REGULAR MULTILINEAR OPERATORS ON $C(K)$ SPACES

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Abstract. The purpose of this paper is to characterize the class of *regular* continuous multilinear operators on a product of $C(K)$ spaces, with values in an arbitrary Banach space. This class has been considered recently by several authors (see, f.i., [3], [8], [9]) in connections with problems of factorization of polynomials and holomorphic mappings. We also obtain several characterizations of the compact dispersed spaces K in terms of polynomials and multilinear forms defined on $C(K)$.

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1. Introduction and Notations

Let K be a compact Hausdorff space. $C(K)$ will be the space of scalar valued continuous functions on K, Σ will denote the σ-algebra of the Borel sets of K and $B(\Sigma)$ will stand for the space of Σ-measurable functions on K which are the uniform limit of elements of Σ-simple functions.

As it is well known, the Riesz representation theorem gives a representation of the operators on $C(K)$ as integrals with respect to Radon measures, and this has been very fruitfully used in the study of the properties of $C(K)$ spaces. In a series of papers (see specially [6], [7]), Dobrakov developed a theory of *polymeasures* , functions defined on a product of σ-algebras which are separately measures, that can be used to obtain a Riesz-type representation theorem for multilinear operators defined on a product of $C(K)$ spaces.

Before going any further, we shall clear out our notation: If X is a Banach space, X^* will denote its topological dual and B_X its closed unit ball. $\mathcal{L}^k(E_1 \dots, E_k; Y)$ will be the Banach space of all the continuous k-linear mappings from $E_1 \times \cdots \times E_k$ into Y, and $\mathcal{P}(^k X; Y)$ the space of continuous k-homogeneous polynomials from X to Y, i.e., the class of mappings $P : X \to Y$ of the form $P(x) = T(x, \dots, x)$, for some $T \in \mathcal{L}^k(X, \dots X; Y)$. When $Y = \mathbb{K}$, we will omit it. We shall use the convention $\overset{[i]}{\dots}$ to mean that the i-th coordinate is not involved.

We shall denote the semivariation of a measure μ by $\|\mu\|$ and also the semivariation of a polymeasure γ by $\|\gamma\|$ (for the general theory of polymeasures see [6], or [14]). It seems convenient to recall here that a polymeasure is called *regular* if it is separately regular and it is called *countably additive* if it is separately countably additive. We will denote the set of the bounded semivariation polymeasures defined in $\Sigma_1 \times \cdots \times \Sigma_k$ with values in X as $bpm(\Sigma_1, \dots, \Sigma_k; X)$. $rcapm(\Sigma_1, \dots, \Sigma_k; X)$ stands for the subset of the regular countably additive polymeasures and bsv-$\omega^* - rcapm(\Sigma_1, \dots, \Sigma_k; X^*)$ for the subset of $bpm(\Sigma_1, \dots, \Sigma_k; X^*)$ composed of those polymeasures that verify that for each $x \in X$, $x \circ \gamma \in rcapm(\Sigma_1, \dots, \Sigma_k; \mathbb{K})$.

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As customary we will call $rca(\Sigma; X)$ the set of regular countably additive measures from Σ into X.

With these notations at hand we can state for further references the following theorem from [4], which extends and completes previous results of Pelczynski ([11]) and Dobrakov ([7]):

Theorem 1.1. ([4]) *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. Then there is a unique $\overline{T} \in \mathcal{L}^k(B(\Sigma_1), \dots, B(\Sigma_k), X^{**})$ which extends T and is $\omega^* - \omega^*$ separately continuous (the ω^*- topology that we consider in $B(\Sigma_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). Besides, we have*

1. $\|T\| = \|\overline{T}\|$.

2. For every $(g_1, \overset{[i]}{\dots}, g_k) \in B(\Sigma_1) \times \overset{[i]}{\dots} \times B(\Sigma_k)$ there is a unique X^{**}-valued bounded ω^*-Radon measure $\gamma_{g_1, \overset{[i]}{\dots}, g_k}$ on K_i (i.e., a X^{**}-valued finitely additive bounded vector measure on the Borel subsets of K_i, such that for every $x^* \in X^*$, $x^* \circ \gamma_{g_1, \overset{[i]}{\dots}, g_k}$ is a Radon measure on K_k), verifying

$$\int g_i d\gamma_{g_1, \overset{[i]}{\dots}, g_k} = \overline{T}(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_k), \ \forall g_i \in B(\Sigma_i).$$

3. $\overline{T}$ is and $\omega^* - \omega^*$ sequentially continuous (i.e., if $(g_i^n)_{n \in \mathbb{N}} \subset B(\Sigma_i)$, for $i = 1, \dots k$, and $g_i^n \xrightarrow{\omega^*} g_i$, then $\lim_{n \to \infty} \overline{T}(g_1^n, \dots, g_k^n) = \overline{T}(g_1, \dots, g_k)$ in the $\sigma(X^{**}, X^*)$ topology.

Besides, if we define $\gamma : B(\Sigma_1) \times \dots \times B(\Sigma_k) \mapsto X^{**}$ as

$$\gamma(A_1, \dots, A_k) := \overline{T}(\chi_{A_1}, \dots \chi_{A_k}),$$

then γ is a polymeasure of bounded semivariation that verifies

(a) $\|T\| = \|\gamma\|$.

(b) $T(f_1, \dots, f_k) = \int (f_1, \dots, f_k) d\gamma \quad (f_i \in C(K_i))$

(c) For every $x^* \in X^*$, $x^* \circ \gamma$ is a regular (scalar) polymeasure and the map $x^* \mapsto x^* \circ \gamma$ is continuous for corresponding weak-$*$ topologies in X^* and $(C(K_1) \hat{\otimes} \cdots \hat{\otimes} C(K_k))^*$.

Conversely, if $\gamma : \Sigma_1 \times \dots \times \Sigma_k \mapsto X^{**}$ is a polymeasure which verifies (c), then it has finite semivariation and formula (b) defines a k-linear continuous operator from $C(K_1) \times \dots \times C(K_k)$ into X for which (a) holds.

Therefore the correspondence $T \leftrightarrow \gamma$ is an isometric isomorphism between $\mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ and the polymeasures in bsv-ω^*- $rcapm(\Sigma_1 \times \dots \times \Sigma_k; X^{**})$ that verify condition c).

Our aim now is to exploit both representation theories, measures and polymeasures, to study the multilinear operators on $C(K)$ spaces. In this paper we present some results in this direction.

2. The Main Results

The following definition can be found in [6] or in [14]

Definition 2.1. *A polymeasure $\gamma : \Sigma_1 \times \dots \times \Sigma_k \mapsto X$ is said to be uniform in the i^{th} variable if it is countably additive and the measures*

$$\left\{ \gamma(A_1, \dots, A_{i-1}, \cdot, A_{i+1}, \dots A_k) \in ca(\Sigma_i; X) : (A_1, \overset{[i]}{\dots}, A_k) \in \Sigma_1 \times \overset{[i]}{\cdots} \times \Sigma_k \right\}$$

are uniformly countably additive.

A polymeasure is said to be uniform if it is uniform in every variable.

It is easy to check that given a natural number r, $1 < r < k$ and r indexes $1 \leq j(1) < j(2) < \ldots < j(r) \leq k$, and given fixed $h_{j(p)} \in B(\Sigma_{j(p)})$, $p = 1 \ldots r$, we can construct the multilinear operator

$$T_{h_{j(1)},\ldots h_{j(r)}} : \prod_{\substack{1\leq q\leq k \\ q\notin(j(1)\ldots j(r))}} C(K_q) \mapsto X$$

defined as $T_{h_{j(1)},\ldots h_{j(r)}}(h_{q(1)},\ldots,h_{q(k-r)}) := \overline{T}(h_1,\ldots,h_k)$ whose associated polymeasure we will call $\gamma_{h_{j(1)},\ldots h_{j(r)}}$.

Given a bounded polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \mapsto X$ and a fixed number i, $1 \leq i \leq k$, we can construct in a natural way the measure $\phi_i : \Sigma_i \mapsto bpm(\Sigma_1, \overset{[i]}{\cdots}, \Sigma_k; X)$ defined as $\phi_i(A_i) := \gamma_{A_i}$. The fact that ϕ_i is bounded, indeed $\|\phi_i\| = \|\gamma\|$, and the following lemma are easy to check.

Lemma 2.2. *With the above notation, a countably additive polymeasure γ is uniform in the i^{th} variable if and only if ϕ_i is countably additive. The same is true if in this statement "countably additive" is replaced by "regular".*

Let $E_1,\ldots,E_k,X$ be Banach spaces. Each $T \in \mathcal{L}^k(E_1,\ldots,E_k;X)$ generates in a natural way k linear operators

$$T_i : E_i \mapsto \mathcal{L}^{k-1}(E_1, \overset{[i]}{\ldots}, E_k; X), \ \ i = 1,\ldots,k$$

defined as $T_i(x_i)(x_1, \overset{[i]}{\ldots}, x_k) := T(x_1,\ldots,x_k)$ for each $x_j \in E_j$, $j = 1,\ldots,k$.

We will state now a definition:

Definition 2.3. *A k-linear mapping $T \in \mathcal{L}^k(E_1,\ldots,E_k;X)$ is said to be* regular *if every mapping T_i above defined is weakly compact.*

When X is the scalar field, the above definition was given in [3]. In general, given an operator ideal $\mathcal{U}$, we can define the *$\mathcal{U}$-regular* k-linear mappings as those such that the corresponding T_i belong to $\mathcal{U}$ for every $1 \leq i \leq k$. When $\mathcal{U}$ is the ideal of compact operators, such mappings have been considered in [8], and for a general closed injective operator ideal $\mathcal{U}$ in [9]. In every case a non-linear version of the factorization theorem of Davies, Figiel, Johnson and Pelczynsky (see [5, pgs. 250, 259]) through operators in $\mathcal{U}$ is obtained for such multilinear mappings. These results are then applied to get some factorization theorems for holomorphic mappings.

We are ready now to prove the following characterization of the uniform polymeasures.

Theorem 2.4. *Let $K_1,\ldots,K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1),\ldots,C(K_k);X)$. Let $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \to X^{**}$ be the polymeasure associated to it according to theorem 1.1. Then γ is uniform if and only if T is regular. Besides, in that case the measures ϕ_i defined before lemma 2.2 are the measures canonically associated to the operators T_i.*

Proof. Let us first assume that γ is uniform (in particular this means that γ is regular countably additive and therefore X-valued, see [7]). According to lemma 2.2 this means that for each $i = 1,\ldots,k$, $\phi_i \in rca(\Sigma_i; rcapm(\Sigma_1, \overset{[i]}{\ldots} \Sigma_k; X))$. Since $rcapm(\Sigma_1, \overset{[i]}{\ldots} \Sigma_k; X) \subset \mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X)$ (*cfr.* theorem 1.1) we get that $\phi_i \in rca(\Sigma_i; \mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X))$. Then we can consider the operator $H_{\phi_i} \in \mathcal{L}(C(K_i); \mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X))$ associated to ϕ_i by the Riesz representation theorem (vector valued case; see, f.i. [5, Theorem VI.2.1]). Since ϕ_i is countably additive we know that H_{ϕ_i} is weakly compact ([5, Theorem VI.2.5]). We consider now $H_{\phi_i}^{**}$, the bitranspose of H_{ϕ_i}. Since H_{ϕ_i} is weakly compact we get that $H_{\phi_i}^{**}$ is $\mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X)$-valued. It is easy to see that for every $A_i \in \Sigma_i$, and for every $(f_1, \overset{[i]}{\ldots}, f_k) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k)$,

$$\begin{aligned} H_{\phi_i}^{**}(A_i)(f_1, \overset{[i]}{\ldots}, f_k) &= < \phi_i(\chi_{A_i}), (f_1, \overset{[i]}{\ldots}, f_k) >= \int (f_1, \overset{[i]}{\ldots}, f_k) d\gamma_{A_i} \\ &= \overline{T}(f_1, \ldots f_{i-1}, \chi_{A_i}, f_{i+1}, \ldots f_k). \end{aligned}$$

Therefore,

$$H_{\phi_i}^{**}(g_i)(f_1, \overset{[i]}{\ldots}, f_k) = \overline{T}(f_1, \ldots f_{i-1}, g_i, f_{i+1}, \ldots f_k),$$

for every Σ_i-simple function g_i and for every $(f_1, \overset{[i]}{\ldots}, f_k) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k)$. From continuity, we get the same relation for every $g_i \in B(\Sigma_i)$. In particular, when we choose $f_i \in C(K_i)$ we get

$$\begin{aligned} &H_{\phi_i}^{**}(f_i)(f_1, \overset{[i]}{\ldots}, f_k) = \overline{T}(f_1, \ldots f_{i-1}, f_i, f_{i+1}, \ldots f_k) \\ &= T(f_1, \ldots f_{i-1}, f_i, f_{i+1}, \ldots f_k) = T_i(f_i)(f_1, \overset{[i]}{\ldots}, f_k). \end{aligned}$$

Obviously this means that $T_i = H_{\phi_i}$ and, therefore, that T_i is weakly compact.

Let us now assume that T is regular. Then, for every $i = 1 \ldots k$, $T_i \in \mathcal{L}(C(K_i); \mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X)$ is weakly compact and so the measure μ_i associated to it by the Riesz representation theorem is countably additive and $\mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X)$-valued ([5, Theorem VI.2.5]). We will check now that for every $i = 1 \ldots k$, $\mu_i = \phi_i$. Then, the proof will be finished just by looking at lemma 2.2.

Let T_i^{**} be the bitranspose of T_i. For each $A_i \in \Sigma_i$ let $(f_i^\alpha)_{\alpha \in I}$ be a net in $C(K_i)$ such that $f_i^\alpha \overset{\omega^*}{\to} \chi_{A_i}$. T_i^{**} is known to be ω^*-ω^* continuous; being T_i weakly compact we get that T_i^{**} is $\mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X)$-valued. Both of these facts together imply that $(T_i^{**}(f_i^\alpha))_{\alpha \in I}$ converges weakly to $T_i^{**}(\chi_{A_i})$. For fixed $(f_1, \overset{[i]}{\ldots}, f_k) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k)$ and $x^* \in X^*$, the linear form

$$\theta : \mathcal{L}^{k-1}(C(K_1), \overset{[i]}{\ldots}, C(K_k); X) \mapsto \mathbb{K}$$

defined as $\theta(S) := \langle S(f_1, \overset{[i]}{\ldots}, f_k), x^* \rangle$ is clearly continuous and therefore

$$\theta(T_i^{**}(f_i^\alpha)) \to \theta(T_i^{**}(\chi_{A_i})) = \langle T_i^{**}(\chi_{A_i})(f_1, \overset{[i]}{\ldots}, f_k), x^* \rangle.$$

Besides,

$$\theta(T_i^{**}(f_i^\alpha)) = \langle T_i^{**}(f_i^\alpha)(f_1, \overset{[i]}{\ldots}, f_k), x^* \rangle = \langle T(f_1, \ldots f_{i-1}, f_i^\alpha, f_{i+1}, \ldots f_k), x^* \rangle.$$

Since $\overline{T}$ is separately ω^*-ω^* continuous we get that this last expression converges to $\langle \overline{T}(f_1, \ldots f_{i-1}, \chi_{A_i}, f_{i+1}, \ldots, f_k), x^* \rangle$. So we have obtained that for every $x^* \in X^*$, $\langle \overline{T}(f_1, \ldots f_{i-1}, \chi_{A_i}, f_{i+1}, \ldots, f_k), x^* \rangle = \langle T_i^{**}(\chi_{A_i})(f_1, \overset{[i]}{\ldots}, f_k), x^* \rangle$. Therefore for every $A_i \in \Sigma_i$ and for every $(f_1, \overset{[i]}{\ldots}, f_k) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k)$,

$$\overline{T}(f_1, \ldots f_{i-1}, \chi_{A_i}, f_{i+1}, \ldots, f_k) = T_i^{**}(\chi_{A_i})(f_1, \overset{[i]}{\ldots}, f_k) = \mu_i(A_i)(f_1, \overset{[i]}{\ldots}, f_k).$$

But clearly

$$\overline{T}(f_1,\dots f_{i-1},\chi_{A_i},f_{i+1},\dots,f_k)=\int(f_1,\overset{[i]}{\dots},f_k)d\gamma_{A_i}=\phi_i(A_i)(f_1,\overset{[i]}{\dots},f_k).$$

From here it follows that $\mu_i=\phi_i$ and the proof is over. □

Since every operator from $C(K_1)$ to $C(K_2)^*$ is weakly compact (*cfr.* [5, Theorem VI-2-15], f.i.), we get immediately the following result (see [6]):

Corollary 2.5. *Every regular countably additive scalar bimeasure $\gamma:\Sigma_1\times\Sigma_2\to\mathbb{K}$ is uniform.*

From the above theorem we can derive the following propositions, useful to decide whether a polymeasure is or is not uniform. Previously we will need a lemma.

Lemma 2.6. *Let $T:C(K_1)\times\cdots\times C(K_k)\mapsto X$ be a regular k- linear operator. Let $(f_i^n)_{n\in\mathbb{N}}\subset C(K_i)$ be a weakly null sequence and let $((g_1^n)_{n\in\mathbb{N}},\overset{[i]}{\dots},(g_k^n)_{n\in\mathbb{N}})\subset B(\Sigma_1)\times\overset{[i]}{\dots}\times B(\Sigma_k)$ be bounded sequences. Then, with the notation of theorem 1.1, $\overline{T}(g_1^n,\dots g_{i-1}^n,f_i^n,g_{i+1}^n,\dots,g_k^n)$ converges in norm to zero.*

Proof. If T is regular, then the above defined operator T_i is weakly compact and therefore completely continuous, by the Dunford-Pettis property of $C(K_i)$. This means that $\|T_i(f_i^n)\|\to 0$. We observe now that, due to the uniqueness of the extension (1.1), for every $(g_1,\overset{[i]}{\dots},g_k)\subset B(\Sigma_1)\times\overset{[i]}{\dots}\times B(\Sigma_k)$ and for every $f_i\in C(K_i)$, we have $\overline{T_i(f_i)}(g_1,\overset{[i]}{\dots},g_k)=\overline{T}(g_1,\dots g_{i-1},f_i,g_{i+1},\dots,g_k)$. By the equality of the norms of the operator and its extension, we can write $\|\overline{T_i(f_i^n)}\|\to 0$. This can also be written as

$$\sup_{g_j\in B_{B(\Sigma_j)}}\|\overline{T_i(f_i^n)}(g_1,\overset{[i]}{\dots},g_k)\|\to 0,$$

which means that

$$\sup_{g_j\in B_{B(\Sigma_j)}}\|\overline{T}(g_1,\dots g_{i-1},f_i^n,g_{i+1},\dots,g_k)\|\to 0$$

and finishes the proof. □

Proposition 2.7. *A regular countably additive polymeasure $\gamma:\Sigma_1\times\cdots\times\Sigma_k\mapsto X$ is uniform in the $\imath^{th}$ variable if and only if the measures*

$$\{\gamma_{g_1,\overset{[i]}{\dots},g_k}:(g_1,\overset{[i]}{\dots},g_k)\in B(\Sigma_1)\times\overset{[i]}{\cdots}\times B(\Sigma_k),\|g_j\|\le 1\}$$

are uniformly countably additive.

Proof. One of the implications is clear. For the other, let us suppose that γ is uniform in the $\imath^{th}$ variable. Were the measures $\{\gamma_{g_1,\overset{[i]}{\dots},g_k};(g_1,\overset{[i]}{\dots},g_k)\in B(\Sigma_1)\times\overset{[i]}{\cdots}\times B(\Sigma_k)\}$ not uniformly countably additive, then there would exist $\epsilon>0$, a sequence $(A_i^n)_{n\in\mathbb{N}}\subset\Sigma_i$ of disjoint open sets and sequences $((g_1^n)_{n\in\mathbb{N}},\overset{[i]}{\dots},(g_k^n)_{n\in\mathbb{N}})\subset B(\Sigma_1)\times\overset{[i]}{\dots}\times B(\Sigma_k)$ with $\|g_j^n\|\le 1$ for each $n\in\mathbb{N}$ and for each $j=1\overset{[i]}{\dots}k$, such that $\|\gamma_{g_1,\overset{[i]}{\dots},g_k}(A_i^n)\|>\epsilon$. Then for each $n\in\mathbb{N}$ there would exist $f_i^n\in C(K_i)$ with $\operatorname{supp}f_i^n\subset A_i^n$ and $\|f_i^n\|\le 1$ such that $\|\int f_i^n d\gamma_{g_1,\overset{[i]}{\dots},g_k}\|>\epsilon$, and this in contradiction with lemma 2.6, since the sequence f_i^n converges weakly to 0. □

Proposition 2.8. *A regular countably additive polymeasure* $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \mapsto X$ *is uniform in the* i^{th} *variable if and only if the measures*

$$\{\gamma_{f_1,\overset{[i]}{\ldots},f_k}; (f_1, \overset{[i]}{\ldots}, f_k) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k), \|f_j\| \leq 1\}$$

are uniformly countably additive.

Proof. In one direction the result follows from the previous proposition. For the other, we will suppose without loss of generality that $i = k$. Let us suppose that the measures $\{\gamma_{f_1,\ldots,f_{k-1}}; (f_1, \ldots, f_{k-1}) \in C(K_1) \times \cdots \times C(K_{k-1}), \|f_j\| \leq 1\}$ are uniformly countably additive. If γ is not uniform in the k^{th} variable then there exist a sequence $A_k^n \subset \Sigma_k$ of disjoint open sets and sequences $(A_j^n)_{n\in\mathbb{N}} \subset \Sigma_j$ for $j = 1 \ldots k-1$ such that $\|\gamma(A_1^n, \ldots, A_k^n,)\| > \epsilon$. Since γ is regular, $\gamma(\cdot, A_2^n, \ldots A_k^n)$ is regular for each $n \in \mathbb{N}$ and therefore there exists a function $f_1^n \in C(K_1)$ with $\|f_1^n\| \leq 1$ such that $\| \int f_1^n d\gamma_{A_2^n,\ldots,A_k^n}\| > \epsilon$. Now $\gamma_{f_1^n,\cdot,\chi_{A_3^n},\ldots,\chi_{A_k^n}}$ is also regular and therefore there exists a function $f_2^n \in C(K_2)$ with $\|f_2^n\| \leq 1$ such that $\| \int f_2^n d\gamma_{f_1^n,\chi_{A_3^n},\ldots,\chi_{A_k^n}}\| > \epsilon$. Continuing in the same way we obtain $k-1$ sequences of norm one functions $f_j^n \subset C(K_j)$, $j = 1 \ldots k-1$ such that $\|\gamma_{f_1^n,\ldots,f_{k-1}^n}(A_k^n)\| > \epsilon$ which contradicts the hypothesis. □

3. Polymeasures on compact dispersed spaces

Recall that a compact Hausdorff space is said to be *dispersed* if it does not contain any non empty perfect set. In [12] a deep insight is given into the structure of dispersed spaces, proving among other results that K is dispersed if and only if $C(K)$ contains no copy of ℓ_1, if and only if $C(K)^*$ contains no copy of L_1. Also, in this case $C(K)^*$ can be identified with $\ell_1(\Gamma)$ for some Γ.

Some (if not all) of the following results are probably known, but we have not been able to find an explicit reference.

Theorem 3.1. *For a compact Hausdorff space K, the following statements are equivalent:*

a) K is dispersed.

b_o) For every $k \geq 1$, the space $\mathcal{L}^k(C(K))$ is Schur.

b_1) For some $k \geq 2$, the space $\mathcal{L}^k(C(K))$ is Schur.

b_2) For some $k \geq 2$, the space $\mathcal{P}(^kC(K))$ is Schur.

b_3) For every $k \geq 2$, the space $\mathcal{P}(^kC(K))$ is Schur.

c_o) For every $k \geq 1$, the space $\mathcal{L}^k(C(K))$ is weakly sequentially complete.

c_1), c_2), c_3): Same statements as b_1), b_2), b_3), replacing Schur by weakly sequentially complete.

d_o) For every $k \geq 1$, $\mathcal{L}^k(C(K))$ contains no copy of ℓ_∞.

d_1), d_2), d_3): Same statements as b_1), b_2), b_3), replacing Schur by the non containment of ℓ_∞.

e) For every $k \geq 1$, $\mathcal{L}^k(C(K))$ contains no copy of c_o.

e_1), e_2), e_3): Same statements as d_1), d_2), d_3), replacing ℓ_∞ by c_o.

Proof. Since $\mathcal{L}^k(C(K))$ is a dual space for every $k \geq 1$, every (d) statement is equivalent to the corresponding (e) statement. Also clearly $b_i) \Rightarrow c_i) \Rightarrow d_i)$, for every i, $b_o) \Rightarrow b_1) \Rightarrow b_2)$ and $b_o) \Rightarrow b_3) \Rightarrow b_2)$. Therefore, it rests to prove a) $\Rightarrow$ b_o) and $e_2) \Rightarrow$ a).

a) $\Rightarrow$ b$_o$): We shall prove it by induction on k. For $k = 1$, it is clear since $C(K)^* \approx \ell_1(\Gamma)$. Suppose now that

$$\mathcal{L}^k(C(K)) = \left(\widehat{\bigotimes}_{\pi}^{k} C(K)\right)^* := X^*$$

(*cfr.* [5, Corollary VIII.2.2]) is Schur. Then

$$\mathcal{L}^{k+1}(C(K)) = \mathcal{L}(C(K); X^*) = \left(C(K)\hat{\otimes}_\pi X\right)^*.$$

Since $C(K)$ contains no copy of ℓ_1 and has the Dunford-Pettis property, by the induction hypothesis it follows that all members of the last space are compact operators. Hence, since $C(K)^*$ has the approximation property,

$$\mathcal{L}^{k+1}(C(K)) = C(K)^* \hat{\otimes}_\epsilon X^*$$

([5, Theorem VIII.3.6]), which is a Schur space, since this property is stable by taking injective tensor products (*cfr.* f.i. [13]).

e$_2$) $\Rightarrow$ a): If K is not dispersed, $C(K)^* \supset L_1 \supset \ell_2$. Consequently

$$\ell_2 \hat{\otimes}_\epsilon \ell_2 \subset C(K)^* \hat{\otimes}_\epsilon C(K)^* \subset \left(C(K)\hat{\otimes}_\pi C(K)\right)^*$$

(topological inclusions), and it is well known that if (e_n) is the canonical basis of ℓ_2, then $(e_n \otimes e_n)$ is equivalent to the canonical basis of c_o (*cfr.*, f.i., [10]). This means that $\mathcal{P}(^2C(K))$ contains a copy of c_o. Since $\mathcal{P}(^2C(K))$ is a (complemented) subspace of $\mathcal{P}(^kC(K))$, for every $k \geq 2$, it follows that the latter space contains a copy of c_o, too. $\square$

As we mention in corollary 2.5, every scalar regular bimeasure on a compact Hausdorff space is always uniform. This is not true for arbitrary polymeasures, as the following example from [2] shows: The 3-linear map $T : C([0,1]) \times C([0,1]) \times C([0,1]) \to \mathbb{C}$ defined by

$$T(f,g,h) := \sum_{i=1}^{\infty} f(\frac{1}{2^i}) \int_0^1 g r_i \, dx \int_0^1 h r_i \, dx,$$

where r_i is the standard ith Rademacher function, is not regular. See [2] for details.

In the next theorem we show that the uniformity of all the k-polymeasures for some (every) $k \geq 3$, characterizes the compact dispersed spaces. We shall denote by $\mathcal{K}(X;Y)$ and $\mathcal{W}(X;Y)$ the compact and weakly compact operators between X and Y, respectively.

Theorem 3.2. *For a compact Hausdorff space K the following statements are equivalent*

a) K is dispersed.

f) For every (some) $k \geq 2$, $\mathcal{L}\left(C(K); \mathcal{L}^k(C(K))\right) = \mathcal{K}\left(C(K); \mathcal{L}^k(C(K))\right)$.

g) For every (some) $k \geq 2$, $\mathcal{L}\left(C(K); \mathcal{L}^k(C(K))\right) = \mathcal{W}\left(C(K); \mathcal{L}^k(C(K))\right)$.

h) For every (some) $k \geq 3$, any scalar regular k-polymeasure on the product of the Borel σ-algebra of K, is uniform.

Proof. a) $\Rightarrow$ f) was included in the proof of a) $\Rightarrow$ b$_o$) in theorem 3.1, and clearly f) $\Rightarrow$ g). The equivalence of (g) and (h) follows from theorem 2.4. Finally, let us prove that (g) implies (a): Let $k \geq 3$. If K is not dispersed, $C(K)$ is infinite dimensional and thus contains a copy of c_o ([5, Corollary VI.2.16]). On the other hand, by theorem 3.1, $\mathcal{L}^{k-1}(C(K))$ contains a copy of ℓ_∞. By the injectivity of this space, the inclusion map from c_o into ℓ_∞ can be extended to the whole space $C(K)$, providing in this way a non weakly compact operator in $\mathcal{L}\left(C(K);\mathcal{L}^{k-1}(C(K))\right)$. $\square$

The equivalence of (a), (f) and (g) has been also obtained in [1], although with a different and, in our opinion, more involved proof.

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