Optics and Lasers in Engineering 41 (2004) 127–137

OPTICS and LASERS in ENGINEERING

Measurement of retardation in digital photoelasticity by load stepping using a sinusoidal least-squares fitting

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Received 20 February 2002; received in revised form 20 May 2002; accepted 4 June 2002

Abstract

The use of digital photoelasticity permits us to determine the distribution of principal stress difference by means of the analysis of a photoelastic fringe pattern using a phase measurement method. However, conventional phase measurement methods for fringe pattern analysis require the application of an unwrapping process which commonly fails in the presence of discontinuities. To alleviate this problem, load-stepping methods have been developed. We present an alternative load-stepping algorithm that is based on a nonlinear sinusoidal least-squares fitting. The description of this technique together with its verification on simulated and real experiments are presented in this work.

Keywords: Digital photoelasticity; Retardation; Load-stepping; Non-linear sinusoidal least-squares fitting

1. Introduction

Photoelasticity is a widely used method for stress analysis to test engineering structures [1,2]. In this method of analysis the central instrument is the so-called polariscope that is used to study the polarisation state of the light after crossing the specimen under test. In photoelasticity, phase measuring algorithms are commonly used to analyse the stress from a photoelastic fringe pattern. Several methods based on phase-measuring techniques for analysing photoelastic fringe patterns have been

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proposed [3–6]. Some phase-shifting methods [3–5] require monochromatic illumination and a series of compensated fringe patterns to compute the phase. The Fourier technique [6,7] is used to introduce a carrier frequency that makes the processing simpler; however, apart from the limitations of this method due to the quartz wedge introduced in the optical set-up and the influence of isoclinics, it can be used only if the orientation of the carrier fringes coincides with the principal stress directions in the specimen. A limitation of both the above-mentioned methods is that the phase value at each image point is wrapped in the range $-\pi$ to π radians so that to calculate the stress, a further processing called phase unwrapping must be carried out.

The phase unwrapping used in phase measuring techniques becomes difficult when discontinuities are present in the phase data, for example, when the specimen has cracks, notches or pieces in contact. Recently, load-stepping methods have been developed to overcome such a problem [8–13]. A particular characteristic of a load-stepping method is that the phase is evaluated independently at each image point which permits its application on discontinuous phase fields. The variation of load on the specimen implies a variation of the irradiance at each image point. Analysing this variation of irradiance we can determine the load–stress relation. If a linear variation of load is introduced, the fluctuation of irradiance at each image point has the form of a sinusoidal function whose frequency defines the slope of the linear load–stress relation.

The works reported in Refs. [8–10] mix the characteristics of the load-stepping method with the phase-shifting methodology. The load-stepping method presented by Ng [11] uses the Fourier algorithm to calculate the isochromatic parameter. This method requires incremental loads to generate a collection of fringe images. The Fourier algorithm applied by Ng, however, has an important drawback: it does not consider the errors introduced when the sinusoidal function of the irradiance (generated by the load variation) does not correspond to an integer number of cycles, especially if it is less than one. This is because of the discrete nature of the FFT algorithm which just considers integers as multiples of the fundamental frequency.

We present an alternative load-stepping method to determine the isochromatic parameter. Our method is based on a sinusoidal least-squares fitting (SLSF). In this way the algorithm will determine the frequency that best fits to the irradiance in a continuous way even in the case that the signal had a number of cycles less than one.

This work is organised in the following way. In Section 2 we describe the algorithm. We make a discussion of the Fourier method in Section 3. A numerical experiment and a comparison with the Fourier method appear in Section 4. Results of our method with real experiments are presented in Section 5. The conclusions are given in Section 6.

2. Description of the algorithm

Let us consider using the configuration of a diffuse-light circular polariscope as shown in Fig. 1. In this configuration circularly polarised light is used to obtain the

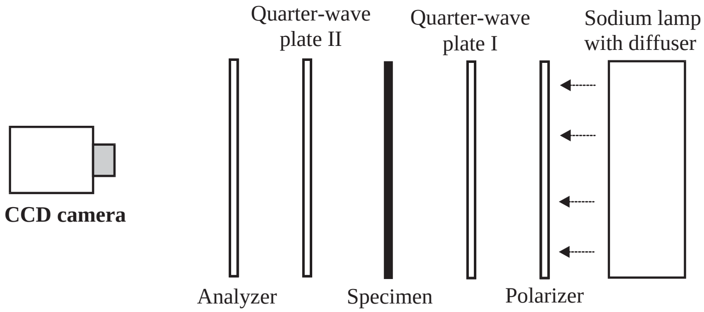

Fig. 1. Configuration of the circular polariscope for load-stepping photoelasticity.

stress information. In this case the analyzer makes an angle of $\pm\pi/4$ rad relative to the axes of plate II. The intensities of the dark and bright field isochromatics can be represented by Eqs. (1a) and (1b), respectively (see for example [11]).

$$I_{\mathrm{d}}(x,y;F) = a(x,y) + b(x,y)\sin^2\left(\frac{\delta(x,y;F)}{2}\right), \tag{1a}$$

$$I_{\mathrm{b}}(x,y;F) = a(x,y) + b(x,y)\cos^2\left(\frac{\delta(x,y;F)}{2}\right). \tag{1b}$$

In both equations, a and b represent the background intensity and the amplitude modulation, respectively. The variable δ represents the phase retardation that depends on the object load F and the spatial coordinates x and y. Performing the subtraction of (1a) from (1b) we will obtain

$$I(x,y;F) = I_{\mathrm{b}}(x,y;F) - I_{\mathrm{d}}(x,y;F) = b(x,y)\cos(\delta(x,y;F)). \tag{2}$$

The phase retardation is related with the stress as [2]

$$\delta = \frac{2\pi h}{\lambda}C(\sigma_1 - \sigma_2), \tag{3}$$

where h represents the thickness of the specimen, λ the wavelength used, and C the stress optic coefficient; σ_1 and σ_2 stand for the principal stresses that are linearly related with the applied force F and depend on the shape of the specimen.

If we rewrite Eq. (3) as $\delta(x,y;F) = \beta(x,y)F$, where $\beta = 2\pi hC(\sigma_1 - \sigma_2)/(\lambda F)$ and supposing that we make a linear load increment in such a way that we can represent the sequence as $F_n = \mu(n-1)$ where $n = 1,2,3,\ldots,N$, $\mu = F_{\max}/(N-1)$ and $F_{\max}$ represents the maximum load to be applied; we can now represent Eq. (2) as a set of images for n different loads as

$$I_n = b\cos(\beta F_n), \quad n = 1,2,3,\ldots,N. \tag{4}$$

In this equation, the spatial dependence is present but it was dropped for simplicity. We can observe that, to evaluate the retardation, we just have to determine the value of β for every point of the image, that is, the frequency of the cosine function in Eq. (4).

We may consider the determination of parameters b and β as a nonlinear least-squares problem, that is, find the values b and β that best fit the model given by Eq. (4) to the data $[F_n, I_n]$. In this case the least-squares problem is defined as [14]

$$\text{minimise} \quad U(\beta) = \sum_{n=1}^{N} [\cos(\beta F_n) - \hat{I}_n]^2. \tag{5}$$

For simplicity, we use $\hat{I}_n(b = 1)$ that is a normalised version of I_n which facilitates the minimisation operation. In our experiments, we obtained good results using a single normalisation based on the determination of the amplitude of the signal.

Minimisation of U as given by Eq. (5) is a nonlinear problem that eliminates the possibility of applying direct methods. Function (5) can be minimised by applying an iterative algorithm such as the Newton method [14].

$$\beta^{k+1} = \beta^k - \frac{U'(\beta^k)}{U''(\beta^k)}, \tag{6}$$

where U' and U'' represent the first and second derivatives, respectively.

Once we calculate β for every image point, we can compute the retardation with the desired value of load F. It can be observed that the one-dimensional unwrapping process is not required as with the Fourier one if the SLSF method is applied.

An important detail when the SLSF method was applied in the experiments is that the image subtraction described in Eq. (2) would not be necessary if we had used just the bright field represented by Eq. (1b). In this case we used a simple algorithm to normalise the signal in the range $[-1,1]$ to eliminate the a and b terms of Eq. (1b). Using just the bright fields we did not introduce noticeable errors in the calculation. This is an important advantage because we reduce the quantity of images and the movements of the analyzer in the polariscope.

In load stepping it is important to have it into a count that if the extension of the signal represented by (4) has a large number of cycles (larger range of load increments) the measurement will have a bigger accuracy. On the other hand, the sampling must be done with phase increments of less than π radians [11], that is, with a sampling bigger than the Nyquist frequency. As the phase depends on the stress difference, it is not possible to predict if the phase will be sampled correctly, however, if we capture the image sequence starting from the maximum load to the decrease of the force applied, we can estimate the minimum number of images multiplying by two the number of fringes in the field.

3. Discussion

In the load-stepping method by Ng [11], the parameter of interest is the phase of the sinusoidal signal generated when the load is incremented. The load-stepping method proposed by Ng detects the phase by use of the discrete Fourier transform. In this technique one of the side lobes of the Fourier spectrum of the signal is filtered out, then, an inverse Fourier transformation is applied to obtain a complex function.

In this way, the phase is computed using the arctan function. Finally, the continuous phase is obtained applying a phase unwrapping process.

If we consider a sinusoidal signal with frequency ω_0, its discrete Fourier transform is a pair of impulses at $+\omega_0$ and $-\omega_0$. In the treatment of sinusoidal signals using the DFT, windowing plays an important role because it smears or broadens the impulses in the theoretical Fourier representation and thus the exact frequency is less sharply defined. The spectral sampling inherent in the DFT is also important because it has the effect of giving a misleading or inaccurate shape of the true spectrum of the sinusoidal signal [15].

We will analyse the Fourier procedure to detect the phase of a sinusoidal signal. Supposing that we can represent the signal as the sequence

$$\begin{aligned} I_n = \cos\left(\frac{2\pi}{N}k_c n\right) &= \frac{1}{2}\exp\left(\mathrm{i}\frac{2\pi}{N}k_c n\right) \\ &+ \frac{1}{2}\exp\left(-\mathrm{i}\frac{2\pi}{N}k_c n\right), \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{7}$$

where k_c defines the number of cycles. Then the discrete Fourier transform of this sequence is represented by

$$\begin{aligned} \tilde{I}_k &= \sum_{n=0}^{N-1} I_n \exp\left(-\mathrm{i}\frac{2\pi}{N}kn\right) \\ &= \frac{1}{2}\sum_{n=0}^{N-1}\exp\left(-\mathrm{i}\frac{2\pi}{N}(k-k_c)n\right) \\ &+ \frac{1}{2}\sum_{n=0}^{N-1}\exp\left(-\mathrm{i}\frac{2\pi}{N}(k+k_c)n\right). \end{aligned} \tag{8}$$

When the second term of the right side of the equation is filtered out we obtain

$$\tilde{I}_k^{\mathrm{Filtered}} = \frac{1}{2}\sum_{n=0}^{N-1}\exp\left(-\mathrm{i}\frac{2\pi}{N}(k-k_c)n\right), \tag{9}$$

and the inverse Fourier transform of this expression is represented as

$$I_n^{\mathrm{Filtered}} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{I}_k^{\mathrm{Filtered}}\exp\left(\mathrm{i}\frac{2\pi}{N}kn\right). \tag{10}$$

Finally, the phase of the signal is computed using

$$\phi_n = \tan^{-1}\left(\frac{\mathrm{Im}\{I_n^{\mathrm{Filtered}}\}}{\mathrm{Re}\{I_n^{\mathrm{Filtered}}\}}\right). \tag{11}$$

To avoid errors caused by the addition of harmonics of different frequency, we are particularly interested in the case when I_n^{Filtered} represents a single complex exponential signal. Analysing Eq. (9) we can see that undesirable harmonics vanishes when k_c = integer which is the following particular case:

$$\tilde{I}_k^{\text{Filtered}} \approx \begin{cases} \dfrac{N}{2} & \text{if } (k = k_c), \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

It has been analysed why the Fourier method can introduce errors when a non-proper windowing is used, that is, when the extension of the signal under test does not correspond to an integer number of cycles. To diminish this error Quiroga and Gómez-Pedrero [16] proposed a preprocessing algorithm based on interpolation and cropping to reduce the error when the Fourier algorithm is applied in temporal evaluation of fringe patterns with carrier. An important improvement can be obtained by applying this algorithm in load-stepping photoelasticity, however, this improved method cannot evaluate the phase when the signal has less than one cycle.

4. Numerical experiments

As mentioned above, the load-stepping method proposed by Ng [11] does not consider an important error source caused by the application of the Fourier method, that is, when the extension of the signal represented by Eq. (4) is not an integer number. A more critical problem is when the extension is less than one cycle. In this case, for the discrete Fourier algorithm, the fundamental frequency cannot be determined properly. On the other hand, the application of the SLSF method detects the frequency of the signal in a continuous way even when the signal extension has less than one cycle.

To test our method we have simulated a photoelastic experiment using the model of a circular disc under diametral compression to evaluate the relative retardation. The dimensions of the disc were of 25 mm radius and 5 mm thickness. The value of the stress-optic coefficient and the wavelength were $51.6 \times 10^{-12}\,\text{m}^2/\text{N}$ and $546 \times 10^{-9}\,\text{m}$, respectively. The image size in the simulation was 200×200 pixels with 256 grey levels. The loads simulated ranged from 0 to 430 N at increments of 10 N. The image of the bright field for $n = 44$ (430 N) is shown in Fig. 2.

Fig. 3 shows the phase retardation along lines A–B of Fig. 2. In the graphic, the theoretical phase retardation is compared with the obtained one using the Fourier algorithm [16] and with the SLSF. It can be seen that the Fourier algorithm cannot work properly in zones of low stress, that is, where the signal extension is less than one cycle. As an example, in Fig. 4 we can observe that in this zone (point C of Fig. 2) the signal extension does not have a complete cycle which produces an error on applying the Fourier algorithm. The maximum error in this experiment using the proposed method was 0.006 rad. It can be seen in the results that, theoretically, there is no minimum phase retardation that this method can detect, however, in practice, it will depend mainly on the experimental conditions.

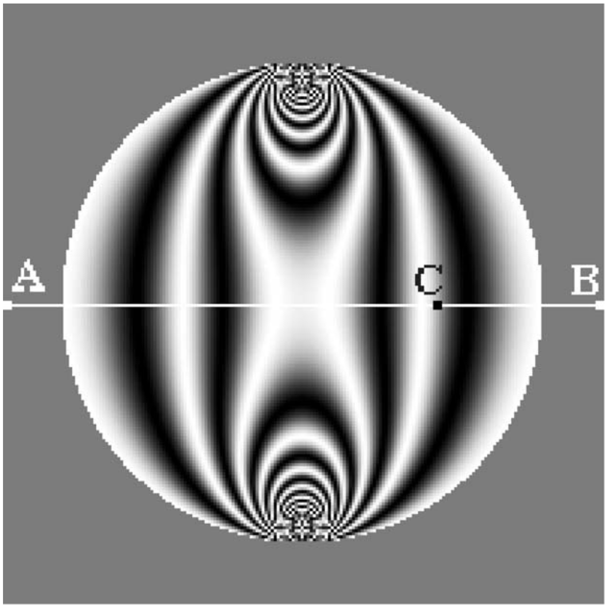

Fig. 2. Bright field for $n = 44$ (430 N) of the simulated experiment.

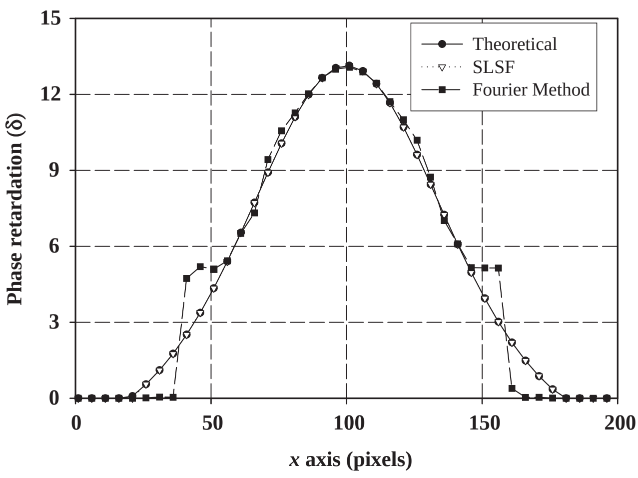

Fig. 3. A profile of the theoretical relative retardation compared with that obtained using the SLSF and the Fourier method.

5. Results with real experiments

We tested the proposed method with images obtained in a real experiment. The description of the specimen under analysis is shown in Fig. 5. The value of the

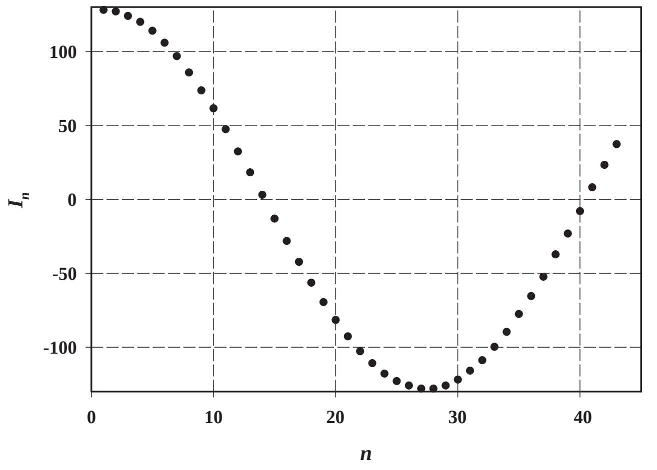

Fig. 4. Intensity variation in point C of Fig. 2. It can be seen that the signal has no complete cycle.

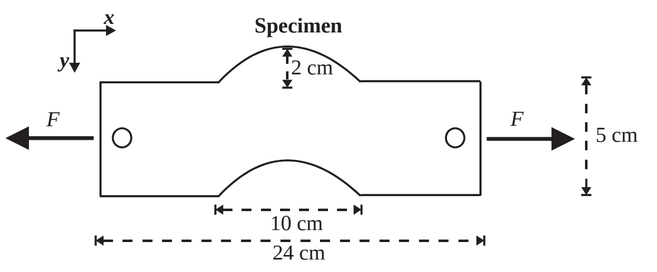

Fig. 5. Description of the transparent object under analysis for a real experiment. The thickness in this case is $h = 3.18\,\mathrm{mm}$.

stress-optic coefficient of the material was $C = 84.1 \times 10^{-12}\,\mathrm{m^2/N}$. We used a TIEDEMANN polariscope model AE-131 in the circular configuration as described in Section 2. The light source used was a sodium lamp of $\lambda = 589\,\mathrm{nm}$. The CCD camera was a JAI model CV-M10 with 50 mm focal length lens. The image size used was of 256×150 pixels with 256 grey levels. We computed the phase retardation using a sequence of images with load ranging from 0 to 430 N at increments of 10 N.

Fig. 6 shows the bright field generated when a load of 430 N was applied on the specimen. Of course, the greater the force applied, the higher is the number of fringes. Fig. 7 shows the mesh of the phase retardation obtained with the SLSF method for a load of 430 N. A profile of the phase retardation along lines A–B in Fig. 6, recovered with the SLSF and the Fourier method, is shown in Fig. 8. It can be observed that in zones of low frequency the Fourier method could not compute the phase properly.

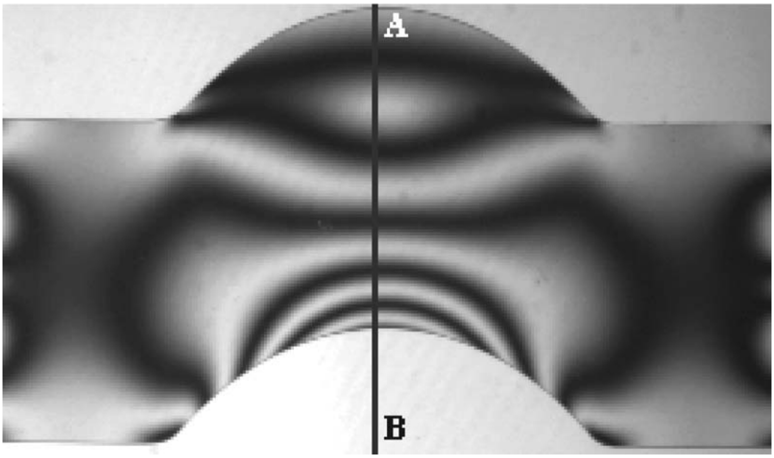

Fig. 6. Bright field isochromatic generated when a load of 430 N was applied on the specimen.

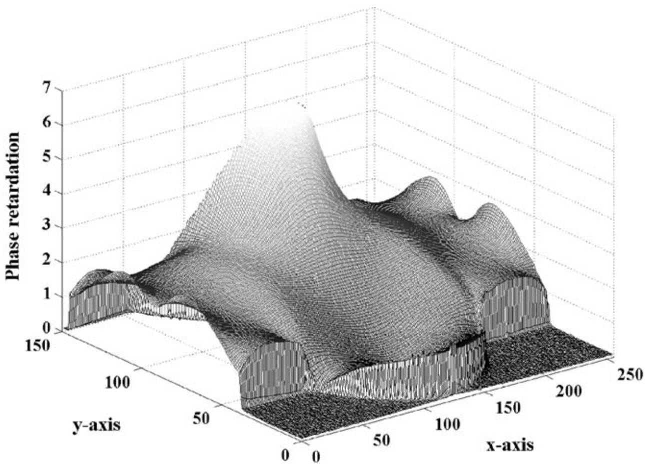

Fig. 7. Distribution of the phase retardation obtained with the SLSF method.

6. Conclusions

We have proposed a load-stepping algorithm to determine the retardation in digital photoelasticity. The proposed method has the advantage compared with the Fourier one that it is not strongly influenced by the conditions of symmetry of the signal; furthermore, it can properly process the signal when it has less than one cycle. We have made a simulated experiment to verify the performance of the proposed algorithm and to compare it with the Fourier algorithm. We have also tested our method on images obtained in a real experiment.

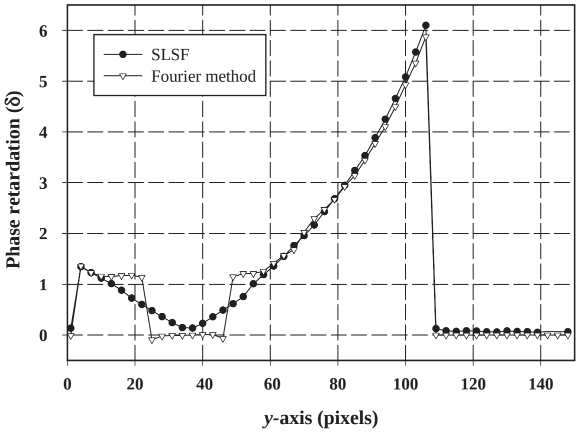

Fig. 8. Profile of the phase retardation along lines A–B of Fig. 6, recovered with the SLSF and the Fourier method. It can be observed that Fourier method cannot properly evaluate the retardation in low frequency zones.

Acknowledgements

We acknowledge the support of the Consejo Nacional de Ciencia y Tecnología (México) and the Centro de Ingeniería y Desarrollo Industrial (Querétaro, México) for the postdoctoral grant of J. Villa. We also wish to thank the financial support of the European Union, project INDUCE, BRPR-CT97-0805 and Universidad Complutense de Madrid, project PR48/01-9858.

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