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A new approach for measuring productivity gaps across groups: an application with historic footballers from FC Barcelona and Real Madrid

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ABSTRACT

The assessment of the productivity gap between two or more groups of production units that operate an equivalent production process in different settings is a common challenge in the field of production analysis. These groups need to determine whether they should keep operating in the same manner to sustain their better performance levels or decide which managerial practices should be altered to close the gap with respect to best practices in other groups. The aim of this paper is to propose a new tool for measuring the productivity gap across different groups based on the use of a synthetic reference group uniformly distributed within a unit hypercube. Besides, as the productivity gap can be decomposed into two components, efficiency gap and global technical gap, this method fulfills the circular test and is independent of a base-reference technology. Thus, it outperforms other traditional approaches. To illustrate the usefulness of this methodology, we analyze the productivity gap between Spanish football's two biggest rival clubs: FC Barcelona and Real Madrid. We use data on the performance of the most important players in the history of both clubs, divided according to their position on the pitch. The period of analysis covers the last 70 years, from the beginning of European competitions to the present day, i. e., from the 1954/55 to the 2023/24 season. Results show that, to date, Real Madrid's goalkeepers and midfielders were more productive than those playing for FC Barcelona, there is a tie for defenders, while the productivity of FC Barcelona's forwards was higher than those playing for Real Madrid.

1. Introduction

Research on the measurement of efficiency and productivity focuses on studying the performance of a set of decision-making units (DMUs) considering their ability to manage the inputs under their control to produce the maximum output feasible. In this framework, an issue that arises in many empirical studies is how to evaluate productivity and/or efficiency gaps between DMUs operating under different programs or belonging to different groups. These groups can be the result of many different factors such as ownership, regulation, resource endowments, economic infrastructures, geographical locations or even the comparison of treated with control group units in the context of a randomized experiment [53].

In the above settings, the most frequent method used in empirical studies is the meta-frontier approach [12,13,48]. This method is based on group analysis introduced in Charnes et al.'s seminal work [20], which proposed a separate evaluation of the units belonging to different groups or programs, followed by another evaluation considering all the units together assuming that the linear combination of DMUs belonging to different groups is feasible. This approach has some similarities with the global Malmquist productivity index introduced by Pastor and Lovell [49], which also relies on a single reference technology and has led to multiple extensions and applications[1]. However, if the groups to which the evaluated units belong have distinctive inherent characteristics, it may make more sense from an economic point of view to use a feasible real reference rather than a benchmark constructed from a mix of the technologies characterizing the different evaluated groups [4]. This is consistent with Epure et al. [28] who argue that the reason for not combining technologies when performing benchmarking analysis is that the benchmark should be well determined and easy to identify.

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[1] Some examples can be found in Afsharian and Ahn [1], Afsharian and Podinovski [2], Camanho et al. [17] or Walheer [65].

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In this vein, an attractive alternative approach is the one proposed by Camanho and Dyson [16], who introduced a Malmquist-type index (hereinafter CDMI) to provide an average indicator of the relative productivity gap of two or more groups of DMUs without mixing technologies. This index facilitates the comparison between the group technologies that researchers are interested in evaluating and also allows the performance gap to be decomposed into two components: the efficiency gap and the technical gap. Although this method has been successfully applied in multiple empirical studies to provide performance comparisons of groups of units operating in different sectors (see, for example, [32,44,62,64]) the CDMI does not satisfy the circularity test [33]. Subsequently, Aparicio and Santín [6] developed an alternative version of this method based on the base-period Malmquist index developed by Berg et al. [14]. This approach defines a 'reference technology', which facilitates the interpretation of productivity gaps and its components, allows for comparisons over time and satisfies the circularity property. However, these authors note that the selection of a reference technology remains arbitrary, and, therefore, the results may be substantially different depending on the selected reference [4].

In this paper, we propose a solution to address both problems, which involves the use of a common reference group composed of a large set of uniformly distributed synthetic DMUs for making group comparisons. To do this, we rely on recent developments by Aparicio and Santín [7,8], who propose the use of virtual units generated at random within a unit hypercube to derive a new decomposition of the traditional Malmquist productivity index. This approach can be adapted to the specific context of the analysis of units belonging to different groups or programs to derive a decomposition of the productivity gap into an efficiency gap and a global technical gap. The proposed method has several key advantages over other alternative approaches. First, the calculated productivity gap and its components are simultaneously circular and independent of arbitrary reference technologies. Second, the average technical gaps among groups are calculated globally, encompassing the frontiers of the different evaluated groups or programs. Third, the dimensions of the reference group can be easily adapted to compare groups in any empirical problem.

To illustrate the usefulness of this novel method, we assess the performance of footballers who have played for the two clubs with probably the greatest historical rivalry in European football: FC Barcelona and Real Madrid. These two Spanish clubs have a very particular idiosyncrasy that makes them clearly different from each other in terms of their playing style, the way in which they manage their signing policy, their approach to talent management and even with respect to political and ideological issues [51,57]. Therefore, when assessing footballers (DMUs) who have played for each club, we consider that productivity evaluation should be performed by keeping the two technologies separate.

The proposed empirical analysis is framed within a line of research that has grown over the last two decades. This consists of the use of frontier techniques to benchmark the performance of players or teams (clubs or franchises) in professional sports [15,22]. This literature was originally developed in the American setting and, therefore, mainly focuses on their major leagues (NBA, NFL, MLB and NHL) (e.g., [26,39,41,46,54,59,60]). However, quite a few empirical studies using this approach can also be found in the context of European football ([21,23,29,34,36–38,68])[2].

Most of these studies focus on analyzing the performance of professional teams, while studies evaluating player performance are less common. Tiedemann et al. [63] use a meta-frontier approach to assess the performance of football players in the German *Bundesliga* divided into three groups according to their position on the field (defenders, midfielders and forwards) covering the 2002/03 to 2008/09 seasons. Similarly, Santín [58] also evaluates the performance of the best players in Real Madrid's history distinguishing between four blocks of players according to their position on the field (goalkeepers, defenders, midfielders and forwards). In this case, players are assessed separately using a super efficiency data envelopment analysis (DEA) model. Qaiser et al. [56] also assess the performance of individual forwards and midfielders playing in the English Premier League during the 2020/21 season, ranking them according to the efficiency scores estimated using different DEA approaches. Finally, Ivanović et al. [42] estimate a novel measure of the performance of football players participating in the 2021/22 UEFA Champions League by incorporating the team effect into the formulation of the original DEA mathematical model while calculating efficiency scores[3]. Thus, to the best of our knowledge, no previous study has focused on analyzing the comparative performance of historic players from two major clubs by field position.

The rest of the paper is structured as follows. Section 2 reviews some basic ideas of previous approaches addressing the analysis of the performance gap between groups or programs. Section 3 introduces the methodological advances recently developed to decompose the Malmquist productivity index using virtual units within a hypercube and explains how they can be adapted to measuring productivity gaps between different groups. Moreover, the new methodology is illustrated with a simple numerical example. Section 4 describes the main characteristics of data and variables used in the empirical analysis. Section 5 reports the main results, and, finally, Section 6 outlines some concluding remarks.

2. Background

This section provides a quick overview of earlier Malmquist index-based methods that can be applied to group comparisons. It also introduces the basic notation that will be used in the explanations of the different methods that are presented in this paper.

2.1. Camanho-Dyson Malmquist index (CDMI) approach

Let us assume that we have observed a first group of N DMUs ($j = 1, \ldots, N$) belonging to group A and a second group of M DMUs ($i = 1, \ldots, M$) in group B. DMUs in both groups use a vector of K inputs $x^A, x^B \in R^K_+$ to produce the same vector of Q outputs $y^A, y^B \in R^Q_+$ in the same time period t.

The technology used by DMUs in group A to produce y^A from x^A is defined as $T^A = \{(x^A, y^A) \in R^{K+Q}_+ : x^A \, can produce \, y^A\}$. For a given DMU j in group A denoted as (x^A_j, y^A_j), $D^A(x^A_j, y^A_j) = \min_\theta\{\theta > 0 : (x^A_j, y^A_j/\theta) \in T^A\}$ should be defined in this setting as the Shephard output distance function, calculated from observation j in group A to the frontier technology of DMUs within group A denoted as T^A. Likewise, the production technology of DMUs in group B is defined as $T^B = \{(x^B, y^B) \in R^{K+Q}_+ : x^B \, can produce \, y^B\}$. The output distance function assumes that inputs are given and proportionally expands the output vector as far as possible to reach the production frontier. In this setting, DEA is a well-known tool for estimating T^A and T^B technologies. Particularly, under constant returns-to-scale (CRS) technology [19], T^A_c is estimated using DEA as

[2] Kulikova and Goshunova [45] provide a comprehensive review of parametric and nonparametric methods that can be applied to measure efficiency in this framework.

[3] In this study, the empirical analysis is first performed jointly for all players regardless of their position on the field and then separately by position (goalkeepers are not included).

$$T_c^A = \left\{ \begin{array}{l} (x^A, y^A) \in R_+^{K+Q} : y_r^A \leq \sum_{j=1}^{N} \lambda_j y_{jr}^A, \forall r = 1, ..., Q, \\ x_s^A \geq \sum_{j=1}^{N} \lambda_j x_{js}^A, \forall s = 1, ..., K, \lambda_j \geq 0, \forall j = 1, ..., N \end{array} \right\} \tag{1}$$

In DEA, any Shephard output distance function for a DMU *o* in group A can be estimated as the inverse of the optimal value of a linear programming model for θ_o. For example, $D_c^A(x_o^A, y_o^A)$ is calculated as follows:

$$CDMI^{AB} = \underbrace{\frac{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{1/N}}{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{1/M}}}_{Efficiency\ \ Gap\ \ EG^{AB}} \cdot \underbrace{\left[\frac{\left(\prod_{j=1}^{N} D_c^B\left(x_j^A, y_j^A\right)\right)^{1/N}}{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{1/N}} \cdot \frac{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{1/M}}{\left(\prod_{i=1}^{M} D_c^A\left(x_i^B, y_i^B\right)\right)^{1/M}}\right]^{1/2}}_{Technical\ \ Gap\ \ TG^{AB}} \tag{5}$$

$$\begin{array}{ll} \left(D_c^A\left(x_o^A, y_o^A\right)\right)^{-1} = \max_{\lambda, \theta_o} \theta_o & \\ s.t. \quad \sum_{j=1}^{N} \lambda_j x_{js}^A \leq x_{os}^A, & s = 1, \ldots, K \\ \quad \sum_{j=1}^{N} \lambda_j y_{jr}^A \geq \theta_o y_{or}^A, & r = 1, ..., Q \\ \lambda_j \geq 0, & j = 1, \ldots, N \end{array} \tag{2}$$

Therefore, $D_c^B\left(x_j^A, y_j^A\right)$ represents the Shephard output distance function from DMU *j* in group A with respect to the CRS production frontier defined by the *M* DMUs belonging to group *B*. Within this framework, the adjacent Malmquist productivity index is the most popular approach for evaluating the productivity change of a set of DMUs over time when market prices are not available [18,30]. The output-oriented Malmquist productivity index for a production unit operating in two periods *t* and *t+1* is defined using output distance functions as:

$$MI_o^{t,t+1} = \left[\frac{D_o^t\left(x_o^{t+1}, y_o^{t+1}\right)}{D_o^t\left(x_o^t, y_o^t\right)} \cdot \frac{D_o^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_o^{t+1}\left(x_o^t, y_o^t\right)}\right]^{1/2} \tag{3}$$

Equation (3) was decomposed by Färe et al. [30] into an efficiency change and a technical change as:

$$MI_o^{t,t+1} = EC_o^{t,t+1} \cdot TC_o^{t,t+1} = \underbrace{\frac{D_o^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_o^t\left(x_o^t, y_o^t\right)}}_{Efficiency\ \ Change} \cdot \underbrace{\left[\frac{D_o^t\left(x_o^t, y_o^t\right)}{D_o^{t+1}\left(x_o^t, y_o^t\right)} \cdot \frac{D_o^t\left(x_o^{t+1}, y_o^{t+1}\right)}{D_o^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}\right]^{1/2}}_{Technical\ \ Change} \tag{4}$$

Note that a $MI_o^{t,t+1} > 1$ implies a total factor productivity gain from *t* to *t+1,* while a $MI_o^{t,t+1} < 1$ is interpreted as a technical regress. The first ratio in (4), denoted as $EC_o^{t,t+1}$, is the efficiency change that captures productivity changes due to the efficiency component. When $EC_o^{t,t+1} > 1$ ($EC_o^{t,t+1} < 1$), the technical efficiency of DMU *o* in period *t+1* was higher (lower) than in period *t*. The second component (in square brackets) represents the technical change $TC_o^{t,t+1}$ or frontier shift in period *t+1* with respect to period *t* and is interpreted in the same way.

Camanho and Dyson [16] had the idea of adapting the decomposition of the adjacent Malmquist index by substituting the time periods *t* and *t+1* in (5) for the geometric average performance of the DMUs belonging to the two groups to be compared, say *A* and *B,* leading to a new Camanho-Dyson Malmquist index (CDMI). The CDMI for measuring the productivity gap and its decomposition is defined as

The first ratio EG^{AB} outside square brackets in Eq. (5) measures the efficiency gap as the geometric average of technical efficiencies of units belonging to group A with respect to their own technology divided by the geometric average distance of DMUs from group B also measured relative to their own frontier. Therefore, an$EG^{AB} > 1$ ($EG^{AB} < 1$) indicates that the average technical efficiency of DMUs in group A is higher (lower) than DMUs in group B. The term TG^{AB}, inside the square brackets, evaluates the technical gap between the frontiers of the two analyzed groups. When $TG^{AB} > 1$, the production technology in group A dominates the frontier in group B and vice versa when $TG^{AB} < 1$. Camanho and Dyson [[16], p.36] claim that the main advantage of their approach is that: "*in contrast with earlier methods, our method makes comparisons relative to groups-specific frontiers only, without pooling the DMUs together to form a common frontier*".

Within the axiomatic test approach (see, for example, [33] or [3]), a desirable property for any productivity index used to carry out multilateral spatial comparisons is that it can be chained (circularity). In this context, this means that a productivity gap index (*I*) computed for three groups of production units, *A, B* and *C,* passes the circular test when ${}^{A}I^{B} \cdot {}^{B}I^{C} = {}^{A}I^{C}$. The main limitation of the CDMI is that, since it is based on the adjacent Malmquist index, it also inherits its properties. Particularly, the CDMI does not satisfy the circularity test for more than two groups because of the technical gap component [3]. This means that for three groups of DMUs —A, B and C—, $CDMI^{AC} \neq CDMI^{AB} \cdot CDMI^{BC}$ because $TG^{AC} \neq TG^{AB} \cdot TG^{BC}$.

2.2. Aparicio-Santín Malmquist index (ASMI) approach

To fulfil the circularity test, Aparicio and Santín [6] proposed a different approach for measuring the productivity gaps of two or more groups. Rather than using the adjacent Malmquist index as Camanho and Dyson [16] did, they adapted the base-period Malmquist index introduced by Berg et al. [14]. Instead of using one of the periods as the reference technology, Aparicio and Santín [6] adopted the technology of a reference group as the base technology. Therefore, the Aparicio and Santín Malmquist index (ASMI) for comparing the performance of DMUs from groups A and B in a single time period with respect to the reference technology R, calculates T_c^R by re-writing Berg et al.'s Eq. 5 [14]. To do this, the reference technology T_c^R is obtained using DEA from the technology of the reference group R composed of P DMUs ($p = 1,\dots,P$). Therefore, $ASMI^{AB}(R)$ is defined as

$$\mathrm{ASMI}^{\mathrm{AB}}(R) = \frac{\left(\prod_{j=1}^{N} D_c^R\left(x_j^A, y_j^A\right)\right)^{1/N}}{\left(\prod_{i=1}^{M} D_c^R\left(x_i^B, y_i^B\right)\right)^{1/M}}. \tag{6}$$

Note that the Shephard output distance function for a given DMU o in group A with respect to the reference technology $D_c^R(x_o^A, y_o^A)$ is calculated as follows:

$$\begin{aligned} \left(D_c^R\left(x_o^A, y_o^A\right)\right)^{-1} &= \max_{\lambda,\theta_o} \theta_o \\ s.t. \quad & \sum_{p=1}^{P} \lambda_p x_{ps}^P \le x_{os}^A, \quad s = 1,\dots,K \\ & \sum_{p=1}^{P} \lambda_p y_{pr}^P \ge \theta_o y_{or}^A, \quad r = 1,\dots,Q \\ & \lambda_p \ge 0, \quad p = 1,\dots,P \end{aligned} \tag{7}$$

Furthermore, Eq. (6) can be decomposed into an efficiency gap and a technical gap as follows:

$$\mathrm{ASMI}^{\mathrm{AB}}(R) = \underbrace{\frac{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{1/N}}{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{1/M}}}_{Efficiency\ Gap\ EG^{AB}} \cdot \underbrace{\left[\frac{\left(\prod_{j=1}^{N} D_c^R\left(x_j^A, y_j^A\right)\right)^{1/N}}{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{1/N}} \cdot \frac{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{1/M}}{\left(\prod_{i=1}^{M} D_c^R\left(x_i^B, y_i^B\right)\right)^{1/M}}\right]}_{Technical\ Gap\ TG^{AB}(R)} \tag{8}$$

The efficiency gap EG^{AB} in Eq. (8) is equal to the term in Eq. (5), while the ratio $TG^{AB}(R)$ evaluates the technical gap between the frontiers of the two analyzed groups, A and B, measured on the base reference technology (R). Values of $TG^{AB}(R)$ that are greater (less) than one mean that group A is more (less) productive than group B or, in other words, can produce more (less) output with the same input levels. Interestingly, this approach can be extended over different periods, satisfying the circularity property and allowing multilateral comparisons over time [5]. Concerning this approach and the reference choice, Camanho et al. [17] claim that when there is no specific reason to select a particular reference group in comparisons involving several groups, the metafrontier technology can be used as the reference technology in this setting.

It is worth noting that this approach also inherits the properties of Berg et al.'s index [14], and this means that the final productivity gap results among the groups are conditioned to the arbitrary choice of the reference group R. This constitutes a major limitation, as practitioners measuring productivity gaps nowadays face a trade-off between circularity and reference technology independence. The use of the ASMI rather than the CDMI has the advantage of satisfying the circularity test, but the downside of being reference dependent. On the other hand, the CDMI satisfies reference technology independence, but the circularity property will not hold.

3. Novel methodological developments

In this section, we present a new index for measuring productivity gaps inspired by the concept of unit-hypercube introduced by Aparicio and Santín [7,8] to globally characterize the technology of a multi-input multi-output production setting. In geometry, a cube is a three-dimensional solid object bounded by six equal squares, with right angles between pairs of intersecting faces and pairs of intersecting edges. A hypercube is the generalization of the idea of a cube (n=3) in an n-dimensional space (see [24], for a more detailed description). The unit- hypercube is a hypercube whose line segments between two connected vertices have a length of one unit.

The second building block in developing the new index is the unit-invariance property of efficiency measures calculated using a DEA radial model [47]. This property states that the radial component of the efficiency measure in (2) remains unchanged when any input or output is multiplied by a positive scalar. In practice, defining a unit hypercube in conjunction with the unit-invariance property of DEA radial models allows us to embed a set of production technologies that use K inputs$x = (x_1, \dots, x_k) \in R_+^K$ to produce Q outputs $y = \left(y_1, \dots, y_q\right) \in R_+^Q$, within a unit-hypercube of dimension H, where $H = K + Q$. This can be done by dividing the input and output values by the maximum observed value of each dimension in all DMUs under analysis. As a result, the input and output values of all DMUs in all groups are scaled to lie between zero and one $x, y \in [0, 1]$.

Bearing in mind these ideas, we can artificially generate a synthetic 'reference group' as a set of L synthetic DMUs, $l = 1, \dots, L$, uniformly distributed within the unit-hypercube H. In this framework, the number of dimensions $H = K + Q$ of the 'reference group' adapts to the empirical problem to be evaluated. Thus, x_{ls} denotes the amount of the s-th input, $s = 1, \dots, K$, used by the l-th synthetic observation, and y_{lr} denotes the quantity of the r-th output, $r = 1, \dots, Q$, generated for the l-th unit. In vector notation, we use (x_l, y_l) to denote the input-output bundle corresponding to the l-th virtual unit. Therefore, the 'reference group' is composed of L synthetic DMUs uniformly distributed within the unit-hypercube of dimension H and is denoted as $\Delta = \left\{\left(x_l^H, y_l^H\right)\right\}_{L=1}^{L^{K+Q}}$.

To better understand how the reference group Δ is distributed, Fig. 1 illustrates a set of synthetic DMUs that, for the sake of simplicity, transform a single input (x) equal to one unit for all DMUs into two outputs (y_2 and y_1), where $y_2, y_1 \in (0,1)$. In this example, the reference group Δ is composed of an arbitrarily large number of synthetic DMUs, say L=1,000,000, uniformly distributed within the unit-hypercube of dimension H=2, which is a square with a line segment equal to one in this two-dimensional example.

Using this setting, Aparicio and Santín [7] observed that previous research on productivity measurement (see [9,10]; and [55]) often misinterprets the technical change of an industry when assessing productivity change over time. Specifically, the geometric average of individual technological changes between two periods, calculated using firm-level data, is frequently misinterpreted. This problem arises because the shift of the frontier over time is a global phenomenon where the technical progress or regress should be evaluated considering the whole surface of both production frontiers. In this framework, Aparicio and Santín [7] proposed using L synthetic DMUs uniformly distributed within a unit hypercube of dimension H to decompose the technical change in (4) into global and local technical change as follows

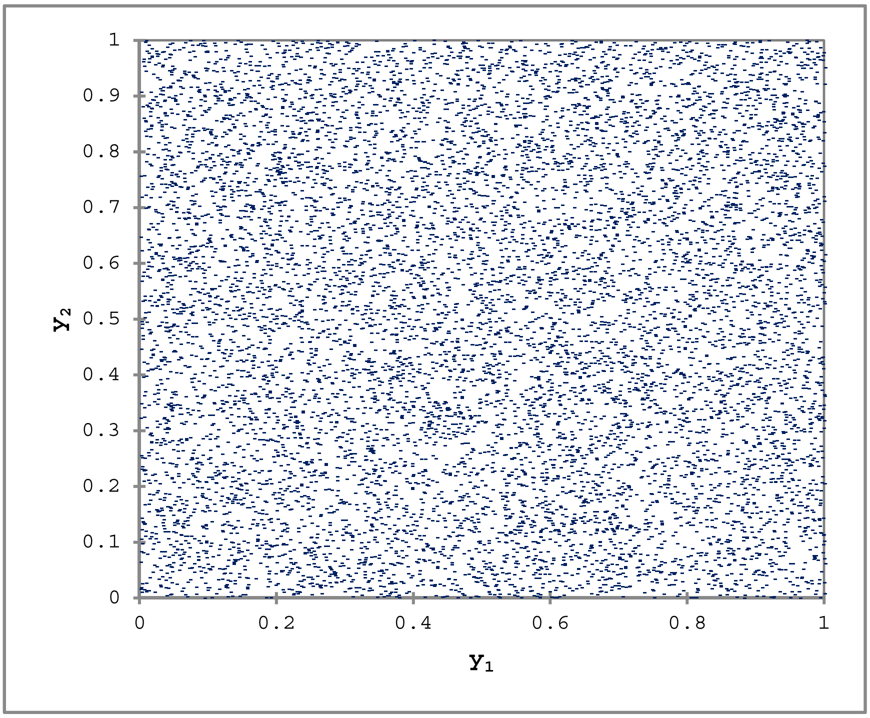


Fig. 1. The reference group Δ composed of 1,000,000 synthetic DMUs uniformly distributed within a two-dimensional unit-hypercube.

$$MI_o^{t,t+1} = EC_o^{t,t+1} \cdot GTC_H^{t,t+1} \cdot LTC_H^{t,t+1} = EC_o^{t,t+1} \cdot \underbrace{\left(\prod_{l=1}^{L} \frac{D_o^t\left(x_l^H, y_l^H\right)}{D_o^{t+1}\left(x_l^H, y_l^H\right)} \right)^{\frac{1}{L}}}_{Global\ Technical\ Change\ GTC_H^{t,t+1}} \cdot \underbrace{\frac{TC_o^{t,t+1}}{GTC_H^{t,t+1}}}_{LTC_H^{t,t+1}} \tag{9}$$

$$PGI^{AB}(H) = \underbrace{\frac{\dfrac{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{\frac{1}{n}}}{\left(\prod_{l=1}^{L} D_c^A\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}{\dfrac{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{\frac{1}{m}}}{\left(\prod_{l=1}^{L} D_c^B\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}}_{PGI^{AB}(H):\ Productivity\ Gap} = \underbrace{\frac{\left(\prod_{j=1}^{N} D_c^A\left(x_j^A, y_j^A\right)\right)^{\frac{1}{n}}}{\left(\prod_{i=1}^{M} D_c^B\left(x_i^B, y_i^B\right)\right)^{\frac{1}{m}}}}_{EG^{AB}:\ Efficiency\ Gap} \cdot \underbrace{\frac{\left(\prod_{l=1}^{L} D_c^B\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^A\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}_{GTG^{AB}(H):\ Global\ Technical\ Gap} \tag{10}$$

The component $GTC_H^{t,t+1}$ averages the technical changes between t and $t+1$ of the L synthetic units. A $GTC_H^{t,t+1} > 1$ indicates that the production technology of this industry at $t+1$ experienced a technical progress in global terms relative to the production frontier at t. The $LTC_o^{t,t+1}$ component is the measurement of the relative position of the evaluated DMU with respect to the synthetic average global technical change and is interpreted as the technical change experienced by each DMU once the $GTC_H^{t,t+1}$, common for the entire industry, is discounted.

3.1. *Measuring productivity gaps using the reference group*

Assume now that we want to compare the performance of a set of groups $(A, B, \ldots, Z)$ each consisting of different DMUs that use K inputs to produce Q outputs. Taking advantage of the unit-invariant property, the next step is to locate all DMUs from all groups within a unit-hypercube of dimension H, where $H = K + Q$ by normalizing each input and output value with respect to the maximum observed value in its corresponding dimension across all groups under analysis. In this way, the transformed input and output values for all groups are naturally constrained to lie between zero and one.

Previous approaches in the literature aimed at achieving circularity without resorting to a meta-frontier have employed an arbitrary 'reference technology' [3,5,6,14,17] or an arbitrary 'reference unit' [28,31,66,67]. To address the trade-off between circularity and independence from a reference technology, we propose in this work the use of a large number of synthetic DMUs uniformly distributed within the unit hypercube H as a common "reference group" for groups comparisons. This approach offers two important advantages over existing methods. First, since the reference group remains the same across all groups being compared, it ensures circularity without relying on an arbitrary 'reference technology'. Second, the reference group can be easily generated and adapted to any number of dimensions $H = K + Q$, making it suitable for analyzing a wide range of empirical problems.

Let us assume that $\Delta = \{(x_l^H, y_l^H)\}_{L=1}^{L^{K+Q}}$ is the set of L synthetic production units uniformly generated and distributed inside a unit hypercube H. Therefore, our proposal for measuring the productivity gap of two groups is

Our productivity gap index, $PGI^{AB}(H)$, measures the average performance of each group (A and B) relative to the average performance of the 'reference group', evaluated with respect to their respective production technologies. The numerator compares the average performance of DMUs in group A with respect to the average performance of DMUs in the reference group given the production technology of group A. Similarly, the denominator compares the performance of DMUs in group B with respect to the average performance of DMUs in the reference group given the production technology of B. Therefore, when $PGI^{AB}(H) > 1$, on average, DMUs in group A are, on average, more productive that their counterparts in group B. Conversely, when $PGI^{AB}(H) < 1$ group B outperforms group A on average.

The index is naturally decomposed in an efficiency gap and a global technical gap[4]. Note that the efficiency gap in Eq. (10) is equal to the terms in Eqs. (5) and (8). The global technical gap component $GTG^{AB}(H)$ captures the overall frontier gap between the two groups. When $GTG^{AB}(H) > 1$, on average, the technology in group A dominates the production frontier in group B and vice versa when $GTG^{AB}(H) < 1$. In the same vein as Aparicio and Santín [7], this method of measuring the technical gap avoids any potential bias there might be when DMUs are concentrated in certain regions of, rather than uniformly distributed over, the production set. In sum, $GTG^{AB}(H)$ averages the global frontier gap between A and B evaluated for every synthetic DMU in Δ, projected against the technologies of groups A and B.

The use of this approach has multiple advantages. Firstly, as we mentioned, the reference group can be easily adapted from one empirical problem to another varying the number of dimensions H of the unit hypercube. Secondly, the use of a synthetic reference group avoids having to resort to arbitrary units or technologies, meaning that the choice of the reference group will always be the same in all empirical problems. Thirdly, the average technical gaps among groups, understood as a global phenomenon affecting all the frontiers of the evaluated groups and not as the geometric average of technical gaps observed in the empirical sample, are calculated globally. Finally, the new productivity gap index and its components pass the circular test and are independent of *ad hoc* reference technologies for any empirical problem. This makes it possible to perform robust multilateral comparisons for three or more groups. The new index is circular because its two components are also circular. Camanho and Dyson [[16]. p. 40] demonstrate that the efficiency gap component in (11) satisfies the circularity test. The $GTG^{AB}(H)$ component also meets the circularity property. It is easy to check that for three groups A, B and C, the equality, $GTG^{AC}(H) = GTG^{AB}(H) \cdot GTG^{BC}(H)$ can be verified as follows

$$\underbrace{\frac{\left(\prod_{l=1}^{L} D_c^C(x_l^H, y_l^H)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^A(x_l^H, y_l^H)\right)^{\frac{1}{L}}}}_{GTG^{AC}(H):\ Global\ Technical\ Gap} = \underbrace{\frac{\left(\prod_{l=1}^{L} D_c^B(x_l^H, y_l^H)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^A(x_l^H, y_l^H)\right)^{\frac{1}{L}}}}_{GTG^{AB}(H):\ Global\ Technical\ Gap} \cdot \underbrace{\frac{\left(\prod_{l=1}^{L} D_c^C(x_l^H, y_l^H)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^B(x_l^H, y_l^H)\right)^{\frac{1}{L}}}}_{GTG^{BC}(H):\ Global\ Technical\ Gap} \quad (11)$$

This indicates that the global technical gap index between two groups, A and C, can be obtained by chaining both groups (A and C) to any other group or groups (e.g., group B), using as reference the same set of synthetic units $\Delta = \{(x_l^H, y_l^H)\}_{L=1}^{L^{K+Q}}$ uniformly distributed within a unit hypercube of dimension H.

3.2. A numerical example

This section illustrates the new index proposed in this paper by adapting the numerical example introduced by Camanho et al., [[17], p. 1193][5]. Suppose we have three groups —A, B, and C—each consisting of three DMUs that employ three distinct CRS technologies to produce two outputsy $= (y_1, y_2)$ using one equal input (x). Table 1 reports the raw data, as well as the output data normalized by the maximum observed values to scale the variables within the [0, 1] interval[6], and the distances associated to the efficiency evaluation by comparison with respect to the three production frontiers T_c^A, T_c^B and T_c^C and the metafrontier $T_c^M = T_c^A \cup T_c^B \cup T_c^C$.

In Table 1, the efficiencies computed considering the production frontier of the own group, $D_c^A(x_a, y_a)$, $D_c^B(x_b, y_b)$ and $D_c^C(x_c, y_c)$ reveal that DMUs 1 (a1, b1 and c1) and 3 (A3, B3 and C3) are fully efficient within each group, while DMUs 2 (a2, b2 and c2) are inefficient. Moreover, the distances with respect to the metafrontier $D_c^M(x, y)$ point out that the metafrontier is built on the performance of DMUs b3 and c1.

For comparison purposes, now we modify the Camanho's et al. [17] data reported in Table 1 (hereafter Example 1) to construct a second example. Specifically, we slightly adjust one of the two outputs of the three inefficient DMUs (a2, b2 and c2) to the following values $y_{a2} = (3, 5.9)$; $y_{b2} = (3.4, 4.8)$ and $y_{c2} = (7.1, 10)$, while keeping the remaining DMUs at their original values. Table 2 extends Table 1 by presenting the updated data and efficiency scores corresponding to Example 2 for these DMUs.

Fig. 2 depicts the performance of the DMUs and shows the production frontiers for the two examples using the normalized data. First, note that the production frontiers of the three groups and the metafrontier coincide in both examples. This implies that the theoretical technical gaps among the groups should be the same in the two examples. Second, DMUs a2, b2 and c2 are inefficient in both examples. However, in Example 2 DMUs a2 and b2 move leftwards while c2 moves upwards with respect to their counterparts in Example 1. Third, the two graphs in Fig. 2 represent the production frontiers within the unit hypercube illustrated in Fig. 1, which in this case corresponds to a square with a line segment equal to one. Furthermore, by overlapping Fig. 1 within each of the two graphs in Fig. 2 we can also represent the reference group of L synthetic units to measure the new PGI for the three groups.

The next step consists of applying Eqs. 5, (8), and (10) to compute

[4] Note that this decomposition does not include the LTC in Equation (9). The LTC component is an individual-specific term that is meaningful when assessing how a single production unit, observed over two periods, experiences the average GTC. In the context of group comparisons, however, we assess the average performance of the units directly using geometric means of average efficiencies and technological positions and, for this reason, the LTC component, being an individual-specific term, becomes irrelevant.

[5] The R code used to replicate the results of this section is provided in the Annex.

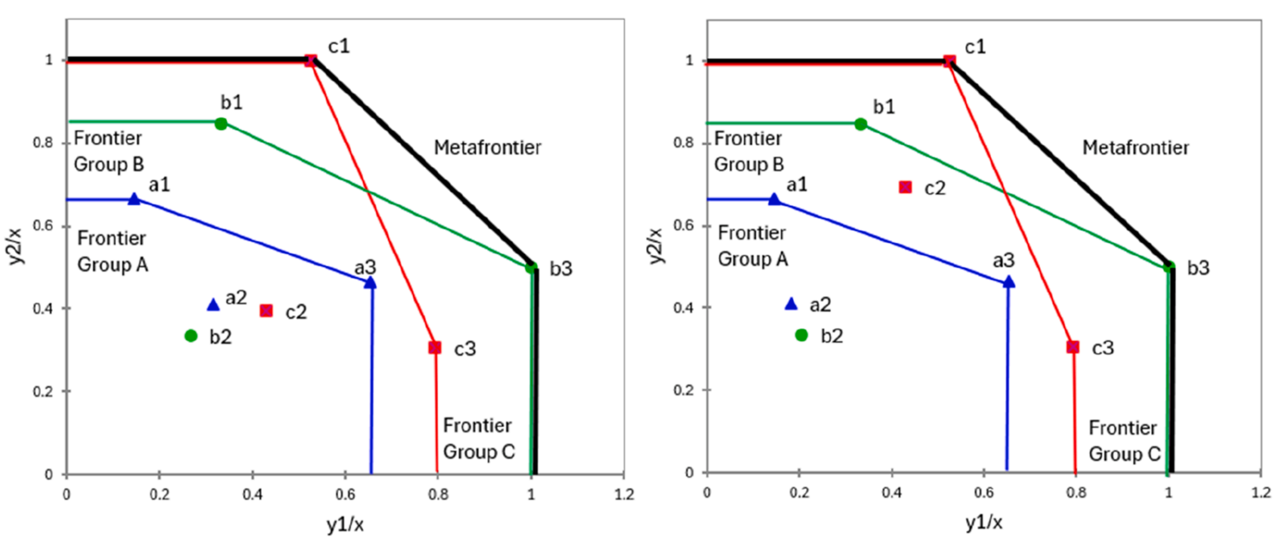
[6] In this specific example the original input is equal to one for all DMUs and this means that it is already normalized within the [0,1] interval. However, in any other alternative empirical example, the inputs must be also normalized using the maximum observed input values.

Table 1
Data and efficiency scores for three groups consisting of three DMUs.

Group	DMU	*Production Data*			*Normalized Data*		*Efficiency scores*			
		x	y_1	y_2	y_{1n}	y_{2n}	$D_c^M(x,y)$	$D_c^A(x,y)$	$D_c^B(x,y)$	$D_c^C(x,y)$
	a1	1	2.4	9.6	0.15	0.67	0.6667	1	0.7869	0.6667
A	a2	1	5.2	5.9	0.32	0.41	0.4770	0.7379	0.5622	0.5185
	a3	1	10.8	6.7	0.65	0.47	0.7431	1	0.7897	0.9143
	b1	1	5.5	12.2	0.33	0.85	0.8472	1.3519	1	0.8472
B	b2	1	4.4	4.8	0.27	0.33	0.3951	0.6059	0.4626	0.4331
	b3	1	16.5	7.2	1.00	0.50	1	1.5278	1	1.3081
	c1	1	8.7	14.4	0.53	1.00	1	1.6688	1.2486	1
C	c2	1	7.1	5.7	0.43	0.40	0.5463	0.7816	0.6073	0.6390
	c3	1	13.1	4.4	0.79	0.31	0.7939	1.2130	0.7939	1
	Max.	1	16.5	14.4						

Table 2
New data and efficiency scores for A2, B2 and C2 in Example 2.

Group	DMU	*Production Data*			*Normalised Data*		*Efficiency scores*			
		x	y_1	y_2	y_{1n}	y_{2n}	$D_c^M(x,y)$	$D_c^A(x,y)$	$D_c^B(x,y)$	$D_c^C(x,y)$
A	a2	1	3	5.9	0.18	0.41	0.4097	0.6651	0.4941	0.4097
B	b2	1	3.4	4.8	0.21	0.33	0.3539	0.5728	0.4317	0.3666
C	c2	1	7.1	10	0.43	0.69	0.7380	1.1940	0.8998	0.7648



Example 1. Camanho's et al. (2021) data

Example 2: Camanho's et al. (2021) modified data

Fig. 2. Production frontiers in the two numerical examples.

and examine the differences among the four productivity indices[7]: the CDMI, the adjusted CDMI[8], the ASMI with the metafrontier as reference technology plus the new productivity index proposed in this paper for group comparisons, alongside the components of these indices. The results are summarized in Table 3, where first we can observe that the efficiency gap EG^{g_1,g_2} is calculated in the same way for the four indices in both examples. Clearly, the efficiency gaps in the two examples differ because the efficiency values of DMUs 2 in Tables 1 and 2 are different. Second, the results of Example 1 are similar for all four indices and their associated technical changes. The mean technical gaps between groups A and B (and between A and C) are all greater than one, ranging from 1.3121 to 1.3638 (and from 1.2845 to 1.3518, respectively). This indicates that, on average, firms in group A employ inferior technology for converting inputs into outputs compared to those in groups B and C. Additionally, the technical gaps between groups B and C, as calculated by the four indices, range from 0.9753 to 1.0004, suggesting that group C's technology appears to be slightly superior to that of group B.

The results of Example 2, compared with Example 1, clearly demonstrate the advantages of the new proposed methodology. It is worth noting that, although the positions of the four frontiers remain unchanged between the two examples, the technical gap values in Example 2 differ from their counterparts in Example 1 when applying the three previous methodologies. This result is counterintuitive, as the new index is the only one that maintains the same technical gap values GTG^{g_1,g_2} in both examples. Even in such a simple illustration, the distri-

[7] To replicate this numerical example, we provide the R code used for calculating the results of this section in the Annex.

[8] To address the circularity problem in group comparisons, Camanho and Dyson [[16], p. 40] also introduced an adjusted technical change, inspired in the EKS method (see [27] and [61] for details), suitable for situations where the circular property is required for comparisons involving more than two groups.

bution of inefficient units within the production sets can significantly distort the technical gap calculation under the previous methodologies, potentially leading to misleading conclusions. This is particularly evident when comparing groups B and C. In Example 1, group C's technology appears to dominate or at least match that of group B (technical gap between 0.9789 and 1.0004), whereas in Example 2, the conclusion is reversed (technical gap between 1.0332 and 1.0480), despite the distances between the three frontiers remaining unchanged in both examples.

3.3. A step forward: measuring the influence of superefficient DMUs on the global technical gap

The discussion so far has centered on comparing the productivity gaps between DMUs in two separate groups. Another interesting question is how to evaluate the impact of the arrival of a new superefficient DMU on the global technical gap measured between the two groups. Andersen and Petersen [69] introduced the super-efficiency DEA model to benchmark efficient units. The main idea behind the super efficiency concept is to calculate a new efficiency measure for an efficient unit with the linear combination of all other units in the group except itself.

The output-oriented CCR model for calculating super-efficiency is similar to (2), although the DMU under evaluation (DMU o) is removed from both the input and output constraints and omitted from the reference set as follows:

$$\begin{aligned} \left(D_c^A\left(x_o^A, y_o^A\right)\right)^{-1} &= \max_{\lambda,\theta_o} \theta_o \\ s.t. \quad & \sum_{j=1}^{N} \lambda_j x_{js}^A \le x_{os}^A, \quad s = 1,\ldots,K \\ & \sum_{j=1}^{N} \lambda_j y_{jr}^A \ge \theta_o y_{or}^A, \quad r = 1,\ldots,Q \\ & \lambda_j \ge 0, \quad \lambda_o = 0 \quad j = 1,\ldots,N; j \ne o \end{aligned} \tag{12}$$

The DMU under evaluation is super-efficient in Eq. (12) when the efficiency score θ_o is lower than one, indicating the extent to which it could proportionately reduce its current outputs to reach the production frontier defined by DMUs in group A. A lower index represents a higher super-efficiency. The measure of super-efficiency is useful for ranking efficient DMUs and for determining which DMUs are most influential in shaping the best practice frontier in different groups. Nevertheless, a pending question is to compare the relative impact that the most influential DMUs have on their own production technologies.

To do this, consider now two DMUs, a and b, belonging to groups A and B, respectively. Assume also that a and b are the most superefficient DMUs in their respective groups. In this framework, we can define the technologies $T^{A'}$ and $T^{B'}$ of two hypothetical groups A' and B' composed of the $N-1$ and $M-1$ DMUs after removing a and b from the original N and M DMUs in each group, respectively, as in (1). Using our setting, we can further decompose $GTG_t^{AB}(H)$ in Eq. (10) to analyze the influence of both superefficient DMUs in their respective groups as follows:

$$GTG^{AB}(H) = \underbrace{\frac{\left(\prod_{l=1}^{L} D_c^B\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^A\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}_{GTG^{AB}(H):\ Global\ Technical\ Gap} = \underbrace{\frac{\left(\prod_{l=1}^{L} D_c^{B'}\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^{A'}\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}_{STG^{A'B'}(H):\ Super\ Technical\ Gap} \cdot \underbrace{\frac{\dfrac{\left(\prod_{l=1}^{L} D_c^B\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^{B'}\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}{\dfrac{\left(\prod_{l=1}^{L} D_c^A\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}{\left(\prod_{l=1}^{L} D_c^{A'}\left(x_l^H, y_l^H\right)\right)^{\frac{1}{L}}}}}_{SRTG^{A'B'}(H):\ Super\ Relative\ Technical\ Gap} \tag{13}$$

In Equation (13), the global technical gap between the two groups $GTG^{AB}(H)$ is decomposed into two components: a super technical gap $STG^{A'B'}(H)$, where A' and B' denotes the two groups of DMUs in A and B after removing the most superefficient units in each group and a super relative technical gap $STG^{A'B'}(H)$. The component $STG^{A'B'}(H)$ captures the overall frontier gap between the two groups, after removing the most superefficient DMUs from each group and is interpreted as in Eq. (10). Values of $STG^{A'B'}(H) > 1$ $\left(STG^{A'B'}(H) < 1\right)$ mean that the production technology in group A' is more (less) productive than group B'. The comparison of $GTG^{AB}(H)$ and $STG^{A'B'}(H)$ reveals the comparative importance of both superefficient DMUs in their respective technologies.

This gap is explained by the relative importance of each superefficient unit within its corresponding group. The two ratios represented in the numerator and the denominator of $SRTG^{A'B'}(H)$ in Equation (13) are by construction less than one and indicate the relative importance of both superefficient DMUs with respect to their own group B and group A technologies. The lower each ratio is, the greater the influence of the super-efficient DMU on the shape of the technology will be. Therefore, when $SRTG^{A'B'}(H) > 1$, the most superefficient unit in group A is more influential in its group than its counterpart in group B, and the opposite applies when $SRTG^{A'B'}(H) < 1$.

4. Data and variables

In this empirical application, our groups are composed of sets of football players grouped by their field position who have played for Spain's two greatest clubs in terms of the number of titles won: FC Barcelona and Real Madrid. The data about the players were gathered from the official websites of both football clubs[9], although this information has been supplemented with data available on Wikipedia in some specific cases. Both databases include information on all the players who have played at least one game for each club. However, we considered that our empirical analysis should only include the most relevant footballers.

Since this may be a somewhat subjective decision, we decided to establish the criterion that a player must have played at least one hundred official games with the club for inclusion in the sample. Although this decision excludes some iconic players from the analysis[10], it ensures that all players in the sample have played a significant role in the history of each club. Players who are still members of the squads of both teams are also excluded[11], as are footballers who played in the first half of the 20th century before the European competitions came into existence. This is designed to ensure greater homogeneity in the composition of the sample regarding the number of trophies won. Therefore, the study covers a total period of seventy years, spanning 1955 to 2024.

The players are classified into four different groups according to their position on the field (goalkeepers, defenders, midfielders and forwards). Thus, we can evaluate their performance separately[12]. The data refers to the period in which the footballers played for each club only, i.e., their statistics for other clubs or their national teams are disregarded[13]. After making some minor adjustments to the composition of the original dataset, which was slightly unbalanced in favor of Barcelona in some positions[14], the final selection includes a total of 10 goalkeepers, 35 defenders, 36 midfielders and 30 forwards per club. The full list of players included in the final sample for each club is reported in Table A included in the Appendix.

Following Santín [58], we have selected five variables to conduct our empirical analysis. The only input included is the number of seasons played. By standardizing the input to the number of seasons spent at the club, players are compared relative to the same temporal opportunity to generate outputs. This ensures that their potential impact on each club's history is evaluated fairly. Moreover, this value serves as a proxy for the degree of familiarity with the particularities of the club and its playing style. Thus, longer tenures often reflect that the player has the manager's confidence and is better adapted to the club's dynamics. Regarding the output variables, we selected four different indicators with the aim of capturing both individual contributions and collective success: (i) the number of official games played; (ii) the number of goals scored (except for goalkeepers); (iii) the number of national titles won, and (iv) the number of international titles won.

The first variable captures a multidimensional aspect of player performance that combines reliability, consistency, and level of integration into the team. It is commonly used in studies analyzing the performance of football players with frontier techniques (e.g., [50]). Although this variable does not account for whether the players were part of the starting lineup or came on as substitutes playing a limited number of minutes, the mere participation in a game denotes that they are relevant to the team strategy.

The second selected output (goals scored) reflects the players' ability to achieve the primary objective of the game, making it the most direct and tangible measure of a player's impact on the result of a game. The use of this variable is also supported by previous literature on the measurement of player performance in different settings (e.g., [35,40,42,43,63]). This indicator is especially relevant for forwards and midfielders, although it can also be important for defenders with significant participation in strategy moves (e.g., corners or free kicks) or as penalty shooters. However, this variable will not be considered when evaluating goalkeepers[15].

The third and fourth variables provide a meaningful measure of each player's contribution to team achievements. Specifically, we distinguish between national titles, which include the Spanish league tournament (*La Liga*), the Spanish Cup (*Copa del Rey*), and the Spanish Super Cup (*Supercopa*), and international titles such as the European Cup (UEFA Champions League), the UEFA Cup (now UEFA Europe League), Intercontinental Cup or FIFA Club World Cup, the UEFA European Super Cup and other former competitions (the UEFA Cup Winners' Cup, the Latin Cup and the Inter-Cities Fair Cup). All these trophies represent the culmination of a team's success over a season or competition and serve as a benchmark of excellence for individual players. The variable representing national trophies won indicates dominance specifically within the Spanish context. International trophies imply competition against the best clubs and players from all over Europe and the world.

Table 4 presents the descriptive statistics for the five variables considered in the empirical analysis disaggregated by the four main on-field playing positions and for each club. As shown in the table, players from both clubs display several similarities in terms of the number of seasons and games played, although Real Madrid forwards exhibit a higher average goal-scoring performance, likely due to having participated in more games and seasons. Moreover, Real Madrid players clearly outperform their Barcelona counterparts in terms of international titles across all positions, while the number of domestic titles remains much more balanced between players from the two clubs.

5. Results

In this section, we present the indices representing the estimated productivity gaps for players belonging to each club using the approach described in Section 3.1, that is, using the $L = 1{,}000{,}000$ synthetic DMUs generated at random within the unit-hypercube H as a reference group for making the comparisons between Real Madrid players (from now on group A) and FC Barcelona players (from now on group B). Subsequently, we also report the results derived from the analysis focused on assessing the productivity gap between superefficient DMUs belonging to the two different groups introduced in Section 3.2.

[9] https://players.fcbarcelona.com/en and https://www.realmadrid.com/es-ES/el-club/historia/futbol/primer-equipo-masculino/jugadores, respectively.

[10] For instance, Maradona, Romario or Ronaldo for FC Barcelona, as well as Laudrup, Schuster or Van Nistelrooy for Real Madrid.

[11] According to this criterion, players like Kroos, who retired at the end of the 2023–2024 season, are included in the sample. In contrast, footballers who clearly meet the requirement of 100 games played but are still playing for Barcelona and Real Madrid, like Ter Stegen, Lewandowski, Carvajal or Courtois, are excluded.

[12] To determine their position, we used the criteria published on each club's website, which may not be the same in the case of some players who have played for both teams. For example, Figo and Luis Enrique are considered forwards by FC Barcelona and midfielders by Real Madrid.

[13] When a player has played for both clubs (e.g., Figo, Celades, Schuster or Luis Enrique), the values of the selected variables corresponding to each team are considered separately as if they were different players.

[14] For players that meet the one hundred games criteria, we eliminated from the sample the players with the lowest efficiency scores after running an initial DEA using the same variables selected to perform the empirical analysis. This guarantees a fairer comparison.

[15] An alternative indicator could have been the number of saves recorded by each goalkeeper. However, we opted not to use this variable for two main reasons. First, such data are available only for a limited subset of goalkeepers in our sample (those playing from the 1990s onward). Second, a high number of saves may introduce an implicit bias, as it is often more reflective of a team's defensive weaknesses than of the goalkeeper's individual performance.

Table 3
Comparison of results with alternative approaches for both examples.

	Camanho's et al [17] data			Camanho's et al [17] modified data		
	Example 1			**Example 2**		
Groups (g_1,g_2)*	A - B	A - C	B - C	A - B	A - C	B – C
EG^{g_1,g_2}	0.8559	0.9532	1.1137	0.8658	1.0477	1.2101
$CDMI^{g_1,g_2}$	1.1444	1.2769	1.0959	1.1651	1.4942	1.2541
TG^{g_1,g_2}	1.3371	1.3397	0.9840	1.3456	1.4262	1.0364
$CDMIadj^{g_1,g_2}$	1.1565	1.2885	1.1142	1.1789	1.4949	1.2680
$TGadj^{g_1,g_2}$	1.3513	1.3518	1.0004	1.3616	1.4268	1.0479
$ASMI(M)^{g_1,g_2}$	1.1230	1.2243	1.0902	1.1389	1.4238	1.2502
$TG(M)^{g_1,g_2}$	1.3121	1.2845	0.9789	1.3154	1.3590	1.0332
PGI^{g_1,g_2}	1.1672	1.2679	1.0862	1.1808	1.3936	1.1802
GTG^{g_1,g_2}	1.3638	1.3302	0.9753	1.3638	1.3302	0.9753

* EG – Efficiency Gap; CDMI – Camanho&Dyson Malmquist Index; TG – Technical Gap; CDMIadj – adjusted CDMI; ASMI(M) – Aparicio&Santin Malmquist index (metafrontier); TG (M) – Technical Gap (metafrontier); PGI – New Productivity Gap Index; GTG – Global Technical Gap.

Table 4
Descriptive statistics for players in different positions and clubs.

	Position	Seasons		Games		Goals		National titles		International titles	
		Mean	*SD*	*Mean*	*SD*	*Mean*	*SD*	*Mean*	*SD*	*Mean*	*SD*
Barcelona FC	Goalkeepers	10.3	4.6	273.6	138.0	-	-	6.9	4.3	2.2	2.0
	Defenders	9.4	4.3	295.4	146.7	15.5	20.3	6.6	4.8	2.2	2.1
	Midfielders	9.0	4.1	281.8	165.7	37.0	28.7	6.8	5.9	2.7	2.6
	Forwards	6.1	3.3	222.9	122.3	100.8	115.1	5.4	4.7	2.1	2.3
Real Madrid	Goalkeepers	10.5	4.7	272.9	194.9	-	-	7.7	3.8	3.6	3.7
	Defenders	9.1	4.1	292.1	173.6	16.4	28.5	7	3.5	3.6	3.8
	Midfielders	7.6	3.9	260.7	133.8	35.3	37.4	5.5	3.3	3.0	3.4
	Forwards	8.2	4.4	297.2	183.4	134.7	115.2	6.5	3.6	4.0	4.0

5.1. *Comparison between groups of football players in both clubs*

As mentioned above, these indices are reported for the different groups of evaluated players according to their position on the field to facilitate the interpretation of the results. Note that values greater than one indicate that Real Madrid's players perform better ($A>B$), while values lower than one denote that the productivity of FC Barcelona's players is better ($B>A$). Table 5 shows the estimated productivity gaps $PGI_t^{AB}(H)$ for each group together with its components: the efficiency gap EG_t^{AB} and the global technical gap $GTG_t^{AB}(H)$. After normalizing the input and the outputs by the maximum observed values we calculate these values assuming an output orientation. For comparison purposes, we also provide the CDMI and its decomposition.

First, according to $PGI^{AB}(H)$ values, we find that Real Madrid's goalkeepers clearly outperform FC Barcelona's players by almost 36%, while there are hardly any differences (0.058% in favor of Real Madrid) in the overall performance of defenders belonging to these clubs. For midfielders, Real Madrid players also have a slight edge, although the difference (2.46%) is quite small. The only group for which there is a productivity gap in favor of FC Barcelona players is the forwards (4.42%).

Second, Real Madrid players' edge over FC Barcelona was mainly due to the efficiency gap EG^{AB}, with values clearly higher than unity for all groups of players, except for forwards. Regarding the second driver for explaining productivity, the global technical gap $GTG^{AB}(H)$ is higher for all groups of Barcelona players, except for goalkeepers. This means that, generally, Real Madrid players' performance is more homogeneous and is closer to its own production frontier than for *Barça* players. However, the players who made the difference at FC Barcelona in their position pushed up the frontier above the players who defined the best practice technology at Real Madrid. The only exception is for goalkeepers (forwards), where Real Madrid's (Barcelona's) players outperform FC Barcelona's (Real Madrid's) in both dimensions.

As mentioned previously, $PGI^{AB}(H)$ and $GTG^{AB}(H)$ are circular and independent of a reference technology. This means that we can use circularity to compare the productivity gaps among positions within the

Table 5
Estimated productivity gaps for players belonging to both clubs.

	The productivity gap index and its decomposition as per Eq. (11)					Camanho and Dyson [16] as per Eq. (6)*	
	$PGI^{AB}(H)$	EG^{AB}*	$GTG^{AB}(H)$	Real Madrid Ratio**	FC Barcelona Ratio**	$CDMI^{AB}$	TG^{AB}
Goalkeepers	1.3586	1.2707	1.0692	0.5430	0.3997	1.2232	0.9626
Defenders	1.0058	1.1249	0.8941	0.4040	0.4017	1.0509	0.9343
Midfielders	1.0246	1.2385	0.8273	0.4108	0.4009	1.0997	0.8880
Forwards	0.9558	0.9668	0.9887	0.4180	0.4373	0.9302	0.9621

* The efficiency gap EG^{AB} matches in both (6) and (11).
** These ratios are the numerator (Real Madrid) and the denominator (FC Barcelona) in the first term on the right-hand side in (11) used to calculate the productivity gap $PGI^{AB}(H)$.

same team. To do this, we can use the ratios reported in Table 5 for players belonging to FC Barcelona and Real Madrid. The above ratios capture the productivity of each group with respect to the reference group calculated with the synthetic units. According to this information, we can conclude that the mean productivity of Real Madrid midfielders (0.4108) was slightly lower than what was observed for forwards (0.4180) over time. As a result, their productivity gap (0.4108/0.4180) was equal to 0.9828, i.e., only 1.72% lower than for the forwards. In the case of FC Barcelona, this distance between midfielders (0.4009) and forwards (0.4373) is larger and represents a productivity gap of 8.32% in favor of FC Barcelona's forwards. This information is useful to conclude that the best productivity achieved, regardless of team and field position, was for Real Madrid goalkeepers, followed by FC Barcelona forwards.

Note that the values calculated using the CDMI approach in Table 5 also indicate that, based on the advantage of its goalkeepers, defenders and midfielders, which offsets the better performance of FC Barcelona's forwards, the productivity of Real Madrid players is better. However, the productivity gap values are slightly different, with Madrid goalkeepers having a smaller, and defenders and midfielders a greater edge (22%, 5% and 10%, respectively). For forwards, the difference in favor of Barcelona players is greater (7%). Another difference is that the $TG^{AB} < 1$ in the Camanho and Dyson [16] specification is always greater for FC Barcelona, because this measure is calculated locally from the technical changes of the observed DMUs.

Fig. 3 shows the kernel density plots of the global technical gap with a view to exploring the shape of the distribution of the technical gaps of the synthetic DMUs within the unit-hypercube. This illustration is useful for understanding why, unlike the comparison of goalkeepers, the average technical gap TG^{AB} and global technical change $GTG^{AB}(H)$ do not match or even change sign. Fig. 3a shows the kernel distribution of the technical gaps between the goalkeepers' technologies calculated using 1,000,000 synthetic DMUs. We observe that the distribution is smooth but with highly concentrated values below the unit value. The intuition here is that the technology of FC Barcelona dominates Real Madrid for more than 50% of the projections of synthetic DMUs against the two best practice frontiers. However, the long tail to the right of the distribution above one indicates that the technical gaps in favor of Real Madrid in this area of technology are on average much larger and compensate for the gaps below one. The advantage of computing a global technical change is that we compare the technical gap globally and not only by averaging the observed technical gaps as in the CDMI.

By contrast, the shape of the distribution of the defenders (Fig. 3b) is more complex, with a bimodal and much more heterogeneous structure. The technical change for more than 75% of synthetic production units. Although there is a sizeable concentration of synthetic DMUs with a technical change close to one, this is not enough to offset the global technical gap in favor of FC Barcelona. The structure of the distributions of midfielders and forwards is clearly multimodal, although very different from each other. Thus, there are several concentrations of values between 0.65 and 0.8 for midfielders (Fig. 3c), where even the third quartile is less than one, leading to a global technical gap of 17.27% in favor of FC Barcelona midfielders. For the forwards (Fig. 3d), there are many values below 0.8 and above 1.2, with the median at a value very close to unity, although the difference is on average slightly (1.13%) in favor of FC Barcelona. Overall, it is worth noting that the two technologies intersect each other in all cases.

5.2. Superefficient football players

After analyzing the differences between the players belonging to both groups (clubs) by position, we report below the results for the assessment of the super efficiency of the players by their position within their teams. In the specific context of our example, this analysis provides two main results. First, this allows us to compare the performance of the best players in each category (goalkeepers, defenders, midfielders and

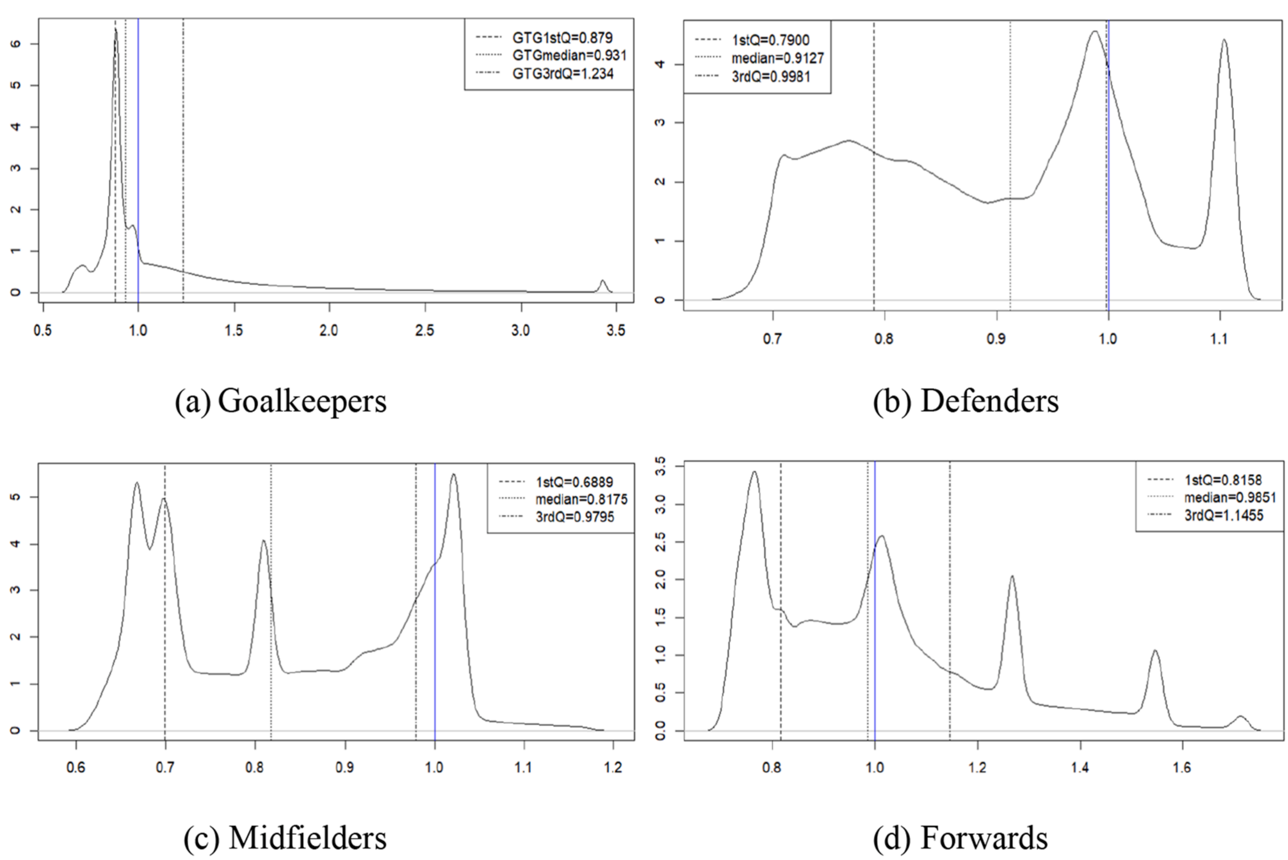


Fig. 3. Kernel density plots of GTC for different groups of players.

forwards) throughout the history of these two great clubs according to our metrics. Second, we can identify the key players on each team by position on the field, comparing their relative influence on the current technologies available and determine whether they marginally contributed to bridging the technological gap between the technologies of the two clubs.

To do this, we first solve Eq. (12) separately for each group of players (goalkeepers, defenders, midfielders and forwards). Tables 6–9 report the estimated values for the top performers in each category. Regarding the interpretation of the values, it is important to note that, with an output orientation, values below one reflect that the player outperforms all other footballers playing in the same position (superefficient), and the lower the super efficiency score the better the player's performance as compared with other peers. Likewise, values above one indicate inefficient behavior, i.e., the higher the value, the higher the inefficiency level.

Starting with the goalkeepers, the most efficient players in the history of each club are Carlos Busquets and Keylor Navas, respectively. The main reason for this result is that FC Barcelona and Real Madrid managed to win many titles in the seasons that both players played at these clubs. The super efficiency score for Keylor Navas is much higher than other Real Madrid goalkeepers, far exceeding that of other players like Iker Casillas or Paco Buyo who won more titles, albeit over a much longer period. The case of Carlos Busquets is curious, since he is the only top performer to have come from FC Barcelona's youth academy. He won most of his titles without playing very many games, as he was the second goalkeeper for most of his career. What makes him a very efficient player, however, is that he was able to win many titles in fewer seasons than others who played more games and won more titles, such as Víctor Valdés or Andoni Zubizarreta.

Among the defenders, two players clearly stand out above the rest: Ronald Koeman and Fernando Hierro. They are two very special players, as they both had a great goalscoring ability, which is not common for players in this position. This characteristic, combined with the large number of titles that they both won, makes their super-efficiency scores much higher (lower values) than the rest. Koeman's score is much better than for other FC Barcelona defenders, as he achieved his results in fewer seasons. Other players with great goalscoring records and a much longer list of titles, such as Gerard Piqué or Sergio Ramos, are ranked much lower in the classification because they achieved these results after spending many more seasons at their respective clubs.

If we examine the ranking of midfielders, we again find a similar pattern to what was observed for goalkeepers. The players at the top of both rankings have in common that they all won many titles in a small number of seasons. Of these, the most efficient is Seydou Keita, followed by Cesc Fábregas and James Rodriguez. They surpass other legends, such as Sergio Busquets or Xavi at FC Barcelona or Kroos and Michel at Real Madrid, who won more titles across more seasons at their clubs. Likewise, we also found that the number of goals scored does not seem to be such a relevant factor in determining the values of the super efficiency scores for midfielders. Another interesting result is that Real Madrid́s top ten midfielders are superefficient with scores below one.

For the forwards, as expected, the players at the top of the rankings are Lionel Messi and Cristiano Ronaldo, considered to be the best two players in the history of their respective clubs. In this case, their records in terms of goals scored and titles won are so much better than all other forwards that they are ranked at the top, despite having been at their respective clubs for many seasons. The lowest super efficiency score is recorded by Cristiano, who achieved his incredible records by outperforming players of the caliber of Hugo Sánchez or Gareth Bale, who triumphed in a smaller number of seasons. Messi is also special because he ranks as FC Barcelona's most superefficient player after spending 17 seasons at the club, whereas the next most superefficient Barcelona forwards (e.g., Luis Suárez, David Villa or Luis Figo) played for the club for much less longer.

Finally, we would like to highlight that the productivity measure, i.e., the ratio between output and input indicators, is different from a measure of effectiveness, which is based on output levels only. This means that some footballers perceived by fans as charismatic legends that have outstanding goalscoring and title-winning records, like Raúl or Benzema at Real Madrid or Iniesta and Sergio Busquets at FC Barcelona, were surpassed by other apparently lower-profile players for the very reason that their goalscoring and title-winning runs took place over a smaller number of seasons.

5.3. The impact of the most superefficient footballers on the global technical gap

As we saw in Section 3.2, we can analyze the relative impact of the most superefficient players of each team by decomposing the global technical gap $GTG_t^{AB}(H)$ into the super technical gap $STG_t^{A'B'}(H)$ and the super relative technical gap $SRTG_t^{A'B'}(H)$ using Equation (13). This is shown in Table 10, which compares the relative impact of the most superefficient goalkeepers (Carlos Busquets vs Keylor Navas), defenders (Koeman vs Hierro), midfielders (Keita vs James) and forwards (Messi vs Cristiano) of each team throughout history.

The super technical gap reported in the third column of Table 10 measures what the global technical gap between the two teams would be like without the presence of the most superefficient player of each team. This gives an overview of which players have been most influential in explaining the existing difference in the global technical gap between the two clubs in each field position. In this regard, the most striking result is found for defenders. Without Ronald Koeman and Fernando Hierro, the best practice frontier of Real Madrid's defenders would be,

Table 6
Top 5 goalkeepers in the history of each club.

FC Barcelona						Real Madrid					
	Seasons	Games	National Titles	International Titles	Score		Seasons	Games	National Titles	International Titles	Score
Carlos Busquets	7	126	12	4	68.1%	Keylor Navas	5	162	2	10	30.77%
Valdés	12	540	14	7	79.8%	Casillas	16	725	11	8	86.83%
Zubizarreta	8	410	8	3	94.4%	Buyo	11	454	12	0	86.95%
Hesp	3	146	3	1	103.0%	Vicente Train	4	100	5	0	87.27%
Ramallets	16	388	11	2	176.3%	Agustín	10	123	9	2	126.90%

Table 7
Top 10 defenders in the history of each club.

FC Barcelona							Real Madrid						
Footballer	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**	Footballer	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**
Koeman	6	267	90	8	2	23.56%	Hierro	12	601	127	10	6	59.65%
Alves	9	409	23	14	9	80.50%	Varane	10	360	17	7	11	86.40%
Abidal	6	194	2	9	6	100.00%	Solana	6	177	2	9	1	87.74%
Piqué	15	617	53	22	9	104.67%	Nacho	14	364	16	11	15	96.41%
Frank de Boer	5	215	14	1	0	105.61%	Tendillo	5	146	12	7	0	98.34%
Jordi Alba	11	460	27	15	3	108.65%	Gordillo	7	254	27	9	1	99.28%
Adriano	7	192	17	10	6	108.78%	Roberto Carlos	11	527	69	7	6	101.82%
Van Bronckhorst	4	155	7	4	1	117.28%	Sergio Ramos	16	671	101	11	11	106.45%
Reiziger	7	262	0	3	0	121.42%	Cannavaro	3	118	1	3	0	108.11%
Márquez	7	245	13	8	4	129.84%	Marcelo	16	546	38	13	12	108.54%

Table 8
Top 10 midfielders in the history of each club.

FC Barcelona							Real Madrid						
Footballer	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**	Footballer	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**
Keita	4	189	22	8	6	63.93%	James R.	4	125	37	3	6	77.74%
Cesc	3	152	43	4	2	72.49%	Makelele	3	145	2	4	3	82.00%
Rakitic	6	311	36	10	3	95.24%	Figo	5	245	58	4	3	88.51%
Cocu	6	297	38	1	0	104.54%	Kroos	10	465	28	9	14	89.26%
Sergio Busquets	16	725	19	23	9	113.02%	Ozil	3	159	27	3	0	92.75%
Víctor	7	316	25	3	1	114.82%	Seedorf	3	159	20	2	2	94.45%
Xavi	18	780	87	17	8	119.30%	Michel	12	559	130	11	2	96.90%
Touré Yaya	3	118	6	4	3	123.24%	Netzer	3	100	13	4	0	97.79%
L. Suárez Miram.	7	176	80	0	2	125.42%	Di María	4	190	36	4	2	97.86%
Schuster	8	241	91	5	1	126.01%	Martín Vázquez	10	342	47	13	2	98.54%

Table 9
Top 10 forwards in the history of each club.

FC Barcelona							Real Madrid						
Footballer	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**	**Footballer**	**Seasons**	**Games**	**Goals**	**National Titles**	**International Titles**	**Score**
Messi	17	779	672	25	10	83.48%	Cristiano Ronaldo	9	438	451	6	10	56.46%
Luis Suárez	6	284	198	10	3	92.43%	Hugo Sánchez	7	282	208	9	1	75.71%
Villa	3	120	48	5	3	93.26%	Bale	7	258	106	7	12	81.70%
Figo	5	249	45	5	0	95.05%	Asensio	7	286	61	7	10	97.09%
Alexis Sánchez	3	142	47	4	2	96.22%	Robinho	3	137	35	3	0	97.44%
Bojan	4	164	41	6	4	98.55%	Benzema	14	648	354	11	14	100.72%
Henry	3	121	49	3	3	99.42%	Higuaín	6	264	121	6	0	101.98%
Giovanni	3	109	35	5	2	100.00%	Raúl	16	741	323	10	6	105.08%
Pedro	9	324	99	14	9	100.00%	Zamorano	4	173	101	3	0	109.33%
Rivaldo	5	235	130	3	0	102.12%	Butragueño	12	463	217	12	2	111.77%

on average, 24.48% more productive than for FC Barcelona's defence. This means that, thanks to his incredible goalscoring record, Koeman is crucial to explaining the current technological dominance of FC Barcelona's defenders. Thus, the column headed by "impact on FC Barcelona" shows that this player improved the technology of the defenders of his team by 35.90%, while Fernando Hierro does so by only 10.75%. In short, the super relative technical gap shows that Koeman was 28.27% more influential on his team than Hierro.

We observe a similar pattern for Keylor Navas at Real Madrid. Without him and Carlos Busquets, FC Barcelona's goalkeepers would have a 9.83% higher global technical gap than Real Madrid's goalkeepers. The technical progress generated by Keylor on Real Madrid's productive frontier was 19.57%. This means that he was 18.57% more influential on Real Madrid's technology than Carlos Busquets was on FC Barcelona's. Likewise, Keita was also more influential than James, as, without them, the global technical gap of Real Madrid's midfielders would have been 8.22% instead of the current 17.27%.

Finally, Cristiano Ronaldo's achievements playing for Real Madrid have significantly reduced the global technical advantage that the technology of Barcelona's forwards have over Real Madrid's. Without Messi and Cristiano, the super technical gap would increase from the current 1.13% to 8.79%. While Lionel Messi has been one of the most award-winning and productive players of all time, he was able to shift the technology of FC Barcelona's forwards up by only 6.67% as compared to Cristiano's 13.86%, which implies that Cristiano's super relative technical gap was 9.39% higher than Messi's.

It is important to note that the methodology proposed in this paper, while rooted in the productivity analysis tradition, does not rely on estimating a classical production function. Instead, it employs DEA to construct best-practice frontiers based on observed data from the evaluated groups. DEA does not assume specific functional forms or transformation properties, making it particularly well-suited for assessing performance in multi-input, multi-output contexts—such as sports, healthcare, education, or justice—where market prices for outputs are absent and traditional production assumptions may not hold. In the specific context of football players, for example, inputs and outputs are inherently multidimensional, and the notion of a well-defined production function—defined as the maximum output attainable given input quantities ([52], p. 129)—is debatable.

6. Conclusions

This paper introduces a novel methodology for assessing productivity gaps between groups using a common reference group composed of a huge number of synthetic observations uniformly generated at random within a unit hypercube. The proposed approach provides a decomposition of the productivity gap into efficiency and global technical gap components, while ensuring circularity and independence from an arbitrary reference technology. With the development of this methodology, we contribute to expanding the existing literature by addressing some of the main limitations of traditional meta-frontier approaches, offering a more flexible tool for cross-group technology comparisons. In addition, we also introduce an approach for assessing productivity gaps based on superefficient units that allow us to extend the decomposition of productivity gaps by isolating the effect of the most influential units in shaping the frontier of each group. This enables a finer-grained assessment of the role that the best performers play in shifting the production frontiers within their own groups.

From an empirical perspective, our study reports a comparative analysis of the productivity of football players from two of the most significant clubs in European football (FC Barcelona and Real Madrid), distinguishing between goalkeepers, defenders, midfielders, and forwards. Using a dataset spanning 70 years that includes the leading players at both clubs, our findings reveal distinct performance patterns by position. Specifically, Real Madrid's goalkeepers and midfielders were more productive than FC Barcelona's, while there was no

Table 10
Most influential football players on the global technical gap by field position.

	$GTG_t^{AB}(H)$	$STG_t^{A'B'}(H)$	$SRTG_t^{A'B'}(H)$	Impact* on FC Barcelona	Impact* on Real Madrid
Busquets vs Keylor	1.0692	0.9017	1.1857	0.9656	0.8143
Koeman vs Hierro	0.8941	1.2448	0.7183	0.6410	0.8925
Keita vs James	0.8273	0.9178	0.9014	0.8874	0.9844
Messi vs Cristiano	0.9887	0.9121	1.0839	0.9337	0.8614

* This impact is the score of the numerator (FC Barcelona) and the denominator (Real Madrid) whose ratio defines the $SRTG_t^{AB}(H)$ in Eq. (13).

significant difference for defenders. In contrast, FC Barcelona's forwards outperformed their Real Madrid counterparts.

The super efficiency analysis provides deeper insights into the impact of the most influential players that shape the productivity frontiers of both clubs. Of the goalkeepers, Keylor Navas stands out as Real Madrid's most superefficient player in this position, while Carlos Busquets surprisingly emerges as FC Barcelona's top-ranked keeper. In defense, Ronald Koeman and Fernando Hierro are the most influential players for their respective teams, with Koeman's exceptional goal-scoring ability making him particularly decisive in shifting Barcelona's defensive frontier. In midfield, Seydou Keita and James Rodríguez rank as the most superefficient players, demonstrating that winning a large number of titles in a relatively short period of time makes a player very productive. For forwards, Cristiano Ronaldo and Lionel Messi dominate their respective rankings, with Cristiano being more influential in narrowing the global technical gap between the forwards from both clubs. These findings demonstrate that, beyond general productivity trends, individual units can have a substantial effect on the performance of their groups, highlighting the importance of considering the role of superefficient units when conducting comparative analyses.

The empirical analysis presented in this study provides valuable insights into the comparative performance of historic football players from FC Barcelona and Real Madrid. However, we must be aware that the results are based on a small number of indicators, mainly due to restrictions on the information available on older players. Historical records often lack detailed performance metrics, restricting the scope of the evaluation. Nevertheless, with advancements in big data analytics, future research could incorporate a broader set of performance indicators, such as minutes played, assists, turnovers, successful passes, dribbles, tackles or ball recoveries. Expanding the data set in this way would allow for a more comprehensive assessment of the performance of players and, therefore, allow for a more refined comparative analysis between groups of players (or teams). Additionally, the proposed methodology has practical applications beyond historical comparisons, particularly for player recruitment. By simulating the impact of replacing one player with another, managers could use this technique to assess whether a specific transfer would improve their team's global technical gap, optimizing squad composition and enhancing decision-making in the transfer market.

Finally, we would like to mention some possible extensions of the proposed approach from the methodological point of view that could be addressed in the future to improve its robustness and expand its applicability to other frameworks. First, estimating confidence intervals for the new productivity gap index and its components (the efficiency gap and the global technical gap) would enable statistical inference, which is particularly valuable for assessing whether the observed productivity differences are statistically significant or merely attributable to sample variability. Second, the methodology could be extended to a dynamic setting, enabling the evaluation of productivity gaps over time. This adaptation would facilitate the analysis of how the performance of different groups evolves, capturing structural changes and long-term trends in efficiency and global technical gaps. Third, the extension of the model to accommodate variable returns to scale (VRS) would allow for a sharper understanding of how scale efficiency differences influence productivity gaps between groups. Four, the cross-efficiency approach in Data Envelopment Analysis (DEA), introduced by Doyle and Green [25], can also be employed as an alternative method to identify top performers and assess their influence by considering not only a DMU's self-evaluation, but also its performance when evaluated using the weights derived from other DMUs.

CRediT authorship contribution statement

José M. Cordero: Writing – original draft, Resources, Investigation, Formal analysis, Data curation, Conceptualization. **Daniel Santín:** Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

"We have nothing to declare"

APPENDIX

Table A
Players included in the sample for each club.

FC Barcelona		**Real Madrid**	
Goalkeepers		**Goalkeepers**	
Artola	Sadurní	Agustín	Illgner
Busquets	Urruticoetxea	Betancort	Juanito Alonso
Hesp	Valdés	Buyo	Keylor Navas
Ramallets	Velasco	Casillas	Miguel Ángel
Reina	Zubizarreta	García Remón	Vicente Train
Defenders		**Defenders**	
Abelardo	Nadal	Albiol	Michel Salgado
Abidal	Oleguer	Alkorta	Miera
Adriano	Olmo	Arbeloa	Nacho
Alexanco	Piqué	Benito	Navarro
Alves	Puyol	Camacho	Pachín
Bartra	Ramos	Calpe	Pavón

(continued on next page)

Table A (*continued*)

Belletti	Reiziger	Cannavaro	Pepe
Benítez	Rifé	Chendo	Raúl Bravo
De Boer, Frank	Sánchez	Coentrao	Roberto Carlos
Eladio	Semedo	De Felipe	San José
Ferrer	Sergi Barjuan	Gordillo	Sanchís (Father)
Gallego	Serna	Helguera	Sanchís (Son)
Gerardo	Soler	Hierro	Santamaría
Jordi Alba	Sylvinho	Isidro	Sergio Ramos
Julio Alberto	Torres	Karanka	Solana
Koeman	Umititi	Lasa	Tendillo
Márquez	Van Bronckhorst	Marcelo	Varane
Migueli		Marquitos	
Midfielders		**Midfielders**	
Amor	Juan Carlos	Beckham	Michel
Asensi	Keita	Breitner	Miguel Muñoz
Bakero	Luis Suárez Miramontes	Casemiro	Milla
Busquets	Marcial	Celades (RM)	Molowny
Calderé	Mascherano	Del Bosque	Netzer
Celades (FCB)	Neeskens	Di María	Ozil
Cesc	Pereda	Figo (RM)	Pirri
Cocu	Rakitic	Gallego	Redondo
De la Peña	Robert	Guti	Santisteban
Deco	Schuster (FCB)	Isco	Savio
Esteban	Segarra	James Rodríguez	Seedorf
Eusebio	Sergi Roberto	José Luis	Solari
Flotats	Thiago	Khedira	Stielike
Fusté	Touré Yaya	Kroos	Velázquez
Gensana	Urbano	Luis Enrique (RM)	Xabi Alonso
Gerard	Vergés	Makélélé	Zidane
Guardiola	Víctor	Martín Vázquez	Zárraga
Iniesta	Xavi	McManaman	Zoco
Forwards		**Forwards**	
Alexis Sánchez	Luis Enrique (FCB)	Aguilar	Joseito
Beguiristain	Luis Suárez	Alfonso	Juanito
Bojan	Marcos	Amancio	Kaká
Carrasco	Messi	Amavisca	Kopa
Cruyff	Neymar	Asensio	Mijatovic
Eto'o	Overmars	Bale	Morientes
Evaristo Macedo	Pedro Rodríguez	Benzema	Puskas
Figo (FCB)	Quini	Butragueño	Raúl
Giovanni	Rivaldo	Cristiano Ronaldo	Rial
Giuly	Ronaldinho	DiStefano	Robinho
Goikoetxea	Salinas	Gento	Ronaldo
Henry	Saviola	Grosso	Santillana
Kluivert	Simonsen	Higuaín	Suker
Laudrup	Stoichkov	Hugo Sánchez	Valdano
Lineker	Villa	Isidro	Zamorano

Annex. R code to replicate the numerical example in Section 3.2

This code illustrates the new index for measuring productivity gaps

among groups of production units together with its decomposition library(Benchmarking) library(uniformly) library(optimbase)

#To ensure replication we set a seed set.seed(18072007)

Table 1. Production data from three groups A, B and C. Each group is composed by three

DMUs. Data are retrieved from the numerical example in Camanho et al. [17]

EXAMPLE 1 xa<-matrix(c(1,1,1), ncol = 1) xb<-matrix(c(1,1,1), ncol = 1) xc<-matrix(c(1,1,1), ncol = 1) y2a<-matrix(c(9.6,5.9,6.7), ncol = 1) y1a<-matrix(c(2.4,5.2,10.8), ncol = 1) y2b<-matrix(c(12.2,4.8,7.2), ncol = 1) y1b<-matrix(c(5.5,4.4,16.5), ncol = 1) y2c<-matrix(c(14.4,5.7,4.4), ncol = 1) y1c<-matrix(c(8.7,7.1,13.1), ncol = 1)

Build a matrix with all values of each output and input from the groups. y2=matrix(c(y2a, y2b, y2c), nrow=3,ncol=3) y1=matrix(c(y1a, y1b, y1c), nrow=3,ncol=3) x=matrix(c(xa, xb, xc), nrow=3,ncol=3)

Seek the maximum values for each variable in the 2 periods. y2max=max(y2) y1max=max(y1) xmax=max(x)

Normalize the variables by the maximum observed values to embed all data

within the unit-hypercube y2an=y2a/y2max y2bn=y2b/y2max y2cn=y2c/y2max y1an=y1a/y1max y1bn=y1b/y1max y1cn=y1c/y1max

In this example all input values are equal to one, so the normalization

is not necessary but it is shown for future applications. xan=xa/xmax xbn=xb/xmax xcn=xc/xmax

We build a matrix for outputs and inputs of each group and the metafrontier yan<-matrix(c(y2an, y1an), nrow=3,ncol=2) ybn<-matrix(c(y2bn, y1bn), nrow=3,ncol=2) ycn<-matrix(c(y2cn, y1cn), nrow=3,ncol=2) xan=matrix(c(xan), nrow=3,ncol=1) xbn=matrix(c(xbn), nrow=3,ncol=1) xcn=matrix(c(xcn), nrow=3,ncol=1) yg<-rbind(yan,ybn,ycn) xg<-rbind(xan,xbn,xcn)

Table 3. Calculate the Camanho & Dyson Malmquist index (CDMI)

Note that the notation in the paper uses Shephard's measures while

# the library 'Benchmarking' uses Farrell's measures. For this reason we run
# the 'input orientation' for obtaining the 'output orientation' interpretation.
# We run the CDMI for the example 1 data.
# Efficiency Gaps deaaa1=dea(xan, yan, RTS="crs", ORIENTATION="in", XREF=NULL, YREF=NULL) deabb1=dea(xbn, ybn, RTS="crs", ORIENTATION="in", XREF=NULL, YREF=NULL) deacc1=dea(xcn, ycn, RTS="crs", ORIENTATION="in", XREF=NULL, YREF=NULL) teA1=10^(mean (log10(deaaa1$eff),na.rm=TRUE)) teB1=10^(mean(log10(deabb1$eff),na.rm=TRUE)) teC1=10^(mean(log10(deacc1$eff),na.rm=TRUE))
EGBA1=teB1/teA1
EGCA1=teC1/teA1
EGCB1=teC1/teB1
# CDMI. Technical gap deaab1=dea(xan, yan, RTS="crs", ORIENTATION="in", XREF=xbn, YREF=ybn) deaac1=dea(xan, yan, RTS="crs", ORIENTATION="in", XREF=xcn, YREF=ycn) deaba1=dea(xbn, ybn, RTS="crs", ORIENTATION="in", XREF=xan, YREF=yan) deabc1=dea(xbn, ybn, RTS="crs", ORIENTATION="in", XREF=xcn, YREF=ycn) deaca1=dea(xcn, ycn, RTS="crs", ORIENTATION="in", XREF=xan, YREF=yan) deacb1=dea (xcn, ycn, RTS="crs", ORIENTATION="in", XREF=xbn, YREF=ybn) gmdeaaa1=10^(mean(log10(deaaa1$eff),na.rm=TRUE)) gmdeabb1=10^(mean (log10(deabb1$eff),na.rm=TRUE)) gmdeacc1=10^(mean(log10(deacc1$eff),na.rm=TRUE)) gmdeaab1=10^(mean(log10(deaab1$eff),na.rm=TRUE)) gmdeaac1=10^(mean(log10(deaac1$eff),na.rm=TRUE)) gmdeaba1=10^(mean(log10(deaba1$eff),na.rm=TRUE)) gmdeabc1=10^ (mean(log10(deabc1$eff),na.rm=TRUE)) gmdeacb1=10^(mean(log10(deacb1$eff),na.rm=TRUE)) gmdeaca1=10^(mean(log10(deaca1$eff),na.rm=TRUE))
TGBA1=sqrt ((gmdeaaa1/gmdeaab1)*(gmdeaba1/gmdeabb1))
TGCA1=sqrt ((gmdeaaa1/gmdeaac1)*(gmdeaca1/gmdeacc1))
TGCB1=sqrt ((gmdeabb1/gmdeabc1)*(gmdeacb1/gmdeacc1))
# CDMI=EG* TG
CDBA1=EGBA1 *TGBA1
CDCA1=EGCA1 *TGCA1
CDCB1=EGCB1 *TGCB1
# We run the adjacent CDMI values together with its adjusted TC component.
# See Camanho and Dyson [[16], p. 40] for details.
TGBA1adj= ((gmdeaaa1/gmdeaab1)*(gmdeaba1/gmdeabb1)*(gmdeaca1/gmdeacb1))^(1/3)
TGCA1adj= ((gmdeaaa1/gmdeaac1)*(gmdeaba1/gmdeabc1)*(gmdeaca1/gmdeacc1))^(1/3)
TGCB1adj= ((gmdeaab1/gmdeaac1)*(gmdeabb1/gmdeabc1)*(gmdeacb1/gmdeacc1))^(1/3)
CDBA1adj= EGBA1*TGBA1adj
CDCA1adj= EGCA1*TGCA1adj
CDCB1adj= EGCB1*TGCB1adj
# Now we calculate the Aparicio & Santín Malmquist Index (ASMI) using
# as 'reference technology' the metafrontier. deaam1= dea(xan, yan, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg) deabm1= dea(xbn, ybn, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg) deacm1= dea(xcn, ycn, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg) gmdeaam1= 10^(mean(log10(deaam1$eff),na.rm= TRUE)) gmdeabm1= 10^(mean(log10(deabm1$eff),na.rm= TRUE)) gmdeacm1= 10^(mean (log10(deacm1$eff),na.rm= TRUE))
# Productivity gaps and its components according to the metafrontier
# Productivity gap
PGBA1meta= gmdeabm1/gmdeaam1
PGCA1meta= gmdeacm1/gmdeaam1
PGCB1meta= gmdeacm1/gmdeabm1
# Technical gap (the efficiency gap coincides with the one in Camanho & Dyson)
TGBA1meta= PGBA1meta*(1/EGBA1)
TGCA1meta= PGCA1meta*(1/EGCA1)
TGCB1meta= PGCB1meta*(1/EGCB1)
GLOBAL TECHNICAL GAP
# Generate a big L number of synthetic DMUs uniformly distributed within
# the unit-hypercube of dimension H. H= k+q
# k= number of inputs; q = number of outputs. H = unit-hypercube size
# The library 'uniformly' allows to generate the unit hypercube H in one step.
# Another possibility is generating uniform variables U(0;1) por each input and output. k= 1 q= 2
H= k+q
L= 1000000 simH<-runif_in_cube(L, H)
Hh<-abs(simH)
Xh<-Hh[,1]
Yh<-Hh[,2:3]
# Project the L synthetic DMUs within the hypercube H against the
# empirical technologies of groups A, B and C. Computation time will depend on L hcubeA1= dea(Xh, Yh, RTS= "crs", ORIENTATION= "in", XREF= xan, YREF= yan) hcubeB1= dea(Xh, Yh, RTS= "crs", ORIENTATION= "in", XREF= xbn, YREF= ybn) hcubeC1= dea(Xh, Yh, RTS= "crs", ORIENTATION= "in", XREF= xcn, YREF= ycn)
# The GTG is the geometric mean of all the TGs computed for the synthetic DMUs
GlobalA1= 10^(mean(log10(hcubeA1$eff),na.rm= TRUE))
GlobalB1= 10^(mean(log10(hcubeB1$eff),na.rm= TRUE))
GlobalC1= 10^(mean(log10(hcubeC1$eff),na.rm= TRUE))
GTGBA1= GlobalA1/GlobalB1

GTGCA1= GlobalA1/GlobalC1
GTGCB1= GlobalB1/GlobalC1
PGBA1= GTGBA1*EGBA1
PGCA1= GTGCA1*EGCA1
PGCB1= GTGCB1*EGCB1
EXAMPLE 1 RESULTS
# The Efficiency gap coincides in all models
EG1= cbind(EGBA1,EGCA1,EGCB1)
# [16] results and adjusted CDMI for example 1
CDMI1= cbind(CDBA1,CDCA1,CDCB1)
TG1= cbind(TGBA1,TGCA1,TGCB1)
CDMI1adj= cbind(CDBA1adj,CDCA1adj,CDCB1adj)
TG1adj= cbind(TGBA1adj,TGCA1adj,TGCB1adj)
# Camanho et al. [17]. Calculate the ASMI using the metafrontier
# as reference technology
ASMI1= cbind(PGBA1meta,PGCA1meta,PGCB1meta)
TGASMI1= cbind(TGBA1meta,TGCA1meta,TGCB1meta)
# This paper. The new Productivity Gap Index and the Global Technical Change.
PGI1= cbind(PGBA1,PGCA1,PGCB1)
GTG1= cbind(GTGBA1,GTGCA1,GTGCB1)
# Results of Table 3, example 1 example1_results= list(EG1 = EG1, CDMI1 = CDMI1, TG1 = TG1, CDMI1adj = CDMI1adj,
TG1adj = TG1adj, ASMI1 = ASMI1, TGASMI1 = TGASMI1, PGI1 = PGI1,
GTG1 = GTG1)
# EXAMPLE 2
# New output 1 values for DMUs a2, b2 and new output 2 value for DMU c2
# This variation constitutes the example.
# Table 2. Substitute y1a, y1b and y2c by these new values: y1a2<-matrix(c(2.4,3,10.8), ncol = 1) y1b2<-matrix(c(5.5,3.4,16.5), ncol = 1) y2c2<-matrix(c(14.4,10,4.4), ncol = 1)
# Maximum values are the same in the two examples.
# Normalize the new information by the maximum observed values
# to embed all data within the unit-hypercube y1an2= y1a2/y1max y1bn2= y1b2/y1max y2cn2= y2c2/y2max
# We build a matrix for the new outputs of each group and the metafrontier yan2<-matrix(c(y2an, y1an2), nrow= 3,ncol= 2) ybn2<-matrix(c(y2bn, y1bn2), nrow= 3,ncol= 2) ycn2<-matrix(c(y2cn2, y1cn), nrow= 3,ncol= 2) yg2<-rbind(yan2,ybn2,ycn2)
# Table3. Calculate the CDMI for the example 2 data.
# Efficiency Gap deaaa2= dea(xan, yan2, RTS= "crs", ORIENTATION= "in", XREF= NULL, YREF= NULL) deabb2= dea(xbn, ybn2, RTS= "crs", ORIENTATION= "in", XREF= NULL, YREF= NULL) deacc2= dea(xcn, ycn2, RTS= "crs", ORIENTATION= "in", XREF= NULL, YREF= NULL) teA2= 10^(mean(log10(deaaa2$eff),na.rm= TRUE)) teB2= 10^(mean(log10(deabb2$eff),na.rm= TRUE)) teC2= 10^(mean(log10(deacc2$eff),na.rm= TRUE))
EGBA2= teB2/teA2
EGCA2= teC2/teA2
EGCB2= teC2/teB2
# CDMI. Technical gap deaab2= dea(xan, yan2, RTS= "crs", ORIENTATION= "in", XREF= xbn, YREF= ybn2) deaac2= dea(xan, yan2, RTS= "crs", ORIENTATION= "in", XREF= xcn, YREF= ycn2) deaba2= dea(xbn, ybn2, RTS= "crs", ORIENTATION= "in", XREF= xan, YREF= yan2) deabc2= dea(xbn, ybn2, RTS= "crs", ORIENTATION= "in", XREF= xcn, YREF= ycn2) deaca2= dea(xcn, ycn2, RTS= "crs", ORIENTATION= "in", XREF= xan, YREF= yan2) deacb2= dea(xcn, ycn2, RTS= "crs", ORIENTATION= "in", XREF= xbn, YREF= ybn2) gmdeaaa2= 10^(mean(log10(deaaa2$eff),na.rm= TRUE)) gmdeabb2= 10^(mean(log10(deabb2$eff),na.rm= TRUE)) gmdeacc2= 10^(mean(log10(deacc2$eff),na.rm= TRUE)) gmdeaab2= 10^(mean(log10(deaab2$eff),na.rm= TRUE)) gmdeaac2= 10^(mean(log10(deaac2$eff),na.rm= TRUE)) gmdeaba2= 10^(mean(log10(deaba2$eff),na.rm= TRUE)) gmdeabc2= 10^(mean(log10(deabc2$eff),na.rm= TRUE)) gmdeacb2= 10^(mean(log10(deacb2$eff),na.rm= TRUE)) gmdeaca2= 10^(mean(log10(deaca2$eff),na.rm= TRUE))
TGBA2= sqrt((gmdeaaa2/gmdeaab2)*(gmdeaba2/gmdeabb2))
TGCA2= sqrt((gmdeaaa2/gmdeaac2)*(gmdeaca2/gmdeacc2))
TGCB2= sqrt((gmdeabb2/gmdeabc2)*(gmdeacb2/gmdeacc2))
# CDMI= EG*TG
CDBA2= EGBA2*TGBA2
CDCA2= EGCA2*TGCA2
CDCB2= EGCB2*TGCB2
# We run the adjCDMI values together with the TG component.
TGBA2adj= ((gmdeaaa2/gmdeaab2)*(gmdeaba2/gmdeabb2)*(gmdeaca2/gmdeacb2))^(1/3)
TGCA2adj= ((gmdeaaa2/gmdeaac2)*(gmdeaba2/gmdeabc2)*(gmdeaca2/gmdeacc2))^(1/3)
TGCB2adj= ((gmdeaab2/gmdeaac2)*(gmdeabb2/gmdeabc2)*(gmdeacb2/gmdeacc2))^(1/3)
CDBA2adj= EGBA2*TGBA2adj
CDCA2adj= EGCA2*TGCA2adj
CDCB2adj= EGCB2*TGCB2adj
# Now we run the ASMI using as reference technology the metafrontier deaam2= dea(xan, yan2, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg2) deabm2= dea(xbn, ybn2, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg2) deacm2= dea(xcn, ycn2, RTS= "crs", ORIENTATION= "in", XREF= xg, YREF= yg2) gmdeaam2= 10^(mean(log10(deaam2$eff),na.rm= TRUE)) gmdeabm2= 10^(mean(log10(deabm2$eff),na.rm= TRUE)) gmdeacm2= 10^(mean(log10(deacm2$eff),na.rm= TRUE))

```
# Productivity gaps and its components according to the metafrontier
# Productivity gap
PGBA2meta= gmdeabm2/gmdeaam2
PGCA2meta= gmdeacm2/gmdeaam2
PGCB2meta= gmdeacm2/gmdeabm2
# Technical gap (the efficiency gap coincides with the one in Camanho & Dyson)
TGBA2meta= PGBA2meta*(1/EGBA2)
TGCA2meta= PGCA2meta*(1/EGCA2)
TGCB2meta= PGCB2meta*(1/EGCB2)
### GLOBAL TECHNICAL GAP
# Project the L synthetic DMUs within the hypercube H against the
# empirical technologies of groups A, B and C.
# Computation time will depend on L hcubeA2= dea(Xh, Yh, RTS= "crs", ORIENTATION= "in", XREF= xan, YREF= yan2) hcubeB2= dea(Xh, Yh,
RTS= "crs", ORIENTATION= "in", XREF= xbn, YREF= ybn2) hcubeC2= dea(Xh, Yh, RTS= "crs", ORIENTATION= "in", XREF= xcn, YREF= ycn2)
# The GTC is the geometric mean of all the TCs computed for the synthetic DMUs
GlobalA2= 10^(mean(log10(hcubeA2$eff),na.rm= TRUE))
GlobalB2= 10^(mean(log10(hcubeB2$eff),na.rm= TRUE))
GlobalC2= 10^(mean(log10(hcubeC2$eff),na.rm= TRUE))
GTGBA2= GlobalA2/GlobalB2
GTGCA2= GlobalA2/GlobalC2
GTGCB2= GlobalB2/GlobalC2
PGBA2= GTGBA2*EGBA2
PGCA2= GTGCA2*EGCA2
PGCB2= GTGCB2*EGCB2
# The Efficiency gap coincides in all models
EG2= cbind(EGBA2,EGCA2,EGCB2)
# [16] results and adjusted CDMI for example 1
CDMI2= cbind(CDBA2,CDCA2,CDCB2)
TG2= cbind(TGBA2,TGCA2,TGCB2)
CDMIadj2= cbind(CDBA2adj,CDCA2adj,CDCB2adj)
TGadj2= cbind(TGBA2adj,TGCA2adj,TGCB2adj)
# Camanho et al. [17]. Calculate the pseudo-panel Malmquist index
# using the metafrontier as reference technology
ASMI2= cbind(PGBA2meta,PGCA2meta,PGCB2meta)
TGASMI2= cbind(TGBA2meta,TGCA2meta,TGCB2meta)
# This paper. The new Productivity Gap Index and the Global Technical Change
PGI2= cbind(PGBA2,PGCA2,PGCB2)
GTG2= cbind(GTGBA2,GTGCA2,GTGCB2)
# Results of Table 3, example 2 example2_results= list(EG2 = EG2, CDMI2 = CDMI2, TG2 = TG2, CDMIadj2 = CDMIadj2,
TGadj2 = TGadj2, ASMI2 = ASMI2, TGASMI2 = TGASMI2,
PGI2 = PGI2, GTG2 = GTG2)
```

Data availability

Data will be made available on request.

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