



p -Groups of automorphisms of compact non-orientable Riemann surfaces

E. Bujalance¹ · F. J. Cirre¹ · J. M. Gamboa²

Received: 3 February 2021 / Accepted: 10 July 2021
© The Author(s) 2021, corrected publication 2021

Abstract

We study p -groups of automorphisms of compact non-orientable Riemann surfaces of topological genus $g \geq 3$. We obtain upper bounds of the order of such groups in terms of p , g and the minimal number of generators of the group. We also determine those values of g for which these bounds are sharp. Furthermore, the same kind of results are obtained when the p -group acts as the full automorphism group of the surface.

Keywords Non-orientable compact Riemann surfaces · p -Groups · NEC groups · Cyclic and dihedral groups · Riemann–Hurwitz formula

Mathematics Subject Classification Primary 30F10; Secondary 14H55

Contents

1	Introduction
2	Preliminaries
3	Case p does not divide $g - 2$
4	Case G cyclic
5	Case G dihedral
6	General case
	References

E. Bujalance and F. J. Cirre partially supported by Ministerio de Ciencia, Innovación y Universidades (PGC2018-096454-B-I-00). J. M. Gamboa supported by Ministerio de Economía, Industria y Competitividad (MINECO MTM2017-82105-P and Grupos UCM 910444).

✉ J. M. Gamboa
jmgamboa@ucm.es

E. Bujalance
eb@mat.uned.es

F. J. Cirre
jcirre@mat.uned.es

¹ Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED, c/Senda del Rey s/n, 28040 Madrid, Spain

² Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

1 Introduction

A fundamental problem in all areas of Mathematics is to decide whether two objects X and Y are equivalent. An approach to obtain negative answers is to check that their groups of automorphisms $\text{Aut}(X)$ and $\text{Aut}(Y)$ are not isomorphic. This applies, in particular, to the category of compact Klein surfaces introduced in [1], which explains, among many other reasons, that the study of their automorphism groups constitutes a central problem in the theory of Klein surfaces.

By Poincaré's uniformization theorem [24], Klein surfaces whose algebraic genus is greater than one are quotients of the upper-half complex plane \mathcal{H} under the action of a *non-Euclidean crystallographic* (NEC in short) group Λ . This leads to use combinatorial methods in the study of automorphism groups of Klein surfaces since they can be written as quotients Γ/Λ , where Γ is another NEC group containing Λ as a normal subgroup. We will use this approach in this paper. A general reference containing the basics of this procedure is the book [8].

Placing constraints on the surface (*e.g.*, to be orientable or not, with or without boundary), on the action of the group (preserving or reversing orientation) and on the group (cyclic, abelian, dihedral, p -group, etc.), breaks the general problem of determining automorphism groups of compact Klein surfaces into many different cases. Orientable Klein surfaces without boundary are classical Riemann surfaces. Their automorphism groups have been extensively studied in a combinatorial way, see for instance, the pioneer work [14] on cyclic groups by Harvey, or the survey on this topic [7].

In this paper we study p -groups of automorphisms of compact non-orientable Klein surfaces with empty boundary. We call them *non-orientable Riemann surfaces*, following the seminal work [25] by Singerman in the study of automorphism groups of this type of surfaces, see also [6]. Our goal is to obtain upper bounds of the orders of these groups. There exist several precedents of this kind of results in the literature, mainly concerned either with classical Riemann surfaces or with bordered surfaces, orientable or not. A sharp bound of its order in the orientable case was obtained by Zomorrodian in [28], who later on computed in [29] presentations for the groups occurring in the lower central series of p -groups acting on classical Riemann surfaces. A deeper study of the action of p -groups on classical Riemann surfaces is due to Kulkarni in [17], see also the extended study by Tucker in [27]. As a by-product, Kulkarni rediscovered Zomorrodian's bounds using a completely different method. These bounds involve the minimum number of generators of the groups. Much later, and using Kulkarni's results, Talu obtained in [26] an exact formula for the so called *minimum stable reduced genus* of a large class of finite abelian p -groups. As to the structure of p -groups of automorphisms of classical Riemann surfaces it is worthwhile mentioning the results by Hidalgo in [15,16].

Nilpotent groups acting on Klein surfaces with non-empty boundary were studied by May in [19]. He obtained bounds of the order of such groups, and it is worth pointing out that they are reached for 2-groups and for orientable surfaces. For non-orientable surfaces the bound was sharpened in [10], where the case of surfaces with empty boundary is also considered. Later on, May and Zimmerman determined in a series of papers, [20–22], the smallest genus of all classical Riemann surfaces on which a given group acts as a group of orientation-preserving automorphisms. Some of our results in this paper can be seen as the non-orientable counterpart of [22], where the authors are just concerned with p -groups in case p is odd. Closely related are the results in [9] where, based on previous work by

Kulkarni and Maclachlan in [18], the minimum topological genus of bordered Klein surfaces admitting a prime-power automorphism is calculated.

The contents of the paper are the following. In Sect. 2 we recall the main results on NEC groups and groups of automorphisms of Klein surfaces to be used in the paper. Some results on extendability of group actions are also included. In Sect. 3 we prove that if a p -group G acts as a group of automorphisms on a compact non-orientable Riemann surface of topological genus $g \geq 3$ and $g - 2$ is not a multiple of p then G is either cyclic or dihedral. Sects. 4 and 5 deal with cyclic and dihedral groups respectively, without the above restriction on g . In both cases we obtain upper bounds of the order of G in terms of p and g . Finally, in Sect. 6 we consider the general case of non-cyclic p -groups, and obtain upper bounds of the order of the group in terms of p , g and the minimal number of generators of the group. In each section, the same kind of results are obtained when the corresponding group acts as the full group of all automorphisms of the surface. The bounds are attained by infinitely many values of g and also not attained by infinitely many values of g , except for non-cyclic 2-groups generated by two elements. In this last unsolved case, it is an interesting problem to find sharp bounds attained by infinitely many values of g .

All throughout the paper, g will denote an integer greater than or equal to 3, and S a compact non-orientable Riemann surface of topological genus g .

2 Preliminaries

Let \mathcal{H} denote the upper complex half plane. A *non-Euclidean crystallographic group* (shortly NEC group) is a discrete subgroup Γ of the group $\text{Aut}(\mathcal{H})$ of automorphisms of \mathcal{H} such that the quotient \mathcal{H}/Γ is compact. Let X be a compact Klein surface of *algebraic genus* bigger than one (see the definition below). By Poincaré’s uniformization theorem [24], there exists an NEC group Λ such that $X = \mathcal{H}/\Lambda$. In addition, Λ can be chosen to be a *surface group*, meaning that it has no nonidentity orientation preserving elements of finite order. Each group G of automorphisms of X is finite and isomorphic to the quotient Γ/Λ , where Γ is an NEC group containing Λ as a normal subgroup. Then X/G and \mathcal{H}/Γ are equivalent in the category of compact Klein surfaces. The algebraic structure of Γ is encoded by its *signature*

$$\sigma(\Gamma) := (\gamma; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}). \tag{2.1}$$

The above signature indicates that the surface \mathcal{H}/Γ has topological genus γ , its boundary has k connected components and \mathcal{H}/Γ is orientable if the sign of $\sigma(\Gamma)$ is “+” and non-orientable otherwise. The integer $\eta\gamma + k - 1$, where $\eta := 2$ if \mathcal{H}/Γ is orientable and $\eta := 1$ otherwise, is the algebraic genus of \mathcal{H}/Γ .

Each m_i is called a *proper period*, each n_{ij} is called a *link period*, and each tuple $(n_{i1}, \dots, n_{is_i})$ is called a *period cycle* of Γ . It is well-known that Γ can be generated by one of the following two sets of elements:

$$\begin{aligned} &\{x_\ell, e_i, c_{ij}, a_t, b_t \mid 1 \leq \ell \leq r, 1 \leq i \leq k, 0 \leq j \leq s_i, 1 \leq t \leq \gamma\}, \\ &\{x_\ell, e_i, c_{ij}, d_t \mid 1 \leq \ell \leq r, 1 \leq i \leq k, 0 \leq j \leq s_i, 1 \leq t \leq \gamma\}. \end{aligned}$$

The first set generates Γ if sign $(\sigma(\Gamma)) = “+”$ and the second set generates Γ if sign $(\sigma(\Gamma)) = “-”$. These sets are called *canonical systems of generators* of Γ . Sometimes we commit an obvious abuse of language by saying that an element of such a set is a *canonical generator* of Γ .

In either case, the canonical generators satisfy the relations

$$\{x_\ell^{m_\ell} = 1, c_{i s_i} = e_i^{-1} c_{i 0} e_i, c_{i, j-1}^2 = c_{i j}^2 = (c_{i, j-1} c_{i j})^{n_{ij}} = 1\},$$

where $1 \leq \ell \leq r, 1 \leq i \leq k, \text{ and } 0 \leq j \leq s_i, \text{ and the additional relation (usually called the long relation)}$

$$\prod_{\ell=1}^r x_\ell \cdot \prod_{i=1}^k e_i \cdot \prod_{t=1}^\gamma a_t b_t a_t^{-1} b_t^{-1} = 1 \text{ if sign } (\sigma(\Gamma)) = \text{“+”},$$

$$\prod_{\ell=1}^r x_\ell \cdot \prod_{i=1}^k e_i \cdot \prod_{t=1}^\gamma d_t^2 = 1 \text{ if sign } (\sigma(\Gamma)) = \text{“−”}.$$

The transformations c_{ij} and d_t are reflections and glide reflections respectively. The other canonical generators preserve the orientation of \mathcal{H} . Therefore, an orientation-reversing element of Γ is a product of canonical generators containing an odd number of reflections and glide reflections.

The *reduced area* of a group Γ whose signature is (2.1) is the rational number

$$\bar{\mu}(\Gamma) := \eta\gamma + k - 2 + \sum_{\ell=1}^r \left(1 - \frac{1}{m_\ell}\right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=0}^{s_i} \left(1 - \frac{1}{n_{ij}}\right), \tag{2.2}$$

where $\eta = 2$ if $\text{sign } (\sigma(\Gamma)) = \text{“+”}$ and $\eta = 1$ if $\text{sign } (\sigma(\Gamma)) = \text{“−”}$. There exists an NEC group Γ whose signature is (2.1) if and only if the right hand side of (2.2) is positive. To abbreviate we define the *reduced area of a signature* σ , denoted by $\bar{\mu}(\sigma)$, as the reduced area of an NEC group whose signature is σ .

If Λ is a surface NEC group then its signature has the form

$$\sigma(\Lambda) = (\gamma'; \pm; [-]; \{(-), k', (-)\}),$$

and its reduced area is $\bar{\mu}(\Lambda) = \eta'\gamma' + k' - 2$. Applying the Riemann–Hurwitz formula to the natural projection $X := \mathcal{H}/\Lambda \rightarrow X/G = \mathcal{H}/\Gamma$ we get $\bar{\mu}(\Lambda) = \bar{\mu}(\Gamma) \cdot [\Gamma : \Lambda]$, that is,

$$\eta'\gamma' + k' - 2 = |G| \cdot \left(\eta\gamma + k - 2 + \sum_{\ell=1}^r \left(1 - \frac{1}{m_\ell}\right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=0}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right), \tag{2.3}$$

where $|G| = [\Gamma : \Lambda]$ denotes the order of G .

A way to construct compact non-orientable Riemann surfaces of topological genus $g \geq 3$ admitting a given group G as a group of automorphisms is to define an epimorphism $\theta : \Gamma \rightarrow G$ from an NEC group Γ onto G whose kernel has signature

$$\sigma(\ker \theta) = (g; -; [-]; \{-\}).$$

Observe that Γ may have elements of finite order but $\ker \theta$ is torsion-free. So the epimorphism θ must preserve the orders of the elements of Γ of finite order. The sign in $\sigma(\ker \theta)$ is “−” if the conditions in [8, Thm. 2.1.2] if p is odd, or in [8, Thm. 2.1.3] if $p = 2$ are fulfilled. In this last case the conditions depend on the existence of *non-orientable words of Γ with respect to $\ker \theta$* . If $\ker \theta$ has the above signature then $S := \mathcal{H}/\ker \theta$ is a non-orientable Riemann surface of topological genus g admitting G as a group of automorphisms. We say that θ is a *non-orientable surface kernel epimorphism*. We also say that *the group G acts with signature $\sigma(\Gamma)$* .

In this article we are also concerned with the problem of finding upper bounds of the order of p -groups acting as the full group $\text{Aut}(S)$ of all automorphisms of a surface S . To that end we recall briefly the notions of *maximal NEC signature* and *maximal NEC group*.

Let Γ be an NEC group containing a surface NEC group Λ as a normal subgroup. Then the group $G := \Gamma/\Lambda$ acts as a group of automorphisms on the compact surface $S := \mathcal{H}/\Lambda$. Suppose that G does not coincide with the full automorphism group $\text{Aut}(S)$ of S . Then there exists an NEC group Γ' containing Γ and normalising Λ , so Γ'/Λ is a group of automorphisms of S larger than G . If Γ and Γ' have the same Teichmüller dimension, then their signatures appear in the lists of finite index inclusions of NEC groups given by the first author [2], see also [8, Thm. 2.4.7], for the case where Γ is a normal subgroup of Γ' , and by Estévez and Izquierdo [13, Table 4] for the case where Γ is not normal in Γ' .

If a pair (σ, σ') of signatures occurs in any of these lists, then for any NEC group Γ with signature $\sigma(\Gamma) = \sigma$ there exists another NEC group Γ' with signature $\sigma(\Gamma') = \sigma'$ containing Γ with finite index. In this case, if the surface subgroup Λ is normal in Γ' then the action of $G = \Gamma/\Lambda$ on $S = \mathcal{H}/\Lambda$ is extendable to the action of the larger group Γ'/Λ , whereas if Λ is not normal in Γ' then no such extension via Γ' is possible.

On the other hand, if a signature σ does not occur as the first entry of a pair in any of the above lists then it is said that σ is a *maximal signature*. This means, see [8, Thm. 5.1], that for every NEC group Γ' containing an NEC group Γ with signature σ , the equality of the dimensions of the Teichmüller spaces of Γ and Γ' implies the equality $\Gamma = \Gamma'$. In such a case it is proved in [8, Thm. 5.1.2] that there exists a *maximal NEC group* Γ with signature $\sigma(\Gamma) = \sigma$, i.e., Γ is not a subgroup of finite index of any other NEC group. Therefore, the action of $G = \Gamma/\Lambda$ cannot be extended, and $G = \text{Aut}(S)$.

The next result, which follows from [4, Thm 2.1], deals with the case of cyclic groups. It will be useful in Section 4.

Lemma 2.1 *Let G be a cyclic group acting with signature $(2; -; [t]; \{-\})$, $(1; -; [t]; \{(-)\})$, $(0; +; [t]; \{(-), (-)\})$, $(1; -; [t, u]; \{-\})$ or $(0; +; [t, u]; \{(-)\})$ on a non-orientable Riemann surface S . Then G is not the full group $\text{Aut}(S)$.*

3 Case p does not divide $g - 2$

A careful examination of Riemann–Hurwitz formula (2.3), as done by Kulkarni in [17] and Tucker in [27], provides the following result on the structure of p -groups of automorphisms of non-orientable surfaces.

Proposition 3.1 *Let G be a p -group of automorphisms of a compact non-orientable Riemann surface S of topological genus g . Assume that p does not divide $g - 2$.*

- (1) *If p is odd then G is cyclic.*
- (2) *If $p = 2$ then G is either cyclic or dihedral.*

Proof Let us write $S = \mathcal{H}/\Lambda$ where Λ is a surface NEC group with $\sigma(\Lambda) = (g; -; [-]; \{-\})$ and $G = \Gamma/\Lambda$ for an NEC group Γ containing Λ as a normal subgroup. Let us write $|G| := p^\alpha$ for a positive integer α , and let $\theta : \Gamma \rightarrow G$ be a non-orientable surface kernel epimorphism with $\Lambda = \ker \theta$.

(1) This is a consequence of [17, Prop. 3.1.(a)], but we give here a self-contained proof whose arguments will be used later. As p is odd and Λ has no torsion elements, the group Γ contains no element of even order. Therefore it contains no reflection but it contains some

non-orientable word. Thus $\text{sign}(\Gamma) = “-”$ and $\sigma(\Gamma) = (\gamma; -; [m_1, \dots, m_r]; \{-\})$ with $\gamma > 0$ and $m_i := p^{\alpha_i}$ where $1 \leq \alpha_i \leq \alpha$. Riemann–Hurwitz formula (2.3) then yields

$$g - 2 = p^\alpha \left(\gamma - 2 + \sum_{i=1}^r \left(1 - \frac{1}{p^{\alpha_i}} \right) \right) = p^\alpha (\gamma - 2) + \sum_{i=1}^r (p^\alpha - p^{\alpha - \alpha_i}).$$

Suppose that $r = 0$ or $\alpha > \alpha_i$ for all $i = 1, \dots, r$. Then the right hand side of the equality would be divisible by p . This is impossible because, by hypothesis, $g - 2$ is not. Therefore there exists an index $i_0 \in \{1, \dots, r\}$ such that $\alpha_{i_0} = \alpha$ and this implies that G is generated by the image of the elliptic element $x_{i_0} \in \Gamma$, of order $p^{\alpha_{i_0}} = p^\alpha$. Thus, G is cyclic.

(2) If $p = 2$ then either $\sigma(\Gamma) = (\gamma; -; [m_1, \dots, m_r]; \{-\})$ or

$$\sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}),$$

with $m_i := 2^{\alpha_i} \leq 2^\alpha$ in both cases, and $n_{ij} := 2^{\alpha_{ij}}$ for some $1 \leq \alpha_{ij} \leq \alpha$ in the second one. Actually, $\alpha_{ij} \leq \alpha - 1$ since otherwise the images of the corresponding pair of canonical reflections would generate a dihedral group of order $2 \cdot 2^\alpha > |G|$.

For the first signature the same arguments employed in case (1), now with $p = 2$, allow us to conclude that G is cyclic. For the second one, Riemann–Hurwitz formula yields

$$g - 2 = 2^\alpha (\eta\gamma + k - 2) + \sum_{i=1}^r (2^\alpha - 2^{\alpha - \alpha_i}) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} (2^\alpha - 2^{\alpha - \alpha_{ij}}),$$

where $\eta = 1$ or 2 . As in the preceding discussion, if there exists $i_0 \in \{1, \dots, r\}$ such that $\alpha_{i_0} = \alpha$ then G is cyclic.

If there does not exist such i_0 then the sum $\sum_{i=1}^r (2^\alpha - 2^{\alpha - \alpha_i})$ is even, and since the left-hand-side $g - 2$ is odd, the double sum $\frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} (2^\alpha - 2^{\alpha - \alpha_{ij}})$ is odd. Therefore, there exist i_0, j_0 such that $\alpha_{i_0 j_0} = \alpha - 1$. The corresponding pair of canonical reflections generates a dihedral group of order $2 \cdot 2^{\alpha_{i_0 j_0}} = 2^\alpha = |G|$, so the group G is dihedral. \square

Remarks 3.2 (1) Even if p divides $g - 2$ there exist compact non-orientable Riemann surfaces S of topological genus g admitting p -groups of automorphisms which are either cyclic or dihedral. For example, for $p = 3$ consider an NEC group Γ_1 whose signature is $\sigma(\Gamma_1) = (3; -; [3, 3]; \{-\})$ generated by three glide reflections d_1, d_2 and d_3 and two elliptic elements x_1 and x_2 of order 3. There exists a non-orientable surface kernel epimorphism $\theta : \Gamma_1 \rightarrow \mathbb{Z}_9$ induced by the assignment

$$\theta(d_1) = \bar{0}, \quad \theta(d_2) = \bar{1}, \quad \theta(d_3) = \bar{8}, \quad \theta(x_1) = \bar{3}, \quad \theta(x_2) = \bar{6}.$$

The surface $S := \mathcal{H}/\Lambda$, where $\Lambda := \ker \theta$, is non-orientable because Λ contains the non-orientable word d_1 . In addition, let g be the topological genus of S . Then $\sigma(\Lambda) = (g; -; [-]; \{-\})$ and $3 \mid (g - 2)$ because

$$g - 2 = \bar{\mu}(\Lambda) = |\mathbb{Z}_9| \cdot \bar{\mu}(\Gamma_1) = 9 \cdot \left(3 - 2 + 2 \left(1 - \frac{1}{3} \right) \right) = 21.$$

(2) Let us show next that there exist cyclic and dihedral 2-groups acting as automorphism groups on non-orientable surfaces of even topological genus g .

Consider first an NEC group Γ_2 whose signature is $\sigma(\Gamma_2) = (3; -; [2, 2]; \{-\})$ generated by three glide reflections d_1, d_2 and d_3 and two elliptic elements x_1 and x_2 of order 2. There exists a non-orientable surface kernel epimorphism $\theta : \Gamma_2 \rightarrow \mathbb{Z}_4$ induced by the assignment

$$\theta(d_1) = \bar{0}, \quad \theta(d_2) = \bar{1}, \quad \theta(d_3) = \bar{3}, \quad \theta(x_1) = \theta(x_2) = \bar{2}.$$

The surface $S := \mathcal{H}/\Lambda$, where $\Lambda := \ker \theta$, is non-orientable because Λ contains the non-orientable word d_1 . In addition, let g be the topological genus of S . Then $\sigma(\Lambda) = (g; -; [-]; \{-\})$ and $2|(g - 2)$ because

$$g - 2 = \bar{\mu}(\Lambda) = |\mathbb{Z}_4| \cdot \bar{\mu}(\Gamma_2) = 4 \cdot \left(3 - 2 + 2 \left(1 - \frac{1}{2} \right) \right) = 8.$$

Finally, consider the non-orientable surface kernel epimorphism $\theta : \Gamma_2 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathcal{D}_2$ induced by the assignment

$$\theta(d_1) = (\bar{0}, \bar{0}), \quad \theta(d_2) = (\bar{1}, \bar{0}), \quad \theta(d_3) = (\bar{0}, \bar{1}), \quad \theta(x_1) = \theta(x_2) = (\bar{1}, \bar{0}).$$

The surface $S := \mathcal{H}/\Lambda$, where $\Lambda := \ker \theta$, has even topological genus 10 and it is non-orientable since Λ contains the non-orientable word d_1 .

(3) There exist non-orientable Riemann surfaces of topological genus $g \equiv 2 \pmod{p}$ with non-cyclic and non-dihedral p -groups of automorphisms, whether p is odd or $p = 2$. Examples can be found in the proof of Theorem 6.1.

4 Case G cyclic

Theorem 4.1 *Let p be a prime integer and let G be a cyclic p -group of automorphisms of a compact non-orientable Riemann surface S of topological genus g . Then*

$$|G| \leq \frac{p(g - 1)}{p - 1}.$$

In addition, the bound is attained if and only if $(g - 1)/(p - 1)$ is a p -power.

Proof (i) Case p odd. We first show that if $(g - 1)/(p - 1)$ is a p -power, say $p^{\alpha-1}$, then there exists a compact non-orientable Riemann surface of topological genus g on which $G := \mathbb{Z}_{p^\alpha}$ acts, showing therefore that the value $|G| = p(g - 1)/(p - 1)$ is attained. For that, let us consider an NEC group Γ_1 with signature

$$(1; -; [p^\alpha, p]; \{-\})$$

and canonical set of generators $\{d_1, x_1, x_2\}$. Let $\theta : \Gamma_1 \rightarrow \mathbb{Z}_{p^\alpha}$ be the epimorphism given by $\theta(d_1) = \bar{a}$, $\theta(x_1) = \bar{1}$ and $\theta(x_2) = \bar{-p^{\alpha-1}}$, where $a := (p^{\alpha-1} - 1)/2$. As $\theta(x_1)$ and $\theta(x_2)$ have orders p^α and p respectively, $\ker \theta$ is torsion-free. Furthermore, it contains the non-orientable word $d_1^{p^\alpha}$. So $S := \mathcal{H}/\ker \theta$ is a compact non-orientable Riemann surface on which \mathbb{Z}_{p^α} acts. Its topological genus g satisfies, by the Riemann–Hurwitz formula, $g - 2 = |G|(1 - \frac{1}{p^\alpha} - \frac{1}{p})$, so $g = p^\alpha - p^{\alpha-1} + 1$ and $|G| = p(g - 1)/(p - 1)$, as claimed.

We now show that this upper bound cannot be improved. Let S be a compact non-orientable Riemann surface of topological genus g with a cyclic p -group \mathbb{Z}_{p^β} of automorphisms acting on it. Let $\theta : \Gamma \rightarrow \mathbb{Z}_{p^\beta}$ be the corresponding non-orientable surface kernel epimorphism, with $S = \mathcal{H}/\ker \theta$. As p is odd, Γ contains no element of even order and, in particular, it contains no reflection. Then $\text{sign}(\Gamma) = \text{“-”}$ because Γ contains orientation reversing elements, that is, $\sigma(\Gamma) = (\gamma; -; [m_1, \dots, m_r]; \{-\})$ with $\gamma > 0$ and $m_i := p^{\beta_i}$ with $1 \leq \beta_i \leq \beta$. Riemann–Hurwitz formula (2.3) yields

$$g - 2 = |G| \cdot (\gamma - 2) + |G| \cdot \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right).$$

If $\gamma > 2$ then

$$|G| \leq |G| \cdot (\gamma - 2) = g - 2 - |G| \cdot \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \leq g - 2.$$

So the bound $\frac{p(g-1)}{p-1}$ is not attained.

If $\gamma = 2$ then $r > 0$ for $\sigma(\Gamma)$ to have positive area. The smallest reduced area of Γ is attained for $r = 1$ and $m_1 = p$, and it equals $1 - 1/p$. So

$$g - 2 = |G| \cdot \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \geq |G| \cdot \left(1 - \frac{1}{p}\right),$$

and therefore $|G| \leq \frac{p(g-2)}{p-1}$. Also in this case the bound $\frac{p(g-1)}{p-1}$ is not attained.

If $\gamma = 1$ then $r > 1$ for $\sigma(\Gamma)$ to have positive area. We claim that at least one elliptic generator has maximal order p^β . Otherwise, the images $\theta(x_1), \dots, \theta(x_r)$ under the non-orientable surface kernel epimorphism $\theta : \Gamma \rightarrow \mathbb{Z}_{p^\beta}$ would belong to the unique subgroup of \mathbb{Z}_{p^β} of order $p^{\beta-1}$. The same would happen to $\theta(d_1)^2 = (\theta(x_1) \cdots \theta(x_r))^{-1}$, and also to $\theta(d_1)$ because the order of $\theta(d_1)$ is odd. Hence, θ would not be onto. Therefore we may assume, without loss of generality, that $m_1 := p^\beta$, that is, $\sigma(\Gamma) = (1; -; [p^\beta, m_2, \dots, m_r]; \{-\})$. The smallest reduced area of Γ is attained for $r = 2$ and $m_2 = p$, and it equals $1 - 1/p^\beta - 1/p$. Thus the largest value of $|G|$ would satisfy

$$g - 2 = |G| \cdot \left(1 - \frac{1}{p^\beta} - \frac{1}{p}\right) = |G| \cdot \left(1 - \frac{1}{p}\right) - 1,$$

that is, $|G| = \frac{p(g-1)}{p-1}$, which is the bound given in the statement. Observe that g has to be of the form $p^\beta - p^{\beta-1} + 1$. This proves the proposition when p is odd.

(ii) Case $p = 2$. As above, we first show that the bound $2(g - 1)$ in the statement is attained if g is of the form $g = 2^{\alpha-1} + 1$. For that, let us consider an NEC group Γ_2 with signature

$$(0; +; [2^\alpha, 2]; \{-\})$$

and canonical set of generators $\{x_1, x_2, c_0\}$. Let $\theta : \Gamma_2 \rightarrow \mathbb{Z}_{2^\alpha}$ be the epimorphism given by $\theta(x_1) = \bar{1}$ and $\theta(x_2) = \theta(c_0) = \bar{2}^{\alpha-1}$. Clearly, $\ker \theta$ is torsion-free and contains the non-orientable word $x_2 c_0$. So $S := \mathcal{H} / \ker \theta$ is a compact non-orientable Riemann surface on which \mathbb{Z}_{2^α} acts. By the Riemann–Hurwitz formula, its topological genus is $g = 2^{\alpha-1} + 1 = \frac{|G|}{2} + 1$. So $|G| = 2(g - 1)$, as claimed.

We now show that this upper bound cannot be improved. Let S be a compact non-orientable Riemann surface of topological genus g with a cyclic 2-group \mathbb{Z}_{2^β} of automorphisms acting on it. Let $\theta : \Gamma \rightarrow \mathbb{Z}_{2^\beta}$ be the corresponding non-orientable surface kernel epimorphism, with $S = \mathcal{H} / \ker \theta$. The signature of Γ has no link period since otherwise the images by θ of the corresponding pair of canonical reflections would generate a dihedral subgroup within \mathbb{Z}_{2^β} .

Assume first that Γ contains no reflection. Then the signature of Γ is $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ with $\gamma > 0$ and $m_i = 2^{\beta_i}$ where $1 \leq \beta_i \leq \beta$. If $\gamma \geq 2$ then the same arguments as in the case p odd show that the bound $2(g - 1)$ is not attained. So we assume that $\gamma = 1$. By the Riemann–Hurwitz formula, if $|G| \geq 2(g - 1)$ then $\bar{\mu}(\Gamma) \leq (g - 2)/(2g - 2) < 1/2$, that is,

$$-1 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) < \frac{1}{2}.$$

Since $1 - 1/m_i \geq 1/2$ for each i , and since $r \geq 2$ for $\sigma(\Gamma)$ to have positive area, it follows that the unique admissible solution to this inequality (up to a relabelling of the periods) is $r = 2$, $m_1 := 2^{\beta_1}$ and $m_2 := 2$. So $\sigma(\Gamma) = (1; -, [2^{\beta_1}, 2]; \{-\})$. We claim that m_1 has to attain the maximal value 2^β . Suppose, to get a contradiction, that $\beta_1 < \beta$. Then the images of x_1 and x_2 by θ would be the classes in \mathbb{Z}_{2^β} of even numbers. So the image of the canonical glide reflection d_1 would be the class of an odd number for it to generate \mathbb{Z}_{2^β} . Now, any non-orientable word $w \in \Gamma$ contains an odd number of occurrences of d_1 (because the other two canonical generators preserve orientation). So its image $\theta(w)$ in \mathbb{Z}_{2^β} is the class of an odd number and therefore cannot be trivial. This prevents $\ker \theta$ from having non-orientable words, a contradiction. Therefore $m_1 = 2^\beta$, as claimed. So $\sigma(\Gamma) = (1; -, [2^\beta, 2]; \{-\})$ and the Riemann–Hurwitz formula yields

$$g - 2 = |G| \cdot \bar{\mu}(1; -, [2^\beta, 2]; \{-\}) = |G| \left(\frac{1}{2} - \frac{1}{|G|}\right) = \frac{|G|}{2} - 1.$$

Hence $|G| = 2(g - 1)$, and $g - 1$ is a 2-power. The bound is not improved.

We assume now that Γ contains some reflection, so $\sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \dots, k, (-)\})$, with $k > 0$. Again, if $|G| \geq 2(g - 1)$ then $\bar{\mu}(\Gamma) \leq (g - 2)/(2g - 2) < 1/2$, that is,

$$\eta\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) < \frac{1}{2}.$$

Hence $\eta\gamma + k = 1$ or 2 (recall that $k > 0$).

(ii.1) If $\eta\gamma + k = 2$ then $r > 0$ for $\sigma(\Gamma)$ to have positive area. The smallest reduced area is attained for $r = 1$ and $m_1 = 2$, and it equals $1/2$. Consequently, $g - 2 \geq \frac{|G|}{2}$, and thus the largest bound is not attained for this type of signatures.

(ii.2) If $\eta\gamma + k = 1$ then $\sigma(\Gamma) = (0; +; [m_1, \dots, m_r]; \{(-)\})$ with $r \geq 2$. Since Γ is generated by a reflection and elliptic elements, at least one of the latter has maximal order 2^β . We may assume that $m_1 := 2^\beta$. The smallest reduced area is attained for $r = 2$ and $m_2 := 2$ and it equals $\bar{\mu}(\Gamma) = \frac{1}{2} - \frac{1}{2^\beta}$. Thus the largest value of $|G|$ satisfies

$$g - 2 = |G| \cdot \left(\frac{1}{2} - \frac{1}{2^\beta}\right) = \frac{|G|}{2} - 1,$$

that is, $|G| = 2(g - 1)$, which is again the bound given in the statement. Observe that $g - 1$ has to be of the form $2^{\beta-1}$. This proves the proposition when $p = 2$. □

Remarks 4.2 (1) The action of the largest cyclic 2-group \mathbb{Z}_{2^α} of automorphisms of a non-orientable Riemann surface is unique. By this we mean that the signature with which it acts is unique and the corresponding non-orientable surface kernel epimorphism is also unique, up to an automorphism of \mathbb{Z}_{2^α} . To prove this claim, observe that the above proof of Theorem 4.1 shows that the signatures with which \mathbb{Z}_{2^α} may act are $(0; +; [2^\alpha, 2]; \{(-)\})$ and $(1; -, [2^\alpha, 2]; \{-\})$. We claim that the second signature has to be ruled out. Suppose, to get a contradiction, that there exists a non-orientable surface kernel epimorphism $\theta : \Gamma \rightarrow \mathbb{Z}_{2^\alpha}$ from an NEC group with this signature. Since $\theta(x_1)$ generates the whole group, we may assume, without loss of generality, that $\theta(x_1) = \bar{1}$. Let us write $\theta(d_1) = \bar{a}$. Since $\theta(x_2) = \bar{2}^{\alpha-1}$ (the unique element of order 2 in \mathbb{Z}_{2^α}), it follows from the long relation $x_1 x_2 d_1^2 = 1$ that

$\bar{1} + \bar{2}^{\alpha-1} + 2\bar{a} = \bar{0}$. This is impossible because the left hand side of the equality is the class of an odd number. This shows that \mathbb{Z}_{2^α} only acts with signature $(0; +; [2^\alpha, 2]; \{-\})$. Furthermore, the non-orientable surface kernel epimorphism is unique since $\theta(x_2) = \theta(c_0) = \bar{2}^{\alpha-1}$ (the unique element of order 2) and, up to an automorphism of \mathbb{Z}_{2^α} , we may assume $\theta(x_1) = \bar{1}$.

(2) We claim that a non-orientable surface with the largest cyclic 2-group $G := \mathbb{Z}_{2(g-1)}$ of automorphisms is hyperelliptic. Denote $g - 1 := 2^{\alpha-1}$. To show our claim we apply Theorem 2.2.4 in [8] together with Riemann–Hurwitz formula to obtain that the preimage $\theta^{-1}(\langle \bar{2}^{\alpha-1} \rangle)$ of the unique involution in G has signature $(1; -; [2, .^s., 2]; \{-\})$. Since $\theta^{-1}(\langle \bar{2}^{\alpha-1} \rangle)$ contains $\ker \theta$ with index two, it follows from [11, Thm. 2.2] that the surface $S := \mathcal{H}/\ker \theta$ is hyperelliptic.

We can describe S by means of algebraic equations. To that end we consider the double covering S^+ of S , see [1] or [8, Construction 0.1.12]. It is a compact Riemann surface admitting an antianalytic involution τ such that $S = S^+/\langle \tau \rangle$. Every group G of automorphisms of S is isomorphic to a group G_{S^+} of conformal automorphisms of S^+ which commute with τ . The topological genus g^* of S^+ is the algebraic genus of S . As S is non-orientable and its boundary is empty, $g^* = g - 1$ in this situation. Hyperelliptic Riemann surfaces admitting antianalytic involutions are studied in detail in [5]. It follows from Theorem 3.4.7.d) in [5] that S^+ and τ can be described as follows:

$$S^+ : y^2 = x(x^{2g-2} - \lambda x^{g-1} + 1) \quad \text{for some real } \lambda \neq \pm 2, \text{ and } \tau(x, y) := \left(\frac{-1}{x}, \frac{\bar{y}i}{x^g} \right).$$

The cyclic group $\mathbb{Z}_{2(g-1)}$ is generated by the automorphism

$$f(x, y) = (x\xi_{2g-2}^2, y\xi_{2g-2}),$$

where $\xi_{2g-2} := e^{i\pi/(g-1)}$. The equation describes the family of non-orientable surfaces with the largest cyclic 2-group of automorphisms as a uniparametric family of real dimension one. On the other hand, observe that $\langle f \rangle = \mathbb{Z}_{2(g-1)}$ is not the full group of automorphisms of the Klein surface $S = S^+/\langle \tau \rangle$ since the involution given by $h(x, y) = (1/x, y/x^g)$ is also an automorphism of S^+ which commutes with τ . We will discuss this question in Proposition 4.3 with more generality.

If p is odd then the above proof of Theorem 4.1 shows that $(1; -; [p^\alpha, p]; \{-\})$ is the unique signature with which \mathbb{Z}_{p^α} may act. But, unlike the case $p = 2$, the epimorphism is not unique. In fact, although we may assume that $\theta(x_1) = \bar{1}$, the image $\theta(x_2)$ may be any of the $p - 1$ classes $k\bar{p}^{\alpha-1}$ of elements of order p , where $k \in \{1, \dots, p - 1\}$. For each choice of $\theta(x_2)$ we define $\theta(d_1) = \bar{a}$ where $a := (-1 - kp^{\alpha-1})/2$. So there are $p - 1$ inequivalent epimorphisms.

Next we obtain upper bounds for the order of those cyclic groups acting as the full automorphism group of a compact non-orientable Riemann surface.

Proposition 4.3 *Let G be a cyclic p -group that is the full automorphism group of a compact non-orientable Riemann surface of topological genus g . Then*

$$|G| \leq \frac{p(g-1)}{2(p-1)}.$$

The bound is attained if and only if $(g - 1)/(2p - 2)$ is a p -power.

Proof (i) Case p odd. Suppose that the cyclic group $G := \mathbb{Z}_{p^\alpha}$ acts as the full automorphism group of a non-orientable Riemann surface. By Lemma 2.1 and Theorems 3.5, 3.6 and 3.7 in

[3], the maximum value for $|G|$ is reached by a non-orientable surface kernel epimorphism $\theta : \Gamma_1 \rightarrow \mathbb{Z}_{p^\alpha}$, where

$$\sigma(\Gamma_1) = (1; -; [p, p, p^\alpha], \{-\}).$$

Explicitly, if $\{d, x_1, x_2, x_3\}$ is a canonical set of generators of Γ_1 , where d is a glide reflection and x_1, x_2 and x_3 are elliptic elements, then the epimorphism θ is induced by the assignment

$$\theta(d) = \bar{1}, \quad \theta(x_1) = \bar{p}^{\alpha-1}, \quad \theta(x_2) = -\bar{p}^{\alpha-1}, \quad \theta(x_3) = -\bar{2}.$$

Note that $\theta(d)$ generates \mathbb{Z}_{p^α} , so θ is surjective, and $S := \mathcal{H}/\ker \theta$ is a non-orientable surface since $\ker \theta$ contains the non-orientable word d^{p^α} . Let g be the topological genus of S . Then,

$$g - 2 = \bar{\mu}(\ker \theta) = |G| \cdot \bar{\mu}(\Gamma_1) = p^\alpha \left(-1 + 2 \left(1 - \frac{1}{p} \right) + 1 - \frac{1}{p^\alpha} \right) = 2p^{\alpha-1}(p - 1) - 1,$$

that is, $g - 1 = 2(p - 1) \cdot |G|/p$. Thus,

$$|G| \leq \frac{p(g - 1)}{2(p - 1)},$$

as claimed. It follows from the proof that the equality holds if and only if $g = 2p^{\alpha-1}(p - 1) + 1$. In addition, since $(1; -; [p^\alpha, p, p]; \{-\})$ is a maximal NEC signature, see [8, Thm. 2.4.7], the NEC group Γ_1 can be chosen maximal by [8, Thm. 5.1.2], so G is the full group of automorphisms of the surface S .

(ii) Case $p = 2$. We first show that the bound $g - 1$ in the statement is attained if $g - 1$ is a 2-power, say 2^α .

For that, let Γ_2 be a maximal NEC group with maximal signature $(0; +; [2^\alpha, 2, 2]; \{(-)\})$, and let $\{x_1, x_2, x_3, c\}$ be a canonical set of generators of Γ_2 . Let $\theta : \Gamma_2 \rightarrow \mathbb{Z}_{2^\alpha}$ be the epimorphism defined by $\theta(x_1) = \bar{1}, \theta(x_2) = \theta(x_3) = \theta(c) = \bar{2}^{\alpha-1}$. It is easy to see that θ is a non-orientable surface kernel epimorphism, so $S := \mathcal{H}/\ker \theta$ is a non-orientable Riemann surface on which \mathbb{Z}_{2^α} acts. The maximality of Γ_2 assures that $G = \mathbb{Z}_{2^\alpha}$ is the full group $\text{Aut}(S)$ of automorphisms of S . Its topological genus g satisfies $g - 2 = 2^\alpha(1 - \frac{1}{2^\alpha})$. So $|G| = g - 1$, as claimed.

We now show that this bound cannot be improved. Let S be a compact non-orientable Riemann surface of topological genus g whose full group of automorphisms is $\text{Aut}(S) = \mathbb{Z}_{2^\beta}$. Let $\theta : \Gamma \rightarrow \mathbb{Z}_{2^\beta}$ be the corresponding non-orientable surface kernel epimorphism, with $S = \mathcal{H}/\ker \theta$. Observe that the signature of Γ has no link period, so $\sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(-)^k\})$ with $m_i := 2^{\beta_i}$ and $1 \leq \beta_i \leq \beta$. By the Riemann–Hurwitz formula, if $|\text{Aut}(S)| \geq g - 1$ then $\bar{\mu}(\Gamma) \leq (g - 2)/(g - 1) < 1$, that is,

$$1 > \eta\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \geq \eta\gamma + k - 2 + \frac{r}{2},$$

where in the last inequality we have used that $1 - \frac{1}{m_i} \geq \frac{1}{2}$. Hence $\eta\gamma + k = 1$ or 2 (it cannot be $\eta\gamma + k = 0$ because Γ has orientation reversing elements).

If $\eta\gamma + k = 2$ then $r = 1$ ($r \neq 0$ for Γ to have positive area) and $(\eta, \gamma, k) \neq (2, 1, 0)$ (for Γ to have orientation reversing elements). The unique possible NEC signatures are $(2; -; [t]; \{-\})$, $(1; -; [t]; \{(-)\})$ or $(0; +; [t]; \{(-), (-)\})$. But a cyclic group acting with any of these signatures on a non-orientable surface cannot be its full group of automorphisms, by Lemma 2.1.

If $\eta\gamma + k = 1$ then $r = 2$ or 3 ($r \neq 0, 1$ for Γ to have positive area). The value $r = 2$ gives signature either $(1; -; [t, u]; \{-\})$ or $(0; +; [t, u]; \{(-)\})$. Again, Lemma 2.1 prevents

a cyclic group acting with any of these signatures to be the full group of automorphisms. It remains to deal with signatures $(1; -; [m_1, m_2, m_3]; \{-\})$ and $(0; +; [m_1, m_2, m_3]; \{(-)\})$.

The same arguments as in the proof of Theorem 4.1 show that in both cases a period, say m_1 , has to attain the maximal value 2^β . The smallest reduced area is attained for $m_2 = m_3 = 2$ and it equals $\bar{\mu}(\Gamma) = 1 - \frac{1}{2^\beta}$. Thus the largest value of $|G|$ satisfies

$$g - 2 = |G| \cdot \left(1 - \frac{1}{2^\beta}\right) = |G| - 1.$$

Hence $|G| = g - 1$, which is the bound given in the statement. Observe that $g - 1$ has to be of the form 2^β . This shows the proposition in the case $p = 2$. □

5 Case G dihedral

In this section we consider dihedral p -groups, and then obviously $p = 2$. We start by setting some notation.

Notation 5.1 Let n be an even integer and let us denote the dihedral group of order $2n$ by \mathcal{D}_n . We will use the following presentation

$$\mathcal{D}_n := \langle \rho, \tau \mid \rho^n = \tau^2 = (\tau\rho)^2 = 1 \rangle.$$

Its elements of order two are $\rho^{n/2}$, which generates the center of the group, and the elements $\tau\rho^i$ for $i = 0, \dots, n - 1$. Two commuting involutions in \mathcal{D}_n are of the form either $\{\rho^{n/2}, \tau\rho^i\}$ for some $i \in \{0, \dots, n - 1\}$, or $\{\tau\rho^i, \tau\rho^{n/2+i}\}$ for some $i \in \{0, \dots, n/2 - 1\}$.

Theorem 5.2 (1) *Let G be a dihedral 2-group of automorphisms of a compact non-orientable Riemann surface of topological genus g . Then $|G| \leq 4(g - 1)$. The bound is attained if and only if $g - 1$ is a 2-power.*

(2) *Let G be a dihedral 2-group that acts as the full automorphism group of a compact non-orientable Riemann surface of topological genus g . Then $|G| \leq 4(g - 1)$ and the bound is attained if and only if $g - 1$ is a 2-power.*

Proof (1) We first show that if $g - 1$ is a 2-power, say $2^{\alpha-1}$, then there exists a compact non-orientable Riemann surface of topological genus g on which $G := \mathcal{D}_{2^\alpha}$ acts, showing therefore that the value $|G| = 4(g - 1)$ is attained. For that, let us consider an NEC group Γ_1 with signature $(0; +; [-]; \{(2, 2, 2, 2s)\})$ where $s := g - 1$. With the notations in 5.1 let us consider the epimorphism $\theta : \Gamma_1 \rightarrow \mathcal{D}_{2s}$ induced by the assignment

$$\theta(c_0) = \tau, \quad \theta(c_1) = \rho^s, \quad \theta(c_2) = \rho^{s-1}\tau, \quad \theta(c_3) = \tau\rho,$$

where $\{c_0, c_1, c_2, c_3\}$ is a canonical set of generating reflections of Γ_1 .

It is clearly surjective and its kernel is torsion-free and contains the non-orientable word $(c_0c_3)^s c_1$. Therefore $\mathcal{H}/\ker \theta$ is a non-orientable Riemann surface, whose topological genus satisfies, by the Riemann–Hurwitz formula,

$$g - 2 = |G| \cdot \left(-1 + \frac{1}{2} \left(\frac{3}{2} + 1 - \frac{1}{2s}\right)\right) = 4s \left(\frac{1}{4} - \frac{1}{4s}\right) = s - 1,$$

so $g = s + 1$. This shows our claim.

We now show that this bound cannot be improved. Suppose, on the contrary, that a dihedral 2-group $G := \Gamma/\Lambda$ of order $|G| > 4(g - 1)$ acts on a non-orientable Riemann surface \mathcal{H}/Λ of topological genus g . Since $\overline{\mu}(\Lambda) = \overline{\mu}(g; -, [-]; \{-\}) = g - 2$, we get

$$\overline{\mu}(\Gamma) = \frac{\overline{\mu}(\Lambda)}{|G|} = \frac{g - 2}{|G|} < \frac{g - 2}{4(g - 1)} = \frac{1}{4} - \frac{1}{4(g - 1)} < \frac{1}{4}.$$

Let us write

$$\sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

with $m_i := 2^{\alpha_i} \leq |G|/2$ and $n_{ij} := 2^{\alpha_{ij}} \leq |G|/2$ because $|G|/2$ is the maximum order of the elements of G . Since $m_i \geq 2$ and $n_{ij} \geq 2$ we get $1 - \frac{1}{m_i} \geq \frac{1}{2}$ and $1 - \frac{1}{n_{ij}} \geq \frac{1}{2}$, so

$$\frac{1}{4} > \overline{\mu}(\Gamma) = \eta\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i,j} \left(1 - \frac{1}{n_{ij}}\right) \geq \eta\gamma + k - 2 + \frac{r}{2} + \frac{t}{4},$$

where t is the total number of link periods. Thus

$$\eta\gamma + k + \frac{r}{2} + \frac{t}{4} < \frac{9}{4}.$$

Hence $\eta\gamma + k \leq 2$, and in fact $\eta\gamma + k \in \{1, 2\}$ since the values $\gamma = k = 0$ do not yield a proper NEC group. If $\eta\gamma + k = 2$ then $\frac{r}{2} + \frac{t}{4} < \frac{1}{4}$, so $r = t = 0$, which does not produce positive area. Therefore $\eta\gamma + k = 1$. This implies $\frac{r}{2} + \frac{t}{4} < \frac{5}{4}$. If $k = 0$ then $t = 0$, so $r \leq 2$. The values $r = 0$ and 1 produce negative area. For $r = 2$ we have $\overline{\mu}(\Gamma) = 1 - \frac{1}{m_1} - \frac{1}{m_2}$. It is easy to see that there is no pair $(m_1, m_2) = (2^{\alpha_1}, 2^{\alpha_2})$ satisfying $0 < \overline{\mu}(\Gamma) < \frac{1}{4}$. So $k = 1$. The pairs (r, t) satisfying $2r + t < 5$ are

$$(r, t) = (2, 0), (1, 2), (1, 1), (1, 0), (0, 4), (0, 3), (0, 2), (0, 1), (0, 0).$$

We discard the pairs $(1, 0)$, $(0, 2)$, $(0, 1)$ and $(0, 0)$ because they do not produce positive area. An easy case by case analysis shows that the unique possible signatures of Γ are the following:

- (i) $(0; +; [2]; \{(2, 2^a)\})$ with $a > 1$.
- (ii) $(0; +; [2^a]; \{(2)\})$ with $a > 2$.
- (iii) $(0; +; [4]; \{(2^a)\})$ with $a > 1$.
- (iv) $(0; +; [-]; \{(2, 2, 2, 2^a)\})$ with $a > 1$.
- (v) $(0; +; [-]; \{(4, 4, 2^a)\})$ with $a \geq 2$.
- (vi) $(0; +; [-]; \{(2, 2^{a+1}, 2^{b+1})\})$ with $1 < a \leq b$.

We will show that none of these signatures provides a dihedral 2-group of order larger than $4(g - 1)$ acting on a non-orientable Riemann surface of topological genus g .

(i) Suppose that there exists a non-orientable surface kernel epimorphism $\theta : \Gamma \rightarrow \mathcal{D}_{2s}$ with $\sigma(\Gamma) = (0; +; [2]; \{(2, 2^a)\})$ for some $a > 1$, and let $\{x_1, c_0, c_1, c_2\}$ be a canonical set of generators of Γ . As $c_2 = x_1c_0x_1$, the generator c_2 is superfluous, and so the image \mathcal{D}_{2s} is generated by the three involutions $\theta(x_1)$, $\theta(c_0)$ and $\theta(c_1)$. The involution $\theta(c_0)$ cannot be the central element ρ^s since otherwise $\theta(c_2) = \theta(x_1)\theta(c_0)\theta(x_1) = \theta(c_0) = \rho^s$ and $\theta(c_1)\theta(c_2)$ would not have order $2^a > 2$. So $\theta(c_0) = \tau\rho^i$ for some $0 \leq i \leq s$, and we may assume, up to an automorphism in \mathcal{D}_{2s} , that $\theta(c_0) = \tau$.

As $\theta(c_1)$ commutes with $\theta(c_0)$, either $\theta(c_1) = \rho^s$ or $\theta(c_1) = \tau\rho^s$. The first possibility is ruled out because $\theta(c_1)\theta(c_2)$ has order $2^a > 2$. So $\theta(c_1) = \tau\rho^s$.

Finally, the involution $\theta(x_1)$ cannot be ρ^s since otherwise θ would not be onto. So $\theta(x_1) = \tau\rho^i$ with i odd for θ to be onto. Let $w := w(x_1, c_0, c_1)$ be a non-orientable word in Γ . This means that the total number of occurrences of c_0 and c_1 in w is odd. If the number of

occurrences of x_1 is even then the total number of occurrences of τ in $\theta(w)$ is odd. This prevents $\theta(w)$ from being the identity. If the number of occurrences of x_1 is odd then the total number of occurrences of ρ in $\theta(w)$ is odd because s is even. This prevents $\theta(w)$ from being the identity. Therefore, $\ker \theta$ contains no non-orientable word, a contradiction.

(ii) and (iii) We study just case (iii) since case (ii) is analogous. Suppose there exists a non-orientable surface kernel epimorphism $\theta : \Gamma \rightarrow \mathcal{D}_{2^s}$ with $\sigma(\Gamma) = (0; +; [4]; \{(2^a)\})$ for some $a > 1$. The group Γ is generated by the elliptic element x_1 of order 4 and the reflection c_0 satisfying $\text{ord}(c_0 x_1 c_0 x_1^{-1}) = 2^a$. Notice that $\langle \rho^{s/2} \rangle$ is the only cyclic subgroup of \mathcal{D}_{2^s} of order 4. Therefore $\theta(x_1) \in \langle \rho^{s/2} \rangle$ and $\theta(c_0)$ is an involution of the form $\tau \rho^i$. A non-orientable word $w := w(x_1, c_0)$ in Γ has an odd number of occurrences of c_0 and so $\theta(w)$ has an odd number of occurrences of τ . Thus $\theta(w) \neq 1$ and $\ker \theta$ contains no non-orientable word, a contradiction.

(iv) We constructed above a non-orientable surface kernel epimorphism from an NEC group Γ with signature $(0; +; [-]; \{(2, 2, 2, 2^a)\})$ onto the dihedral group \mathcal{D}_{2^a} of order 2^{a+1} which, by the Riemann–Hurwitz formula, equals $4(g - 1)$. We now show that 2^{a+1} is the largest possible order of a dihedral group acting with this signature on a non-orientable Riemann surface.

Suppose, to get a contradiction, that there exists such an epimorphism $\theta : \Gamma \rightarrow \mathcal{D}_{2^b}$ with $b > a$. The involution $\theta(c_0)$ cannot be the central element $z := \rho^{2^{b-1}}$ in \mathcal{D}_{2^b} because $\theta(c_0)$ does not commute with $\theta(c_3)$, since both are involutions whose product is not an involution. So we may assume, up to an automorphism of \mathcal{D}_{2^b} , that $\theta(c_0) = \tau$. Since $\theta(c_1)$ commutes with $\theta(c_0)$, either $\theta(c_1) = z$ or $\theta(c_1) = \tau z$. Since $\theta(c_3)\theta(c_0)$ has order $2^a > 2$, it is $\theta(c_3) = \tau \rho^i$ for some i such that $\text{gcd}(2^b, i) = 2^{b-a}$. In particular, i is even. Finally, $\theta(c_2)$ has to be of the form $\tau \rho^j$ for some odd j since otherwise θ would not be onto. But any such involution does not commute with $\theta(c_3)$, see Notation 5.1, a contradiction.

(v) and (vi) Let c_0, c_1 and c_2 be the canonical reflections generating an NEC group Γ with signature either $(0; +; [-]; \{(4, 4, 2^a)\})$ with $a \geq 2$, or $(0; +; [-]; \{(2, 2^{a+1}, 2^{b+1})\})$ with $1 < a \leq b$. Suppose there exists a non-orientable surface kernel epimorphism $\theta : \Gamma \rightarrow \mathcal{D}_{2^s}$. If each involution $\theta(c_i)$ is of the form $\tau \rho^j$ then $\ker \theta$ does not contain a non-orientable word. Thus $\theta(c_i) = \rho^s$ for some $i = 0, 1, 2$. So the products $\theta(c_i c_{i+1})$ and $\theta(c_{i-1} c_i)$ would have order 2, and this is false since there is no pair of link periods in $\sigma(\Gamma)$ equal to 2.

(2) This statement follows straightforwardly from part (1) since, by [8, Thm. 2.4.7] and [13, Table 4], the signature $(0; +; [-]; \{(2, 2, 2, 2^a)\})$ providing the bound $4(g - 1)$ is a maximal signature. So the group Γ can be chosen as a maximal NEC group, by [8, Thm. 5.1.2], and this assures that $G := \Gamma/\Lambda$ is the full automorphism group of the surface \mathcal{H}/Λ . \square

6 General case

In Sect. 3 we studied the case when p does not divide $g - 2$, where g is the topological genus of the non-orientable surface, and showed in Proposition 3.1 that any p -group of automorphisms is either cyclic or dihedral. In this section we remove this arithmetic restriction and consider non-cyclic p -groups G acting on non-orientable surfaces. If p is odd then $|G| \leq p(g - 2)/(p - 2)$, and the bound is attained for infinitely many values of g , as [10, Thm. 5.3] shows. In this section we refine this bound in the sense that we determine it in terms of the minimal number of generators of G . We describe the infinitely many values of g for which the bound is attained. The case $p = 2$ is also considered.

Theorem 6.1 *Let G be a non-cyclic p -group of automorphisms of a compact non-orientable Riemann surface of topological genus g , and let $\ell \geq 2$ be the minimum number of generators of G .*

(1) *If p is odd then*

$$|G| \leq \frac{p(g - 2)}{\ell(p - 1) - p}.$$

The bound is attained if and only if $(g - 2)/(\ell(p - 1) - p)$ is a p -power.

(2) *If $p = 2$ then*

$$|G| \leq \begin{cases} 8(g - 2) & \text{if } \ell = 2 \text{ or } 3, \\ \frac{4(g - 2)}{\ell - 3} & \text{if } \ell > 3. \end{cases}$$

If $\ell = 2$ then the bound is attained if $g = 3$, if $\ell = 3$ then it is attained if and only if $g - 2$ is a 2-power, and if $\ell > 3$ then the bound is attained if and only if $(g - 2)/(\ell - 3)$ is a 2-power.

Proof (1) We first show that if $(g - 2)/(\ell(p - 1) - p)$ is a p -power, say p^{n-1} , then there exist a non-orientable Riemann surface S of topological genus g and a non-cyclic p -group G of order p^n generated by ℓ elements but not less such that $G < \text{Aut}(S)$. This will show that the bound in the statement is attained.

If $p \geq 5$ then it follows from Theorem 2.0.1 in [28] that for each integer $m > 1$ there exists a p -group H_m of order p^m generated by two elements a and b of order p whose product also has order p . Let G be the direct product $G := \mathbb{Z}_p \oplus \overset{\ell-2}{\oplus} \mathbb{Z}_p \oplus H_{n-\ell+2}$, which is generated by ℓ elements but not less, and has order p^n . For each $i \in \{1, \dots, \ell - 2\}$ we denote

$$u_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}_p \oplus \overset{\ell-2}{\oplus} \mathbb{Z}_p, \text{ with } 1 \text{ in the } i\text{-th place.} \tag{6.1}$$

Let Γ_1 be an NEC group with signature $(1; -; [p, \overset{\ell}{\cdot}, p]; \{-\})$ and let d, x_1, \dots, x_ℓ be a canonical set of generators of Γ_1 , where d is a glide reflection and each x_i is an elliptic element. Then the assignment

$$\begin{aligned} x_i &\mapsto (u_i, 1) \text{ for } i = 1, \dots, \ell - 2, & x_{\ell-1} &\mapsto (0, \overset{\ell-2}{\cdot}, 0, a), & x_\ell &\mapsto (0, \overset{\ell-2}{\cdot}, 0, b), \\ d &\mapsto ((p - 1)/2, \overset{\ell-2}{\cdot}, (p - 1)/2, (ab)^{(p-1)/2}) \end{aligned}$$

defines an epimorphism $\theta : \Gamma_1 \rightarrow G$ whose kernel is torsion-free and contains the non-orientable word d^p . This makes G a non-cyclic p -group of automorphisms of the non-orientable Riemann surface $S := \mathcal{H}/\ker \theta$. Its topological genus g satisfies $(g - 2)/(\ell(p - 1) - p) = |G|/p$, which is a p -power, as claimed.

If $p = 3$ then for each integer $m > 1$ there exists a 3-group H_m of order 3^m generated by two elements a and b of order 3 whose product has order 3. To show this claim we use the description by Coxeter in [12] of the $(l, m \mid n, k)$ and $(l, m, n; q)$ groups. The group $(3, 3 \mid 3, 3^t)$, with presentation $\langle a, b \mid a^3 = b^3 = (ab)^3 = (a^{-1}b)^{3^t} = 1 \rangle$, has order $3(3^t)^2 = 3^{2t+1}$ [12, Section 1.6], whilst the group $(3, 3, 3; 3^t)$, with presentation $\langle a, b \mid a^3 = b^3 = (ab)^3 = (a^{-1}b^{-1}ab)^{3^t} = 1 \rangle$, has order $9(3^t)^2 = 3^{2t+2}$ [12, Section 2.7]. Putting these two families together we get, for each $m > 1$ a group H_m with the above features. Let G be the direct product $G := \mathbb{Z}_3 \oplus \overset{\ell-2}{\oplus} \mathbb{Z}_3 \oplus H_{n-\ell+2}$, which is generated by ℓ elements but not less, and has order 3^n . For each $i \in \{1, \dots, \ell - 2\}$ we denote u_i as in (6.1). Let Γ_2 be an NEC group with signature $(1; -; [3, \overset{\ell}{\cdot}, 3]; \{-\})$ and let d, x_1, \dots, x_ℓ be a canonical set of generators of Γ_2 . Then the assignment

$$x_i \mapsto (u_i, 1) \text{ for } i = 1, \dots, \ell - 2, \quad x_{\ell-1} \mapsto (0, \ell-2, 0, a), \quad x_\ell \mapsto (0, \ell-2, 0, b),$$

$$d \mapsto (u_1, \ell-2, u_{\ell-2}, ab)$$

defines an epimorphism $\theta : \Gamma_2 \rightarrow G$ whose kernel is torsion-free and contains the non-orientable word d^3 . This makes G a non-cyclic 3-group of automorphisms of the non-orientable Riemann surface $S := \mathcal{H}/\ker \theta$. Its topological genus g satisfies $(g-2)/(2\ell-3) = |G|/3$, which is a 3-power, as claimed.

Let us show that the bound is sharp for $p \geq 3$. Let G be a non-cyclic p -group of automorphisms of order p^α of a compact non-orientable Riemann surface S of topological genus g , and let $\ell \geq 2$ be the minimum number of generators of G . Let us write $S := \mathcal{H}/\Lambda$ and $G := \Gamma/\Lambda$ where $\sigma(\Lambda) = (g; -; [-]; \{-\})$. All reduces to prove that

$$\bar{\mu}(\Gamma) \geq \bar{\mu}(1; -; [p, \dots, p]; \{-\}) = -1 + \frac{\ell(p-1)}{p}.$$

As G has no elements of even order, the same happens to Γ . Consequently,

$$\sigma(\Gamma) = (\gamma; -; [m_1, \dots, m_r]; \{-\}),$$

where $m_i := p^{\alpha_i}$ for some integer α_i with $1 \leq \alpha_i \leq \alpha$. Notice that the minimum number of generators of Γ equals $\gamma + r - 1$ if $r \geq 1$ and γ if $r = 0$. So $\gamma + r - 1 \geq \ell$ if $r \geq 1$, and $\gamma \geq \ell$ if $r = 0$. Furthermore, we claim that if $r = 0$ then $\gamma \geq \ell + 1$. In fact, let us suppose, to get a contradiction, that $\sigma(\Gamma) = (\gamma; -; [-]; \{-\})$ with $\gamma \leq \ell$.

Let d_1, \dots, d_γ be a canonical set of generating glide reflections of Γ . Since $|G|$ is odd the equalities $\langle \theta(d_i)^2 \rangle = \langle \theta(d_i) \rangle$ hold for $i = 1, \dots, \gamma$. In addition, $\theta(d_\gamma)^2 = (\theta(d_1)^2 \dots \theta(d_{\gamma-1})^2)^{-1}$, so $\langle \theta(d_\gamma) \rangle = \langle \theta(d_\gamma)^2 \rangle$ is a subgroup of $\langle \theta(d_1)^2, \dots, \theta(d_{\gamma-1})^2 \rangle = \langle \theta(d_1), \dots, \theta(d_{\gamma-1}) \rangle$, i.e., $\theta(d_\gamma)$ is superfluous to generate the whole group G and therefore G can be generated just by $\gamma - 1 \leq \ell - 1$ elements, a contradiction. Thus $\gamma \geq \ell + 1$ if $r = 0$. So we may write $\gamma \geq \ell + 1 - r$ for any value of r . In order to show that $\bar{\mu}(\Gamma) \geq -1 + \ell(p-1)/p$ we distinguish two cases:

(i) If $r > \ell$ then, using that each $m_i \geq p$ and that $\gamma \geq 1$ we get

$$\begin{aligned} \bar{\mu}(\Gamma) &= \gamma - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \geq \gamma - 2 + r \left(\frac{p-1}{p}\right) \\ &\geq -1 + r \left(\frac{p-1}{p}\right) > -1 + \ell \left(\frac{p-1}{p}\right). \end{aligned}$$

(ii) If $r \leq \ell$ then, using that each $m_i \geq p$ and that $\gamma \geq \ell + 1 - r$ we get

$$\begin{aligned} \bar{\mu}(\Gamma) &= \gamma - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \geq \gamma - 2 + r \left(\frac{p-1}{p}\right) \\ &= \gamma - 2 + r - \frac{r}{p} \geq \ell - 1 - \frac{r}{p} \geq \ell - 1 - \frac{\ell}{p} = -1 + \ell \left(\frac{p-1}{p}\right). \end{aligned}$$

(2) Let S^+ be the double covering of S . As said in Remark 4.2, it is a compact Riemann surface of topological genus $g^* := g - 1$ admitting an antianalytic involution τ such that $S = S^+/\langle \tau \rangle$. Every group G of automorphisms of S is isomorphic to a group G_{S^+} of conformal automorphisms of S^+ which commute with τ .

Assume first that $\ell \geq 3$. Let G be a 2-group of automorphisms of S generated by ℓ elements but not less. As said above, G is isomorphic to a group G_{S^+} of automorphisms of

S^+ . By [17, Prop. 3.4], the order of G_{S^+} is upperly bounded by $4(g^* - 1)/(\ell - 3)$ if $\ell > 3$, and by $8(g^* - 1)$ if $\ell = 3$. Consequently,

$$|G| = |G_{S^+}| \leq \begin{cases} \frac{4(g^* - 1)}{\ell - 3} = \frac{4(g - 2)}{\ell - 3} & \text{if } \ell > 3, \\ 8(g^* - 1) = 8(g - 2) & \text{if } \ell = 3. \end{cases}$$

If $\ell = 2$ then the order of G_{S^+} is upperly bounded by $16(g^* - 1)$. However, this bound is not attained by $|G|$. In fact, since 2-groups are nilpotent, it follows from the study of nilpotent groups on non-orientable surfaces done in [10], that the order of every 2-group of automorphisms of a non-orientable surface of topological genus g is upperly bounded by $8(g - 2)$, see [10, Thm. 4.2]. In addition, the bound is attained for $g = 3$ by the dihedral group \mathcal{D}_4 , generated by $\ell = 2$ elements, acting with signature $(0; +; [-]; \{(2, 2, 2, 4)\})$, see [10, Section 4].

We now show that if $\ell > 2$ then the bound is attained for infinitely many values of g . Assume first that $\ell > 3$. We claim that if $(g - 2)/(\ell - 3)$ is a 2-power, say 2^{n-2} , then there exist a non-orientable Riemann surface S of topological genus g and a non-cyclic 2-group G of order 2^n generated by ℓ elements but not less such that $G < \text{Aut}(S)$.

Let $\mathcal{D}_{2^{n-\ell+1}}$ be the dihedral group of order $2^{n-\ell+2}$, generated by the involutions τ and $\tau\rho$, and let G be the direct product $G := \mathbb{Z}_2 \oplus \overset{\ell-2}{\mathbb{Z}_2} \oplus \mathbb{Z}_2 \oplus \mathcal{D}_{2^{n-\ell+1}}$. This is a group of order 2^n generated by ℓ elements but not less. For each $i \in \{1, \dots, \ell-2\}$ we define $u_i \in \mathbb{Z}_2 \oplus \overset{\ell-2}{\mathbb{Z}_2} \oplus \mathbb{Z}_2$ as in (6.1). Let Γ_1 be an NEC group Γ with signature $(0; +; [-], \{(2, \overset{\ell+1}{2}\})$ and canonical generators c_0, \dots, c_ℓ . We define the epimorphism $\theta : \Gamma_1 \rightarrow G$ induced by the assignment

$$\begin{aligned} c_0 &\mapsto (u_1, 1), & c_1 &\mapsto (0, \overset{\ell-2}{\mathbb{Z}_2}, 0, \tau), & c_2 &\mapsto (u_2, 1), \\ c_3 &\mapsto (0, \overset{\ell-2}{\mathbb{Z}_2}, 0, \tau\rho), & c_4 &\mapsto (u_1 + u_2, 1), \\ c_i &\mapsto (u_{i-2}, 1) \text{ for } i \geq 5. \end{aligned}$$

It is indeed an epimorphism, whose kernel is torsion free and contains the non-orientable word $c_0c_2c_4$. So $S := \mathcal{H}/\ker \theta$ is a compact non-orientable Riemann surface admitting G as a group of automorphisms. Its topological genus satisfies $g - 2 = |G|(-1 + \frac{\ell+1}{4})$, so $g = 2^{n-2}(\ell - 3) + 2$, as claimed.

We finally consider the case $\ell = 3$. If $g - 2$ is a 2-power, say 2^{n-3} , then there exist a non-orientable Riemann surface S of topological genus g and a non-cyclic group G of order 2^n generated by 3 elements but not less such that $G < \text{Aut}(S)$. To show this claim we consider, for each positive integer m , the group G with presentation

$$\begin{aligned} G &= \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 \\ &= (ad)^2c = (ac)^2 = (bd)^{2m} = 1 \rangle. \end{aligned}$$

Observe that $c = (ad)^2$ is a central element of order two, and that a centralizes $(bd)^2$. The first claim is obvious, and for the second one we use the first: $a(bd)^2a = badbda = b(ad)^{-3}bda = bda(da)^2bda = bdab(da)^2da = bdba(da)^3 = bdbd$. It follows that G is an extension of its cyclic normal subgroup N of order m generated by $(bd)^2$ by the group $\mathcal{D}_4 \times C_2 := \langle aN, dN \rangle \times \langle bN \rangle$ of order 16, and as such, G has order $16m$. Choosing $m := 2^{n-4}$ makes G a non-cyclic group of order 2^n . In addition, G is generated by three elements since the generator c is superfluous. It cannot be generated by two elements since its quotient group $\mathcal{D}_4 \times C_2$ cannot be generated by two elements.

Let Γ_1 be an NEC group with signature $(0; +; [-]; \{(2, 2, 2, 4)\})$ and let $\{c_0, c_1, c_2, c_3\}$ be a canonical set of generators of Γ_1 . It is clear that the assignment $\theta : \Gamma_1 \rightarrow G$ given by

$$c_0 \mapsto a, \quad c_1 \mapsto b, \quad c_2 \mapsto c, \quad c_3 \mapsto d$$

is an epimorphism. Since its kernel, which is torsion-free, contains the non-orientable word $(c_0c_3)^2c_2$, the orbit space $S := \mathcal{H}/\ker \theta$ is a compact non-orientable Riemann surface admitting G as a group of automorphisms. Its topological genus g satisfies $g - 2 = |G| \cdot \frac{1}{8}$, that is, $|G| = 8(g - 2)$, as claimed. \square

Corollary 6.2 *Let p and g be integers greater than 2 such that p is prime. Let G be a p -group of automorphisms of a compact non-orientable Riemann surface of topological genus g . Then*

$$|G| \leq \begin{cases} \frac{p(g-1)}{p-1} & \text{if } p \geq g, \\ \frac{p(g-2)}{p-2} & \text{if } p < g. \end{cases}$$

Proof Let G be a p -group acting on a compact non-orientable Riemann surface of topological genus $g \geq 3$ and let ℓ be the minimum number of generators of G . If G is not a cyclic group then it follows from Theorem 6.1 that

$$|G| \leq \frac{p(g-2)}{\ell(p-1)-p} \leq \frac{p(g-2)}{2(p-1)-p} = \frac{p(g-2)}{p-2}.$$

If G is cyclic then it follows from Theorem 4.1 that $|G| \leq \frac{p(g-1)}{p-1}$. So we are led to compare both bounds. A straightforward computation shows that

$$\frac{p(g-1)}{p-1} \geq \frac{p(g-2)}{p-2} \quad \text{if and only if } p \geq g.$$

This finishes the proof. \square

In order to show that the bounds given in Theorem 6.1 are attained, we have constructed non-orientable surface kernel epimorphisms from NEC groups Γ with signatures $(1; -; [p, \dots, p]; \{-\})$, $(0; +; [-]; \{(2, \ell+1, 2)\})$ and $(0; +; [-]; \{(2, 2, 2, 4)\})$ onto some p -groups G . It turns out that if $\ell \geq 3$ then these signatures are maximal, and so Γ can be chosen to be a maximal NEC group. For such a choice, the action of the p -group G cannot be extended, so G is the full group of automorphisms of the surface. This shows the following proposition.

Proposition 6.3 *Let G be a non-cyclic p -group that acts as the full automorphism group of a compact non-orientable Riemann surface S of topological genus g . Assume that the minimum number of generators of G is $\ell \geq 3$.*

- (1) *If p is odd then $|G| \leq \frac{p(g-2)}{\ell(p-1)-p}$.*
- (2) *If $p = 2$ then*

$$|G| \leq \begin{cases} \frac{4(g-2)}{\ell-3} & \text{if } \ell > 3, \\ 8(g-2) & \text{if } \ell = 3. \end{cases} \tag{6.2}$$

Remark 6.4 The category $\mathcal{F}_{\mathbb{R}}$ of algebraic function fields in one variable over \mathbb{R} is functorially equivalent to the category \mathcal{K} of compact Klein surfaces, see the Appendix of [8]. This

functorial equivalence maps each surface $S \in \mathcal{K}$ to its field $\mathcal{M}(S)$ of meromorphic functions on S , and it is proved in the Appendix of [8] that if S is non-orientable with empty boundary, then -1 is not a square in $\mathcal{M}(S)$ but it is a sum of squares in $\mathcal{M}(S)$. Indeed, it follows from the main result in [23] that in this case -1 is a sum of two squares in $\mathcal{M}(S)$.

Thus, the results in this paper provide information about the structure and upper bounds of the order of *p*-groups of automorphisms of those fields $F \in \mathcal{F}_{\mathbb{R}}$ satisfying that -1 is a sum of two squares but it is not a square. These fields F are said to have *level 2* (originally $\text{Stufe}(F) = 2$).

Acknowledgements We would like to thank the anonymous referee for pointing out a mistake in the proof of Theorem 6.1, and for correcting it.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Alling, N.L., Greenleaf, N.: Foundations of the Theory of Klein Surfaces. Lecture Notes in Mathematics, vol. 219. Springer, Berlin (1971)
2. Bujalance, E.: Normal N.E.C. signatures. III. J. Math. **26**, 519–530 (1982)
3. Bujalance, E.: Cyclic groups of automorphisms of compact non-orientable Klein surfaces without boundary. Pac. J. Math. (2) **109**, 279–289 (1983)
4. Bujalance, E., Cirre, F.J., Conder, M.D.E.: Extensions of finite cyclic group actions on non-orientable surfaces. Trans. Am. Math. Soc. **365**(8), 4209–4227 (2013)
5. Bujalance, E., Cirre, F.J., Gamboa, J.M., Gromadzki, G.: Symmetry types of hyperelliptic Riemann surfaces. Mém. Soc. Math. Fr. (N.S.) **86**, 122 (2001)
6. Bujalance, E., Cirre, F.J., Etayo, J.J., Gromadzki, G., Martínez, E.: Automorphism groups of compact non-orientable Riemann surfaces. In: Groups St Andrews 2013. London Mathematical Society Lecture Note Series, vol. 422, pp. 183–193. Cambridge University Press, Cambridge (2015)
7. Bujalance, E., Cirre, F.J., Gromadzki, G.: A survey of research inspired by Harvey’s theorem on cyclic groups of automorphisms. In: Geometry of Riemann Surfaces. London Mathematical Society Lecture Note Series, vol. 368, pp. 15–37. Cambridge University Press, Cambridge (2010)
8. Bujalance, E., Etayo, J.J., Gamboa, J.M., Gromadzki, G.: Automorphism Groups of Compact Bordered Klein Surfaces. Lecture Notes in Mathematics, vol. 1439. Springer, Berlin (1990)
9. Bujalance, E., Gamboa, J.M., Maclachlan, C.: Minimum topological genus of compact bordered Klein surfaces admitting a prime-power automorphism. Glasg. Math. J. (3) **37**, 221–232 (1995)
10. Bujalance, E., Gromadzki, G.: On nilpotent groups of automorphisms of Klein surfaces. Proc. Am. Math. Soc. (3) **108**, 749–759 (1990)
11. Bujalance, J.A.: Hyperelliptic compact nonorientable Klein surfaces without boundary. Kodai Math. J. **12**(1), 1–8 (1989)
12. Coxeter, H.S.M.: The abstract groups $G^{m,n,p}$. Trans. Am. Math. Soc. **45**(1), 73–150 (1939)
13. Estévez, J.L., Izquierdo, M.: Non-normal pairs of non-Euclidean crystallographic groups. Bull. Lond. Math. Soc. **38**, 113–123 (2006)
14. Harvey, W.J.: Cyclic groups of automorphisms of a compact Riemann surface. Q. J. Math. **17**, 86–97 (1966)
15. Hidalgo, R.A.: Conjugacy classes of automorphisms *p*-groups. Bull. Korean Math. Soc. **48**(4), 847–851 (2011)
16. Hidalgo, R.A.: *p*-groups acting on Riemann surfaces. J. Pure Appl. Algebra **222**(12), 4173–4188 (2018)

17. Kulkarni, R.S.: Symmetries of surfaces. *Topology* **26**(2), 195–203 (1987)
18. Kulkarni, R.S., Maclachlan, C.: Cyclic p -groups of symmetries of surfaces. *Glasg. Math. J.* **33**(2), 213–221 (1991)
19. May, C.L.: Nilpotent automorphism groups of bordered Klein surfaces. *Proc. Am. Math. Soc.* **101**(2), 287–292 (1987)
20. May, C.L., Zimmerman, J.: The 2-groups of odd strong symmetric genus. *J. Algebra Appl.* **9**(3), 465–481 (2010)
21. May, C.L., Zimmerman, J.: The symmetric genus of 2-groups. *Glasg. Math. J.* **55**(1), 9–21 (2013)
22. May, C.L., Zimmerman, J.: The symmetric genus of p -groups. *Commun. Algebra* **42**(10), 4402–4413 (2014)
23. Pfister, A.: Zur Darstellung von -1 als Summe von Quadraten in einem Körper. *J. Lond. Math. Soc.* **40**, 159–165 (1965)
24. Poincaré, H.: Sur l’uniformization des fonctions analytiques. *Acta Math.* **31**, 1–63 (1908)
25. Singerman, D.: Automorphisms of compact non-orientable Riemann surfaces. *Glasg. Math. J.* **12**, 50–59 (1971)
26. Talu, Y.: Abelian p -groups of symmetries of surfaces. *Taiwan. J. Math.* **15**(3), 1129–1140 (2011)
27. Tucker, T.: Finite groups acting on all but finitely many closed surfaces: Kulkarni revisited. *Acta Math. Univ. Comen. (N.S.)* **88**(2), 329–340 (2019)
28. Zomorrodian, R.: Classification of p -groups of automorphisms of Riemann surfaces and their lower central series. *Glasg. Math. J.* **29**(2), 237–244 (1987)
29. Zomorrodian, R.: The lower central series for the symmetry p -groups of a compact orientable surface. *JP J. Geom. Topol.* **3**(2), 113–147 (2003)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.