\sigma_3 :=$ CAD	3	$\frac{1}{(1-t^{1/p})(1-t^{1/r})(1-t)}$	$\sigma_{10} :=$ CA	2	$\frac{1}{(1-t^{1/p})(1-t^{1/r})}$
$\sigma_4 :=$ AD	3	$\frac{1}{(1-t^{1/p})(1-t)}$	$\sigma_{11} :=$ A	1	$\frac{1}{(1-t^{1/p})}$
$\sigma_5 :=$ BD	3	$\frac{1}{(1-t^{1/q})(1-t)}$	$\sigma_{12} :=$ B	1	$\frac{1}{(1-t^{1/q})}$
$\sigma_6 :=$ CD	3	$\frac{1}{(1-t^{1/r})(1-t)}$	$\sigma_{13} :=$ C	1	$\frac{1}{(1-t^{1/r})}$
$\sigma_7 :=$ D	3	$\frac{1}{(1-t)}$			

Table 4.1: Poincaré series of the 13 faces of $N(f)$.

Thus, by using Steenbrink's Theorem 4.14 it is a straightforward computation to obtain the Poincaré series

$$p_{H_f}(t) = t\left(-2 + t + \frac{1-t}{1-t^{1/p}} + \frac{1-t}{1-t^{1/q}} + \frac{1-t}{1-t^{1/r}}\right)$$

Moreover, thanks to Kouchnirenko [92, Lemme 2.9] we can also compute the Milnor number as $\mu = p_{H_f}(1)$. To do so, let us first note that for any $l \in \mathbb{N}$

$$\frac{1-t}{1-t^{1/l}} = 1 + \sum_{k=1}^{l-1} t^{k/l}.$$

Then,

$$p_{H_f}(1) = -2 + 1 + 1 + p - 1 + 1 + q - 1 + 1 + r - 1 = p + q + r - 1.$$

4.3 Irreducible plane curve singularities

Let us now return for a moment to the case of irreducible plane curve singularities. Since K. Saito established the limit distribution in the case of irreducible plane curves, our goal in the section is to study the function Φ_f in a more detailed way.

Suppose first that C has a single Puiseux pair (p, q). Then f is Newton non-degenerate with Newton diagram $N(f)$ consisting of a single line segment $[(p, 0), (q, 0)]$ and defines an irreducible plane curve singularity $C = f^{-1}(0)$. The function f is semiquasihomogeneous of weighted degree 1 with respect to weights

$$w_0 = \frac{1}{p}, \quad w_1 = \frac{1}{q}, \quad d := pq, \tag{4.8}$$

on variables z_0, z_1 and can be written explicitly as

$$f(z_0, z_1) = z_0^p + z_1^q + \sum_{iq+jp>d} a_{i,j} z_0^i z_1^j.$$

As we mentioned in Chapter 1, the normalization $\tilde{C} \twoheadrightarrow C$ allows to define a discrete valuation $\nu\colon \tilde{\mathcal{O}}_C \to \mathbb{N}$, $\nu(t) = 1$ from which we defined the value semigroup $\Gamma(C) = \langle p, q\rangle$. In the case of irreducible plane curve singularities, the conductor of the value semigroup equals the Milnor number (see [28, Prop.1.2.1.1)]). In the case of irreducible plane curve singularities with one Puiseux pair the formula for the conductor given in Equation (1.2) reads as

$$\mu = c(\Gamma) = (p - 1)(q - 1). \tag{4.9}$$

On the other hand, the Gorenstein property of C is reflected by the symmetry between elements and gaps

$$\begin{aligned} S &\xrightarrow{1:1} \mathbb{Z} \setminus S, \\ a &\longmapsto \mu - 1 - a. \end{aligned} \tag{4.10}$$

The normalized valuation ν/d induces the filtration $\mathcal{O}_C$ defined by weights $w = (w_0, w_1)$ on z_0, z_1. Since C is Newton non degenerate, this filtration is precisely the Newton filtration $\mathcal{N}$. We can then factorize $\mathcal{O}_C \to \mathbb{C}\{t\}$ as

$$\begin{array}{ccc} \mathcal{O}_C & \hookrightarrow & \mathbb{C}\{t\} \\ \downarrow & & \downarrow \\ \mathcal{O}/\langle z_0 \frac{\partial f}{\partial z_0}, z_1 \frac{\partial f}{\partial z_1}\rangle & \longrightarrow & \mathbb{C}\{t\}/\langle t^d\rangle \end{array}$$

and to use (4.4) in order to obtain the Newton filtered inclusion

$$\mathcal{O}/\mathcal{N}_{1+w_0+w_1}\mathcal{O} \xrightarrow[\cong]{dz_0 \wedge dz_1} \Omega_f/\mathcal{N}_1 \Omega_f \hookrightarrow \mathbb{C}\{t\}/\langle t^d\rangle.$$

This identifies the corresponding ranges of exponents and of values in the semigroup by means of

$$\begin{aligned} \{\alpha \in \{\alpha_1, \dots, \alpha_\mu\} \mid \alpha < 1 + w_0 + w_1\} &\overset{1:1}{\longleftrightarrow} S/\langle d\rangle, \\ \alpha &\longmapsto d\alpha - p - q, \\ \tfrac{k}{d} + w_0 + w_1 &\longleftarrow k. \end{aligned} \tag{4.11}$$

The smallest exponent w_0+w_1 corresponds to $0 \in S$, and the gap $\mu-1$ of S defining the Gorenstein symmetry (4.10) corresponds to the non-exponent 1. It follows that the number of exponents less than 1 can be written explicitly as

$$|\{(i,j) \in \mathbb{N}^2 \mid \frac{i+1}{p} + \frac{j+1}{q} < 1\}| = |\mathbb{N} \setminus S| = \frac{\mu}{2}. \tag{4.12}$$

Under (4.10) the gap $1 \in \mathbb{N} \setminus S$ is the mirror of $\mu - 2 \in S$ and corresponds to the largest spectral number $1 - w_0 w_1 < 1$ by (4.11).

After these preparations we are ready to give the following

Proposition 4.16. *If $f(z_0, z_1)$ has a single Puiseux pair (p,q), then*

(a) $\Phi_f(\frac{1}{p} + \frac{1}{q}) > 0$ unless $p = 2$ and $q \in \{3, 5\}$, with $\lim_{p\to\infty} \Phi_f(\frac{1}{p} + \frac{1}{q}) = 0$,

(b) $\Phi_f(1 - \frac{1}{pq}) < 0$ with $\lim_{p\to\infty} \Phi_f(1 - \frac{1}{pq}) = 0$.

Proof. (a) Using (4.8) and (4.9) we compute

$$\begin{aligned} \mu\Phi_f(w_0 + w_1) &= \frac{\mu}{2}(w_0 + w_1)^2 - 1 = \frac{(p-1)(q-1)}{2}\left(\frac{1}{p} + \frac{1}{q}\right)^2 - 1 \\ &= \frac{(p-1)(q-1)(p+q)^2 - 2p^2q^2}{2p^2q^2} \\ &= \frac{(pq - p - q + 1)(p^2 + 2pq + q^2) - 2p^2q^2}{2p^2q^2} \\ &= \frac{2pq + p^3q - 3p^2q - p^3 + p^2 + pq^3 - 3pq^2 - q^3 + q^2}{2p^2q^2} \\ &= \frac{1}{pq} + \frac{pq - 3q - p + 1}{2q^2} + \frac{pq - 3p - q + 1}{2p^2}. \end{aligned} \tag{4.13}$$

If $p \geq 4$ and $q \geq 5$, then (4.13) is positive since

$$\begin{aligned} pq - 3q - p + 1 &= p(q-1) - 3q + 1 \geq 4q - 4 - 3q + 1 \geq q - 3 > 0, \\ pq - 3p - q + 1 &= p(q-3) - q + 1 \geq 4q - 12 - q + 1 \geq 3q - 11 > 0. \end{aligned}$$

If $p = 3$, then (4.13) becomes

$$\frac{1}{3q} - \frac{2}{2q^2} + \frac{2q-8}{18} = \frac{2q^3 - 8q^2 + 6q - 18}{18q^2},$$

which is positive if $q \geq 4$. Finally, if $p = 2$, then (4.13) becomes

$$\frac{1}{2q} - \frac{q+1}{2q^2} + \frac{q-5}{8} = \frac{q^3 - 5q^2 - 4}{8q^2},$$

which is positive if $q \geq 6$, but negative if $q \in \{3, 5\}$.

Let us now show that $\lim_{p\to\infty} \Phi_f(\frac{1}{p} + \frac{1}{q}) = 0$. First, we observe that $q = lp + s$ with $l \geq 1$ and $s \geq 0$ since $p < q$. Now,

$$\lim_{p\to\infty} \Phi_f(\frac{1}{p} + \frac{1}{q}) = \frac{\mu}{2}(w_0 + w_1)^2 - 1 = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{lp+s}\right)^2 - \frac{1}{(lp+s-1)(p-1)} = 0.$$

(b) Using (4.12) and (4.8) we compute

$$\Phi_f(1 - w_0 w_1) = \frac{1}{2}(1 - w_0 w_1)^2 - \frac{1}{2} = -\frac{1}{2d^2} = -\frac{1}{2p^2q^2} < 0,$$

which tends to 0 for $p \to \infty$. □

Example 4.17. The description of the exponents in terms of the value semigroup $\Gamma = \langle p, q\rangle$ of C provided by the identification (4.11) allows us to visualize the graph of Φ_f as a difference. The following Figure 4.1 shows the continuous distribution $\int_0^r N_1(s)ds = x^2/2$ versus the discrete distribution of exponents for $\Gamma = \langle 5, 9\rangle$.

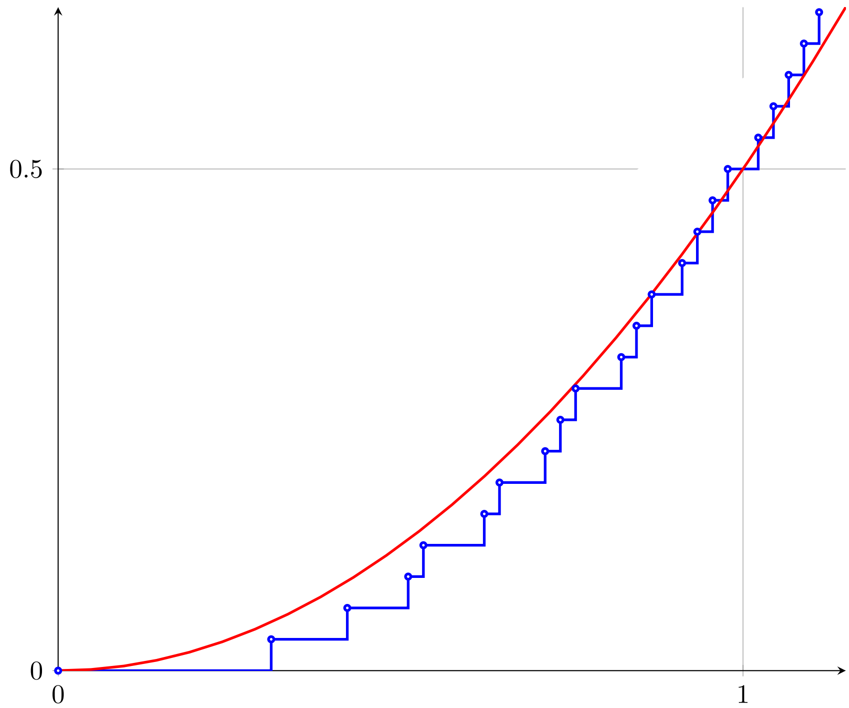


Figure 4.1: The function Φ_f as a difference for $S = \langle 5, 9\rangle$.

One can easily observe the facts given by Proposition 4.16 in terms of the relative position of both graphics in Figure 4.1, i.e. $\Phi_f(1/5+1/9) = (14/45)^2/2 - 1/32 \approx 0.017 > 0$ and $\Phi_f(1-1/45) = (44/45)^2 - 16/32 \approx -0.022 < 0$.

Interesting observations are also the following ones:

(a) $\Phi_f(1 - 1/pq)$ is not in general the first negative value of the function Φ_f. In this case, the first negative value of the function Φ_f is $\Phi_f(42/45)$.

(b) It is a general fact that the discrete distribution $\frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)ds$ is stair-shaped so it can be thought as a certain interpolation of the continuous distribution. K. Saito's limit distribution can be then interpreted as: when the number of exponents tends to infinity, then the distribution of exponents is the continuous distribution $\int_0^r N_1(s)ds$.

This example gives a taste of how difficult could be to provide answers to K. Saito's problem about the study of dominating values. Even in this easy example it is difficult to characterize the first value for which the function Φ_f is negative. It would be certainly interesting to show the sign of the function Φ_f for certain general important values of the exponents. In this direction, we will now focus on the smallest exponent.

For general irreducible plane curve singularities, Igusa [82, Thm. 1] showed that the *log canonical threshold*, which equals the smaller value of an exponent, depends only on multiplicity and the first Puiseux exponent (see also [95, Proof of Thm. 1.1]). It thus equals $\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}$ where $\overline{\beta}_0,\overline{\beta}_1$ are the two smallest minimal generators of the value semigroup. We are going to show in Theorem 4.18 that the statement of Proposition 4.16 (a) remains valid in this extended generality.

Before to prove Theorem 4.18 let us first recall the main notations about irreducible plane curve singularities introduced in Chapter 1 Section 1.1.2. Consider an irreducible plane curve singularity $C=f^{-1}(0)$ with arbitrary number g of Puiseux pairs. As we showed in Chapter 1 Section 1.1.2, the value semigroup $\Gamma(C)$ is minimally generated by g natural numbers denoted by $\overline{\beta}_0<\overline{\beta}_1<\cdots<\overline{\beta}_g$. Recall that we denoted

$$e_i := \gcd(\overline{\beta}_0,\overline{\beta}_1,\dots,\overline{\beta}_i),\quad n_i := \frac{e_{i-1}}{e_i},\quad q_i := \frac{\overline{\beta}_i}{e_i} \tag{4.14}$$

for $i=0,\dots,g$. Moreover, these greatest common divisors form a strictly decreasing sequence

$$\overline{\beta}_0 = e_0 > e_1 > \cdots > e_g = 1. \tag{4.15}$$

On the other hand, from the recursive expression of the minimal generators of the semigroup in terms of the Puiseux characteristic exponents given by Equation (1.1) one can deduce the following inequalities

$$n_{i-1}\overline{\beta}_{i-1} < \overline{\beta}_i \tag{4.16}$$

for $i=1,\dots,g$.

As we already pointed out, in the case of irreducible plane curve singularities the conductor of $\Gamma(C)$ equals the Milnor number, so we can use Equation (1.2) to obtain

$$\mu = c(\Gamma) = \sum_{i=1}^{g}\beta_i(e_{i-1}-e_i)-\beta_0+1. \tag{4.17}$$

In contrast, A'Campo showed the following alternative formula for the Milnor number in terms of the Milnor numbers μ_i associated to the Newton pairs (n_i,q_i) (see [2, Thm. 3.(ii)])

$$\mu = \sum_{i=1}^{g} e_i\mu_i,\quad \mu_i := (n_i-1)(q_i-1). \tag{4.18}$$

Finally, we can prove the main Theorem of this section.

Theorem 4.18. *For any irreducible plane curve singularity* $C = f^{-1}(0)$ *with value semigroup different from* $\langle 2,3\rangle$ *and* $\langle 2,5\rangle$*, we have* $\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}) > 0$*. In other words, the squared log canonical threshold is bounded by*

$$\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right)^2 > \frac{2}{\mu}.$$

Moreover, $\lim_{n_g\to\infty}\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}) = 0$.

Proof. The case where $g=1$ is covered by Proposition 4.16.(a).

Using (4.15) and that the sequence of Puiseux characteristic exponents is of the form $1 \le \beta_0 < \beta_1 < \cdots < \beta_g$, we find the following lower bound for the Milnor number

$$\begin{aligned}\mu &= -\beta_g e_g + \sum_{i=1}^{g-1}(\beta_{i+1}-\beta_i)e_i + \beta_1 e_0 - \beta_0 + 1 \\ &\geq -\beta_g e_g + \beta_1 e_0 - \beta_0 + 1 \\ &= -\beta_g + \beta_0(\beta_1 - 1) + 1 \\ &> -\beta_g + \beta_0\beta_1 - \beta_0.\end{aligned} \tag{4.19}$$

Suppose first that $g \geq 3$. Using (4.17) and (4.19), we compute

$$\begin{aligned}(\beta_0+\beta_1)^2\mu - 2\beta_0^2\beta_1^2 &= (\beta_0^2 + 2\beta_0\beta_1 + \beta_1^2)\mu - 2\beta_0^2\beta_1^2 \\ &> \sum_{i=1}^{g}\beta_0^2\beta_i(e_{i-1}-e_i) - \beta_0^3 + \beta_0^2 \\ &+ \sum_{i=1}^{g}\beta_1^2\beta_i(e_{i-1}-e_i) - \beta_0\beta_1^2 + \beta_1^2 \\ &- 2\beta_0\beta_1\beta_g - 2\beta_0^2\beta_1 \\ &> \sum_{i=1}^{g-1}\beta_0^2\beta_i + \sum_{i=1}^{g-1}\beta_1^2\beta_i - \beta_0^3 - \beta_0\beta_1^2 \\ &+ (\beta_0-\beta_1)^2\beta_g - 2\beta_0^2\beta_1 \\ &> 2\beta_0^2\beta_1 + 2\beta_1^3 - \beta_0^3 - \beta_0\beta_1^2 - 2\beta_0^2\beta_1 > 0.\end{aligned}$$

It follows that

$$\Phi_f\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right) = \frac{1}{2}\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right)^2 - \frac{1}{\mu} = \frac{(\beta_0+\beta_1)^2\mu - 2\beta_0^2\beta_1^2}{2\beta_0^2\beta_1^2\mu} > 0.$$

Suppose now that $g=2$. By (4.18), (4.14), (4.15) and (4.16),

$$e_1\mu_1 = (n_1-1)(\overline{\beta}_1 - e_1) = n_1\overline{\beta}_1 - \overline{\beta}_1 - e_0 + e_1 < n_1\overline{\beta}_1 \leq \overline{\beta}_2 - 1 = q_2 - 1$$

and hence

$$\begin{aligned}\mu - e_1^2\mu_1 &= e_1\mu_1 + e_2\mu_2 - e_1^2\mu_1 \\ &> e_1(1 + e_2(n_2-1) - e_1)\mu_1 \\ &= e_1(1 + e_1 - e_2 - e_1)\mu_1 \\ &= e_1(1-e_2)\mu_1 = 0.\end{aligned} \tag{4.20}$$

If $(n_1, q_1) \notin \{(2,3),(2,5)\}$, then by (4.14), Proposition 4.16 and (4.20)

$$\Phi_f\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right)=\frac{1}{e_1^2}\frac{1}{2}\left(\frac{1}{n_1}+\frac{1}{q_1}\right)^2-\frac{1}{\mu}>\frac{1}{e_1^2}\frac{1}{\mu_1}-\frac{1}{\mu}=\frac{\mu-e_1^2\mu_1}{e_1^2\mu_1\mu}>0.$$

Otherwise, we have $n_1 = 2 \leq e_1 = n_2$ and (4.16) yields $q_2 > 2e_1q_1$. Using (4.18) it follows that

$$\mu=(e_1-1)(q_2-1)+e_1(q_1-1)>(e_1-1)(2e_1q_1-1)+e_1(q_1-1)=2q_1e_1^2-(q_1+2)e_1+1$$

and hence

$$(2+q_1)^2\mu-8e_1^2q_1^2=\begin{cases}78e_1^2-125e_1+25 & \text{if } q=3,\\ 290e_1^2-343e_1+49 & \text{if } q=5.\end{cases}$$

In both cases $e_1 \geq 2$ implies

$$\Phi_f\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right)=\frac{(n_1+q_1)^2\mu-2e_1^2n_1^2q_1^2}{2e_1^2n_1^2q_1^2\mu}=\frac{(2+q_1)^2\mu-8e_1^2q_1^2}{8e_1^2q_1^2\mu}>0.$$

To finish let us show that $\lim_{n_g\to\infty}\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1})=0$. As in the proof of Proposition 4.16, we can write $\overline{\beta}_1 = l\overline{\beta}_0 + s$ with $l \geq 1$ and $\overline{\beta}_0 - 1 \geq s \geq 0$. Since by definition $n_g = e_{g-1}$, Equation (4.15) allows also to write $\overline{\beta}_0 = jn_g + r'$ with $j \geq 1$ and $n_g - 1 \geq r' \geq 0$. Finally, observe that A'Campo's formula for the Milnor number (4.18) implies that $\lim_{n_g\to\infty}\frac{1}{\mu}=0$. Therefore,

$$\lim_{n_g\to\infty}\Phi_f(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1})=\lim_{n_g\to\infty}\frac{1}{2}\left(\frac{1}{\overline{\beta}_0}+\frac{1}{\overline{\beta}_1}\right)^2-\lim_{n_g\to\infty}\frac{1}{\mu}=0. \qquad \square$$

In particular, Theorem 4.18 provides a quite surprising constraint on the first Puiseux pair of an irreducible plane curve singularity with a given Milnor number.

4.4 Limit spectral distribution

In this section we return to the general setup of Section 4.2 to stablish K. Saito's limit distribution for Newton non-degenerate hypersurface singularities. Our approach is to use the existence of a regular subdivision of the Newton fan and to mimic the argument of K. Saito to prove the quasi-homogeneous case of Theorem 4.2 (see [146, (2.2),(3.7)]). Let us first recall the proof of the quasi-homogeneous case of Theorem 4.2 ([146, (3.7) Example 1]).

Let $f(z_0, \ldots, z_n)$ be a quasi-homogeneous germ of function of degree 1 with respect to the weights $w_0, \ldots, w_n$ and isolated singularity at $\mathbf{0}$, i.e. $f(z_0,\ldots,z_n) = f(\mathbf{z}) = \sum a_\alpha \mathbf{z}^\alpha$ where $\alpha = (\alpha_0, \ldots, \alpha_n)$, $\mathbf{z} = z_0^{\alpha_0}\cdots z_n^{\alpha_\mathbf{n}}$, $a_\alpha \neq 0$ and $\sum w_i\alpha_i = 1$. Steenbrink [160, (5.11)] showed that the Poincaré series of H_f, where H_f is defined by Equation (4.5), has the following form

$$p_{H_f}(t)=\prod_{i=0}^{n}\frac{t-t^{w_i}}{t^{w_i}-1.}$$

Following K. Saito [146, (3.7)] we first observe that identifying $T = \exp(2\pi it)$ we have

$$\begin{aligned}\lim_{w\to 0}\frac{w}{w-1}\frac{T-T^w}{T^w-1} &= \lim_{w\to 0}\frac{w}{1-w}\frac{\exp(2\pi it)-\exp(2\pi itw)}{\exp(2\pi itw)-1} \\ &= \lim_{w\to 0}\frac{\exp(2\pi it)-\exp(2\pi itw)-2\pi itw\exp(2\pi itw)}{1-\exp(2\pi itw)+2\pi it(1-w)\exp(2\pi itw)} \\ &= \frac{\exp(2\pi it)-1}{2\pi it} = \frac{\exp(\pi it)}{\pi t}\frac{\exp(\pi it)-\exp(-\pi it)}{2i} \\ &= \frac{\exp(\pi it)}{\pi t}\sin(\pi t) = \mathcal{F}(\varphi)(t).\end{aligned} \tag{4.21}$$

Now, recall that the Milnor number of a quasi-homogeneous function with isolated singularity can be computed as $\mu = \prod_{i=0}^{n}(w_i^{-1}-1)$. Thus,

$$\lim_{w\to 0}\chi_f(t) = \lim_{w\to 0}\frac{p_H(T)}{\mu} = \prod_{i=0}^{n}\lim_{w_i\to 0}\frac{w_i}{w_i-1}\frac{T-T^{w_i}}{T^{w_i}-1} = \mathcal{F}(\varphi)^{n+1}(t) = \mathcal{F}(N_{n+1})(t).$$

For our purpose we adapt the calculation (4.21) as follows.

Lemma 4.19. $\lim_{w\to 0} w\frac{1-T}{1-T^w} = \mathcal{F}(\varphi)(t)$.

Proof. Using L'Hôpital's rule in the second step and (4.21), we compute

$$\begin{aligned}\lim_{w\to 0} w\frac{1-T}{1-T^w} &= \lim_{w\to 0}\frac{w\cdot(1-\exp(2\pi it))}{1-\exp(2\pi itw)} \\ &= \lim_{w\to 0}\frac{1-\exp(2\pi it)}{-2\pi it\exp(2\pi itw)} \\ &= \frac{1-\exp(2\pi it)}{-2\pi it} = \mathcal{F}(\varphi)(t). \qquad \square\end{aligned} \tag{4.22}$$

Assume now that $f\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ is a Newton non-degenerate isolated hypersurface singularity. Let us denote by $\Gamma := N(f)$ its Newton diagram. By Steenbrink's Theorem 4.14, to compute p_{H_f} is equivalent to compute the Poincaré series of the graded algebras A_σ associated to each of the faces $\sigma\in\Gamma$.

If σ is a regular rational cone we can easily compute p_{A_σ}. Let $w_0^\sigma,\ldots,w_k^\sigma$ be the weights of the basis of $\mathbb{Q}_{\geq 0}\tau\cap\mathbb{Z}$, i.e. the minimal system of generators of the monoid $\mathbb{Q}_{\geq 0}\tau\cap\mathbb{Z}$. Since σ is a rational regular cone then $A_\sigma\simeq\mathbb{C}[y_0,\ldots,y_k]$ with $\mathrm{wt}(y)=w_i$ (see Subsection 1.4.1). Thus, the Poincaré series reads

$$p_{A_\sigma}(t) = \frac{1}{\prod_{i=0}^{k}(1-t^{w_i})}. \tag{4.23}$$

Our aim now is to mimic K. Saito's argument for quasi-homogeneous functions to stablish the limit distribution for Newton non-degenerate hypersurface singularities. The argument is based in the fact that by Theorem 1.77, the Newton diagram has a regular subdivision that will allow us to compute the p_{A_σ} appearing in Steenbrink's Theorem 4.13 by means of Equation (4.23).

We are now ready to stablish K. Saito's limit distribution for Newton non-degenerate hypersurface singularities.

Theorem 4.20. *For a fixed Newton diagram Γ, consider the Newton diagrams $\varpi\Gamma$ obtained from Γ by scaling with the factor ϖ. Then we have*

$$\lim_{\varpi\to\infty}\chi_{f_\varpi}=\mathcal{F}(N_{n+1}),\tag{4.24}$$

where the limit runs over all Newton non-degenerate f_ϖ of $n+1$ variables with Newton diagram $\varpi\Gamma$.

Proof. By Theorem 1.77, Γ has a subdivision $\tilde{\Gamma}$ corresponding to a regular subdivision of its fan of cones. For any $\tau\in\tilde{\Gamma}$ let $w_0^\tau,\dots,w_k^\tau$ be the weights of the basis of $\mathbb{Q}_{\geq 0}\tau\cap\mathbb{Z}$. Then

$$p_{A_\sigma}(t)=\sum_{\tilde{\Gamma}\ni\tau\leq\sigma}(-1)^{d(\sigma)-d(\tau)}p_{A_\tau}(t),\quad p_{A_\tau}(t)=\prod_{j=0}^{\dim\tau}\frac{1}{(1-t^{w_j^\tau})}.$$

Substituting into Steenbrink's formula from Theorem 4.14 yields

$$p_{H_f}(t)=\sum_{\tilde{\Gamma}\ni\tau\leq\sigma\in\Gamma}(-1)^{n+1-d(\tau)}\frac{(1-t)^{k(\sigma)}}{\prod_{j=0}^{\dim\tau}(1-t^{w_j^\tau})}.$$

Passing to $\varpi\Gamma$, w_j^τ is preplaced by εw_j^τ where $\varepsilon\varpi=1$ and hence

$$p_{H_{f_\varpi}}(t)=\sum_{\tilde{\Gamma}\ni\tau\leq\sigma\in\Gamma}(-1)^{n+1-d(\tau)}\frac{(1-t)^{k(\sigma)}}{\prod_{j=0}^{\dim\tau}(1-t^{\varepsilon w_j^\tau})}.$$

By Theorem 4.13, we can decompose the Poincaré series into a sum of the Poincaré series associated to the cones of a regular subdivision of Γ

$$\begin{aligned}\lim_{\varpi\to\infty}\chi_{f_\varpi}(t)&=\lim_{\varpi\to\infty}\frac{p_{H_{f_\omega}}(T)}{\mu_{f_\varpi}}\\&=\sum_{\tilde{\Gamma}\ni\tau\leq\sigma\in\Gamma}(-1)^{n+1-d(\tau)}\lim_{\varpi\to\infty}\frac{1}{\mu_{f_\varpi}}\frac{(1-t)^{k(\sigma)}}{\prod_{j=0}^{\dim\tau}(1-t^{\varepsilon w_j^\tau})}\end{aligned}\tag{4.25}$$

where μ_{f_ϖ} can be computed by means of Kouchnirenko's Theorem 1.86 as

$$\mu_{f_\varpi}=\sum_{j=0}^{n+1}(-1)^{n+1-j}j!\varpi^jV_j.\tag{4.26}$$

Fix $\tilde{\Gamma}\ni\tau\leq\sigma\in\Gamma$. Let $V(\tau)$ be the $d(\tau)$-dimensional volume of the convex hull of $\tau\cup\{0\}$. Note that

$$\sum_{\substack{\tau\in\tilde{\Gamma}\\ d(\tau)=n+1}}V(\tau)=V_{n+1},\quad 1/V(\tau)=d(\tau)!\prod_{j=0}^{d(\tau)}w_j^\tau.\tag{4.27}$$

The summand in (4.25) indexed by τ is then computed using (4.27), Lemma 4.19 and (4.26):

$$\begin{aligned}
&\lim_{\varpi\to\infty}\frac{1}{\mu_{f_\varpi}}\frac{(1-T)^{k(\sigma)}}{\prod_{j=0}^{\dim\tau}(1-T^{\varepsilon w_j^\tau})}\\
&=\lim_{\varpi\to\infty}\frac{\prod_{j=0}^{\dim\tau}\frac{\varpi}{w_j^\tau}}{\mu_{f_\varpi}}(1-T)^{k(\sigma)-d(\tau)}\prod_{j=0}^{\dim\tau}\lim_{\varepsilon\to 0}\varepsilon w_j^\tau\frac{1-T}{1-T^{\varepsilon w_j^\tau}}\\
&=\lim_{\varpi\to\infty}\frac{(\prod_{j=0}^{\dim\tau}\frac{1}{w_j^\tau})\varpi^{d(\tau)}}{\mu_{f_\varpi}}(1-T)^{k(\sigma)-d(\tau)}\left(\lim_{w\to 0}w\frac{1-T}{1-T^w}\right)^{d(\tau)}\\
&=\lim_{\varpi\to\infty}\frac{(\prod_{j=0}^{\dim\tau}\frac{1}{w_j^\tau})\varpi^{d(\tau)}}{\mu_{f_\varpi}}(1-T)^{k(\sigma)-d(\tau)}\mathcal{F}(N_{d(\tau)})(t)\\
&=\begin{cases}\frac{V(\tau)}{V_{n+1}}\mathcal{F}(N_{n+1})(t) & \text{if } d(\tau)=n+1,\\ 0 & \text{if } d(\tau)<n+1.\end{cases}
\end{aligned}$$

The claim now follows by substituting into (4.25) and applying (4.27). □

Let us see with an example how the proof of Theorem 4.20 works.

Example 4.21. Let us consider the $T_{p,q,r}$ singularity type computed in Example 4.15. A scaling factor ϖ of the Newton diagram give rise to the equation

$$x^{\varpi p}+y^{\varpi q}+z^{\varpi r}+x^{\varpi}y^{\varpi}z^{\varpi}.$$

Let us denote by $\sigma_i(\varpi)$ the faces of the scaled Newton diagram, i.e. $\sigma_i(1)=\sigma_i$ in the table 4.1. Let us denote by

$$w_1=\frac{1}{\varpi p},\quad w_2=\frac{1}{\varpi q},\quad w_3=\frac{1}{\varpi}\quad w_4=\frac{1}{\varpi r}.$$

First of all, we recall that we can compute the Milnor number with the help of Kouchnirenko's Theorem 1.86. Then,

$$\mu=3!V_3-2!V_2+V_1-1=\varpi^3(pq+pr+qr)-\varpi^2(pq+pr+qr)-\varpi(p+q+r)-1. \tag{4.28}$$

Observe that all the cones σ_i are regular, so are $\sigma_i(\varpi)$. To compute $p_H(t)$ we will use Steenbrink's formula 4.14. We start with $\sigma_1(\varpi)$. With the help of Table 4.1, it is easy to see that

$$\frac{(1-t)^3 p_{A_{\sigma_1(\varpi)}}}{3!V_{\sigma_1(\varpi)}}=w_1w_2w_3\frac{1-t}{1-t^{w_1}}\frac{1-t}{1-t^{w_2}}\frac{1-t}{1-t^{w_3}}.$$

Then, we can apply Lemma 4.19 to obtain

$$\lim_{\varpi\to\infty}\frac{(1-t)^3 p_{A_{\sigma_1(\varpi)}}}{3!V_{\sigma_1(\varpi)}}=(\mathcal{F}(\varphi)(t))^3.$$

Similar computations lead to obtain $\lim_{\varpi\to\infty}\frac{(1-t)^3 p_{A_{\sigma_i(\varpi)}}}{3!V_{\sigma_i(\varpi)}}=(\mathcal{F}(\varphi)(t))^3$ for $i=2,3$. Thus,

$$\begin{aligned}
\lim_{\varpi\to\infty}\frac{(1-t)^3(p_{A_{\sigma_1(\varpi)}}+p_{A_{\sigma_2(\varpi)}}+p_{A_{\sigma_3(\varpi)}})}{\mu}&=\sum_{i=1}^{3}\lim_{\varpi\to\infty}\frac{3!V_{\sigma_i(\varpi)}}{\mu}\frac{(1-t)^3 p_{A_{\sigma_i(\varpi)}}}{3!V_{\sigma_i(\varpi)}} \\
&=\sum_{i=1}^{3}\left(\lim_{\varpi\to\infty}\frac{3!V_{\sigma_i(\varpi)}}{\mu}\right)(\mathcal{F}(\varphi)(t))^3.
\end{aligned}\tag{4.29}$$

Now, observe that $3!V_{\sigma_1(\varpi)} = \varpi^3 pq$, $3!V_{\sigma_2(\varpi)} = \varpi^3 qr$ and $3!V_{\sigma_3(\varpi)} = \varpi^3 pr$. Thus, by equation (4.28)

$$\sum_{i=1}^{3} \lim_{\varpi\to\infty} \frac{3!V_{\sigma_i(\varpi)}}{\mu} = \lim_{\varpi\to\infty} \frac{\varpi^3(pq+pr+qr)}{\mu} = 1. \tag{4.30}$$

For the σ_i faces with $i = 8, 9, 10$, we have $2!V_{\sigma_8(\varpi)} = w_1^{-1}w_2^{-1} = \varpi^2 pq$, $2!V_{\sigma_9(\varpi)} = w_2^{-1}w_4^{-1} = \varpi^2 qr$ and $2!V_{\sigma_{10}(\varpi)} = w_1^{-1}w_4^{-1} = \varpi^2 pr$. We can apply again Lemma 4.19 to obtain

$$\lim_{\varpi\to\infty} \frac{(1-t)^2 p_{A_{\sigma_8(\varpi)}}}{2!V_{\sigma_8(\varpi)}} = \lim_{\varpi\to\infty} \frac{(1-t)^2 p_{A_{\sigma_9(\varpi)}}}{2!V_{\sigma_9(\varpi)}} = \lim_{\varpi\to\infty} \frac{(1-t)^2 p_{A_{\sigma_{10}(\varpi)}}}{2!V_{\sigma_{10}(\varpi)}} = (\mathcal{F}(\varphi)(t))^2.$$

Therefore, a similar computation as the ones in Equation (4.29) and Equation (4.30) give

$$\lim_{\varpi\to\infty} \frac{(1-t)^2(\sum_{i=8}^{10} p_{A_{\sigma_i(\varpi)}})}{\mu} = \sum_{i=8}^{10} \left(\lim_{\varpi\to\infty} \frac{2!V_{\sigma_i(\varpi)}}{\mu} \right) (\mathcal{F}(\varphi)(t))^2 = 0. \tag{4.31}$$

On the other hand, the faces σ_i for $i = 4, 5, 6, 7$ have a different behaviour due to the dimension of the maximal hyperplane containing them. Observe that in this case $V_{\sigma_i(\varpi)} = 0$ since $d(\sigma) < k(\sigma)$. We apply Lemma 4.19 to obtain

$$\begin{aligned} \lim_{\varpi\to\infty} w_1 w_3 (1-t)^3 p_{A_{\sigma_4}} &= \lim_{\varpi\to\infty} w_2 w_3 (1-t)^3 p_{A_{\sigma_5}} = \\ &= \lim_{\varpi\to\infty} w_4 w_3 (1-t)^3 p_{A_{\sigma_6}} = (1-\exp(2\pi i\tau))(\mathcal{F}(\varphi)(t))^2 \end{aligned}$$

Also,

$$\lim_{\varpi\to\infty} w_3 (1-t)^3 p_{A_{\sigma_7}} = (1-\exp(2\pi i\tau))^2(\mathcal{F}(\varphi)(t)).$$

Observe that in any case we have at most the product of two different weights, so they behave at most like ϖ^2. Thus,

$$\lim_{\varpi\to\infty} \frac{(1-t)^2(\sum_{i=4}^{6} p_{A_{\sigma_i(\varpi)}})}{\mu} = \lim_{\varpi\to\infty} \frac{(1-t)p_{A_{\sigma_7(\varpi)}}}{\mu} = 0. \tag{4.32}$$

Therefore, as showed in the proof of Theorem 4.20, any face with $d(\sigma) < 3$ satisfies $\lim_{\varpi\to\infty}(1-t)^{k(\sigma)} p_{A_{\sigma(\varpi)}}/\mu = 0$ and

$$\lim_{\varpi\to\infty} \frac{p_{H_f}(t)}{\mu} = \sum_{i=1}^{3} \left(\lim_{\varpi\to\infty} \frac{3!V_{\sigma_i(\varpi)}}{\mu} \right) (\mathcal{F}(\varphi)(t))^3 = (\mathcal{F}(\varphi)(t))^3.$$

To finish, let us discuss a while the problem about taking an "adequate "limit. At the beginning of this section, we recalled K. Saito's proof about the limit distribution of a quasi-homogeneous function. In that case, one could obtain the limit as independent limits on each of the weights. Thus, it would be natural to ask if a similar argument could be done for non-degenerate singularities, i.e. to assign a scaling factor for each of the facets and then to do the limit independently. The problem of this approach is that we cannot, in general, provide independent scaling factors for each of the facets in such a way the Newton diagram is non degenerate regardless of the scaling factors. Our approach has been then to scale the whole Newton polygon in order to assure that the non-degeneracy condition is preserved.

It is also worth it to say that in all the cases, i.e. Newton non-degenerate hypersurfaces, quasi-homogeneous hypersurfaces and irreducible plane curves, the established limit also implies that the Milnor number and the multiplicity tends to infinity. It would be natural to ask if one can just ask for the limit distribution when the Milnor number, which is equal to the number of exponents, tends to infinity. However, it is unclear up to whether different ways to make the Milnor number tends to infinity also provide the same limit distribution.

Part II

Numerical Semigroups and modules over them

Chapter 5
The Wilf function of a numerical semigroup

The "money-changing problem" asks for those sums of money we can change if we have coins of e different values, say $a_1, \ldots, a_e$. In 1978, H. Wilf [185] presented an algorithm to solve this problem; he named it "circle-of-lights algorithm". The idea is quite basic: we consider a circle of a_e lights labeled as $0, 1, \ldots, a_e - 1$, where all of them are "off" except for the one representing 0. We start turning on lights clockwise. During the first round we turn on the lights corresponding to the values $\{a_1, a_2, \ldots, a_e\}$. After that, the algorithm runs turning on bulbs representing the possible combinations of numbers $x = x_0 a_0 + \cdots + x_e a_e$. Observe that the representation of x is given modulo a_e. If the possible values of coins $\{a_1, a_2, \ldots, a_e\}$ are coprime, then the algorithm terminates after finitely many steps. In this case the "money-changing problem" can be easily translated in terms of numerical semigroups.

As pointed out in Definition 0.1, a numerical semigroup Γ has finite complement as a subset of the natural numbers. The elements in this complement are said to be the gaps of the semigroup, and the number $g(\Gamma)$ of all of them is called the *genus* of Γ. The fact that $g(\Gamma) < \infty$ implies that Γ is finitely generated, and it is not difficult to see that every numerical semigroup has a unique system of minimal generators (see for example [142, Theorem 2.7]); its cardinality is called the *embedding dimension* of Γ, written $e(\Gamma)$. Observe that in the "circle-of-lights algorithm" the generators of the semigroup correspond to coin values $\{a_1, \ldots, a_e\}$ and the bulbs that are "off" correspond to the gaps of the semigroup generated by the possible values of coins (see [185, II, p. 563]). The biggest gap with respect to the natural ordering is called the *Frobenius number* of Γ, say $F(\Gamma)$, and $c(\Gamma) := F(\Gamma) + 1$ is called the *conductor* of Γ. Finally, it is also customary to consider the *delta-invariant* of Γ, which is defined to be the cardinality $\delta(\Gamma)$ of the set $\{x \in \Gamma : x < F(\Gamma)\}$.

Numerical semigroups became notorious in Mathematics due to the Frobenius problem, which asks for a closed formula for the Frobenius number or, equivalently, for the conductor of a numerical semigroup. In particular, the "circle-of-lights algorithm"provides a partial answer to the Frobenius problem, i.e. for a set of generators of the semigroup this is an algorithm which computes the conductor of the semigroup (see [185, p. 564]). Moreover, the "circle-of-lights algorithm"computes $c(\Gamma)$ with time complexity $O(e(\Gamma)c(\Gamma))$ (see [126, p. 833]). Based on this complexity H. Wilf [185] posed the following question, nowadays known as Wilf's conjecture 5.1:

Conjecture 5.1. [1] For any numerical semigroup Γ, its conductor, embedding dimension, number of gaps and delta-invariant are related by means of the inequalities

[1] The original formulation of Wilf [185] was presented in the following form: consider a set of $a_1, \ldots, a_k$ natural numbers such that $\gcd(a_1, \ldots, a_k) = 1$. Let Ω be the number of values that cannot be represented as sum $x_1 a_1 + \cdots + x_k a_k$ and let $\chi - 1$ be the maximal number that cannot be represented as such a sum. Is it true that for a fixed k the fraction $\Omega/\chi < 1 - (1/k)$. A careful reading of Wilf original paper shows that his question is strongly based in his computational approach to Frobenius problem.

$$\frac{|\mathbb{N} \setminus \Gamma|}{c(\Gamma)} \leq 1 - \frac{1}{e(\Gamma)}, \quad \text{or equivalently} \quad c(\Gamma) \leq e(\Gamma) \cdot \delta(\Gamma). \tag{5.1}$$

The Wilf conjecture is known to be true in several cases, see e.g. Delgado [36], Dobbs and Matthews [47], Eliahou [51], Eliahou and Fromentin [52], Fromentin and Hivert [60], Kaplan [83], Moscariello and Sammartano [119] and Sammartano [153]. However this is still an open question, and in the meanwhile several related problems have been treated in order to gain a better understanding of the conjecture (see for example [37, 38, 9]). For a recent account of the conjecture, we refer the reader to the survey [37].

A. Nijenhuis [126] proposed an alternative method that improves the time complexity of Wilf's algorithm. More recently, S. Böcker and Z. Lipták [20] improved the time complexity of Nijenhuis' algorithm to $O(e(\Gamma)a_1)$ with interesting applications to interpreting mass spectrometry peaks. Based on those improvements, as well as on the huge number of examples provided by many of the above mentioned references [40, 36, 51, 83, 47], one realizes that the inequality in the Wilf conjecture 5.1 seems to be far from being tight. Having this in mind, the underlying meaning of Wilf's conjecture reveals to be the finding of an upper bound for the quotient $c(\Gamma)/\delta(\Gamma)$ and, after that, the comparison of this bound with the embedding dimension of the semigroup.

This idea leads us to not directly consider Wilf's conjecture but a general inequality of the type $k\delta(\Gamma) > c(\Gamma)$ with $k \in \mathbb{Z}$; we call this inequality a *Wilf-type inequality.* In this general setting, it makes sense to define the *Wilf function* associated to a numerical semigroup as the map $W_\Gamma : \mathbb{N} \to \mathbb{Z}$ given by

$$k \mapsto k\delta(\Gamma) - c(\Gamma).$$

For $k = e(\Gamma)$, the value $W_\Gamma(e(\Gamma))$ is called the *Wilf number* of Γ, see [51], [8]; also [37, p. 45]. Of course, Conjecture 5.1 may be trivially rewritten in terms of the Wilf number as $W_\Gamma(e(\Gamma)) \geq 0$. Moreover, by the previous discussion, it becomes natural to try to find the best upper bound, say μ_Γ, for the quotient $c(\Gamma)/\delta(\Gamma)$ (see end of Section 5.2).

On the other hand, inspired by Fröberg, Gottlieb and Häggkvist [59], Moscariello and Sammartano [119] asked for the equality in (5.1), and they proposed the following conjecture:

Conjecture 5.2 (Frögohämosa-conjecture). Let $\Gamma \neq \mathbb{N}$ be a numerical semigroup. The equality $c(\Gamma) = e(\Gamma) \cdot \delta(\Gamma)$ holds if and only if Γ has embedding dimension 2 or there exist $m, q \in \mathbb{N} \setminus \{0\}$ with $m > 1$ such that

$$\Gamma = W_{m,q} := \{0, m, 2m, 3m, \ldots, (q-1)m, qm, qm+1, qm+2, \ldots\}.$$

Observe that in the case $\Gamma = \mathbb{N}$ the equality holds trivially. Also, for $\Gamma \neq \mathbb{N}$, it is easy to check that the semigroups of the form $\Gamma = \langle a, b\rangle$ with $\gcd(a,b) = 1$ and $\Gamma = W_{m,q}$ satisfy the equality $c(\Gamma) = e(\Gamma) \cdot \delta(\Gamma)$. Therefore the difficult part of Conjecture 5.2 is the converse statement. Although many numerical experiments have been done, there is no hint to prove Conjecture 5.2 in its whole generality. Moreover, there is no philosophical reason explaining why the semigroups occurring in Conjecture 5.2 are exactly those. In the first part of this Chapter we will then propose an possible explanation for the reason why the semigroups involved in the Frögohämosa-conjecture are precisely the ones appearing and no others.

Our argument is based on Theorem 5.30. Theorem 5.30 allows us to define the semigroups appearing in the Frögohämosa-conjecture as those for which the Wilf function $W_\Gamma(k)$ has an extreme behaviour, where *extreme* is precisely defined by the conditions either $W_\Gamma(k) \leq 0$ for all $2 \leq k \leq m$ or minimal embedding dimension among those satisfying $W_\Gamma(k) \geq 0$ for all $2 \leq k \leq m$ and $W_\Gamma(2) = 0$. This constitutes a complete new point of view and we hope that this argument will shed some light on the understanding of both the Wilf conjecture and the Frögohämosa-conjecture.

In this regard, it is natural to unify the treatment of the Wilf and the Frögohämosa conjectures through a conjecture we call *Wilf archimedean conjecture*:

Conjecture 5.3 (Wilf archimedean conjecture). For any numerical semigroup Γ not having neither minimal nor maximal embedding dimension, we have

$$W_{\Gamma}(e) \geq 1.$$

Therefore, it is easily seen that the Wilf archimedean conjecture implies both Wilf's and Frögohämosa's conjectures, cf. Proposition 5.23.

One of those nearby problems to Wilf's conjecture is related to another interesting number that can be associated to a semigroup, the Eliahou number.

Eliahou [51] has been able to relate the Wilf conjecture to an invariant $E(\Gamma)$, now called the Eliahou number associated to a semigroup (cf. [36]; see also Section 5.4) such that $e(\Gamma) \cdot \delta(\Gamma) - c(\Gamma) \geq E(\Gamma)$. Therefore, any numerical semigroup with positive Eliahou number satisfies the Wilf conjecture. Unfortunately, there exist numerical semigroups with negative Eliahou number [36, 51, 52] and thus Wilf's conjecture is reduced to study those semigroups with negative Eliahou number.

The characterization of the numerical semigroups with negative Eliahou number is a huge challenge, as observed e.g. in [36, 38, 52], and very few general properties about them are known. One of the main goals of the second part of this Chapter is to establish necessary conditions for a numerical semigroup in order to have negative Eliahou number, see Section 5.4. In particular, these conditions will allow us to show some new examples of numerical semigroups with negative Eliahou number, see 5.42. To do so, we make use of two recent techniques introduced in the study of the Wilf conjecture.

On the one hand, the Wilf function introduced by the author and Moyano-Fernández in [9] which, as we already mentioned, is going to be studied in the first part of this chapter. Our approach to the study of Eliahou number in the second part of this Chapter is the following: Eliahou reduced the problem to the study of the Wilf number for semigroups with negative Eliahou number. If we denote by e_s the number of minimal generators less than $c(\Gamma)$, our approach will change the study of the semigroups with negative Eliahou number to the investigation of semigroups with $\mu_{\Gamma} \geq e_s$. Concretely, we will show in Theorem 5.34 that the Eliahou number is bounded below by $W_{\Gamma}(e_s)$. Thus, the negativity of the Eliahou number implies $\mu_{\Gamma} > e_s$. To find examples with $\mu_{\Gamma} > e_s$ and positive Eliahou number is not difficult as Example 5.35 shows; in contrast with the fact that the semigroups with negative Eliahou number seem to be rare [36, 52]. Therefore, the condition $\mu_{\Gamma} > e_s$ may lead to an easier characterization of the interesting family of semigroups to investigate.

On the other hand, in a recent preprint Rosales et al. [141] introduce the concept of concentration of a numerical semigroup: set $\operatorname{next}_{\Gamma}(s) := \min\{x \in \Gamma \mid s < x\}$; the *concentration of a numerical semigroup* is then defined as

$$\mathbf{C}(\Gamma) = \max\{\operatorname{next}_{\Gamma}(s) - s \mid s \in \Gamma \setminus \{0\}\}.$$

If we call $m(\Gamma) := \min(\Gamma \setminus \{0\})$ the *multiplicity* of the semigroup, it is clear that a numerical semigroup with concentration 1 is of the form $\{0, m(\Gamma), \rightarrow\}$, where the arrow $\rightarrow$ means that from $m(\Gamma)$ on all natural numbers belong to the set. The numerical semigroups with concentration 2 have been characterized by Rosales et al. [141].

As already mentioned, the essence of Wilf's conjecture may be expressed as how often do elements of Γ occur in the integral interval $[0, c] \cap \mathbb{N}$. From this viewpoint, it is natural to ask for semigroups with fixed concentration satisfying the Wilf conjecture. One should obviously expect that smaller concentration should lead to a higher frequency of occurrence of elements of Γ in the integral interval $[0, c] \cap \mathbb{N}$; this is indeed the case as shown in our Theorem 5.37.

5.1 Basic facts about numerical semigroups

Let Γ be a numerical semigroup generated by $a_1, \ldots, a_e$; this fact will be expressed by writing $\Gamma = \langle a_1, \ldots, a_e \rangle$. We will say that $a_1, \ldots, a_e$ are the *minimal system of generators* if $0 \neq a_1 < a_2 < \cdots < a_e$ and $a_{i+1} \notin \langle a_1, \ldots, a_i \rangle$. It is easy to see that any numerical semigroup admits a unique finite system of generators [144, Thm. 2.7]. From now on, we will always assume that $\Gamma = \langle a_1, \ldots, a_e \rangle$ means $\{a_1, \ldots, a_e\}$ is the minimal system of generators. We call $m(\Gamma) := \min(\Gamma \setminus \{0\})$ the *multiplicity* of the semigroup; it is a trivial observation that $m(\Gamma) = a_1$.

Remark 5.4. For generalities and topics related with numerical semigroups the reader is referred to the book of Rosales and García-Sánchez [142] and the book of Ramírez Alfonsín [138].

Sometimes, it is also useful to work with a remarkable system of generators —by no means minimal— that can be attached to a numerical semigroup Γ: let $s \in \Gamma \setminus \{0\}$, the *Apéry set* of Γ with respect to s is defined to be the set

$$\mathrm{Ap}(\Gamma, s) = \{w \in \Gamma \,:\, w - s \notin \Gamma\},$$

see Apéry [12], or also Kunz and Herzog [94, Lemma 4.2]. The Apéry set satisfies the following property.

Lemma 5.5. *[144, Lem. 2.4]* $\mathrm{Ap}(\Gamma, s) = \{0 = w(0), w(1), \ldots, w(s-1)\}$ *is such that* $w(i)$ *is the least element of* Γ *congruent with* i *modulo* s *for all* $i \in \{0, \ldots, s-1\}$.

In particular, the previous lemma implies that the cardinality of $\mathrm{Ap}(\Gamma, s)$ is s, and that $\mathrm{Ap}(\Gamma, s) = \{w_0 < w_1 < \cdots < w_{s-1}\}$ where $w_i = \min\{z \in \Gamma : z \equiv i \bmod s\}$; obviously, $w_0 = 0$. We will always consider the particular case $s = m := m(\Gamma)$, for which $w_1 = a_2$ and $w_{m-1} = c - 1 + a_1 = c + m - 1$. Moreover, the Apéry set with respect to the multiplicity m has the following property.

Lemma 5.6. *Let* $\Gamma = \langle a_1, \ldots, a_e \rangle$ *be the numerical semigroup minimally generated by* $\{a_1, \ldots, a_e\}$. *We denote* $a_1 := m$, *then*

$$\{a_2, \ldots, a_e\} \subset \mathrm{Ap}(\Gamma, m) = \{w \in \Gamma \,:\, w - m \notin \Gamma\}.$$

Proof. If $a_j \notin \mathrm{Ap}(\Gamma, m)$ then $a_j = \nu + m$ with $\nu \in \Gamma$ and $\nu < a_j$, which contradicts the fact that a_j is a minimal generator of Γ. □

A numerical semigroup Γ can be endowed with the following partial ordering: for every $s, t \in \Gamma$ we set $s \preceq t$ if and only if there exists $u \in \Gamma$ such that $s + u = t$. Thus we define the following two subsets of the Apéry set of Γ with respect to the multiplicity $m = m(\Gamma)$:

$$\min \mathrm{Ap}(\Gamma, m) := \{w \in \mathrm{Ap}(\Gamma, m) \setminus \{0\} \mid w \text{ is minimal with respect to } \preceq\},$$

$$\max \mathrm{Ap}(\Gamma, m) := \{w \in \mathrm{Ap}(\Gamma, m) \setminus \{0\} \mid w \text{ is maximal with respect to } \preceq\}.$$

The latter set leads to the definition of the *type* of Γ as the cardinality $t(\Gamma) := |\max \mathrm{Ap}(\Gamma, m)|$. An important property of the type of a semigroup was proven by Fröberg, Gottlieb, Häggkvistin:

Proposition 5.7. *[59, Theorem 22] Let Γ be a numerical semigroup, then*

$$c(\Gamma) \leq \delta(\Gamma) \cdot (t(\Gamma) + 1).$$

In particular, any numerical semigroup satisfying $t(\Gamma) + 1 \leq e(\Gamma)$ satisfies Wilf's conjecture 5.1. From the proof of [153, Lemma 6] we deduce the following result as an example of numerical semigroup satisfying $t(\Gamma) + 1 \leq e(\Gamma)$:

Lemma 5.8. *If Γ is a numerical semigroup satisfying $1 \leq m(\Gamma) - e(\Gamma) \leq 2$, then $t(\Gamma) \leq e(\Gamma) - 2$.*

Proof. Let us assume that Γ is minimally generated by $m, g_2, \ldots, g_{e(\Gamma)}$. Under the hypothesis we have two cases, namely: either $e(\Gamma) = m - 1$, and so there exists $u \in \Gamma$ such that $\mathrm{Ap}(\Gamma, m) \setminus \{0\} = \{g_2, \ldots, g_{e(\Gamma)}, u\}$, or $e(\Gamma) = m - 2$, and then we have that the non-zero elements of the Apéry set are $\mathrm{Ap}(\Gamma, m) \setminus \{0\} = \{g_2, \ldots, g_{e(\Gamma)}, u, v\}$ for some $u, v \in \Gamma$. In both cases there exists $i \in \{2, \ldots, e(\Gamma)\}$ such that $g_i \preceq u$, since otherwise it would be $u \preceq g_2$, therefore $g_i \notin \max \mathrm{Ap}(\Gamma, m)$ and $t(\Gamma) \leq e(\Gamma) - 2$. □

Remark 5.9. Not every numerical semigroup satisfies the property that $t(\Gamma)+1 \leq e(\Gamma)$, for example $\Gamma = \langle 213, 216, 226, 227\rangle$ is a numerical semigroup with $t(\Gamma) = 14$ and $e(\Gamma) = 4$, as one can check with GAP [39, 61].
Observe also that Γ belongs to the family given in [59] defined as $\Gamma_{s,n} = \langle s, s+3, s+3n+1, s+3n+2\rangle$ for $n \geq 2$, $r \geq 3n + 2$ and $s = r(3n + 2) + 3$. However in [59] they wrongly claim, as the previous example shows, that this family has type $t(\Gamma_{s,n}) = 2n + 3$.

Given a numerical semigroup Γ, we will be concerned with the study of the integral interval $[0, c+m]$; of course, this interval is meant to be $[0, c+m] \cap \mathbb{N}$, but we leave out the intersection with $\mathbb{N}$ in order to discharge the notation, and we will assume all along this chapter that all occurring intervals are integral. In analogy to [119], we will make a partition of this interval in subintervals of length $m - 1$, say

$$I_\alpha := [\alpha m, (\alpha + 1)m - 1], \text{ for } \alpha = 0, 1, \ldots, \left\lfloor \frac{c + m}{m} \right\rfloor.$$

Let $L := \lfloor \frac{c-1}{m} \rfloor = \lfloor \frac{w_{m-1}}{m} \rfloor - 1$ denote the integer part of the quotient between the conductor of Γ minus 1 and its multiplicity. Hence, we can write $c - 1 = Lm + \rho'$ with $0 \leq \rho' \leq m - 1$ and $\rho' \neq 0$ because $c - 1 \neq \Gamma$. Therefore, we can rewrite $c = Lm + \rho$ with $\rho = \rho' + 1$ and $2 \leq \rho \leq m$. Thus we have in particular the following identity.

Lemma 5.10. *Let Γ be a numerical semigroup with conductor c and multiplicity m, and set $L := \lfloor \frac{c-1}{m} \rfloor$. Then,*

$$L = \begin{cases} \lfloor \frac{c}{m} \rfloor & \textit{if } c \textit{ is not a multiple of } m, \\ \frac{c}{m} - 1 & \textit{if } c \textit{ is a multiple of } m. \end{cases}$$

Following the notation of [153] and [119], for $j = 1, \ldots, m - 1$ we define

$$\eta_j = |\{\alpha \in \mathbb{N} : |I_\alpha \cap \Gamma| = j\}| \quad \text{and} \quad n_\alpha = |\{s \in \Gamma \cap I_\alpha : s < F\}|.$$

The number η_j can be computed from the Apéry set $\mathrm{Ap}(\Gamma, m)$ in the following way:

Lemma 5.11. *[153, Proposition 13], For any $j = 1, \dots, m-1$ we have*

$$\eta_j = \left\lfloor \frac{w_j}{m} \right\rfloor - \left\lfloor \frac{w_{j-1}}{m} \right\rfloor.$$

Moreover, it is easy to see that the numbers n_α satisfy the following properties:

Lemma 5.12. *[153, Proposition 9] The numbers n_α satisfy the following properties:*

(i) For $\alpha = 0, \dots, L$, we have that $1 \leq n_\alpha = |\Gamma \cap I_\alpha| \leq m-1$.
(ii) If $0 \leq \alpha < \beta \leq L-1$, then $n_\alpha \leq n_\beta$.
(iii) $\delta(\Gamma) = n_0 + n_1 + \cdots + n_L$.

Therefore we can compute $\delta(\Gamma)$ in terms of the Apéry set as follows:

Proposition 5.13. *Let Γ be a numerical semigroup, then*

$$\delta(\Gamma) = m\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1} \left\lfloor \frac{w_j}{m} \right\rfloor + \rho - m.$$

Proof. First of all, observe that $\delta(\Gamma) = n_0 + \cdots + n_L$ by Lemma 5.12. Thus, an extensive use of the statement in Lemma 5.11 shows that

$$\begin{aligned}
\delta =& \sum_{j=0}^{L} n_j = \sum_{j=1}^{m-1} (\eta_j \cdot j) + \rho - m = \sum_{j=1}^{m-1} \left(\sum_{i=j}^{m-1} \eta_i \right) + \rho - m \\
=& \sum_{j=1}^{m-1} \left(\sum_{i=j}^{m-1} \left\lfloor \frac{w_i}{m} \right\rfloor - \left\lfloor \frac{w_{i-1}}{m} \right\rfloor \right) + \rho - m = \sum_{j=1}^{m-1} \left(\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_{j-1}}{m} \right\rfloor \right) + \rho - m \\
=& (m-1)\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=1}^{m-2} \left\lfloor \frac{w_j}{m} \right\rfloor + \rho - m \\
=& m\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1} \left\lfloor \frac{w_j}{m} \right\rfloor + \rho - m \qquad \square
\end{aligned}$$

Observe that if we have to our disposal a Wilf type inequality then the ratio $c(\Gamma)/\delta(\Gamma) \leq k$. The classification of those $k \in \mathbb{Z}$ such that k satisfies a Wilf type inequality is therefore crucial for our purpose. With the previous notation, one can find the following characterization for a k satisfying a Wilf type inequality.

Proposition 5.14. *Let $k \in \mathbb{Z}$. Preserving notation as above, a numerical semigroup Γ satisfies the inequality $c(\Gamma) \leq k\delta(\Gamma)$ if and only if*

$$\sum_{j=0}^{L} (kn_j - m(\Gamma)) + m(\Gamma) - \rho \geq 0. \tag{5.2}$$

Proof. Using Lemma 5.12, it is easily checked that

$$c(\Gamma) \leq k\delta(\Gamma) \iff Lm(\Gamma) + \rho \leq k \sum_{j=0}^{L} n_j \iff \sum_{j=0}^{L} m(\Gamma) + \rho - m(\Gamma) \leq k \sum_{j=0}^{L} n_j$$
$$\iff \sum_{j=0}^{L} (kn_j - m(\Gamma)) + m(\Gamma) - \rho \geq 0. \quad \square$$

Remark 5.15. Proposition 5.14 is an analogue to [153, Proposition 10] but considering an integer k instead of $e(\Gamma)$ and the summation running up to L. In fact, notice that the inequality (5.2) can be reformulated as

$$\sum_{j=0}^{L-1} (kn_j - m(\Gamma)) + (n_L k - \rho) \geq 0.$$

5.2 Wilf function of a numerical semigroup

In this section we are going to introduce a new instrument for the study of the Wilf conjecture. Moreover, we will see in Section 5.4 that it is useful to study the Eliahou number of a numerical semigroup. Our approach to Wilf's conjecture takes into consideration the study of the behaviour of the map

$$\begin{aligned} W_\Gamma : \mathbb{N} &\to \mathbb{Z} \\ k &\mapsto W_\Gamma(k) := k\delta(\Gamma) - c(\Gamma). \end{aligned}$$

The function W_Γ will be called the *Wilf function* of the semigroup Γ. As already mentioned, for $k = e(\Gamma) = e$, the nonnegativity $W_\Gamma(e) \geq 0$ expresses thus the statement of Wilf's conjecture; indeed, the Wilf function contributes to the understanding of Wilf's conjecture and Frögohämosa-conjecture, as we shall see in Section 5.3. For the moment we will investigate this function along the remainder of the current section.

First of all, we recall that a numerical semigroup Γ is said to be *symmetric* if for every $z \in \mathbb{Z}$ one has that $z \in \Gamma \iff F(\Gamma) - z \notin \Gamma$. This is equivalent to say that $c(\Gamma) = 2g(\Gamma) = 2\delta(\Gamma)$, see e.g. [142, Corollary 4.5]. In particular, the Wilf function is nonpositive for $k = 2$:

Proposition 5.16. *Let Γ be a numerical semigroup, then $W_\Gamma(2) \leq 0$, and the equality holds if and only if Γ is symmetric.*

Proof. Since $c(\Gamma) \leq 2\delta(\Gamma)$, then the first assertion is clear. The equality holds in virtue of [158, p. 80, Proposition 7]. $\square$

On the other hand, we have been able to prove that the Wilf function $W_\Gamma(k)$ is nonnegative for $k = m$:

Theorem 5.17. *Let Γ be a numerical semigroup of multiplicity m, then $W_\Gamma(m) \geq 0$, and the equality holds if and only if $\Gamma = \langle m, qm+1, \ldots, qm+(m-1) \rangle$ for integers m, q such that $m > 1$ and $q > 0$.*

Proof. The fact that $W_\Gamma(m) \geq 0$ follows easily from Proposition 5.14 with $k = m$:

$$\sum_{j=0}^{L-1}(n_j m - m) + n_L m - \rho = m\sum_{j=0}^{L}(n_j - 1) + m - \rho \geq 0$$

together with the fact that $n_j \geq 1$ and $\rho \leq m$.

In order to prove the characterization of the equality, the converse is clear: If $\Gamma = \langle m, qm+1, \ldots, qm+(m-1)\rangle$, then $W_\Gamma(m) = m\delta - c = mq - qm = 0$. So let us prove the direct implication, and assume the existence of a semigroup $\Gamma = \langle a_1 = m, a_2, \ldots, a_e\rangle$ with a_i minimal generators such that $W_\Gamma(m) = m\delta - c = 0$. Recall that by Proposition 5.13

$$\delta = m\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor + \rho - m.$$

Now we appeal to the writing $c = Lm + \rho$, with $L = \left\lfloor \frac{w_{m-1}-m}{m}\right\rfloor = \left\lfloor \frac{w_{m-1}}{m}\right\rfloor - 1$ and $2 \leq \rho \leq m$. By the previous equalities we deduce

$$\begin{aligned} 0 = m\delta - c =& m\left(m\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor + \rho - m\right) - Lm - \rho \\ =& m\left(m\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor + \rho - m\right) - m\left(\left\lfloor \frac{w_{m-1}}{m}\right\rfloor - 1\right) - \rho \\ =& m\left((m-1)\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \sum_{j=0}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor + 1 + \rho - m\right) - \rho. \end{aligned}$$

For the sake of simplicity set $A := (m-1)\left\lfloor \frac{w_{m-1}}{m}\right\rfloor - \sum_{j=0}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor + 1 + \rho - m$, then the previous reasoning show that $\rho = A \cdot m$, i.e., ρ is a multiple of m that varies in the range $2 \leq \rho \leq m$, therefore it must be $A = 1$ and $\rho = m$.

We are now in a position to show that all Apéry elements have the same integral part if we divide them by the multiplicity of the semigroup. To this aim, we observe that, since $A = 1$ and $\rho = m$, we have

$$(m-1)\left\lfloor \frac{w_{m-1}}{m}\right\rfloor - \sum_{j=1}^{m-1}\left\lfloor \frac{w_j}{m}\right\rfloor = 0.$$

From this,

$$(m-2)\left\lfloor \frac{w_{m-1}}{m}\right\rfloor = \sum_{j=1}^{m-2}\left\lfloor \frac{w_j}{m}\right\rfloor.$$

Moreover, since the Apéry set is ordered by $w_0 < w_1 < \ldots, w_{m-1}$ it is clear that

$$\sum_{j=1}^{m-2}\left\lfloor \frac{w_j}{m}\right\rfloor \leq (m-2)\left\lfloor \frac{w_{m-1}}{m}\right\rfloor,$$

since for any $j = 1, \ldots, m-1$ it holds that $0 \leq \left\lfloor \frac{w_j}{m}\right\rfloor \leq \left\lfloor \frac{w_{m-1}}{m}\right\rfloor$. Altogether we obtain

$$\left\lfloor \frac{w_{m-1}}{m} \right\rfloor = \left\lfloor \frac{w_j}{m} \right\rfloor \text{ for any } j = 1, \ldots, m-1.$$

We want now to inspect the form of the minimal generators $a_2, \ldots, a_e$. Starting with a_2, since this coincides with w_1, we have

$$\left\lfloor \frac{a_2}{m} \right\rfloor = \left\lfloor \frac{w_{m-1}}{m} \right\rfloor = L+1,$$

where the last equality holds by the writing $c = Lm + \rho = (L+1)m$ under our proven condition $\rho = m$. Therefore there exists an integer α_2 such that

$$a_2 = (L+1)m + \alpha_2 \text{ with } 1 \leq \alpha_2 \leq m-1.$$

Moreover, by Lemma 5.6 there exists $j_2 \in \{2, \ldots, e-1\}$ such that $a_3 = w_{j_2}$ and again there is an integer α_3 such that

$$a_3 = (L+1)m + \alpha_3 \text{ with } 1 \leq \alpha_3 \leq m-1.$$

Since $a_2 < a_3$, it is obvious that $1 \leq \alpha_2 < \alpha_3$.

An easy reasoning by induction provides the shape of each minimal generator of Γ, namely

$$a_i = (L+1)m + \alpha_i \text{ with } 1 \leq \alpha_i \leq m-1,$$

for any $i = 2, \ldots, e$. It remains thus to prove that $e = m$ and $\alpha_i = i-1$ for $i = 2, \ldots, e = m$.

On the contrary, suppose that $e < m$. From what has already been proven, we may write

$$a_2 = w_1 < a_3 = w_{j_2} < a_4 = w_{j_3} < \cdots < a_e = w_{j_{e-1}} = w_{e-1} < w_e < \cdots < w_{m-1}, \tag{5.3}$$

where $\{j_2, j_3, \ldots, j_{e-1}\} = \{2, 3, \ldots, e-2\}$. Without loss of generality we may assume that $j_i = i$ for any $i = 2, \ldots, e-2$. Therefore w_e is not a minimal generator of Γ, although it belongs itself to Γ. This guarantees the existence of nonnegative integers $b_0, b_1, \ldots, b_{e-1}$ such that

$$\begin{aligned} w_e &= b_0 m + b_1 w_1 + b_2 w_2 + \cdots + b_{e-1} w_{e-1} \\ &= b_0 m + \sum_{j=1}^{e-1} \Big((L+1)m + \alpha_{j+1} \Big) b_j \\ &= b_0 m + \sum_{j=1}^{e-1} (L+1) m b_j + \sum_{j=1}^{e-1} \alpha_{j+1} b_j \\ &= \Big(b_0 + \sum_{j=1}^{e-1} (L+1) b_j \Big) m + \sum_{j=1}^{e-1} \alpha_{j+1} b_j. \end{aligned}$$

Observe that the right-hand side summation is positive, since $b_j \geq 0$ and $\alpha_{j+1} > 0$ for any $j = 1, \ldots, e-1$. There must thus exist $i_0 \in \{1, \ldots, e-1\}$ such that $b_{i_0} \neq 0$. This yields the writing

$$w_e = b_{i_0}(L+1)m + \Big(b_0 + \sum_{\substack{j=1 \\ j \neq i_0}}^{e-1} b_j (L+1) \Big) m + \sum_{j=1}^{e-1} \alpha_{j+1} b_j.$$

On the other hand, we may write

$$w_e = (L+1)m + \epsilon \text{ with } \alpha_{e-1} < \epsilon \leq m-1$$

(the cases $\epsilon \leq \alpha_{e-1}$ are all excluded in view of equation (5.3)). This implies that $b_j = 0$ for any $j \in \{1, \ldots, e-1\}$, $j \neq i_0$, and moreover $b_{i_0} = 1$. By the above,

$$w_e = (L+1)m + \alpha_{i_0} = w_{i_0 - 1} < w_e,$$

a contradiction. Therefore $e = m$.

Our next and last claim is that $\alpha_i = i - 1$ for any $i = 2, \ldots, m$; but this is easy, since there are $m-1$ values $\alpha_2 < \alpha_3 < \cdots < \alpha_m$ satisfying $1 \leq \alpha_i \leq m-1$ each of them.

Hence $\Gamma = \langle m, (L+1)m+1, \ldots, (L+1)m+m-1 \rangle$ with $m \geq 2$ and $L+1 \geq 1$, and the proof is complete. □

The Figure 5.1 sketches the behaviour of the Wilf function.

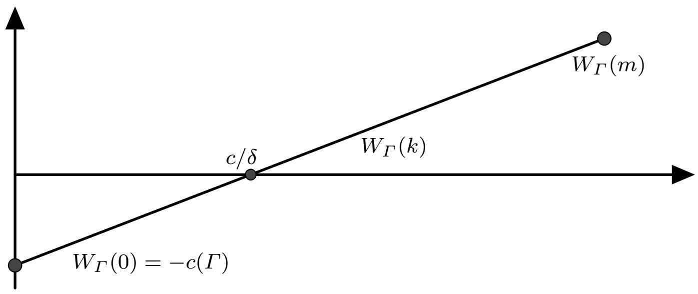


Figure 5.1: Sketch of the graph of the function $W_\Gamma(k)$.

In view of Figure 5.1, Theorem 5.17 together with Proposition 5.16 suggest that the interesting range through which we must let k run is precisely $2 \leq k \leq m$; the case $k = 1$ leads to $\Gamma = \mathbb{N}$ and is therefore irrelevant for our purpose. Wilf's question concerns the value $k = e$ that belongs certainly to this range. But in general, $k = e$ is not the minimal value making $W_\Gamma(k)$ nonnegative. This means, the Wilf number $W_\Gamma(e)$ does not yield in general a sharp bound for the positivity of the Wilf function. From this point of view, it would be interesting to investigate the constant

$$\mu_\Gamma := \min\{k \in \mathbb{N} : W_\Gamma(k) \geq 0\}, \tag{5.4}$$

where obviously $2 \leq \mu_\Gamma \leq m$. In other words, as emphasized by Figure 5.1, the function W_Γ considered as a real function has a root between 2 and m, and we propose to find the minimal integer value over the root so that the function is positive. Let us look at some examples.

Example 5.18. Before presenting the example we establish the following standard notation: write $S = \langle x_1, \ldots, x_s \rangle_r$ for the minimal semigroup that contains $\{x_1, \ldots, x_s\}$ and all the integers greater than or equal to r. This notation is widely used e.g. by Delgado in [36].

Consider the numerical semigroups

$$
\begin{aligned}
S_1 =& \langle 162, 1114, 1115 \rangle_{9879} \\
S_2 =& \langle 222, 1532, 1533 \rangle_{16647} \\
S_3 =& \langle 172, 327, 328 \rangle_{3437} \\
S_4 =& \langle 88, 100, 102 \rangle_{566} \\
S_5 =& \langle 88, 100, 343, 345, 346, 351, 361, 679, 680, 681, 687, 693 \rangle_{700}
\end{aligned}
$$

For each $i = 1, \ldots, 5$, we present in Table 5.1 the embedding dimension $e_i = e(S_i)$, the delta-invariant $\delta_i = \delta(S_i)$, the conductor $c_i = c(S_i)$, the Wilf number $W_i(e_i) = W_{S_i}(e_i)$ as well as $\mu_i = \mu_{S_i}$, the value $W_i(\mu_i)$ and the difference $\Delta_i := e_i - \mu_i$ for the semigroup S_i:

i	δ_i	c_i	e_i	μ_i	Δ_i	$W_i(e_i)$	$W_i(\mu_i)$
1	1109	9879	110	**9**	101	112111	102
2	1935	16647	147	**9**	138	267798	768
3	505	3437	97	**7**	90	45548	98
4	63	566	63	**9**	54	3403	1
5	100	700	51	**7**	44	5100	0

Table 5.1: Invariants of some semigroups.

These examples show how far is in some cases the bound given by the Wilf conjecture (i.e. the bound associated with the embedding dimension) to be sharp. They also motivate the careful study of the constant μ_Γ we have just introduced.

In addition, a companion question to that of the investigation of the constant μ_Γ is the following:

Question 5.19. To ask for properties of, and to characterize those numerical semigroups Γ with fixed μ_Γ. In particular, to characterize those semigroups with $\mu_\Gamma = e(\Gamma)$.

The value $\mu_\Gamma = 2$ is already known: it is equivalent to the case of Γ symmetric, which in purely algebraic terms means the Gorenstein property. We wonder whether the constant μ_Γ encloses any other stimulating feature.

To finish this section, let us present an easy procedure that allows us to find a numerical semigroup satisfying that $W(\mu_\Gamma) = 0$. As a first observation, we have that $W(\mu_\Gamma) = 0$ implies $\mu_\Gamma = c/\delta(\Gamma)$. Choose a semigroup $\Gamma = \langle a_1, \ldots, a_e \rangle$ with conductor $c(\Gamma)$ and embedding dimension $e = e(\Gamma)$. If $c(\Gamma)$ is a multiple of $\delta(\Gamma)$, we are done; otherwise, if $\delta(\Gamma)$ does not divide $c(\Gamma)$, we require the existence of a divisor $d \neq 1$ of $c(\Gamma)$ with the property that the number of elements of the semigroup in the integer interval $[c-1-a_1, c-1]$ is less than $a_1 - d - 1 + \delta(\Gamma)$; notice that it must hold $d - \delta(\Gamma) < a_1 - 1$. If this number does exist, then we consider the set $G = \{g_1, \ldots, g_{d-\delta(\Gamma)}\}$ consisting of $d - \delta(\Gamma)$ gaps such that $g_i + a_1 > c(\Gamma)$ and $g_i \neq F(\Gamma)$ for all $i = 1, \ldots, d - \delta(\Gamma)$. Therefore, we build the semigroup Γ' minimally generated by the generators of Γ and the gaps in G, namely $\Gamma' = \langle a_1, \ldots, a_e, g_1, \ldots, g_{d-\delta(\Gamma)} \rangle$. Now it is trivial to see that $\delta(\Gamma') = d$ and $c(\Gamma') = c(\Gamma)$, as desired.

To see an example, with the notation of Example 5.18 consider the numerical semigroup

$$S = \langle 88, 100, 343, 345, 346, 351, 361 \rangle_{700}.$$

This semigroup has embedding dimension $e(S) = 52$ and multiplicity $a_1 = 88$, delta-invariant $\delta(S) = 95$ and conductor $c(S) = 700$. The number of semigroup elements in the integer interval

$[611, 699]$ is 34, so it is possible to find a divisor $d \neq 1$ of $c(S)$ such that $34 < 88 - d - 1 + \delta(S)$, namely $d = 100$. In this case we find $d - \delta(S) = 5$ gaps of S different from the Frobenius number of S and bigger than the difference $c(S) - a_1$, e.g. $679, 680, 681, 687, 693$ and define a numerical semigroup S' minimally generated by the minimal generators of S plus the gaps $679, 680, 681, 687, 693$ such that $\delta(S') = d = 100$ and with the same conductor as S. This semigroup S' has therefore $\mu_{S'} = c(S')/\delta(S') = 700/100 = 7$.

Finally, observe that we can use the properties of the type of a semigroup to give an upper bound for the invariant μ_Γ.

Proposition 5.20. *Let Γ be a numerical semigroup. Then, $\mu_\Gamma \leq t(\Gamma) + 1$.*

Proof. Proposition 5.7 can be stated as $W_\Gamma(t(\Gamma)+1) \geq 0$ for any numerical semigroup. In particular this means that $\mu_\Gamma \leq t(\Gamma) + 1$. □

Remark 5.21. Observe that since $t(\Gamma) \leq m - 1$, the previous proposition provides an alternative proof to the inequality $W_\Gamma(m) \geq 0$.

5.3 The conjectures of Wilf and Frögohämosa

In the previous section, we have pointed out the well known fact that the Wilf number seems to be far from being 0 except in some explicit cases. The non-sharpness of $W_\Gamma(e)$ is closely related to Frögohämosa-conjecture; in this section, we are going to provide a unified treatment of both problems. From this unification of both conjectures we are going to be able to prove Frögohämosa-conjecture in several cases.

Conjecture 5.22 (Wilf archimedean conjecture). For any numerical semigroup Γ such that Γ does not have nor minimal neither maximal embedding dimension, we have

$$W_\Gamma(e) \geq 1.$$

Obviously, the Wilf archimedean conjecture implies both Wilf and Frögogämosa-conjectures.

Proposition 5.23. *For any numerical semigroup Γ,*

1. *Wilf archimedean conjecture $\Rightarrow$ Frögohämosa-conjecture;*
2. *Wilf archimedean conjecture $\Rightarrow$ Wilf's conjecture.*

Proof. The cases for maximal and minimal embedding dimension Frögohämosa-conjecture and Wilf's conjecture are true by Theorem 5.17 and Proposition 5.16. Therefore, let us assume that Wilf archimedean conjecture is true, then $W_\Gamma(e) \geq 1$. This means that the Wilf function at the embedding dimesion is strictly positive for any numerical semigroup which does not have minimal neither maximal embedding dimension. In particular, Frögohämosa-conjecture and Wilf's conjecture are true. □

Moreover, from the careful reading of [153, Proof of Theorem 18] we are able to show the following

Theorem 5.24. *Wilf archimedean conjecture is true for any numerical semigroup with $e > m/2$.*

Proof. We consider first the cases $e = m-1$ and $e = m-2$; since the Wilf function is increasing, Lemma 5.8 yields the inequality $W_\Gamma(t(\Gamma)+2) \leq W_\Gamma(e(\Gamma))$. By Proposition 5.7 it holds that $W_\Gamma(t(\Gamma)+1) \geq 0$. Therefore,

$$0 \leq W_\Gamma(t(\Gamma)+1) < W_\Gamma(t(\Gamma)+2) \leq W_\Gamma(e(\Gamma)),$$

which implies $W_\Gamma(e(\Gamma)) \geq \delta(\Gamma) > 1$.

Consider next the remainder of the cases, i.e. $m - e \geq 3$ and $e > m/2$. For the semigroups with these invariants, the proof of [153, Theorem 18] shows the following:

$$W_\Gamma(e) \geq (\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_2}{m} \right\rfloor - \left\lfloor \frac{w_1}{m} \right\rfloor - 1)(3e-m) + \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) + n_L e - \rho.$$

For this bound, the proof of [153, Theorem 18] distinguishes four cases. We may be even sharper in these estimations as we assume $e > m/2$. More precisely, if m is even then $4e - 2m \geq 4$, and if m is odd then $4e - 2m \geq 2$ since we are assuming the strict inequality $e > m/2$ and $e \in \mathbb{N}$. Having this in mind, we turn our attention to the four cases considered in the proof of [153, Theorem 18]:

Case 1: If $\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_2}{m} \right\rfloor - \left\lfloor \frac{w_1}{m} \right\rfloor - 1 = -1$ and $n_L \geq 3$,

$$W_\Gamma(e) \geq \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) + (n_L - 3)e + m - \rho \geq 2\left\lfloor \frac{w_1}{m} \right\rfloor + m - \rho \geq 1.$$

Case 2: If $\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_2}{m} \right\rfloor - \left\lfloor \frac{w_1}{m} \right\rfloor - 1 \geq 0$ and $n_L \geq 2$,

$$W_\Gamma(e) > \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) + (n_L e - m) + m - \rho \geq 2\left\lfloor \frac{w_1}{m} \right\rfloor + m - \rho \geq 1,$$

since $n_L e \geq 2m/2 = m$.

Case 3: If $\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_2}{m} \right\rfloor - \left\lfloor \frac{w_1}{m} \right\rfloor - 1 \geq 0$, $n_L \geq 1$ and $n_{L-1} \geq 4$,

$$W_\Gamma(e) > e + \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) + (n_L e - m) + m - \rho \geq 2\left\lfloor \frac{w_1}{m} \right\rfloor + m - \rho \geq 1,$$

Case 4: If $\left\lfloor \frac{w_{m-1}}{m} \right\rfloor - \left\lfloor \frac{w_2}{m} \right\rfloor - \left\lfloor \frac{w_1}{m} \right\rfloor - 1 \geq 0$, $n_L \geq 1$, $n_{L-1} = 3$, $m - e = 3$ and $\rho \leq m-2$

$$W_\Gamma(e) > \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) + e - \rho \geq \left\lfloor \frac{w_1}{m} \right\rfloor (4e-2m) - 1 \geq 2\left\lfloor \frac{w_1}{m} \right\rfloor - 1 \geq 1,$$

since $e - \rho = e - m + m - \rho \geq -3 + 2 = -1$. □

In the proof of [119, Theorem 1], Moscariello and Sammartano showed that, under the hypothesis of [119, Theorem 1], the strict inequality in Wilf's conjecture holds.

Proposition 5.25. *Let Γ be a numerical semigroup with embedding dimension e and multiplicity m. Let us denote by $U := \lceil m/e \rceil$. Wilf archimedean conjecture holds if the prime factors of m are greater than or equal to U and*

$$m \geq \frac{U(3U^2 - U - 4)(3U^2 - U - 2)}{8(U-2)}.$$

Proof. By [119, Theorem 1], $W_\Gamma(e) > 0$. Thus, $W_\Gamma(e) > 1$ since $W_\Gamma(e) \in \mathbb{N}$. □

Before continuing, let us define the following remarkable families of numerical semigroups.

Definition 5.26. A *pseudo-symmetric numerical semigroup* is a numerical semigroup whose Frobenius number $F(\Gamma) := c(\Gamma) - 1$ is even and $x \in \mathbb{N} \setminus \Gamma$ implies that either $F(\Gamma) - x \in \Gamma$ or $x = F(\Gamma)/2$ (see also [142, Proposition 4.4]).

Definition 5.27. Let p be an even positive integer. The family of numerical semigroups $S(p)$ defined by Delgado, in [36, Sect. 3], are those semigroups containing the numbers

$$\begin{aligned} \nu = \nu(p) =& \frac{p}{2}(\frac{p}{2} + 4) + 2 \\ \gamma = \gamma(p) =& 2\nu(p) - (\frac{p}{2} + 4) \end{aligned}$$

as well as $\gamma + 1$ and all the integers greater than or equal to $p\nu$. We call such numerical semigroups *Delgado numerical semigroups.*

Definition 5.28. Let $m, a, b, n \in \mathbb{N} \setminus \{0\}$ satisfy $n \geq 3$ and $(3m + 1)/2 \leq a < b \leq (5m - 1)/3$.

Let $A \subset \mathbb{N} \setminus \{0\}$ be a subset of cardinality $|A| = n - 1$ with $\min(A) = a$, $\max(A) = b$ and inducing a B_3 set in $\mathbb{Z}/m\mathbb{Z}$; where a B_3 set (see also [52, Section 3.2]) is defined as a subset B of an abelian group G such that

$$|3B| \leq \binom{|B| + 2}{2}.$$

With the previous notation let us consider $S = \langle \{m\} \cup A \rangle_{4m}$ the minimal semigroup containing $\{m\} \cup A$ and all natural numbers bigger or equal than $4m$. We call to such a semigroup *Eliahou-Fromentin numerical semigroups.*

There are many other cases where the Wilf conjecture is known to be true. However, we have chosen the previous semigroups since similar arguments as those of Theorem 5.24 and Proposition 5.25 show that the Wilf archimedean conjecture is also true.

Theorem 5.29. *Let Γ be a numerical semigroup such that Γ does not have minimal neither maximal embedding dimension. Then, Wilf archimedean conjecture is true for the following families:*

(1) Symmetric numerical semigroups.

(2) Pseudo-symmetric numerical semigroups with $c(\Gamma) \geq 2$.

(3) Eliahou-Fromentin numerical semigroups.

(4) Numerical semigroups with $c(\Gamma) \leq 3m(\Gamma)$.

(5) Numerical semigroups with $|\mathbb{N} \setminus \Gamma| \leq 60$.

(6) Delgado numerical semigroups.

Proof. (1) is immediate from Proposition 5.16 since a symmetric numerical semigroup has $W_\Gamma(2) = 0$.

(2) follows from [47, Proof of Proposition 2.2]; they show that a pseudo-symmetric numerical semigroup satisfies $e(\Gamma)\delta(\Gamma) \geq 3(c(\Gamma) - 1)/2$.

(3) follows from [52, Theorem 4.1] Eliahou-Fromentin semigroups satisfy $W_\Gamma(e) \geq 9$.

(4) The case $c \leq 2m(\Gamma)$ comes from [83, Proposition 26] and the case $2m < c \leq 3m$ by [51, Remark 6.6].

(5) has been checked computationally [60].

(6) By [36, Proposition 6]. □

As a general conclusion of this section, we would like to emphasize the importance of the constant μ_Γ. Observe that in most of the previous cases $\mu_\Gamma \neq e(\Gamma)$. Moreover, Table 5.1 shows that the difference $e(\Gamma) - \mu_\Gamma$ can be extremely big. However, observe that the remaining cases of Wilf archimedean conjecture, i.e. those numerical semigroups with $e(\Gamma) < m(\Gamma)/2$ and low multiplicity, impose that the difference $e(\Gamma) - \mu_\Gamma$ must be "small". This shows how important is to have a good control over the invariant μ_Γ.

Moreover, the Wilf function is conceptually very useful: it allows us to guess an explanation of the exceptional numerical semigroups appearing in Frögohämosa-conjecture, i.e. those with $W_\Gamma(e) = 0$. Observe that the numerical semigroups appearing as candidates for being the only ones with $W_\Gamma(e) = 0$ in Frögohämosa-conjecture are precisely those for which the Wilf function has a extreme behaviour.

Obviously, since $W_\Gamma(0)$ and $W_\Gamma(1)$ are negative for every numerical semigroup $\Gamma \neq \mathbb{N}$, the first integer where the function could be positive is 2; as we have previously shown, $W_\Gamma(m) \geq 0$ for any numerical semigroup. Therefore to say that 2 and the multiplicity of Γ are extreme values for $W_\Gamma(k)$ means the following:

Theorem 5.30. *Let Γ be a numerical semigroup. Then,*

1. *$\Gamma = \mathbb{N}$ if and only if $W_\Gamma(k) \geq 0$ for $1 \leq k \leq m$.*
2. *$\Gamma = \langle m, qm+1, \ldots, qm+(m-1)\rangle$ for $q \geq 1$ if and only if $W_\Gamma(k) \leq 0$ for all $1 \leq k \leq m$.*
3. *$\Gamma = \langle a, b\rangle$ with $\gcd(a,b) = 1$ is the numerical semigroup with minimal embedding dimension between those satisfying $W_\Gamma(k) \geq 0$ for all $2 \leq k \leq m$ and $W_\Gamma(2) = 0$.*

Proof. The statement (1) is clear. The second assertion is equivalent to Theorem 5.17 and the fact that the Wilf function is strictly increasing. Finally, for the assertion (3) let us assume that $W_\Gamma(k) \geq 0$ for all $2 \leq k \leq m$ and $W_\Gamma(2) = 0$. By Proposition 5.16 the numerical semigroups satisfying those conditions are exactly the family of symmetric numerical semigroups. In particular, the symmetric numerical semigroups with minimal embedding dimension are those of type $\Gamma = \langle a, b\rangle$ with $\gcd(a,b) = 1$. □

In Figure 5.2 and Figure 5.3 we try to illustrate this phenomenon for both semigroups $\Gamma = \langle \alpha, \beta\rangle$ of embedding dimension two and the semigroups with maximal embedding dimension of the special form $\Gamma = \langle m, qm+1, \ldots, qm+(m-1)\rangle$.

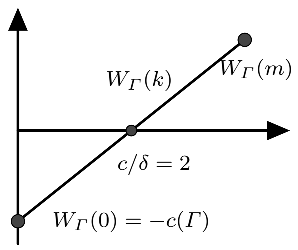


Figure 5.2: Sketch of the graph of a symmetric numerical semigroup $W_\Gamma(k)$ with $e = 2$.

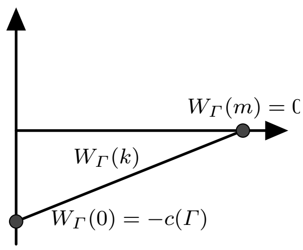


Figure 5.3: Graph of the function $W_\Gamma(k)$ for $\Gamma = \langle m, qm+1, \ldots, qm+(m-1)\rangle$

We finish the section with the observation that the vanishing of the Wilf function imposes a very strong condition on Γ:

Proposition 5.31. *Let Γ be a numerical semigroup. If $W_\Gamma(k) = 0$ for some k, then $k\delta \le (L+1)m$, hence $k \le m$.*

Proof. The first statement is clear by the above reasonings. Moreover, by the statement (iii) in Lemma 5.12 we have $\delta = \sum_{j=0}^{L} n_j \ge L+1$, it follows that $(L+1)k \le \delta k \le (L+1)m$ and so $k \le m$. □

The conclusion $k \le m$ in Proposition 5.31 follows also from the application of the Darboux property to the Wilf function, as it is easily deduced by an inspection of Figure 5.1.

5.4 On the negativity of the Eliahou number

An ultimate tool towards the solution of the Wilf conjecture seems to be the Eliahou number, whose definition will be recalled in the sequel. Let $q := q(\Gamma) = \lceil \frac{c(\Gamma)}{m(\Gamma)} \rceil$ be the q-number of Γ. We set

$$\nu(\Gamma) = \nu = qm - c, \quad \mathrm{small}(\Gamma) = |\{s \in \Gamma : s < c\}|, \quad G(\Gamma) = G := \{a_1, \dots, a_e\}.$$

In contrast with the partition proposed by Sammartano, which we have introduced in Section 5.1, Eliahou [51] defined the following partition of the interval $[-\nu, c+m]$:

$$J_\alpha := [\alpha m - \nu, (\alpha+1)m - \nu] \quad \text{for } \alpha = 0, 1, \dots, q.$$

The main advantage of Eliahou's partition is that the last subinterval is $J_q = [c, c+m]$.

Set $p_q := J_q \cap G$ and $d_q := [c, c+m] \setminus p_q$. Let us denote by $e_s := |G \cap \mathrm{small}(\Gamma)|$ resp. $e_c := |p_q|$ the number of minimal generators of the semigroup which are smaller than the conductor resp. bigger than the conductor. Obviously, $e(\Gamma) = e_s + e_c$. Eliahou introduced the following invariant [51], named the *Eliahou number* of Γ by Delgado in [36]:

$$E(\Gamma) = e_s \delta(\Gamma) - q|d_q| + \nu.$$

The Eliahou number plays an important role in Wilf's conjecture in virtue of the following [51, Proposition 3.11]:

Theorem 5.32 (Eliahou). *Let Γ be a numerical semigroup, then $W_\Gamma(e) \ge E(\Gamma)$.*

It is an important result the fact that negative Eliahou numbers can be effectively attained, see [36, Corollary 14, Corollary 35]:

Theorem 5.33 (Delgado). *For any $z \in \mathbb{Z}$ there exists a numerical semigroup with Eliahou number $E(\Gamma) = z$. In particular, there exist numerical semigroups with arbitrarily negative Eliahou number.*

The negativity of Eliahou number poses an interesting question within the theory of numerical semigroups: the semigroups having negative Eliahou number seem to be rare and infrequent, as already observed in several works [36, 38, 52]. We will first present a lower bound for the Eliahou number in terms of the Wilf function. This contrasts with the fact that the Eliahou number attains any integer value and allows us to provide a necessary condition for its negativity in terms of the Wilf function. In addition, we continue the section investigating Eliahou numbers in semigroups with fixed concentration. This will allow us to provide a necessary condition for its negativity in terms of the concentration.

5.4.1 Eliahou number vs Wilf function

As we have seen in Theorem 5.33, Delgado showed that the Eliahou number can attain any integer value. The main problem for the computation of the Eliahou number is that the Eliahou partition J_α defining Eliahou number does not coincide with the one defined by Sammartano I_α, and Sammartano's partition allows an easier calculation of δ, as Proposition 5.13 witnesses.

Our main idea in this subsection is to give a range of the possible values of the Eliahou number by considering the Wilf function; this will allow us to check only the properties of Wilf function in order to study semigroups with a prescribed Eliahou number.

Theorem 5.34. *Let Γ be a numerical semigroup with embedding dimension e. We have the inequalities*

$$W_\Gamma(e) \geq E(\Gamma) \geq W_\Gamma(e_s).$$

Proof. The first inequality is due to Eliahou [51, Proposition 3.11]. The second inequality is deduced from the fact that $|d_q| \leq m$, so that

$$W_\Gamma(e) \geq E(\Gamma) = e_s\delta(\Gamma) - q|d_q| + \nu \geq e_s\delta(\Gamma) - qm + \nu = W_\Gamma(e_s). \quad \square$$

Example 5.35. According to the computations done with the functions in GAP [61], the numerical semigroup $\Gamma := \langle 30, 42, 51\rangle_{290}$ has $W_\Gamma(e_s) < 0$, $\mu_\Gamma = 5$, $c = 290$, $e = 23$, $\delta = 65$ and $W_\Gamma(\mu_\Gamma) = 35 < E(\Gamma) = 105 < W_\Gamma(e) = 1205$.

Theorem 5.34 yields a necessary condition for the negativity of Eliahou number in terms of the Wilf function.

Theorem 5.36. *Let Γ be a numerical semigroup with Eliahou number $E(\Gamma) < 0$. Then,*

$$W_\Gamma(e) < e_c\delta.$$

In particular, $\mu_\Gamma > e_s$.

Proof. We begin by observing that $W_\Gamma(1) = -\sum_{j=0}^{m-1} \left\lfloor \frac{w_j}{m} \right\rfloor$; this follows by Proposition 5.13 and by the fact that $c = Lm + \rho$ with $L = \lfloor \frac{w_{m-1}}{m} \rfloor - 1$.

On the other hand, from the linearity of Wilf function we have $W_\Gamma(e_s) = (e_s - 1)\delta + W_\Gamma(1)$. Moreover, since $E(\Gamma) < 0$, Theorem 5.34 implies $W_\Gamma(e_s) < 0$. All this together yields

$$W_\Gamma(e-1) = W_\Gamma(e_s+e_c-1) = (e_s-1)\delta + W_\Gamma(e_c) < -W_\Gamma(1) + W_\Gamma(e_c) = (e_c-1)\delta,$$

which establishes the desired inequality. □

5.4.2 Positivity of the Eliahou number associated to the concentration

Once we have shown a necessary condition for the negativity of the Eliahou number obtained thanks to the Wilf function, our purpose now is to give a necessary condition for the negativity of Eliahou number in terms of the concentration of the semigroup. To do so, we first need to prove the following.

Theorem 5.37. *Let Γ be a numerical semigroup with multiplicity m and concentration $\mathbf{C}(\Gamma) = k$. Write $c = Lm + \rho$ with $2 \leq \rho \leq m$, and assume that $c > 2m$. If $m/k^2 > (L+1)/(L-1)$, then $E(\Gamma) \geq 0$.*

Proof. First of all, observe that Lemma 5.10 implies that $L+1 = \lceil c/m \rceil = q$. Also $|d_q| \leq m$. Therefore

$$E(\Gamma) \geq e_s\delta(\Gamma) - (L+1)m.$$

On the other hand, Proposition 5.43 together with Proposition 5.46 give us

$$e_s\delta(\Gamma) - (L+1)m \geq \left(\frac{m}{k}\right)\left(\frac{(L-1)m+\rho+k}{k}\right) - (L+1)m.$$

Since $\rho, k \geq 0$, the claim follows from the hypothesis $m/k^2 > (L+1)/(L-1)$. □

Theorem 5.37 gives us an easy-to-handle condition which implies the positivity of the Eliahou number. In contrast to the examples of negative Eliahou number given by Delgado [36], Eliahou [51], and Fromentin [52], our condition only assumes the knowledge of the multiplicity, the concentration and the conductor of the semigroup. In this way we do not need to compute neither the embedding dimension nor the δ-invariant in our case. This leads to the following necessary condition for a semigroup to be a semigroup with negative Eliahou number.

Corollary 5.38. *Let Γ be a numerical semigroup with multiplicity m and concentration $\mathbf{C}(\Gamma) = k$, and write $c = Lm + \rho$ with $2 \leq \rho \leq m$. If $E(\Gamma) < 0$, then $m/k^2 < (L+1)/(L-1)$.*

Remark 5.39. It is not difficult to check that all the semigroups defined by Delgado in [36] that have negative Eliahou number satisfy the inequality $m/k^2 < (L+1)/(L-1)$ and $k < m$.

Here it is natural to ask whether the condition $m/k^2 < (L+1)/(L-1)$ is too restrictive. This seems not to be the case: it is quite easy to construct numerical semigroups satisfying the mentioned inequality. The general trick to find them is to observe that $1 < (L+1)/(L-1) < 2$ if $L \geq 4$ and $(L+1)/(L-1) \geq 2$ if $1 \leq L \leq 3$. Now, we have two options: either we choose a big multiplicity in order to allow bigger concentrations, or we choose directly small concentrations. Let us illustrate this behaviour with some examples computed with the aid of GAP [61, 39]:

Example 5.40. Let $A := \{1000 + 25 \cdot k \mid 0 \leq k \leq 39\}$. Let Γ be the numerical semigroup minimally generated by $A \cup \{1507, 1899, 13765, 13790, 13815\}$. The multiplicity of Γ is $m(\Gamma) = 1000$, the conductor is $c = 13741 = 13 \cdot 1000 + 741$, and the concentration $\mathbf{C}(\Gamma) = 25$. Thus $L = 13$ and the conditions of Theorem 5.37 are fulfilled, therefore Γ has positive Eliahou number.

Example 5.41. Let us consider the numerical semigroup defined by

$$\Gamma = \langle 50, 55, 60, 65, 70, 73, 77, 81, 86, 91, 96, 194, 199 \rangle.$$

We see that $c = 190$ and it has concentration $\mathbf{C}(\Gamma) = 5$. Then it fulfils the hypothesis of Theorem 5.37 and so $E(\Gamma) > 0$. Moreover, since $\delta = 66 > 50$ it satisfies the conditions of Proposition 5.48 so $W(6) \geq 0$. An easy computation shows that $E(\Gamma) = 544$ and $W(6) = 206$. It is also easily seen that $\mu_\Gamma = 3$.

On the other hand, the type of Γ is 17, and it is neither symmetric nor pseudo-symmetric, according to the computations done with the routines in GAP [61]. Moreover, it is easily checked that it does not fulfils any of the conditions of the main theorems of [119, 153].

5.4.3 Examples of semigroups with negative Eliahou number

Not many examples of numerical semigroups with negative Eliahou number are known. Some of them already appeared in Eliahou's paper [51]. Those are the unique numerical semigroups with negative Eliahou number and $c - \delta \leq 60$. Later, Delgado [36, Sections 3 and 4] provided several families of numerical semigroups with negative Eliahou number and $e_s = 3$. In fact, these families offer examples with arbitrarily large negative Eliahou number. Moreover, Delgado showed a few examples with $e_s = 4, 5$ in [36, Tables 6 and 7]. More recently, Eliahou and Fromentin [52] presented new families of numerical semigroups with negative Eliahou number, all of them with $c = 4m$.

It is not difficult to check that all the examples provided by Delgado, Eliahou and Fromentin satisfy the conditions of Theorem 5.36 and Corollary 5.38. We were wondering whether these necessary conditions may help to find new examples of numerical semigroups with negative Eliahou number. This is the case; in fact we present now a few of them: it is straightforward to check that they do not belong to the above collections of Delgado resp. Eliahou and Fromentin [51, 52], since in our examples we have $e_s = 4$ and $c \geq 5m$; to the best of the authors' knowledge, these are not mentioned in the literature.

Example 5.42. In Table 5.2 we show eight numerical semigroups with negative Eliahou number, $e_s = 4$ and concentrations $70, 100$.

Γ	$E(\Gamma)$	$\mathbf{C}(\Gamma)$	e_i	μ_i	$W_i(e_i)$	$W_i(\mu_i)$
$\langle 100, 170, 171, 176 \rangle_{599}$	-1	70	71	**13**	2880	38
$\langle 100, 270, 272, 275 \rangle_{998}$	-2	100	70	**15**	4882	52
$\langle 100, 270, 271, 175 \rangle_{999}$	-3	100	70	**12**	4881	9
$\langle 100, 270, 273, 275 \rangle_{1000}$	-4	100	70	**12**	4880	8
$\langle 100, 170, 173, 174 \rangle_{597}$	-5	70	70	**13**	2833	40
$\langle 100, 170, 172, 175 \rangle_{598}$	-6	70	70	**13**	2832	39
$\langle 100, 170, 173, 175 \rangle_{599}$	-7	70	70	**13**	2831	38
$\langle 100, 170, 172, 175 \rangle_{600}$	-8	70	70	**13**	2830	37

Table 5.2: Some semigroups with negative Eliahou number.

Different combinations of the minimal generators and conductors of the examples of Table 5.2 allowed us to find 36 numerical semigroups with Eliahou number within the interval $[-8,-1]$. Those semigroups are of two types:

T1 $\langle 100, 170, a, b\rangle_c$ with $a, b \in [171, 176]$, $c \in [597, 600]$. These semigroups have $e_s \in \{3, 4\}$ and $c > 5m$.

T2 $\langle 100, 270, a, b\rangle_c$ with $a, b \in [271, 276]$, $c \in [997, 1000]$. These semigroups have $e_s \in \{3, 4\}$ and $c > 9m$.

There are 18 numerical semigroups of type 1 and negative Eliahou number and 18 numerical semigroups of type 2 and negative Eliahou number. All of them can be computed with the help of GAP [61, 39].

5.5 Positivity of the Wilf function associated to the concentration

Let us come back to the invariant μ_Γ. This section is devoted to use the notion of concentration to show upper bounds for the invariant μ_Γ.

To do so, we are first going to give estimates for the δ-invariant and the embedding dimension in terms of the multiplicity of the semigroup and its concentration.

Proposition 5.43. *Let Γ be a numerical semigroup with concentration $\mathbf{C}(\Gamma) = k$ and $c = Lm + \rho$, then*

$$\delta(\Gamma) \geq \frac{(L-1)m + \rho}{k} + 1.$$

Proof. With the notation of Section 5.1, let us write I_α for the subintervals of the form $I_\alpha := [\alpha m, (\alpha+1)m - 1]$ appearing in Sammartano's partition of the interval $[0, c+m]$. Let us denote by $A_\alpha := I_\alpha \cap \Gamma$. For $1 \leq \alpha \leq L-1$, let us consider the set

$$A'_\alpha := \{b_1 := \alpha m < \cdots < b_s := (\alpha+1)m \mid b_i \in \Gamma\}.$$

Thus, $|A_\alpha| = |A'_\alpha| - 1 = s - 1$. On the other hand, since we are assuming concentration k, we have that

$$m = (b_s - b_{s-1}) + \cdots + (b_2 - b_1) \leq k(s-1).$$

Hence, $|A_\alpha| = s - 1 \geq \frac{m}{k}$.

A simple observation shows that $|A_0| = 1$, and so

$$\delta(\Gamma) = 1 + \sum_{\alpha=1}^{L-1} |A_\alpha| + (|B_L| - 1),$$

where $B_L := \{x_1 := Lm < \cdots < x_t := c = Lm + \rho\}$.

Again, since the concentration is assumed to be k, we have $|B_L| - 1 \geq \rho/k$. Therefore

$$\delta(\Gamma) = 1 + \sum_{\alpha=1}^{L-1} |A_\alpha| + |B_L| \geq \frac{(L-1)m + \rho}{k} + 1. \quad \square$$

Remark 5.44. Observe that the bound in Proposition 5.43 is sharp: consider

$$\Gamma = W_{m,q} = \langle m, qm+1, \ldots, qm+(m-1)\rangle$$

for integers m, q such that $m > 1$ and $q > 0$. These semigroups have concentration $\mathbf{C}(\Gamma) = k = m$, and moreover $\rho = m$ and $L = q-1$. Also it is easy to check that $\delta = L+1$. Thus,

$$L+1 = \delta \geq \frac{(L-1)m+m}{m} + 1 = L+1.$$

By Theorem 5.30, the semigroups $W_{m,q}$ are indeed very interesting in the context of Wilf's conjecture, since they are extreme cases of the conjecture.

Remark 5.45. In the particular case of $\mathbf{C}(\Gamma) = 2$ and $c > 2m$, Rosales et al. [141, Lemma 2] show the inequality $\delta \geq m/2+2$. Observe that their case constitute now a particular case of our Proposition 5.43.

Proposition 5.46. *Let Γ be a numerical semigroup with concentration $\mathbf{C}(\Gamma) = k$ and conductor $c > 2m$, then $e_s \geq m/k$. In particular, the embedding dimension is bounded below by m/k, i.e. $e \geq m/k$.*

Proof. Any element of the interval $I_1 \cap \Gamma$ is a minimal generator of the semigroup. Hence $e \geq |I_1 \cap \Gamma| \geq m/k$, where the last inequality holds by the same arguments as those in the proof of Proposition 5.43. □

The nonnegativity of the Wilf function can be related to the concentration of the semigroup in the following manner.

Proposition 5.47. *Let Γ be a numerical semigroup with concentration $\mathbf{C}(\Gamma) = k$, then $W_\Gamma(2k) \geq 0$. In particular, $2k \geq \mu_\Gamma$.*

Proof. Let us write $c = Lm + \rho$ with $L := \lfloor \frac{c-1}{m} \rfloor$ and $2 \leq \rho \leq m$. By Proposition 5.43 we have $k\delta(\Gamma) \geq (L-1)m+\rho$. Moreover, since $G \subset \Gamma \backslash \{0\}$ and $e = |G|$, Proposition 5.46 implies $k\delta(\Gamma) \geq m$. Therefore

$$W_\Gamma(2k) = 2k\delta(\Gamma) - c \geq (L-1)m + \rho + m - c = 0. \quad \square$$

The inequalities in Proposition 5.47 can be improved by adding additional hypothesis:

Proposition 5.48. *Let Γ be a numerical semigroup with concentration $\mathbf{C}(\Gamma) = k$.*

If $\delta(\Gamma) \geq m-k$, then $W_\Gamma(k+1) \geq 0$. In particular, $k+1 \geq \mu_\Gamma$.

Proof. Let us write $c = Lm + \rho$ with $L := \lfloor \frac{c-1}{m} \rfloor$ and $2 \leq \rho \leq m$. By Proposition 5.43 we have $k\delta(\Gamma) \geq (L-1)m + \rho + k$. Since $\delta(\Gamma) \geq m-k$ by hypothesis, the claim follows. □

5.6 Highly dense numerical semigroups

Finally, let us present a new class of numerical semigroups satisfying Wilf's conjecture. To do so, we will make use of the results of Section 5.5. In addition, they also have positive Eliahou number under certain restrictions. We need first to define the notion of *highly dense* numerical semigroup:

Definition 5.49. We say that Γ is highly dense if one of the following two conditions is satisfied:

1. Γ has concentration less or equal than 2.
2. Γ has concentration less or equal than $e(\Gamma)/2$ and $4 \leq e(\Gamma)$.

Examples 5.40 and 5.41 show already highly dense numerical semigroups with concentration $\mathbf{C}(\Gamma) = 5$ resp. $\mathbf{C}(\Gamma) = 25$. We employ the terminology *highly dense* due to the fact that small concentrations lead to higher number of elements of the numerical semigroup in the interval $[0, c]$, as Proposition 5.43 shows.

Definition 5.49 and the discussion of Section 5.5 lead to the following

Proposition 5.50. *Let Γ be a highly dense numerical semigroup. Then $W_\Gamma(e) \geq 0$.*

Proof. This is an straightforward consequence of Proposition 5.47. □

Therefore, highly dense numerical semigroups provide a new family of numerical semigroups satisfying Wilf's conjecture. Moreover, we can use Theorem 5.34 to show that —under an additional hypothesis— highly dense numerical semigroups have positive Eliahou number.

Corollary 5.51. *Let Γ be a numerical semigroup with $\mathbf{C}(\Gamma) = k \geq 2$, with conductor $c > 2m$ and satisfying $e_s \geq 2k$. Then $E(\Gamma) \geq 0$.*

In particular, any highly dense numerical semigroup with $e_s \geq 2k$ has positive Eliahou number.

Proof. By Theorem 5.34 we have $E(\Gamma) \geq W_\Gamma(e_s)$. Since $e_s \geq 2k$, the linearity of the Wilf function together with Proposition 5.47 shows

$$E(\Gamma) \geq W_\Gamma(e_s) \geq W_\Gamma(2k) \geq 0. \quad \square$$

Corollary 5.52. *Let Γ be a numerical semigroup with concentration $k \geq 2$, with $e_s \geq k+1$, delta-invariant $\delta(\Gamma) \geq m - k$, and conductor $c > 2m$. Then $E(\Gamma) \geq 0$.*

In particular, any highly dense numerical semigroup with $e_s \geq k+1$ and $\delta(\Gamma) \geq m-k$ has positive Eliahou number.

Proof. By Theorem 5.34 we have $E(\Gamma) \geq W_\Gamma(e_s)$. Since $e_s \geq k+1$ and $\delta(\Gamma) \geq m-k$, the linearity of the Wilf function together with Proposition 5.48 shows

$$E(\Gamma) \geq W_\Gamma(e_s) \geq W_\Gamma(k+1) \geq 0. \quad \square$$

Chapter 6

An extension of the Wilf conjecture to semimodules over a numerical semigroup

In the previous chapter, the Wilf function of a numerical semigroup Γ has revealed to be an useful tool in the understanding of Wilf's conjecture and its related topics. Thus, through the analysis of the Wilf function we have introduced the invariant μ_Γ which plays a key role as refinement of Wilf's conjecture. Even more, we have been able to provide new examples of numerical semigroups satisfying Wilf's conjecture. All those results are based on some specific properties of the elements of a numerical in the interval $[0, c + m(\Gamma)]$. However, there are another structure related to the gaps of Γ that one can take into account in this challenging problem.

Pieces of information about Γ are also encoded in its semimodules. A Γ-semimodule Δ is a non-empty subset of $\mathbb{Z}$ that is bounded below and satisfies $\Delta + \Gamma \subseteq \Delta$. In analogy to the case of a numerical semigroup, a Γ-semimodule Δ is finitely generated, has a unique system of minimal generators and therefore it possesses invariants such as conductor, Frobenius number, embedding dimension, or delta-invariant. Moreover, as we will see in Section 6.1 the non–zero minimal generators of Δ are gaps of Γ.

Hence it is possible –and natural– to consider the Wilf function associated to a Γ-semimodule Δ, say $W_\Delta(k)$. Thus, it is reasonable to ask for a possible extension of Wilf's conjecture in the case of a Γ–semimodule Δ. In this case, the definition of the Wilf function for a semimodule is key in order to provide a good generalization of Wilf's conjecture for semimodules over a numerical semigroup as we will see in Section 6.2. The main problem in this context is that the natural generalization of Wilf's conjecture, i.e. $W_\Delta(e(\Delta)) \geq 0$, does not work. This is because $W_\Delta(e(\Delta))$ depends on the generators of Δ as one can see for example in Proposition 6.30. Therefore, it is a challenging question to find the minimal k such that $W_\Delta(k) \geq 0$ under certain restriction. We have encoded the main questions in Question 6.28. In this way, Question 6.28 can be considered as an extension of the Wilf conjecture to semimodules.

One the other hand, it is natural to ask if the Wilf function of a Γ–semimodule can provide some insight about Wilf's conjecture. In this direction, in subsection 6.2.1 we study the properties of the Wilf function of a semimodule with two generators which leads us to propose the following conjecture.

Conjecture 6.1 (Bound conjecture). Let Γ be a numerical semigroup. There exists a semimodule Δ minimally generated by $\{0, g\}$ for a gap g of Γ, such that $W_\Delta(2) \geq -W_\Gamma(e(\Gamma))$.

This viewpoint might bring some knowledge in order to solve the fascinating and involved Wilf's conjecture, since we are also able to prove

Theorem 6.2. *Bound conjecture* $\implies$ *Wilf's conjecture.*

It is certainly remarkable that the Wilf function associated to a Γ-semimodule is related to the Wilf conjecture. We hope that this new viewpoint may be helpful for the understanding of the Wilf conjecture.

This lead us to delve into the Wilf number of concrete Γ-semimodules. In particular, the second part of this chapter focuses in the study of numerical semigroups minimally generated by two elements and their Γ-semimodules generated by just two elements, i.e. generated by 0 and a gap of the semigroup.

In the case $\Gamma = \langle \alpha, \beta \rangle$, Moyano-Fernández together with Uliczka [121] proved that every Γ–semimodule corresponds to a certain lattice path of the lattice $\mathbb{N}^2$. The lattice path representation has already led us to a formula for the conductor $c(\Delta)$ of those Γ–semimodules, see [7] and Section 6.1.1. In particular, this allows us to explicitly compute the Wilf number associated to a Γ–semimodule. Moreover, in the case that Δ is generated by $[0, g]$ for $g \in \mathbb{N} \setminus \Gamma$ we see that $W(\Delta)$ only depends on g (see Proposition 6.30) so in this case it will be referred to as the Wilf number associated to g. Finally, we observe that the Wilf number provides a beautiful symmetry on the set of gaps (see Sect. 6.3.1) which motivates our definition of supersymmetric gaps. Moreover, if some of the generators of Γ is even, then there exist some gaps whose Wilf number vanishes and that are invariant under several operations; this motivates the name self-symmetric gaps (see Sections 6.3 and 6.3.1).

In Section 6.3 we are going to introduce the concept of *supersymmetric gap* and *self-symmetric gap* of a numerical semigroup with two generators. Let us denote SG resp. SSG the set of supersymmetric resp. self-symmetric gaps. We prove that the set $\mathsf{SG} \cup \mathsf{SSG}$ completely determines the semigroup Γ (see Theorem 6.38). Our construction lies on the application of certain affine linear transformations to the sets $\mathsf{SG}, \mathsf{SSG}$ represented in the lattice $\mathbb{N}^2$; we call the process of representation of Γ via those transformations *polyomino game*.

Moreover, we can compare the set $\mathsf{SG} \cup \mathsf{SSG}$ with another relevant set of gaps defined by Rosales et al. [143] in 2004. They defined the concept of fundamental gaps of a numerical semigroup as an alternative way to represent a numerical semigroup as the set

$$\mathcal{FG} = \mathcal{FG}(\Gamma) := \{g \in \mathbb{N} \setminus \Gamma : \ \{2g, 3g\} \subset \Gamma\}.$$

From $\mathcal{FG}$ one can define the set $\mathcal{D}(\mathcal{FG}) := \{x \in \mathbb{N} : \ x | x_i \text{ for some } \ x_i \in \mathcal{FG}\}$ and see that $\Gamma = \mathbb{N} \setminus \mathcal{D}(\mathcal{FG})$ provides an alternative representation of Γ. Moreover, $\mathcal{FG}$ is the smallest subset of $\mathbb{N} \setminus \Gamma$ that H–determines the semigroup Γ: a subset X of $\mathbb{N}$ is said to H-determine Γ if Γ is the maximum (with respect to the set inclusion) numerical semigroup such that X is a subset of $\mathbb{N} \setminus \Gamma$ (see [143]; also Section 6.3.2). We will see that the set $\mathsf{SG} \cup \mathsf{SSG}$ presents advantages over the set $\mathcal{FG}$: it is not contained in $\mathcal{FG}$ and its cardinality is less than or equal to the cardinality of the set of fundamental gaps whenever $\alpha > 2$ (and in the case $\Gamma = \langle 2, 3 \rangle$), see Section 6.3.2.

To conclude, we discuss some issues regarding the possible extensions of the concepts of supersymmetric and self-symmetric gaps to the general case when Γ is a numerical semigroup with an arbitrary number of generators (see Subsection 6.3.3). More concretely, we propose a general definition for symmetric gaps (see Definition 6.49) and we ask if this definition allows us to introduce the concepts of supersymmetric and self-symmetric gaps for a semigroup with any number of generators. We hope that—if this extension succeeds—these new concepts could be helpful to the solution of the Wilf conjecture. Moreover, we show that in fact the properties of the supersymmetric and self-symmetric gaps are useful to solve a particular case of the extension of Wilf's conjecture proposed in Question 6.28; this is the content of Theorem 6.54.

6.1 Semimodules of a numerical semigroup

Over a numerical semigroup Γ it is possible to define a module structure in analogy to ring theory: A non-empty subset $\Delta \subseteq \mathbb{Z}$ is said to be a Γ-semimodule if $\Delta + \Gamma \subseteq \Delta$, where the set $\Delta + \Gamma$ is understood as all possible sums $a + \gamma$ with $a \in \Delta$, $\gamma \in \Gamma$. The set of all Γ-semimodules with respect to this addition has a structure of additive binoid, with Γ as a neutral element, and $\mathbb{N}$ as an absorbent element; notice that, in particular, Γ itself is a Γ-semimodule.

A system of generators of Δ is a subset $\mathcal{E} \subseteq \Delta$ with

$$\bigcup_{x \in \mathcal{E}} (x + \Gamma) = \Delta.$$

It is called minimal if no proper subset of $\mathcal{E}$ generates Δ. Every Γ-semimodule Δ is finitely generated, and possesses a *minimal* system of generators. Minimal systems of generators of Γ-semimodules containing 0 are well-understood: they are of the form $I = [g_0 = 0, g_1, \dots, g_r]$ where $|g_i - g_j|$ is a gap of Γ for every $i, j \in \{0, \dots r\}$ with $i \neq j$, see [121]. Notice that in particular g_i is a gap of Γ for every $i \in \{1, \dots, r\}$, and the *embedding dimension* $r =: e(\Delta)$ of Δ is bounded by $0 \leq r \leq m(\Gamma) - 1$, cf. [121]. For $\Delta = \Gamma$, the embedding dimension $e(\Gamma)$ of Γ is bounded below by 2 and above by the multiplicity $m(\Gamma)$ of the semigroup, cf. [142, Proposition 2.10]. If $e(\Gamma) = 2$ resp. $e(\Gamma) = m(\Gamma)$ we say that Γ has minimal resp. maximal embedding dimension.

In analogy to the case of a numerical semigroup, the elements in the set $\mathbb{N} \setminus \Delta$, which is finite, are called gaps of Δ. The cardinality $g(\Delta)$ of the set of gaps of Δ is called the genus of Δ. The maximal gap with respect to the usual total ordering in $\mathbb{Z}$ is called the *Frobenius number* of Δ, written $F(\Delta)$. The number $c(\Delta) := F(\Delta) + 1$ is called the conductor of Δ. Moreover, the delta-invariant of Γ is defined to be

$$\delta(\Delta) := |\{x \in \Delta : x < c(\Delta)\}|.$$

In all these invariants, the dependency of Δ will be dropped out from the notation whenever no risk of confusion arises.

Continuing with the analogy with the semigroup case, there is another kind of system of generators—not minimal—for a semimodule Δ of Γ relative to $s \in \Gamma \setminus \{0\}$: this is the set of the s smallest elements in Δ in each of the s classes modulo s, namely the set $\Delta \setminus (s + \Delta)$, and is called the Apéry set of Δ with respect to s; we write $\mathrm{Ap}(\Delta, s)$. Moreover, a formula for the conductor in terms of $\mathrm{Ap}(\Delta, s)$ for $s \in \Gamma \setminus \{0\}$ is easily deduced.

Proposition 6.3. *Let Δ be a Γ-semimodule. For any $s \in \Gamma \setminus \{0\}$ we have that*

$$c(\Delta) - 1 = \max_{\leq_{\mathbb{N}}} \mathrm{Ap}(\Delta, s) - s.$$

Proof. The equality follows as in the case $\Delta = \Gamma$, see e.g.[21, Lemma 3]. □

Two Γ-semimodules Δ and Δ' are called isomorphic if there is an integer n such that $x \mapsto x + n$ is a bijection from Δ to Δ'; we write then $\Delta \cong \Delta'$. For every Γ-semimodule Δ there is a unique semimodule $\Delta' \cong \Delta$ containing 0; such a semimodule is called *normalized*. The Γ-semimodule

$$\Delta^\circ := \{x - \min \Delta : x \in \Delta\}$$

is called the normalization of Δ; this is the unique Γ-semimodule isomorphic to Δ and containing 0. Moreover, the minimal system of generators $\{x_0 = 0, \dots, x_n\}$ of a normalized Γ-semimodule is a Γ-lean set, i.e. it satisfies that

$$|x_i - x_j| \notin \Gamma \quad \text{for any} \quad 0 \le i < j \le n,$$

and conversely, every Γ-lean set of $\mathbb{N}$ minimally generates a normalized Γ-semimodule; we will write then $[x_0 = 0, \dots, x_n]$. Hence there is a bijection between the set of isomorphism classes of Γ-semimodules and the set of Γ-lean sets of $\mathbb{N}$; see Sect. 2 in [121] for the proofs of those statements.

The dual Δ^* of a Γ-semimodule Δ is defined to be

$$\Delta^* := \mathrm{Hom}_{\Gamma}(\Delta, \Gamma) = \{x \in \mathbb{N} : x + \Delta \subseteq \Gamma\},$$

cf. [122, p. 677]. A Γ-semimodule is said to be *selfdual* if $\Delta = \Delta^*$.

In addition, we define the set of syzygies of a Γ-semimodule $\Delta = \Delta_I$ with minimal set of generators $I = [g_0, \dots, g_n]$ as

$$\mathrm{Syz}(\Delta) := \bigcup_{i,j \in I, i \neq j} \Big((\Gamma + g_i) \cap (\Gamma + g_j) \Big).$$

As we will see in the next subsection, this set is a key tool to provide a formula for the conductor of a semimodule of a numerical semigroup with two generators.

6.1.1 A formula for the conductor of a semimodule of a numerical semigroup with two generators

In this subsection we will focus on numerical semigroups of the form $\Gamma = \langle \alpha, \beta \rangle$, with $\alpha, \beta \in \mathbb{N}$, $\gcd(\alpha, \beta) = 1$ and $\alpha < \beta$. All along this subsection we will assume that Γ is of the form $\Gamma = \langle \alpha, \beta \rangle$.

The conductor of numerical semigroup $\Gamma = \langle \alpha, \beta \rangle$ can be computed from the minimal generators as $c = c(\Gamma) = (\alpha - 1)(\beta - 1)$. The gaps of $\langle \alpha, \beta \rangle$ are also easy to describe: they admit a representation $\alpha\beta - a\alpha - b\beta$, where $a \in \,]0, \beta - 1] \cap \mathbb{N}$ and $b \in \,]0, \alpha - 1] \cap \mathbb{N}$, see Rosales [140, Lemma 1]. This writing yields a map from the set of gaps of $\langle \alpha, \beta \rangle$ to $\mathbb{N}^2$ given by $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$, which allows us to identify a gap with a point in the lattice $\mathcal{L} = \mathbb{N}^2$; since the gaps are positive numbers, the point lies inside the triangle with vertices $(0,0), (0,\alpha), (\beta, 0)$. Let us denote by LG the image of the map $\alpha\beta - a\alpha - b\beta \mapsto (a, b)$, i.e. the points of $\mathcal{L}$ inside the triangle of vertices $(0,0), (0,\alpha), (\beta, 0)$.

Following Moyano and Uliczka [121], we will use the notation $e := \alpha\beta - a(e)\alpha - b(e)\beta$ for a gap e of the semigroup $\langle \alpha, \beta \rangle$; if the gap is subscripted as e_i then we write $a_i = a(e_i)$ and $b_i = b(e_i)$.

Let us denote by $\le$ the total ordering in $\mathbb{N}$, if needed we will denote it by $\le_{\mathbb{N}}$ to emphasize that it is the natural order. Moyano and Uliczka [121] consider the following partial ordering $\preceq$ on the set of gaps:

Definition 6.4. Given two gaps e_1, e_2 of $\langle \alpha, \beta \rangle$, we define

$$e_1 \preceq e_2 \iff a_1 \le a_2 \;\wedge\; b_1 \ge b_2 \quad \text{and} \quad e_1 \prec e_2 \iff a_1 < a_2 \;\wedge\; b_1 > b_2.$$

Let $\mathcal{E} = \{0, e_1, \dots, e_n\}$ be a subset of $\mathbb{N}$ with gaps $e_i = \alpha\beta - a_i\alpha - b_i\beta$ of $\langle \alpha, \beta \rangle$ for every $i = 1, \dots, n$ such that $a_1 < a_2 < \cdots < a_n$. Moreover, Moyano and Uliczka [121, Corollary 3.3] show that $\mathcal{E}$ is $\langle \alpha, \beta \rangle$-lean if and only if $b_1 > b_2 > \cdots > b_n$. This simple fact leads to an identification, by means of [121, Lemma 3.4], between an $\langle \alpha, \beta \rangle$-lean set and a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the line joining these two points, where the lattice points identified with the gaps in $\mathcal{E}$ mark the turns from the x-direction to the y-direction; these turns will be called ES-turns for abbreviation. Figure 6.1 shows the lattice path corresponding to the $\langle 5,7\rangle$-lean set $[0,9,11,8]$.

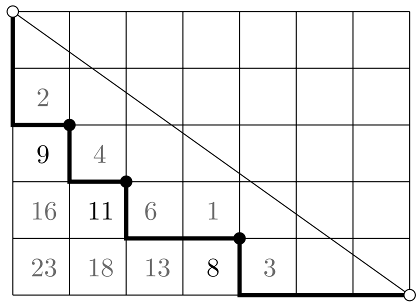


Figure 6.1: Lattice path for the $\langle 5,7\rangle$-lean set $[0,9,11,8]$.

Let $g_0=0,g_1,\dots,g_n$ be the minimal system of generators of a $\langle\alpha,\beta\rangle$-semimodule $\varDelta$. From now on, we will assume that the indexing in the minimal set of generators of $\varDelta$ is such that $g_0=0\preceq g_1\preceq\cdots\preceq g_n$. Under this assumption, Moyano and Uliczka [122, Theorem 2.5] gave an explicit formula for the minimal generators of the dual semimodule $\varDelta^*$ in terms of those of $\varDelta$:

$$\varDelta^*=(\varGamma+a_1\alpha)\cup\bigcup_{k=1}^{n-1}(\varGamma+a_{k+1}\alpha+b_k\beta)\cup(\varGamma+b_n\beta). \tag{6.1}$$

Also, the semimodule $\mathrm{Syz}(\varDelta)$ of syzygies of $\varDelta$ can be characterized from the minimal set of generators of $\varDelta$ as follows:

Proposition 6.5. *[121, Theorem 4.2] Let $\varDelta$ be a $\varGamma$-semimodule with minimal system of generators $[g_0=0,g_1,\dots,g_s]$. Assume that the minimal system of generators is ordered with the gap-order, i.e. $g_0\prec g_1\prec\cdots\prec g_s$. Then the syzygy of $\varDelta$ is the set*

$$\mathrm{Syz}(\varDelta)=\bigcup_{0\le k<j\le s}\Big((\varGamma+g_k)\cap(\varGamma+g_j)\Big)=\bigcup_{k=0}^{s}(\varGamma+h_k),$$

where $h_1,\dots,h_{s-1}$ are gaps of $\varGamma$, $h_0,h_s\le\alpha\beta$, and

$$\begin{aligned}
&h_k\equiv g_k \bmod \beta,\ h_k>g_k \quad \textit{for} \quad k=0,\dots,s\\
&h_k\equiv g_{k+1} \bmod \alpha,\ h_k>g_{k+1} \quad \textit{for} \quad k=0,\dots,s-1\\
&h_s\equiv 0 \bmod \alpha,\ \text{and } h_s>0.
\end{aligned}$$

Remark 6.6. Observe that the congruence conditions for the generators of the syzygy module $[h_0,\dots,h_s]$ give us explicit expressions for h_i in terms of the coordinates of the minimal system of generators of $\varDelta$. Assume that we denote the minimal system of generators of $\varDelta$ is denoted by $\{g_0=0,g_1,\dots,g_s\}$ and for any $i=1,\dots,s$ we write $g_i=\alpha\beta-a_i\alpha-b_i\beta$ and $g_0\prec g_1\prec\cdots\prec g_s$. Then,

$$h_i=\alpha\beta-a_{i-1}\alpha-b_i\beta.$$

Example 6.7. Consider again $\varGamma=\langle 5,7\rangle$ and the semimodule $\varDelta$ with minimal system of generators $[0,9,11,8]$ and lattice path as in Figure 6.1. Then the $\langle 5,7\rangle$-semimodule $\mathrm{Syz}(\varDelta)$ is minimally generated by $h_3=15,h_2=18,h_1=16,h_0=14$.

In particular, $J = [h_0, \ldots, h_n]$ is a minimal system of generators of the semimodule $\Delta_J = \mathrm{Syz}(\Delta)$, hence $h_0 \preceq h_1 \preceq \cdots \preceq h_n$. Therefore it is easily seen that the SE-turns of the lattice path associated to Δ can be identified with the minimal set of generators of the syzygy module (we call SE-turns to the turns from the y–direction to the x–direction). After that, Moyano and Uliczka [121] associate to any Γ-semimodule Δ a lean couple (I, J) where I is a minimal set of generators of Δ and J a minimal set of generators of $\mathrm{Syz}(\Delta)$; or equivalently a lattice path.

The syzygies lead also to the concept of *fixed point* for a semimodule:

Definition 6.8. An $\langle\alpha,\beta\rangle$-semimodule Δ_I with associated Γ-lean couple (I, J) is said to be a $\langle\alpha,\beta\rangle$-fixed point (or simply a fixed point if the semigroup is clear from the context) if the semimodule $(\Delta_J)^\circ$ admits I again as a minimal system of generators.

Remark 6.9. The name *fixed point* has been chosen because of the following reason: it refers to the orbits of period 1 of the Picard sequence associated to the map $f = h \circ \mathrm{Syz}$, where Syz is the map $\Delta_I \mapsto \Delta_J$ and h is the normalization map for $\Delta_J = \mathrm{Syz}(\Delta_I)$; this is further explained in [121, Sect. 5].

Before presenting the announced formula for the conductor of a Γ-semimodule, we need to prove the following technical result.

Lemma 6.10. *Let $\Gamma = \langle\alpha,\beta\rangle$.Let Δ be a Γ-semimodule with associated lean set $[I, J]$, with $I = \{g_0 = 0, g_1, \ldots, g_n\}$ and $J = \{h_0, \ldots, h_n\}$. Then, for any $h \in J$ we have $h - \alpha - \beta \notin \Delta$.*

Proof. Consider $h \in J$ such that that $g_i \prec h \prec g_{i+1}$. Let us denote (a_j, b_j) resp. (a_{j+1}, b_{j+1}) the coordinates of g_j resp. g_{j+1} in the lattice $\mathcal{L}$; then the element h is represented in the lattice path as (a_j, b_{j+1}), see Proposition 6.5. By contradiction, assume that $h - \alpha - \beta \in \Delta$; then there exists a gap $g \in I$ together with two integers $\nu_1, \nu_2 \in \mathbb{N}$ such that

$$h - \alpha - \beta = \nu_1\alpha + \nu_2\beta + g. \tag{6.2}$$

Observe that $h - \alpha - \beta \notin \Gamma$ since otherwise $h - \alpha - \beta = \nu \in \Gamma$ which implies $h = \nu + \alpha + \beta \in \Gamma$. Therefore, $h - \alpha - \beta$ is a gap of Γ and we may also write

$$h - \alpha - \beta = \alpha\beta - (a_j + 1)\alpha - (b_{j+1} + 1)\beta.$$

The writing of g as $g = \alpha\beta - a\alpha - b\beta$ is unique whenever $(a, b) \in \mathcal{L}$, therefore

$$a_j + 1 = a - \nu_1, \quad b_{j+1} + 1 = b - \nu_2.$$

These equalities yield the condition $a_j < a$ and $b_{j+1} < b$. But the unique minimal generator which fulfills such conditions is g_{i+1}; however, h cannot be expressed as $h = g_{i+1} + \nu + \alpha + \beta$ since h is represented in the lattice path as (a_j, b_{j+1}). □

After this lemma we are able to proof the following formula for the conductor of a Γ–semimodule.

Theorem 6.11. *Let $\Gamma = \langle\alpha,\beta\rangle$.Let Δ be a Γ-semimodule with associated lean set $[I, J]$ as above, and let $M := \max_{\leq_{\mathbb{N}}}\{h \in J\}$ denote the biggest (with respect to the order of the natural numbers) minimal generator of* $\mathrm{Syz}(\Delta)$*. Then*

$$c(\Delta) = M - \alpha - \beta + 1.$$

In particular, if (m_1, m_2) *are the coordinates of the point representing* M *in the lattice* $\mathcal{L}$*, we have*

$$c(\Delta) = c(\Gamma) - m_1\alpha - m_2\beta.$$

Proof. Since $c(\Delta) - 1$ is the Frobenius number of the Γ–semimodule Δ, it is enough to check that (i) $M - \alpha - \beta \notin \Delta$, and (ii) if $\ell \notin \Delta$, then $\ell \leq M - \alpha - \beta$. The statement (i) is clear by Lemma 6.10, since $M \in J$. To see (ii), consider an element $\ell \notin \Delta$, which in particular means $\ell \notin \Gamma$. So we can associate to ℓ a point (a, b) in the lattice $\mathcal{L}$. Moreover, ℓ is upon and not contained in the lattice path associated to I. This means that there exists some $j \in J$ with coordinates (j_1, j_2) in the lattice path such that $a > j_1$ and $b > j_2$, otherwise ℓ would be an element of Δ, since the elements represented by lattice points on and under the lattice path belong to Δ. Therefore, $a \geq j_1 + 1$ and $b \geq j_2 + 1$. Thus, from the representation of ℓ and j as gaps we can check that

$$\ell = \alpha\beta - a\alpha - b\beta \leq_{\mathbb{N}} \alpha\beta - (j_1 + 1)\alpha - (j_2 + 1)\beta = j - \alpha - \beta.$$

Hence, since $M = \max_{\leq_{\mathbb{N}}}\{h \in J\}$ and $M \in J$, we have that $M - \alpha - \beta \geq_{\mathbb{N}} \ell$ for any $\ell \notin \Delta$, which proves (ii).

Finally, since M can be represented as a lattice point $(m_1, m_2) \in \mathcal{L}$, we have

$$c(\Delta) = M - \alpha - \beta + 1 = \alpha\beta - m_1\alpha - m_2\beta - \alpha - \beta + 1 = c(\Gamma) - m_1\alpha - m_2\beta. \qquad \square$$

Example 6.12. In the case of the semigroup $\langle 5, 7\rangle$ and the Γ–semimodule minimally generated by $[0, 9, 11, 8]$. One can see in Figure 6.1 that the maximal syzygy is $M = 18$. Thus the conductor of the semimodule is 7, as Figure 6.1 shows.

Notice that for the particular case of $\Delta = \Gamma$ we have $M = \alpha\beta$, and we recover the well-known formula $c(\Gamma) = \alpha\beta - \alpha - \beta + 1$. The value M can be easily characterized in terms of the Apéry set of Δ with respect to $\alpha + \beta$:

Proposition 6.13. *Let* $M := \max_{\leq_{\mathbb{N}}}\{h \in J\}$ *be the biggest minimal generator of the syzygy module with respect to the natural ordering of* $\mathbb{N}$ *as above, then*

$$M = \max_{\leq_{\mathbb{N}}} \mathrm{Ap}(\Delta, \alpha + \beta).$$

Proof. This is a consequence of Proposition 6.3 for $s = \alpha + \beta \in \langle \alpha, \beta\rangle$. $\square$

A straightforward consequence of Theorem 6.11 is the following.

Corollary 6.14. *Let* Δ *be a* Γ *semimodule. Then* $c(\Gamma) - c(\Delta) \in \Gamma$*.*

We conclude this subsection rewriting the formula of Theorem 6.11 in terms of the dual Γ–semimodule of Δ. An important fact about the dual semimodule is that the minimal set of generators of $\mathrm{Syz}(\Delta)$ is in bijection with the minimal set of generators of Δ^*:

Lemma 6.15. *[122, Lemma 6.1], The minimal sets of generators of* Δ^* *and* $\mathrm{Syz}(\Delta)$ *are in correspondence via the map* $x \mapsto \alpha\beta - x$*.*

In particular, this bijection together with Theorem 6.11 allows us to compute the conductor of the semimodule Δ in terms of the minimal generators of Δ^* in a natural way:

Corollary 6.16. *Let Δ be a Γ-semimodule, and let Δ^* be its dual, minimally generated by $x_0, \dots, x_n$. Then*

$$c(\Delta) = \alpha\beta - \min_{\leq_{\mathbb{N}}}\{x_0, \dots, x_n\} - \alpha - \beta + 1.$$

Proof. By Theorem 6.11 we have that $c(\Delta) = \max_{\leq_{\mathbb{N}}}\{h \in J\} - \alpha - \beta + 1$, where J is a minimal set of generators of $\mathrm{Syz}(\Delta)$. Lemma 6.15 yields the equality

$$\min_{\leq_{\mathbb{N}}}\{x_0, x_1, \dots, x_n\} = \alpha\beta - \max_{\leq_{\mathbb{N}}}\{h \in J\},$$

which allows us to conclude. □

Example 6.17. By [122, Theorem 2.5], the minimal generators of the dual of the $\langle 5, 7\rangle$-semimodule Δ_I are given by $[20, 17, 19, 21]$; notice that, for the explicit calculation, the mentioned theorem requires the reverse ordering $\succeq$ instead of the ordering $\preceq$ we use here. The minimum of this set is 17, therefore by Corollary 6.16 we have $c(\Delta) = 35 - 17 - 12 + 1 = 7$, as computed in Example 6.12.

6.2 Wilf number of a Γ–semimodule

Let us return to the case of Γ being a numerical semigroup with an arbitrary number of minimal generators. The notion of *Wilf number* of a Γ–semimodule can be defined in perfect analogy to the numerical semigroup case as follows.

Definition 6.18. Let Δ be a Γ-semimodule, then the *Wilf number of Δ* is defined to be

$$W(\Delta) = e(\Delta) \cdot \delta(\Delta) - c(\Delta).$$

Observe that in the special case $\Delta = \Gamma$, we have $W(\Gamma) = W_\Gamma(e(\Gamma))$.

A particular case of a Γ-semigroup containing 0 is that minimally generated by 0 and a gap g of Γ. Following the notation used in the previous sections, if we write $I = [0, g]$ for this minimal system of generators, and Δ_I for the Γ-semimodule minimally generated by I, then we define the *Wilf number of a gap $g \in \mathbb{N} \setminus \Gamma$* by assigning the Wilf number of Δ_I to g, namely

$$W(g) := W(\Delta_I) = 2\delta(\Delta_I) - c(\Delta_I).$$

6.2.1 Wilf number of a gap

As we will see in Section 6.3, the Wilf number $W(g)$ associated to a gap g of Γ seems to show intrinsic properties of the semigroup itself. Based on this idea, we wonder whether the Wilf number of a gap would help to give an answer to Wilf's conjecture 5.1. The manuscript [8] shows indeed some evidences so that $W(g)$ can take both positive and negative values; but this is bounded as follows:

Theorem 6.19. *Let Γ be a numerical semigroup, and let g be a gap of Γ, then*

$$\max(W(g)) \leq W_\Gamma(4).$$

Proof. Let $I = [0, g]$ be the minimal system of generators of the Γ-semimodule Δ_I. Since $\delta(\Delta_I) \leq \delta(\Gamma) + \delta(\Gamma) - (c(\Gamma) - c(\Delta_I))$, we have

$$\begin{aligned} W(g) =& 2\delta(\Delta_I) - c(\Delta_I) \\ \leq& 4\delta(\Gamma) - 2(c(\Gamma) - c(\Delta_I)) - c(\Gamma) + (c(\Gamma) - c(\Delta_I)) \\ =& 4\delta(\Gamma) - c(\Gamma) - (c(\Gamma) - c(\Delta_I)). \end{aligned}$$

As $c(\Gamma) - c(\Delta_I) \geq 0$, we conclude that $W(g) \leq 4\delta(\Gamma) - c(\Gamma) = W_\Gamma(4)$. □

In addition, not only the maximum but the whole range of possible values for the Wilf number of a gap is bounded:

Proposition 6.20. *Let g be a gap of a numerical semigroup Γ, then*

$$\max(W(g)) - \min(W(g)) < 2\delta(\Gamma).$$

Furthermore, if Γ is symmetric, then the range $\max(W(g)) - \min(W(g))$ *is strictly bounded above by the conductor $c(\Gamma)$ of Γ.*

Proof. Let $I = [0, g]$ be the minimal system of generators of the Γ-semimodule Δ_I. We know that

$$\max(W(g)) \leq 4\delta(\Gamma) - c(\Gamma) - (c(\Gamma) - c(\Delta_I)),$$

cf. proof of Theorem 6.19. Moreover $\min(W(g)) \geq 2\delta(\Gamma) + 2 - c(\Gamma) - (c(\Gamma) - c(\Delta_I))$, hence a straightforward computation shows that

$$\max(W(g)) - \min(W(g)) \leq 2\delta(\Gamma) - 2 < 2\delta(\Gamma).$$

If Γ is symmetric, then $c(\Gamma) = 2\delta(\Gamma)$ (see e.g. [142, Corollary 4.5]), and the second claim follows. □

Given a semigroup Γ with embedding dimension $e(\Gamma) \geq 2$, one can try to use the properties of the Wilf number of any gap to deduce as much as possible information leading to the solution of the Wilf conjecture. To do so, recall that $W_\Gamma(k)$ is an increasing function of k. Therefore, the following corollary is straightforward:

Corollary 6.21. *Let Γ be a numerical semigroup with $e(\Gamma) \geq 4$. If there exists $g \in \mathbb{N} \setminus \Gamma$ such that $W(g) \geq 0$, then $W_\Gamma(k) \geq 0$ for any $k \geq 4$.*

In particular, Γ satisfies Wilf's conjecture 5.1.

Proof. If there exists a gap $g \in \mathbb{N} \setminus \Gamma$ such that $W(g) \geq 0$ then $\max(W(g)) \geq 0$. Therefore, by Theorem 6.19 we have that $0 \leq \max(W(g)) \leq W_\Gamma(4)$. Thus, $W_\Gamma(k) \geq 0$ for any $k \geq 4$ since the Wilf function is increasing in k. □

However, the existence of a positive value of the Wilf number of a gap is not guaranteed. Moreover, we can obtain a necessary condition for a semigroup Γ to have a gap with positive Wilf number: we use Theorem 6.19 together with the invariant μ_Γ (see Section 5.2) to see the following

Corollary 6.22. *Let Γ be a numerical semigroup with $\mu_\Gamma > 4$. Then, $W(g) < 0$ for any $g \in \mathbb{N} \setminus \Gamma$.*

Proof. Since $\mu_\Gamma > 4$ it holds that $W_\Gamma(4) < 0$. The statement is then a direct consequence of Theorem 6.19. □

In particular, Corollary 6.22 shows that all the semigroups of Example 5.18 provide examples of numerical semigroups with $W(g) < 0$ for all the gaps. Moreover, it gives a necessary condition for a semigroup to have a gap with positive Wilf number is $\mu_\Gamma \leq 4$. Thus, it is natural to ask:

Question 6.23. Is there always a gap g with $W(g) \geq 0$ for any numerical semigroup with $\mu_\Gamma \leq 4$?

After that, we wonder if there exists a uniform lower bound for the Wilf number of a gap. In this direction, numerical examples lead us to propose the following conjecture for a lower bound of the Wilf number of a gap:

Conjecture (Bound conjecture 6.1). *For any semigroup Γ, the following inequality holds*

$$\min(W(g)) \geq -W_\Gamma(e(\Gamma))$$

Example 6.24. Let us show some examples which have motivated Bound Conjecture 6.1. With the notation of Example 5.18, consider the numerical semigroups

$\Gamma_1 = \langle 88, 100, 343, 527, 679\rangle_{700}$, $\Gamma_2 = \langle 97, 128, 234, 437\rangle_{800}$ and $\Gamma_3 = \langle 27, 97, 99, 131, 245\rangle$.

For each $i = 1, \ldots, 5$, we present in Table 6.1 the Wilf number of Γ $W(\Gamma) = W_\Gamma(e(\Gamma))$ and the minimal value of $W(g)$ for $g \in \mathbb{N} \setminus \Gamma$, $\min(W(g))$:

i	$W(\Gamma_i)$	$\min(W(g))$
1	3096	−597
2	4597	−668
3	752	−114

Table 6.1: Some semigroups which fulfill Bound Conjecture 6.1

Here it is important to notice that a positive answer to Conjecture 6.1 would solve the Wilf conjecture:

Theorem 6.25. *The Bound conjecture 6.1 implies the Wilf conjecture.*

Proof. Since Wilf's conjecture is known to be true for semigroups with $e(\Gamma) < 4$ [47], let us assume $e(\Gamma) \geq 4$. By Theorem 6.19 we have $\max(W(g)) \leq W_\Gamma(4) \leq W_\Gamma(e(\Gamma))$, since $W_\Gamma(k)$ is increasing in k. Therefore, if the Bound conjecture 6.1 is true, then

$$W_\Gamma(e) \geq \max(W(g)) \geq \min(W(g)) \geq -W_\Gamma(e(\Gamma)).$$

Hence $|W(g)| \leq W_\Gamma(e(\Gamma))$ which implies $W_\Gamma(e(\Gamma)) \geq 0$. □

Remark 6.26. Moreover, for a semigroup Γ with embedding dimension e, the proof of the bound conjecture 6.1 would be enough to prove the existence of a gap g of Γ and of an integer k such that $k \leq e - 2$ and $W(g) \geq -W_\Gamma(k)$. In such a case, one would have

$$\min(W(g)) \geq \max(W(g)) - 2\delta(\Gamma) \geq -W_\Gamma(k) - 2\delta(\Gamma) \geq -W_\Gamma(e),$$

which implies the claim.

In view of Theorem 6.25, it might be useful to understand the behaviour of the Wilf function for an arbitrary Γ-semimodule. This study is precisely the content of the next subsection.

6.2.2 *Wilf function of a Γ-semimodule with an arbitrary number of minimal generators*

Let Δ be a Γ-semimodule of embedding dimension $e(\Delta)$. We define the *Wilf function of a Γ-semimodule* as the function $W_\Delta : \mathbb{N} \to \mathbb{Z}$ given by

$$W_\Delta(k) = k\delta(\Delta) - c(\Delta).$$

This is the natural generalization of the Wilf function of a semigroup. From this point of view, a generalization of the Wilf conjecture would seem to be natural. However, for $k = e(\Delta)$ this function has in general a chaotic behaviour. The following example illustrate that.

Example 6.27. For $\Gamma = \langle 6, 8, 35\rangle$, let $\Delta_{[0,g]}$ be the Γ-semimodule minimally generated by $[0, g]$ with g a gap of Γ. Table 6.2 presents the values taken by the Wilf function $W_{\Delta_I}(2)$:

g	$W_{\Delta_I}(2)$	g	$W_{\Delta_I}(2)$	g	$W_{\Delta_I}(2)$	g	$W_{\Delta_I}(2)$	g	$W_{\Delta_I}(2)$
1	0	7	0	15	0	25	4	37	2
2	-2	9	0	17	0	27	0	39	4
3	0	10	2	19	0	29	0	45	4
4	0	11	0	21	0	31	4		
5	0	13	0	23	0	33	2		

Table 6.2: Gaps and their Wilf numbers for $\Gamma = \langle 6, 8, 35\rangle$.

Taking this into account, we propose several questions:

Question 6.28. Let Γ be a numerical semigroup with any number of minimal generators and Δ be a Γ–semimodule with any number of minimal generators.

(1) Find a characterization of

$$\widetilde{\mu}_\Gamma := \min\{k \in \mathbb{N} : W_\Delta(k) \geq 0 \text{ for all } \Gamma\text{-semimodules } \Delta\}.$$

(2) Is $\widetilde{\mu}_\Gamma$ related to any invariant of Γ?

(3) For all Γ-semimodules Δ with e minimal generators, characterize

$$\widetilde{\mu}_{\Gamma,\mathrm{e}} := \min\{k \in \mathbb{N} : W_\Delta(k) \geq 0 \text{ for all } \Delta \text{ with e minimal generators}\}.$$

(4) Can $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)}$ be computed from $\widetilde{\mu}_\Gamma$?

(5) For a fixed Γ, describe those Γ–semimodules Δ such that $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)} \leq \mathrm{e}(\Delta)$.

(6) Find a characterization of those numerical semigroups Γ such that for any Γ–semimodule Δ one has $\widetilde{\mu}_{\Gamma,\mathrm{e}(\Delta)} \leq \mathrm{e}(\Delta)$, i.e. $W_\Delta(\mathrm{e}(\Delta)) \geq 0$.

These questions arise naturally after the whole discussion, however the possible solutions to the items in Question 6.28 are far from being trivial as we will see in subsection 6.3.4. More concretely, in subsection 6.3.4 a solution to part (3) of Question 6.28 in the case of $\Gamma = \langle \alpha, \beta \rangle$ and $e(\Delta) = 2$.

6.3 On the structure of gaps of $\boldsymbol{\Gamma = \langle \alpha, \beta \rangle}$

We move now to the case where $\Gamma = \langle \alpha, \beta \rangle$. In this case we are going to study in detail the Wilf number of a gap. In particular, we will show a closed formula for it. Here $W(g)$ only depends on the gap g as a consequence of the formula for the conductor of a Γ–semimodule given in Theorem 6.11.

Before providing a formula for the Wilf number of a gap, let us first prove the following technical result.

Proposition 6.29. *Let $\Gamma = \langle \alpha, \beta \rangle$. Let $I = [g_0 = 0, g_1, \ldots, g_n]$ be the minimal system of generators of a Γ-semimodule Δ ordered as $0 \prec g_1 \prec \ldots \prec g_n$; recall the writing $g_i = \alpha\beta - a_i\alpha - b_i\beta$, and set $a_0 = b_0 = 0$. Then*

$$\delta(\Delta) = c(\Delta) - \delta(\Gamma) + \sum_{i=0}^{n} (a_{i+1} - a_i) b_{i+1} = c(\Delta) - \delta(\Gamma) + \sum_{i=0}^{n} (b_i - b_{i+1}) a_{i+1}.$$

Furthermore, for any $I = [0, g_i]$ with $i = 1, \ldots, n$, we have

$$\delta(\Delta_I) = c(\Delta_I) - \delta(\Gamma) + a_i b_i.$$

Proof. Since $\delta(\Gamma) = |\mathbb{N} \setminus \Gamma|$, it is easily deduced from the lattice path representation of Δ_I that

$$|\mathbb{N} \setminus \Delta_I| = \delta(\Gamma) - \sum_{i=0}^{n} (a_{i+1} - a_i) b_{i+1} = \sum_{i=0}^{n} (b_i - b_{i+1}) a_{i+1}.$$

The definition of $\delta(\Delta_I)$ implies the claim. □

Therefore, we can compute explicitly the Wilf number of a gap of $\langle \alpha, \beta \rangle$:

Proposition 6.30. *Let $\Gamma = \langle \alpha, \beta \rangle$. Let $g = \alpha\beta - a\alpha - b\beta$ be a gap of Γ. Let us denote by $[h_0, h_1]$ the minimal system of generators of the Γ-semimodule* $\mathrm{Syz}(\Delta_{[0,g]})$. *Then*

$$-W(g) = \begin{cases} a\alpha - 2ab \text{ if } & \min\{h_0, h_1\} = \alpha\beta - b\beta, \\ b\beta - 2ab \text{ if } & \min\{h_0, h_1\} = \alpha\beta - a\alpha. \end{cases}$$

Proof. Consider the Γ–semimodule Δ_I generated by $I = [0, g]$. From the representation of the lattice path we know that $h_0 = \alpha\beta - b\beta$ and $h_1 = \alpha\beta - a\alpha$. Let us first assume $\min\{h_0, h_1\} = h_0$, thus $\max\{h_0, h_1\} = h_1 = \alpha\beta - a\alpha$. Therefore, by Theorem 6.11 we have

$$c(\Delta_I) = \alpha\beta - a\alpha - \alpha - \beta + 1.$$

Lemma 6.29 yields

$$c(\Delta_I) - 2\delta(\Delta_I) = -c(\Delta_I) + 2\delta(\Gamma) - 2ab = c(\Gamma) - c(\Delta_I) - 2ab = a\alpha - 2ab.$$

The same reasoning applies after exchanging the roles of a and b to obtain the second case of the formula. □

We are interested in the case in which the Wilf number of a gap is zero:

Theorem 6.31. *Let $\Gamma = \langle \alpha, \beta \rangle$ and let $g = \alpha\beta - a\alpha - b\beta$ be a gap of Γ. Consider the Γ-semimodule $\Delta = \Delta_I$ minimally generated by $I = [0, g]$. The following statements are equivalent:*

1. *$W(g) = 0$;*
2. *either $\alpha = 2b$ or $\beta = 2a$;*
3. *Δ is a fixed point;*
4. *Δ is selfdual;*
5. *Δ is symmetric, i.e. for every $x \in \Delta$ we have that $c(\Delta) - 1 - x \notin \Delta$ if and only if $x \in \Delta$.*

Proof. (1) $\iff$ (2) is obvious by Proposition 6.30.

(2) $\iff$ (3): If $\alpha = 2b$, then $g = \alpha\beta - a\alpha - b\beta = b\beta - a\alpha = h_1 - h_0$, and this is positive since g is a gap, hence $\mathrm{Syz}(\Delta_{[0,g]}) = \Delta_{[h_0,h_1]} = \Delta_{[0,h_1-h_0]} = \Delta_{[0,g]}$. *Mutatis mutandis*, if $\beta = 2a$, then Δ is a fixed point. Conversely, assume without loss of generality that $h_0 < h_1$; since $\alpha\beta - a\alpha - b\beta = g = h_1 - h_0 = b\beta - a\alpha$, it is easily seen that $\alpha = 2b$.

(3) $\iff$ (4): First we observe that the number of semimodules Δ_I with $I = [0, g]$ which are fixed points coincide with the number of selfdual modules of that form, as a direct application of [121, Theorem 5.5], as well as Proposition 4.1 and Theorem 4.4 in [122]. Conversely, every selfdual semimodule is a fixed point: For $\Delta = \Delta_I$ with $I = [0, g]$, eq. (6.1) implies that Δ^* is minimally generated by $[a_1\alpha, b_1\beta]$; the selfduality implies that $a_1 = 0$ and $b_1\beta = \alpha\beta - b_1\beta$, hence $I = [0, \alpha\beta - b_1\beta]$. On the other hand, $\mathrm{Syz}(\Delta_I)$ is minimally generated by $[a_1\alpha, b_1\beta] = [0, \alpha\beta - b_1\beta]$, that equals its own normalization. Therefore Δ_I coincides with the normalization of $\mathrm{Syz}(\Delta_I)$ and Δ_I is a fixed point.

(4) $\iff$ (5): this is a consequence of Proposition 4 in [169], and Theorem 2.11 together with Proposition 3.8 in [120]. □

For $\langle \alpha, \beta \rangle$-semimodules Δ with $\mathrm{ed}(\Delta) > 2$, Theorem 6.31 is no longer true: for instance the $\langle 5, 8 \rangle$-semimodule minimally generated by the lean set $I = [0, 4, 6, 7]$ has Wilf number $W(\Delta_I) = 4\delta(\Delta_I) - c(\Delta_I) = 4 \cdot 1 - 4 = 0$ and it is not a fixed point. Moreover, let us consider the numerical semigroup $\Gamma = \langle 10, 14, 27 \rangle$. This semigroup is both symmetric and complete intersection, however if we consider the Γ–semimodule generated by $I = [0, 9]$ then $W(\Delta_I) = 0$ and $I = [0, 9]$ is neither a fixed point nor symmetric. On the other hand, if we consider $\Gamma = \langle 10, 14, 29 \rangle$ every Γ–semimodule with Wilf number equal to zero is a fixed point.

Therefore, we cannot expect a generalization of Theorem 6.31 for numerical semigroups Γ with more than two minimal generators just by imposing the condition of symmetric or complete intersection. This encourages us to propose the following question.

Question 6.32. Given a numerical semigroup Γ with $\mathrm{ed}(\Gamma) > 2$, does there exist a class of numerical semigroups for which any of the equivalences of Theorem 6.31 remain true?

6.3.1 Supersymmetry of the gap set with respect to the Wilf number

This section is devoted to introduce and to develop the concept of supersymmetric gaps. This notion is based on certain symmetries encoded in the Wilf number of a gap. A first observation is that a Γ–semimodule generated by $[0, g]$ has its syzygy module with two minimal generators. Moreover, by the formulas for the minimal set of generators of the syzygy module the following lemma is a straightforward computation.

Lemma 6.33. *Let $g = \alpha\beta - a\alpha - b\beta$ be a gap of $\Gamma = \langle \alpha, \beta \rangle$. Let $[h_0, h_1]$ be the minimal set of generators of* $\mathrm{Syz}(\Delta_{[0,g]})$*. Then,*

1. *If $b > \lfloor \frac{\alpha}{2} \rfloor$ and $a \leq \lfloor \frac{\beta}{2} \rfloor$ then* $\min\{h_0, h_1\} = \alpha\beta - b\beta$.
2. *If $b \leq \lfloor \frac{\alpha}{2} \rfloor$ and $a > \lfloor \frac{\beta}{2} \rfloor$ then* $\min\{h_0, h_1\} = \alpha\beta - a\alpha$.

This lemma allows us to describe the behavior of the set of gaps with respect to the Wilf number. First, observe that any integral point inside the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ represents a gap g of Γ with expression $g = \alpha\beta - a\alpha - b\beta$ and $b > \lfloor \frac{\alpha}{2} \rfloor$. Hence this gap has Wilf number $-W(g) = a\alpha - 2ab$. Now, let us consider the symmetric point to g with respect to the reflection along the line $y = \lfloor \frac{\alpha}{2} \rfloor$. This reflection is given by the map $(a, b) \mapsto (a, \alpha - b)$. Therefore, we have

Lemma 6.34. *If (a, b) is an integral point inside the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ then*

$$-W(a, b) = W(a, \alpha - b).$$

Proof. By Lemma 6.33 and Proposition 6.30 we have that $-W(a, b) = a\alpha - 2ab$. Now, let us denote $g_{\mathrm{sym}} = \alpha\beta - a\alpha - (\alpha - b)\beta$ the symmetric gap with respect to the reflection $(a, b) \mapsto (a, \alpha - b)$. Let us consider the minimal set of generators $[h'_0, h'_1]$ of $\mathrm{Syz}(\Delta_{[0,g_{\mathrm{sym}}]})$. It is thus clear that $\min\{h'_0, h'_1\} = \alpha\beta - b\beta$, since

$$h'_1 - h'_0 = \alpha\beta - a\alpha - \alpha\beta + (\alpha - b)\beta = \alpha\beta - a\alpha - b\beta = g > 0,$$

and the proof follows. □

An analogous situation occurs when considering the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$. In this case, the map $(a, b) \mapsto (\beta - a, b)$ yields the following result:

Lemma 6.35. *If (a, b) is an integral point inside the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$, then*

$$-W(a, b) = W(\beta - a, b).$$

In particular, the set of fixed points of each of the previous symmetries is exactly the set of points with $W(a, b) = 0$. As we have seen in Theorem 6.31 those are exactly fixed points of the orbits of the associated lattice path, i.e. of the associated semimodule.

The previous discussion leads to the following definition:

Definition 6.36. Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let us denote by $\mathcal{T}_r$ the set of points of $\mathcal{L}$ inside (and not in the border of) the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ and $\mathcal{T}_u$ the set of points of $\mathcal{L}$ inside (and not in the border of) the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$. The set of *supersymmetric gaps* is defined to be

$$\mathsf{SG} := \begin{cases} \mathcal{T}_u & \text{if} \quad |\mathcal{T}_u| < |\mathcal{T}_r| \\ \mathcal{T}_r & \text{if} \quad |\mathcal{T}_r| < |\mathcal{T}_u|. \end{cases}$$

We also define the set of *self-symmetric gaps*

$$\mathsf{SSG} := \{ g \in \mathbb{N} \setminus \Gamma : \; W(g) = 0 \}.$$

Example 6.37. For the semigroup $\Gamma = \langle 7, 8 \rangle$ we have that the set $\mathcal{T}_r$ consists of the gaps $5, 13, 6$, and the set $\mathcal{T}_u$ is made up with the gaps $1, 9, 17, 2, 10, 3$. Hence $\mathsf{SG} = \mathcal{T}_r$, this is represented in the shaded part of Figure 6.2. It is easily checked that $\mathsf{SSG} = \{4, 12, 20\}$. This is represented in the striped part of Figure 6.2.

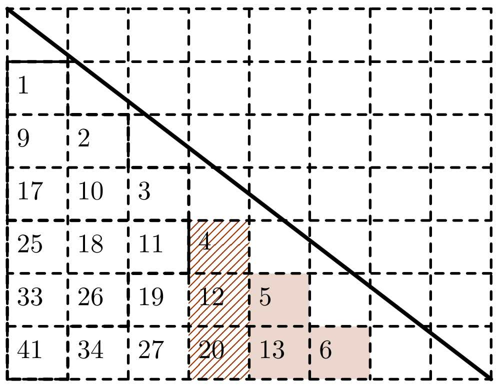


Figure 6.2: Lattice representation of the gap set $\mathbb{N} \setminus \Gamma$. The shaded set is SG and the striped one is SSG.

At this point, we are able to prove one of the main results of this section.

Theorem 6.38. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Then the set $\mathsf{SG} \cup \mathsf{SSG}$ of supersymmetric and self-symmetric gaps completely determines the set of gaps of Γ. In particular, it determines Γ itself.*

Proof. With the notation of Definition 6.36, consider the symmetry $s_\alpha : \mathcal{T}_u \to LG$ along the line $y = \lfloor \frac{\alpha}{2} \rfloor$ defined by $(a, b) \mapsto (a, \alpha - b)$, as well as the symmetry $s_\beta : \mathcal{T}_r \to LG$ along the line $x = \lfloor \frac{\beta}{2} \rfloor$ defined by $(a, b) \mapsto (\beta - a, b)$. First we are going to show that $s_\alpha(\mathcal{T}_u) \cap s_\beta(\mathcal{T}_r) = \emptyset$. Consider $(a, b) \in \mathcal{T}_u$ then

$$(a, b) \mapsto (a, \alpha - b) \mapsto (\beta - a, \alpha - b),$$

where $\alpha\beta - (\beta - a)\alpha - (\alpha - b)\beta = a\alpha + b\beta - \alpha\beta < 0$, since $\alpha\beta - a\alpha - b\beta$ is the representation of a gap. Therefore, $s_\beta^{-1}(s_\alpha(\mathcal{T}_u)) = \emptyset$. Analogously, it can be shown that $s_\alpha^{-1}(s_\beta(\mathcal{T}_u)) = \emptyset$.

Now, let $B(\mathcal{T}_u)$ resp. $B(\mathcal{T}_r)$ be the border points of sets $\mathcal{T}_u$ resp. $\mathcal{T}_r$, i.e. those points such that $(a, b) \in \mathcal{T}_u$ resp. $\mathcal{T}_r$ and $(a + 1, b) \notin LG$ or $(a, b + 1) \notin LG$. Moreover, let $RB(\mathcal{T}_u)$ resp. $RB(\mathcal{T}_r)$

denote the set of border points of the type $(a+1, b) \notin LG$. Observe that those points determine ES–turns, hence the borders $B(\mathcal{T}_u)$ and $B(\mathcal{T}_r)$ are determined by $RB(\mathcal{T}_u)$ and $RB(\mathcal{T}_r)$.

Let us denote by $\tau : \mathcal{L} \to \mathcal{L}$ the translation defined by $(a, b) \mapsto (a+1, b)$. We claim that

$$s_\beta^{-1}(\tau(s_\alpha(RB(\mathcal{T}_u)))) = RB(\mathcal{T}_r).$$

Indeed, consider the point $(a, b) \in \mathcal{T}_u$, then $s_\beta^{-1}(\tau(s_\alpha((a, b)))) = (\beta - a - 1, \alpha - a) \in RB(\mathcal{T}_r)$ due to the fact that $\alpha\beta - (\beta - a - 1)\alpha - b\beta > 0$ and $\alpha\beta - (\beta - a)\alpha - b\beta < 0$. A similar reasoning allows us to prove the equality

$$s_\alpha^{-1}(\tau^{-1}(s_\beta(RB(\mathcal{T}_r)))) = RB(\mathcal{T}_u).$$

The proof will finish by distinguishing three cases concerning the parity of α and β. Let us start with the easiest one and assume that α, β are both odd. By Theorem 6.31 there are no gaps with $W(g) = 0$. Thus, we have a configuration as in Figure 6.3.

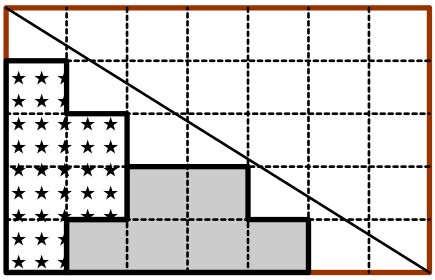


Figure 6.3: The sets $\mathcal{T}_u \cup s_\alpha(\mathcal{T}_u)$ (starred) and $\mathcal{T}_r \cup s_\beta(\mathcal{T}_r)$ (shaded).

In fact, it is easily checked that

$$B(\mathcal{T}_u) \cup B(\mathcal{T}_r) \cup s_\beta(B(\mathcal{T}_r)) \supseteq B(LG),$$

and the sets fit as shown in Figure 6.3.

Next we assume that α is even (the case β even follows analogously). By Theorem 6.31 the set SSG consists of exactly those gaps given by the lattice points $(a, \alpha/2)$ with $1 \leq a \leq \lfloor \beta/2 \rfloor$. Let $B(\mathsf{SSG})$ denote the set of border points in SSG, then we have a configuration as in Figure 6.4.

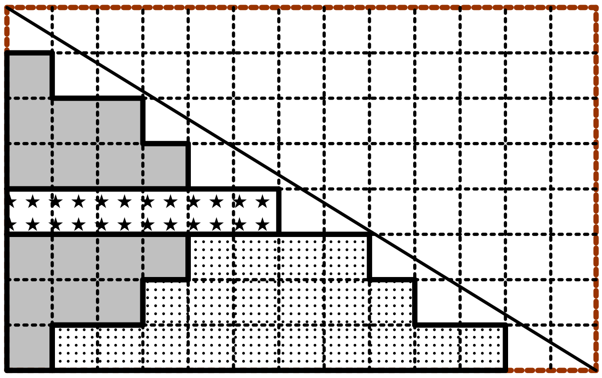


Figure 6.4: Sets $\mathcal{T}_u \cup s_\alpha(\mathcal{T}_u)$ (shaded), SSG (starred), and $\mathcal{T}_r \cup s_\beta(\mathcal{T}_r)$ (dotted).

So it is easily seen that

$$B(\mathcal{T}_u) \cup B(\mathsf{SSG}) \cup B(\mathcal{T}_r) \cup s_\beta(B(\mathcal{T}_r)) \supseteq B(LG).$$

All this together shows that the union of the triangles $\mathcal{T}_u, \mathcal{T}_r$, their images and the set of self-symmetric gaps build a partition of the set of gaps into disjoint sets

$$\mathbb{N} \setminus \Gamma = \mathcal{T}_u \bigsqcup s_\alpha(\mathcal{T}_u) \bigsqcup \mathsf{SSG} \bigsqcup \mathcal{T}_r \bigsqcup s_\beta(\mathcal{T}_r).$$

We are finished as soon as the procedure to recover $\mathbb{N} \setminus \Gamma$ —hence Γ— from the set $\mathsf{SG} \cup \mathsf{SSG}$ will be given.

Let us assume that $\mathsf{SG} = \mathcal{T}_u$ (similarly for $\mathcal{T}_r$). Thus, we have

$$s_\beta(B(\mathcal{T}_r)) = \tau(s_\alpha(B(\mathsf{SG}))).$$

We distinguish two cases:

1. If α, β are both odd, then we can recover $s_\beta(\mathcal{T}_r)$ as the polyomino corresponding to the complement of $s_\alpha(\mathsf{SG})$ in the lattice square with vertices $(0,0), (\lfloor\frac{\beta}{2}\rfloor, 0), (\lfloor\frac{\beta}{2}\rfloor, \lfloor\frac{\alpha}{2}\rfloor)$, and $(0, \lfloor\frac{\alpha}{2}\rfloor)$.

2. If α or β is even, then consider the polyomino $\mathsf{SSG} \cup s_\alpha(\mathsf{SG})$. Thus, we can recover $s_\beta(\mathcal{T}_r)$ as the polyomino corresponding to the complement of $s_\alpha(\mathsf{SG}) \cup \mathsf{SSG}$ in the lattice square with vertices $(0,0), (\lfloor\frac{\beta}{2}\rfloor, 0), (\lfloor\frac{\beta}{2}\rfloor, \lfloor\frac{\alpha}{2}\rfloor)$, and $(0, \lfloor\frac{\alpha}{2}\rfloor)$.

Observe that, if $\mathsf{SG} = \mathcal{T}_r$, then the roles of α, β in (2) and (3) need to be exchange.

In short, we have checked that in all cases we can obtain $s_\beta(\mathcal{T}_r)$ resp. $s_\alpha(\mathcal{T}_u)$, hence $\mathcal{T}_r$ resp. $\mathcal{T}_u$ from certain linear transformations of the set $\mathsf{SG} \cup \mathsf{SSG}$ in the lattice. Therefore, by the previous partition of the set of gaps we can reconstruct completely the set of gaps from the set $\mathsf{SG} \cup \mathsf{SSG}$. □

The proof of Theorem 6.38 shows in particular that SG and SSG are polyominoes, and that we can obtain the whole set LG making operations with them. These necessary operations which allow us to obtain the set LG from SG and SSG will be called *polyomino game*. We illustrate both the polyomino game and the proof of Theorem 6.38 with an example.

Example 6.39. Let $\Gamma = \langle 7, 8\rangle$. We start with SG which in this case is $\mathcal{T}_r$. Then the set of gaps represented in $\mathcal{T}_r$ is $\{5, 6, 13\}$ (see Figure 6.5). Now, we consider the polyomino $s_\beta(\mathsf{SG}) \cup \mathsf{SSG}$ which represents $\{4, 12, 19, 20, 27, 34\}$ inside the rectangle of vertices $(0, 0), (4, 0), (4, 3), (0, 3)$ (see Figure 6.6).

After that, we consider the complement in the rectangle of vertices $(0, 0), (4, 0), (4, 3), (0, 3)$ of the polyomino $s_\beta(\mathsf{SG}) \cup \mathsf{SSG}$ which represents the set of gaps $\{41, 33, 26, 25, 18, 11\}$ and we apply the map s_α as we can see in Figure 6.7. Finally, we put all polyominoes together to give rise to $\mathbb{N} \setminus \Gamma$ as shown in Figure 6.8.

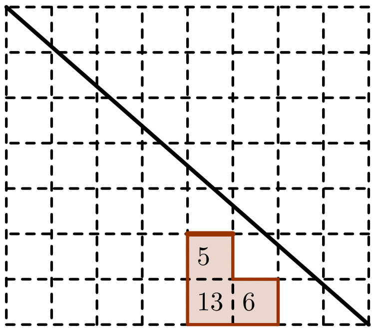


Figure 6.5: The set SG (shaded).

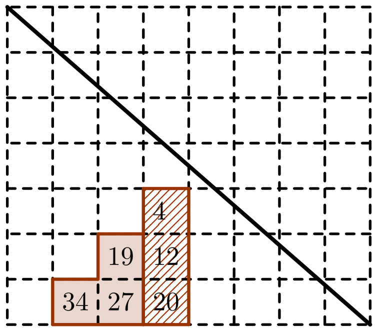


Figure 6.6: The sets $s_\beta(\mathsf{SG})$ (shaded) and SSG (striped).

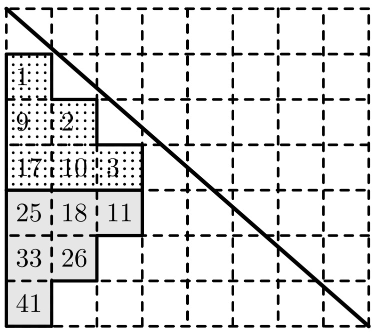


Figure 6.7: The set $\mathcal{T}_u \cup s_\alpha(\mathcal{T}_u)$.

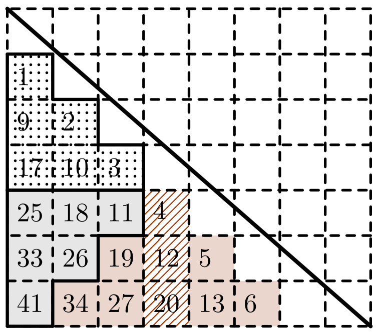


Figure 6.8: Lattice representation of the gap set $\mathbb{N} \setminus \Gamma$.

Since the polyominoes are lattice polygons, we are able to present formulas for the cardinal of the sets of supersymmetric and self-symmetric gaps.

Proposition 6.40. *Let $\Gamma = \langle \alpha, \beta\rangle$ be a numerical semigroup. Then*

$$|\mathsf{SSG}| = \begin{cases} 0 & \text{if } \alpha, \beta \text{ are odd} \\ (\beta - 1)/2 & \text{if } \alpha \text{ even} \\ (\alpha - 1)/2 & \text{if } \beta \text{ even} \end{cases}$$

Moreover,

$$|\mathsf{SG}| = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor\frac{\alpha}{2}\rfloor-1} \left\lfloor \frac{j\beta}{\alpha} \right\rfloor & \text{if } \mathsf{SG} = \mathcal{T}_u \\ \displaystyle\sum_{j=h}^{\alpha-1} \left(\left\lfloor \frac{j\beta}{\alpha} \right\rfloor - \left\lfloor \frac{\beta}{2} \right\rfloor \right) & \text{if } \mathsf{SG} = \mathcal{T}_r \end{cases}$$

where $h = \frac{\alpha}{2} + 1$ *if* α *is even, and* $h = \left\lfloor \frac{\alpha}{2} \right\rfloor$ *if* α *is odd.*

Proof. The formula for $|\mathsf{SSG}|$ is a direct consequence of Theorem 6.31. So let us prove the formula for $|\mathsf{SG}|$. We start by showing that $(a,b) \in RB(\mathcal{T}_u)$ resp. $(a,b) \in RB(\mathcal{T}_r)$ if it is of the form $(\lfloor j\beta/\alpha \rfloor, \alpha - j)$ with $j = 1, \ldots, \lfloor \alpha/2 \rfloor - 1$, resp. $j = h, \ldots, \alpha - 1$, where $h = \alpha/2 + 1$ if α is even, and $h = \lfloor \alpha/2 \rfloor$ if α is odd. Obviously, $(\lfloor j\beta/\alpha \rfloor, \alpha - j)$ lies always on the right-hand sided border, since

$$\alpha\beta - \lfloor j\beta/\alpha \rfloor \alpha - (\alpha - j)\beta \geq 0 \quad \text{and} \quad \alpha\beta - (\lfloor j\beta/\alpha \rfloor + 1)\alpha - (\alpha - j)\beta \leq 0.$$

Now, observe that by definition the points on $RB(\mathcal{T}_u)$ have second coordinate varying from $\alpha - 1$ to $\alpha - \lfloor \alpha/2 \rfloor + 1$. For the points in $RB(\mathcal{T}_r)$ we need to distinguish two cases: if α is even, then the points with second coordinate $\alpha - \alpha/2$ are self-symmetric gaps so they do not belong to $\mathcal{T}_r$ and we need to start the summation running from $\alpha/2 + 1$ on. If α is odd, then there are no self-symmetric gaps of the previous form. The unique self-symmetric gaps may be those with coordinates $(\beta/2, \lfloor \alpha/2 \rfloor)$, but if one of them is actually a border point, then it adds zero in the summation. □

6.3.2 Fundamental gaps vs supersymmetric gaps and self-symmetric gaps

The fundamental gaps for semigroups of the form $\Gamma = \langle \alpha, \beta \rangle$ are explicitly described by Rosales in [140, Theorem 9]. As part of the proof, he characterized the elements $x \in \mathbb{N} \setminus \Gamma$ such that $2x \in \Gamma$. From this characterization we are able to prove the following.

Proposition 6.41. *Let* $\Gamma = \langle \alpha, \beta \rangle$*, and let* $x \in \mathbb{N} \setminus \Gamma$ *be a gap of* Γ*. Then the following are equivalent:*

1. $2x \in \Gamma$;
2. $x = \alpha\beta - a\alpha - b\beta$ *with* $1 \leq a \leq \beta/2$ *and* $1 \leq b \leq \alpha/2$;
3. $W(x) \leq 0$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is [140, Proposition 4], and (2) $\Leftrightarrow$ (3) is a straightforward computation from the formula given in Proposition 6.30. □

In particular, non-positive Wilf number is a necessary condition for a gap to be a fundamental gap.

Corollary 6.42. *Let* $\Gamma = \langle \alpha, \beta \rangle$*, and let* $x \in \mathbb{N} \setminus \Gamma$ *be a gap of* Γ*. If* $x \in \mathcal{FG}(\Gamma)$*, then* $W(x) \leq 0$*.*

Notice that the converse is not true: consider the semigroup $\Gamma = \langle 8, 13 \rangle$ and take the gap 25, then $W(25) = -9 < 0$ but $25 \notin \mathcal{FG}(\Gamma)$.

We recall that a subset X of the set of nonnegative integers H–determines a numerical semigroup Γ if Γ is the maximal numerical semigroup with respect to set inclusion such that $X \subset \mathbb{N} \setminus \Gamma$. Under this description of Γ, the set of fundamental gaps is the smallest subset H–determining Γ. Moreover, Rosales et al. [143] proved the following important result about minimality of the fundamental gaps with respect the H–determinacy.

Proposition 6.43. *[143, Corollary 7] Let Γ be a numerical semigroup and let be $X \subset \mathbb{N} \setminus \Gamma$. The set X H–determines Γ if and only if $\mathcal{FG}(\Gamma) \subset X$.*

On the other hand, we have proven in Theorem 6.38 that $\mathsf{SG} \cup \mathsf{SSG}$ completely determines Γ. In this way, it is natural to compare $\mathsf{SG} \cup \mathsf{SSG}$ with $\mathcal{FG}(\Gamma)$. However, this comparison is not set-theoretically possible since we do not have inclusion relations, i.e. $\mathsf{SG} \cap \mathcal{FG}(\Gamma) = \emptyset$ and $\mathsf{SSG} \subset \{x \in \mathbb{N} \setminus \Gamma : \ 2x \in \Gamma\}$ but in general $\mathsf{SSG} \subsetneq \mathcal{FG}(\Gamma)$ as Example 6.46 shows; this example also shows that if $\mathsf{SG} = \mathcal{T}_u$ resp. $\mathcal{T}_r$ then $s_\alpha(\mathsf{SG})$ resp. $s_\beta(\mathsf{SG})$ does not need to be contained in $\mathcal{FG}(\Gamma)$.

This means that, in general, the set $\mathsf{SG} \cup \mathsf{SSG}$ does not H–determines Γ, but it determines Γ in the sense that Γ can be recovered from $\mathsf{SG} \cup \mathsf{SSG}$. Moreover, the polyomino game cannot be recovered from the set of fundamental gaps. Then, the most we can do is to compare the cardinality of both sets:

Proposition 6.44. *[8, Proposition 4.11] Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with $\alpha > 2$, then*

$$|\mathsf{SG} \cup \mathsf{SSG}| \leq |\mathcal{FG}(\Gamma)|.$$

Proof. We will distinguish three cases depending on the parity of the semigroup generators.

Case (A): If α and β are both of them odd numbers, then $\mathsf{SSG} = \emptyset$ and

$$\min\{|\tau_u|, |\tau_r|\} \leq \frac{1}{8}(\alpha-1)(\beta-1). \tag{6.3}$$

In order to prove Equation (6.3) we observe that

$$|\mathbb{N} \setminus \Gamma| = \sum_{j=1}^{\alpha-1} \left\lfloor \frac{j\beta}{\alpha} \right\rfloor = \frac{1}{2}(\alpha-1)(\beta-1) = 2\left\lfloor \frac{\alpha}{2} \right\rfloor \left\lfloor \frac{\beta}{2} \right\rfloor = 2 \cdot |\{x \in \mathbb{N} \setminus \Gamma : 2x \in \Gamma\}|.$$

Therefore, $|\tau_u| + |\tau_r| = |\mathbb{N} \setminus \Gamma| - \frac{1}{4}(\alpha-1)(\beta-1) = \frac{1}{4}(\alpha-1)(\beta-1)$. This give us directly Equation (6.3). Now, in view of [140, Corollary 11] it is enough to show that

$$\frac{1}{4}(\alpha-1)(\beta-1) - \left\lceil \frac{\alpha-3}{6} \right\rceil \left\lceil \frac{\beta-3}{6} \right\rceil \geq \frac{1}{8}(\alpha-1)(\beta-1),$$

which is equivalent to the inequality

$$\frac{1}{8}(\alpha-1)(\beta-1) \geq \left\lceil \frac{\alpha-3}{6} \right\rceil \left\lceil \frac{\beta-3}{6} \right\rceil.$$

This is true: since

$$\left\lceil \frac{\alpha-3}{6} \right\rceil \left\lceil \frac{\beta-3}{6} \right\rceil = \Big(\left\lfloor \frac{\alpha-3}{6} \right\rfloor + 1\Big)\Big(\left\lfloor \frac{\beta-3}{6} \right\rfloor + 1\Big) \leq \frac{1}{36}(\alpha\beta + 3\alpha + 3\beta + 9),$$

we just need to realize that

$$\frac{1}{36}(\alpha\beta + 3\alpha + 3\beta + 9) \leq \frac{1}{8}(\alpha\beta - \alpha - \beta + 1),$$

which leads to the inequality

$$7\alpha\beta - 15\alpha - 15\beta - 9 \geq 0.$$

This holds for $\alpha = 3$ and $\beta \geq 11$ odd as well as for any $\alpha \geq 5$ odd and $\beta > \alpha$ odd. The cases $\alpha = 3, \beta = 5$ and $\alpha = 3, \beta = 7$ must be treated separately, see Figure 6.9. In the first case, an easy computation shows that $|\mathcal{FG}(\Gamma)| = 2$ and $|\tau_u| = 1$, $|\tau_r| = 1$, and the result follows; in the second case, we have that $|\mathcal{FG}(\Gamma)| = 3$, and $|\tau_u| = 2$, $|\tau_r| = 1$, so the result remains also true.

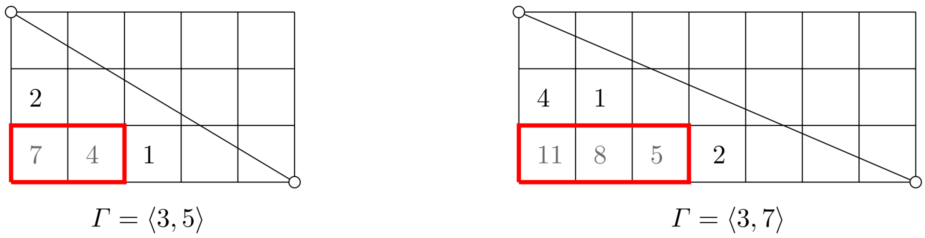


Figure 6.9: Special cases in (A) of Proposition 6.44.

Case (B): If $\alpha > 2$ is even and $\beta > \alpha$ is odd, then $|\mathsf{SSG}| = \left\lfloor \frac{\beta}{2} \right\rfloor$ and also

$$|\tau_u| + |\tau_r| = \frac{1}{4}(\alpha - 1)(\beta - 1).$$

By reasoning as in Case (A), it suffices to prove that

$$\frac{1}{4}\alpha(\beta - 1) - \left\lceil \frac{\alpha - 3}{6} \right\rceil \left\lceil \frac{\beta - 3}{6} \right\rceil \geq \frac{1}{8}(\alpha - 1)(\beta - 1) + \frac{1}{2}\left\lfloor \frac{\beta}{2} \right\rfloor,$$

again by [140, Corollary 11]. But this leads us to Case (A) since

$$\frac{1}{4}\alpha(\beta - 1) - \frac{1}{8}(\alpha - 1)(\beta - 1) = \frac{1}{8}(\alpha - 1)(\beta - 1) + \frac{\beta - 1}{4} \quad \text{and} \quad \frac{\beta - 1}{4} = \frac{1}{2}\left\lfloor \frac{\beta}{2} \right\rfloor.$$

Case (C): If $\alpha \geq 3$ is odd and $\beta > \alpha$ is even, we may repeat *mutatis mutandis* the argument in Case (B), and the result follows. □

Remark 6.45. Observe that for $\alpha = 2, \beta = 3$ the statement of Proposition 6.44 holds; but this is no longer true for $\alpha = 2$ and any $\beta \geq 3$ odd, since in that case $|\tau_u| = |\tau_r| = 0$ and $|\mathcal{FG}(\Gamma)| = \frac{\beta-1}{2} - \left\lceil \frac{\beta-3}{6} \right\rceil > 0$.

Finally, let us show in an example how all the important sets presented in this section look like.

Example 6.46. Consider the numerical semigroup $\Gamma = \langle 8, 13 \rangle$. In this case $\mathsf{SG} = \mathcal{T}_u$ as we can see in Figure 6.10. This figure is labelled in the following manner: as usual, every lattice cell represents the gap of Γ given by (a, b), where these are the coordinates of the upper-right corner of the cell. Every cell is endowed with two numbers: the one lying on the bottom of the cell is just the corresponding gap, while the number on the top of the cell is the Wilf number of the gap.

The figure also presents a filling code: we have shadowed the set of fundamental gaps and dotted the set of self-symmetric gaps. This makes it clear that self-symmetric gaps are not fully contained in the set of fundamental gaps neither the images by s_α, s_β of the triangles $\mathcal{T}_u, \mathcal{T}_r$. The red rectangle contains the gaps x such that $2x \in \Gamma$, cf. Proposition 6.41. The polyominoes corresponding to $\mathcal{T}_r \cup s_\alpha(\mathcal{T}_r)$ and $\mathcal{T}_u \cup s_\beta(\mathcal{T}_u)$ are also distinguishable.

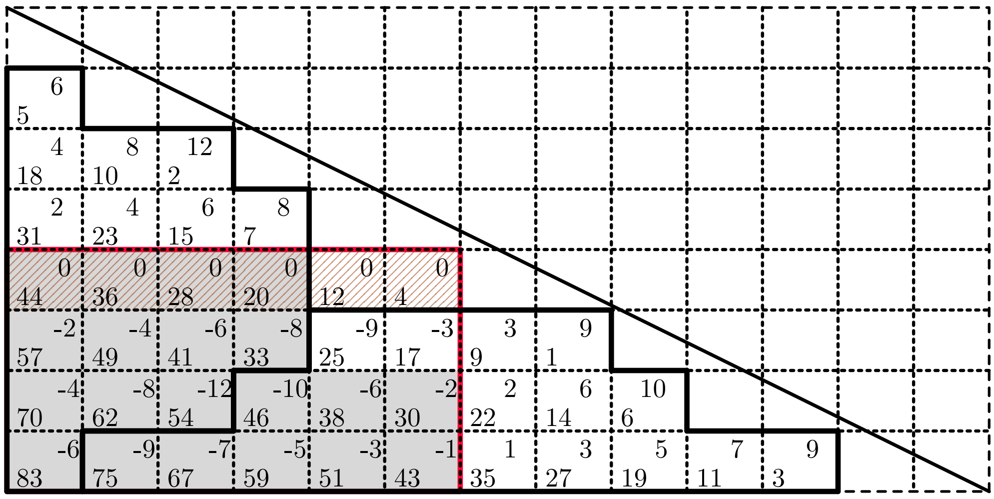


Figure 6.10: Polyomino game for the semigroup $\langle 8, 13 \rangle$.

6.3.3 Remarks on the concepts of supersymmetric and self-symmetric gaps

We would like to finish the section with a brief discussion about the possible extension of the concepts of supersymmetric and self-symmetric gaps to the case of a numerical semigroup Γ with arbitrary embedding dimension.

We remark first that our definition of supersymmetry does not coincide with the one given by Fröberg, Gottlieb and Häggkvist in [59]. Their notion lies on a lattice representation of each of the Apéry sets with respect to all minimal generators of the semigroup in such a way that supersymmetry in the sense of [59, Lemma 15] means symmetry plus uniqueness of the concerned lattice representation; recall that the Apéry set of Γ with respect to a nonzero element $s \in \Gamma$ is defined to be $\{w \in \Gamma : w - s \notin \Gamma\}$. Moreover, our notion is defined from the lattice representation of the set of gaps of the semigroup together with its properties with respect to the Wilf numbers.

The extension of the notions of self-symmetric and supersymmetric gaps to higher embedding dimensions is trickier. The following example shows us that we cannot expect a definition through only the sign of the Wilf number of the concerned gap, since for this example all gaps have negative Wilf number; this means, the sign here is not the important issue, but the absolute value.

Example 6.47. Let $\Gamma = \langle 4, 6, 13 \rangle$. This is a symmetric semigroup which is the value semigroup of the complex plane curve singularity given by the equation $f(x, y) = x^{13} + 4x^8 y - x^6 + 2x^3 y^2 - y^4$; see [94] for the definition of value semigroup of a curve. The set of gaps of Γ is $G = \{1, 2, 3, 5, 7, 9, 11, 15\}$. We list the minimal system of generators for those Γ-semimodules Δ with embedding dimension 2, as well as the corresponding minimal generating system for $\Delta_J = \mathrm{Syz}(\Delta)$ and its normalized Δ_J°, together with $W(\Delta_I)$:

Δ_I	$\Delta_J = \mathrm{Syz}(\Delta_I)$	Δ_I°	$W(\Delta_I)$
$[0,1]$	$[13,14]$	$[0,1]$	0
$[0,2]$	$[6,8]$	$[0,2]$	0
$[0,3]$	$[13,16]$	$[0,3]$	0
$[0,5]$	$[13,18]$	$[0,5]$	0
$[0,7]$	$[13,20]$	$[0,7]$	0
$[0,9]$	$[13,22]$	$[0,9]$	0
$[0,11]$	$[17,19,24]$	$[0,2,7]$	-2
$[0,15]$	$[19,21,28]$	[0,2,9]	-2

In addition, the example $\Gamma = \langle 10, 14, 27\rangle$ after Theorem 6.31 shows that if we want to extend the concept of self-symmetric gap we cannot only focus on Wilf number zero; it seems that the notion of supersymmetry is deeper.

Moreover, observe that the symmetries under consideration imply the following property:

Proposition 6.48. *Let $\Gamma = \langle \alpha, \beta\rangle$ be a numerical semigroup. With the previous notation,*

(1) If $\alpha\beta - a\alpha - b\beta = g \in \mathcal{T}_u$, then $c(\Delta_{[0,g]}) = c(\Delta_{[0,s_\alpha(g)]}) = c(\Gamma) - a\alpha$.

(2) If $\alpha\beta - a\alpha - b\beta = g \in \mathcal{T}_r$, then $c(\Delta_{[0,g]}) = c(\Delta_{[0,s_\beta(g)]}) = c(\Gamma) - b\beta$.

Proof. We will prove only (1), and (2) follows *mutatis mutandis.*

Consider the gap $\alpha\beta - a\alpha - b\beta = g \in \mathcal{T}_u$. Let $[h_0, h_1]$ be the minimal set of generators of $\mathrm{Syz}(\Delta_{[0,g]})$. Then by Lemma 6.33

$$\min\{h_0, h_1\} = \alpha\beta - b\beta,$$

and Theorem 6.11 implies that $c(\Delta_{[0,g]}) = c(\Gamma) - a\alpha$.

Consider now $s_\alpha(g) = \alpha\beta - a\alpha - (\alpha - b)\beta$, and denote by

$$[h'_0 = \alpha\beta - a\alpha, h'_1 = \alpha\beta - (\alpha - b)\beta]$$

the minimal set of generators of $\mathrm{Syz}(\Delta_{[0,s_\alpha(g)]})$. In order to finish it would be enough to prove that $\min\{h'_0, h'_1\} = h'_1$; but this is immediate, as

$$h'_0 - h'_1 = \alpha\beta - a\alpha - \alpha\beta + (\alpha - b)\beta = \alpha\beta - a\alpha - b\beta = g > 0.$$

The previous discussion together with Proposition 6.48 lead us to propose the following definition of supersymmetry of gaps for a numerical semigroup of arbitrary embedding dimension.

Definition 6.49. Let $\Gamma = \langle x_1, \ldots, x_n\rangle$ be a numerical semigroup minimally generated by $x_1, \ldots, x_n$. We define on the set of gaps $G := \mathbb{N} \setminus \Gamma$ the relation

$$g_1 \sim_c g_2 \Longleftrightarrow c(g_1) = c(g_2) \quad \text{for any } g_1, g_2 \in G.$$

This is in fact an equivalence relation which thus provides a partition of the set gaps into equivalence classes. This partition will be called the *gap conductor partition* of G. We say that two gaps g_1, g_2 are *candidates to be supersymmetric* if $g_1 \sim_c g_2$. In addition, we say that two gaps g_1, g_2 are *supersymmetric* if $g_1 \sim_c g_2$ and $|W(g_1)| = |W(g_2)|$ and we will say that g is *self-symmetric* if there is no other g' such that $g \sim_c g'$.

Remark 6.50. In the case of a numerical semigroup with embedding dimension 2, the sets of supersymmetric gaps and self-symmetric gaps are the set of elements that represent the equivalence classes of a special set of supersymmetric and self-symmetric gaps in terms of the previous definition.

Remark 6.51. We are giving a definition of two gaps to be supersymmetric. This definition has no relation and has not to be mixed with the symmetry properties of the semigroup in the sense of [94].

In the special case $\Gamma = \langle \alpha, \beta \rangle$, two gaps g_1, g_2 of Γ are supersymmetric if and only if $s_\alpha(g_1) = g_2$ or $s_\beta(g_1) = g_2$. Moreover, there is no three different supersymmetric gaps; i.e. either g_1 is supersymmetric to a unique $g_2 \neq g_1$ or g_1 is self-symmetric and then it is its own supersymmetric point. Therefore, Definition 6.49 allows us to extent the properties of the lattice symmetries to purely algebraic properties of the gaps. This discussion leads to pose the following closing questions:

Question 6.52. Given a symmetric numerical semigroup $\Gamma = \langle x_1, \ldots, x_n \rangle$, we ask:

(1) For $n > 2$, does there exist a subset of the set of gaps such that the supersymmetry property defined in Definition 6.49 allows to recover the whole semigroup from this set?
(2) Does there exist a lattice representation in $\mathbb{Z}^n$ of the set of gaps such that —in analogy with the case $n = 2$— it can be made up from the sets of self-symmetric and supersymmetric gaps together with some affine transformation of them?

6.3.4 Wilf function of a semimodule with two generators

To finish, we are going to apply the new tools introduced in the previous section, i.e. the notion of supersymmetry, to study the topics of Subsections 6.2.1 and 6.2.2. A first result is the following corollary of Proposition 6.20.

Corollary 6.53. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup and let $g \in \mathbb{N} \setminus \Gamma$ be a gap. Then*

$$\max(W(g)) = -\min(W(g)) < \delta(\Gamma).$$

Proof. Since the Wilf number of a gap satisfies the symmetries of Lemmas 6.34 and 6.35 then it is obvious that $\max(W(g)) = -\min(W(g))$. Therefore, Proposition 6.20 reads

$$\max(W(g)) - \min(W(g)) = 2\max(W(g)) < 2\delta(\Gamma). \qquad \square$$

Thanks to the supersymmetry properties we can culminate the chapter with a partial answer of Question 6.28 (3) in the case of a Γ–semimodule Δ with $\mathrm{ed}(\Delta) = 2$.

Theorem 6.54. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with two generators. Let $g \in \mathbb{N} \setminus \Gamma$ be a gap of the semigroup. Let $\Delta = [0, g]$ be the Γ–semimodule associated to g. Then,*

$$W_\Delta(3) \geq 0.$$

In particular, $\widetilde{\mu}_{\Gamma,2} = 3$.

Proof. Recall that by the results explained in Section 6.3.1, all the values of $W_\Delta(2)$ are in the set

$$\{0\} \cup \{\pm W(g) : \ g \in \mathcal{T}_r \cup \mathcal{T}_u\}.$$

Moreover,

1. If $g \in \mathcal{T}_u$ then $-W(g) = a\alpha - 2ab < 0$.
2. If $g \in \mathcal{T}_r$ then $-W(g) = b\beta - 2ab < 0$.

Now, let us show that $-W(g) \leq \delta(\Delta)$ for any $g \in \mathbb{N} \setminus \Gamma$. Since $\delta(\Delta) > 0$ it is enough to prove it for those $g \notin \mathcal{T}_u \cup \mathcal{T}_r$. First of all, recall that we have the following expression for $\delta(\Delta_{[0,g]})$ given by Proposition 6.29

$$\delta(\Delta_{[0,g]}) = c(\Delta_{[0,g]}) - \delta(\Gamma) + ab.$$

Let us consider $g \notin \mathcal{T}_u \cup \mathcal{T}_r$, then $g = \alpha\beta - a\alpha - b\beta$ with $1 \leq a \leq \beta/2$ and $1 \leq b \leq \alpha/2$. Let us assume that $\min\{\Gamma \cap (\Gamma + g)\} = \alpha\beta - b\beta$. In such a case, we know by Proposition 6.48 that $c(\Delta_{[0,g]}) = c(\Gamma) - a\alpha$. Therefore, the previous expression read as

$$\delta(\Delta_{[0,g]}) = c(\Gamma) - a\alpha - \delta(\Gamma) + ab = \delta(\Gamma) - a\alpha + ab.$$

Thus,

$$-W(g) < \delta(\Delta_{[0,g]}) \Leftrightarrow a\alpha - 2ab < \delta(\Gamma) - a\alpha + ab \Leftrightarrow 2a\alpha - 3ab < \delta(\Gamma).$$

Since $g = \alpha\beta - a\alpha - b\beta$ with $1 \leq a \leq \beta/2$ and $1 \leq b \leq \alpha/2$, it is easy to check that:

$$2a\alpha - 3ab \leq \frac{\alpha\beta}{4} \leq \delta(\Gamma) = \frac{\alpha\beta - \alpha - \beta + 1}{2} \Leftrightarrow \alpha\beta - 2(\alpha + \beta) + 2 \geq 0.$$

Now, the last inequality is true if $3 \leq \alpha < \beta$. Similarly, we obtain the same last inequality and thus the same final condition if we consider $\min\{\Gamma \cap (\Gamma + g)\} = \alpha\beta - a\alpha$ instead of $\min\{\Gamma \cap (\Gamma + g)\} = \alpha\beta - b\beta$.

Hence, let us consider $\Gamma = \langle \alpha, \beta \rangle$ with $3 \leq \alpha < \beta$. Then, we have that any $g \in \mathbb{N} \setminus \Gamma$ fulfills $-W(g) < \delta(\Delta)$. Thus,

$$W_{\Delta_{[0,g]}}(3) = W(g) + \delta(\Delta_{[0,g]}) \geq W(g) - W(g) = 0.$$

To finish the proof it remains to show that the statement of the theorem also holds for the case $\Gamma = \langle 2, \beta \rangle$ with $\gcd(\alpha, \beta) = 1$. But for this case $W(g) = 0$ for all $g \in \mathbb{N} \setminus \Gamma$ by Theorem 6.31. □

Remark 6.55. Let us clarify the assertion of the proof where we claim that if $3 \leq \alpha < \beta$ then $\alpha\beta - 2(\alpha + \beta) + 2 \geq 0$. Let us consider the real variable function $F(x, y) = xy - 2(x + y) + 2$. The function $F(x, y)$ has a saddle point at $(2, 2)$ and it is positive within the region

$$(x - 2)(y - 2) \geq 2.$$

This can be easily seen if we look at the surface defined by $F(x, y) - z = 0$ is a hyperbolic paraboloid which is represented in Figure 6.11.

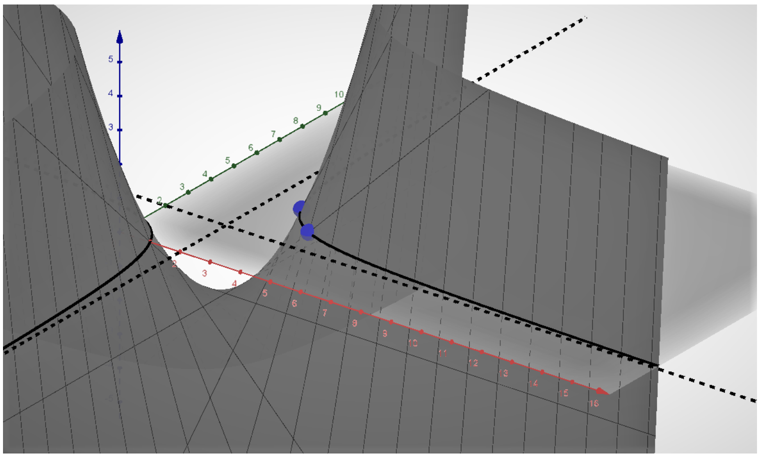


Figure 6.11: Representation of the surface $xy - 2(x + y) + 2$.

Since we are working with a numerical semigroup, we only need to take care about the values of $F(x, y)$ with $x \geq 0$, $y \geq 0$ and $x, y \in \mathbb{N}$. Observe that under those conditions the corresponding branch of the hyperbola $(x - 2)(y - 2) = 2$ has only two points with $(x, y) \in \mathbb{N}^2$, namely $(3, 4)$ and $(4, 3)$. Moreover, since this hyperbola is symmetric with respect to bisection of the first positive quadrant, it is enough to look for points with $x \leq y$. Therefore, any point $(x, y) \in \mathbb{N}^2$ such that $3 \leq x < y$ represents a semigroup $\Gamma = \langle x, y \rangle$ satisfying the required inequality.

Theorem 6.54 shows that the items in Question 6.28 are far from being trivial. This is because —even if we have a nice description for the Wilf function— we need to exploit the constitutive properties of the gaps of the numerical semigroups, such as an adequate expression for them in terms of the minimal generators of the semigroup. It seems therefore to be a challenge to even guess the candidates for the invariants introduced in Question 6.28.

In conclusion, we think that the issues presented in Question 6.28 may offer a fruitful research topic to be investigated in the future.

Chapter 7

Modules over the local ring of a curve with one Puiseux pair

In the case of an irreducible plane curve singularity C with any number of Puiseux pairs defined by a germ of function f, we have seen in Section 1.1.2 that one has naturally a discrete valuation $v : R := \mathbb{C}\{x,y\}/(f) \to \mathbb{N} \cup \{\infty\}$. Moreover, the value set of the local ring of the curve $\Gamma(C) := v(R) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ has a natural structure of numerical semigroup and it constitutes a topological invariant of the singularity. Also, if we denote by $n_i := \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{i-1})/gcd(\overline{\beta}_0, \ldots, \overline{\beta}_i)$ we know that $\Gamma(C)$ satisfy the following properties

(1) $n_i\overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$,

(2) $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$ for all $i = 1, \ldots, g$.

In 1972, Bresinsky [23, Theorem 2] and Teissier [165, Chap. I. 3.2] independently proved that for any numerical semigroup Γ satisfying conditions (1) and (2) there exist a plane branch $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ such that $\Gamma = \Gamma(C)$, (see Theorem 3.16). The analytic counterpart of Γ is the value set of Kähler differentials. The value set of Kähler differentials of an irreducible plane curve singularity is defined as $\Delta' = v(Rdx + Rdy)$ and it was proved by Delorme in 1978 [42] that it is an analytic invariant. Moreover, one can easily check that Δ' has a natural structure of Γ–semimodule, its normalization is given by $\Delta = \Delta' - (\overline{\beta}_0 - 1)$, where n the multiplicity of C at the singular point, and $\Delta = v(R + Rdy/dx)$. Obviously, Δ is the normalization of the value set associated to the fractional ideal $(f, \partial f/\partial x, \partial f/\partial y)$ as a result of the following equalities $R = \mathbb{C}\{x,y\}/(f)$ and $(\partial f/\partial x)(t)/((\partial f/\partial y)(t)) = dy(t)/dx(t)$, where t is the parameter of a Puiseux parameterization $R \hookrightarrow \overline{R} = \mathbb{C}\{t\}$ of C.

In this chapter, we will focus in the case of $C : f(x,y) = 0$ being the germ of an irreducible plane curve singularity with one Puiseux pair, i.e. a germ of plane curve with equation

$$f(x,y) = x^p - y^q + \sum_{iq+jp>pq} a_{i,j}x^iy^j,$$

where $\gcd(p,q) = 1$. As we have seen in Section 1.1.2, in this particular case the semigroup of values of C is minimally generated by p and q, i.e. $\Gamma = \langle p, q \rangle := \{n \in \mathbb{N} | \; n = ap + bq\}$. The main goal of this chapter is to provide an extension to Γ–semimodules of Bresinsky-Teissier Theorem 3.16. Our starting point is the definition of a proper non-empty class of Γ–semimodules which we call *increasing semimodules* (See Definition 7.3), which extract the essence of the combinatorial properties of the Γ–semimodules associated to the value set of the module of Kähler differentials. In particular, thanks to Delorme's results [42, Lemma 12] we can easily check that the set of values of the module of Kähler differentials is an increasing semimodule. Obviously, from our combinatorial definition the class of increasing semimodules is larger then the class of value sets of modules of Kähler differentials (see Section 7.2). In Section 7.2 we deal with the combinatorial properties of the class of increasing semimodules. More concretely, from the lattice path representation of the set of gaps of a numerical semigroup Γ with two generators introduced in Section 6.1, we have been able to provide a geometric method to construct the set of all possible increasing semimodules of a given Γ. As a consequence, we prove that the set of all possible increasing semimodules of a given Γ has a natural tree structure (see Section 7.2.1).

Our main contribution in this Chapter is precisely Theorem 7.16, which for a given increasing semimodule L provides a constructive proof of the existence of an irreducible plane curve with semigroup Γ and a module $R + Rz$ over its local ring R such that $v(R + Rz) = L$. Unfortunately, in this full generality the converse of our Theorem 7.16 is still a work in progress. Fortunately, thanks to Delorme's results [42, Lemma 12] the converse statement of Theorem 7.16 holds for the particular set of increasing semimodules whose first generator is $p - q$, assuming $p < q$ (Corollary 7.22). In particular, this implies that the value sets of Kähler differentials can be computed from the combinatorics of the increasing semimodules over the semigroup of the curve. We illustrate this fact with a final example obtained from the implementation of our constructive proof of Theorem 7.16 in MATHEMATICA code.

7.1 Value set of the module of Kähler differentials

Let C be an irreducible plane curve singularity defined by an equation

$$f(x,y) = x^\beta - y^\alpha + \sum_{i\alpha + j\beta > \alpha\beta} a_{i,j} x^i y^j,$$

with $\alpha < \beta$ and $\gcd(\alpha, \beta) = 1$. Let us denote by $R = \mathbb{C}\{x,y\}/(f)$ the local ring of the irreducible plane curve singularity with one Puiseux pair and by $R \hookrightarrow \overline{R} = \mathbb{C}\{t\}$ its normalization. Recall that to any $\mathbb{C}$-algebra A we can associate the A-module $\Omega_A := I/I^2$, where I is the kernel of the diagonal surjection $A \bigotimes_{\mathbb{C}} A \to A$; we call Ω_A the module of Kähler differentials. The injection $R \hookrightarrow \overline{R}$ induces a morphism of R-modules $\varphi : \Omega_R \to \Omega_{\overline{R}}$. Now, the $\overline{R}$-module $\Omega_{\overline{R}}$ is free of rank 1 and it is generated by $dt := t \otimes 1 - 1 \otimes t$. On the other hand, the R-module Ω_R has two generators, namely

$$dx = \alpha t^{\alpha-1} \quad \text{and} \quad dy = \beta t^{\beta-1} + \sum_{j\geq\beta} j a_j t^{j-1}.$$

In this way we can see that $\varphi(\Omega_R)$ is a free sub-$\overline{R}$-module of rank 1 generated by $\alpha t^{\alpha-1}$.

We can define the set of values of the module of Kähler differentials as $\Delta' := v(\varphi(\Omega_R))$. It is trivial to check that Δ' is a non-normalized Γ–semimodule. Since $\min \Delta' = \alpha - 1$ its normalization is defined as

$$\Delta = (\Delta')^\circ = v\Big(R + R\frac{dy}{dx}\Big).$$

From now on, we will deal with the normalized set of values Δ. Let us recall a few notation from Delorme's paper [42]. Let us denote by $g_0 = 0 < g_1 < \cdots < g_s$ the minimal set of generators of Δ and $g_{s+1} = \infty > g_s$. Also set

$$E_i := \bigcup_{0\leq j\leq i} (\Gamma + g_j), \; E_s := \Delta \text{ and } \; u_i := \min\{E_{i-1} \cap (\Gamma + g_i).\}$$

Delorme proves the following result

Theorem 7.1. *Given a plane branch with one characteristic exponent and under the previous notation, set $w_0 = 1$, $w_1 = dy/dx$ and $w_{s+1} = 0$. Then for all $1 \leq i \leq s$ there exists a relation*

$$\omega_{i+1} = \sum_{0\leq j\leq i} F_{j,i} \omega_j$$

where $F_{j,i} \in R$, $u_i = v(F_{i,i}) + g_i = \inf_j\{v(F_{j,i}) + g_j\}$ *and* $v(\omega_{i+1}) = g_{i+1}$.

Proof. See [42, Lemma 12 (b)]. □

Here it is necessary to make two important remarks. The first one is the fact that in this case $g_1 = \beta - \alpha$. The second one is that, as a consequence of Theorem 7.1, the value set of Kähler differentials has the property that for all $0 \leq i \leq s$ one has $g_{i+1} > u_i$. Moreover, it is easy to check that this property is no longer true if we increase the number of Puiseux pairs of the irreducible plane curve singularity as the following example shows:

Example 7.2. Let us consider an irreducible plane curve singularity with semigroup $\langle 4, 6, 17\rangle$ with equation

$$f(x, y) = (y^2 - x^3)^2 - x^7 y.$$

With the help of SINGULAR [35], we can compute the minimal generators of the module of Kähler differentials by computing a standard basis of the ideal $(f, \partial f/\partial x, \partial f/\partial y)$. Once we have the standard basis, it is easy to check that the normalized set of values is minimally generated by $\{g_0 = 0, g_1 = 2, g_2 = 11, g_3 = 13\}$. A straightforward computation shows that $u_2 = \min\{E_1 \cap \Gamma + g_2\} = 17 > g_3$.

7.2 Increasing semimodules

Based on the combinatorial properties of the value set of Kähler differentials of a plane branch with one Puiseux pair determined by Delorme's Theorem 7.1 it is natural to consider the following class of Γ–semimodules.

Definition 7.3. Let Γ be a numerical semigroup minimally generated by $\langle \alpha, \beta\rangle$. A Γ-semimodule L is called an *increasing semimodule* if it satisfies the following property:

$$\begin{aligned}
&\text{If } L \text{ has minimal set of generators } \{g_0 = 0, g_1, \ldots, g_s\}\\
&\text{and we put } g_{s+1} = \infty,\ u_0 = 0,\\
&\text{then for all } 0 \leq i \leq s \text{ we have } g_{i+1} > u_i,\\
&\text{where } u_i = \min\{(\Gamma + g_i) \cap E_{i-1}\} \text{ for } 1 \leq i \leq s\\
&\text{and } E_i = \bigcup_{0 \leq j \leq i} (\Gamma + g_j) \text{ for } 0 \leq i \leq s.
\end{aligned} \qquad (✠)$$

Remark 7.4. Our definition is in the setting of a numerical semigroup minimally generated by two elements since, as we observed in the previous section, the value sets of Kähler differentials for curve singularities with more than one Puiseux pair are not in general increasing semimodules. However, it would be certainly interesting to explore the properties of increasing semimodules defined over a general numerical semigroup with any number of minimal generators.

Before continuing, let us mention the easy result that the class of increasing semimodules is non-empty:

Lemma 7.5. *Any normalized semimodule with two generators is increasing.*

Proof. Because of the minimal set of generators is of the form $\{0, g\}$ with $g \in \mathbb{N} \setminus \Gamma$, condition (✠) trivially holds since $g_2 = \infty$ and $u_0 = 0$. □

Also, an immediate consequence of Delorme's Theorem 7.1 is the following

Corollary 7.6. *Let $\Delta = v(R + Rdy/dx)$ where R is the local ring of an irreducible plane curve singularity with one Puiseux pair (α, β). Then Δ is an increasing Γ–semimodule, where $\Gamma = \langle \alpha, \beta \rangle$.*

Proof. By Theorem 7.1, the generators w_i of $R + Rdy/dx$ satisfy $\omega_{i+1} = \sum_{0 \leq j \leq i} F_{j,i}\omega_j$ where $F_{j,i} \in R$, $u_i = v(F_{i,i}) + g_i = \inf_j\{v(F_{j,i}) + g_j\}$ and $v(\omega_{i+1}) = g_{i+1}$. Since $v(w_{i+1}) \geq \min\{v(F_{j,i}) + v(w_j)\}$ the claim follows. □

Moreover, the set of increasing semimodules is a *proper* non-empty subset in the set of Γ-semimodules: consider for instance the Γ–semimodule minimally generated by $\{0, 6, 8, 9\}$ (see Figure 6.1); this is not an increasing semimodule since $\inf(\langle 5, 7\rangle \cap (\langle 5, 7\rangle + 6)) = 20 > 8$.

Finally, we recall a few properties of the u_i's previously defined which will be very useful in the sequel. These properties were already given by Delorme [42, Lemma 10] in a different context:

Lemma 7.7. *[42, Lemma 10] Let $p, q \in \mathbb{Z}$ be such that $|p - q| \notin \Gamma$. We set*

$$u := \min_{\leq_{\mathbb{N}}}\{(\Gamma + p) \cap (\Gamma + q)\}$$

as well as $\bar{u} := u + c(\Gamma) - \alpha\beta$, $v := p + q + \alpha\beta - u$ and $\bar{v} := v + c(\Gamma) - \alpha\beta$. Then we have:

1. $(\Gamma + p) \cap (\Gamma + q) = (\Gamma + u) \cup (\Gamma + v)$,
2. $(\Gamma + p) \cup (\Gamma + q) = (\Gamma + u - \alpha\beta) \cap (\Gamma + v - \alpha\beta)$,
3. $\mathbb{N} + \bar{v} \subset (\Gamma + p) \cup (\Gamma + q)$.
4. $(\mathbb{N} + \bar{u}) \cap ((\Gamma + p) \cup (\Gamma + q)) = (\mathbb{N} + \bar{u}) \cap (\Gamma + v - \alpha\beta)$.

To finish, the following Lemma provides an important property of an increasing Γ–semimodule.

Lemma 7.8. *Let be $\Gamma = \langle \alpha, \beta \rangle$. Let L be an increasing Γ-semimodule with $\{g_0 = 0, g_1, \ldots, g_s\}$, we set $g_{s+1} = \infty$ and $\bar{u}_i := u_i + c(\Gamma) - \alpha\beta$. Then, for any $i = 0, \ldots, s - 1$ there exists an element $c_i \in (-\Gamma)$, namely $c_i = c_{i-1} + g_i - u_i$, such that*

$$(\mathbb{N} + \bar{u}_i) \cap E_i = (\mathbb{N} + \bar{u}_i) \cap (\Gamma + c_i).$$

Proof. We proceed by induction on s. The case $s = 1$ is easily deduced from Lemma 7.7 with $p = g_0 = 0$, $q = g_1$; then $u_1 = \min\{(\Gamma + g_1) \cap \Gamma\} \in \Gamma$, therefore it is enough to consider

$$\begin{aligned} c_1 &= v - \alpha\beta = g_0 + g_1 + \alpha\beta - u_1 - \alpha\beta \\ &= g_1 - u_1 \in (-\Gamma). \end{aligned}$$

Since $\overline{u}_s < u_s < g_{s+1}$, we have $g_{s+1} \in \mathbb{N} + \overline{u}_s$; this together with the fact that $g_{s+1} \notin E_s$ implies that $g_{s+1} \notin \mathbb{N} + c_s$ by induction hypothesis. Therefore $g_{s+1} - c_s \notin \Gamma$ and we apply Lemma 7.7 (4) with $p = g_{s+1}$ and $q = c_s$, so that $u_{s+1} = \min\{(\Gamma + g_{s+1}) \cap E_s\}$. If we set $c_{s+1} = g_{s+1} - u_{s+1} + c_s$, then Lemma 7.7 again yields the equality

$$(\mathbb{N} + \overline{u}_{s+1}) \cap E_{s+1} = (\mathbb{N} + \overline{u}_{s+1}) \cap (\Gamma + c_{s+1}),$$

as desired. □

Remark 7.9. Lemma 7.8 was proven by Delorme [42, Lemma 12 (a)] for the case of an increasing semimodule with $g_1 = \beta - \alpha$.

Remark 7.10. From the proof of Lemma 7.8, we observe that Lemma 7.8 still holds if we substitute the assumption $g_{i+1} > u_i$ for all i by the hypothesis $g_{i+1} > \overline{u}_i$ for all i.

7.2.1 Lattice paths of increasing semimodules

To conclude the section, we are going to show a procedure to construct any lattice path associated to an increasing semimodule. Recall that, as we explained in Section 6.1, the gaps of $\langle\alpha,\beta\rangle$ are easy to describe: they admit a representation $\alpha\beta - a\alpha - b\beta$, where $a \in]0,\beta-1]\cap\mathbb{N}$ and $b \in]0,\alpha-1]\cap\mathbb{N}$, see Rosales [140, Lemma 1]. This writing yields a map from the set of gaps of $\langle\alpha,\beta\rangle$ to $\mathbb{N}^2$ given by $\alpha\beta - a\alpha - b\beta \mapsto (a,b)$, which allows to identify a gap with a point in the lattice $\mathcal{L}=\mathbb{N}^2$; since the gaps are positive numbers, the point lies inside the triangle with vertices $(0,0),(0,\alpha),(\beta,0)$. As in Section 6.1, following Moyano and Uliczka [121], we will use the notation $g := \alpha\beta - a(g)\alpha - b(g)\beta$ for a gap g of the semigroup $\langle\alpha,\beta\rangle$; if the gap is subscripted as g_i then we write $a_i = a(g_i)$ and $b_i = b(g_i)$. Recall that we denote by $\leq$ the total ordering in $\mathbb{N}$, if needed we will denote it by $\leq_{\mathbb{N}}$ to emphasize that it is the natural order and that Moyano and Uliczka [121] (see also Definition 6.4) consider the following partial ordering $\preceq$ on the set of gaps:

$$g_1 \preceq g_2 \iff a_1 \leq a_2 \quad\wedge\quad b_1 \geq b_2 \quad \text{and} \quad g_1 \prec g_2 \iff a_1 < a_2 \quad\wedge\quad b_1 > b_2.$$

Let $\mathcal{E} = \{0, g_1, \dots, g_n\}$ be a subset of $\mathbb{N}$ with gaps $g_i = \alpha\beta - a_i\alpha - b_i\beta$ of $\langle\alpha,\beta\rangle$ for every $i = 1,\dots,n$ such that $a_1 < a_2 < \cdots < a_n$. Moreover, Moyano and Uliczka [121, Corollary 3.3] show that $\mathcal{E}$ is a minimal set of generators of a normalized $\langle\alpha,\beta\rangle$–semimodule if and only if $b_1 > b_2 > \cdots > b_n$. This simple fact leads to an identification, by means of [121, Lemma 3.4], between minimal system of generators of $\langle\alpha,\beta\rangle$–semimodules and a lattice path with steps downwards and to the right from $(0,\alpha)$ to $(\beta,0)$ not crossing the line joining these two points, where the lattice points identified with the gaps in $\mathcal{E}$ mark the turns from the x-direction to the y-direction; these turns will be called ES-turns for abbreviation.

Let $g_0 = 0, g_1, \dots, g_n$ be the minimal system of generators of a $\langle\alpha,\beta\rangle$-semimodule Δ. Let us assume for a moment that the indexing in the minimal set of generators of Δ is such that $g_0 = 0 \preceq g_1 \preceq \cdots \preceq g_n$. Under this assumption, as mentioned in Proposition 6.5, Moyano and Uliczka [121, Theorem 4.2] gave an explicit formula for the minimal generators of the syzygy semimodule $\mathrm{Syz}(\Delta)$ in terms of those of Δ:

$$\mathrm{Syz}(\Delta) = \bigcup_{0\leq k<j\leq s} \Big((\Gamma + g_k) \cap (\Gamma + g_j) \Big) = \bigcup_{k=0}^{s} (\Gamma + h_k),$$

where $h_1, \dots, h_{s-1}$ are gaps of Γ, $h_0, h_s \leq \alpha\beta$, and

$$\begin{aligned} &h_k \equiv g_k \bmod \beta,\ h_k > g_k \quad \text{for} \quad k = 0,\dots,s \\ &h_k \equiv g_{k+1} \bmod \alpha,\ h_k > g_{k+1} \quad \text{for} \quad k = 0,\dots,s-1 \\ &h_s \equiv 0 \bmod \alpha, \text{ and } h_s > 0. \end{aligned}$$

Remark 7.11. Observe that the congruence conditions for the generators of the syzygy module $[h_0,\dots,h_s]$ give us explicit expressions for h_i in terms of the coordinates of the minimal system of generators of Δ. Assume that we denote the minimal system of generators of Δ is denoted by $\{g_0 = 0, g_1, \dots, g_s\}$ and for any $i = 1,\dots,s$ we write $g_i = \alpha\beta - a_i\alpha - b_i\beta$ and $g_0 \prec g_1 \prec \cdots \prec g_s$. Then,

$$h_i = \alpha\beta - a_{i-1}\alpha - b_i\beta.$$

In particular, $J = [h_0, \dots, h_n]$ is a minimal system of generators of the semimodule $\Delta_J = \mathrm{Syz}(\Delta)$, hence $h_0 \preceq h_1 \preceq \cdots \preceq h_n$. Therefore it is easily seen (see also Moyano and Uliczka [121]) that the

SE-turns of the lattice path associated to Δ can be identified with the minimal set of generators of the syzygy module (we call SE-turns to the turns from the y–direction to the x–direction).

For now on we will assume that any minimal system of generators $g_0 = 0 <_{\mathbb{N}} g_1 <_{\mathbb{N}} \cdots <_{\mathbb{N}} g_s$ of a Γ–semimodule is order by the natural order. An easy consequence of Lemma 7.5 is that any lattice path with a unique ES-turn is an increasing semimodule $\Delta^{(1)}$ generated by $g_0 = 0$, $g_1 \in \mathbb{N} \setminus \Gamma$. Let us write $J = \{h_0, h_1\}$ for the minimal set of generators of the syzygy semimodule $\mathrm{Syz}(\Delta^{(1)})$. Therefore, it is a straightforward computation to check that $\min\{h_0, h_1\} = \min\{\Gamma \cap (\Gamma + g_1)\}$.

Write $u_1 = \min\{h_0, h_1\}$. There exists an increasing semimodule $\Delta^{(2)}$ with three generators containing $\Delta^{(1)}$ if and only if there is an element $g_2 \in \mathbb{N} \setminus \Delta^{(1)}$ with $g_2 > u_1$ such that $\Delta^{(2)} = \Delta^{(1)} \cup (\Gamma + g_2)$. Since u_1 is a generator of the syzygy module and g_2 must be a gap, then $g_2 > u_1$ means

$$g_2 \in \{(a,b) \in \mathbb{N}^2 : \ u_1 < \alpha\beta - a\alpha - b\beta\}.$$

On the other hand, condition $g_2 \in \mathbb{N} \setminus \Delta^{(1)}$ means that g_2 is a point above the lattice path associated to $\Delta^{(1)}$. Let us denote by $L(\Delta^{(1)})^+$ the region above the lattice path associated to $\Delta^{(1)}$. Hence the existence of $\Delta^{(2)}$ is equivalent to

$$L(\Delta^{(1)})^+ \cap \{(a,b) \in \mathbb{N}^2 : \ u_1 < \alpha\beta - a\alpha - b\beta\} \neq \emptyset.$$

So, let us assume that we start with $\Delta^{(i-1)}$ minimally generated by $I = \{g_0 = 0, g_1, \ldots, g_{i-1}\}$. Consider $u_{i-1} = \min\{(\Gamma + g_{i-1}) \cap E_{i-2}\}$. Observe that, since the u_i's coincide with SE-turns by construction, the syzygy module $\mathrm{Syz}(\Delta^{(i-1)})$ is minimally generated by

$$\{u_1, \ldots, u_{i-1}, M\},$$

where $M = c(\Delta^{(i-1)}) + \alpha + \beta - 1$ by Theorem 6.11. We observe that we are ordering g_i here by the natural order in $\mathbb{N}$, and this ordering does not necessarily coincide with the order $\prec$. So, the indices in the minimal set of the syzygies may not coincide with those in Proposition 6.5.

As before, let us denote $L(\Delta^{(i-1)})^+$ the region above the lattice path associated to $\Delta^{(i-1)}$, then, there is a $g_i \in \mathbb{N} \setminus \Gamma$ with $g_i > u_{i-1}$ if and only if

$$L(\Delta^{(i-1)})^+ \cap \{(a,b) \in \mathbb{N}^2 : \ u_{i-1} < \alpha\beta - a\alpha - b\beta\} \neq \emptyset.$$

Observe that the previous construction give us the following immediate consequence:

Proposition 7.12. *Let $\Gamma = \langle \alpha, \beta \rangle$ and et Δ be an increasing Γ–semimodule minimally generated by $\{g_0 = 0 <_{\mathbb{N}} \cdots <_{\mathbb{N}} g_s\}$. Then, $u_1 <_{\mathbb{N}} \cdots <_{\mathbb{N}} u_s <_{\mathbb{N}} M := c(\Delta) + \alpha + \beta - 1$ are the minimal set of generators of* $\mathrm{Syz}(\Delta)$.

Remark 7.13. Propostion 7.12 shows that, in constrast to Proposition 6.5, the minimal set of generators of the syzygy semimodule of an increasing semimodule can be obtained with the natural order. Obviously the labeled may differ from the order established in the lattice path.

Example 7.14. To see how the ordering in the labeling of the minimal generators of the syzygy semimodule may differ, let us consider $\Gamma = \langle 7, 9 \rangle$ and the increasing Γ–semimodule Δ constructed in Example 7.15 with minimal set of generators $[0, 5, 20, 31]$. Observe that $\mathrm{Syz}(\Delta)$ is minimally generated by $u_1 = 14, u_2 = 27, u_3 = 38$ and $c(\Delta) + \alpha + \beta - 1 = 40$. If we order the minimal generators of the syzygy module with the order $\prec$ then we have $h_0 = 14, h_1 = 40, h_2 = 38, h_3 = 27$.

The previous construction encloses a rooted tree structure over the set of incresing semimodules in terms of its first non zero minimal generator; the root corresponds to the semimodule associated to a gap of the semigroup, and represents the unique Γ–semimodule of the form $\Delta = [0, g]$. We assign this to the level 0 of the tree. The next level represent the possible increasing Γ-semimodules with three generators $[0, g_1, g_2]$ with $g_1 = g$; hence the number of leaves at this level is

$$|L(\Delta^{(1)})^+ \cap \{(a,b) \in \mathbb{N}^2 : \ u_1 < \alpha\beta - a\alpha - b\beta\}|.$$

In general the number of nodes at a level k represent the number of increasing Γ-semimodules with g as first non zero generator and $k+2$ minimal generators. To each node at level k we attach exactly

$$|L(\Delta^{(k+1)})^+ \cap \{(a,b) \in \mathbb{N}^2 : \ u_{k+1} < \alpha\beta - a\alpha - b\beta\}|$$

leaves. Obviously, this tree representation is finite; observe that $k \leq \alpha$. Let us show the procedure with an example:

Example 7.15. Let us consider the semigroup $\Gamma = \langle 7, 9\rangle$. We are going to construct all possible increasing Γ-semimodules with $g_1 = 5$. Since $u_1 = 14$, the first step of the above procedure says that there are 8 increasing Γ-semimodules with $g_1 = 5$ and 3 minimal generators, see Figure 7.1.

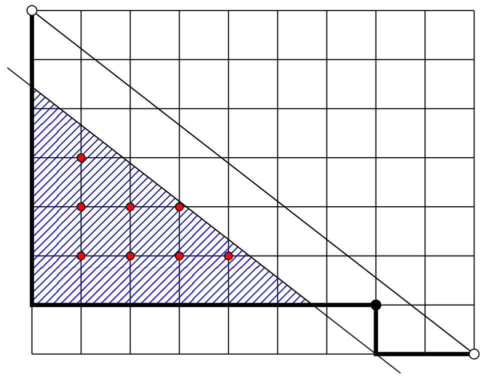


Figure 7.1: Lattice path for the $\langle 7, 9\rangle$-lean set $[0, 5]$ and the eight candidates for g_2 in red.

As a second step, let us choose $g_2 = 20$ as minimal generator. Then, $u_2 = 27$ and the following figure shows that there is only one increasing Γ-semimodule with $g_1 = 5$, $g_2 = 20$ and 4 minimal generators in total, namely $\Delta = [0, 5, 20, 31]$. Which has $u_3 = 38$.

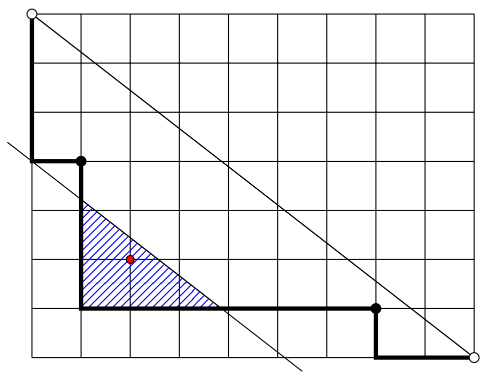


Figure 7.2: Lattice path for the $\langle 7, 9\rangle$-lean set $[0, 5, 20]$ and candidate for g_3 in red.

Finally, it is an easy computation to see that the set of increasing Γ-semimodules with $g_1 = 5$ has the following tree structure, Figure 7.3.

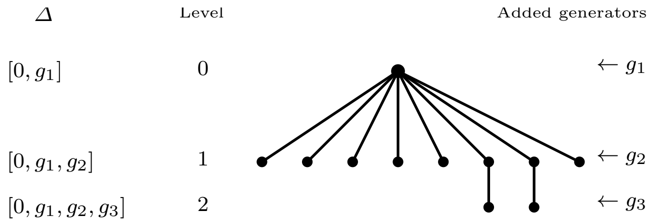


Figure 7.3: Tree of increasing Γ-semimodules with first non zero generator $g = 5$.

7.3 Realization of increasing semimodules as value set of *R*-modules

In this section we will prove the following

Theorem 7.16. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with $\alpha < \beta$. Let L be an increasing Γ-semimodule, and set $b := c(\Gamma) - \beta - 1$. Then there exist a tuple $(a_1, \ldots, a_b) \in \mathbb{C}^b$ and $z \in \overline{R}$ such that $L = v(R + zR)$, where R is the local ring of the germ of plane curve singularity defined by the Puiseux parameterization*

$$C : \begin{cases} x(t) := t^{\alpha} \\ y(t) := t^{\beta} + \sum_{i=1}^{b} a_i t^{i+\beta}. \end{cases}$$

Proof. We first introduce some notation and definitions. Let us denote by $g_0 = 0 < g_1 < \cdots < g_s$ the minimal set of generators of L. Also set

$$\begin{aligned} E_i &:= \bigcup_{0 \le j \le i} (\Gamma + g_j), \text{ and } E_s := L \\ u_i &:= \min\{E_{i-1} \cap (\Gamma + g_i)\} \\ \sigma_{i+1} &:= \sum_{1 \le j \le i} (g_{j+1} - u_j) \quad \text{with } \sigma_0 = \sigma_1 = 0, \quad \text{for } i = 1, \ldots, s \\ h_i &:= g_i - \sigma_i, \quad \text{for } i = 1, \ldots, s+1 \\ I_i &:= [\sigma_{i-1}, \sigma_i] \cap \mathbb{N}, \quad \text{for } i = 2, \ldots, s+1 \end{aligned}$$

Let us consider the polynomial ring $\mathbb{C}[X_1, \ldots, X_b, T]$. We consider formal elements $Y, z \in \mathbb{C}[X_1, \ldots, X_b, T]$ of the form

$$Y = T^{\beta}\Big(1 + \sum_{i=1}^{b} X_i T^i\Big), \quad z = T^{g_1}\Big(c_0 + \sum_{i=1}^{b} c_i X_i T^i\Big).$$

Step 1: We set $U^0_{\sigma_0} = 1$, $U^1_{\sigma_1} = z$. We will prove the existence of a family of polynomials

$$\Big\{\{\{U^i_j\}_{j \in I_i}\}_{i=2,\ldots,s+1}\Big\} \subset \mathbb{C}[X_1, \ldots, X_b, T]$$

such that

$$U_j^i = \Big(X_j \prod_{k=2}^{i} A_k c_{k,j} + V_j\Big) T^{j+h_i} + \sum_{r=j+1}^{b} \Big(X_r \prod_{k=2}^{i} A_k c_{k,r} + V_r\Big) T^{r+h_i} + \text{h.o.t.}$$

with $V_r \in \mathbb{C}[X_1, \ldots, X_{r-1}]$ if $r \geq j$, $A_k \in \mathbb{C}[X_1, \ldots, X_{\sigma_i - 1}]$ and $c_{k,r} \in \mathbb{C}$.

Step 2: We consider the polynomials U_j^i as polynomials in the variable T and we observe that $A_k = \mathrm{lc}_T(U^k_{\sigma_k})$ the leading coefficient as polynomial in T. Set $\omega_k := U^k_{\sigma_k}$ for $k = 0, \ldots, s$. By construction, we have that $\mathrm{ord}_T(\omega_k) = g_k$ if $A_k \neq 0$ for $k = 0, \ldots, s$ and for all $j \in (\sigma_{k-1}, \sigma_k)$ we have $\mathrm{lc}_T(U_j^k) = 0$. We want to show that the system of polynomial equations defined by

$$\begin{cases} A_k \neq 0, & \text{for all } k \\ \mathrm{lc}_T(U_j^k) = 0, & \text{for all } k \text{ and all } j \in (\sigma_{k-1}, \sigma_k) \\ \mathrm{lc}_T(U^{s+1}_{\sigma_{s+1}}) = 0. & \end{cases} \tag{7.1}$$

has a non-trivial compatible solution $(a_1, \ldots, a_b) \in \mathbb{C}^b$.

Step 3: Finally, let us take a solution $(a_1, \ldots, a_b) \in \mathbb{C}^b$ of the system (7.1). We can consider the ring morphism defined by

$$\begin{array}{ccc} \mathbb{C}[X_1, \ldots, X_b, T] & \xrightarrow{ev} & \overline{R} = \mathbb{C}\{t\} \\ X_i & \mapsto & a_i \\ T & \mapsto & t. \end{array}$$

Therefore, we can define the germ ξ of plane curve singularity given by the following Puiseux parameterization:

$$\xi : \begin{cases} x(t) := t^\alpha \\ y(t) := ev(Y) = t^\beta + \sum_{i=1}^{b} a_i t^{i+\beta}. \end{cases}$$

If R stands for the local ring of the curve C, then $\Gamma = v(R)$. Moreover, it is easy to check that, by construction, the set $\{ev(\omega_k)\}$ is a minimal set of generators of the R-module $R + zR$. Therefore, we have $v(R + zR) = L$, since $\mathrm{ord}_T(\omega_k) = \mathrm{ord}_t(ev(\omega_k))$.

We conclude proving Steps 1 and 2.

Proof of Step 1: we apply induction. Define

$$U^0_{\sigma_0} := 1 \quad U^1_{\sigma_1} := T^{g_1}(c_0 + \sum_{i \geq 1} c_i X_i T^i)$$

It is easily checked that $U^0_{\sigma_0}, U^1_{\sigma_1}$ are of the required form. Let be $\epsilon = e_1\alpha + e_2\beta \in \Gamma$, and write $P(\epsilon) := T^{e_1\alpha} Y^{e_2}$. Let us assume that for $i < k < s$ there exists this family of polynomials. We are going to construct $\{U_j^k\}$ for $j \in I_k$; first, we define $U^k_{\sigma_{k-1}} := P(u_{k-1} - g_{k-1}) U^{k-1}_{\sigma_{k-1}}$. Since by induction hypothesis $U^{k-1}_{\sigma_{k-1}}$ is of the desired form, so is $U^k_{\sigma_{k-1}}$. Now, for $j \leq \sigma_k$ we define the corresponding U_j^k recursively:

- If $h_k + j \notin E_{k-1}$ we put
$$U^k_{j+1} := U^k_j - \mathrm{lc}_T(U^k_j) T^{h_i + j}.$$
- If $h_k + j \in E_m \cap (\mathbb{N} \setminus E_m)$ for some $m < k$ we set
$$U^k_{j+1} := \mathrm{lc}_T(U^m_{\sigma_m}) U^k_j - \mathrm{lc}_T(U^k_j) U^m_{\sigma_m}.$$

Finally, it is a straightforward computation to check that in both cases U^k_{j+1} has the desired form.

Proof of Step 2: Observe that, by construction,

$$A_k = \prod_{j<k} A_j X_{\sigma_k} + V_{\sigma_k} \quad \text{with } V_{\sigma_k} \in \mathbb{C}[X_1, \ldots, X_{\sigma_k - 1}]$$

so that the condition $A_k \neq 0$ is equivalent to $X_{\sigma_k} \neq \frac{V_{\sigma_k}}{\prod_{j<k} A_j}$. Also observe that for $\ell \in (\sigma_{k-1}, \sigma_k)$ by definition

$$\mathrm{lc}_T(U^k_\ell) = \prod_{j<k} c_{j,\ell} A_j X_\ell + V_\ell \quad \text{with } V_\ell \in \mathbb{C}[X_1, \ldots, X_{\ell-1}]$$

so that $\mathrm{lc}_T(U^k_\ell) = 0$ is equivalent to $X_\ell = \frac{V_\ell}{\prod_{j<k} c_{j,\ell} A_j}$.

Therefore, the system (7.1) can be rewritten as

$$\begin{cases} X_{\sigma_k} \neq \frac{V_{s_k}}{\prod_{j<k} A_j} & \text{for all } k = 2, \ldots, s \\ X_\ell = \frac{V_\ell}{\prod_{j<k} c_{j,\ell} A_j} & \text{for all } k = 2, \ldots, s+1 \text{ and for all } \ell \in (\sigma_{k-1}, \sigma_k) \\ X_{\sigma_{s+1}} = \frac{V_{\sigma_{s+1}}}{\prod_{j<s+1} c_{j,\sigma_{s+1}} A_j}. \end{cases} \tag{7.2}$$

Finally, observe that $(\sigma_{i-1}, \sigma_i) \cap (\sigma_{j-1}, \sigma_j) = \emptyset$ if $i \neq j$. Since every isolated variable in the system is different, we can solve the system in the following recursive way. We start with $A_0, A_1 \in \mathbb{C}^*$, and for $\ell \in (\sigma_1, \sigma_2) = (0, g_2 - u_1)$ we have $X_\ell = 0$. Thus, $X_{\sigma_2} \neq 0$. Let us denote $a_{\sigma_2} = X_{\sigma_2} \in \mathbb{C}^*$. After that, since V_ℓ depends on variables of lower index than ℓ a recursive reasoning solves the system. □

The systems (7.1) and (7.2) have more equations than needed in order to obtain the required semimodule; the reason is that, if $j \in (\sigma_{k-1}, \sigma_k)$ is such that $h_k + j \in E_{k-1}$, then U^k_j cannot be an element with $v(U^k_j) = g_k$, since $v(U^k_j) = h_k + j \in E_{k-1}$. Moreover, the last condition provides an element with $\sigma_{s+1} + h_{s+1} = c_{s+1} + \alpha\beta$. If we apply Lemma 7.8, it is trivial to see that $\mathbb{N} + c_s + \alpha\beta \subset E_s$, since $\mathbb{N} + c_s + \alpha\beta \subset \mathbb{N} + \overline{u}_s$; thus we can eliminate the last condition as well. As a result of that we may replace the system of equations given in the proof with the following:

$$\begin{cases} X_{\sigma_k} \neq \frac{V_{\sigma_k}}{\prod_{j<k} A_j} & \text{for all } k = 2, \ldots, s \\ X_\ell = \frac{V_\ell}{\prod_{j<k} c_{j,\ell} A_j} & \text{for } k = 2, \ldots, s+1 \text{ and for all } \ell \in (\sigma_{k-1}, \sigma_k) \\ & \text{such that } h_k + \ell \in \mathbb{N} \setminus E_{k-1} \end{cases} \tag{7.3}$$

Remark 7.17. We have programmed with MATHEMATICA software the computations of all possible increasing Γ–semimodules with $g_1 = \beta - \alpha$ and the corresponding system of Equations 7.2, see Appendix A.

Remark 7.18. A result of Zariski [197, Chap. VI Prop. 2.1] shows that any plane branch with one Puiseux pair is isomorphic to a deformation of the monomial curve given by $t \mapsto (t^\alpha, t^\beta)$. The system of equations (7.3) together with the proof of Theorem 7.16 show that if $(a_1, \ldots, a_b) \in \mathbb{C}$ is a solution of this system, then the increasing Γ-semimodule L associated to the equation system (7.3) induces the deformation of the parameterization of the monomial curve defined by

$$\begin{cases} x(t) := t^\alpha \\ y(t) := t^\beta + \sum_{i=1}^{b} a_i t^{i+\beta}. \end{cases}$$

7.4 Revisiting Kähler differentials for one Puiseux pair

Coming back to the set of values of Kähler differentials, the combination of Delorme's Theorem 7.1 and Theorem 7.16 provide that all combinatorially possible value set of Kähler differentials can be realized. This fact together with the constructive proof of Theorem 7.16 provide us some advantages with respect to some results existing in the literature.

In the sequel, we will call $(\beta-\alpha)$*–increasing* Γ*-semimodule* to an increasing Γ-semimodule with $g_0=0$, $g_1=\beta-\alpha$. Observe that the Γ–semimodule of values of Kähler differentials, Δ, is a particular example of $(\beta-\alpha)$–increasing Γ-semimodule. Recall that the set Δ is an analytic invariant of the curve ξ, as deduced from [42, §4].

Remark 7.19. Observe that Δ is the normalization of the value set associated to the fractional ideal $(f,\partial f/\partial x,\partial f/\partial y)$: a straightforward computation yields that

$$\frac{\partial f/\partial x(t)}{\partial f/\partial y(t)}=\frac{dy(t)}{dx(t)}.$$

Moreover, Hefez and Hernandes [77, Theorem 2.1] prove the following general result:

Theorem 7.20. *[77, Theorem 2.1] Let $\Gamma=\langle\overline{\beta}_0,\ldots,\overline{\beta}_g\rangle$ be the semigroup of values of a plane branch ξ. Then, a Puiseux parameterization of ξ is analytically equivalent to either $(t^{\overline{\beta}_0},t^{\overline{\beta}_1})$ or*

$$\left(t^{\overline{\beta}_0},t^{\overline{\beta}_1}+t^{\lambda}+\sum_{\substack{i>\lambda\\ i\notin\Delta'-\overline{\beta}_0}}a_it^i\right),$$

where λ is its Zariski invariant and Δ' is the non-normalized set of orders of differentials of the branch. Moreover, if φ and φ' are Puiseux parameterizations of the previous form with the same Γ and Δ' then they are analytically equivalent if and only if there is $r\in\mathbb{C}^*$ such that $r^{\lambda-\overline{\beta}_1}=1$ and $a_i=r^{i-\overline{\beta}_1}a'_i$.

Remark 7.21. Theorem 7.20 was already stated by Delorme in [42, Proposition 6] under two restrictions (Condition (CE) and (CU) of [42, Proposition 6]) for the Δ' set. Therefore, Hefez and Hernandes' theorem 7.20 could be stated as: the restrictions of Delorme for the Δ' set can be eliminated.

From the computational point of view, Hefez and Hernandes [76] provided a method to compute all possible Δ sets appearing in Theorem 7.20. However, this method is essentially brute force in the computation of a standard basis with variable coefficients for the module of Kähler differentials. In the case of one Puiseux pair, our constructive Theorem 7.16 provides a method which actually computes all possible Δ set with much less computational cost than the standard bases process implemented by Hefez and Hernandes. Moreover the following Corollary to our Theorem 7.16 shows that in fact all combinatorial possibilities for the possible value sets of Kähler differentials are realizable.

Corollary 7.22. *Let $\Gamma=\langle\alpha,\beta\rangle$ be a numerical semigroup. Then, L is an increasing Γ–semimodule with first non-zero minimal generator equal to $\beta-\alpha$ if and only if L can be realized as the set of values of Kähler differentials of some irreducible plane curve singularity with value semigroup Γ.*

Proof. Direct consequence of Theorem 7.1 and Theorem 7.16. □

Example 7.23. Let us consider the numerical semigroup $\Gamma = \langle 5, 12\rangle$. Following the tree structure explained in Section 7.2.1, all possible $\beta - \alpha$–increasing Γ–semimodules have the following tree structure:

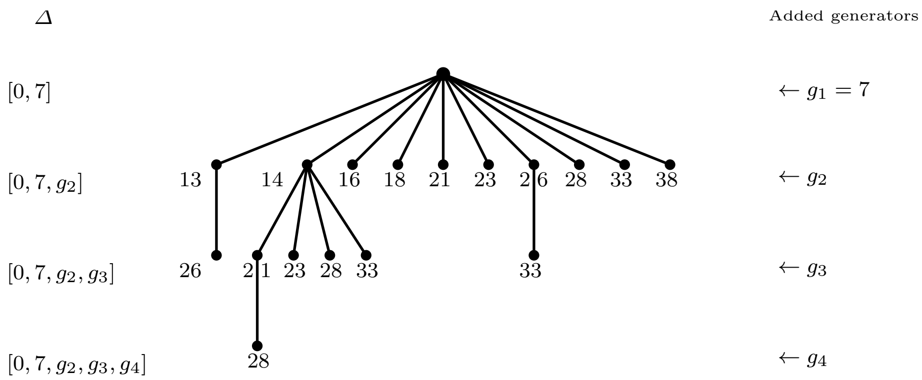


Figure 7.4: Tree of $\beta - \alpha$–increasing Γ-semimodules.

Recall that according with Zariski [197], we can always consider a short Puiseux series of the form:

$$\begin{aligned} y(t) =& t^{12} + c_{13}t^{13} + c_{14}t^{14} + c_{16}t^{16} + c_{18}t^{18} + c_{19}t^{19} + c_{21}t^{21} \\ &+ c_{23}t^{23} + c_{26}t^{26} + c_{28}t^{28} + c_{31}t^{31} + c_{33}t^{33} + c_{38}t^{38}. \end{aligned}$$

With this Puiseux series our program computes all the $\beta - \alpha$–increasing Γ–semimodules and their associated conditions according to the proof of Theorem 7.16. In the previous tree we can see all possible $\beta - \alpha$–increasing Γ–semimodules so let us show the conditions according to the system of equations (7.3).

- Conditions for the Kähler semimodule $\{0, 7\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = c_{26} = c_{28} = c_{33} = c_{38} = 0.$$

- Conditions for the Kähler semimodule $\{0, 7, 13\}$:

$$\begin{aligned} &c_{13} \neq 0 \\ &c_{13} + 2c_{14} = 0. \end{aligned}$$

- Conditions for the Kähler semimodule $\{0, 7, 14\}$:

$$\begin{aligned} &c_{13} = 0 \\ &c_{14} \neq 0 \\ &48c_{16} = 52c_{14}^2 \\ &\frac{72c_{18}}{25} = \frac{266c_{14}^3}{75} \\ &264c_{14}c_{23} = 816c_{14}^2c_{21} \\ &-\frac{6422c_{14}^9}{16875} - \frac{15308c_{14}^6c_{18}}{1875} + \frac{4272c_{14}^3c_{18}^2}{625} - \frac{14592c_{14}^2c_{26}}{625} + \frac{4608c_{14}c_{28}}{625} - \frac{11664c_{21}^2}{625} = 0. \end{aligned}$$

- Conditions for the Kähler semimodule $\{0, 7, 16\}$:

$$\begin{aligned} &c_{13} = c_{14} = 0 \\ &c_{16} \neq 0. \end{aligned}$$

- Conditions for the Kähler semimodule $\{0, 7, 18\}$:

$$c_{13} = c_{14} = c_{16} = 0$$
$$c_{18} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 21\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = 0$$
$$c_{21} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 23\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = 0$$
$$c_{23} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 26\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = c_{28} = 0$$
$$c_{26} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 28\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = c_{26} = 0$$
$$c_{28} \neq 0.$$

- Conditions for the Kähler semimodule$\{7, 33\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = c_{26} = c_{28} = 0$$
$$c_{33} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 38\}$:

$$c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = c_{26} = c_{28} = c_{33} = 0$$
$$c_{38} \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 13, 26\}$:

$$c_{13} \neq 0$$
$$c_{13}^2 + 2c_{14} \neq 0.$$

  Here we would like to remark that this is the $\beta - \alpha$-increasing Γ-semimodule associated to the generic component of the moduli space [42], since there are no imposed zero conditions.

- Conditions for the Kähler semimodule $\{7, 14, 21\}$:

$$c_{13} = 0$$
$$48c_{14}^4 + 144c_{14}c_{18} - 192c_{16}^2 = 0$$
$$c_{14} \neq 0$$
$$48c_{16} - 52c_{14}^2 \neq 0.$$

- Conditions for the Kähler semimodule $\{7, 14, 23\}$:

$$\begin{aligned}
&c_{13} = 0\\
&48c_{16} - 52c_{14}^2 = 0\\
&c_{14} \neq 0\\
&-28c_{14}^3 - 56c_{14}c_{16} + 72c_{18} \neq 0.
\end{aligned}$$

- Conditions for the Kähler semimodule $\{7, 14, 28\}$:

$$\begin{aligned}
&c_{13} = 0\\
&48c_{16} - 52c_{14}^2 = 0\\
&-28c_{14}^3 - 56c_{14}c_{16} + 72c_{18} = 0\\
&c_{14} \neq 0\\
&-348c_{14}^2c_{21} + 264c_{14}c_{23} - 432c_{16}c_{21} \neq 0.
\end{aligned}$$

- Conditions for the Kähler semimodule $\{7, 14, 33\}$:

$$\begin{aligned}
&c_{13} = 0\\
&48c_{16} - 52c_{14}^2 = 0\\
&-28c_{14}^3 - 56c_{14}c_{16} + 72c_{18} = 0\\
&-348c_{14}^2c_{21} + 264c_{14}c_{23} - 432c_{16}c_{21} = 0\\
&c_{14} \neq 0\\
&-\frac{418c_{14}^9}{6075} - \frac{11664c_{21}^2}{625} - \frac{14592c_{14}^2c_{26}}{625} + \frac{4608c_{14}c_{28}}{625} \neq 0.
\end{aligned}$$

- Conditions for the Kähler semimodule $\{7, 26, 33\}$:

$$\begin{aligned}
&c_{13} = c_{14} = c_{16} = c_{18} = c_{21} = c_{23} = 0\\
&c_{26} \neq 0\\
&c_{28} \neq 0.
\end{aligned}$$

- Conditions for the Kähler semimodule $\{7, 14, 21, 28\}$:

$$\begin{aligned}
&c_{13} = 0\\
&c_{14} \neq 0\\
&48c_{16} - 52c_{14}^2 \neq 0\\
&48c_{14}^4 + 144c_{14}c_{18} - 192c_{16}^2 = 0
\end{aligned}$$

Appendix A

Implementation of some results of Chapter 7

This appendix contains the implementation in MATHEMATICA of the constructive proof of the following Theorem in the case of an increasing $\langle \alpha, \beta \rangle$–semimodule with $g_0 =, g_1 = \beta - \alpha$.

Theorem. *7.16 Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup with $\alpha < \beta$. Let L be an increasing Γ-semimodule, and set $b := c(\Gamma) - \beta - 1$. Then there exist a tuple $(a_1, \ldots, a_b) \in \mathbb{C}^b$ and $z \in \overline{R}$ such that $L = v(R + zR)$, where R is the local ring of the germ of plane curve singularity defined by the Puiseux parameterization*

$$C : \begin{cases} x(t) := t^{\alpha} \\ y(t) := t^{\beta} + \displaystyle\sum_{i=1}^{b} a_i t^{i+\beta}. \end{cases}$$

Our implementation uses the minimal system of equations (7.3), so it provides the minimal system of conditions in order to construct the parameterization of the curve.

Also it contains some implemations to compute all possible increasing $\langle \alpha, \beta \rangle$–semimodules withouth any assumption of its minimal generators. As a consequence those implementations allow to compute all possible set of Kähler differentials associated to an equisingular class $\langle \alpha, \beta \rangle$ and the conditions of the coefficients of its parameterization.

```
(* DESCRIPTION OF THE PACKAGE kahlerstratification.M: this package computes
the conditions on the coefficients of a Puiseux series to have a given
semimodule of kahler differentials. It only works for the case of one
Puiseux pair.
*)

BeginPackage["kahlerstratification`"]

(* Main functions
*)

CharacteristicQ::usage = "Characteristic[n,M] returns 1 if M/n is the set
of characteristic exponents of a plane curve singularity of multiplicity n;
returns 0 if M/n is not a characteristic set. In the first case returns
also a list of generators for the semigroup of the singularity and also
returns the order of singularity."

SemiGroup::usage = "SemiGroup[L,r] gives the intersection of the semigroup
generated by the list L with the interval [0,r]"
KConditions::usage="KConditions[M,K] gives a list with the conditions on
the coefficients of a Puiseux series to have K as Kahler monoid"
Increasingsemi::usage="Increasingsemi[M] gives the number of increasing
semimodules of the semigroup M and the list of them"
KahlerMonoids::usage="gives the number of kahler incresing semimodules"
TjurinaMonoids::usage="gives the number of kahler incresing semimodules"
(* Auxiliary functions to have an easy flow
*)

ES::usage = "ES[L,M] returns the list formed by l+m with l belonging to
L and m belonging to M"

PRJ1::usage = "PRJ1[A] returns the first component of a vector A
with three entries"

PRJ2::usage = "PRJ2[A] returns the second component of a vector A
with three entries"

PRJ3::usage = "PRJ3[A] returns the third component of a vector A
with three entries"

Prj1::usage = "Prj1[A] returns the list of first components of a list
of vectors with three entries"

Prj2::usage = "Prj2[A] returns the list of second components of a
list of vectors with three entries"

Prj3::usage = "Prj3[A] returns the list of third components of a
```

```
list of vectors with three entries"

Clean::usage = "Clean[L] Deletes all empty sublists of L"

ExponentsQ::usage = "ExponentsQ[n, II] returns the characteristic
 exponents from the set of all the exponents II/n"

Delorme::usage = "FooBar"

DelormeAB::usage = "FooBar"

LExponent::usage= "FooBar"
```

```
Begin["Private`"]

SemiGroup[L_,d_]:=
Module[{a,b,i,W,G,J,K,S},
  W=Table[i,{i,0,d}];
  S=Union[L];
  a=Length[S]+1;
  b=Ceiling[d/S[[1]]]+1;
  G=Intersection[W,Table[S[[1]]*i,{i,0,b}]];
  For[J=2,J<a,J++,
    For[K=0,K<b,K++,
      G=Intersection[W,Union[G,G+(S[[J]]*K)]];
    ];
  ];
  Return[G]
]

CharacteristicQ[n_,M_]:=
  Module[{S,k,G,K,r,s,T},
    S=Union[M,{}];
    k=Length[S];
    G=Range[k];
    If[n>=S[[1]],Return[{0,"n>S[[1]]"}]];
    If[Mod[S[[1]],n]==0,Return[{0,"Mod[S[[1]],n]=0"}]];
    r=GCD[n,M[[1]]];
    G[[1]]=r;
    s=M[[1]];
    For[K=2,K<k+1,K++,
      If[Mod[M[[K]],r]==0,Return[{0,"Mod[M[[K]],r]*0",K,r}]];
      G[[K]]=GCD[r,M[[K]]];
      r=G[[K]];
      s=GCD[s,M[[K]]];
    ];
    s=GCD[n,s];
    If[s>1,Return[{0,"gcd >1"}]];
```

```
    r=M[[1]]*(n-G[[1]]);
    If[k==1,Return[{1,{n,M[[1]]},(r-n+1)/2}]];
    r=r+(M[[2]]*(G[[1]]-G[[2]]));
    If[k==2,
      Return[{1,{n,M[[1]],((n-G[[1]])*M[[1]]/G[[1]])+M[[2]]},(r-n+1)/2}]];
    T=Range[k];
    T[[1]]=M[[1]];
    T[[2]]=((n-G[[1]])*M[[1]]/G[[1]])+M[[2]];
    For[K=3,K<k+1,K++,
      T[[K]]=((T[[K-1]]-M[[K-1]])*G[[K-2]]/G[[K-1]])+M[[K]]
      +((G[[K-2]]-G[[K-1]])/G[[K-1]])*M[[K-1]];
    r=r+(M[[K]]*(G[[K-1]]-G[[K]]));
    ];
    T=Union[{n},T];
    Return[{1,T,((r-n)+1)/2}];
  ]

ES[L_,M_]:=Union[Flatten[Outer[Plus,L,M]]]
PRJ1[A_]:=A[[1]]
PRJ2[A_]:=A[[2]]
PRJ3[A_]:=A[[3]]
Prj1[A_]:=Map[PRJ1,A]
Prj2[A_]:=Map[PRJ2,A]
Prj3[A_]:=Map[PRJ3,A]

Clean[L_]:=Module[{AX},
  AX={};
  For[i=1,i<=Length[L],i++,
    If[L[[i]]=={},,AppendTo[AX,L[[i]]]]];
  ];
  Return[AX];
]
```

```
ExponentsQ[n_,I_]:=Module[{ni,mi,M},
  ni=n;
  M={};
  While[ni!=1,
    mi=Select[I,Mod[#,ni]!=0&,1][[1]];
    ni=GCD[ni,mi];
    AppendTo[M,mi];
  ];
  Return[M];
]

Delorme[n_,M_]:=Module[{CH,c,Gamma,Gi,Ei,uj,gj,cj,sj,NN,i,j,Γ},
    CH=CharacteristicQ[n,M];
    If[CH[[1]]==0,Return["Error: this is not a Characteristic Set"]];
    c=2*CH[[3]]; NN:=Join[{0},Range[c]];
```

```
    Γ=CH[[2]]; Gamma=SemiGroup[Γ,c];

    For[i=2,i<=Length[Γ],i++,
        Gi={0};
        Ei=ES[Gamma,Gi];
        gj=Γ[[i]]-n;
        cj=0;
        sj=n;
        uj=M[[1]];
        While[gj!=Infinity,
            sj=sj+gj-uj;
            uj=Min[Intersection[ES[Gamma,{gj}],Ei]];
            cj=cj+gj-uj;

            Gi=Join[Gi,{gj}];
            Ei=ES[Gamma,Gi];

            gj=Min[Intersection[NN+uj,Complement[NN,Ei]]];

    ];
    ];

    Return[{Gi,cj+sj+n*M[[1]]-uj}];
]

LExponent[n_,M_,K_]:=Module[{CH,c,Gamma,Gi,Ei,uj,gj,cj,sj,NN,i,j,Γ,S,A},
    CH=CharacteristicQ[n,M];
    If[CH[[1]]==0,Return["Error: this is not a Characteristic Set"]];
    c=2*CH[[3]]; NN:=Join[{0},Range[c]];
    Γ=CH[[2]]; Gamma=SemiGroup[Γ,c];
Gi={0};
Ei=ES[Gamma,Gi];

        cj=0;
        sj=n;
        uj=M[[1]];
A={};
S={};
    For[i=1,i<=Length[K],i++,
        gj=K[[i]];


            sj=sj+gj-uj;
        AppendTo[S,sj];
            uj=Min[Intersection[ES[Gamma,{gj}],Ei]];
            AppendTo[A,uj];
            cj=cj+gj-uj;
```

```
            Gi=Join[Gi,{gj}];
            Ei=ES[Gamma,Gi];


    ];

    Return[{A,cj+sj+n*M[[1]]-uj,S}];
]
```

```
KConditions[M_,K_]:=Module[{AUX4,AUX3,AUX2,c,r,S,NN,Parf,SSet,b,t,ulist,
slist,hj,coef,i,s,w1,CON,EI,rel,relnc,OMEGA,j,u,mon,R,UU,AUX,w,a},
    c=(M[[1]]-1)*(M[[2]]-1);
    S=SemiGroup[M,c];
    NN=Join[{0},Range[c]];
    Parf=LExponent[M[[1]],{M[[2]]},K]; (*Computes list of ui's and sigmas
of the proof of the constructive theorem*)
    slist=Union[Parf[[3]],{Parf[[2]]}];
    b=c-1-M[[2]];
    ulist=Parf[[1]];
    (*We create the list of coefficients for the Puiseux series*)
    coef={};
    For[i=0,i<b,i++;
                AppendTo[coef,ToExpression[StringJoin["a"<>ToString[i]]]]];

    AUX2=Union[S,S+K[[1]]]; (*Auxiliary semimodule to define a
                                 short parameterization*)
    s=t^M[[2]];
    For[i=M[[2]],i<b+M[[2]],i++;If[MemberQ[AUX2,i],
    s=s,s=s+ToExpression[StringJoin["c"<>ToString[i]]]*t^i]];
    (*construction of the series corresponding to dy/dx*)
    w1=(M[[2]]/M[[1]])*t^(M[[2]]-M[[1]]);
    For[i=M[[2]],i<b+M[[2]],i++;
    If[MemberQ[AUX2,i],w1=w1,
w1=w1+(i/M[[1]])*ToExpression[StringJoin["c"<>ToString[i]]]*t^(i-M[[1]])]];


CON={}; (*list where we are going to put conditions that must be equal to 0*)
EI=S; (*first step in the construction of the kahler semimodule*)
rel={};
relnc={}; (*list where we are going to put conditions that must be non 0*)
OMEGA={w1}; (*list where we are going to put the expression of the
                    differentials generating the Kahler semimodule K*)
    For[i=1,i<Length[ulist],i++,
        u=ulist[[i]];
(*we obtain all possible {a,b} such that aM[[1]]+bM[[2]]=nu and u=nu+K[[i]]*)
        mon=FrobeniusSolve[M,u-K[[i]]];
(*construction of a series of order nu, with u=nu+K[[i]]*)
        R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
```

```
        (*construction of a series of order u*)
        UU=Expand[R*OMEGA[[i]]];
        j=1;


        (*kahler semimodule up to the generator K[[i]]*)
        EI=Union[EI,S+K[[i]]];
        (*set of gaps which can be candidates to be a new generator of the
            kahler semimodule*)
        SSet=Complement[NN,EI];
        AUX=S;
            a=0;
        While[!MemberQ[AUX,u],
                a=a+1;
                AUX=Union[AUX,S+K[[a]]];
                  ];
(*a indicates that u+j=nu+K[[a]] with nu in the semigroup.
If a=0 then u+j is in the semigroup*)
(*We create a new UU*)

            If[a==0,
                mon=FrobeniusSolve[M,u];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[UU-Coefficient[UU,t^(u)]*R],
                mon=FrobeniusSolve[M,u-K[[a]]];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[Coefficient[OMEGA[[a]],
                t^K[[a]]]*UU-Coefficient[UU,t^(u)]*R*OMEGA[[a]]];
                    ];

(*In case K[[i+1]]-u-1 is bigger than one, i.e. K[[i+1]] is not equal
to u+1. Then, we maybe need to imposse conditions and we go into the loop *)
        While[j<=K[[i+1]]-u-1,
            AUX=S;
            a=0;
(*If UU is a possible diferential with order less than K[[i+1]]
then the initial term must be equal to zero in order to not have it as a
 generator. Thus, we must include the leading coefficient in the list of
conditions that must be equal to zero, CON. If j does not belong to SSet
then u+j belong to EI. Therefore, there is no need to imposse conditions *)
            If[MemberQ[SSet,j+u], AppendTo[CON,Coefficient[UU,t^(u+j)]],
                CON=CON;
                ];
            (*Recursion to create a new UU: if u+j not in EI
then elimination is trivial. Otherwise elimination must use the previous UU*)
            If[!MemberQ[EI,u+j],
(*Trivial elimination*)
            UU=Expand[UU-Coefficient[UU,t^(u+j)]*t^(u+j)] ,
```

```
            While[!MemberQ[AUX,u+j],
                a=a+1;
                AUX=Union[AUX,S+K[[a]]];
                  ];
(*a indicates that u+j=nu+K[[a]] with nu in the semigroup*)

            (*We create a new UU*)
            If[a==0,
                mon=FrobeniusSolve[M,u+j];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[UU-Coefficient[UU,t^(u+j)]*R],
                mon=FrobeniusSolve[M,u+j-K[[a]]];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[Coefficient[OMEGA[[a]],
                t^K[[a]]]*UU-Coefficient[UU,t^(u+j)]*R*OMEGA[[a]]];
                    ];
                ];
            AppendTo[rel,UU];

            j=j+1;
            ];
            (*The last UU in this elimination process is a UU of
order K[[i+1]] thus we need to imposse the leading coefficient to be non
zero and we set this series as a new generator of the kahler module
--as we see in the proof of the constructive theorem*)
        w=Expand[UU];
        AppendTo[OMEGA,w];
        AppendTo[relnc,Coefficient[OMEGA[[i+1]],t^K[[i+1]]]]
(*The leading coefficient of the series of order K[[i+1]] must be non
zero in order to be a generator. Thus we add the term to relnc*)
            ];

(*Elimination after the last u must be done in order to not have
more generators than the ones in K. We imposse conditions that to be zero
if some candidate to new generator appear.
All the procedures has already been explained*)
    u=Last[ulist];
        AUX4={};
        mon=FrobeniusSolve[M,u-Last[K]];
        R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];

        UU=Expand[R*Last[OMEGA]];

        j=1;
        hj=ulist[[Length[ulist]]]-slist[[Length[slist]-1]];
        EI=Union[EI,S+Last[K]];
        SSet=Complement[NN,EI];
```

```
AUX=S;
            a=0;
        While[!MemberQ[AUX,u],
                a=a+1;
                AUX=Union[AUX,S+K[[a]]];
                  ];


            If[a==0,
                mon=FrobeniusSolve[M,u];

                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[UU-Coefficient[UU,t^(u)]*R],
                mon=FrobeniusSolve[M,u-K[[a]]];

                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[Coefficient[OMEGA[[a]],
                t^K[[a]]]*UU-Coefficient[UU,t^(u)]*R*OMEGA[[a]]];
                    ];
AppendTo[rel,UU];



        While[u+j<=Max[Complement[Range[c],EI]],

            AUX=S;
            a=0;
            If[MemberQ[SSet,u+j],AppendTo[CON,Coefficient[UU,t^(u+j)]];
                                AppendTo[AUX4,Coefficient[UU,t^(u+j)]],
                CON=CON;
                ];


            If[!MemberQ[EI,u+j],
            UU=Expand[UU-Coefficient[UU,t^(u+j)]*t^(u+j)],
            While[!MemberQ[AUX,u+j],
                a=a+1;
                AUX=Union[AUX,S+K[[a]]];
                  ];


            If[a==0,
                mon=FrobeniusSolve[M,u+j];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
                UU=Expand[UU-Coefficient[UU,t^(u+j)]*R],
                mon=FrobeniusSolve[M,u+j-K[[a]]];
                R=Expand[t^(M[[1]]*mon[[1]][[1]])*s^(mon[[1]][[2]])];
```

```
                UU=Expand[Coefficient[OMEGA[[a]],
                    t^K[[a]]]*UU-Coefficient[UU,t^(u+j)]*R*OMEGA[[a]]];
                    ];
                ];
            AppendTo[rel,UU];
            j=j+1;
            ];





    Return[{CON,relnc,AUX4}];
]
```

```
Increasingsemi[M_]:=Module[{c,L,Gaps,NN,u,Aux,Aux2,Aux3,i,j,Aux4,Q,s,S},
    c=(M[[1]]-1)*(M[[2]]-1);
    S=SemiGroup[M,c];
    NN=Range[0,c];
    Gaps=Complement[NN,S];
    L={};
    For[i=0,i<Length[Gaps],i++;AppendTo[L,{0,Gaps[[i]]}]];
    Aux4={};
    s=1;
    While[Length[Aux4]!=Length[L],
        Aux4=L;
        Q={};
        For[i=0,i<Length[L],i++;If[Length[L[[i]]]>s,AppendTo[Q,L[[i]]]]];
        For[i=0,i<Length[Q],i++;
        Aux2=Delete[Q[[i]],Length[Q[[i]]]];
        Aux3=ES[S,Aux2];

        u=Min[Intersection[Aux3,S+Q[[i]][[Length[Q[[i]]]]]]];

        Aux=ES[S,Q[[i]]];For[j=0,j<Length[Complement[NN,Aux]],j++;
        If[Complement[NN,Aux][[j]]>u,AppendTo[L,Union[Q[[i]],
            {Complement[NN,Aux][[j]]}]]]]]];
            s=s+1];

    Return[{Length[L],L}]

]
```

```
KahlerMonoids[M_]:=Module[{L,K,i},
    L=Increasingsemi[M][[2]];
    K={};
    For[i=0,i<Length[L],i++;
        If[L[[i]][[2]]==M[[2]]-M[[1]],AppendTo[K,Delete[L[[i]],1]]]];

Return[{Length[K],K}]

]

TjurinaMonoids[M_]:=Module[{S,K,L,Aux1,Aux2,Aux3,c,i,j},
    c=(M[[1]]-1)(M[[2]]-1);
    S=SemiGroup[M,c];
    K=KahlerMonoids[M][[2]];
    L={};
    For[i=0,i<Length[K],i++;
            Aux1=Union[S,ES[S,K[[i]]]];
            Aux2=Complement[Aux1+M[[1]],S];
            j=1;
            Aux3={};
            While[Aux2[[j]]<c,AppendTo[Aux3,Aux2[[j]]];j=j+1];
            AppendTo[L,{c-Length[Aux3],K[[i]]}];
            ];
    Return[L]



]
```

```
MinimalMonoid[n_,M_,Λ_]:=Module[{CH,c,Λs,Λmin,Gamma,Λi,i},
    CH=CharacteristicQ[n,M];
    If[CH[[1]]==0,Return["Error: this is not a Characteristic Set"]];
    c=2*CH[[3]];
    Gamma=SemiGroup[CH[[2]],c];
    Λs=Sort[Λ];

    Λmin={0};
    For[i=1,i<Length[Λ],i++,
        Λi=ES[Gamma,Join[Λmin]];
        If[!MemberQ[Λi,Λs[[i+1]]],AppendTo[Λmin,Λs[[i+1]]]];
    ];

    Return[Λmin];
]

End[]
EndPackage[]
```

Appendix B

The miniversal deformation of an isolated complete intersection singularity

In 1958, Kodaira and Spencer developed a whole theory of deformations of complex structures in their foundational work [89]. The key ingredient is the construction of a homomorphism between the sheaf of germs of differentiable vector fields on a connected manifold M into a certain cohomology sheaf. To study deformations of a complex structure, they consider a complex structure which depends differentiably on a parameter moving on M; so that the complex structure can be regarded as a fiber over M. As they exposed, this homomorphism may be considered to be a measure of dependence of the complex structure on the parameter. The restriction of this homomorphism to a fixed complex structure is a linear map which is nowadays called the Kodaira-Spencer map and its image represents the infinitesimal deformations of the complex structure. In this way, it was a natural question to ask under what conditions the Kodaira-Spencer map is an isomorphism. A remarkable answer was given by Kodaira, Nirenberg and Spencer in [88] by using the theory of elliptic partial differential equations.

In 1964, Grauert and Kerner in [66] established the theory of deformations of complex isolated singularities in analogy to the work of Kodaira and Spencer. In particular, they characterized rigid singularities in terms of the cancellation of a certain Ext module. In fact, one can see in Palamodov's survey [129, pg.135–140] a good dictionary to translate one world into another. Before the result of Grauert [65] about the existence of miniversal/semiuniversal/effective versal deformations of singularities in 1972, Tjurina's work [166] can be seen as the inflection point of deformation theory of singularities. Tjurina's Theorem is an exact analogy to the Theorem of Kodaira-Nirenberg-Spencer [88] which completes the initial steps performed by Grauert and Kerner in [66].

Nowadays, after Schlessinger results on formal deformation theory (see [71, Appendix C]), the language of categories is a usual way to talk about deformation of singularities. However, we are going to try to overview Tjurina's paper [166] in a down to earth language. First, let us introduce the definition of deformation of complex space germs.

Definition B.1. Let $(X, \mathbf{0}), (S, 0)$ be complex space germs. A *deformation of* $(X, \mathbf{0})$ *over* $(S, 0)$ consists of a flat morphism $\pi : (\mathcal{X}, \mathbf{0}) \to (S, 0)$ of complex germs together with an isomorphism $(X, \mathbf{0}) \simeq (\pi^{-1}(0), \mathbf{0})$.

We denote by $\mathcal{X}_s := \pi^{-1}(s)$ to the fibers of the morphism. We call *base space* to $(S, 0)$ and *special fiber* to the fiber above 0.

In brief, a deformation of a singularity is a germ of flat morphism whose special fiber is isomorphic to the singular germ. In a short way, deformations are usually expressed as a couple (i, π) where $i : (X, \mathbf{0}) \hookrightarrow (\mathcal{X}, \mathbf{0})$ is an embedding and $\pi : (\mathcal{X}, \mathbf{0}) \to (S, 0)$ is a flat morphism.

In analogy with the smooth case, the aim is to construct a $\mathbb{C}$–linear map which acts as the differential of the deformation map. In the spirit of differential geometry such a map should be a map between tangent spaces. Also, it should analyze the infinitesimal structure of the deformation.

Let us consider $(X, \mathcal{O}_X)$ an analytic space in an open set $U \subset \mathbb{C}^n$, where $\mathcal{O}_X$, $\mathcal{O}_U$ denote the corresponding structural sheaves. Assume that, we have local coordinates $(x_1, \dots, x_n)$ in $\mathbb{C}^n$. We usually keep in mind $\mathcal{O}_U = \mathbb{C}\{x_1, \dots, x_n\}$. Assume that X is defined by a sheaf ideal $I_X := (f_1, \dots, f_l)\mathcal{O}_U$.

Remark B.2. As a general notation, we will put $A|_X$ to denote the restriction of a space A to the space X.

Definition B.3. The sheaf $\Omega := (dx_1, \dots, dx_n)\mathcal{O}_U$ is called *sheaf of germs of* 1*–dimensional holomorphic forms on* U. Let us consider the subsheaf of Ω defined by $\Omega' := \{\omega = \sum_{i=1}^{l} h_i df_i + \sum_{j=1}^{n} g_j dx_j | \ h_i \in \mathcal{O}_U, \ g_j \in I_X\}$. We define the *module of absolute differentials of* X of $\Omega_X := \frac{\Omega}{\Omega'}|_X$. Recall that Ω_X is a coherent $\mathcal{O}_X$-module which is invariant of the defining equations of X as it is shown by Teissier in [164, 2.2 §2] (see also [75, 8 of Chapter II]).

In a natural way, given a morphism $\pi : Y \to T$ of analytic spaces, there is an induced homomorphism $d\pi : \Omega_T \to \Omega_Y$ which allows to define the *module of relative differentials* as the cokernel of $d\pi$, $\Omega_{Y|T} := \Omega_Y / d\pi(\Omega_T)$.

Now, let us consider a deformation of $(X_0, \mathcal{O}_{X_0})$, i.e. $(X_0, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathcal{X}, \mathbf{0}) \overset{\pi}{\to} (S, 0)$. Then the definition of the module of relative differentials give rise to the following exact sequence:

$$0 \to d\pi(\Omega_S) \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}|S} \to 0.$$

Following [66, §3], let us denote by $\mathcal{T}^*(S) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(d\pi(\Omega_S), \mathcal{O}_{X_0})$. Then, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\Omega_{\mathcal{X}|S}, \mathcal{O}_{X_0}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\Omega_{\mathcal{X}}, \mathcal{O}_{X_0}) \to \mathcal{T}^*(S) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}}}(\Omega_{\mathcal{X}|S}, \mathcal{O}_{X_0}). \tag{B.1}$$

Proposition B.4. *[166, Prop 3.1]*

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}}}(\Omega_{\mathcal{X}|S}, \mathcal{O}_{X_0})|_{X_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Finally, if we denote by T_0S the linear tangent space over $\mathbb{C}$ at $0 \in S$ we can identify (see [166, (3.9) pg. 977]) $\mathcal{T}^*(S)|_{X_0} = T_0S \otimes_{\mathbb{C}} \mathcal{O}_{X_0}$. After that, Tjurina gives the precise definition of the Kodaira-Spencer map associated to a deformation.

Definition B.5. Let $(X_0, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathcal{X}, \mathbf{0}) \overset{\pi}{\to} (S, 0)$ be a deformation of a complex analytic space $(X, \mathcal{O}_X)$. Let us denote by T_0S the linear tangent space over $\mathbb{C}$ at $0 \in S$. The *Kodaira-Spencer map of the deformation* (i, π) is the $\mathbb{C}$–linear map

$$\varphi : T_0S \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$$

induced by the restriction of the exact sequence (B.1) to X_0 together with the inclusion $T_0S \subset \mathcal{T}^*(S)|_{X_0}$.

Remark B.6. Compare this definition of the Kodaira-Spencer map with the one given in [108, 6.B] and [71, Lemma 1.20].

It is important to notice that the deformations of (X, x) form a category (See [71, subsec. 1.1 Chap. II]). To see that, we need to define the morphisms of this category.

Definition B.7. A *morphism of deformations* from (i, π) to (j, ρ) is a pair of morphisms (ϕ, ψ) such that the following diagram commutes:

$$\begin{array}{ccccc} (X,x) & \xrightarrow{i} & (\mathcal{X},x) & \xrightarrow{\pi} & (S,s) \\ & \searrow^{j} & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ & & (\mathcal{Y},y) & \xrightarrow{\rho} & (T,t) \end{array}$$

Two deformations over the same base space are *isomorphic* if there exists a morphism (ϕ, Id) with ϕ being an isomorphism.

Also, we can construct a morphism of deformations only from the map ψ. Let us consider (i, π) a deformation with base (S, s) and $\beta : (T, t) \to (S, s)$ a closed morphism of germs. Then the fiber product induces a deformation $\beta^*(i, \pi) := (\beta^* i, \beta^* \pi)$

$$\begin{array}{ccccc} (X,x) & \xrightarrow{i} & (\mathcal{X},x) & \xrightarrow{\pi} & (S,s) \\ & \searrow^{\beta^* i} & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \beta} \\ & & (\mathcal{X},x)\times_{(S,s)}(T,t) & \xrightarrow{\beta^*\pi} & (T,t). \end{array}$$

We say that the deformation $\beta^*(i, \pi)$ is *induced by the base change* β.

Finally, the following proposition gives sense to the sentence that Kodaira-Spencer map behaves well if we change the deformation.

Proposition B.8. *[166, Prop 3.2] Let* $(X_0, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathcal{X}, \mathbf{x}) \overset{\pi}{\to} (S, s)$ *and* $(X_0, \mathbf{0}) \overset{j}{\hookrightarrow} (\mathcal{Y}, \mathbf{y}) \overset{\rho}{\to} (S', s')$ *be two deformations. Let* (ϕ, ψ) *be a morphism of deformations from* (i, π) *to* (j, ρ)*. If* φ, φ' *are the corresponding Kodaira-Spencer maps of* $(i, \pi), (j, \rho)$ *then there exists a commutative diagram*

$$\begin{array}{ccc} T_sS & \xrightarrow{\varphi} & \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \\ \downarrow{\scriptstyle d\psi} & \nearrow_{\varphi'} & \\ T_{s'}S' & & \end{array}$$

where $d\psi : T_sS \to T_{s'}S'$ *is the map between tangent spaces induced by* $\psi : (S, s) \to (S', s')$*.*

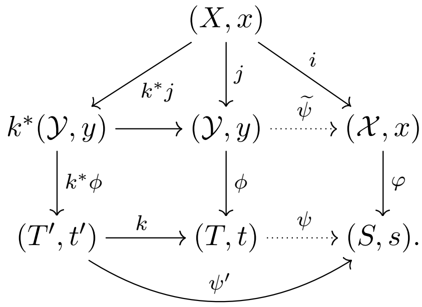
Thus, it is natural to ask whether this map is an isomorphism. This question was solved by Tjurina in the case of normal isolated singularities.

Theorem B.9. *[166, Thm. 4.1] Let* X_0 *be a complex space with normal isolated singularity* x_0*. If* $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})|_{x_0} = 0$ *then there exist a deformation* (i, π) *such that the Kodaira-Spencer map* φ *is an isomorphism.*

Remark B.10. One can check that Theorem B.9 is the analog to the Theorem of Kodaira-Nirenberg-Spencer [88] in this setting; see also [166, pg. 968].

In fact, Theorem B.9 gives a precise construction of the deformation whose Kodaira-Spencer map is an isomorphism. Before to continue let us give the following definitions.

Definition B.11. Let $i : (X, x) \hookrightarrow (\mathcal{X}, x)$ and $j : (X, x) \hookrightarrow (\mathcal{Y}, y)$ be two different embeddings of the singularity in some complex analytic spaces. Let $\varphi : (\mathcal{X}, x) \to (S, s)$ and $\phi : (\mathcal{Y}, y) \to (T, t)$ be two different deformations and $(T',t') \xrightarrow{k} (T,t)$ a closed embedding. We say that the deformation (i,φ) is *versal* if for any morphism $\psi' : (T',t) \to (S,s)$ such that $((\psi')^*i, (\psi')^*\varphi)$ is isomorphic as deformation to $(k^*j, k^*\phi)$ then there exist morphisms $\psi, \widetilde{\psi}$ which make the following diagram commutative:



A versal deformation is called *miniversal or semiuniversal* if the Zariski tangent map $T(\psi) : T_{(T,t)} \to T_{(S,s)}$ is uniquely determined by (i,φ) and (j,ϕ).

The previous definition can be summarized in the following way: the deformation is called versal if any other deformation results from it by base change. It is called miniversal if it is versal and S has minimal possible dimension. Therefore, it is not obvious whether a versal or miniversal deformation of a complex space exist or not. This question was completely solved by Grauert in 1972:

Theorem B.12. *[65, Thm. pg. 198] If (X,x) is a germ of a reduced complex analytic space with isolated singularity then there exist a versal deformation.*

In particular, the following Theorem of Flenner together with Grauert Theorem show that for isolated singularities there exists a semiuniversal deformation.

Theorem B.13. *[57, Satz 5.2] If a complex space has a versal deformation then there also exist a semiuniversal deformation.*

Moreover, if there exist a miniversal deformation it is unique up to analytic isomorphism (see [71, Lemma 1.12]). This is a quite natural property due to the condition of minimality of the dimension of the base space. Apart from its existence, one do not know a lot about how it looks like the miniversal deformation in the general situation. However, one can give a precise description of the miniversal deformation in certain cases thanks to the Kodaira-Spencer map. This result was given by Tjurina in 1969:

Theorem B.14. *[166, Thm. 6.1] Let X_0 be a complex space with normal isolated singularity x_0. Suppose that X_0 has a deformation (i,π) whose Kodaira-Spencer map*

$$\varphi : T_0 S \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$$

is an isomorphism. Then the deformation (i,π) is miniversal.

Therefore, Theorem B.9 together with Theorem B.14 lead to the following important consequence.

Theorem B.15. *[166, Thm. 0.1] Let X_0 be a complex space with normal isolated singularity x_0. If $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})|_{x_0} = 0$ then it has a unique (up to isomorphism) miniversal deformation $(X_0, \mathbf{0}) \overset{i}{\hookrightarrow} (\mathcal{X}, \mathbf{x}) \xrightarrow{\pi} (S,s)$ such that*

$$\dim_{\mathbb{C}}(S) = \dim_{\mathbb{C}}(\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})).$$

To finish this section let us see how to apply Theorem B.15 to construct the miniversal deformation of an isolated complete intersection, e.g. hypersurface singularity.

Let us consider $(X_0, \mathcal{O}_{X_0})$ a germ of an isolated complete intersection singularity in an open set $U \subset \mathbb{C}^N$. Assume that, we have local coordinates $(x_1, \ldots, x_N)$ in $\mathbb{C}^N$ and that X is defined by a sheaf ideal $I_X := (f_1, \ldots, f_k)\mathcal{O}_U$. Since X_0 is an ICIS, following [166, §1] we can construct a free resolution by $\mathcal{O}_U$–modules of the module of Kähler differentials of the form

$$0 \to \mathcal{O}^k_{X_0} \xrightarrow{d_1} \mathcal{O}^N_{X_0} \xrightarrow{d_0} \Omega_{X_0} \to 0$$

where the maps d_0, d_1 are defined as

$$(h_1, \ldots, h_N) \overset{d_0}{\longmapsto} \sum_i h_i dx_i \quad \text{and} \quad (g_1, \ldots, g_k) \overset{d_1}{\longmapsto} (\sum_i g_i \frac{df_i}{dx_1}, \ldots, \sum_i g_i \frac{df_i}{dx_N}).$$

After applying $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\bullet, \mathcal{O}_{X_0})$ to the free resolution, we obtain that $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) = 0$. Moreover, from the derived exact sequence together with [166, Prop. 1.5] we obtain the following isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \cong \frac{\mathcal{O}^k_U}{\left(\frac{\partial f_j}{\partial x_i}\right) \mathcal{O}^N_U + (f_1, \ldots, f_k)\mathcal{O}^k_U}, \tag{B.2}$$

where $\left(\frac{\partial f_j}{\partial x_i}\right) : \mathcal{O}^N_U \to \mathcal{O}^k_U$ is the Jacobian matrix. For the historical reasons exposed here, the $\mathbb{C}$–algebra $\frac{\mathcal{O}^k_U}{\left(\frac{\partial f_j}{\partial x_i}\right)\mathcal{O}^N_U + (f_1, \ldots, f_k)\mathcal{O}^k_U}$ is called in the bibliography *Tjurina algebra* of the complete intersection X.

Therefore, applying Theorem B.15 we obtain a precise description of the miniversal deformation of an isolated complete intersection singularity.

Theorem B.16 (Tjurina). *Let us consider $(X_0, \mathbf{0}) \subset (\mathbb{C}^N, \mathbf{0})$ be an isolated complete intersection singularity. Assume that X_0 is defined by the ideal $I_X := (f) = (f_1, \ldots, f_k)$ and denote by $Df := (\frac{\partial f_j}{\partial x_i})$ the Jacobian matrix. Let $g_1 \ldots, g_\tau \in \mathcal{O}^k_{(\mathbb{C}^N, \mathbf{0})}$, with $g_i = (g^1_i, \ldots, g^k_i)$, be a $\mathbb{C}$–basis for the finite dimensional $\mathbb{C}$–algebra*

$$\frac{\mathcal{O}^k_{(\mathbb{C}^n, \mathbf{0})}}{\left(\frac{\partial f_j}{\partial x_i}\right) \mathcal{O}^N_{(\mathbb{C}^n, \mathbf{0})} + (f_1, \ldots, f_k)\mathcal{O}^k_{(\mathbb{C}^n, \mathbf{0})}},$$

and define $F = (F_1, \ldots, F_k)$:

$$\begin{aligned} F_1(\mathbf{x}, \mathbf{t}) &= f_1(\mathbf{x}) + \textstyle\sum_{j=1}^{\tau} t_j g^1_j(\mathbf{x}), \\ &\vdots \qquad\qquad \vdots \\ F_k(\mathbf{x}, \mathbf{t}) &= f_k(\mathbf{x}) + \textstyle\sum_{j=1}^{\tau} t_j g^k_j(\mathbf{x}). \end{aligned}$$

If we define $(\mathcal{X}, \mathbf{0}) := V(F_1, \ldots, F_k) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})$ the zero set of F, then $(X, \mathbf{0}) \xrightarrow{i} (\mathcal{X}, \mathbf{0}) \xrightarrow{\phi} (\mathbb{C}^\tau, \mathbf{0})$ is the miniversal deformation of $(X, \mathbf{0})$, with i being induced by the inclusion $(\mathbb{C}^n, \mathbf{0}) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0})$ and ϕ by the projection $(\mathbb{C}^n \times \mathbb{C}^\tau, \mathbf{0}) \to (\mathbb{C}^\tau, \mathbf{0})$.

Proof. Applying Theorem B.15 and Equation (B.2) (see also [71, Chap. II Thm. 1.16]). □

Observe that in the case of a hypersurface singularity we have a simpler expression.

Theorem B.17 (Tjurina). *Let* $(X,0) \subset (\mathbb{C}^n,0)$ *be an isolated hypersurface singularity defined by* $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *and* $g_1,\ldots,g_\tau \in \mathcal{O}_{\mathbb{C}^n,0}$ *be a* $\mathbb{C}$*–basis of the Tjurina algebra* T_f*. If we set,*

$$F(x,\mathbf{t}) := f(x) + \sum_{j=1}^{\tau} t_j g_j(x), \quad (\mathcal{X},0) := V(F) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, 0),$$

then $(X,0) \hookrightarrow (\mathcal{X},0) \xrightarrow{\varphi} (\mathbb{C}^\tau,0)$*, with* φ *the second projection, is a miniversal deformation of* $(X,0)$*.*

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